7. $x\,y'' + y' + xy = 0$ about $x = 0$

Ans. $y = A\left[1 - \frac{1}{2^2}x^2 + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \ldots\ldots\right]$

$+ B\left[y_1 \log x + a_0\left\{\frac{x^2}{2^2} - \frac{1}{2^2\cdot 4^2}\left(1+\frac{1}{2}\right)x^4 + \frac{1}{2^2\cdot 4^2\cdot 6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^6 + \ldots\ldots\right\}\right]$

8. $x\,(1 - x)\,y'' + 4y' + 2y = 0$ (*A.M.I.E.T.E., Summer 2001, 2000*)

Ans. $y_1 = a_0\left[1 - \frac{x}{2} + \frac{x^2}{10}\right]$, $y_2 = a_0\,x^{-3}\,[\,1 - 5x + 10\,x^2 - 10\,x^3 + 5x^4 - x^5\,]$

9. $x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + x^2y = 0$ (*A.M.I.E.T.E., Summer 2001, 2000, Winter 2003*)

[**Hint** $y = \Sigma\,a_k\,(m + k0\,\,(m + k + 4)\,\,x^{m+k} + \Sigma a_k\,x^{m+k+2}$]

Indicial eq. $a_0\,\,m\,(m + 4) = 0$

Ans. $= a_0\left[1 - \frac{x^2}{12} + \frac{x^4}{364} - \ldots\ldots\right] + b_0\,x^{-4}\,\log x\left[1 - \frac{x^4}{16} - \frac{x^6}{16}\ldots\ldots\ldots\right] + b_0x^{-}2\left[\frac{1}{4} + \frac{x^2}{64}\,\,\ldots.\right]$

8.6 BESSEL'S EQUATION

The differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\,y = 0$$

is called the *Bessel's differential equation,* and particular solutions of this equation are called Bessel's functions of order *n*.

8.7 SOLUTION OF BESSEL'S EQUATION

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\,y = 0. \qquad \ldots(1)$$

Let $\qquad y = \Sigma\,a_r\,x^{m+r}$ or $y = a_0\,x^m + a_1\,x^{m+1} + a_2\,x^{m+2} + \ldots \qquad \ldots(2)$

so that $\frac{dy}{dx} = \Sigma\,a_r\,(m + r)\,x^{m+r-1}$ and $\frac{d^2y}{dx^2} = \Sigma\,a_r\,(m + r)\,(m + r - 1)\,x^{m+r-2}$

Substituting these values in the equation, we have

$$x^2\,\Sigma\,a_r\,(m + r)\,(m + r - 1)\,x^{m+r-2} + x\,\Sigma\,a_r\,(m + r)\,x^{m+r-1} + (x^2 - n^2)\,\Sigma\,a_r\,x^{m+'r} = 0$$

or $\quad \Sigma\,a_r\,(m + r)\,(m + r - 1)\,x^{m+r} + \Sigma\,a_r\,(m + r\}\,x^{m+r} + \Sigma\,a_rx^{m+r+2} - n^2\,\Sigma\,a_r\,x^{m+r} = 0$

or $\quad \Sigma\,a_r\,[(m + r)\,(m + r - 1) + (m + r) - n^2]\,x^{m+r} + \Sigma\,a_r\,x^{m+r+2} = 0$

or $\quad \Sigma\,a_r\,[(m + r)^2 - n^2]\,x^{m+r} + \Sigma\,a_r\,x^{m+r+2} = 0.$

Equating the coefficient of x^m to zero, we get

$$a_0\,[(m + 0)^2 - n^2] = 0. \qquad (r = 0)$$

or $\quad m^2 = n^2$ *i.e.* $m = n \qquad a_0 \neq 0$

Equating the coefficient of x^{m+1} $\qquad r = 1$

$$a_1\,[(m + 1)^2 - n^2] = 0 \;\; i.e. \;\; a_1 = 0. \qquad \text{since } (m + 1)^2 - n^2 \neq 0$$

