METHOD OF VARIATION OF PARAMETERS

To find particular integral of $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=X$

General solution = complementary function + particular integral.

Working Rule

Step 1. Find out the $C.F.$ *i.e.*, $A\,y_1+B\,y_2$

Step 2. Particular integral = $u\,y_1+v\,y_2$

Step 3. Find u and v by the formulae

$$u=\int\frac{-y_2\,X}{y_1\,y_2'-y_1'\,y_2}\,dx,\quad v=\int\frac{y_1\,X}{y_1\,y_2'-y_1'\,y_2}\,dx$$

Example *Solve* $\frac{d^2y}{dx^2} + y = \text{cosec}\, x.$

Solution. $(D^2 + 1)\, y = \text{cosec}\, x$

A.E. is $m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$

$$\text{C.F.} = A \cos x + B \sin x$$

Here $y_1 = \cos x, \qquad y_2 = \sin x$

$$\text{P.I.} = y_1\, u + y_2\, v$$

where
$$u = \int \frac{-y_2 \,.\, \text{cosec}\, x\, dx}{y_1 \,.\, y'_2 - y'_1 \,.\, y_2} = \int \frac{-\sin x \,.\, \text{cosec}\, x\, dx}{\cos x\,(\cos x) - (-\sin x\,)\,(\sin x)}$$

$$= \int \frac{-\sin x \,.\, \frac{1}{\sin x}\, dx}{\cos^2 x + \sin^2 x} = -\int dx = -x$$

$$v = \int \frac{y_1 \,.\, X\, dx}{y_1 \,.\, y'_2 - y'_1\, y_2} = \int \frac{\cos x \,.\, \text{cosec}\, x\, dx}{\cos x\,(\cos x) - (-\sin x\,)\,(\sin x)}$$

$$= \int \frac{\cos x \,.\, \frac{1}{\sin x}}{\cos^2 x + \sin^2 x}\, dx = \int \frac{\cot x\, dx}{1} = \log \sin x$$

$$P.I. = uy_1 + vy_2 = -x \cos x + \sin x\, (\log \sin x)$$

General solution = C.F. + P.I.

$$y = A \cos x + B \sin x - x \cos x + \sin x\, .(\log \sin x)$$

Example *Apply the mehtod of variation of parameters to solve* $\frac{d^2y}{dx^2}+y=\tan x$

Solution. We have, $\frac{d^2y}{dx^2}+y=\tan x$

$$(D^2+1)y=\tan x$$

A.E. is $m^2=-1$ or $m=\pm i$

C. F. $y=A\cos x+B\sin x$

Here, $y_1=\cos x,\qquad y_2=\sin x$

$$y_1 . y'_2-y'_1 . y_2=\cos x\,(\cos x)-(-\sin x)\sin x=\cos^2 x+\sin^2 x=1$$

P. I. $=u.\,y_1+v.\,y_2$ where

$$u=\int\frac{-y_2\tan x}{y_1\cdot y'_2-y'_1\cdot y_2}\,dx=-\int\frac{\sin x\tan x}{1}\,dx=-\int\frac{\sin^2 x}{\cos x}\,dx=-\int\frac{1-\cos^2 x}{\cos x}\,dx$$

$$=\int(\cos x-\sec x)\,dx=\sin x-\log(\sec x+\tan x)$$

$$v=\int\frac{y_1\tan x}{y_1\cdot y'_2-y'_1\cdot y_2}\,dx=\int\frac{\cos x\cdot\tan x}{1}\,dx=\int\sin x\,dx=-\cos x$$

P. I. $=u.\,y_1+v.\,y_2$

$$=[\sin x-\log(\sec x+\tan x)]\cos x-\cos x\sin x=-\cos x\log(\sec x+\tan x)$$

Complete solution is

$$y=A\cos x+B\sin x-\cos x\log(\sec x+\tan x)$$ **Ans.**

Example *Solve by method of variation of parameters:* $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}$

Solution. $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}$

A. E. is $(m^2 - 1) = 0$

$$m^2 = 1, \quad m = \pm 1$$

$$C.F. = C_1 e^x + C_2 e^{-x}$$

$\therefore$ $P.I. = uy_1 + vy_2$

Here, $y_1 = e^x, \quad y_2 = e^{-x}$

and $y_1 \cdot y'_2 - y'_1 \cdot y_2 = -e^x \, . \, e^{-x} - e^x . e^{-x} = -2$

$$u = \int \frac{-y_2 X}{y_1 . y'_2 - y'_1 \cdot y_2} dx = -\int \frac{e^{-x}}{-2} \times \frac{2}{1+e^x} dx$$

$$= \int \frac{e^{-x}}{1+e^x} dx = \int \frac{dx}{e^x (1+e^x)} = \int \left(\frac{1}{e^x} - \frac{1}{1+e^x} \right) dx$$

$$= \int e^{-x} dx - \int \frac{e^{-x}}{e^{-x}+1} dx = -e^{-x} + \log (e^{-x} + 1)$$

$$v = \int \frac{y_1 X}{y_1 . y'_2 - y'_1 . y_2} dx = \int \frac{e^x}{-2} \frac{2}{1+e^x} dx = -\int \frac{e^x}{1+e^x} dx = -\log (1+e^x)$$

$$\text{P.I.} = u . y_1 + v . y_2 = [-e^{-x} + \log (e^{-x}+1)] e^x - e^{-x} \log (1+e^x)$$

$$= -1 + e^x \log (e^{-x}+1) - e^{-x} \log (e^x+1)$$

Complete solution $= y = C_1 e^x + C_2 e^{-x} - 1 + e^x \log (e^{-x}+1) - e^{-x} \log (e^x + 1)$ **Ans.**

Solve the following equations by variation of parameters method.

1. $\frac{d^2y}{dx^2} - 4y = e^{2x}$ **Ans.** $y = C_1e^{2x} + C_2e^{-2x} + \frac{x}{4}e^{2x} - \frac{e^{2x}}{16}$

2. $\frac{d^2y}{dx^2} + y = \sin x$ **Ans.** $y = C_1 \cos x + C_2 \sin x - \frac{x}{2}\cos x + \frac{1}{4}\sin x$

3. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin x$ **Ans.** $y = C_1e^{x} + C_2e^{2x} + \frac{1}{10}(3\cos x + \sin x)$

4. $\frac{d^2y}{dx^2} + y = \sec x \tan x$ **Ans.** $y = C_1 \cos x + C_2 \sin x + x\cos x + \sin x \log \sec x - \sin x$

THANKS TO ALL

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