Differential Equations (DE)

Course Instructor:
Dr. Md. Mustafizur Rahman
Professor
Department of Mathematics, BUET

Contact details:	Tel:	
	Mobile:	01552556549
	E-mail:	mmustafizurrahman@math.buet.ac.bd

Learning Objectives

At the end of this lecture, you will be able to do the following:

1. Recognize and classify differential equations.
2. Identify order and degree of differential equations.
3. Distinguish the Initial and Boundary-Value Problems
4. Find ordinary differential equations from the basic.
5. Know about the solution of differential equations.
6. Identify research problems where differential equations can be used to model the system.
7. Apply differential equations in every day life

Introduction to differential equations

Differential Equation:

A differential equation is, in simpler terms, a statement of equality having a derivative or differentials.

An equation involving differentials or differential co-efficient is called a differential equation.

For examples, $\frac{d^2y}{dx^2}=0$ and $y\,dx+x\,dy=0$ are two differential equations.

Ordinary Differential Equation (ODE):

If a differential equation contains one dependent variable and one independent variable, then the differential equation is called ordinary differential equation.

For example, $(i)\,\frac{dy}{dx}=x\,sinx$, $(ii)\,4\frac{d^2y}{dx^2}+6y=\tan x$.

Identities such as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f^2(x)\right)=2f(x)f'(x), \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}y}=1$$

are not differential equations

Introduction to differential equations

Partial Differential Equation:

If there are two or more independent variables, so that the derivatives are partial, then the differential equation is called partial differential equation.

For example, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$ $\qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Some Problems Leading to Ordinary Differential Equations:

Before we develop methods for the solution of ordinary differential equations, it will be helpful to examine some simple geometrical and physical problems that lead to ODEs. There are many such problems, so we only consider some representative examples

A Geometrical Problem: Orthogonal Trajectories

A curve that intersects every member of a one-parameter family of curves orthogonally (at right angles) is called an orthogonal trajectory of the family. A geometrical problem that often occurs is how to find a family of curves that form orthogonal trajectories to a given family

Two typical families of orthogonal trajectories are illustrated in Fig. 1, and if these curves are related to steady state heat flow, family 1 could represent the isotherms and family 2 the heat flow lines.

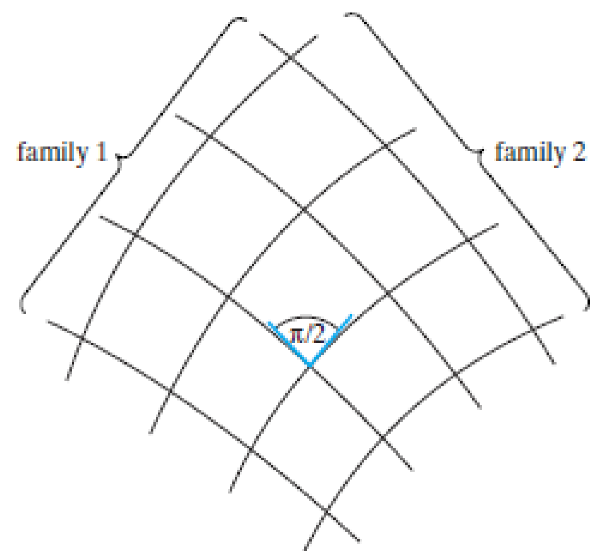


FIGURE1: Two typical families of orthogonal trajectories

Some Problems Leading to Ordinary Differential Equations:

Two specific examples of families of orthogonal trajectories are shown in Fig. 2, where in case (a) the curves are given by $x^2 + y^2 = c^2$ and $y = k x$ (with c and k real). The first equation describes a family of concentric circles centered on the origin, and the second family that forms their orthogonal trajectories comprises all the straight lines that pass through the origin. In case (b) the curves are given by $x^2 - y^2 = c$ and $xy = k$ (with c and k real), where the two families of curves are families of mutually orthogonal rectangular hyperbolas.

