First Order Ordinary Differential Equations

First order and first degree ODE

The most general first order and first degree ODE is $\frac{dy}{dx} = f(x, y)$.

The integration of this equation gives its general solution. Direct integration is not always possible. The methods of identification to make this ODE integrable are the objective of the rest of this chapter. The following procedures will help in this respect.

- **Separation of variables**
- **Homogeneous equations**
- **Linear equations**
- **Reducible to Linear equations (Bernoulli's equations)**
- **Exact equation**

Separation of variables

A first-order ODE is separable if it can be written in the form:

$$P(x,y)+Q(x,y)\frac{dy}{dx}=0$$

Or, $P(x,y)\,dx+Q(x,y)\,dy=0$

Where, $P(x,y)$ and $Q(x,y)$ are functions of x and y.

If these equations can be written as

$$f_1(x)+f_2(y)\frac{dy}{dx}=0$$

Or, $f_1(x)\,dx+f_2(y)\,dy=0$

by some algebraic manipulation, then the variables are said to be separated. Integrating on both sides,

$$\int f_1(x)\,dx+\int f_2(y)\,dy=c$$

Or, $F_1(x)+F_2(y)=c$

Separation of variables

Problem#1. Solve the ODE $\left(x^2+1\right)\left(y^2-1\right)dx+x\,y\,dy=0$

Solution: Given $\left(x^2+1\right)\left(y^2-1\right)dx+x\,y\,dy=0 \;\ldots\;\ldots\;\ldots\;(i)$

Dividing both sides of Eq. (i) by $\left(y^2-1\right)x$, we get

$$\frac{\left(x^2+1\right)}{x}\,dx+\frac{y}{y^2-1}\,dy=0$$

Integrating, $\int\frac{\left(x^2+1\right)}{x}\,dx+\int\frac{y}{y^2-1}\,dy=c_1$

$$\text{or, } \int\left(x+\frac{1}{x}\right)dx+\int\frac{y}{y^2-1}\,\frac{d\left(y^2-1\right)}{2y}=c_1$$

$$\text{or, } \frac{x^2}{2}+\ln(x)+\frac{1}{2}\ln\left(y^2-1\right)=c_1$$

Or, $\ln\left(y^2-1\right)=2c_1-x^2-\ln x^2$

or, $y^2-1=e^{2c_1-x^2-\ln x^2}$ $\qquad\therefore y^2=1+\frac{1}{x^2}c\,e^{-x^2}$

Separation of variables

Problem#2. Solve the ODE $\frac{dy}{dx} = e^{x+y} + x^2 e^{x^3+y}$

Solution: Given $\frac{dy}{dx} = e^{x+y} + x^2 e^{x^3+y}$ ... ... (i)

Dividing both sides of Eq. (i) by 'e^y', we get

$$e^{-y}\,dy = e^x\,dx + x^2 e^{x^3}\,dx$$

Integrating, $\int e^{-y}\,dy = \int e^x\,dx + \int x^2 e^{x^3}\,dx$

$$\text{or, } -e^{-y} = e^x + \int x^2 e^{x^3} \frac{d\left(x^3\right)}{3x^2}$$

$$\text{or, } -e^{-y} = e^x + \frac{1}{3}\int e^{x^3}\,d\left(x^3\right)$$

$$\text{or, } -e^{-y} = e^x + \frac{1}{3}e^{x^3} + c$$

Therefore, the general solution is $-e^{-y} = e^x + \frac{1}{3}e^{x^3} + c.$

Separation of variables

Problem#3. Solve the ODE $\frac{dy}{dx}=(4x+y+1)^2$

Solution: Given $\frac{dy}{dx}=(4x+y+1)^2 \quad \ldots \quad \ldots \quad \ldots \quad (i)$

Let $4x+y+1=z$

Then, $4+\frac{dy}{dx}=\frac{dz}{dx}$

Therefore the given equation becomes, $\frac{dz}{dx}-4=z^2$

$$\frac{dz}{z^2+4}=dx$$

Integrating, $\int\frac{1}{z^2+2^2}dz=\int dx$

or, $\frac{1}{2}\tan^{-1}\left(\frac{z}{2}\right)=x+c$

or, $\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right)=x+c.$

Therefore, the general solution is $\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right)=x+c.$

Separation of variables

Problem#4. Solve the IVP $y' + y^2 \sin x = 0,\ y(0) = 1.$

Solution: Given $\dfrac{dy}{dx} + y^2 \sin x = 0$

or, $\dfrac{dy}{dx} = -y^2 \sin x$

or, $y^{-2} dy = -\sin x dx$

Integrating, $\int y^{-2} dy = \int -\sin x dx$

or, $-y^{-1} = \cos x + c \quad \ldots \quad \ldots \quad \ldots \quad (i)$

Now putting $y(0) = 1$ [i.e. $x = 0$ and $y = 1$] in Eq.(i), we get

$-1^{-1} = \cos 0 + c$

or, $-1 = 1 + c$

$\therefore c = -2$

Now putting $c = -2$ in Eq.(i), we get $-\dfrac{1}{y} = \cos x - 2.$

Therefore, the general solution is $y = \dfrac{1}{2 - \cos x}.$

Separation of variables

1. Put the following equation in separated form. Do not integrate.

a) $\dfrac{dy}{dx} = \dfrac{x^2y - 4y}{x+4}$

b) $\dfrac{dy}{dx} = \sec(y)e^{x-y}(1+x)$

c) $\dfrac{dy}{dx} = \dfrac{xy}{(x+1)(y+1)}$

d) $\dfrac{d\theta}{dt} + \sin\theta = 0$

Separation of variables

Solution to the problem 1 is:

1.

a) $\frac{1}{y}\,dy = \frac{x^2-4}{x+4}\,dx$

b) $e^y \cos y\,dy = e^x(1+x)\,dx$

c) $\frac{y+1}{y}\,dy = \frac{x}{(x+1)}\,dx$

d) $\frac{1}{\sin\theta}\,d\theta = -dt$

Practice quiz: Separable first-order ode

1. The solution of $y' = \sqrt{xy}$ with initial value $y(1) = 0$ is given by

a) $y(x) = \frac{1}{9}(x^{1/2} - 1)^2$

b) $y(x) = \frac{1}{9}(x - 1)^2$

c) $y(x) = \frac{1}{9}(x^{3/2} - 1)^2$

d) $y(x) = \frac{1}{9}(x^2 - 1)^2$

2. The solution of $y^2 - xy' = 0$ with initial value $y(1) = 1$ is given by

a) $y(x) = \dfrac{1}{1 - \ln x}$

b) $y(x) = \dfrac{1}{1 - 2\ln x}$

c) $y(x) = \dfrac{1}{1 + \ln x}$

d) $y(x) = \dfrac{1}{1 + 2\ln x}$

3. The solution of $y' + (\sin x)y = 0$ with initial value $y(\pi/2) = 1$ is given by

a) $y(x) = e^{\sin x}$

b) $y(x) = e^{\cos x}$

c) $y(x) = e^{1-\sin x}$

d) $y(x) = e^{1-\cos x}$

Solutions to the Practice quiz: Separable first-order odes

1. c. $\dfrac{dy}{dx} = x^{1/2}y^{1/2}, \quad y(1) = 0.$

$$\int_0^y \frac{dy}{y^{1/2}} = \int_1^x x^{1/2}\,dx; \quad 2y^{1/2} = \frac{2}{3}(x^{3/2}-1); \quad y = \frac{(x^{3/2}-1)^2}{9}.$$

2. a. $x\dfrac{dy}{dx} = y^2, \quad y(1) = 1.$

$$\int_1^y \frac{dy}{y^2} = \int_1^x \frac{dx}{x}; \quad -(\frac{1}{y}-1) = \ln x; \quad \frac{1}{y} = 1 - \ln x; \quad y = \frac{1}{1-\ln x}.$$

3. b. $\dfrac{dy}{dx} = -(\sin x)y, \quad y(\pi/2) = 1.$

$$\int_1^y \frac{dy}{y} = -\int_{\pi/2}^x \sin x\,dx; \quad \ln y = \cos x; \quad y = e^{\cos x}.$$

Self Study

1. **Solve the ODE** $\frac{x}{y}(dx+dy)+(x+y)(y\,dx-x\,dy)=0.$

2. **Solve the** following separable first-order equations:

$(i)\ \frac{dy}{dx}=4x\sqrt{y},$ with $y(0)=1.$

$(ii)\ \frac{dx}{dt}=x(1-x),$ with $x(0)=x_0$ and $0\le x_0\le 1.$

3. **Solve the ODE** $y\sqrt{1+x^2}\,dy-x\sqrt{1+y^2}\,dx=0.$

4. **Solve the ODE** $3e^{2x}\sec^2 3y\,dy+2\left(e^{2x}-1\right)\tan 3y\,dx=0.$

5. **Solve the ODE** $xy^4dx+\left(y^2+2\right)e^{-3x}dy=0.$

Homogeneous Equations (of degree zero)

Form of equation: $\dfrac{dy}{dx} = f(x, y)$

$f(x,y)$ is a function homogeneous of degree zero,
i.e., $f(tx,ty) = t^0 f(x,y) = f(x,y)$.

Homogeneous function: A function homogeneous of degree n can be defined as

$$f(tx,ty) = t^n f(x,y).$$

Example : solve $\dfrac{dy}{dx} = -\dfrac{x^2+y^2}{2xy}$

$$\text{Let, } f(x,y) = \frac{x^2+y^2}{2xy}$$

$$\therefore f(tx,ty) = \frac{(tx)^2+(ty)^2}{2(tx)(ty)} = \frac{t^2(x^2+y^2)}{t^2(2xy)} = t^0\frac{(x^2+y^2)}{(2xy)} = t^0 f(x,y)$$

Therefore *f(x,y)* is a homogeneous function of degree zero

Homogeneous Equation

Method of Solution: Substituting $y = vx, \frac{dy}{dx} = v + x\frac{dv}{dx}$, reduces the equation in form separable in variables v and x. Integrate both sides and then substituting the value of v will give the solution.