Equating the coefficient of x^{m+r+2} to zero, to find relation in successive coefficients, we get

$$a_{r+2}\,[(m + r + 2)^2 - n^2] + a_r = 0 \qquad \text{or} \qquad a_{r+2} = -\frac{1}{(m + r + 2)^2 - n^2}\cdot a_r$$

Therefore $$a_3 = a_5 = a_7 = \ldots = 0, \text{ since } a_1 = 0$$

If $r = 0$, $$a_2 = -\frac{1}{(m+2)^2 - n^2} a_0$$

If $r = 2$, $$a_4 = -\frac{1}{(m+4)^2 - n^2} a_2 = \frac{1}{[(m+2)^2 - n^2]\,[(m+4)^2 - n^2)]} a_0$$ and so on

On substituting the values of the coefficients in (2) we have

$$y = a_0 x^m - \frac{a_0}{(m+2)^2 - n^2} x^{m+2} + \frac{a_0}{[(m+2)^2 - n^2]\,[(m+4)^2 - n^2]} x^{m+4} + \ldots$$

$$y = a_0 x^m \left[1 - \frac{1}{(m+2)^2 - n^2} x^2 + \frac{1}{[(m+2)^2 - n^2]\,[(m+4)^2 - n^2]} x^4 - \ldots \right]$$

For $m = n$

$$y = a_0 x^n \left[1 - \frac{1}{4\,(n+1)} x^2 + \frac{1}{4^2 \cdot 2\,!\,(n+1)\,(n+2)} x^4 - \ldots \right]$$

where a_0 is an arbitrary constant.

For $m = -n$

$$y = a_0 x^{-n} \left[1 - \frac{1}{4\,(-n+1)} x^2 + \frac{1}{4^2 \cdot 2\,!\,(-n+1)\,(-n+2)} x^4 - \ldots \right]$$

8.8 BESSEL FUNCTIONS, $J_n(x)$

The Bessel's equation is $$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0 \qquad \ldots(1)$$

Solution of (1) is

$$y = a_0 x^n \left[1 - \frac{x^2}{2 \cdot 2\,(n+1)} + \frac{x^4}{2.4.2^2\,(n+1)\,(n+2)} - \ldots + (-1)^r \frac{x^{2r}}{(2^r\, r\,!) \cdot 2^r\,(n+1)\,(n+2) \ldots (n+r)} + \ldots \right]$$

$$= a_0 x^n \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{2^{2r} \cdot r\,!\,(n+1)\,(n+2) \ldots (n+r)}$$

where a_0 is an arbitrary constant.

If $$a_0 = \frac{1}{2^n \Gamma(n+1)}$$

The above solution is called Bessel's, function denoted by $J_n(x)$.

Thus $$J_n(x) = \frac{1}{2^n \Gamma(n+1)} \Sigma\, (-1)^r \frac{x^{n+2r}}{2^{2r} \cdot r\,!\,(n+1)\,(n+2) \ldots (n+r)}$$

$$(\Gamma\, n+1 = n\,!)$$

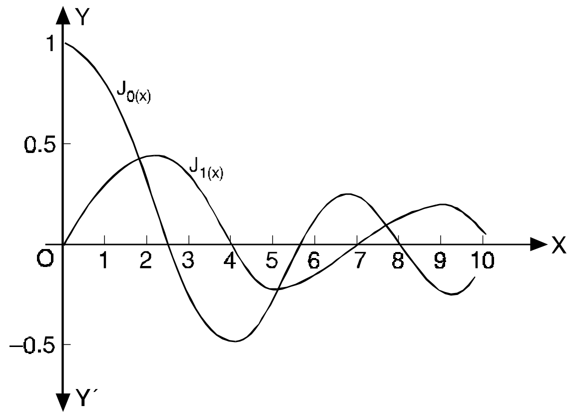
$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{ \frac{1}{\Gamma(n+1)} - \frac{1}{\lfloor 1\, \Gamma(n+2)} \left(\frac{x}{2}\right)^2 + \frac{1}{\lfloor 2\, \Gamma(n+3)} \left(\frac{x}{2}\right)^4 - \frac{1}{\lfloor 3\, \Gamma(n+4)} \left(\frac{x}{2}\right)^6 + \ldots \right\}$$

or $$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,(n+r)\,!} \left(\frac{x}{2}\right)^{n+2r} \qquad \ldots(2)$$