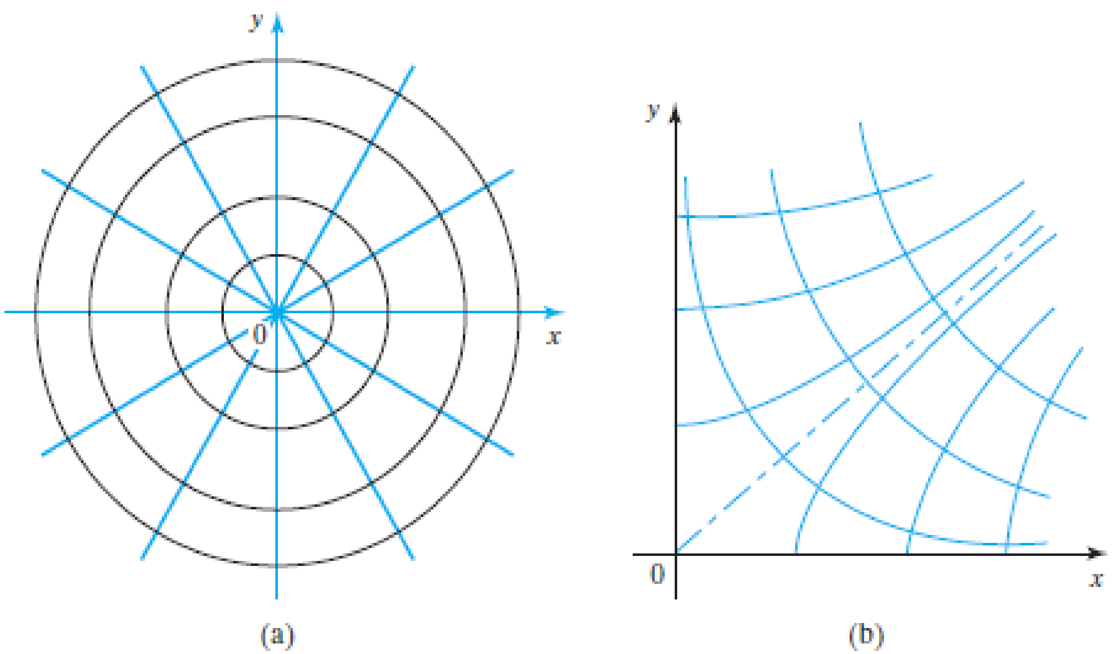


FIGURE 2: Specific examples of orthogonal trajectories.

Some Problems Leading to Ordinary Differential Equations:

Chemical Reaction Rates and Radioactive Decay

In many circumstances, for a limited period of time, the rate of reaction of a chemical process can be considered to be proportional only to the amount Q of the chemical that is present at a given time t. The differential equation governing such a process then has the form

$$\frac{dQ}{dt} = KQ,$$

where k ≥ 0 is a constant of proportionality. This is a homogeneous linear first order differential equation

Some Problems Leading to Ordinary Differential Equations:

The Logistic Equation: Population Growth

In the study of phenomena involving the rate of increase of a quantity of interest, it often happens that the rate is influenced both by the amount of the quantity that is present at any given instant of time and by the limitation of a resource that is necessary to enable an increase to occur. Such a situation arises in a population of animals that compete for limited food resources, leading to the so-called predator–prey situations where an animal (the predator) feeds on another species (the prey) with the effect that overfeeding leads to starvation. This in turn leads to a reduction in the number of predators that in turn can lead to a recovery of the food stock. Similar situations arise in manufacturing when there is competition for scarce resources, and in a variety of similar situations.

To model the situation we let P represent the amount of the quantity of interest present at a given time t, and M represent the amount of resources available at the start. Then a simple model for this process is provided by the differential equation

$dP/dt = k P(M - P)$, in which k is a constant of proportionality.