Another Definition

A homogeneous equation of the first order and degree is onewhich can be written in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \quad \dots \ \dots \ \dots (*)$$

For a solution of this equation,

$$\text{Let } \frac{y}{x} = v, \text{ i.e. } y = vx$$

$$\text{or, } \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting this into (*),

$$\text{or, } v + x\frac{dv}{dx} = f(v)$$

Homogeneous Equation

$$\text{or, } v + x\frac{dv}{dx} = f(v)$$

$$\text{or, } x\frac{dv}{dx} = f(v) - v$$

$$\text{or, } \frac{dv}{f(v)-v} = \frac{dx}{x} \quad \left[\text{ separating variables}\right]$$

$$\text{or, } \ln x = \int \frac{dv}{f(v)-v} + c$$

which is a solution of the general equation ofthe first order and degree $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

Homogeneous Equation

1. **Solve** : $\dfrac{dy}{dx} = \dfrac{y}{x}$

Solution: Let $\dfrac{y}{x} = v$, i.e. $y = vx$

$$\text{or, } \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting this into (1), $v + x\dfrac{dv}{dx} = \dfrac{vx}{x}$

$$\text{or, } v + x\frac{dv}{dx} = v$$

$$\text{or, } x\frac{dv}{dx} = 0$$

$$\text{or, } x\frac{dv}{dx} = 0$$

$$\text{or, } dv = 0$$

Which is the required solution

Integrating, $v = c$ i.e. $\dfrac{y}{x} = c$ $\Rightarrow y = cx$

Homogeneous Equation

2. **Solve** : $\left(x^2+y^2\right)dx+2xy\,dy=0$

$$\text{or, } \frac{dy}{dx}=-\frac{x^2+y^2}{2xy}$$

For a solution of this equation,

$$\text{Let } \frac{y}{x}=v, \text{ i.e. } y=vx$$

$$\text{or, } \frac{dy}{dx}=v+x\frac{dv}{dx}$$

Substituting this into (1),

$$v+x\frac{dv}{dx}=-\frac{x^2+v^2x^2}{2x.vx}$$

$$\text{or, } v+x\frac{dv}{dx}=-\frac{1+v^2}{2v}$$

Homogeneous Equation

$$\text{or, } v + x\frac{dv}{dx} = -\frac{1+v^2}{2v}$$

$$\text{or, } x\frac{dv}{dx} = -\frac{1+v^2}{2v} - v$$

$$\text{or, } x\frac{dv}{dx} = -\frac{1+3v^2}{2v}$$

$$\text{or, } -\frac{2v}{1+3v^2}dv = \frac{1}{x}dx$$

$$\text{Integrating, } -\int\frac{2v}{1+3v^2}dv = \int\frac{1}{x}dx$$

$$\text{Or, } -\int\frac{2v}{1+3v^2}\frac{d\left(1+3v^2\right)}{6v} = \int\frac{1}{x}dx$$

$$\text{Or, } -\frac{1}{3}\ln\left(1+3v^2\right) + \ln c = \ln(x)$$

$$\text{Or, } \ln(x) + \frac{1}{3}\ln\left(1+3v^2\right) = \ln c$$

$$\text{Or, } \ln(x) + \ln\left(1+3v^2\right)^{1/3} = \ln c$$

$$\text{Or, } \ln\left\{(x)\left(1+3v^2\right)^{1/3}\right\} = \ln c$$

$$\text{Or, } x\left(1+3v^2\right)^{1/3} = c$$

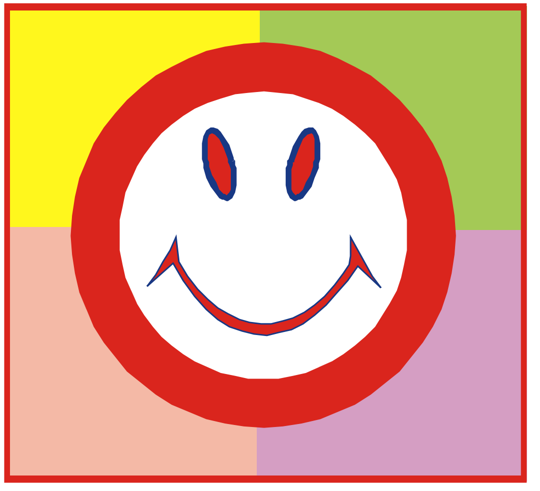
$$\text{Or, } x\left(1+3\frac{y^2}{x^2}\right)^{1/3} = c, \quad \because v = \frac{y}{x}$$

Self Study

1. **Solve the initial value problem** $(x+y)dy - ydx = 0$, $y(0)=1$

2. **Solve** $(x^2+y^2)dx - 2xy\,dy = 0.$

3. **Solve** $\dfrac{dy}{dx} = \dfrac{x-y}{x+y}$



Thank you

Dr. M. M. Rahman,
Professor of Mathematics