There are different forms of solutions of partial differential equation. For simplicity, we will consider two independent variables. But this can be extended to any number of variables. Solutions of partial differential equations appear in almost four forms, namely

1. Complete solution; 2. Particular solution
3. Singular solution and 4. General solution.

Complete Solution- The differential equation formed by eliminating arbitrary constants a, b from

$$z = ax^3 + by^3 + ab \quad \text{———} (1) \quad \text{is}$$

$$3px^3y^2 + 2qy^3x^2 + pq = 9x^2y^2z \quad \text{———} (2)$$

Equation (1) contains two arbitrary constants equal to the number of independent variables of differential equation (2) and is called the complete solution of (2).

Particular Integral: Assigning particular values for arbitrary constants, particular integrals of the differential equation are obtained.

$z = x^3 + y^3 + 1$ is a particular integral of (2) which is obtained by letting $a = 1$ and $b = 1$ in the complete solution.

Singular solution- Differentiating (1) w.r.to a and b we get

$$0 = x^3 + b \quad \text{or} \quad b = -x^3$$

$$0 = y^3 + a \quad \text{or} \quad a = -y^3$$

and eliminate a and b we obtain

$$z = -x^3y^3 - x^3y^3 + x^3y^3$$

or $z + x^3y^3 = 0$ is the singular solution.

General solution- General solution does not contain any arbitrary constant but is different from singular solution. In equation (1) making b a function of a, say b = a, we obtain

$$z = a(x^3 + y^3 + a)$$

Differenting w.r.to a,

$$0 = x^3 + y^3 + 2a$$

$$a = -\frac{1}{2}(x^3 + y^3)$$

Eliminating a we have

$$z = -\frac{1}{2}(x^3+y^3)\left[x^3+y^3 - \frac{1}{2}(x^3+y^3)\right]$$

$$= -\frac{1}{2}(x^3+y^3)\,\frac{1}{2}(x^3+y^3)$$

$$4z + (x^3+y^3)^2 = 0$$

which is a general solution and is different from singular solution.

Note: An arbitrary functional relation between a and b is of the form $b = \phi(a)$. Here the elimination becomes rather difficult. So we can choose any convenient relation between the arbitrary constants. Hence there are infinite number of general solutions.

Linear partial differential equation of 1st order-

A partial differential equation involving partial derivatives p and q only and not higher partial derivatives is called a first order partial differential equation.

If the degree of p and q is unity, then it is called a linear partial differential equation of order one. Thus $px^2+qy^2=z^2$ is a linear equation of first order.

On the other hand, if the degrees of p and q are higher than the first, then the equation is called non-linear

$p^2x^2+pqxy+q^2y^2=z^2$ is a non-linear first order partial differential equation.

Lagrange's Linear Equation- A Linear differential equation of the form $Pp+Qq=R$, where P, Q, R are functions of x, y, z is known as Lagrange's linear partial differential equation.

Algorithm to solve 1st order linear PDE $Pp+Qq=R$ by Lagrange's Method-

Input: First order linear PDE

Output: Solution of the PDE

Step1: Write down the given first order linear PDE in the standard form

$$Pp+Qq=R \quad \text{--- (1)}$$

Step2: Write down Lagrange's auxiliary equation for (1), namely $\dfrac{dx}{P}=\dfrac{dy}{Q}=\dfrac{dz}{R}$ --- (2)

Step3: Find the two independent solutions of (2) in the form

$$u(x, y, z) = a$$

and

$$v(x, y, z) = b$$

Step4: The general solution or integral of (1) is then written in one of the following three equivalent forms:

$$\varphi(u, v) = 0, \quad u = \varphi(a), \quad v = \varphi(b)$$

Steps5: Stop

Note: The system can be generalized to any number of independent variables.

Example: solve $yzp + zxq = xy$ —— (1)

Auxiliary equations of (1) are

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

From first two $\frac{dx}{y} = \frac{dy}{x}$

Integrating $x^2 - y^2 = c_1$

Similarly from last two, we have

$$\frac{dy}{z} = \frac{dz}{y}$$

$$y^2 - z^2 = c_2$$

Hence the required general solution is

$$\varphi(x^2 - y^2, y^2 - z^2) = 0$$

φ is arbitrary function.

