A partial differential equation is one which contains partial differential coefficients, independent variables and dependent variable.

The independent variables will be denoted by x and y and the dependent variable by z

The partial differential coefficients are denoted as follows

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q$$

$$\frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = s, \quad \frac{\partial^2 z}{\partial y^2} = t$$

Order and Degree: The order of a PDE is the order of the highest derivative of the equation. The degree of the PDE is the power of the highest derivative of the equation.

e.g. 1. $\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial z}{\partial x} + 6\frac{\partial z}{\partial y} = 0$ is the 2nd order 1st degree PDE.

2. $x^2 p + y^2 q = z^2$ is the 1st order 1st degree

3. $r + (a+b)s + abt = xy$ is the 2nd order and 1st degree.

Formation: A PDE is formed by two methods

i) by eliminating arbitrary constants

ii) by eliminating arbitrary functions

i) Formation of PDE by eliminating arbitrary constants—

Algorithm:

Input: Multivariable function

Output: PDE

Step1: consider $f(x, y, z, a, b) = 0$ ——— (1)

where z is a function of two independent variables x and y and a, b are arbitrary constants.

Step2: Differentiating (1) partially w.r.to x & y

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \quad \text{——— (2)}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} = \frac{\partial f}{\partial f} + q\frac{\partial f}{\partial z} = 0 \quad \text{——— (3)}$$

Step3: Eliminating arbitrary constants from (1) by applying (2) & (3) to get

$$f(x, y, z, p, q) = 0 \quad \text{——— (4)}$$

Step4: Stop

Note: 1) If the number of arbitrary constants is less than or equal to the number of independent variables then the differential equation formed by eliminating of arbitrary constants would be of the first order.

2) If the number of arbitrary constants are more than the number of independent variable than the differential equation will be of the minimum second order.

3) The PDE formed by eliminating arbitrary constants is not always unique.

Examples-

1. Form the PDE from $z = ax^3 + by^3 + ab$.

Soln: Given $z = ax^3 + by^3 + ab$ ——— (1)

Differentiating (1) partially with respect to x & y

$$\frac{\partial z}{\partial x} = p = 3ax^2 \Rightarrow a = \frac{p}{3x^2} \quad \text{——— (2)}$$

$$\frac{\partial z}{\partial y} = q = 3by^2 \Rightarrow b = \frac{q}{3y^2} \quad \text{——— (3)}$$

Putting the values of a and b in (1)

$$z = \frac{p}{3x^2}x^3 + \frac{q}{3y^2}y^3 + \frac{pq}{9x^2y^2}$$

$$\Rightarrow \boxed{9x^2y^2z = 3px^3y^2 + 3qx^2y^3 + pq}$$

2. Form a partial differential equation from

$$x^2+y^2+(z-c)^2=a^2 \quad \text{--------} \ (1)$$

Differentiating (1) partially w.r. to x we get

$$2x+2(z-c)\frac{\partial z}{\partial x}=0$$

$$x+(z-c)p=0 \quad \text{--------} \ (2)$$

Differentiating (1) partially w.r. to y we get

$$2y+2(z-c)\frac{\partial z}{\partial y}=0$$

$$y+(z-c)q=0 \quad \text{--------} \ (3)$$

Eliminating (c) from (2) & (3)

$$(2) \Rightarrow \quad z-c=-\frac{x}{p}$$

$$(3) \Rightarrow \quad y-\frac{x}{p}q=0$$

$$\Rightarrow \boxed{yp-xq=0}.$$

3. Form a partial differential equation from

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{--------} \ (1)$$

Differentiating (1) partially w.r. to x & y, we get

$$\boxed{\frac{x}{a^2}+\frac{z}{c^2}p=0} \quad \text{or} \quad \frac{c^2}{a^2}=-\frac{pz}{x} \quad \text{--------} \ (2)$$

$$\frac{y}{b^2}+\frac{z}{c^2}q=0 \quad \text{or} \quad \frac{c^2}{b^2}=-\frac{qz}{y} \quad \text{--------} \ (3)$$

