First Order Ordinary Differential Equations

First Order Linear Differential Equations

Def: A first order differential equation is said to be ***linear*** if it can be written

$$\boxed{\frac{dy}{dx} + p(x)y = q(x)}$$

OR

$$\boxed{\frac{dx}{dy} + p(y)x = q(y)}$$

First Order Linear Differential Equations

Form of Equation: $\frac{dy}{dx} + p(x)y = q(x)$

Method of Solution: **The integrating Factor is** $I.F. = e^{\int p(x)dx}$

Multiplying the equation with this factor gives,

$$e^{\int p(x)dx}\frac{dy}{dx} + p(x)ye^{\int p(x)dx} = q(x)e^{\int p(x)dx}$$

$$\frac{d}{dx}\left(ye^{\int p(x)dx}\right) = q(x)e^{\int p(x)dx}$$

Integrating w.r.to x, $\int d\left(ye^{\int p(x)dx}\right) = \int q(x)e^{\int p(x)dx}dx$

$$ye^{\int p(x)dx} = \int q(x)e^{\int p(x)dx}dx$$

$$\therefore \mathrm{y}\,(x)(\mathrm{I.F}) = \int q(x) \times (\mathrm{I.F.})\,dx + \mathrm{c}$$

First Order Linear Differential Equations

Equation Form: $$\frac{dy}{dx} + p(x)y = q(x)$$

$$\text{I.F.} = e^{\int p(x)dx}$$

Solution: $$y\,(x)(\text{I.F}) = \int q(x) \times (\text{I.F.})\,dx \; + \text{c}$$

Equation Form: $$\frac{dx}{dy} + p(y)x = q(y)$$

$$\text{I.F.} = e^{\int p(y)dy}$$

Solution: $$\text{x}(y)(\text{I.F}) = \int q(y) \times (\text{I.F.})\,dy \; + \text{c}$$

First Order Linear Differential Equations

1. **Solve** : $\frac{dy}{dx} = \frac{y}{x}$

Solution: Given Eq. can be writen as: $\frac{dy}{dx} - \frac{y}{x} = 0 \quad \dots \quad \dots \quad \dots (i)$

Which is a first-order linear differential equation

Where, $p(x) = -\frac{1}{x}$, and $q(x) = 0$

$$\therefore I.F. \quad = e^{\int p(x)\,dx} = e^{\int -\frac{1}{x}dx} \quad = e^{-\ln x} \quad = e^{\ln\frac{1}{x}} = \frac{1}{x}$$

Therefore, the solution of Eq. (i) is:

$$y(x)(\text{I.F.}) = \int q(x)(\text{I.F.})\,dx$$

$$\text{or, } y(x)\frac{1}{x} = \int 0.e^{\int -\frac{1}{x}dx}dx + c$$

$$\text{or, } y(x)\frac{1}{x} = 0 + c \quad \Rightarrow y = cx$$

Which is the required solution

First Order Linear Differential Equations

2. **Solve** : $\dfrac{dy}{dx} = x - 3y$

Solution: Given Eq. can be writen as: $\dfrac{dy}{dx} + 3y = x \quad \ldots \quad \ldots \quad \ldots \; (i)$

Which is a first-order linear differential equation

Where, $p(x) = 3$, and $q(x) = x$

$$\therefore I.F. \; = e^{\int p(x)\,dx} = e^{\int 3dx} \; = e^{3x}$$

Therefore, the solution of Eq. (i) is:

$$y(x)(\text{I.F.}) = \int q(x)(\text{I.F.})\; dx$$

$$\text{or, } \; y.e^{3x} = \int x.e^{3x} dx + c$$

$$\text{or, } \; y.e^{3x} = x\frac{e^{3x}}{3} - \frac{e^{3x}}{9} + c$$

+	x	$\frac{e^{3x}}{3}$
-	1	$\frac{e^{3x}}{9}$
+	0	

Which is the required solution

Practice quiz: Linear first-order odes

1. The solution of $(1+x^2)y' + 2xy = 2x$ with initial value $y(0) = 0$ is given by

a) $y(x) = \dfrac{x}{1+x}$

b) $y(x) = \dfrac{x}{1+x^2}$

c) $y(x) = \dfrac{x^2}{1+x}$

d) $y(x) = \dfrac{x^2}{1+x^2}$

2. The solution of $x^2y' = 1 - 2xy$ with initial value $y(1) = 2$ is given by

a) $y(x) = \dfrac{1+x}{x}$

b) $y(x) = \dfrac{1+x}{x^2}$

c) $y(x) = \dfrac{1+x^2}{x}$

d) $y(x) = \dfrac{1+x^2}{x^2}$

3. The solution of $y' + \lambda y = a$ with initial value $y(0) = 0$ and $\lambda > 0$ is given by

a) $y(x) = a(1 - e^{\lambda x})$

b) $y(x) = a(1 - e^{-\lambda x})$

c) $y(x) = \dfrac{a}{\lambda}(1 - e^{\lambda x})$

d) $y(x) = \dfrac{a}{\lambda}(1 - e^{-\lambda x})$

Self study

1. Solve: $\dfrac{dy}{dx}+2y=e^{-2x}$, with $y(0)=3/4$.

2. Solve: $\dfrac{dy}{dx}-2xy=x$, with $y(0)=0$.

3. Solve: $\dfrac{dp}{dt}+pt=3t$.

4. Solve the following linear odes:

(i) $\dfrac{dy}{dx}=x-y$, with $y(0)=-1$;

