Linear Differential Equations with Constant Co-efficients

Linear Second Order DEs

The most general linear second order differential equation is in the form.

$$\boxed{p(x)y'' + q(x)y' + r(x)y = g(x)}$$

The constant coefficient linear second order differential equation is

$$\boxed{ay'' + by' + cy = g(x)}$$

where a, b, c are all constants.

Initially we will make our life easier by looking at differential equations with *g(x)*=0.

- When g(x)=0 we call the differential equation **homogeneous**
- When $g(x) \neq 0$ we call the differential equation **non-homogeneous**

Second Order Non-homogeneous Linear Differential Equations

The general solution to the linear differential equation $L(y) = f(x), y = y_h + y_p$ where y_p denotes one solution to the differential equation and y_h is the general solution to the associated homogeneous equation, $L(y) = 0$. Methods for obtaining y_h when the differential equation has constant coefficients are given in previous lectures. In this lecture, we give methods for obtaining a particular solution y_p *once* y_h *is known.*

Method of Undetermined Coefficients

Non-homogeneous Differential Equations

$$\boxed{ay'' + by' + cy = g(x)}$$

1. If $g(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_n x^n$ an n[th] degree polynomial in x.

The form of a particular solution is: $y_p = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_n x^n$

Where, $\alpha_0, \alpha_1, \alpha_2, \ldots .\alpha_n$, are constants to be determined.

2. If $g(x) = ke^{px}$.

The form of a particular solution is: $y_p = Ae^{px}$

Where, A is constant to be determined.

3. If $g(x) = C\sin nx + D\cos nx$.

The form of a particular solution is: $y_p = E\sin nx + F\cos nx$

Where, E and F are constants to be determined.

Non-homogeneous Differential Equations

Table#1: Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

Non-homogeneous Differential Equations

(R.H.S is a Polynomial)

Example#1

Solve

$$\boxed{y''+2y'+y=2x+x^2}$$

Homogeneous equation is:

$$\boxed{y''+2y'+y=0}$$

The characteristic equation is

$$r^2+2r+1=0$$

$$\text{Or, } (r+1)^2=0$$

$$r=-1,-1$$

The complementary function is:

$$y_c=(c_1+c_2x)e^{-x}$$

Let,

$$y_p=p+qx+rx^2$$

$$\therefore y'=q+2rx$$

$$\therefore y''=2r$$

$$(2r)+2(q+2rx)+(p+qx+rx^2)=2x+x^2$$

$$(2r+2q+p)+(4r+q)x+(r)x^2=2x+1x^2$$

Equating coefficients of equal powers of x

$$\begin{cases} r=1 \\ 4r+q=2 \Rightarrow q=-2 \\ 2r+2q+p=0 \Rightarrow p=2 \end{cases}$$

The particular integral is:

$$y_p=2-2x+1x^2$$

$$\begin{aligned} y &= y_c+y_p \\ &= (c_1+c_2x)e^{-x}+2-2x+1x^2 \end{aligned}$$

Non-homogeneous Differential Equations

(R.H.S is a Polynomial)

Example#2

Solve $\boxed{y''-4y'+4y=4x+8x^3}$ → Homogeneous equation is:

$$\boxed{y''-4y'+4y=0}$$

The characteristic equation is

$$r^2-4r+4=0$$

Or, $\left(r-2\right)^2=0$

$$r=2,\ 2$$

The complementary function is:

$$y_c=(c_1+c_2x)e^{2x}$$

Let,

$$y_p=p+qx+rx^2+sx^3 \qquad \therefore y'=q+2rx+3sx^2$$

$$\therefore y''=2r+6sx$$

$$(2r+6sx)-4(q+2rx+3sx^2)+4(p+qx+rx^2+sx^3)=4x+8x^3$$

$$\left(2r-4q+4p\right)+\left(6s-8r+4q\right)x+\left(-12s+4r\right)x^2+\left(4s\right)x^3=4x+8x^3$$

Equating coefficients of equal powers of x

$$\begin{cases} 2r-4q+4p=0 \\ 6s-8r+4q=4 \\ 4r-12s=0 \\ 4s=8 \end{cases}$$

The particular integral is:

$$y_p=7+10x+6x^2+2x^3$$

$$\boxed{\begin{aligned} y&=y_c+y_p \\ &=(c_1+c_2x)e^{2x}+7+10x+6x^2+2x^3 \end{aligned}}$$

Non-homogeneous Differential Equations

(R.H.S is a Polynomial)

Self Study

1. Solve: $y'' - y' - 2y = 4x^2.$

Ans $y_g = y_h + y_p = c_1 e^{2x} + c_2 e^{-x} - 2x^2 + 2x - 3$

2. Solve $y'' - 5y' + 6y = x^2.$

Ans: $y(x) = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{6}x^2 + \frac{5}{18}x + \frac{19}{108}.$

3. Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6.$

Ans: $y = y_c + y_p = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9.$

4. Solve $y'' - 10y' + 25y = 30x + 3$

5. Solve $\frac{1}{4}y'' + y' + y = x^2 - 2x$

Non-homogeneous Differential Equations

(R.H.S is an exponential)

Example#1

Solve $\boxed{y'' - 3y' + 2y = e^{3x}}$

Homogeneous equation is:

$$\boxed{y'' - 3y' + 2y = 0}$$

The characteristic equation is

$$r^2 - 3r + 2 = 0$$

$$\text{Or, } (r-2)(r-1) = 0$$

$$r_1 = 1, \text{ and } r_2 = 2$$

The complementary function is:

$$y_c = c_1 e^{x} + c_2 e^{2x}$$

Let,

$$y_p = Ae^{3x}$$

$$\therefore y' = 3Ae^{3x}$$

$$\therefore y'' = 9Ae^{3x}$$

$$9Ae^{3x} - 3\left(3Ae^{3x}\right) + 2Ae^{3x} = e^{3x}$$

$$(9-9+2)Ae^{3x} = e^{3x}$$

$$A = \frac{1}{2}$$

The particular integral is:

$$y_p = \frac{1}{2}e^{3x}$$

$$\boxed{\begin{aligned} y &= y_c + y_p \\ &= c_1 e^{x} + c_2 e^{2x} + \frac{1}{2}e^{3x} \end{aligned}}$$

Non-homogeneous Differential Equations
(R.H.S is an exponential)