The equation is called the logistic equation, and it is nonlinear because of the presence of the term $-kP^2$ on the right,

Some Problems Leading to Ordinary Differential Equations:

A Differential Equation that Models Damped Oscillations

Mechanical and electrical systems, and control systems in general, can exhibit oscillatory behavior that after an initial disturbance slowly decays to zero. The process producing the decay is a dissipative one that removes energy from the system, and it is called damping. To see the prototype equation that exhibits this phenomenon we need only consider the following very simple mechanical model. A mass M rests on a rough horizontal surface and is attached by a spring of negligible mass to a fixed point. The mass–spring system is caused to oscillate along the line of the spring by being displaced from its equilibrium position by a small amount and then released. Figure 3a shows the system in its equilibrium configuration, and Fig. 3b shows it when the mass has been displaced through a distance x from its rest position

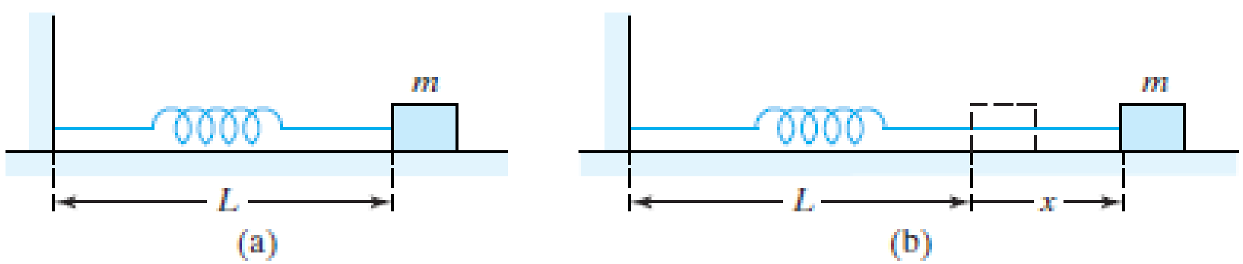


FIGURE 3 Mass–spring system.

Some Problems Leading to Ordinary Differential Equations:

Equating the forces acting along the line of the spring and taking account of the fact that the spring and frictional forces oppose the force due to the acceleration shows the equation of motion to be the homogeneous second order linear equation

$$M\frac{d^2x}{dt^2} = -k\frac{dx}{dt} - px,$$

or

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + bx = 0,$$

where $a = k/M$ and $b = p/M$.

Some Problems Leading to Ordinary Differential Equations:

The Shape of a Suspended Power Line: The Catenary

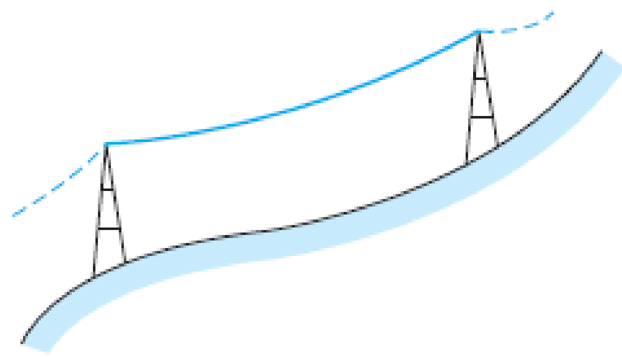


FIGURE 5 Suspended cable.

An analysis of the forces acting on a power line attached to pillars as shown in Fig. 5, or on the suspension cable of a cable car, shows the shape of the cable to be determined by the solution y(x) of the nonlinear differential equation

$$\frac{d^2y}{dx^2} = a\sqrt{1 + (dy/dx)^2}.$$

The shape taken by the cable is called a catenary, after the Latin word catena, meaning chain. Although this equation will not be solved here, it is not difficult to show that its solution is a hyperbolic cosine curve.