(ii) $\dfrac{dy}{dx}=2x(1-y)$, with $y(0)=0$.

First Order Non Linear Differential Equations

Bernoulli Equations

(a) Bernoulli

Form of Equation: $\frac{dy}{dx} + p(x)y = q(x)y^n, n \in \mathbb{R}$.

Method of Solution: The substitution $z = y^{1-n}$ transform the Bernoulli's equation into a linear equation in z. This linear equation, after solving and back substituting the value of z gives the solution of the Bernoulli's equation.

FIGURE 9. (a) Jacob Bernoulli (1654-1705), Swiss mathematician, one of eight Bernoulli mathematicians (all related to each other), Leibniz's (Fig. 1) student and ally against Newton (Fig. 11), discoverer of e and of the Law of Large Numbers. His younger brother Johann was Euler's (Fig. 22) adviser (as well as l'Hôpital's). (b) Jacopo Francesco Riccati (1676-1754),

First Order non Linear Differential Equations

1. **Solve**: $\dfrac{dy}{dx} + 3y = xy^2 \quad \dots \dots \dots (i)$

Solution: Put $z = y^{1-2} \qquad \Rightarrow z = \dfrac{1}{y}$

$$\Rightarrow y = \frac{1}{z}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$$

Equation (i) becomes, $-\dfrac{1}{z^2}\dfrac{dz}{dx} + 3\dfrac{1}{z} = x\dfrac{1}{z^2}$

$$\text{Or, } \frac{dz}{dx} - 3z = -x \quad \dots \dots \dots (ii)$$

Eq. (ii) is a first-order linear differential equation

Where, $p(x) = -3$, and $q(x) = -x$

$$\therefore I.F. \quad = e^{\int p(x)\,dx} = e^{\int -3dx} \quad = e^{-3x}$$

First Order non Linear Differential Equations

$$\therefore I.F. \quad = e^{\int p(x)\,dx} = e^{\int -3dx} \quad = e^{-3x}$$

Therefore, the solution of Eq. (ii) is:

$$z(\mathrm{I.F.}) = \int q(x)(\mathrm{I.F.})\,dx$$

$$\text{or, } z e^{-3x} = \int -x e^{-3x} dx + c$$

$$\text{or, } z e^{-3x} = -\left(x\frac{e^{-3x}}{-3} - \frac{e^{-3x}}{9} \right) + c$$

$$\text{or, } \frac{1}{y} e^{-3x} = \frac{xe^{-3x}}{3} + \frac{e^{-3x}}{9} + c$$

$$\text{or, } \frac{1}{y} = \frac{x}{3} + \frac{1}{9} + ce^{3x} \qquad \Rightarrow y = 1 \Big/ \left(\frac{x}{3} + \frac{1}{9} + ce^{3x} \right)$$

+	x	$\dfrac{e^{-3x}}{-3}$
-	1	$\dfrac{e^{-3x}}{9}$
+	0	

Which is the required solution

First Order non Linear Differential Equations

2. **Solve**: $\dfrac{dy}{dx}+xy=xy^2 \quad \dots \dots \dots (i)$

Solution: Put $z=y^{1-2} \quad \Rightarrow z=\dfrac{1}{y}$

$$\Rightarrow y=\frac{1}{z}$$

$$\therefore \frac{dy}{dx}=-\frac{1}{z^2}\frac{dz}{dx}$$

Equation (i) becomes, $-\dfrac{1}{z^2}\dfrac{dz}{dx}+x\dfrac{1}{z}=x\dfrac{1}{z^2}$

$$\text{Or, } \frac{dz}{dx}-xz=-x \quad \dots \dots \dots (ii)$$

Eq. (ii) is a first-order linear differential equation

Where, $p(x)=-x$, and $q(x)=-x$

$$\therefore I.F. \quad =e^{\int p(x)\,dx}=e^{\int -x dx} \quad =e^{-\frac{x^2}{2}}$$