Example#2

Solve $\boxed{y'' - 3y' + 2y = e^x}$

Homogeneous equation is:

$$\boxed{y'' - 3y' + 2y = 0}$$

Let,

$$y_p = Axe^x$$

$$\therefore y' = A(1+x)e^x$$

$$\therefore y'' = A(2+x)e^x$$

$$A(2+x)e^x - 3A(1+x)e^x + 2Axe^x = e^x$$

$$(2A + Ax - 3A - 3Ax + 2Ax)e^x = e^x$$

$$A = -1$$

The particular integral is:

$$y_p = -xe^x$$

The characteristic equation is

$$r^2 - 3r + 2 = 0$$

$$\text{Or, } (r-2)(r-1) = 0$$

$$r_1 = 1, \text{ and } r_2 = 2$$

The complementary function is:

$$y_c = c_1 e^x + c_2 e^{2x}$$

$$y = y_c + y_p$$

$$= c_1 e^x + c_2 e^{2x} - xe^x$$

Multiplication Rule for Case II *If any y_{p_i} contains terms that duplicate terms in y_c, then that y_{p_i} must be multiplied by x^n, where n is the smallest positive integer that eliminates that duplication.*

Non-homogeneous Differential Equations
(R.H.S is an exponential)

Example#3

Solve

$$\boxed{3y''+10y'-8y=7e^{-4x}}$$

Now let $y_p = Axe^{-4x}$

$$\therefore y' = (1-4x)Ae^{-4x}$$

$$\therefore y'' = (16x-8)Ae^{-4x}$$

$$-24A+10A=7$$

$$A=-\frac{1}{2}$$

The particular integral is:

$$y_p = -\frac{1}{2}xe^{-4x}$$

Homogeneous equation is:

$$\boxed{3y''+10y'-8y=0}$$

The characteristic equation

$$3r^2+10r-8=0$$

$$(3r-2)(r+4)=0$$

$$r_1 = 2/3,\ r_2=-4$$

The complementary function is:

$$y_c = c_1e^{2x/3}+c_2e^{-4x}$$

$$\boxed{\begin{aligned} y &= y_c + y_p \\ &= -\frac{1}{2}xe^{-4x}+c_1e^{2x/3}+c_2e^{-4x}\end{aligned}}$$

Non-homogeneous Differential Equations
(R.H.S is an exponential)

Example#4

Find a particular solution of $y'' - 2y' + y = e^x$.

SOLUTION The complementary function is $y_c = c_1e^x + c_2xe^x$. As in Example 4, the assumption $y_p = Ae^x$ will fail, since it is apparent from y_c that e^x is a solution of the associated homogeneous equation $y'' - 2y' + y = 0$. Moreover, we will not be able to find a particular solution of the form $y_p = Axe^x$, since the term xe^x is also duplicated in y_c. We next try

$$y_p = Ax^2e^x.$$

Substituting into the given differential equation yields $2Ae^x = e^x$, so $A = \frac{1}{2}$. Thus a particular solution is $y_p = \frac{1}{2}x^2e^x$.

Non-homogeneous Differential Equations
(R.H.S is an exponential)

Example#5

Solve $\boxed{3y''-6y'=18}$

Homogeneous equation is:

$$\boxed{3y''-6y'=0}$$

The characteristic equation is

$$3r^2-6r=0$$

$$r_1=0,\ r_2=2$$

The complementary function is:

$$y_c=c_1+c_2e^{2x}$$

Let,

$$y_p=Axe^{0x}$$

$$\therefore y'=A$$

$$\therefore y''=0$$

$$3(0)-6(A)=18$$

$$A=-3$$

The particular integral is:

$$y_p=-3x$$

$$\boxed{\begin{aligned} y&=y_c+y_p\\ &=-3x+c_1+c_2e^{2x}\end{aligned}}$$

Non-homogeneous Differential Equations
(R.H.S is an exponential)

M.M.B Method

$$\boxed{ay'' + by' + cy = ke^{qx}}$$

where k and q are constants.

The form of a particular solution is: $\boxed{y_p = Ae^{qx}}$

The value of A can be calculated by using the following formulae:

$$A = \frac{k}{aq^2 + bq + c}$$ If $q \neq r$, where r is the root of characteristic Eq.

$$A = \frac{kx}{2aq + b}$$ If $q = r$, where r is the root of characteristic Eq.

$$A = \frac{kx^2}{2a}$$ If $q = r_1 = r_2$, where r is the repeated roots.

Non-homogeneous Differential Equations
(R.H.S is an exponential)

Example#1

Solve $$\boxed{y'' - 3y' + 2y = e^{3x}}$$

Homogeneous equation is:

$$\boxed{y'' - 3y' + 2y = 0}$$

The characteristic equation is

$$r^2 - 3r + 2 = 0$$

$$\text{Or, } (r-2)(r-1) = 0$$

$$r_1 = 1, \text{ and } r_2 = 2$$

The complementary function is:

$$y_c = c_1 e^x + c_2 e^{2x}$$

Let,

$$y_p = Ae^{3x}$$

Since $q \neq r$

$$\therefore A = \frac{k}{aq^2 + bq + c}$$

$$A = \frac{1}{1.3^2 + (-3)3 + 2}$$

$$A = \frac{1}{2}$$

The particular integral is:

$$y_p = \frac{1}{2} e^{3x}$$

The general solution is

$$y = y_c + y_p$$

$$= c_1 e^x + c_2 e^{2x} + \frac{1}{2} e^{3x}$$

M.M.B. Method

Non-homogeneous Differential Equations
(R.H.S is an exponential)

Example#2

M.M.B. Method

Solve $\boxed{y'' - 3y' + 2y = e^x}$

Let,

$$y_p = Ae^x$$

Since $q = r_1$

$$\therefore A = \frac{kx}{2aq + b}$$

$$A = \frac{1.x}{2.1.1 + (-3)}$$

$$A = -x$$

The particular integral is:

$$y_p = -xe^x$$

Homogeneous equation is:

$$\boxed{y'' - 3y' + 2y = 0}$$

The characteristic equation is

$$r^2 - 3r + 2 = 0$$

$$\text{Or, } (r-2)(r-1) = 0$$

$r_1 = 1$, and $r_2 = 2$

The complementary function is:

$$y_c = c_1 e^x + c_2 e^{2x}$$

The general solution is

$$y = y_c + y_p$$

$$= c_1 e^x + c_2 e^{2x} - xe^x$$

Non-homogeneous Differential Equations
(R.H.S is an exponential)