Some Problems Leading to Ordinary Differential Equations:

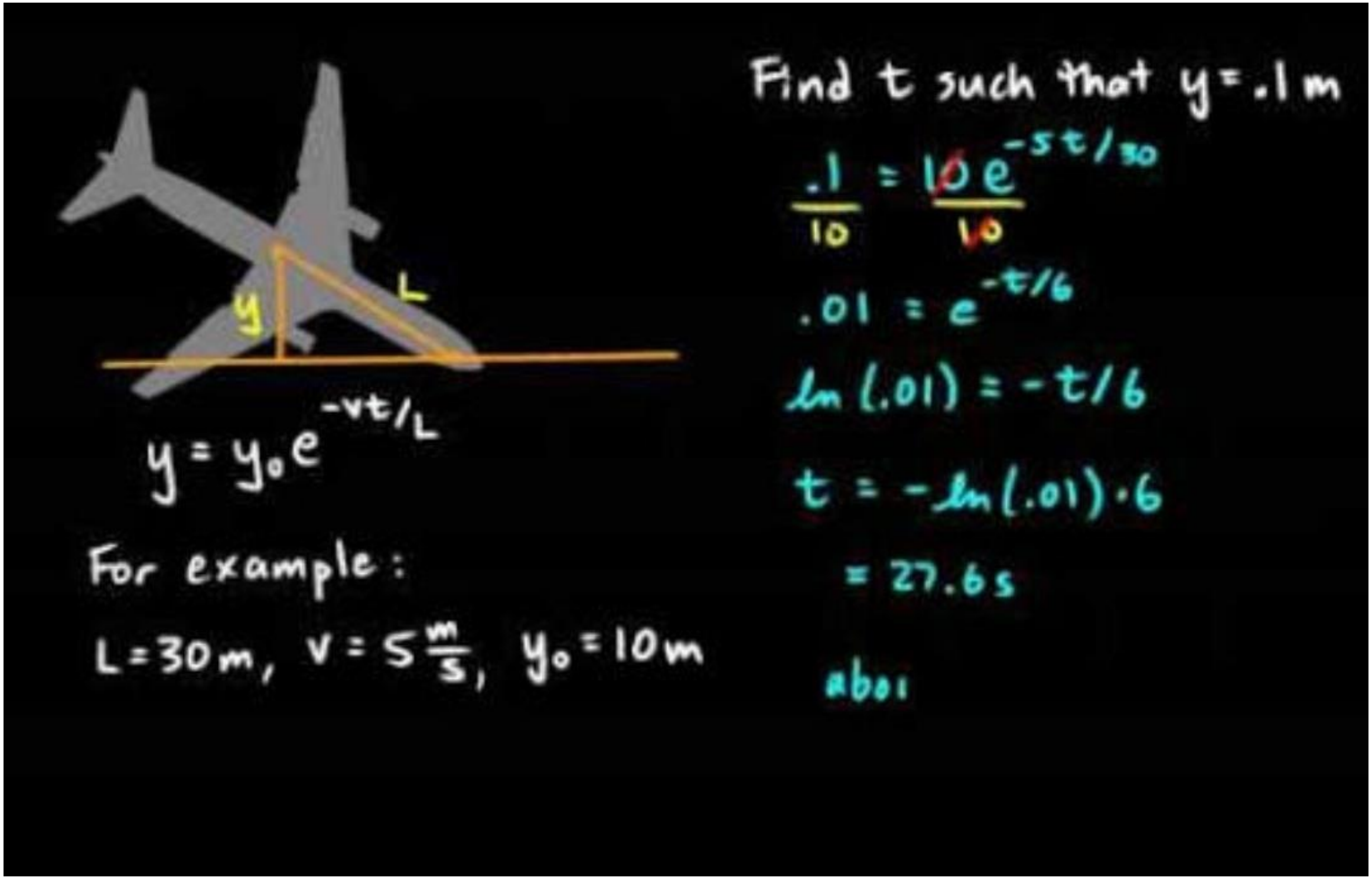
An analysis of the forces and moments acting on a horizontal beam of uniform construction made from a material with Young's modulus E and supported at its two end points, with the moment of inertia of its cross-section about the central horizontal axis of the beam equal to I, leads to the following equation for the vertical deflection y caused by the weight of the beam and any loads it is supporting:

$$\frac{d^2}{dx^2}\left\{\frac{EId^2y/dx^2}{[1+(dy/dx)^2]^{3/2}}\right\} = w(x),$$

Here M(x) is the bending moment that acts to one side of a point x in the beam.

