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HW3 Solution

4.3 a.

$$A := \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

The eigenvalue equation is $\boldsymbol{Ax} = \lambda \boldsymbol{x} \implies (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = 0$. The characteristic polynomial is found by the determinant

$$\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)$$

Thus, the eigenvalue of the matrix $\boldsymbol{A}$ is $\lambda = 1$ with multiplicity of 2.

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{0}$$

The eigenspace is the vector subspace spanned by a standard basis $\boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or span $\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$

b.

$$\boldsymbol{B} := \begin{bmatrix} -2 & 2 \\ 2 & 1 \end{bmatrix}$$

The characteristic polynomial is found by the determinant

$$\begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (-2-\lambda)(1-\lambda) - 4 = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

Thus, the eigenvalues of the matrix $\boldsymbol{B}$ is $\lambda_1 = -3, \lambda_2 = 2$.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{0}, \qquad E_{-3} = \text{span}\left[\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right]$$
$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{0}, \qquad E_{2} = \text{span}\left[\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right]$$

4.6 a.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial is found by the following determinant.

$$\begin{vmatrix} 2-\lambda & 3 & 0 \\ 1 & 4-\lambda & 3 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

Performing Gaussian elimination on the matrix result in

$$\frac{1}{2-\lambda}\begin{vmatrix} 1 & \frac{3}{2-\lambda} & 0 \\ 0 & \frac{\lambda^2-6\lambda+5}{2-\lambda} & 3 \\ 0 & 0 & 1-\lambda \end{vmatrix} = \frac{(\lambda-5)(\lambda-1)(1-\lambda)}{(2-\lambda)^2}$$

Thus the eigenvalues of the matrix is $\lambda_1 = 5, \lambda_2 = 1, \lambda_3 = 1$. The eigenvalue 1 has a multiplicity of 2.

$$\begin{bmatrix} -3 & 3 & 0 & 0 \\ 1 & -1 & 3 & 0 \\ 0 & 0 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_2$$
$$x_3 = 0$$

The eigenspace $E_5 = \text{span}\begin{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}\end{bmatrix}$ with $\dim E_5 = 1$

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The eigenspace $E_1 = \text{span}\begin{bmatrix}\begin{bmatrix}-3\\1\\0\end{bmatrix}\end{bmatrix}$ with $\dim E_1 = 1$.

Thus, the matrix is defective and not diagonalizable.

b.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the eigenvalue equation $\boldsymbol{Ax} = \lambda\boldsymbol{x}$, we get the determinant

$$\begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix}$$

The characteristic polynomial is $(1-\lambda)(-\lambda)^3$ with eigenvalues as its roots $\lambda_1 = 0$ with multiplicity of 3 and $\lambda_2 = 1$. Now we use these eigenvalues to get the eigenvectors.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= 0 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

Thus, eigenspace $E_1 = \operatorname{span}\left[\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right]$ with $\dim E_1 = 1$. Now using the eigenvalue of 0, we get

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -x_2 \\ x_2 &= x_2 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

Thus, the eigenspace $E_0 = \operatorname{span}\left[\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right]$ with $\dim E_0 = 3$. Since the sum of the dimensions of the eigenspaces is 4, the matrix is nondefective and diagonalizable.

4.8 Create a symmetric matrix $\boldsymbol{S} = \boldsymbol{A}^\top \boldsymbol{A}$.

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}.$$

Find the eigenvalues and their corresponding eigenvectors of this matrix.

$$\begin{vmatrix} 13-\lambda & 12 & 2 \\ 12 & 13-\lambda & -2 \\ 2 & -2 & 8-\lambda \end{vmatrix} = -\lambda(\lambda-25)(\lambda-9).$$

The eigenvalues of $\boldsymbol{S}$ is $\lambda_1 = 25, \lambda_2 = 9, \lambda_3 = 0$. The eigenvectors corresponding to these eigenvalues are

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

We see that the eigenvectors are orthogonal. Thus, $\boldsymbol{S}$ can be diagonalized as $\boldsymbol{PDP}^\top$ with each column vectors of $\boldsymbol{P}$ as one of the normalized eigenvectors and $\boldsymbol{D}$ with corresponding eigenvalues on the diagonal.

$$\begin{bmatrix} 1/\sqrt{2} & 1/3\sqrt{2} & -2/3 \\ 1/\sqrt{2} & -1/3\sqrt{2} & 2/3 \\ 0 & 4/3\sqrt{2} & 1/3 \end{bmatrix} \begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/3\sqrt{2} & -1/3\sqrt{2} & 4/3\sqrt{2} \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$

$\boldsymbol{P} = \boldsymbol{V}$ forms the right-singular vectors of the SVD.

The singular values of the SVD is made from the square root of the nonzero eigenvalues.

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$$

The left-singular vectors of the SVD can be found by the following matrix equations.

$$\boldsymbol{u}_1 = \frac{1}{\sigma_1}\boldsymbol{A}\boldsymbol{v}_1 = \frac{1}{5}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\boldsymbol{u}_2 = \frac{1}{\sigma_2}\boldsymbol{A}\boldsymbol{v}_2 = \frac{1}{3}\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1/3\sqrt{2} \\ -1/3\sqrt{2} \\ 4/3\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\boldsymbol{U} = [\boldsymbol{u}_1, \boldsymbol{u}_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Therefore, the SVD of the matrix $\boldsymbol{A} = \boldsymbol{U\Sigma V}^\top$ is

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/3\sqrt{2} & -1/3\sqrt{2} & 4/3\sqrt{2} \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$$