(P-1)

Example 1: Solve $(D^2+3DD'+2D'^2)z = x+y$

Solution: The auxiliary equation of

$(D^2+3DD'+2D'^2)z=0$ is $m^2+3m+2=0$

Now $m^2+3m+2=0$

$\Rightarrow m^2+2m+m+2=0$

$\Rightarrow m(m+2)+1(m+2)=0$

$\Rightarrow (m+1)(m+2)=0$

$m=-1,-2$

$C.F = f_1(y-x)+f_2(y-2x)$

$$P.I = \frac{1}{D^2+3DD'+2D'^2}(x+y)$$

$$= \frac{1}{D^2}\left(1+\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)^{-1}(x+y)$$

$$= \frac{1}{D^2}\left[1-\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)+\cdots\right](x+y)$$

$$= \frac{1}{D^2}\left(x+y-3\frac{1}{D}(1)\right)$$

$$= \frac{1}{D^2}(x+y-3x)$$

$$= \frac{1}{D^2}(y-2x) = \frac{1}{D}(yx-x^2) = y\frac{x^2}{2}-\frac{x^3}{3}$$

Hence the complete solution is

$$z = f_1(y-x)+f_2(y-2x)+\frac{x^2y}{2}-\frac{x^3}{3}$$

Example-2: Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = x+y$

Solution: The given equation can be written in the form $(D^2 + DD' - 6D'^2)z = x+y$

The auxiliary equation is $m^2 + m - 6 = 0$

$\Rightarrow m^2 + 3m - 2m - 6 = 0 \Rightarrow m(m+3) - 2(m+3) = 0$

$\Rightarrow (m-2)(m+3) = 0 \Rightarrow m = 2, -3$

C.F $= f_1(y+2x) + f_2(y-3x)$

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}(x+y)$$

$$= \frac{1}{D^2}\cdot\left(1 + \frac{D'}{D} - 6\frac{D'^2}{D^2}\right)^{-1}(x+y)$$

$$= \frac{1}{D^2}\left\{1 + \left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right)\right\}^{-1}(x+y)$$

$$= \frac{1}{D^2}\left\{1 - \left(\frac{D'}{D} - 6\frac{D'^2}{D^2}\right) + \cdots\right\}(x+y)$$

$$= \frac{1}{D^2}\left\{(x+y) - \frac{1}{D}(1)\right\}$$

$$= \frac{1}{D^2}\{x+y-x\}$$

$$= \frac{1}{D^2}y = y\cdot\frac{1}{D}x = \frac{x^2 y}{2}$$

The complete solution is

$$z = f_1(y+2x) + f_2(y-3x) + \frac{x^2 y}{2}$$

Example-3: solve $(D_x^2 - D_xD_y - 2D_y^2)z = 2x+3y$

Solution: The auxiliary equation is

$$m^2 - m - 2 = 0 \Rightarrow m^2 - 2m + m - 2 = 0$$

$$\Rightarrow m(m-2) + 1(m-2) = 0 \Rightarrow (m-2)(m+1) = 0$$

$$\Rightarrow m = 2, -1$$

C.F $z_c = f_1(y+2x) + f_2(y-x)$

P.I.

$$z_p = \frac{1}{D_x^2 - D_xD_y - 2D_y^2}(2x+3y)$$

$$= \frac{1}{D_x^2}\left[1 - \left(\frac{D_y}{D_x} + 2\frac{D_y^2}{D_x^2}\right)\right]^{-1}(2x+3y)$$

$$= \frac{1}{D_x^2}\left[1 + \left(\frac{D_y}{D_x} + 2\frac{D_y^2}{D_x^2}\right)\right](2x+3y)$$

$$= \frac{1}{D_x^2}\left[2x + 3y + \frac{1}{D_x}3\right] = \frac{1}{D_x^2}\left[2x+3y+3x\right]$$