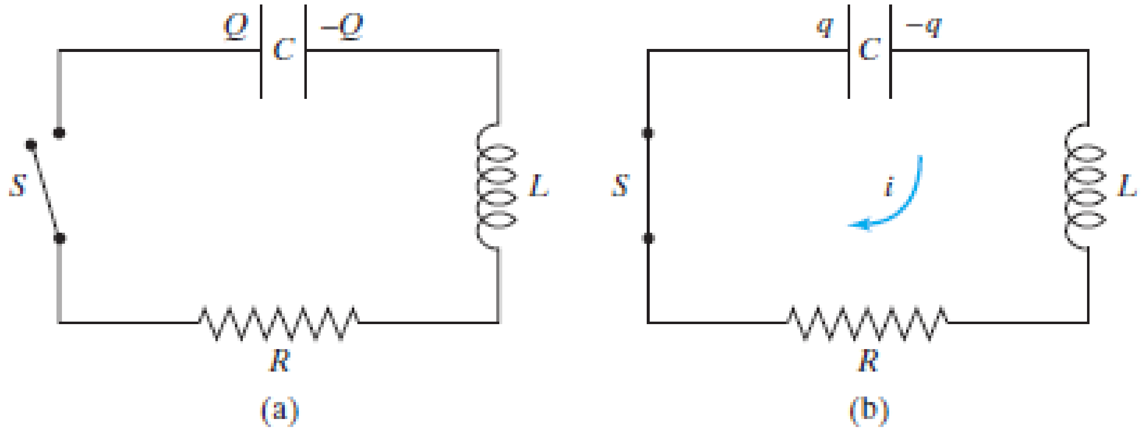
Some Problems Leading to Ordinary Differential Equations:

To control the motion of aero plane, differential equations is used.



Some Problems Leading to Ordinary Differential Equations:

It is used to calculate current flow in circuit in physics.



The respective potential drops in the direction of the arrow across the resistor R, the inductance L, and the capacitor C are V = i R, where i = dq/dt, Ldi/dt, and q/C.

Applying Kirchhoff's law, which requires the sum of the potential drops around the circuit to be zero, gives

