STA107 – Module 1
Introduction to Probability
(with Answers)

Probability Theory

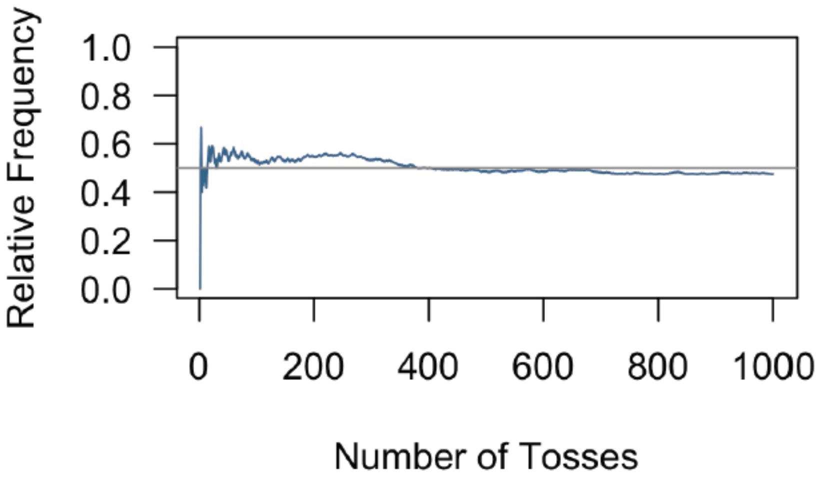
- Probability theory deals with **phenomena whose outcomes is affected by random events**, and therefore **cannot be predicted with certainty**.
- A **random phenomena** is a situation in which we know what outcomes can possibly occur, but we do not know which particular outcome will happen (do not know *exactly* what will happen). But we could know *how likely* is something to happen.
- In a random sample or a randomized experiment, the possible outcomes are known, but it is not certain which will occur.
  - E.g., Experiment: Roll a fair coin. The possible outcome is Head or Tail.

Probability as a Long-Run Relative Frequence

- For a particular possible outcome for a random phenomenon, the probability of that outcome is the proportion of times that outcome would occur in a very long sequence of observations.
- With a random sample or randomized experiment, the ***probability*** that an observation has *a* ***particular outcome*** is the ***proportion of times*** that ***outcome would occur*** in *a* ***very long sequence of observations*.**

The statistics student tossed a coin throughout the Boston Marathon to study the behavior of a chance process in the long run.

The graph illustrates the long-run relative frequency of heads for a fair coin and how it tends to stabilize around half as the number of coin flips increases toward 1000.



Random Experiment

- A **Random Experiment** or a **Chance Experiment** is an experiment (a process) whose **outcome** is **uncertain**, not known in advance (not known beforehand).

Examples:

- Roll of a standard (6-sided) die 
- Tossing a coin 

- Selecting a person at random and asking their age

Trial

- **Each occasion** upon which we **observe a random phenomenon** is called a **trial**.

- **At each trial**, we note the **result of the random phenomena** and call that **trail's outcome**.

- So, trial's outcome is the result of experiment which cannot be decomposed (broken down) to simpler results.

E.g., 

Rolling a 1" vs. "Rolling an even number"

- Thus, a random phenomenon consists of trials and each trial has an outcome.

Sample Space and Events

- A **sample space** of the experiment is **set of all possible outcomes**, denoted by $\boldsymbol{\Omega}$ (or denoted by S)

- **Elements** in Ω are called **sample points** (or simply points) for the experiment and are denoted by $\boldsymbol{\omega}$

- An **event** is an arbitrary collection of outcomes. For example, it is a subset of a sample space.

- We denote events with capital letters A, B, C, etc.

- **Elementary (or simple) event:** An event A = $\{\omega\}$, consisting of a single point, i.e., single outcome $\omega \in \Omega$

- **Compound event:** An event that consists of more than one outcome.

- **Note:** When an experiment is performed, a particular event A is said to occur if the resulting experimental outcome is contained in A. In general, exactly one simple event will occur, but many compound events may occur simultaneously.