If $n = 0, J_0(x) = \Sigma \dfrac{(-1)^r}{(r!)^2}\left(\dfrac{x}{2}\right)^{2r}$ or $J_0(x) = 1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2 \cdot 4^2} - \dfrac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$

If $n = 1$, $J_1(x) = \dfrac{x}{2} - \dfrac{x^3}{2^2 \cdot 4} + \dfrac{x^5}{2^2 \cdot 4^2 \cdot 6} - \ldots$

We draw the graphs of these two functions. Both the functions are oscillatory with a varying period and a decreasing amplitude.



Replacing n by $-n$ in (2), we get

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

General solution of Bessel's Equation is

$$y = AJ_n(x) + BJ_{-n}(x)$$

Example 10. *Prove that*

$$J_{-n}(x) = (-1)^n J_n(x)$$

where n is a positive integer. (*A.M.I.E.T.E., Winter 2001*)

Solution. $$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(r-n+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

$$= \sum_{r=0}^{n-1} \frac{(-1)^r\left(\frac{x}{2}\right)^{-n+2r}}{r!\,\Gamma(-n+r+1)} + \sum_{r=n}^{\infty} \frac{(-1)^r\left(\frac{x}{2}\right)^{-n+2r}}{r!\,\Gamma(-n+r+1)}$$

$$= 0 + \sum_{r=n}^{\infty} \frac{(-1)^r\left(\frac{x}{2}\right)^{-n+2r}}{r!\,\Gamma(-n+r+1)} \qquad \text{since } \Gamma\, -\text{ve integer} = \infty$$

On putting $r = n + k$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k}\left(\frac{x}{2}\right)^{n+2k}}{(n+k)!\,\Gamma(k+1)}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k\left(\frac{x}{2}\right)^{n+2k}}{(n+k)!\,k!}$$

$$= (-1)^n J_n(x)$$ **Proved.**

Example 11. *Prove that*

(*a*) $J_{1/2}(x) = \sqrt{\left(\dfrac{2}{\pi x}\right)}\, \sin x$ (*b*) $J_{-1/2}(x) = \sqrt{\left(\dfrac{2}{\pi x}\right)}\, \cos x$

Solution. *We know that*

$$J_n(x) = \frac{x^n}{2^n \lceil n+1}\left[1 - \frac{x^2}{2.2\,(n+1)} + \frac{x^4}{2.4.2^2\,(n+1)\,(n+2)} \cdots\right] \quad ...(1)$$

(*a*) Substituting $n = \dfrac{1}{2}$ in (1) we obtain

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2}\lceil \frac{1}{2}+1}\left[1 - \frac{x^2}{2.2.\left(\frac{1}{2}+1\right)} + \frac{x^4}{2.4.2^2\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)} \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\lceil 3/2}\left[1 - \frac{x^2}{2.3\,!} + \frac{x^4}{2.3.4.5} \cdots\right] = \frac{\sqrt{x}}{\sqrt{2}\cdot\frac{1}{2}\lceil\frac{1}{2}}\frac{1}{x}\left[x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \cdots\right]$$

$$= \frac{1}{\sqrt{2x}\cdot\frac{1}{2}\sqrt{\pi}}\sin x = \sqrt{\left(\frac{2}{\pi x}\right)}\sin x \qquad \left(\text{since } \lceil\frac{1}{2} = \sqrt{\pi}\right) \quad \textbf{Proved.}$$

(*b*) Again substituting $n = -\dfrac{1}{2}$ in (1), we have

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2}\lceil -\frac{1}{2}+1}\left[1 - \frac{x^2}{2.2.\left(-\frac{1}{2}+1\right)} + \frac{x^4}{2.4.2^2\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}+2\right)} - \cdots\right]$$

$$= \frac{\sqrt{2}}{\sqrt{x}\lceil\frac{1}{2}}\left[1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \cdots\right] = \sqrt{\left(\frac{2}{\pi x}\right)}\cos x \quad \left(\text{since } \lceil\frac{1}{2} = \sqrt{\pi}\right) \quad \textbf{Proved.}$$

