LEGENDRE'S EQUATION

The differential equation $$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad \text{...(1)}$$

is known as Legendre's equation. The above equation can also be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)\,y = 0 \qquad n \in I$$

This equation can be integrated in series of ascending or descending powers of x. *i.e.*, series in ascending or descending powers of x can be found which satisfy the equation (1).

Let the series in descending powers of x be

$$y = x^m(a_0 + a_1 x^{-1} + a_2 x^{-2} + \ldots) \qquad \text{...(2)}$$

or $$y = \sum_{r=0}^{\infty} a_r x^{m-r}$$

so that $$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (m-r)^{m-r-1}$$

and $$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (m-r)(m-r-1)\,x^{m-r-2}$$

Substituting these in (1), we have

$$(1-x^2)\sum_{r=0}^{\infty} a_r (m-r)(m-r-1)\,x^{m-r-2} - 2x\sum_{r=0}^{\infty} a_r (m-r)\,x^{m-r-1} + n(n+1)\sum_{0}^{\infty} a_r x^{m-r} = 0$$

or $$\sum_{r=0}^{\infty} a_r (m-r)(m-r-1)\,x^{m-r-2} + \{n(n+1) - 2(m-r) - (m-r)(m-r-1)\}\,x^{m-r} a_r = 0$$

or $$\sum_{r=0}^{\infty} [(m-r)(m-r-1)\,x^{m-r-2} + \{n(n+1) - (m-r)(m-r+1)\}\,x^{m-r}]\,a_r = 0 \qquad \text{...(3)}$$

The equation (3) is an identity and therefore coefficients of various powers of x must vanish. Now equating to zero the coefficients of x^m from the above we have $(r = 0)$

$$a_0\{n(n+1) - m(m+1)\} = 0$$

But $a_0 \neq 0$, as it is the coefficient of the very first term in the series.

Hence $(n+1) - m(m+1) = 0$ ....(4)

i.e., $n^2 + n - m^2 - m = 0$ or $(n^2 - m^2) + (n - m) = 0$

or $(n - m)(n + m + 1) = 0$

which gives $m = n$ or $m = -n-1$ ...(5)

This is important as it determines the index.

Next, equating to zero the coefficient of x^{m-1} by putting $r = 1$,

$a_1 [n(n+1) - (m-1)m] = 0$

or $a_1 [(m+n)(m-n-1) = 0$

which gives $a_1 = 0$ ...(6)

Since $(m+n)(m-n-1) \neq 0.$ by (5)

Again to find a relation in successive coefficients a_r, etc., equating the coefficient of x^{m-r-2} to zero, we get

$$(m-r)(m-r-1)a_r + [n(n+1) - (m-r-2)(m-r-1)]a_{r+2} = 0$$

$$\begin{aligned} \text{Now } n(n+1) - (m-r-2)(m-r-1) &= n^2 + n - (m-r-1-1)(m-r-1) \\ &= -[(m-r-1)^2 - (m-r-1) - n^2 - n] \\ &= -[(m-r-1+n)(m-r-1-n) - (m-r-1+n)] \\ &= -[(m-r-1+n)(m-r-1-n-1)] \\ &= (m-r+n-1)(m-r+n-2) \end{aligned}$$

or $(m-r)(m-r-1)a_r - (m-r+n-1)(m-r-n-2)a_{r+2} = 0$

or $$a_{r+2} = \frac{(m-r)(m-r-1)}{(m-r+n-1)(m-r-n-2)} a_r \qquad \text{...(7)}$$

Now since $a_1 = a_3 = a_5 = a_7 = \ldots = 0$

For the two values given by (5) there arises following two cases.

Case I: When $m = n$

$$a_{r+2} = -\frac{(n-r)(n-r-1)}{(2n-r-1)(r+2)} a_r \qquad \text{from (7)}$$

so that, $$a_2 = -\frac{n(n-1)}{(2n-1)2} a_0,$$

$$a_4 = -\frac{(n-2)(n-3)}{(2n-3)\times 4} a_2 = \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)2.4} a_0$$

and so on and $a_1 = a_3 = a_5 = \ldots = 0$

Hence the series (2) becomes

$$y = a_0\left[x^n - \frac{n(n-1)}{(2n-1)\cdot 2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)2.4} \cdot x^{n-4} - \ldots\right] \qquad \text{...(8)}$$

which is a solution of (1)

Case II: When $m = -(n+1)$, we have

$$a_{r+2} = \frac{(n+r+1)(n+r+2)}{(r+2)(2n+r+3)} a_r \qquad \text{from (7)}$$

so that
$$a_2 = \frac{(n+1)(n+2)}{2(2n+3)} a_0;$$

$$a_4 = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)} a_0 \qquad \text{and so on.}$$

Hence the series (2) in this case becomes

$$y = a_0 \left[x^{-n-1} + \frac{(n+1)(n+2)}{2 \cdot (2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)} \cdot x^{-n-5} + \dots \right] \quad ...(9)$$

This gives another solution of (1) in a series of descending powers of x.