Example#3

M.M.B. Method

Solve

$$\boxed{3y''+10y'-8y=7e^{-4x}}$$

Let,

$$y_p = Ae^{-4x} \qquad \because q = r_2$$

$$\therefore A = \frac{kx}{2aq+b}$$

$$\therefore A = \frac{7x}{2.3(-4)+10} = \frac{7x}{-14}$$

$$A = -\frac{1}{2}x$$

The particular integral is:

$$y_p = -\frac{1}{2}xe^{-4x}$$

Homogeneous equation is:

$$\boxed{3y''+10y'-8y=0}$$

The characteristic equation is

$$3r^2+10r-8=0$$

$$(3r-2)(r+4)=0$$

$$r_1 = 2/3,\ r_2 = -4$$

The complementary function is:

$$y_c = c_1e^{2x/3}+c_2e^{-4x}$$

The general solution is:

$$y = y_c + y_p$$

$$= -\frac{1}{2}xe^{-4x}+c_1e^{2x/3}+c_2e^{-4x}$$

Non-homogeneous Differential Equations

(R.H.S is an exponential)

Example#4

M.M.B. Method

Solve

$$\boxed{y''-2y'+y=e^{x}}$$

Let,

$$y_p = Ae^{x}$$

$$\because q = r_1 = r_2 = 1$$

$$\therefore A = \frac{kx^2}{2a}$$

$$\therefore A = \frac{1.x^2}{2.1}$$

$$A = \frac{1}{2}x^2$$

The particular integral is:

$$y_p = \frac{1}{2}x^2e^{x}$$

Homogeneous equation is:

$$\boxed{y''-2y'+y=0}$$

The characteristic equation is

$$r^2-2r+1=0$$

$$(r-1)^2=0$$

$$r_1 = r_2 = 1$$

The complementary function is:

$$y_c = (c_1 + c_2x)e^{x}$$

The general solution is:

$$y = y_c + y_p$$

$$= \frac{1}{2}x^2e^{x} + (c_1 + c_2x)e^{x}$$

Non-homogeneous Differential Equations

(R.H.S is an exponential)

Example#5

M.M.B. Method

Solve

$$\boxed{3y''-6y'=18}$$

$$\boxed{\text{Or, } 3y''-6y'=18e^{0x}}$$

Homogeneous equation is:

$$\boxed{3y''-6y'=0}$$

The characteristic equation is

$$3r^2-6r=0$$

$$r_1=0,\ r_2=2$$

The complementary function is:

$$y_c=c_1e^{0x}+c_2e^{2x}$$

Let,

$$y_p=Ae^{0x}$$

Since $q=r_1=0$

$$\therefore A=\frac{kx}{2aq+b}$$

$$\therefore A=\frac{18x}{2.3.0+(-6)}$$

$$A=-3x$$

The particular integral is:

$$y_p=-3xe^{0x}=-3x$$

The general solution is:

$$y=y_c+y_p$$

$$=-3x+c_1+c_2e^{2x}$$

Non-homogeneous Differential Equations

(R.H.S is a Polynomial and exponential)

Self Study

1. Solve $y'' - 16y = 2e^{4x}$

2. Find a particular solution of $y'' - 5y' + 4y = 8e^{x}$.

Ans: a particular solution of the given equation is $y_p = -\frac{8}{3}xe^{x}$.

3. Determine the form of a particular solution of

(a) $y'' - 8y' + 25y = 5x^3e^{-x} - 7e^{-x}$ **(b)** $y'' + 4y = x\cos x$

4. Solve $y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$.

Ans $y = c_1e^{-x} + c_2e^{3x} - \frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x}$.

5. Solve $y'' + 5y' + 6y = e^{-x}$, with $y(0) = 0$, $y'(0) = 0$.

Ans: $y(x) = \frac{1}{2}e^{-x} - e^{-2x} + \frac{1}{2}e^{-3x}$

6. Solve $y'' - 8y' + 25y = e^{x}$

$Ans\ y(x) = e^{4x}(c_1\cos 3x + c_2\sin 3x) + \frac{1}{18}e^{x}.$

Practice quiz: Solving inhomogeneous equations

1. All the questions will consider the differential equation given by

$y'' + 5y' + 6y = 2e^{-x}$. What is the soluion that satisfies $y(0) = 0$ and $y'(0) = 0$?

$(a)\ y(x) = e^{-x}\left(1 - 2e^{-x} + e^{-2x}\right)$

$(b)\ y(x) = e^{x}\left(1 - 4e^{-3x} + 3e^{-4x}\right)$

$(c)\ y(x) = e^{3x}\left(3 - 4e^{-x} + e^{-4x}\right)$

$(d)\ y(x) = e^{3x}\left(1 - 2e^{-x} + e^{-2x}\right)$

2. All the questions will consider the differential equation given by

$y'' + 5y' + 6y = 2e^{-x}$. What is the soluion that satisfies $y(0) = 1$ and $y'(0) = 0$?

$(a)\ y(x) = e^{-x}\left(1 + e^{-x} - e^{-2x}\right)$

$(b)\ y(x) = e^{x}\left(1 - e^{-3x} + e^{-4x}\right)$

$(c)\ y(x) = e^{3x}\left(1 - e^{-x} + e^{-4x}\right)$

$(d)\ y(x) = -e^{3x}\left(1 - e^{-x} - e^{-2x}\right)$

Practice quiz: Solving inhomogeneous equations

3. All the questions will consider the differential equation given by

$y'' + 5y' + 6y = 2e^{-x}$. What is the soluion that satisfies $y(0) = 0$ and $y'(0) = 1$?

$(a)\ y(x) = e^{-x}\left(1 - e^{-x}\right)$

$(b)\ y(x) = e^{x}\left(1 - 3e^{-3x} + 2e^{-4x}\right)$

$(c)\ y(x) = e^{3x}\left(4 - 5e^{-x} + e^{-4x}\right)$

$(d)\ y(x) = e^{3x}\left(2 - 3e^{-x} + e^{-2x}\right)$

R.H.S is an sinnx or cosnx

Non-homogeneous Differential Equations

Example#1 **(R.H.S is an sinnx or cosnx)**

Solve $\boxed{y'' + 5y' + 6y = \sin 2x}$ → The homogeneous Eq. is:

$$\boxed{y'' + 5y' + 6y = 0}$$

The characteristic equation is:

$$r^2 + 5r + 6 = 0$$

$$(r+3)(r+2) = 0$$

$$r_1 = -2, \quad r_2 = -3$$

The complementary function is:

$$y_c = c_1 e^{-2x} + c_2 e^{-3x}$$

Let, $y_p = E\cos(2x) + F\sin(2x)$

$$y' = -2E\sin 2x + 2F\cos 2x$$

$$y'' = -4E\cos 2x - 4F\sin 2x$$

$$(-4E + 10F + 6E)\cos 2x + (-4F - 10E + 6F)\sin 2x = \sin 2x$$

$$2E + 10F = 0$$

$$-10E + 2F = 1$$

$$E = -5/52$$

$$F = 1/52$$

The particular integral is:

$$y_p = \frac{-5}{52}\cos 2x + \frac{1}{52}\sin 2x$$

The general solution is:

$$y = y_c + y_p$$

$$= c_1 e^{-2x} + c_2 e^{-3x} - \frac{5}{52}\cos 2x + \frac{1}{52}\sin 2x$$

Non-homogeneous Differential Equations

(R.H.S is an sinnx or cosnx)