8.9 RECURRENCE FORMULAE

Formula I. $x J_n' = n J_n - x J_{n+1}$

Proof. We know that

$$J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\lceil(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$$

Differentiating with respect to *x,* we get

$$J_n' = \sum \frac{(-1)^r(n+2r)}{r\,!\lceil(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}\frac{1}{2}.$$

$$x J_n' = n\sum \frac{(-1)^r}{r\,!\lceil(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} + x\sum \frac{(-1)^r\cdot 2r}{2\cdot r\,!\lceil(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n J_n + x\sum_{r=1}^{\infty}\frac{(-1)^r}{(r-1)\,!\lceil(n+r+1)}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n J_n + x \sum_{s=0}^{\infty} \frac{(-1)^{s+1}}{s!\,\Gamma(n+s+2)} \left(\frac{x}{2}\right)^{n+2s+1} \qquad \text{Putting } r-1=s$$

$$= n J_n + x \sum_{s=0}^{\infty} \frac{(-1)^{s}}{s!\,\Gamma[(n+1)+s+1]} \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$= n J_n - x J_{n+1}$$ **Proved.**

Formula II. $x J_n' = -n J_n + x J_{n-1}$

Proof. We have $J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$

Differentiating w.r.t. 'x' we get $J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{2}$

$$x J_n' = \sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} = \sum_{r=0}^{\infty} \frac{(-1)^r [(2n+2r)-n]}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r (2n+2r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} - n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r\, 2}{r!\,\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r} - n J_n = x \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma[(n-1)+r+1]} \left(\frac{x}{2}\right)^{(n-1+2r)} - n J_n$$

$$= x J_{n-1} - n J_n$$ **Proved.**

Formula III. $2 J_n' = J_{n-1} - J_{n+1}$

Proof. We know that

$$x J_n' = n J_n - x J_{n+1} \qquad \text{...(1)} \qquad \text{Recurrence formula } I$$

$$x J_n' = -n J_n + x J_{n-1} \qquad \text{...(2)} \qquad \text{Recurrence formula } II$$

Adding (1) and (3), we get

$$2 x J_n' = -x J_{n+1} + x J_{n-1} \quad \text{or} \quad 2 J_n' = J_{n-1} - J_{n+1}$$ **Proved.**

Formula IV. $2 n J_n = x (J_{n-1} + J_{n+1})$

Proof. We know that

$$x J_n' = n J_n - x J_{n-1} \qquad \text{Recurrence formula I}$$

$$x J_n' = -n J_n + x J_{n+1} \qquad \text{Recurrence formula II}$$

Subtracting (2) from (1), we get

$$0 = 2 n J_n - x J_{n+1} - x J_{n-1} \quad \text{or} \quad 2 n J_n = x (J_{n-1} + J_{n+1})$$ **Proved.**

Formula V. $\frac{d}{dx}(x^{-n} \cdot J_n) = -x^{-n} J_{n+1}$

Proof. We know that

$$x\,J_n' = n\,J_n - x\,J_{n+1} \qquad \text{Recurrence formula } I$$

Multiplying by $x^{-n-1,}$ we obtain

$$x^{-n}\,J_n' = n\,x^{-n-1}\,J_n - x^{-n}\,J_{n+1}$$

i.e., $$x^{-n}\,J_n' - n\,x^{-n-1}\,J_n = -\,x^{-n}\,J_{n+1}$$

or $$\frac{d}{dx}(x^{-n}\,J_n) = -\,x^{-n}\,J_{n+1}$$ **Proved.**

Formula VI. $\dfrac{d}{dx}(x^n\,J_n) = x^n\,J_{n-1}$

Proof. We know that

$$x\,J_n' = -\,n\,J_n + x\,J_{n-1} \qquad \text{Recurrence formula } II$$

Multiplying by x^{n-1}, we have

$$x^n\,J_n' = -\,n\,x^{n-1}\,J_n + x^n\,J_{n-1}$$

i.e., $$x^n\,J_n' + n\,x^{n-1}\,J_n = x^n\,J_{n-1}$$

or $$\frac{d}{dx}(x^n\,J_n) = x^n\,J_{n-1}$$ **Proved.**

Example 12. *Prove that* $J_2'(x) = \left(1 - \dfrac{4}{x^2}\right) J_1(x) + \dfrac{2}{x} J_0(x)$ *where* $J_n(x)$ *is the Bessel function of first kind.* (*U.P. III Semester, Summer 2001*)