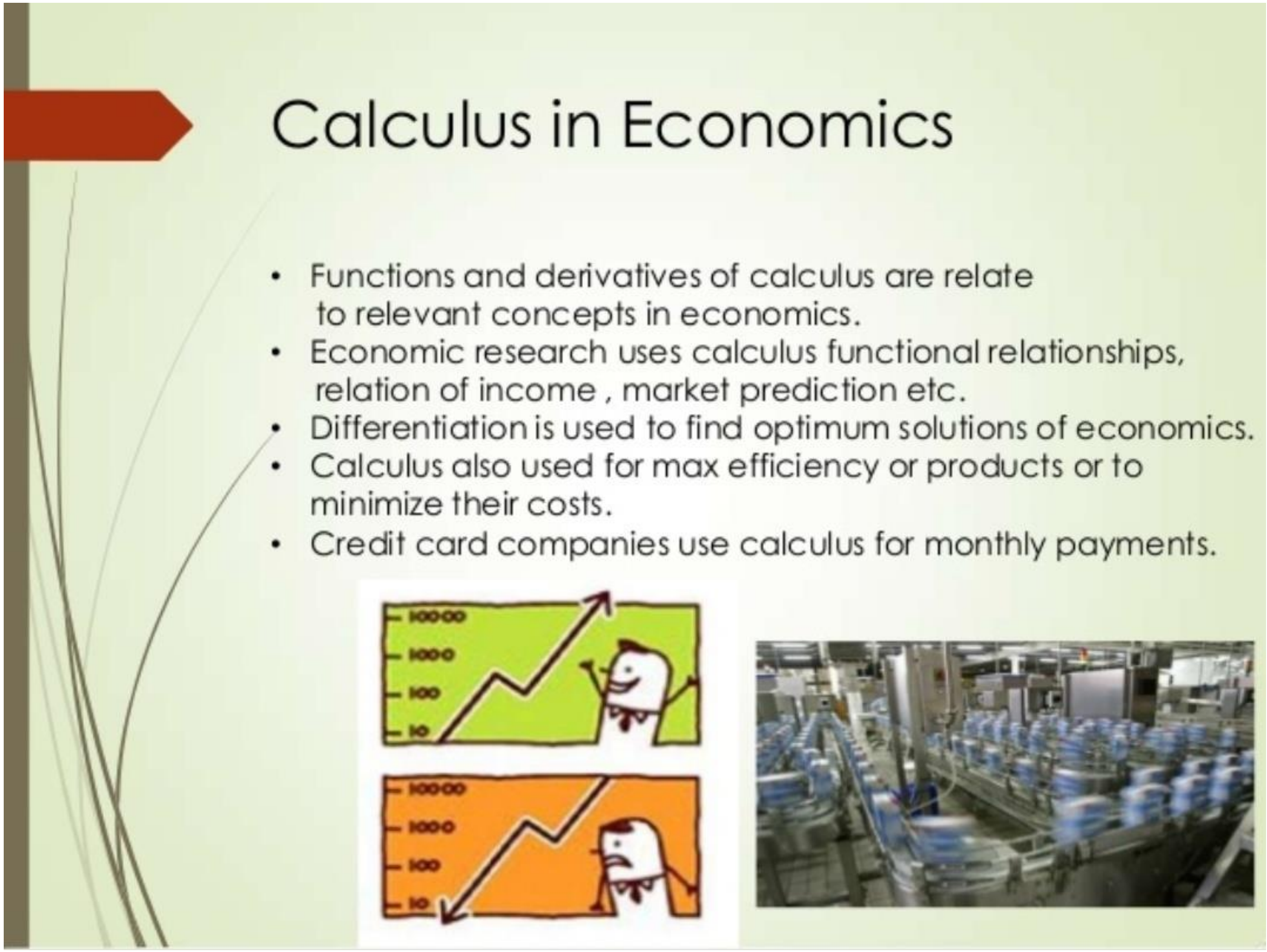
$$L\frac{di}{dt} + Ri + \frac{q}{C} = 0.$$

Eliminating i by using the result i = dq/dt leads to the following homogeneous linear second order equation for q:

$$LC\frac{d^2q}{dt^2} + RC\frac{dq}{dt} + q = 0.$$

Some Problems Leading to Ordinary Differential Equations:

It is also used in Economics



Some Problems Leading to Ordinary Differential Equations:

- In a simple video game involving a jumping motion, a differential equation is used to model the velocity of a character after the command is given to return them to the ground in a simulated gravitational field.
- The electrical equipment we use, it is an outcome of a differential equation.
- Riding a bike, or a car, or any vehicle? The engineers have designed your vehicles system using some sets of differential equation.
- Falling Object
- Newton's Law of Cooling

Order of a differential equation

The **order** of an ordinary differential equations is the order of the highest order derivative

Examples:

$$\frac{dy}{dx} - y = e^x$$

First order ODE

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = \cos(x)$$

Second order ODE

$$\left(\frac{d^3y}{dx^3}\right)^2 - \frac{d^2y}{dx^2} + 2y^4 = 1$$

Third order ODE

Degree of a differential equation

The **degree** of a differential equation is the highest degree of the highest derivative after the equation has been put in the form free from radicals and fraction.

$$Examples:$$

$$\frac{dy}{dx}-(y)^4=e^x$$

1st order 1st degree ODE

$$\left(\frac{d^2y}{dx^2}\right)^3-5\left(\frac{dy}{dx}\right)^4+2y=\cos(x)$$

2nd order 3rd degree ODE

$$\left(\frac{d^3y}{dx^3}\right)^2-\frac{dy}{dx}+2y^4=1$$

33d order 2nd degree ODE

Degree of a differential equation

- What is the **degree** of a differential

$$\sqrt[4]{\frac{dy}{dx}+2x\left(\frac{d^4y}{dx^4}\right)^3}=\sqrt[3]{x-2}\quad ?$$