First Order non Linear Differential Equations

$$\therefore I.F. = e^{\int p(x)\,dx} = e^{\int -x dx} = e^{-\frac{x^2}{2}}$$

Therefore, the solution of Eq. (ii) is:

$$z(\text{I.F.}) = \int q(x)(\text{I.F.})\,dx$$

$$\text{or, } z e^{-\frac{x^2}{2}} = \int -x e^{-\frac{x^2}{2}} dx + c$$

$$\text{or, } z e^{-\frac{x^2}{2}} = \int -x e^{-x^2/2} \frac{d\left(-x^2/2\right)}{-2x/2} + c$$

$$\text{or, } z e^{-\frac{x^2}{2}} = \int e^{-x^2/2} d\left(-x^2/2\right) + c$$

$$\text{or, } \frac{1}{y} e^{-x^2/2} = e^{-x^2/2} + c$$

$$\therefore \mathrm{y} = 1 \Big/ \left(1 + c e^{x^2/2}\right)$$

Which is the required solution

Self study

1. Solve: $$\frac{dy}{dx} + \frac{y}{x} = \left(\log x\right) y^2 \ , \ y\left(1\right)=1$$

2. Solve: $$xy - \frac{dy}{dx} = y^3 . e^{-x^2} .$$

3. Solve: $$\frac{dy}{dx} + y = x y^3$$

Exact Equations

A differential equation

$$M(x,y)dx + N(x,y)dy = 0.$$

is exact if there exists a function $g(x,y)$ such that

$$dg(x,y) = M(x,y)dx + N(x,y)dy.$$

Test for exactness: If $M(x,y)$ and $N(x,y)$ are continuous functions and have continuous first partial derivatives on some rectangle of the xy-plane, then

$$M(x,y)dx + N(x,y)dy = 0$$

is exact if and only if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}.$$

How to obtain an exact equation from a function?

$$f(x,y) = c,$$

$$\frac{\partial f(x,y)}{\partial x}dx + \frac{\partial f(x,y)}{\partial y}dy = 0,$$

$$M(x,y)dx + N(x,y)dy = 0.$$

First Order Exact Differential Equations

Equation Form: $M\left(x,y\right)dx+N(x,y)dy=0$

Test for exactness: $$\frac{\partial M\left(x.y\right)}{\partial y}=\frac{\partial N\left(x.y\right)}{\partial x}$$

General Solution : $\int M\left(x,y\right)dx+\int N_{(Terms\ free\ from\ x)}\ dy\ =0.$

First Order Exact Differential Equations

1. **Solve**: $\left(2x^3+4y\right)dx+\left(4x+y-1\right)dy=0 \; \dots \; \dots \; \dots \left(i\right)$

Solution: Let $M=2x^3+4y$ and $N=4x+y-1.$

$$\therefore \; \frac{\partial M}{\partial y}=4 \quad \text{and} \quad \frac{\partial N}{\partial x}=4.$$

$$\text{Gives, } \; \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}.$$

$\therefore$ Equation (i) is an exact equation.

Therefore the general solution is:

$$\int M\left(x,y\right)dx+\int N_{\left(\text{Terms free from } x\right)} \; dy \; =c.$$

$$\text{Or, } \int\left(2x^3+4y\right)dx+\int\left(y-1\right)\,dy \; =c.$$

$$\text{Or, } \frac{2x^4}{4}+4yx+\frac{y^2}{2}-y=c. \qquad \Rightarrow \frac{x^4}{2}+4xy+\frac{y^2}{2}-y=c.$$

First Order Exact Differential Equations

2. **Solve**: $\left(y^3 - y^2 \sin x - x\right)dx + \left(3xy^2 + 2y\cos x\right)dy = 0 \quad \ldots \quad \ldots \quad \ldots (i)$

Solution: Let $M = y^3 - y^2 \sin x - x$ and $N = 3xy^2 + 2y\cos x$.

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(y^3 - y^2 \sin x - x\right) \text{ and } \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(3xy^2 + 2y\cos x\right)$$

$$\therefore \frac{\partial M}{\partial y} = 3y^2 - 2y\sin x \text{ and } \frac{\partial N}{\partial x} = 3y^2 - 2y\sin x.$$

$$\text{Gives, } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$\therefore$ Equation (i) is an exact equation.

Therefore the general solution is:

$$\int M(x, y)\,dx + \int N_{(\text{Terms free from } x)}\,dy = c.$$

$$\text{Or, } \int\left(y^3 - y^2 \sin x - x\right)dx + \int 0\,dy = c.$$

$$\text{Or, } y^3 x + y^2 \cos x - \frac{x^2}{2} = c.$$

Self study

1. Solve: $\cos(x+y)\,dx+\left(3y^2+2y+\cos(x+y)\right)dy=0.$

2. Solve: $\left(1+2xy^3\right)dx+3x^2y^2dy=0$

3. Solve: $\left(ye^{xy}\cos 2x-2e^{xy}\sin 2x+2x\right)dx+\left(xe^{xy}\cos 2x-3\right)dy=0.$

Thank you

Dr. M. M. Rahman,
Professor of Mathematics