Solution. By recurrence formula *II*

$$x\,J_n' = -\,n\,J_n + x\,J_{n-1} \qquad ...(1)$$

On putting $n = 2$, in (1) we have $x\,J_2' = -\,2\,J_2 + x\,J_1$

or $$J_2' = -\frac{2}{x}J_2 + J_1 \qquad ...(2)$$

By recurrence formula *I*

$$x\,J_n' = n\,J_n - x\,J_{n+1} \qquad ...(3)$$

From (1) and (3) we have $-\,n\,J_n + x\,J_{n-1} = n\,J_n - x\,J_{n+1}$

On putting $n = 1$, $-\,J_1 + x\,J_0 = J_1 - x\,J_2$

or $$-\frac{1}{x}J_1 + J_0 = \frac{1}{x}J_1 - J_2 \quad \text{or} \quad J_2 = \frac{2}{x}J_1 - J_0 \qquad ...(4)$$

Putting the value of J_2 from (4) in (2) we get

$$J_2' = -\frac{2}{x}\left(\frac{2}{x}J_1 - J_0\right) + J_1 = -\frac{4}{x^2}J_1 + \frac{2}{x}J_0 + J_1$$

$$= \left(1 - \frac{4}{x^2}\right)J_1 + \frac{2}{x}J_0$$ **Proved.**

Example 13. *Using the recurrence relations, show that*

$$4\,J_n''(x) = J_{n-2}(x) - 2\,J_n(x) + J_{n+2}(x).$$

Solution. We know that the recurrence formula

$$2\,J_n' = J_{n-1} - J_{n+1} \qquad ...(1)$$

On differentiating again, we have

$$2\,J_n'' = J'_{n-1} - J'_{n+1} \qquad \text{...(2)}$$

Replacing n by $n-1$ and n by $n+1$ in (1) we have

$$2\,J'_{n-1} = J_{n-2} - J_n \quad \text{or} \quad J'_{n-1} = \frac{1}{2} J_{n-2} - \frac{1}{2} J_n \qquad \text{...(3)}$$

$$2\,J'_{n+1} = J_n - J_{n+2} \quad \text{or} \quad J'_{n+1} = \frac{1}{2} J_n - \frac{1}{2} J_{n+2} \qquad \text{...(4)}$$

Putting the values of J'_{n-1} and J'_{n+1} from (3) and (4) in (2) we get

$$2\,J_n'' = \frac{1}{2}\left[J_{n-2} - J_n\right] - \frac{1}{2}\left[J_n - J_{n+2}\right]$$

or $$4\,J_n'' = J_{n-2} - J_n - J_n + J_{n+2}$$

or $$4\,J_n'' = J_{n-2} - 2\,J_n + J_{n+2}$$ **Proved.**

Example 14. *Prove that* $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$ *(A.M.I.E.T.E., Summer 2002)*

Solution.
$$x^n J_n(x) = x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} = \sum \frac{(-1)^r x^{2n+2r}}{r!\,\Gamma(n+r+1)\cdot 2^{n+2r}}$$

$$\frac{d}{dx}[x^n J_n(x)] = \sum \frac{(-1)^r (2n+2r)\, x^{2n+2r-1}}{r!\,\Gamma(n+r+1)\cdot 2^{n+2r}}$$

$$= x^n \sum \frac{(-1)^r (n+r)}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1} = x \sum \frac{(-1)^r}{r!\,\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= x^n \sum \frac{(-1)^r}{r!\,\Gamma(n-1+r+1)} \left(\frac{x}{2}\right)^{\overline{n-1}+2r}$$

$$= x^n J_{n-1}(x)$$ **Proved.**

Similarly we can prove that

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

Example 15. *Prove that* $\frac{d}{dx}(x\,J_n\,J_{n+1}) = x\,(J_n^2 - J_{n+1}^2)$