Solution:

$$\sqrt[4]{\frac{dy}{dx}+2x\left(\frac{d^4y}{dx^4}\right)^3}=\sqrt[3]{x-2}$$

$$\text{or, }\left\{\frac{dy}{dx}+2x\left(\frac{d^4y}{dx^4}\right)^3\right\}^{1/4}=(x-2)^{1/3}$$

$$\text{or, }\left\{\frac{dy}{dx}+2x\left(\frac{d^4y}{dx^4}\right)^3\right\}^{3}=(x-2)^{4}$$

$$\text{or, }\left(\frac{dy}{dx}\right)^3+\ldots\ldots+8x^3\left(\frac{d^4y}{dx^4}\right)^9=(x-2)^4$$

4^{th} order 9^{th} degree ODE

Linear ODE

An ODE is linear if the unknown function and its derivatives appear to power one. No product of the unknown function and/or its derivatives

$$a_n(x)y^n(x)+a_{n-1}(x)y^{n-1}(x)+\cdots+a_1(x)y'(x)+a_0(x)y(x)=g(x)$$

$Examples:$

$$\frac{dy}{dx}-y=e^x$$ Linear ODE

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+2x^2y=\cos(x)$$ Linear ODE

$$\left(\frac{d^2y}{dx^2}\right)^3-\frac{dy}{dx}+\sqrt{y}=1$$ Non-linear ODE

Initial value and Boundary-Value Problems

Initial-Value Problems

- The auxiliary conditions are at one point of the independent variable

$$y''+2y'+y=e^{-2x}$$

$$y(0)=1,\quad y'(0)=2.5$$

same

Boundary-Value Problems

- The auxiliary conditions are not at one point of the independent variable
- More difficult to solve than initial value problem

$$y''+2y'+y=e^{-2x}$$

$$y(0)=1,\quad y(2)=1.5$$

different

Solution of ODE:

General Solution of ODE:

The solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation is called the general solution.

For example, $y = ax + b$ is the general solution of the differentialequation $\frac{d^2y}{dx^2} = 0$

where a and b are arbitrary constants.

Particular Solution of ODE:

If particular values are given to the arbitrary constants in the general solution, then the solution so obtained is called particular solution.

For example, putting $a = 2$, and $b = 3$, a particular solution of the

differential equation $\frac{d^2y}{dx^2} = 0$ is $y = 2x + 3$.

Solution of ODE:

- How to solve differential Equation $\dfrac{d^2y}{dx^2} = 0$?

Solution:

$$\frac{d^2y}{dx^2} = 0$$

$$\text{or, } \frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$$

$$\text{or, } \int d\left(\frac{dy}{dx}\right) = a$$

$$\text{or, } \int du = a, \text{ where } u = \frac{dy}{dx}$$

$$\text{or, } u = a$$

$$\text{or, } \frac{dy}{dx} = a$$

$$\text{or, } \int dy = a\int dx + b \Rightarrow y = ax + b$$

which geometrically represents a straight line.

Or one may solve it in the following way :

$$\frac{d^2y}{dx^2} = 0$$

$$\text{or, } \frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$$

since the derivative of $\dfrac{dy}{dx}$ is zero;

so $\dfrac{dy}{dx} =$ constant a (say)

or $\int dy = a\int dx + b$ or $y = a\,x + b.$

In fact $\dfrac{d^2y}{dx^2} = 0$ is an ODE of order 2 and has as solution

with 2 parameters a and b.