Example#2

Solve $\boxed{y''+y'-6y=52\cos 2x}$

$$y=E\cos 2x+F\sin 2x$$
$$y'=-2(E\sin 2x-F\cos 2x)$$
$$y''=-4(E\cos 2x+F\sin 2x)$$

$$-10E+2F=52$$
$$-2E-10F=0$$

$$E=-5$$
$$F=1$$

The particular integral is:

$$y_p=-5\cos 2x+\sin 2x$$

The homogeneous Eq. is:

$$\boxed{y''+y'-6y=0}$$

The characteristic equation is:

$$r^2+r-6=0$$
$$(r-2)(r+3)=0$$

$$r_1=2,\ r_2=-3$$

The complementary function is:

$$y_c=c_1e^{2x}+c_2e^{-3x}$$

The general solution is:

$$y=y_c+y_p$$
$$=c_1e^{2x}+c_2e^{-3x}-5\cos 2x+\sin 2x$$

Non-homogeneous Differential Equations
(R.H.S is an sinnx or cosnx)

M.M.B Method

$$\boxed{ay'' + by' + cy = C\sin nx + D\cos nx}$$

where C and D are constants.

The form of a particular solution is: $\boxed{y_p = E\sin nx + F\cos nx}$

The value of E and F can be calculated by using the following formulae:

$$E = \frac{(c - n^2 a)C + nbD}{(c - n^2 a)^2 + n^2 b^2}$$

$$F = \frac{(c - n^2 a)D - nbC}{(c - n^2 a)^2 + n^2 b^2}$$

Non-homogeneous Differential Equations

Example#1 **(R.H.S is an sinnx or cosnx)**

Solve $\boxed{y''+5y'+6y=\sin 2x}$ → The homogeneous Eq. is:

$$\boxed{y''+5y'+6y=0}$$

The characteristic equation is:

$$r^2+5r+6=0$$

$$(r+3)(r+2)=0$$

$$r_1=-2,\ r_2=-3$$

The complementary function is:

$$y_c=c_1e^{-2x}+c_2e^{-3x}$$

Let, $y_p=E\sin(2x)+F\cos(2x)$

Here $a=1,\ b=5,\ c=6, n=2,\ C=1,\ \ D=0$.

M.M.B. Method

$$E=\frac{(c-n^2a)C+nbD}{(c-n^2a)^2+n^2b^2}=\frac{(6-2^2.1)1+2.5.0}{(2)^2+2^2.5^2}=\frac{2}{104}=\frac{1}{52}$$

$$F=\frac{(c-n^2a)D-nbC}{(c-n^2a)^2+n^2b^2}=\frac{(6-2^2.1)0-2.5.1}{(6-2^2.1)^2+2^2.5^2}=\frac{-10}{104}=-\frac{5}{52}$$

The particular integral is:

$$y_p=\frac{1}{52}\sin 2x-\frac{5}{52}\cos 2x$$

The general solution is:

$$y=y_c+y_p$$

$$=c_1e^{-2x}+c_2e^{-3x}-\frac{5}{52}\cos 2x+\frac{1}{52}\sin 2x$$

Non-homogeneous Differential Equations

Example#2

(R.H.S is an sinnx or cosnx)

Solve $\boxed{y''+y'-6y=52\cos 2x}$

The homogeneous Eq. is:

$$\boxed{y''+y'-6y=0}$$

Here $a=1,\ b=1,\ c=-6, n=2,\ C=0,\ \ D=52.$

The characteristic equation is:

$$r^2+r-6=0$$

$$(r-2)(r+3)=0$$

$$r_1=2,\ \ r_2=-3$$

The complementary function is:

$$y_c=c_1e^{2x}+c_2e^{-3x}$$

M.M.B. Method

$$E=\frac{(c-n^2a)C+nbD}{(c-n^2a)^2+n^2b^2}=\frac{(-6-2^2.1)0+2.1.52}{(-6-2^2.1)^2+2^2.1^2}=\frac{104}{104}=1$$

$$F=\frac{(c-n^2a)D-nbC}{(c-n^2a)^2+n^2b^2}=\frac{(-6-2^2.1)52-2.1.0}{(-6-2^2.1)^2+2^2.1^2}=\frac{-520}{104}=-5$$

The particular integral is:

$$y_p=-5\cos 2x+\sin 2x$$

The general solution is:

$$y=y_c+y_p$$

$$=c_1e^{2x}+c_2e^{-3x}+\sin 2x-5\cos 2x$$

Non-homogeneous Differential Equations

(R.H.S is an sinnx or cosnx)

Self Study

1. Solve: $y'' - y' + y = 2\sin 3x.$

2. Solve $y'' - 5y' + 6y = \sin(3x+2).$

3. Solve $y'' - y' - 2y = \sin 2x.$

Ans: the general solution is

$$y_g = y_h + y_p = c_1 e^{2x} + c_2 e^{-x} - \frac{3}{20}\sin 2x + \frac{1}{20}\cos 2x\,.$$

4. Solve: $4y'' - 4y' - 3y = \cos 2x$

5. Solve $y'' - y = \sin x$ Ans: $y = c_1 e^{-x} + c_2 e^{x} - \frac{1}{2}\sin x.$

Example#3

Determine the form of a particular solution of

$$y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}.$$

SOLUTION

Corresponding to $3x^2$ we assume $\quad y_{p_1} = Ax^2 + Bx + C.$

Corresponding to $-5\sin 2x$ we assume $\quad y_{p_2} = E\cos 2x + F\sin 2x.$

Corresponding to $7xe^{6x}$ we assume $\quad y_{p_3} = (Gx + H)e^{6x}.$

The assumption for the particular solution is then

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} = Ax^2 + Bx + C + E\cos 2x + F\sin 2x + (Gx + H)e^{6x}.$$

No term in this assumption duplicates a term in $y_c = c_1e^{2x} + c_2e^{7x}$.

Non-homogeneous Differential Equations

Example#4

Determine the form of a particular solution of

(a) $y'' - 8y' + 25y = 5x^3e^{-x} - 7e^{-x}$ **(b)** $y'' + 4y = x\cos x$

SOLUTION **(a)** We can write $g(x) = (5x^3 - 7)e^{-x}$. Using entry 9 in Table 4.4.1 as a model, we assume a particular solution of the form

$$y_p = (Ax^3 + Bx^2 + Cx + E)e^{-x}.$$

Note that there is no duplication between the terms in y_p and the terms in the complementary function $y_c = e^{4x}(c_1\cos 3x + c_2\sin 3x)$.

