Higher Order Homogeneous Differential Equations

Higher Order Linear Differential Equations

The general Higher Order Linear Differential equation of order *n* can be expressed in the form

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + a_3 D^{n-3} y + \dots\dots\dots + a_n y = X$$

Where, $Dy = y',\ D^2 y = y'',\ \dots\dots\dots\ D^n y = y^{(n)}$

and $a_0, a_1, a_2, \dots a_n, X$ are functions of x or constants.

Operators:

An operator is a symbol indicating an operation to be performed. The operator D indicates the derivative with respect to x and generally

$$\frac{d}{dx}, \frac{d^2}{dx^2}$$

etc. are respectively indicated by the operators D, D^2 etc.

Second Order Linear Differential Equations

The general equation can be expressed in the form

$$ay''+by'+cy = g(x)$$

where a, b and c are constant coefficients

Let the dependent variable y be replaced by the sum of the two new variables: y = u + v

Therefore

$$[au''+bu'+cu]+\boxed{[av''+bv'+cv]= g(x)}$$

If v is a particular solution of the original differential equation

$$[au''+bu'+cu]= 0$$

purpose

The general solution of the linear differential equation will be the sum of a "*complementary function*" and a "*particular solution*".

The Complementary Function (solution of the homogeneous equation)

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$$

Let the solution assumed to be: $y=e^{rx}$

$$\text{or, } \left(aD^2+bD+c\right)y=0$$

$$\therefore y'=re^{rx}$$

$$\text{or, } ay''+by'+cy=0$$

$$\therefore y''=r^2e^{rx}$$

$$e^{rx}\underline{(ar^2+br+c)=0}$$

$$\text{Since } e^{rx}\neq 0,$$

$$\text{So, } ar^2+br+c=0$$

is known as characteristic equation

- **Real:** Distinct roots or
- Repeat roots
- **Complex roots**

Real, Distinct Roots to Characteristic Equation

- Let the roots of the characteristic equation be real, distinct and of values r_1 and r_2. Therefore, the solutions of the characteristic equation are:

$$y = e^{r_1 x} \qquad\qquad y = e^{r_2 x}$$

- The general solution will be

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Real, Distinct Roots to Characteristic Equation

- **Example #1** Solve: $y'' - 5y' + 6y = 0$

$$or, \left(D^2 - 5D + 6\right)y = 0, \text{ where } D \equiv \frac{d}{dx}$$

- **Solution:** Let, $y = e^{rx}$

$$\therefore y' = re^{rx}$$

$$\therefore y'' = r^2 e^{rx}$$

Putting these values into equation (i), we get the characteristic $r^2 - 5r + 6 = 0$

$$\text{or, } r^2 - 3r - 2r + 6 = 0$$

$$\text{or, } (r-3)(r-2) = 0$$

$$\therefore r = 2 \text{ and } 3. \quad \Rightarrow \quad r_1 = 2, \; r_2 = 3$$

$$\text{The general solution is: } y = c_1 e^{2x} + c_2 e^{3x}$$

Real, Distinct Roots to Characteristic Equation

- **Example #2** Solve: $y'' + 5y' + 6y = 0$, with $y(0) = 2$, $y'(0) = 3$.
- **Solution:** Let, $y = e^{rx}$

$$\therefore y' = re^{rx}$$

$$\therefore y'' = r^2 e^{rx}$$

Putting these values into equation (i), we get the characteristic

$$r^2 + 5r + 6 = 0$$

$$\text{or, } r^2 + 3r + 2r + 6 = 0$$

$$\text{or, } (r+3)(r+2) = 0$$

$$\therefore r = -2 \text{ and } -3.$$

$$r_1 = -2$$

$$r_2 = -3$$

The general solution is: $y = c_1 e^{-2x} + c_2 e^{-3x}$

Real, Distinct Roots to Characteristic Equation

- **Example#2** Solve: $y'' + 5y' + 6y = 0$, with $y(0) = 2$, $y'(0) = 3$.
- **Solution:** The general solution is: $y(x) = c_1 e^{-2x} + c_2 e^{-3x}$

$$\therefore y'(x) = -2c_1 e^{-2x} - 3c_2 e^{-3x}$$

Applying the initial condition $y(0)=2$, we have

$$y(0) = c_1 e^0 + c_2 e^0$$

$$\Rightarrow 2 = c_1 + c_2 \qquad ....(1)$$

Applying the initial condition $y'(0)=3$, we have

$$y'(0) = -2c_1 e^0 - 3c_2 e^0$$

$$\Rightarrow 3 = -2c_1 - 3c_2 \qquad .....(2)$$

Solving Eq. (1) and (2), we get

$$c_1 = 9, \text{ and } c_2 = -7.$$

$\therefore$ The required solution is: $y(x) = 9e^{-2x} - 7e^{-3x}$.