Practice quiz: Classify differential equations

1. By checking all that apply, classify the following differential equation:

$$\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} = 0$$

a) first order

b) second order

c) third order

d) ordinary

e) partial

f) linear

g) nonlinear

Practice quiz: Classify differential equations

2. By checking all that apply, classify the following differential equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right) = e^{-\psi}$$

a) first order

b) second order

c) ordinary

d) partial

e) linear

f) nonlinear

Practice quiz: Classify differential equations

3. By checking all that apply, classify the following differential equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

a) first order

b) second order

c) ordinary

d) partial

e) linear

f) nonlinear

Practice quiz: Classify differential equations

4. By checking all that apply, classify the following differential equation:

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

a) first order

b) second order

c) ordinary

d) partial

e) linear

f) nonlinear

Practice quiz: Classify differential equations

5. By checking all that apply, classify the following differential equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

a) first order

b) second order

c) ordinary

d) partial

e) linear

f) nonlinear

Practice quiz: Classify differential equations

Solutions to the Practice quiz: Classify differential equations

1. c, d, g. Third order, ordinary, nonlinear.

2. b, c , f. Second order, ordinary, nonlinear.

3. b, d , f. Second order, partial, nonlinear.

4. b, c , e. Second order, ordinary, linear.

5. b, d , e. Second order, partial, linear.

Formation of ODE: **an ordinary differential equation is formed by differentiating the equations and eliminating the arbitrary constants.**

- Form an ODE corresponding to $y = ax + a^2$ also write down the order and degree of the obtained ODE.

Solution: $y = ax + a^2 ....(i)$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = a$$

Now putting the value of $'a'$ into eq. (i) , we get

$$y = x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2$$

Which is the required ODE.

$\therefore$ Orde $= 1$, degree $= 2$.

Formation of ODE:

- Form an ODE corresponding to $y = mx + c$ also write down the order and degree of the obtained ODE.

Solution: Given $y = mx + c \quad \ldots \quad \ldots \quad \ldots \ (i)$

Differentiating Eq. (i) with respect to x, we get

$$\frac{dy}{dx} = m \quad \ldots \quad \ldots \quad \ldots (ii)$$

Differentiating again Eq. (ii) with respect to x, we get

$$\frac{d^2 y}{dx^2} = 0$$

Which is the required ODE.

$\therefore$ Orde $= 2$, degree $= 1$.

Formation of ODE:

- **Form an ODE corresponding to** $y = e^{x}\left(Acosx + Bsinx\right)$.

Solution: Given $y = e^{x}\left(Acosx + Bsinx\right) \ \ldots \ \ldots \ \ldots \ (i)$

Differentiating Eq. (i) with respect to x, we get

$$\frac{dy}{dx} = e^{x}\left(Acosx + Bsinx\right) + e^{x}\left(-Asinx + Bcosx\right)$$

$$= y + e^{x}\left(-Asinx + Bcosx\right) \ \ldots \ \ldots \ \ldots \ (ii)$$

Differentiating again Eq. (ii) with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^{x}\left(-Asinx + B\cos x\right) + e^{x}\left(-Acosx - Bsinx\right)$$

$$= \frac{dy}{dx} + e^{x}\left(-Asinx + B\cos x\right) - e^{x}\left(Acosx + Bsinx\right)$$

$$= \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y$$

$$\therefore \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

(self study)

1. State the order and degree of the following differential equations:

$(a). \frac{d^2y}{dx^2} + \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = 0$; $(b). x\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^{-1} = y^2$; $(c). L\frac{d^2q}{dt^2} + R\frac{dq}{dx} + \frac{q}{c} = Esin\omega t$

$(d). \frac{d^2y}{dx^2} - \left[x + \frac{dy}{dx}\right]^{-1/2} = 1$; $(e). \left(\frac{d^3y}{dx^3}\right)^{3/7} = \left[1 + \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right]^{2/3}$

2. Form an ODE corresponding to the following equations also write down the order and degree of the obtained ODE:

$(1). xy = c_1e^x - c_2e^{-x} + x^2$; $(2). y = e^{mx}(Acosnx + Bsinnx)$; $(3). xy = Ae^x + Be^{-x}$; $(4). y = Acosmx + Bsinmx$; $(5). Ax^2 + By^2 = 1$

THANKS TO ALL

Dr. M. M. Rahman,
Professor, Department of Mathematics, BUET