(b) The function $g(x) = x\cos x$ is similar to entry 11 in Table 4.4.1 except, of course, that we use a linear rather than a quadratic polynomial and $\cos x$ and $\sin x$ instead of $\cos 4x$ and $\sin 4x$ in the form of y_p:

$$y_p = (Ax + B)\cos x + (Cx + E)\sin x.$$

Again observe that there is no duplication of terms between y_p and $y_c = c_1\cos 2x + c_2\sin 2x$. ≡

Example#5

Solve $y''' + y'' = e^x \cos x$.

SOLUTION From the characteristic equation $m^3 + m^2 = 0$ we find $m_1 = m_2 = 0$ and $m_3 = -1$. Hence the complementary function of the equation is $y_c = c_1 + c_2 x + c_3 e^{-x}$. With $g(x) = e^x \cos x$, we see from entry 10 of Table 4.4.1 that we should assume that

$$y_p = Ae^x \cos x + Be^x \sin x.$$

Because there are no functions in y_p that duplicate functions in the complementary solution, we proceed in the usual manner. From

$$y_p''' + y_p'' = (-2A + 4B)e^x \cos x + (-4A - 2B)e^x \sin x = e^x \cos x$$

we get $-2A + 4B = 1$ and $-4A - 2B = 0$. This system gives $A = -\frac{1}{10}$ and $B = \frac{1}{5}$, so a particular solution is $y_p = -\frac{1}{10}e^x \cos x + \frac{1}{5}e^x \sin x$. The general solution of the equation is

$$y = y_c + y_p = c_1 + c_2 x + c_3 e^{-x} - \frac{1}{10}e^x \cos x + \frac{1}{5}e^x \sin x.$$

Example#6

Solve $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$.

SOLUTION The complementary function is $y_c = c_1e^{3x} + c_2xe^{3x}$. And so, based on entries 3 and 7 of Table 4.4.1, the usual assumption for a particular solution would be

$$y_p = \underbrace{Ax^2 + Bx + C}_{y_{p_1}} + \underbrace{Ee^{3x}}_{y_{p_2}}.$$

Inspection of these functions shows that the one term in y_{p_2} is duplicated in y_c. If we multiply y_{p_2} by x, we note that the term xe^{3x} is still part of y_c. But multiplying y_{p_2} by x^2 eliminates all duplications. Thus the operative form of a particular solution is

$$y_p = Ax^2 + Bx + C + Ex^2e^{3x}.$$

Differentiating this last form, substituting into the differential equation, and collecting like terms gives

$$y_p'' - 6y_p' + 9y_p = 9Ax^2 + (-12A + 9B)x + 2A - 6B + 9C + 2Ee^{3x} = 6x^2 + 2 - 12e^{3x}.$$

It follows from this identity that $A = \frac{2}{3}$, $B = \frac{8}{9}$, $C = \frac{2}{3}$, and $E = -6$. Hence the general solution $y = y_c + y_p$ is $y = c_1e^{3x} + c_2xe^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2e^{3x}$.

Example#7

Determine the form of a particular solution of $y^{(4)} + y''' = 1 - x^2e^{-x}$.

SOLUTION Comparing $y_c = c_1 + c_2x + c_3x^2 + c_4e^{-x}$ with our normal assumption for a particular solution

$$y_p = \underbrace{A}_{y_{p_1}} + \underbrace{Bx^2e^{-x} + Cxe^{-x} + Ee^{-x}}_{y_{p_2}},$$

we see that the duplications between y_c and y_p are eliminated when y_{p_1} is multiplied by x^3 and y_{p_2} is multiplied by x. Thus the correct assumption for a particular solution is $y_p = Ax^3 + Bx^3e^{-x} + Cx^2e^{-x} + Exe^{-x}$. ≡

Non-homogeneous Differential Equations

Example#8

Solve $y'' + y = 4x + 10 \sin x$, $y(\pi) = 0$, $y'(\pi) = 2$.

SOLUTION The solution of the associated homogeneous equation $y'' + y = 0$ is $y_c = c_1 \cos x + c_2 \sin x$. Because $g(x) = 4x + 10 \sin x$ is the sum of a linear polynomial and a sine function, our normal assumption for y_p, from entries 2 and 5 of Table 4.4.1, would be the sum of $y_{p_1} = Ax + B$ and $y_{p_2} = C \cos x + E \sin x$:

$$y_p = Ax + B + C \cos x + E \sin x. \tag{5}$$

But there is an obvious duplication of the terms $\cos x$ and $\sin x$ in this assumed form and two terms in the complementary function. This duplication can be eliminated by simply multiplying y_{p_2} by x. Instead of (5) we now use

$$y_p = Ax + B + Cx \cos x + Ex \sin x. \tag{6}$$

Differentiating this expression and substituting the results into the differential equation gives

$$y_p'' + y_p = Ax + B - 2C \sin x + 2E \cos x = 4x + 10 \sin x,$$

and so $A = 4$, $B = 0$, $-2C = 10$, and $2E = 0$. The solutions of the system are immediate: $A = 4$, $B = 0$, $C = -5$, and $E = 0$. Therefore from (6) we obtain $y_p = 4x - 5x \cos x$. The general solution of the given equation is

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x + 4x - 5x \cos x.$$

We now apply the prescribed initial conditions to the general solution of the equation. First, $y(\pi) = c_1 \cos \pi + c_2 \sin \pi + 4\pi - 5\pi \cos \pi = 0$ yields $c_1 = 9\pi$, since $\cos \pi = -1$ and $\sin \pi = 0$. Next, from the derivative

$$y' = -9\pi \sin x + c_2 \cos x + 4 + 5x \sin x - 5 \cos x$$

and $$y'(\pi) = -9\pi \sin \pi + c_2 \cos \pi + 4 + 5\pi \sin \pi - 5 \cos \pi = 2$$

we find $c_2 = 7$. The solution of the initial-value is then

$$y = 9\pi \cos x + 7 \sin x + 4x - 5x \cos x.$$

Practice quiz: Particular solutions

1. A particular solution of $y'' + 3y' + 2y = 2e^{2x}$ is given by

$(a)\ y_p = \frac{1}{6}e^{-2x}$

$(b)\ y_p = \frac{1}{6}e^{2x}$

$(c)\ y_p = -\frac{1}{6}e^{-2x}$

$(d)\ y_p = -\frac{1}{6}e^{2x}$

2. A particular solution of $y'' - y' - 2y = 2\cos 2x$ is given by

$(a)\ y_p = \frac{1}{10}(\cos 2t + 3\sin 2t)$

$(b)\ y_p = -\frac{1}{10}(\cos 2t + 3\sin 2t)$

$(c)\ y_p = \frac{1}{10}(3\cos 2t + \sin 2t)$

$(d)\ y_p = -\frac{1}{10}(3\cos 2t + \sin 2t)$

Practice quiz: Particular solutions

3. A particular solution of $y'' - 3y' + 2y = x + 1$ is given by

$(a)\ y_p = \frac{5}{4}x + \frac{1}{2}$

$(b)\ y_p = \frac{5}{4}x - \frac{1}{2}$

$(c)\ y_p = \frac{1}{2}x + \frac{5}{4}$

$(d)\ y_p = \frac{1}{2}x - \frac{5}{4}$

Self Study

In Problems 1–26 solve the given differential equation by undetermined coefficients.

