Partial Differential Equations in Practical Problems

INTRODUCTION

In practical problems, the following types of equations are generally used :

(i) *Wave equation* : $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

(ii) *One-dimensional heat flow* : $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

(iii) *Two-dimensional heat flow* : $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(iv) *Radio equations* : $-\dfrac{\partial V}{\partial x} = L\dfrac{\partial I}{\partial t}, \quad -\dfrac{\partial I}{\partial x} = C\dfrac{\partial V}{\partial t}$

METHOD OF SEPARATION OF VARIABLES

In this method, we assume that the dependent variable is the product of two functions, each of which involves only one of the independent variables. So two ordinary differential equations are formed.

Example 1:

Obtain the solution of the wave equation using the method of separation of variables.

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Solution. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Let $y = XT$ where X is a function of x only and T is a function of t only.

$$\frac{\partial y}{\partial t} = X\frac{dT}{dt} \quad \text{and} \quad \frac{\partial y}{\partial x} = T\frac{dX}{dx}$$

Since T and X are functions of a single variable only.

$$\frac{\partial^2 y}{\partial t^2} = X\frac{d^2 T}{dt^2} \quad \text{and} \quad \frac{\partial^2 y}{\partial x^2} = T\frac{d^2 X}{dx^2}$$

Substituting these values in the given equation, we get

$$X \frac{d^2T}{dt^2} = c^2T \frac{d^2X}{dx^2}$$

By separating the variables, we get

$$\frac{\frac{d^2T}{dt^2}}{c^2T} = \frac{\frac{d^2X}{dx^2}}{X} = k \qquad \text{(say).}$$

(Each side is constant, since the variables x and y are independent).

$$\frac{d^2T}{dt^2} - k\,c^2T = 0 \qquad \text{and} \qquad \frac{d^2X}{dx^2} - kX = 0$$

Auxiliary equations are

$$m^2 - kc^2 = 0 \quad \text{or} \quad m = \pm c\sqrt{k} \qquad \text{and} \quad m^2 - k = 0 \quad \text{or} \quad m = \pm\sqrt{k}$$

Case 1. If $k > 0$.

$$T = C_1 e^{c\sqrt{k}t} + C_2 e^{-c\sqrt{k}t}$$

$$X = C_3 e^{\sqrt{k}x} + C_4 e^{-\sqrt{k}x}$$

Case 2. If $k < 0$.

$$T = C_5 \cos c\sqrt{k}t + C_6 \sin c\sqrt{k}t$$

$$X = C_7 \cos\sqrt{k}x + C_8 \sin\sqrt{k}x$$

Case 3. If $k = 0$.

$$T = C_9 t + C_{10}$$

$$X = C_{11}x + C_{12}$$

These are the three cases depending upon the particular problems. Here we are dealing h wave motion ($k < 0$).

$$y = TX$$

$$y = (C_5 \cos c\sqrt{k}t + C_6 \sin c\sqrt{k}t) \times (C_7 \cos\sqrt{k}x + c_8 \sin\sqrt{k}x)$$ **Ans.**

Example 2:

The vibrations of an elastic string is governed by the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

The length of the string is π and the ends are fixed. The initial velocity is zero and the initial deflection is u (x, 0) = 2 (sin x + sin 3x). Find the deflection u (x, t) of the vibrating string for t > 0.

Solution. $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \quad u = (c_1 \cos pt + c_2 \sin pt)\ (c_3 \cos px + c_4 \sin px) \qquad ..(1)$$

On putting $x = 0$, $u = 0$ in (1), we get

$$0 = (c_1 \cos pt + c_2 \sin pt)\, c_3 \quad \Rightarrow \quad c_3 = 0$$

On putting $c_3 = 0$ in (1), it reduces

$$u = (c_1 \cos pt + c_2 \sin pt)\, c_4 \sin px \qquad ...(2)$$

$$0 = (\, c_1 \cos pt + c_2 \sin pt\,)\, c_4 \sin p\,\pi$$

$$\sin p\,\pi = 0 = \sin n\,\pi \qquad n = 1, 2, 3, 4, \ldots\ldots$$

$$\therefore \quad p\,\pi = n\,\pi \quad \text{or} \quad p = n$$

On substituting the value of p in (2) we get

$$u = (c_1 \cos n\,t + c_2 \sin n\,t)\ c_4 \sin nx \qquad ...(3)$$

On differentiating (3) w.r.t. "t" we get

$$\frac{du}{dt} = (-\,c_1\, n \sin nt + c_2\, n \cos nt)\, c_4 \sin nx \qquad ...(4)$$

On putting $\frac{du}{dt} = 0,\ t = 0$ in (4) we have

$$0 = (c_2\, n\,)\ (c_4 \sin nx) \quad \Rightarrow \quad c_2 = 0$$

On putting $c_2 = 0$, (3) becomes

$$u = (c_1 \cos nt)\ (c_4\ \sin nx)$$

$$u = c_1\, c_4\ \cos nt \sin nx \qquad ...(5)$$

given $u(x, 0) = 2\ (\sin x + \sin 3x)$

On putting $t = 0$ in (5) we have

$$u\,(x, 0) = c_1\, c_4 \sin nx$$

$$2\,(\sin x + \sin 3x) = c_1\, c_4 \sin nx$$

$$4 \sin 2x \cos x = c_1\, c_4 \sin nx$$

$$c_1\, c_4 = 4 \cos x \qquad 2 = n$$

On substituting the values of $c_1\, c_4$ and $n = 2$, (5) becomes

$$u\,(x, t) = 4 \cos x \cos 2t \sin 2x$$