Linear Homogeneous Partial Differential Equations of nth order with constant Coefficients-

An equation of the type

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = F(x,y) \quad \text{—} (1)$$

is called a homogeneous linear partial differential equation of nth order with constant coefficients. It is called homogeneous because all the terms contain derivatives of the same order.

Putting $\frac{\partial}{\partial x} = D$ (D_x) and $\frac{\partial}{\partial y} = D'$ (D_y), then (1) becomes

$$(a_0 D^n + a_1 D^{n-1} D' + \cdots + a_n D'^n) z = F(x,y) \quad \text{—} (2)$$

or $f(D, D') z = F(x,y)$

Algorithm for finding the complementary function (C.F) of $(a_0 D^2 + a_1 DD' + a_2 D'^2) z = 0$

or $a_0 \frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0$

Input : The given equation

$$(a_0 D^2 + a_1 DD' + a_2 D'^2) z = 0 \quad \text{—} (1)$$

Output : Solution of the equation (1)

Step1 : Put $D = m$ and $D' = 1$ in (1) to get auxiliary equation

$$a_0 m^2 + a_1 m + a_2 = 0 \quad \text{—} (2)$$

Step2 : Solve the auxiliary equation (2)

(P-2)

Case 1: If the roots of the auxiliary equation are real and different; say m_1, m_2 Then $C.F = f_1(y+m_1x)+f_2(y+m_2x)$

Case 2: If the roots are equal and real, say m, then

$$C.F = f_1(y+mx)+x f_2(y+mx)$$

Case 3: If the roots are complex, say m_1 and $\bar{m}_1$, then

$$C.F = f_1(y+m_1x)+f_2(y+\bar{m}_1x)$$

Step 3: Stop.

Example-1: Solve $2\frac{\partial^2 z}{\partial x^2}+5\frac{\partial^2 z}{\partial x\partial y}+2\frac{\partial^2 z}{\partial y^2}=0$

Solution: The given equation can be written as

$$(2D^2+5DD'+2D'^2)z=0 \quad \text{--- (1)}$$

Put $D=m$ and $D'=1$, then the auxiliary equation $2m^2+5m+2=0$ --- (2)

Solve the equation (2)

$$2m^2+4m+m+2=0$$

$$\Rightarrow 2m(m+2)+1(m+2)=0$$

$$\Rightarrow (m+2)(2m+1)=0$$

$$\therefore m=-2,\quad m=-\tfrac{1}{2}$$

Hence the complementary function is

$$CF = f_1(y-2x)+f_2(y-\tfrac{1}{2}x)$$

(P-3)

Example 2: solve $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 0$

Solution: The given equation can be written as $(D^2 - 4DD' + 4D'^2)z = 0$ ——— (1)

The auxiliary equation of (1) is

$$m^2 - 4m + 4 = 0 \text{ ——— (2)} \quad (D = m, D' = 1)$$

$$\Rightarrow (m-2)^2 = 0 \Rightarrow (m-2)(m-2) = 0$$

$$\Rightarrow m = 2, 2$$

$$\therefore C.F = f_1(y+2x) + x f_2(y+2x)$$

Example-3: solve $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$

Solution: The given equation can be written as $(D^4 - D'^4)z = 0$ ——— (1)

The auxiliary equation of (1) is

$$m^4 - 1 = 0$$

$$(m^2)^2 - 1^2 = 0$$

$$(m^2 - 1)(m^2 + 1) = 0$$

$m^2 - 1 = 0$ or $m = \pm 1$ | $m^2 = -1$, $m = \pm i$

$$\therefore C.F = f_1(y+x) + f_2(y-x) + f_3(y+ix) + f_4(y-ix)$$

P-4

Exercises: Solve the following equations:

1. $\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} - 5\frac{\partial^2 z}{\partial y^2} = 0$; $CF = f_1(y+x) + f_2(y-5x)$

2. $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$; $C.F = f_1(2y-x) + f_2(y-2x)$

3. $(D^3 - 4D^2D' + 3DD'^2)z = 0$; $CF = f_1(y) + f_2(y+x) + f_3(y+3x)$

4. $(D^2 + 8DD' + 16D'^2)z = 0$; $CF = f_1(y-4x) + x f_2(y-4x)$