Solution.
$$\frac{d}{dx}(x\,J_n\,J_{n+1}) = J_n\,J_{n+1} + x\frac{d}{dx}(J_n\,J_{n+1})$$

$$= J_n\,J_{n+1} + x\,(J_n\,J'_{n+1} + J_n'\,J_{n+1})$$

$$= J_n\,J_{n+1} + (x\,J_n')\,J_{n+1} + J_n\,(x\,J'_{n+1}) \qquad \text{...(1)}$$

Recurrence formula I, $x\,J_n' = n\,J_n - x\,J_{n+1}$ ...(2)

Recurrence formula II, $x\,J_n' = -n\,J_n + x\,J_{n-1}$

Putting $n+1$ for n, $x\,J'_{n+1} = -(n+1)\,J_{n+1} + x\,J_n$ ...(3)

Putting the values of $x\,J_n'$ and $x\,J'_{n+1}$ from (2) and (3) in (1) we obtain

$$\frac{d}{dx}(x\,J_n\,J_{n+1}) = J_n\,J_{n+1} + (n\,J_n - x\,J_{n+1})\,J_{n+1} + J_n\,[-(n+1)\,J_{n+1} + x\,J_n]$$

$$= (1 + n - n - 1)\,J_n\,J_{n+1} + x\,(J_n^2 - J_{n+1}^2)$$

$$= x\,(J_n^2 - J_{n+1}^2)$$ **Proved.**

Example 16. *Prove that* $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

Solution $x^n J_n(x) = x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r} = \sum \frac{(-1)^2\, x^{2n+2r}}{r\,!\,\Gamma(n+r+1).\, 2^{n+2r}}$

$$\frac{d}{dx}[x^n J_n(x)] = \sum \frac{(-1)^r (2n+2r)\, x^{2n+2r-1}}{r\,!\,\Gamma(n+r+1)\,.\,2^{n+2r}}$$

$$= x^n \sum \frac{(-1)^r (n+r)}{r\,!\,\Gamma(n+r+1)\,.\,2^{n+2r}} \left(\frac{x}{2}\right)^{n+2r-1} = x^n \sum \frac{(-1)^r}{r\,!\,\Gamma(n+r)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= x^n \sum \frac{(-1)^r}{r\,!\,\Gamma(n-1+r+1)} \left(\frac{x}{2}\right)^{n-1+2r}$$

$$= x^n J_{n-1}(x)$$ **Proved.**

Similarly we can prove that

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

$$J_3(x) + 3\,J_0'(x) + 4\,J_0'''(x) = 0$$

Example 17. *Prove that*

$\int J_3(x)\,dx + J_2(x) + \frac{2}{x} J_1(x) = 0$ (*A.M.I.E.T.E., Summer 2000*)

Solution. We know that

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$ (Recurrence Relation V)

Integrating above relation, we get

$$x^{-n} J_n(x) = -\int x^{-n} J_{n+1}(x)\,dx \qquad \text{... (1)}$$

On taking $n = 2$ in (1), we have

$$\int x^{-2} J_3(x)\,dx = -x^{-2} J_2(x) \qquad \text{... (2)}$$

Again $\int J_3(x)\,dx = \int x^2 (x^{-2}) J_3(x)\,dx$

$$= x^2 \int (x^{-2}) J_3(x)\,dx - \int 2x \int (x^{-2} J_3 x)\,dx \qquad \text{... (3)}$$

Putting the value of $\int x^{-2} J_3(x)\,dx$ from (2) in (3), we get

$$\int J_3(x)\,dx = x^2(-x^{-2} J_2) - \int 2x(-x^{-2} J_2)\,dx$$

$$= -J_2 + 2\int x^{-1} J_2\,dx = -J_2 + 2(-x^{-1} J_1)$$

On using (1), again, when $n = 1$