Name: Sanghyeon Kim
Student ID: 20181605

Sogang University
Date: October 1, 2020

HW2 Solution

3.3 The distance between two vectors $\boldsymbol{x}, \boldsymbol{y}$ in an inner product space $(V, \langle \cdot, \cdot \rangle)$ is defined as

$$d(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{x} - \boldsymbol{y}\| = \sqrt{\langle \boldsymbol{x} - \boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y} \rangle}$$

a. The inner product is defined as dot product $\boldsymbol{x}^\top \boldsymbol{y}$

$$\boldsymbol{x} - \boldsymbol{y} = \begin{bmatrix} 1 - (-1) \\ 2 - (-1) \\ 3 - 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

$$\sqrt{(\boldsymbol{x} - \boldsymbol{y})^\top (\boldsymbol{x} - \boldsymbol{y})} = \sqrt{2 \times 2 + 3 \times 3 + 3 \times 3} = \sqrt{22}$$

b. The inner product is defined as $\boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{y}, \quad \boldsymbol{A} := \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$\begin{aligned}
(\boldsymbol{x} - \boldsymbol{y})^\top \boldsymbol{A} (\boldsymbol{x} - \boldsymbol{y}) &= \begin{bmatrix} 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} 7 & 8 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} = 47
\end{aligned}$$

$$\sqrt{(\boldsymbol{x} - \boldsymbol{y})^\top \boldsymbol{A} (\boldsymbol{x} - \boldsymbol{y})} = \sqrt{47}$$

3.8 Set $\boldsymbol{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as $\boldsymbol{u}_1$ of the orthogonal basis. Then project $\boldsymbol{b}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ onto $\boldsymbol{u}_1$ to find

$$\boldsymbol{u}_2 = \boldsymbol{b}_2 - \pi_{\text{span}[\boldsymbol{u}_1]}(\boldsymbol{b}_2)$$

First, note that $\boldsymbol{b}_2$ is projected onto a one-dimensional subspace spanned by $\boldsymbol{u}_1$. Thus the projection can be represented as $\lambda \boldsymbol{u}_1$. Now $\boldsymbol{u}_2$ that is orthogonal to $\boldsymbol{u}_1$ can be found from the following inner product:

$$\begin{aligned}
\langle \boldsymbol{u}_1, \boldsymbol{b}_2 - \lambda \boldsymbol{u}_1 \rangle &= \langle \boldsymbol{u}_1, \boldsymbol{b}_2 \rangle - \lambda \langle \boldsymbol{u}_1, \boldsymbol{u}_1 \rangle \\
&= 0
\end{aligned}$$

From the above inner product, λ is found to be

$$\lambda = \frac{\langle \boldsymbol{u}_1, \boldsymbol{b}_2 \rangle}{\langle \boldsymbol{u}_1, \boldsymbol{u}_1 \rangle}$$

Using dot product as the inner product,

$$\lambda = \frac{\boldsymbol{u}_1^\top \boldsymbol{b}_2}{\|\boldsymbol{u}_1\|^2} = \frac{1}{3}$$

The second orthogonal basis vector $\boldsymbol{u}_2$ is

$$\boldsymbol{b}_2 - \lambda \boldsymbol{u}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -4/3 \\ 5/3 \\ -1/3 \end{bmatrix}$$

To create an orthonormal basis C, $U = \{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ has to be normalized.

$$\frac{\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|} = \frac{\sqrt{3}}{3}\boldsymbol{u}_1 = \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}$$

$$\frac{\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|} = \frac{\sqrt{42}}{14}\boldsymbol{u}_2 = \begin{bmatrix} -2\sqrt{42}/21 \\ 5\sqrt{42}/42 \\ -\sqrt{42}/42 \end{bmatrix}$$

Therefore, ONB $C = (\boldsymbol{c}_1, \boldsymbol{c}_2)$ of U is

$$C = \left(\begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}, \begin{bmatrix} -2\sqrt{42}/21 \\ 5\sqrt{42}/42 \\ -\sqrt{42}/42 \end{bmatrix} \right)$$