1. $y'' + 3y' + 2y = 6$
2. $4y'' + 9y = 15$
3. $y'' - 10y' + 25y = 30x + 3$
4. $y'' + y' - 6y = 2x$
5. $\frac{1}{4}y'' + y' + y = x^2 - 2x$
6. $y'' - 8y' + 20y = 100x^2 - 26xe^x$
7. $y'' + 3y = -48x^2e^{3x}$
8. $4y'' - 4y' - 3y = \cos 2x$
9. $y'' - y' = -3$
10. $y'' + 2y' = 2x + 5 - e^{-2x}$
11. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$
12. $y'' - 16y = 2e^{4x}$
13. $y'' + 4y = 3 \sin 2x$
14. $y'' - 4y = (x^2 - 3) \sin 2x$
15. $y'' + y = 2x \sin x$
16. $y'' - 5y' = 2x^3 - 4x^2 - x + 6$
17. $y'' - 2y' + 5y = e^x \cos 2x$
18. $y'' - 2y' + 2y = e^{2x}(\cos x - 3 \sin x)$
19. $y'' + 2y' + y = \sin x + 3 \cos 2x$
20. $y'' + 2y' - 24y = 16 - (x + 2)e^{4x}$
21. $y''' - 6y'' = 3 - \cos x$
22. $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$
23. $y''' - 3y'' + 3y' - y = x - 4e^x$
24. $y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$
25. $y^{(4)} + 2y'' + y = (x - 1)^2$
26. $y^{(4)} - y'' = 4x + 2xe^{-x}$

In Problems 27–36 solve the given initial-value problem.

27. $y'' + 4y = -2, \quad y(\pi/8) = \frac{1}{2}, y'(\pi/8) = 2$
28. $2y'' + 3y' - 2y = 14x^2 - 4x - 11, \quad y(0) = 0, y'(0) = 0$
29. $5y'' + y' = -6x, \quad y(0) = 0, y'(0) = -10$
30. $y'' + 4y' + 4y = (3 + x)e^{-2x}, \quad y(0) = 2, y'(0) = 5$
31. $y'' + 4y' + 5y = 35e^{-4x}, \quad y(0) = -3, y'(0) = 1$

Inverse operator method

Inverse operator method for finding particular integral

$$\left(a_0D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n\right)y = g(x), \rightarrow (1)$$

The general solution of (1) is $y = y_c + y_p$

complementary function y_c is the solution of corresponding homogeneous differential equation of (1).

i.e. y_c is the soultion of $\left(a_0D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n\right)y = 0.$

Particular integral can be found by any one of the followings:

a) the method of undermined coeffiecients

b) Inverse operator method

$Let\ f(D) = a_0D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n$

$$\therefore (1) \Rightarrow f(D)y = g(x) \Rightarrow y_p = \frac{1}{f(D)}g(x)$$

Inverse operator method for finding particular integral

(R.H.S is a Polynomial)

Type1:

If $g(x) = X$ (polinomial in x),

then $y_p = \dfrac{1}{f(D)} X$,

Need to apply binomial expansion for $\left[f(D) \right]^{-1}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots.$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots.$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots.$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots.$$

Non-homogeneous Differential Equations

(R.H.S is a Polynomial)

Example#1

Solve $\boxed{y''+2y'+y=2x+x^2}$ → Homogeneous equation is:

$$\boxed{y''+2y'+y=0}$$

The characteristic equation is

$$r^2+2r+1=0$$

$$\text{Or, } (r+1)^2=0$$

$$r=-1,-1$$

The complementary function is:

$$y_c=(c_1+c_2x)e^{-x}$$

The given Eq. can be written as:

$$(D^2+2D+1)y=2x+x^2$$

$$\therefore y_p=\frac{1}{(D+1)^2}\left(2x+x^2\right)$$

$$=(1+D)^{-2}\left(2x+x^2\right)$$

$$=\left(1-2D+3D^2-4D^3+.....\right)\left(2x+x^2\right)$$

$$=1.\left(2x+x^2\right)-2\left(2+2x\right)+3\left(0+2\right)$$

$$=2-2x+x^2$$

The particular integral is:

$$y_p=2-2x+1x^2$$

$$y=y_c+y_p$$
$$=(c_1+c_2x)e^{-x}+2-2x+1x^2$$

Non-homogeneous Differential Equations
(R.H.S is a Polynomial)

Example#2

Solve $\boxed{y''-4y'+4y=4x+8x^3}$ → Homogeneous equation is:

$$\boxed{y''-4y'+4y=0}$$

The characteristic equation is

$$r^2-4r+4=0$$

$$\text{Or, } (r-2)^2=0$$

$$r=2,\ 2$$

The complementary function is:

$$y_c=(c_1+c_2x)e^{2x}$$

The given Eq. can be written as:

$$(D^2-4D+4)y=4x+8x^3$$

$$\therefore y_p=\frac{1}{(D-2)^2}\left(4x+8x^3\right)=\frac{1}{4\left(1-\frac{D}{2}\right)^2}\left(4x+8x^3\right)$$

$$=\frac{1}{4}\left(1-\frac{D}{2}\right)^{-2}\left(4x+8x^3\right)$$

$$=\frac{1}{4}\left(1+2\frac{D}{2}+3\frac{D^2}{4}+4\frac{D^3}{8}+\ldots\right)\left(4x+8x^3\right)$$

$$=\frac{1}{4}\left(1+D+\frac{3}{4}D^2+\frac{1}{2}D^3+\ldots\right)\left(4x+8x^3\right)$$

$$=\frac{1}{4}\left(4x+4+8x^3+24x^2+\frac{3}{4}48x+\frac{1}{2}48+\ldots\right)$$

$$=7+10x+6x^2+2x^3$$

The particular integral is:

$$y_p=7+10x+6x^2+2x^3$$

$$\boxed{\begin{aligned} y&=y_c+y_p\\ &=(c_1+c_2x)e^{2x}+7+10x+6x^2+2x^3\end{aligned}}$$