Note. If we want to integrate the Legendre's equation in a series of ascending powers of x, we may proceed by taking

$$y = a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + \dots = \sum_{0}^{\infty} a_r x^{k+r}$$

But integration in descending powers of x is more important than that in ascending powers of x.

LEGENDRE'S POLYNOMIAL $P_n(x)$.

Definition:

The Legendre's Equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0 \quad ...(1)$$

The solution of the above equation in the series of descending powers of x is

$$y = a_0 \left[x^n - \frac{n(n-1)}{(2n-1)2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)2.4} x^{n-4} \dots \right]$$

where a_0 is an arbitrary constant.

Now if n is a positive integer and $a_0 = \frac{1.3.5 \dots (2n-1)}{n!}$ the above solution is $P_n(x)$, so that

$$P_n(x) = \frac{1.3.5 \dots (2n-1)}{n!} \left[x^n - \frac{n(n-1)}{(2n-1) \cdot 2} x^{n-2} + \dots \right]$$

$P_n(x)$ is called the *Legendre's functions of the first kind.*

LEGENDRE'S FUNCTION OF THE SECOND KIND *i.e.* $Q_n(x)$.

Another solution of Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

when n is a positive integer

$$y = a_0\left[x^{-n-1} + \frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-n-3} + \ldots\right]$$

If we take $$a_0 = \frac{n!}{1.3.5\ldots(2n+1)}$$

the above solution is called $Q_n(x)$, so that

$$Q_n(x) = \frac{n!}{1.3.5.\ldots(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-n-3} + \ldots\right]$$

The series for $Q_n(x)$ is a non-terminating series.

GENERAL SOLUTION OF LEGENDRE'S EQUATION

Since $P_n(x)$ and $Q_n(x)$ are two independent solutions of Legendre's equation, therefore the most general solution of Legendre's equation is

$$y = A P_n(x) + B Q_n(x)$$

where A and B are two arbitrary constants.

RODRIGUE'S FORMULA

$$P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Proof. Let $$v = (x^2-1)^n \qquad \ldots(1)$$

Then $$\frac{dv}{dx} = n(x^2-1)^{n-1}(2x)$$

Multiplying both sides by (x^2-1), we get

$$(x^2-1)\frac{dv}{dx} = 2n(x^2-1)^n x.$$

$$(x^2-1)\frac{dv}{dx} = 2nvx \qquad \ldots(2)$$

Now differentiating (2), $(n+1)$ times by Leibnitz's theorem, we have

$$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + {}^{(n+1)}C_1\,(2x)\frac{d^{n+1}v}{dx^{n+1}} + {}^{(n+1)}C_2\,(2)\frac{d^n v}{dx^n} = 2n\left[x\frac{d^{n+1}v}{dx^{n+1}} + {}^{(n+1)}C_1\,(1)\frac{d^n v}{dx^n}\right]$$

or $$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + 2x\,[{}^{n+1}C_1 - n]\frac{d^{n+1}v}{dx^{n+1}} + 2\,[{}^{n+1}C_2 - n.\,{}^{(n+1)}C_1]\frac{d^n v}{dx^n} = 0$$

or $$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}} + 2x\frac{d^{n+1}v}{dx^{n+1}} - n(n+1)\frac{d^n v}{dx^n} = 0 \qquad ...(3)$$

If we put $\frac{d^n v}{dx^n} = y$, (3) becomes

$$(x^2-1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)\,y = 0$$

or $$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0$$

This shows that $y = \frac{d^n v}{dx^n}$ is a solution of Legendre's equation.

$\therefore$ $$C\frac{d^n v}{dx^n} = P_n(x) \qquad ...(4)$$

where C is a constant.

But $$v = (x^2-1)^n = (x+1)^n(x-1)^n$$

so that $$\frac{d^n v}{dx^n} = (x+1)^n\frac{d^n}{dx^n}(x-1)^n + {}^nC_1\cdot n(x+1)^{n-1}\cdot\frac{d^{n-1}}{dx^{n-1}}(x-1)^n + \ldots + (x-1)^n\frac{d^n}{dx^n}(x+1)^n = 0$$

when $x = 1$, $$\frac{d^n v}{dx^n} = 2^n\cdot n\,!$$

All the other terms disappear as $(x-1)$ is a factor in every term except first.