Differenting (2) again partially w.r. to x, we get

$$\frac{1}{a^2}+\frac{1}{c^2}(p^2+zr)=0$$

$$\Rightarrow \frac{c^2}{a^2}=-(p^2+zr)=-\frac{pz}{x} \Rightarrow pz=(p^2+zr)x$$

Similarly, differentiating (2) w.r. to y we get

$$\frac{1}{b^2} + \frac{1}{c^2}(q^2 + zt) = 0$$

$$\Rightarrow \frac{c^2}{b^2} = -(q^2 + zt) = -\frac{qz}{y}$$

$$\Rightarrow qz = yq^2 + yzt$$

Practice-

Form the PDE from the following-

1. $z = (x+a)(y+b)$; $pq = z$
2. $(x-h)^2 + (y-k)^2 + z^2 = a^2$; $z^2(p^2+q^2+1) = a^2$
3. $ax^2 + by^2 + z^2 = 1$; $z(px+qy) = z^2 - 1$

~~(ii)~~ Elimination of arbitrary functions -

Let u, v be two functions of x, y, z connected by the relation $\phi(u,v)=0$ ——— (1)

Differentiating (1) partially w.r.to x and y, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right)=0$$

$$\Rightarrow \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0 \quad \text{——— (2)}$$

and

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0 \quad \text{——— (3)}$$

Eliminating $\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}$ from (2) & (3), we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z} \end{vmatrix}=0$$

$$\Rightarrow p\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)+q\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)+\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)=0 \quad \text{——— (4)}$$

$$\Rightarrow \boxed{Pp + qQ = R}$$

Example– Find the differential equation arising from $\phi(x-y+z, x^2+2y^2-3z^2)=0$

Sol^n: Let $u = x-y+z$, $v = x^2+2y^2-3z^2$

$$\frac{\partial u}{\partial x} = 1;\ \frac{\partial u}{\partial y} = -1,\ \frac{\partial u}{\partial z} = 1;\ \frac{\partial v}{\partial x} = 2x;\ \frac{\partial v}{\partial y} = 4y,\ \frac{\partial v}{\partial z} = -6z$$

Now applying equation (4) we get

$$p(4y-6z) + q(-6z-2x) + (4y+2x) = 0$$

or $\underbrace{p(2y-3z)}_{P} - q\underbrace{(x+3z)}_{Q} + \underbrace{(x+2y)}_{R} = 0$

Example– Form PDE from $z = f(x^2-y^2)$

Solution: $z = f(x^2-y^2)$ ——— (1)

Differentiating (1) partially w.r.to x & y

$$\frac{\partial z}{\partial x} = f'(x^2-y^2)\cdot 2x$$

$$p = f'(x^2-y^2)2x \text{ ——— } (2)$$

$$\frac{\partial z}{\partial y} = f'(x^2-y^2)(-2y)$$

$$q = f'(x^2-y^2)(-2y) \text{ ——— } (3)$$

$$(2)\div(3) \Rightarrow \frac{p}{q} = \frac{-x}{y}$$

$$\Rightarrow yp + xq = 0$$

Example– Eliminate arbitrary function ϕ from the equation $\phi(\tan x + \sin^{-1} y - \log z, e^x - \sec y + z^3) = 0$

Solution: Let $u = \tan x + \sin^{-1} y - \log z$, $v = e^x - \sec y + z^3$

$$\frac{\partial u}{\partial x} = \sec^2 x,\ \frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-y^2}}\ \ \frac{\partial u}{\partial z} = -\frac{1}{z}$$

$$\frac{\partial v}{\partial x} = e^x,\ \frac{\partial v}{\partial y} = -\sec y\tan y,\ \frac{\partial v}{\partial z} = 3z^2$$

Now $\phi(u, v) = 0$

$\cancel{p\left(\frac{1}{z}\sec y\tan y - \frac{1}{\sqrt{1-y^2}}3z^2\right) + q(-\sec^2 x\sec y\tan y}$

Applying eqn 4

$$p\left(\frac{1}{z}\sec y\tan y - \frac{1}{\sqrt{1-y^2}}3z^2\right) + q\left(\sec^2 x\,3z^2 + \frac{1}{z}e^x\right)$$
$$+\left(-\sec^2 x\sec y\tan y - \frac{1}{\sqrt{1-y^2}}e^x\right) = 0$$

Example: Find the differential equations from

$$\varphi(x+y+z,\ x^2+y^2-z^2) = 0$$

Sol^n: Let $u = x+y+z$, $v = x^2+y^2-z^2$

Then the given equation is $\varphi(u,v) = 0$

Differentiating it partially w.r. to x we get

$$\frac{\partial \varphi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \varphi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

$$\frac{\partial \varphi}{\partial u}(1+p) + \frac{\partial \varphi}{\partial v}(2x - 2zp) = 0 \quad \text{——— (1)}$$