Real, Distinct Roots to Characteristic Equation

Self study

1. Solve: $\left(D^2 - 3D + 2\right)y = 0,$ where $\mathrm{D} \equiv \dfrac{d}{dx}$

2. Solve: $\left(D^2 - 7D + 12\right)y = 0,$ where $\mathrm{D} \equiv \dfrac{d}{dx}$

3. Solve the initial value problem $y'' + 4y' + 3y = 0,$
with $y(0) = 1,\ y'(0) = 0.$

Ans: $y(x) = \dfrac{3}{2}e^{-x} - \dfrac{1}{2}e^{-3x}.$

Repeated Roots to Characteristic Equation

Let the roots of the characteristic equation equal (say) $r = r_1 = r_2$.

Consider the equation, $(D-r_1)(D-r_1)y = 0 \quad \ldots\ldots\ldots (i)$

$$\text{Put,} \quad (D-r_1)y = v \quad \ldots \ldots \ldots (ii)$$

Then the equation (i) reduces to, $(D-r_1)v = 0 \ldots \ldots \ldots (ii)$

$$\text{or, } \frac{dv}{dx} - r_1 v = 0$$

$$\text{or, } \frac{dv}{v} = r_1 dx$$

$$\text{or, } \ln v = r_1 x + \ln c$$

$$\text{or, } v = c_1 e^{r_1 x}$$

Put the value of v into Eq. (ii), we get

$$(D-r_1)y = c_1 e^{r_1 x}$$

Repeated Roots to Characteristic Equation

$$(D - r_1)y = c_1 e^{r_1 x}$$

$$\frac{dy}{dx} - r_1 y = c_1 e^{r_1 x}$$

[Which is linear 1^{st} order ODE]

$$\therefore I.F. = e^{\int -r_1\,dx} = e^{-r_1 x}$$

$\therefore$ The general solution is : $y(I.F) = \int (I.F.)\,q(x)\,dx + c_2$

$$y e^{-r_1 x} = \int e^{-r_1 x}\left(c_1 e^{r_1 x}\right)dx + c_2$$

$$\text{or, } y e^{-r_1 x} = \int c_1\,dx + c_2$$

$$\text{or, } y e^{-r_1 x} = c_1 x + c_2$$

$$\boxed{\therefore\ y = (c_1 x + c_2)e^{r_1 x}}$$

Repeated Roots to Characteristic Equation

(Alternative Proof)

- Let the roots of the characteristic equation equal and of value $r_1 = r_2 = r$. Therefore, the solution of the characteristic equation is:

$$y = e^{rx}$$

Let $y = Ve^{rx} \quad \Rightarrow y' = e^{rx}V' + rVe^{rx}$ and $y'' = e^{rx}V'' + 2re^{rx}V' + r^2Ve^{rx}$

where V is a function of x

$$ay'' + by' + cy = 0$$

$$\mathrm{ar}^2 + \mathrm{br} + \mathrm{c} = 0 \qquad 2\mathrm{ar} + \mathrm{b} = 0$$

$$V''(x) = 0 \rightarrow V = cx + d$$

$$\boxed{y = be^{rx} + (cx + d)e^{rx} = \left(c_1 + c_2 x\right)e^{rx}}$$

Repeated Roots to Characteristic Equation

- **Example#1** Solve: $y''-4y'+4y=0$

$$or,\ \left(D^2-4D+4\right)y=0,\ \text{where } D\equiv\frac{d}{dx}$$

- **Solution:** Let, $y=e^{rx}$

$$\therefore y'=re^{rx}$$

$$\therefore y''=r^2e^{rx}$$

Putting these values into equation (i), we get the characteristic equation

$$r^2-4r+4=0$$

$$\text{or, } r^2-2.r.2+2^2=0$$

$$\text{or, } (r-2)^2=0$$

$$\therefore r=2 \text{ and } 2.$$

$$r_1=2$$

$$r_2=2$$

The general solution is: $y=\left(c_1+c_2x\right)e^{2x}$

Repeated Roots to Characteristic Equation

- **Example#2** Solve: $y'' + 2y' + y = 0$, with $y(0) = 1$, and $y'(0) = 0$.
  - **Solution:** **Let,** $y = e^{rx}$