Therefore when $x = 1$, (4) gives

$$C\cdot 2^n\cdot n\,! = P_n(1) = 1 \qquad P_n(1) = 1$$

$$C = \frac{1}{2^n\cdot n\,!}. \qquad ...(5)$$

Substituting the value of C from (1) in (5) we have

$$P_n(x) = \frac{1}{2^n\cdot n\,!}\frac{d^n v}{dx^n}$$

$$P_n(x) = \frac{1}{2^n\lfloor n}\frac{d^n}{dx^n}(x^2-1)^n$$

LEGENDRE POLYNOMIALS

$$P_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n}(x^2-1)^n \qquad \text{(Rodrigue's formula)}$$

If $n = 0$, $\quad P_0(x) = \frac{1}{2^0 \cdot 0!} = 1$

If $n = 1$, $\quad P_1(x) = \frac{1}{2^1 \cdot 1!} \frac{d}{dx}(x^2-1) = \frac{1}{2}(2x) = x$

If $n = 2$, $\quad P_2(x) = \frac{1}{2^2 \cdot 2!} \frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8} \frac{d}{dx}[2(x^2-1)(2x)]$

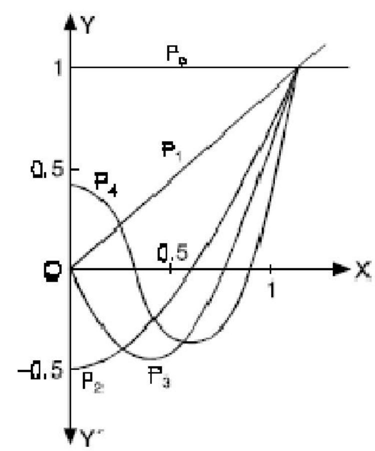
$$= \frac{1}{2}[(x^2-1) \cdot 1 + 2x \cdot x] = \frac{1}{2}(3x^2-1)$$

similarly $\quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$



$$P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n \cdot r!\,(n-r)!\,(n-2r)!} x^{n-2r}$$

where $\quad N = \frac{n}{2}$ if n is even.

$$N = \frac{1}{2}(n-1) \text{ if } n \text{ is odd.}$$

Example *Express $f(x) = 4x^3 + 6x^2 + 7x + 2$ in terms of Legendre Polynomials.*

Solution. Let

$$4x^3 + 6x^2 + 7x + 2 \equiv a P_3(x) + b P_2(x) + c P_1(x) + d P_0(x) \qquad \ldots(1)$$

$$\equiv a\left(\frac{5x^3}{2} - \frac{3x}{2}\right) + b\left(\frac{3x^2}{2} - \frac{1}{2}\right) + c(x) + d(1)$$

$$\equiv \frac{5ax^3}{2} - \frac{3ax}{2} + \frac{3bx^2}{2} - \frac{b}{2} + cx + d$$

$$\equiv \frac{5ax^3}{2} + \frac{3bx^2}{2} + \left(\frac{-3a}{2} + c\right)x - \frac{b}{2} + d.$$

Equating the coefficients of like powers of x, we have

$$4 = \frac{5a}{2}, \quad \text{or} \quad a = \frac{8}{5}$$

$$6 = \frac{3b}{2} \quad \text{or} \quad b = 4$$

$$7 = \frac{-3a}{2} + c \quad \text{or} \quad 7 = \frac{-3}{2}\left(\frac{8}{5}\right) + c \quad \text{or} \quad c = \frac{47}{5}$$

$$2 = \frac{-b}{2} + d \quad \text{or} \quad 2 = \frac{-4}{2} + d \quad \text{or} \quad d = 4$$

Putting the values of a, b, c, d in (1), we get

$$4x^3 + 6x^2 + 7x + 2 = \frac{8}{5} P_3(x) + 4 P_2(x) + \frac{47}{5} P_1(x) + 4 P_0(x) \qquad \textbf{Ans.}$$

ORTHOGONALITY OF LEGENDRE POLYNOMIALS

$$\int_{-1}^{+1} P_m(x) \cdot P_n(x)\, dx = 0 \qquad n \neq m$$

Proof. $P_n(x)$ is a solution of

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \qquad \ldots(1)$$

$P_m(x)$ is the solution of

$$(1 - x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + m(m+1)z = 0 \qquad \ldots(2)$$