Again differencting it partially w.r.to y we get

$$\frac{\partial \varphi}{\partial u}(1+q) + \frac{\partial \varphi}{\partial v}(2y - 2qz) = 0 \quad \text{——— (2)}$$

Eliminating $\frac{\partial \varphi}{\partial u}$ and $\frac{\partial \varphi}{\partial v}$, we get

$$\begin{vmatrix} 1+p & 2x - 2zp \\ 1+q & 2y - 2zq \end{vmatrix} = 0$$

$$2(1+p)(y - zq) - 2(1+q)(x - zp) = 0$$

$$(y+z)p - (x+z)q = x - y$$

Example- Eliminate the arbitrary functions $f(x)$ and $g(x)$ from $z = y f(x) + x g(y)$ —— (1)

Solution: Differentiating (1) partially w.r. to x and y we have

$$\frac{\partial z}{\partial x} = y f'(x) + g(y)$$

$$p = y f'(x) + g(y) \text{ ——— } (2)$$

$$q = f(x) + x g'(y) \text{ ——— } (3)$$

Again differentiating p and q w.r. to x we get

$$r = y f''(x) \quad ; \quad s = f'(x) + g'(y)$$

$$(2) \Rightarrow f'(x) = \frac{p - g(y)}{y} \quad ; \quad g'(y) = \frac{q - f(x)}{x}$$

$$s = \frac{p - g(y)}{y} + \frac{q - f(x)}{x}$$

or $xys = px + qy - [y f(x) + x g(y)]$

$$xys = px + qy - z$$

is the required partial differential equation.

Example: obtain the partial differential equation by eliminating the arbitrary functions f and g from $z = f(x\cos\alpha + y\sin\alpha - at) + g(x\cos\alpha + y\sin\alpha + at)$

Solution:

$$\frac{\partial z}{\partial x} = \cos\alpha\, f'(x\cos\alpha + y\sin\alpha - at) + \cos\alpha\, g(x\cos\alpha + y\sin\alpha + at)$$

$$\frac{\partial^2 z}{\partial x^2} = \cos^2\alpha\, f''(x\cos\alpha + y\sin\alpha - at) + \cos^2\alpha\, g(x\cos\alpha + y\sin\alpha + at)$$

similarly

$$\frac{\partial^2 z}{\partial y^2} = \sin^2\alpha\, f''(x\cos\alpha + y\sin\alpha - at) + \sin^2\alpha\, g(x\cos\alpha + y\sin\alpha + at)$$

$$\frac{\partial^2 z}{\partial t^2} = a^2 f''(x\cos\alpha + y\sin\alpha - at) + a^2 g''(x\cos\alpha + y\sin\alpha + at)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2}$$ is the required differential equation.

<u>Example</u>: Eliminate the functions from $x = f(z) + g(y)$ — (1)

<u>Solution</u>: Differentiating (1) w.r. to x and y, we get

$$1 = p f'(z) \quad \text{——— (2)}$$

$$0 = q f'(z) + g'(y) \quad \text{——— (3)}$$

Differentiating (2) w.r. to x & y

$$0 = p^2 f''(z) + r f'(z) \quad \text{——— (4)}$$

$$0 = pq f''(z) + s f'(z) \quad \text{——— (5)}$$

From (2) $f'(z) = \frac{1}{p}$

From (4) $f''(z) = -\frac{r}{p^2} f'(z) = -\frac{r}{p^3}$

From (5) $0 = -pq\left(\frac{r}{p^3}\right) + \frac{s}{p}$

$\Rightarrow \frac{s}{p} - \frac{pqr}{p^3} = 0$

$\Rightarrow p^2 s - pqr = 0$

$\therefore$ $ps - qr = 0$ is the required PDE.

Example Eliminate constant and function from

$$z = ax^2 + g(y)$$

<u>Solution:</u> $p = 2ax$

$$q = g'(y)$$

$$\frac{\partial p}{\partial x} = 2a$$

$$r = 2a$$

$$\frac{\partial q}{\partial x} = 0$$

$$s = 0$$

$$p = 2ax = 2a \cdot x = rx$$

$$p - xr = 0$$