Inverse operator method for finding particular integral

(R.H.S is a Polynomial)

Example#3

Solve $\boxed{y'' - y = x^2}$

Homogeneous equation is:

$$\boxed{y'' - y = 0}$$

The characteristic equation is

$$r^2 - 1 = 0$$

$$\text{Or, } (r+1)(r-1) = 0$$

$$r_1 = -1, \text{ and } r_2 = 1$$

The complementary function is:

$$y_c = c_1 e^{-x} + c_2 e^{x}$$

The given Eq. can be written as:

$$(\mathrm{D}^2 - 1)y = x^2$$

$$\therefore y_p = \frac{1}{\mathrm{D}^2 - 1} x^2$$

$$= -(1 - D^2)^{-1}(x^2) \quad \xleftarrow{Type1}$$

$$= -\left[1 + \mathrm{D}^2 + \left(\mathrm{D}^2\right)^2 + \ldots\right](x^2)$$

$$= -x^2 - 2$$

The particular integral is:

$$y_p = -x^2 - 2$$

The general solution is

$$y = y_c + y_p$$

$$= c_1 e^{-x} + c_2 e^{x} - x^2 - 2$$

Inverse operator method for finding particular integral
(R.H.S is an exponential)

Type 2:

If $g(x) = e^{ax}$,

then $y_p = \dfrac{1}{f(D)} e^{ax} = \dfrac{1}{f(a)} e^{ax}$; $f(a) \neq 0$.

but if $f(a) = 0$, then $y_p = x \dfrac{1}{f'(D)} e^{ax}$; $f'(a) \neq 0$.

but if $f'(a) = 0$, then $y_p = x^2 \dfrac{1}{f''(D)} e^{ax}$; $f''(a) \neq 0$.

Inverse operator method for finding particular integral

(R.H.S is an exponential)

Example#1

Solve $\boxed{y'' - 3y' + 2y = e^{3x}}$ → Homogeneous equation is:

$$\boxed{y'' - 3y' + 2y = 0}$$

The characteristic equation is

$$r^2 - 3r + 2 = 0$$

Or, $(r-2)(r-1) = 0$

$r_1 = 1$, and $r_2 = 2$

The complementary function is:

$$y_c = c_1 e^{x} + c_2 e^{2x}$$

The given Eq. can be written as:

$$\left(D^2 - 3D + 2\right) y = e^{3x}$$

$$\therefore y_p = \frac{1}{\left(D^2 - 3D + 2\right)} e^{3x}$$

$$= \frac{1}{\left(3^2 - 3.3 + 2\right)} e^{3x} \quad \leftarrow Type\,2$$

$$= \frac{1}{2} e^{3x}$$

The particular integral is:

$$y_p = \frac{1}{2} e^{3x}$$

The general solution is

$$y = y_c + y_p$$

$$= c_1 e^{x} + c_2 e^{2x} + \frac{1}{2} e^{3x}$$

Inverse operator method for finding particular integral
(R.H.S is an exponential)

Example#2

Solve $\boxed{y'' - 3y' + 2y = e^x}$ → Homogeneous equation is:

$$\boxed{y'' - 3y' + 2y = 0}$$

The characteristic equation is

$$r^2 - 3r + 2 = 0$$

$$\text{Or, } (r-2)(r-1) = 0$$

$$r_1 = 1, \text{ and } r_2 = 2$$

The complementary function is:

$$y_c = c_1 e^x + c_2 e^{2x}$$

The given Eq. can be written as:

$$(D^2 - 3D + 2)y = e^x$$

$$\therefore y_p = \frac{1}{(D^2 - 3D + 2)} e^x$$

$$= \frac{x}{(2D - 3)} e^x$$

Type 2 ← (Case Failed)

$$= \frac{x}{(2.1 - 3)} e^{1x}$$

$$= -xe^x$$

The particular integral is:

$$y_p = -xe^x$$

The general solution is

$$y = y_c + y_p$$

$$= c_1 e^x + c_2 e^{2x} - xe^x$$

Inverse operator method for finding particular integral

(R.H.S is an sinnx or cosnx)

Type 3:

If $g(x) = \sin ax$,

then $y_p = \dfrac{1}{f(D^2)} \sin ax = \dfrac{1}{f(-a^2)} \sin ax;\ f(-a^2) \neq 0$

but if $f(-a^2) = 0$, then $y_p = x \dfrac{1}{f'(-a^2)} \sin ax;\ f'(-a^2) \neq 0.$

but if $f'(-a^2) = 0$, then $y_p = x^2 \dfrac{1}{f''(-a^2)} \sin ax;\ f''(-a^2) \neq 0.$

If $g(x) = \cos ax$, the process is same as above.

Non-homogeneous Differential Equations

Example#1 **(R.H.S is an sinnx or cosnx)**

Solve $\boxed{y''+5y'+6y=\sin 2x}$ → The homogeneous Eq. is:

$$\boxed{y''+5y'+6y=0}$$

The characteristic equation is:

$$r^2+5r+6=0$$

$$(r+3)(r+2)=0$$

$$r_1=-2,\quad r_2=-3$$

The complementary function is:

$$y_c=c_1e^{-2x}+c_2e^{-3x}$$

The given Eq. can be written as:

$$\left(D^2+5D+6\right)y=\sin 2x$$

$$\therefore y_p=\frac{1}{D^2+5D+6}\sin 2x \leftarrow \underset{D^2=-a^2=-2^2}{Type\,3}$$

$$=\frac{1}{-2^2+5D+6}\sin 2x \qquad =\frac{1}{2+5D}\sin 2x$$

$$=\frac{2-5D}{\left(4-25D^2\right)}\sin 2x=\frac{2-5D}{\left(4-25\left(-2^2\right)\right)}\sin 2x$$

$$=\frac{1}{104}\left(2-5D\right)\sin 2x \quad =\frac{2}{104}\sin 2x-\frac{5}{104}.2\cos 2x$$

$$=\frac{1}{52}\sin 2x-\frac{5}{52}\cos 2x$$

The particular integral is:

$$y_p=\frac{1}{52}\sin 2x-\frac{5}{52}\cos 2x$$

The general solution is:

$$y=y_c+y_p$$

$$=c_1e^{-2x}+c_2e^{-3x}-\frac{5}{52}\cos 2x+\frac{1}{52}\sin 2x$$

Non-homogeneous Differential Equations

Example#2 **(R.H.S is an sinnx or cosnx)**

Solve $\boxed{y'' + y = \cos x}$ → The homogeneous Eq. is:

$$\boxed{y'' + y = 0}$$

The characteristic equation is:

$$r^2 + 1 = 0$$

$$r_1 = -i, \; r_2 = i$$

The complementary function is:

$$y_c = e^{0.x}\left(c_1 \sin x + c_2 \cos x\right)$$

The given Eq. can be written as:

$$\left(D^2 + 1\right) y = \cos x$$

$$\therefore y_p = \frac{1}{D^2 + 1} \cos x$$

$$= \frac{x}{(2D)} \cos x \quad \leftarrow \; \begin{matrix} Type\,3 \\ \text{(Case Failed)} \end{matrix}$$

$$= \frac{x}{2} \frac{1}{D} \left(\cos x \quad \right)$$

$$= \frac{x}{2} \sin x$$

The particular integral is:

$$y_p = \frac{x}{2} \sin x$$

The general solution is:

$$y = y_c + y_p$$

$$= c_1 \sin x + c_2 \cos x + \frac{x}{2} \sin x$$

Type 4:

If $g(x) = e^{ax} V$,

then $y_p = \dfrac{1}{f(D)} e^{ax} V = e^{ax} \dfrac{1}{f(D+a)} V.$

Non-homogeneous Differential Equations

Example#1

Solve $\boxed{y'' - 9y' + 14y = 7xe^{6x}}$

The given Eq. can be written as:

$$\left(D^2 - 9D + 14\right) y = 7xe^{6x}$$

$$\therefore y_p = \frac{1}{D^2 - 9D + 14}\left(7xe^{6x}\right)$$

$$= 7e^{6x} \frac{1}{(D+6)^2 - 9(D+6) + 14}(x)$$

Type 4

(D≡D+6)

$$= 7e^{6x} \frac{1}{D^2 + 3D - 4}(x)$$

$$= 7e^{6x} \frac{1}{(D+4)(D-1)}(x)$$

$$= 7e^{6x}\left[\frac{1}{5}\frac{1}{(D-1)} - \frac{1}{5}\frac{1}{(D+4)}\right](x)$$

$$= \frac{7e^{6x}}{5}\left[\frac{1}{(D-1)} - \frac{1}{(D+4)}\right](x)$$

The homogeneous Eq. is:

$$\boxed{y'' - 9y' + 14y = 0}$$

The characteristic equation is:

$$r^2 - 9r + 14 = 0$$

$$r_1 = 2, \ r_2 = 7$$

The complementary function is:

$$y_c = c_1 e^{2x} + c_2 e^{9x}$$

Non-homogeneous Differential Equations

Example#1

Solve $\boxed{y'' - 9y' + 14y = 7xe^{6x}}$ → The homogeneous Eq. is:

$$\boxed{y'' - 9y' + 14y = 0}$$

The characteristic equation is:

$$r^2 - 9r + 14 = 0$$

$$r_1 = 2, \quad r_2 = 7$$

The complementary function is:

$$y_c = c_1 e^{2x} + c_2 e^{9x}$$

$$= \frac{7e^{6x}}{5}\left[\frac{1}{(D-1)} - \frac{1}{(D+4)}\right](x)$$

$$= \frac{7e^{6x}}{5}\left[-(1-D)^{-1} - \frac{1}{4}\left(1-\frac{D}{4}\right)^{-1}\right](x)$$

$$= -\frac{7e^{6x}}{5}\left[(1-D)^{-1} + \frac{1}{4}\left(1-\frac{D}{4}\right)^{-1}\right](x)$$

$$= -\frac{7e^{6x}}{5}\left[\left(1+D+D^2+....\right) + \frac{1}{4}\left(1-\frac{D}{4}+\frac{D^2}{4^2}-....\right)\right](x)$$

$$= -\frac{7e^{6x}}{5}\left[(x+1) + \frac{1}{4}\left(x-\frac{1}{4}\right)\right]$$

The particular integral is:

$$y_p = -\frac{7e^{6x}}{5}\left[(x+1) + \frac{1}{4}\left(x-\frac{1}{4}\right)\right]$$

The general solution is:

$$y = y_c + y_p$$

$$= c_1 e^{2x} + c_2 e^{9x} - \frac{7e^{6x}}{5}\left[(x+1) + \frac{1}{4}\left(x-\frac{1}{4}\right)\right]$$

Type 5:

If $g(x) = xV$,

then $y_p = \dfrac{1}{f(D)} xV = x\dfrac{1}{f(D)}V - \dfrac{f'(D)}{(f(D))^2}V.$

Non-homogeneous Differential Equations

Example#1

Solve $\boxed{y'' + 4y = x\cos x}$ $\longrightarrow$ The homogeneous Eq. is:

$$\boxed{y'' + 4y = 0}$$

The characteristic equation is:

$$r^2 + 4 = 0$$

$$r_1 = -2i,\ r_2 = 2i$$

The complementary function is:

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

The given Eq. can be written as:

$$\left(D^2 + 4\right) y = x\cos x$$

$$\therefore y_p = \frac{1}{D^2+4}\left(x\cos x\right)$$

$$= x\frac{1}{D^2+4}\cos x - \frac{2D}{(D^2+4)^2}\cos x \quad \xleftarrow{Type\ 5}$$

$$= x\frac{1}{-1+4}\cos x - \frac{2D}{(-1+4)^2}\cos x \quad \xleftarrow[D^2=-1^2]{Type\ 3}$$

$$= x\frac{1}{3}\cos x - \frac{2D}{9}\cos x$$

$$= \frac{x}{3}\cos x + \frac{2}{9}\sin x$$

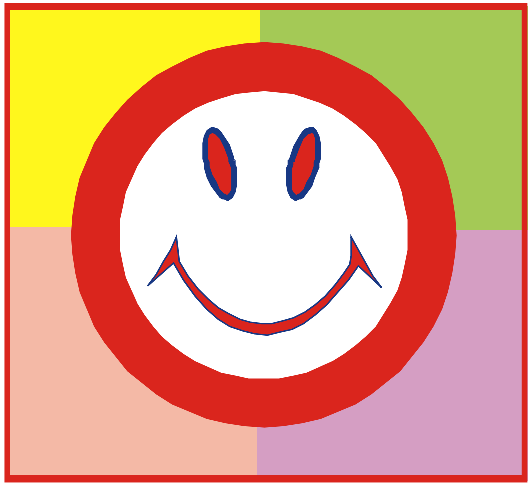
The particular integral is:

$$y_p = \frac{x}{3}\cos x + \frac{2}{9}\sin x$$

The general solution is:

$$y = y_c + y_p$$

$$= c_1\cos 2x + c_2 \sin 2x + \frac{x}{3}\cos x + \frac{2}{9}\sin x$$



Thank you

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