P-1

Roles for finding the Particular Integral (P. I).

Given partial differential equation is

$$f(D, D')\,z = F(x, y)$$

$$P.I. = \frac{1}{f(D, D')} F(x, y)$$

Case 1: When $F(x,y) = e^{ax+by}$ then

$$P.I = \frac{1}{f(D, D')} F(x, y) = \frac{e^{ax+by}}{f(a,b)} \quad \left| \begin{array}{l} \text{put} \\ D = a \\ D' = b \end{array} \right.$$

Provided $f(a,b) \neq 0$

If $f(a,b) = 0$, then differentiate $f(D,D')$ with respect to D partially and multiply the expression by x, so that

$$P.I = x \cdot \frac{1}{f'(D,D')} e^{ax+by} = x \cdot \frac{1}{f'(a,b)} e^{ax+by}$$

Provided $f'(a,b) \neq 0$

If $f'(a,b)$ is also zero, proceed differentiating in this way every time and multiplying by x as long as the derivative of $f(D,D')$ vanishes when $D=a$ and $D'=b$.

Case 2: When $F(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

then $$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

$$= \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax+by)/\cos(ax+by)$$

Provided $f(-a^2, -ab, -b^2) \neq 0$

If $f(-a^2, -ab, -b^2) = 0$, then P.I is obtained in similar way as in the case 1.

Case 3: When $F(x,y) = x^a y^b$, a, b are constants

then $$P.I. = \frac{1}{f(D, D')} x^a y^b = [f(D, D')]^{-1} x^a y^b$$

and to evaluate it expand $[f(D, D')]^{-1}$ in ascending power of D or D' using the binomial theorem and then operate on $x^a y^b$ term by term.

Example 1: solve $(D^2+4DD'+4D'^2)z = e^{2x+y}$

Solution: The auxiliary equation of $(D^2+4DD'+4D'^2)z=0$ is $m^2+4m+4=0$

$\Rightarrow m^2+2\cdot m\cdot 2+2^2=0$

$\Rightarrow (m+2)^2=0$

$\therefore m=-2,-2$

Hence $C.F = f_1(y-2x)+xf_2(y-2x)$

Also $P.I = \dfrac{1}{D^2+4DD'+4D'^2}e^{2x+y}$ $a=2$ $b=1$

$= \dfrac{1}{2^2+4\cdot 2\cdot 1+4\cdot 1^2}e^{2x+y}$

$=\dfrac{1}{16}e^{2x+y}$

Therefore, the complete solution is

$$z=f_1(y-2x)+xf_2(y-2x)+\frac{1}{16}e^{2x+y}$$

Example 2: solve $(D^3-3D^2D'+4D'^3)z=e^{x+2y}$

Solution: The auxiliary equation of $(D^3-3D^2D'+4D'^3)z=0$ is $m^3-3m^2+4=0$

Now $m^3-3m^2+4=0$

$\Rightarrow m^3+m^2-4m^2-4m+4m+4=0$

$\Rightarrow m^2(m+1)-4m(m+1)+4(m+1)=0$

$\Rightarrow (m+1)(m^2-4m+4)=0$

$\Rightarrow (m+1)(m-2)^2=0 \Rightarrow m=-1,2,2$

$$C.F. = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)$$

$$\therefore P.I = \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$$

$D^3 - 3D^2D' + 4D'^3 = 27 \neq 0$

$$= \frac{1}{1-6+32} e^{x+2y} = \frac{1}{27} e^{x+2y}$$

Hence the complete solution is

$$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}$$

Example-3: solve $(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y}$

Solution: The auxiliary equation of $(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = 0$ is $m^3 - 2m^2 - m + 2 = 0$

$$\Rightarrow m^3 - m^2 - m^2 + m - 2m + 2 = 0$$

$$\Rightarrow m^2(m-1) - m(m-1) - 2(m-1) = 0$$

$$\Rightarrow (m-1)(m^2 - m - 2) = 0$$

$$\Rightarrow (m-1)\{m^2 - 2m + m - 2\} = 0$$

$$\Rightarrow (m-1)\{m(m-2) + 1(m-2)\} = 0$$

$$\Rightarrow (m-1)(m+1)(m-2) = 0$$

$$\therefore m = 1, -1, 2$$

$$C.F = f_1(y+x) + f_2(y-x) + f_3(y+2x)$$

$$P.I = \frac{1}{D^3 - 2D^2D' - DD'^2 + 2D'^3} e^{x+y}$$

If $D = 1$ and $D' = 1$, then $D^3 - 2D^2D' - DD'^2 + 2D'^3 = 0$

$3D^2 - 4DD' - D'^2 = 0$

$$\therefore \text{P.I} = \frac{x}{3D^2 - 4DD' - D'^2} e^{x+y}$$

$$= \frac{x e^{x+y}}{3-4-1} = -\frac{1}{2} x e^{x+y}$$

The complete solution is

$$z = f_1(y+x) + f_2(y-x) + f_3(y+2x) - \frac{1}{2} x e^{x+y}$$

Exercise-4: solve the following

1. $(D^2 - 5DD' + 6D'^2) z = e^{x+y}$

$$z = f_1(y+2x) + f_2(y+3x) + \frac{1}{2} e^{x+y}$$

2. $(D^2 - 4DD' + 4D'^2) z = e^{2x+y}$

$$z = f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}$$

3. $(D^2 - 7DD' + 12D'^2) = e^{x-y}$

$$z = f_1(y+3x) + f_2(y+4x) + \frac{1}{20} e^{x-y}$$