$$= \frac{1}{D_x^2}(5x+3y) = \frac{1}{D_x}\left[5 \cdot \frac{x^2}{2} + 3xy\right]$$

$$= \frac{5}{6}x^3 + \frac{3}{2}x^2y$$

$$G.S = z_c + z_p = f_1(y+2x) + f_2(y-x) + \frac{5}{6}x^3 + \frac{3}{2}x^2y.$$

Example-4: solve $(D-D')^2 z = x + \phi(x+y)$

Solution: The auxiliary equation is $(m-1)^2=0$

$\Rightarrow m = 1, 1.$

C.F: $z_c = f_1(y+x) + x f_2(y+x)$

P.I. $$z_p = \frac{1}{(D-D')^2} x + \frac{1}{(D-D')^2}\phi(x+y)$$

$$= \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2} x + x\cdot\frac{1}{2(D-D')}\phi(x+y)$$

$$= \frac{1}{D^2}\left(1+2\frac{D'}{D}\right) x + x^2\cdot\frac{1}{2}\phi(x+y)$$

$$= \frac{1}{D^2} x + \frac{x^2}{2}\phi(x+y)$$

$$= \frac{1}{D}\cdot\frac{x^2}{2} + \frac{x^2}{2}\phi(x+y)$$

$$= \frac{x^3}{6} + \frac{x^2}{2}\phi(x+y)$$

The complete solution is

$$z = f_1(y+x) + x f_2(y+x) + \frac{x^3}{6} + \frac{x^2}{2}\phi(x+y)$$

Example-5: Solve $(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$

Solution: The auxiliary equation is $m^3 - 2m^2 = 0$

$\Rightarrow m^2(m-2) = 0 \Rightarrow m = 0, 0, 2$

C.F $= f_1(y) + x f_2(y) + f_3(y+2x)$

The P.I $$= \frac{1}{D^3 - 2D^2D'} 2e^{2x} + \frac{1}{D^3-2D^2D'} 3x^2y$$

$$= 2\cdot\frac{1}{8-2\cdot4\cdot0} e^{2x} + \frac{1}{D^3}\left(1-\frac{2D'}{D}\right)^{-1} 3x^2y$$

$$= \frac{1}{4}e^{2x} + \frac{1}{D^3}\left(1+\frac{2D'}{D}\right) 3x^2y$$

$$= \frac{1}{4}e^{2x} + \frac{1}{D^3}\left(3x^2y + 6\cdot\frac{1}{D}x^2\right)$$

$$= \frac{1}{4}e^{2x} + \frac{1}{D^2}\left(\cdot x^3y + 2\frac{x^4}{4}\right)$$

$$Z_p = \frac{1}{4}e^{2x} + \frac{1}{D}\left(\frac{x^4}{4}y + \frac{1}{2}\cdot\frac{x^5}{5}\right)$$

$$= \frac{1}{4}e^{2x} + \frac{1}{4}\cdot\frac{x^5}{5}y + \frac{1}{2}\cdot\frac{1}{5}\cdot\frac{x^6}{6}$$

$$= \frac{1}{4}e^{2x} + \frac{1}{20}x^5y + \frac{1}{60}x^6$$

$$= \frac{1}{60}\left(15e^{2x} + 3x^5y + x^6\right)$$

Practice problems:

Solve the following:

1. $(D^2 - D'^2)z = x - y$; $z = f_1(y-x) + f_2(y+x) + \frac{x^3}{6} - \frac{x^2y}{2}$

2. $(D^2 + 3DD' + 2D'^2)z = 12xy$; $z = f_1(y-x) + f_2(y-2x) + 2x^3y - \frac{3}{2}x^4$

3. $(D^2 - DD' - 6D'^2)z = xy$; $z = f_1(y-2x) + f_2(y+3x) + \frac{x^3y}{6} + \frac{1}{24}x^4$

4. $(D^2 + 2DD' + D'^2)z = x^2 + xy + y^2$