- Probabilities are assigned to events!

Example #1

Consider the experiment that involves tossing a thumbtack (push pin) and noting whether it landed point up or point down. List all possible outcomes.

Ω =

Example #1

Consider the experiment that involves tossing a thumbtack (push pin) and noting whether it landed

point up or point down. List all possible outcomes.

The possible outcomes of this experiment might be denoted U (for point up) and D (for point down).

The sample space for this experiment can be abbreviated as **Ω = {U, D}**, where the braces are used to enclose the elements of a set.

Example #2

For experiment, tossing a fair coin once, describe all possible outcomes of the sample space:

Heads (denoted by H) and Tail (denoted by T)

Ω =

Example #2

For experiment, tossing a fair coin once, the sample space consists of two possible outcome:

Heads (denoted by H) and Tail (denoted by T)

$\Omega = \{H, T\}$

Example #3

Roll a standard (6-sided) die. List all possible outcomes.

Ω =

Example #3

Roll a standard (6-sided) die. List all possible outcomes.

The die may land on any face with a number *i* on it, where *i* takes the values 1, 2, 3, 4, 5, 6.

Therefore, this experiment has sample space:

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Example #4

Roll a standard (6-sided) die.

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Let A be the event even number spots.

A =

Example #4

Roll a standard (6-sided) die.

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Let A be the event even number spots.

$A = \{2, 4, 6\}$

This is an example of compound event since the event consists of more than one outcome.

Independent Trials

- When thinking about what happens with combinations of outcomes, things are simplified if the individual trials are independent.

- This means that the outcome of one trial doesn’t influence or change the outcome of another.

For example:

  - Coin flips are independent.
  - Rolling pair of dice are independent.

Example #1

For experiment, tossing a fair coin twice (same as tossing two fair coins), describe all possible outcomes of the sample space:

Ω =

Example #1

For experiment, tossing a fair coin twice (same as tossing two fair coins), the sample space consists of four $(2^2 = 4)$ possible outcome:

Ω = {HH, HT, TH, TT}

Example #2

For experiment, tossing a fair coin three times (same as tossing three fair coins), describe all possible outcomes of the sample space:

Example #2

For experiment, tossing a fair coin three times (same as tossing three fair coins), the sample space consists of eight $(2^3 = 8)$ possible outcome:

Ω = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Example #3

For experiment, tossing a fair coin three times (same as tossing three fair coins), the sample space consists of eight $(2^3 = 8)$ possible outcome:

Ω = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

- Let A be the event “only head appears when we toss three coins”.

  Describe all possible outcomes of event A.

- Let B be the event “exactly one head appears when we toss three coins”.

  Describe all possible outcomes of event B.

Example #3

For experiment, tossing a fair coin three times (same as tossing three fair coins),
the sample space consists of eight possible outcome:

Ω **=** {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

- Let A be the event “only head appears when we toss three coins”.

A = {HHH}

This is an example of **simple event** since the event consists of **exactly one outcome**.