$$\therefore y' = re^{rx}$$

$$\therefore y'' = r^2 e^{rx}$$

Putting these values into equation (i), we get the characteristic equation

$$r^2 + 2r + 1 = 0$$

$$\text{or, } r^2 + 2.r.1 + 1^2 = 0$$

$$\text{or, } (r+1)^2 = 0$$

$$\therefore r = -1 \text{ and } -1.$$

$$\Rightarrow \quad r_1 = -1, \quad r_2 = -1$$

The general solution is: $y(x) = (c_1 + c_2 x)e^{-x}$

Repeated Roots to Characteristic Equation

- **Example#2** Solve: $y'' + 2y' + y = 0$, with $y(0) = 1$, and $y'(0) = 0$.
  - **Solution:** The general solution is: $y(x) = (c_1 + c_2 x)e^{-x}$

$$\therefore y'(x) = (c_1 + c_2 x)(-e^{-x}) + e^{-x}c_2$$

$$= (c_2 - c_1 - c_2 x)e^{-x}$$

Applying the initial condition $y(0)=1$, we have

$$y(0) = (c_1 + c_2 0)e^0$$

$$\Rightarrow 1 = c_1$$

Applying the initial condition $y'(0)=0$, we have

$$y'(0) = (c_2 - c_1 - c_2 0)e^0$$

$$\Rightarrow 0 = c_2 - c_1$$

$$\Rightarrow c_2 = c_1 = 1$$

$\therefore$ The required solution is: $y(x) = (1+x)e^{-x}$.

Repeated Roots to Characteristic Equation

Self Study

1. Solve: $\left(D^2+6D+9\right)y=0$, where $\mathrm{D}\equiv\dfrac{d}{dx}$

2. Solve: $\left(D^3-7D^2+16D-12\right)y=0$, where $\mathrm{D}\equiv\dfrac{d}{dx}$

Ans: $y(x)=(c_1+c_2x)e^{2x}+c_3e^{3x}.$

3. Solve: $y''-2y'+y=0$, with $y(0)=1$, $y'(0)=0$.

Ans: $y(x)=(1-x)e^{x}.$

4. Solve the initial value problem $y'''-6y''+9y'=0$,
with $y(0)=0$, $y'(0)=2$, $y''(0)=-6$.

Complex Roots to Characteristic Equation

Let the roots of the characteristic equation be complex in the form $r = \alpha \pm i\beta$. Therefore, the solution of the characteristic equation is:

$$\boxed{y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x}}$$

$$\text{or, } y = e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right)$$

$$\text{or, } y = e^{\alpha x}\left(c_1\left(\cos\beta x + i\sin\beta x\right) + c_2\left(\cos\beta x - i\sin\beta x\right)\right)$$

$$\text{or, } y = e^{\alpha x}\left(\left(c_1 + c_2\right)\cos\beta x + i\left(c_1 - c_2\right)\sin\beta x\right)$$

$$\text{or, } y = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)$$

Therefore, the geneal solution to the D.E. is:

$$\boxed{y(x) = e^{\alpha x}\left(C_1\cos(\beta x) + C_2\sin(\beta x)\right).}$$

Complex Roots to Characteristic Equation

- **Example#1** Solve: $y'' - 4y' + 5y = 0$

$$or,\ \left(D^2 - 4D + 5\right)y = 0,\ \text{where}\ D \equiv \frac{d}{dx}$$

- **Solution:** Let, $y = e^{rx}$

$$\therefore y' = re^{rx}$$

$$\therefore y'' = r^2 e^{rx}$$

Putting these values into equation (i), we get the characteristic equation

$$r^2 - 4r + 5 = 0$$

$$\therefore \mathrm{r} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4.1.5}}{2.1}$$

$$\text{or, } \mathrm{r} = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} \Rightarrow r = 2 \pm i$$