Example: Solve $(y+z)p + (z+x)q = x+y$ —— (1)

The subsidiary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

By the theory of proportion, we find

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{-(x-y)}$$

$$\Rightarrow \frac{d(x+y+z)}{(x+y+z)} + 2\frac{d(x-y)}{x-y} = 0$$

$$\ln(x+y+z) + 2\ln(x-y) = \ln c$$

$$(x+y+z)(x-y)^2 = c$$

Again we have

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$

$$\ln(x-y) = \ln(y-z) + \ln c_2$$

or $$\frac{x-y}{y-z} = c_2$$

The general solution is

$$\phi\left[(x+y+z)(x-y)^2, \frac{x-y}{y-z}\right] = 0$$

Example: solve $pz - qz = z^2 + (x+y)^2$ ——— (1)

Solution: The subsidiary equations are

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$$

Taking the first and second members, we have

$$dx + dy = 0$$

$$x + y = c_1$$

Taking the first and the last members, we have

$$dx = \frac{z\,dz}{z^2+(x+y)^2}$$

$$2\,dx = \frac{2z\,dz}{z^2+c_1^2}$$

Integrating, we have

$$2x + c_2 = \ln(z^2+c_1^2)$$

$$\Rightarrow \ln(z^2+x^2+y^2+2xy) - 2x = c_2$$

Hence the complete integral is

$$\phi\left[x+y, \ln\{z^2+(x+y)^2\} - 2x\right] = 0$$

Example- solve $p\cos(x+y)+q\sin(x+y)=z$ ——— (1)

Lagrange's auxiliary equations are

$$\frac{dx}{\cos(x+y)}=\frac{dy}{\sin(x+y)}=\frac{dz}{z}$$

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)}=\frac{dx-dy}{\boxed{\cos(x+y)-\sin(x+y)}}$$

or $$\frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}(dx+dy)=dx-dy$$

Integrating

$$\ln[\cos(x+y)+\sin(x+y)]=x-y+\ln a$$

$$\Rightarrow [\cos(x+y)+\sin(x+y)]e^{x-y}=a$$

Again $$\frac{dx+dy}{\cos(x+y)+\sin(x+y)}=\frac{dz}{z}$$

$$\Rightarrow \frac{\frac{1}{\sqrt{2}}(dx+dy)}{\cos(x+y)\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\sin(x+y)}=\frac{dz}{z}$$

$$\Rightarrow \int\frac{\frac{1}{\sqrt{2}}(dx+dy)}{\sin(x+y+\frac{\pi}{4})}=\int\frac{dz}{z}$$ ✓ $\int$ cosec u du

$\rule{5cm}{0.4pt}$ $=\ln z$ =

$\frac{1}{\sqrt{2}}$

Example: Solve $p+3q=5z+\tan(y-3x)$

Solution: Auxiliary equations are

$$\frac{dx}{1}=\frac{dy}{3}=\frac{dz}{5z+\tan(y-3x)}$$

Taking the first two members, we have

$$dy-3dx=0\;;\; y-3x=a$$

Again taking the first and the last members

$$\frac{dx}{1}=\frac{dz}{5z+\tan(y-3x)}=\frac{dz}{5z+\tan a}$$

$$x=\frac{1}{5}\log(5z+\tan a)-\log c_1$$

$$5x=\log(5z+\tan a)-\log b$$

$$\left[5z+\tan(y-3x)\right]e^{-5x}=b$$

The general solution is

$$\phi\left[y-3x,\; e^{-5x}\{5z+\tan(y-3x)\}\right]=0$$

Method of Multipliers –

Let $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ be the auxiliary equations

l, m, n may be constants or functions of x, y, z, then we have

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l\,dx+m\,dy+n\,dz}{lP+Qm+nR}$$

l, m, n are chosen in such a way that $lP+mQ+nR=0$

Thus $l\,dx+m\,dy+n\,dz=0$ Solve it and if the solution is $u=c_1$

Similarly, choose another set of multipliers (l_1, m_1, n_1) and if the second solution is $v=c_2$

