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HW1 Solution

2.2 (a)
- If $a, b \in \mathbb{Z}$, then $a + b \in \mathbb{Z}$. Thus $\overline{a+b} \in \mathbb{Z}_n$. Thus $(\mathbb{Z}_n, \oplus)$ is **closed**.
- Let $\bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}_n$. Then, $(\bar{a} \oplus \bar{b}) \oplus \bar{c} = \overline{a+b} \oplus \bar{c} = \overline{(a+b)+c} = \overline{a+b+c}$ and $\bar{a} \oplus (\bar{b} \oplus \bar{c}) = \bar{a} \oplus \overline{b+c} = \overline{a+(b+c)} = \overline{a+b+c}$. Thus **associativity** is satisfied.
- $\bar{a} \oplus \bar{0} = \overline{a+0}$ and $\bar{0} \oplus \bar{a} = \overline{0+a}$. Thus **neutral element** exists.
- $\bar{a} \oplus \overline{-a} = \bar{0} = \overline{-a} \oplus \bar{a}$. Thus **inverse element** exists.

Therefore, $(\mathbb{Z}_n, \oplus)$ is a group.
Moreover, $\bar{a} \oplus \bar{b} = \overline{a+b}$ and $\bar{b} \oplus \bar{a} = \overline{b+a}$. Since $a + b = b + a$, the group is **abelian**.

(b) From the table, we can deduce that there is no $\bar{a}, \bar{b} \in \mathbb{Z}_5 \setminus \{\bar{0}\}$ such that $\overline{a \times b} = \bar{0}$. Therefore, the set is **closed under** $\otimes$.

×	1	2	3	4
1	1	2	3	4
2	2	4	6	8
3	4	6	9	12
4	4	8	12	16

Also, $\bar{1}$ is the **neutral element**, where $\bar{1} \otimes \bar{a} = \bar{a} \otimes \bar{1} = \bar{a}$
There exist an **inverse element** $\bar{y} \in \mathbb{Z}_5 \setminus \{\bar{0}\}$ for all $\bar{a} \in \mathbb{Z}_5 \setminus \{\bar{0}\}$
- $\bar{1} \otimes \bar{1} = \bar{1}$
- $\bar{2} \otimes \bar{3} = \bar{6} = \bar{1}$
- $\bar{3} \otimes \bar{2} = \bar{6} = \bar{1}$
- $\bar{4} \otimes \bar{4} = \bar{16} = \bar{1}$

The table is symmetrical. Therefore, $\bar{a} \times \bar{b} = \bar{b} \times \bar{a} = \overline{a \times b}$ and the group is **abelian**.

(c) $(\mathbb{Z}_8) \setminus \{\bar{0}\}, \otimes)$ is not closed because $\overline{2 \times 4} = \bar{8} = \bar{0}$ when $n = 8$

(d)

2.6 Inhomogeneous equation system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

can be constructed as an augmented matrix,

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Now Gaussian elimination is done on the matrix by adding negative of (row 1) to row 3.

$$\left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right]$$

The particular solution of the system is found to be

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

The general solution of the system is found using the particular solution of $A\mathbf{x} = \mathbf{b}$ and the solution set of $A\mathbf{x} = \mathbf{0}$. The solution set of $A\mathbf{x} = \mathbf{0}$ is

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, the general solution set is

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

2.19 (a) $ker(\Phi)$ is the solution space of $A_\Phi \mathbf{x} = \mathbf{0}$, which can be found by Gaussian elimination

$$A_\Phi = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, $ker(\Phi) = \{\mathbf{0}\}$. $Im(\Phi)$ is the column space of A_Φ. From the row echelon matrix, we found that the columns of A_Φ are linearly independent. Therefore, $Im(\Phi) = \mathbb{R}^3$.

(b) $\tilde{A}_\Phi = T^{-1} A_\Phi S$. In this case, the standard basis in $\mathbb{R}^3$ has changed to the basis

$$B = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

S is a transformation matrix that represents basis B in terms of standard basis E.

T^{-1} is a transformation matrix that represents basis E in terms of basis B.
S is a matrix that has set of vectors in basis B as its column vectors

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

T^{-1} is found from the solutions of the following three augmented matrices of matrix equations, where each represents each of the vectors in the standard basis E as a linear combination of the vectors in the basis B.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \tag{1}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \tag{2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{3}$$

From the above, T^{-1} is found to be

$$\begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

Now, $\tilde{A}_\Phi$ can be found.

$$\begin{aligned} \tilde{A}_\Phi &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 2 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 & 4 \\ 0 & -4 & 7 \\ 0 & -2 & 4 \end{bmatrix} \end{aligned}$$