The general solution is: $y = e^{2x}\left(c_1 \cos x + c_2 \sin x\right)$

Complex Roots to Characteristic Equation

- **Example#2** Solve: $y'' - 2y' + 5y = 0$ with $y(0) = 1$, and $y'(0) = 0$.
  - **Solution:** **Let,** $y = e^{rx}$

$$\therefore y' = re^{rx}$$

$$\therefore y'' = r^2 e^{rx}$$

Putting these values into equation (i), we get the characteristic equation

$$r^2 - 2r + 5 = 0$$

$$\therefore \mathrm{r} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4.1.5}}{2.1}$$

$$\text{or, } \mathrm{r} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2}$$

$$\Rightarrow r = 1 \pm 2i$$

The general solution is: $y = e^{x}\left(c_1 \cos 2x + c_2 \sin 2x\right)$

Complex Roots to Characteristic Equation

- **Example#2** Solve: $y''-2y'+5y=0$ with $y(0)=1$, and $y'(0)=0$.
  - **Solution:** The general solution is: $y=e^{x}\left(c_1\cos 2x+c_2\sin 2x\right)$

$$\therefore y'(x)=e^{x}\left(c_1\cos 2x+c_2\sin 2x\right)+e^{x}\left(-2c_1\sin 2x+2c_2\cos 2x\right)$$

Applying the initial condition $y(0)=1$, we have

$$y(0)=e^{0}\left(c_1\cos 0+c_2\sin 0\right)$$

$$\Rightarrow 1=c_1$$

Applying the initial condition $y'(0)=0$, we have

$$y'(0)=e^{0}\left(c_1\cos 0+c_2\sin 0\right)+e^{0}\left(-2c_1\sin 0+2c_2\cos 0\right)$$

$$\Rightarrow 0=c_1+2c_2$$

$$\Rightarrow c_2=-\frac{1}{2}c_1=-\frac{1}{2}(1)=-\frac{1}{2}$$

$\therefore$ The required solution is: $y(x)=e^{x}\left(\cos 2x-\frac{1}{2}\sin 2x\right).$

Complex Roots to Characteristic Equation

Self Study

1. Solve: $y'' - 4y' + 13y = 0$, with $y(0) = -1$, $y'(0) = 2$.

Ans: $y(x) = e^{2x}\left(-\cos 3x + \frac{4}{3}\sin 3x\right)$.

2. Solve the initial value problem $y'' - 6y' + 25y = 0$, with $y(0) = -3$, $y'(0) = -1$.

Ans: $y(x) = e^{3x}\left(2\sin 4x - 3\cos 4x\right)$.

3. Solve the initial value problem $y'' + y' + y = 0$, with $y(0) = 1$, $y'(0) = 0$.

Ans: $y(x) = e^{-\frac{x}{2}}\left(\cos\frac{\sqrt{3}}{2}x + \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}x\right)$.

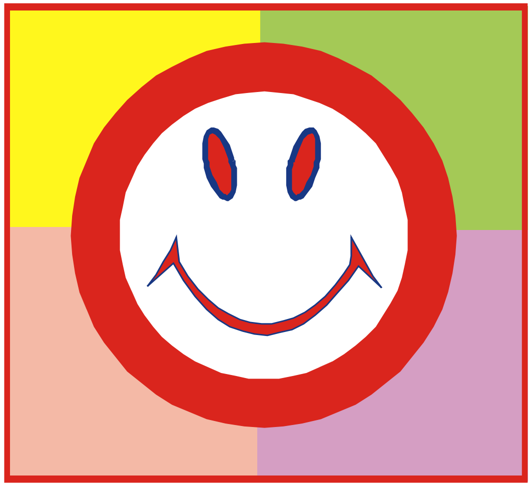
4. Solve: $\left(D^4 - 16\right)y = 0$, where $D \equiv \frac{d}{dx}$ Hints: $r = \pm 2, \pm 2i$

Ans: $y(x) = c_1e^{2x} + c_2e^{-2x} + e^{0x}\left(c_3\cos 2x + c_4\sin 2x\right)$.

The Complementary Function (solution of the homogeneous equation)

Summary

Case	Roots	General Solution
1	Distinct real r_1 and r_2	$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
2	Repeated root $r_1 = r_2 = r$	$y = e^{rx}\left(C_1 + C_2 x\right)$
3	Complex roots $r_{1,2} = \alpha \pm i\beta$	$y(x) = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right).$



Thank you

Dr. M. M. Rahman,
Professor of Mathematics