Required solution is $f(u,v)=0$

Example- Solve $(z^2-2yz-y^2)p+(xy+zx)q=xy-zx$

Solution: The auxiliary equations are

$$\frac{dx}{z^2-2yz-y^2}=\frac{dy}{xy+zx}=\frac{dz}{xy-zx}$$

Taking x, y, z as multipliers, we have each relation

$$=\frac{x\,dx+y\,dy+z\,dz}{0}$$

$$x\,dx+y\,dy+z\,dz=0$$

Integrating

$$x^2+y^2+z^2=a$$

Again taking the last two members, we have

$$\frac{dy}{y+z}=\frac{dz}{y-z}$$

or, $y\,dy-(y\,dz+z\,dy)-z\,dz=0$

Integrating

$$\frac{y^2}{2}-yz-\frac{z^2}{2}=c_1$$

$$y^2-2yz-z^2=b.$$

The general solution

$$x^2+y^2+z^2=\varphi(y^2-2yz-z^2)$$

Example- Solve $(mz-ny)\frac{\partial z}{\partial x}+(nx-lz)\frac{\partial z}{\partial y}=ly-mx$

Solution: The auxiliary equations are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

Using multipliers x, y, z we get

each factor $=\dfrac{x\,dx+y\,dy+z\,dz}{0}$

$\therefore x\,dx+y\,dy+z\,dz=0$

which on integration gives $x^2+y^2+z^2=c_1$ —— ①

Again using multipliers l, m, n we get

each ~~factor~~ fraction $= \dfrac{l\,dx+m\,dy+n\,dz}{0}$

$l\,dx+m\,dy+n\,dz=0$, which, on integration

gives $lx+my+nz=c_2$ —— ②

Hence from ① and (2) the required solution is

$x^2+y^2+z^2=f(lx+my+nz)$.

Example: Find the general solution of

$$x(z^2-y^2)\frac{\partial z}{\partial x}+y(x^2-z^2)\frac{\partial z}{\partial y}=z(y^2-x^2) \quad ①$$

Solution: The auxiliary equations are

$$\frac{dx}{x(z^2-y^2)}=\frac{dy}{y(x^2-z^2)}=\frac{dz}{z(y^2-x^2)} \quad ②$$

using multipliers x, y, z we get, each term of (2) is equal to

$$=\frac{x\,dx+y\,dy+z\,dz}{0} \Rightarrow x\,dx+y\,dy+z\,dz=0$$

on integration $x^2+y^2+z^2=c_1$ —— ③

(2) can be written as

$$\frac{\frac{dx}{x}}{z^2-y^2}=\frac{\frac{dy}{y}}{x^2-z^2}=\frac{\frac{dz}{z}}{y^2-x^2} \quad ④$$

Each term of ④ is $\dfrac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}=0$

$$\therefore \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

$$\ln x+\ln y+\ln z=\ln c_2$$

$$xyz=c_2 \quad ⑤$$

From (3) & (5), the general solution is

$$xyz=f(x+y+z)$$

Example- solve $(x^2-y^2-z^2)p+2xyq=2xz$ —— (1)

Solution: Here the auxiliary equations are

$$\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz} \qquad \text{—— (2)}$$

From the last two members of (2) we have

$$\frac{dy}{y}=\frac{dz}{z}$$

which on integration gives

$$\ln y=\ln z+\ln a \Rightarrow \frac{y}{z}=a \qquad \text{—— (3)}$$

Using multipliers x, y, z we have

$$\text{each } \cancel{\text{factor}} \text{ fraction} = \frac{x\,dx+y\,dy+z\,dz}{x(x^2+y^2+z^2)}$$

or $$\frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2}=\frac{dz}{z}$$

or. $$\ln(x^2+y^2+z^2)=\ln z+\ln b$$

or $$\frac{x^2+y^2+z^2}{z}=b \qquad \text{—— (4)}$$

Hence from (3) and (4), the required solution is

$$x^2+y^2+z^2=zf\left(\frac{y}{z}\right)$$