Multiplying (1) by z and (2) by y and subtracting, we get

$$(1 - x^2)\left[z\frac{d^2y}{dx^2} - y\frac{d^2z}{dx^2}\right] - 2x\left[z\frac{dy}{dx} - y\frac{dz}{dx}\right] + [n(n+1) - m(m+1)]yz = 0$$

$$(1 - x^{2)}\left[\left\{z\frac{d^2y}{dx^2} + \frac{dz}{dx} \times \frac{dy}{dz}\right\} - \left\{\frac{dy}{dx}\frac{dz}{dx} + y\frac{d^2z}{dx^2}\right\} - 2x\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right) + (n-m)(n+m+i)yz = 0\right.$$

or $$\frac{d}{dx}\left[(\mathbf{1} - x^2)\left(z\frac{dy}{dx} - y\frac{dz}{dx}\right)\right] + (n-m)(n+m+1)yz = 0$$

Now integrating from -1 to 1, we get

$$\left[(1-x^2)\left(z\frac{dy}{dx}-y\frac{dz}{dx}\right)\right]_{-1}^{+1}+(n-m)(n+m+1)\int_{-1}^{+1} y.z\,dx = 0.$$

or $$0+(n-m)(n+m+1)\int_{-1}^{+1} y\cdot z\,dx = 0$$

or $$\int_{-1}^{+1} P_n(x)\cdot P_m(x)\,dx = 0 \qquad \text{if } n \neq m$$ **Proved**

Example *Using the Rodrigue's formula for Legendre function, prove that*

$$\int_{-1}^{+1} x^m P_n(x)\,dx = 0, \text{ where } m, n \text{ are positive integers and } m < n.$$

Solution. $$\int_{-1}^{+1} x^m P_n(x)\,dx = \int_{-1}^{+1} x^m \frac{1}{2^n\, n\,!}\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{1}{2^n\, n\,!}\int_{-1}^{+1} x^m \frac{d^n}{dx^n}(x^2-1)^n\,dx$$

On integrating by parts we get

$$= \frac{1}{2^n\, n\,!}\left[\left\{x^m\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right\}_{-1}^{+1} - \int_{-1}^{+1} m\,x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx\right]$$

$$= 0-\frac{m}{2^n\, n\,!}\int_{-1}^{+1} x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx$$

$$\int_{-1}^{+1} x^m P_n(x)\,dx = -\frac{(-1)^2 m\,(m-1)}{2^n\, n\,!}\int_{-1}^{+1} x^{m-2}\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\,dx$$

Integrating $m-2$ times, we get

$$= (-1)^m\,\frac{m\,(m-1)\ldots 1}{2^n\, n\,!}\int_{-1}^{+1}\frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n\,dx$$

$$= \frac{(-1)^m\, m\,!}{2^n\, n\,!}\int_{-1}^{+1}\frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n\,dx$$

$$= \frac{(-1)^m\, m\,!}{2^n\, n\,!}\left[\frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)^n\right]_{-1}^{+1} = 0$$ **Ans.**

RECURRENCE FORMULAE FOR $P_n(x)$

Formula 1. $nP_n = (2n-1)\,x\,P_{n-1} - (n-1)\,P_{n-2}$.

Formula II. $x\,P_n' - P'_{n-1} = nP_n$.

Formula III. $P_n' - x\,P'_{n-1} = nP_{n-1}$

Formula IV. $P'_{n+1} - P'_{n-1} = (2n+1)\,P_n$

Formula V. $(x^2-1)\,P_n' = n\,[x\,P_n - P_{n-1}]$

Formula VI. $(x^2-1)\,P_n' = (n+1)\,(P_{n+1} - x\,P_n)$

Exercise Express in terms of Legendre Polynomials.

$(a)\ 1+x-x^2 \quad (b)\ x^4+x^3+x^2+x+1 \quad (c)\ 1+2x-3x^2+4x^3 \quad (d)\ x^3+1$

Ans. $(a)\ -\frac{2}{3}P_2(x)+P_1(x)+\frac{2}{3}P_0(x)$ $(b)\ \frac{8}{35}P_4(x)+\frac{2}{5}P_3(x)+\frac{26}{21}P_2(x)+\frac{8}{5}P_1(x)+\frac{23}{15}P_0(x)$

$(c)\ \frac{8}{5}P_3(x)-2P_2(x)+\frac{22}{5}P_1(x)$ $(d)\ \frac{2}{5}P_3(x)+\frac{3}{5}P_1(x)+P_0(x)$