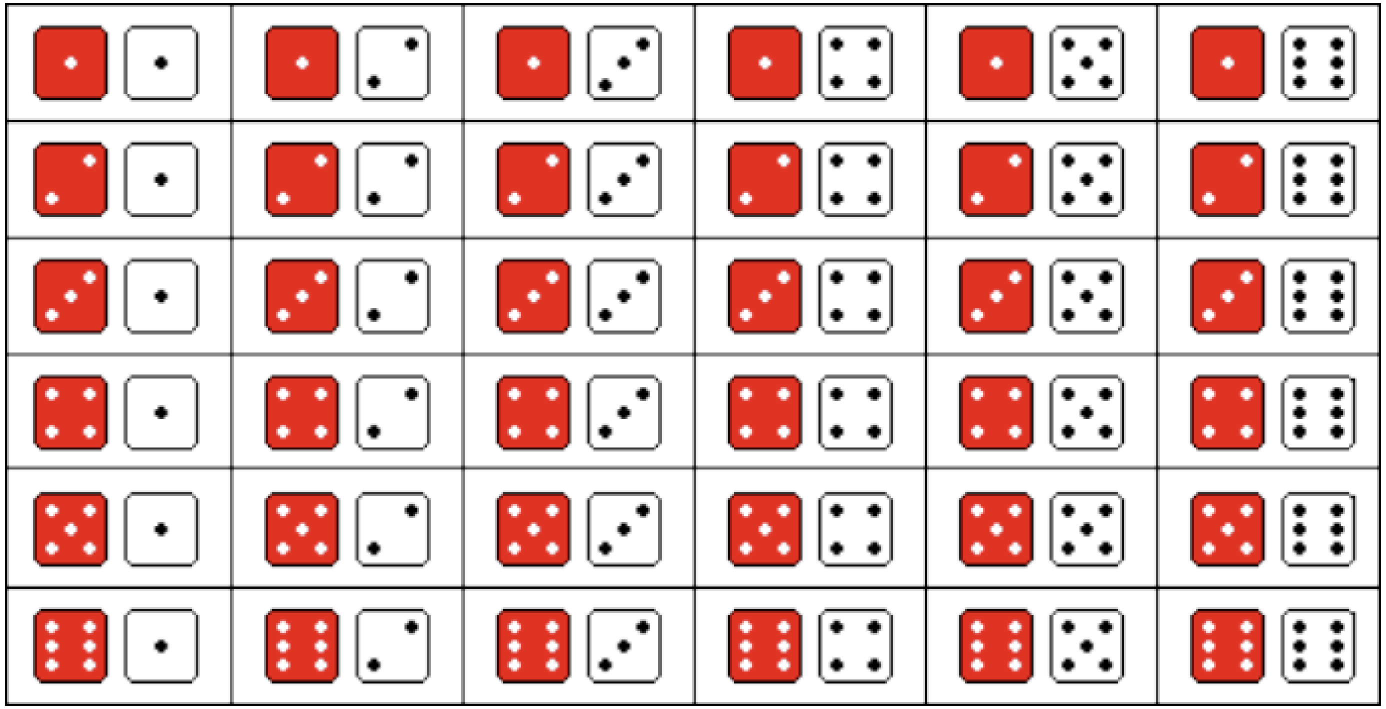
- Let B be the event “exactly one head appears when we toss three coins”

B = {HTT, THT, TTH}

This is an example of **compound event** since the event consists of **more than one outcome**.

Experiment: Rolling Two Six-sided Dice

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.



Experiment: Rolling Two Six-sided Dice

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.

The sample space Ω is given by

$$\left(\begin{array}{c} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{array}\right)$$

There are 36 possible outcomes. $6^2 = 36$

Example #5

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.

The sample space Ω is given by

$$\begin{pmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{pmatrix}$$

Find event E such that “the sum of the two dice is 6”.

Example #5

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.

The sample space Ω is given by

$$\begin{pmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{pmatrix}$$

Find event E such that “the sum of the two dice is 6”.

E = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}

Example #6

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.

The sample space Ω is given by

$$\begin{pmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{pmatrix}$$

Find event F such that “At least one of the dice is a 2.”

Example #6

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.

The sample space Ω is given by

$$\begin{pmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{pmatrix}$$

Find event F such that “At least one of the dice is a 2.”

F = {(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(1,2),(3,2),(4,2),(5,2),(6,2)}

Example #7

Consider an experiment in which each of three vehicles taking a particular highway exist turns left (L) or right (R) at the end of the off-ramp.

a. List all possible outcomes that comprise the sample space.

b. Describe the following events:

Let A be the event that exactly one of the three vehicles turns right.

Let B be the event that at most one of the vehicles turns right.

Let C be the event that all three vehicles turn in the same direction.

Example #7

Consider an experiment in which each of three vehicles taking a particular highway exi
or right (R) at the end of the off-ramp.

a. List all possible outcomes that comprise the sample space.

The sample space consists of eight $(2^3 = 8)$ possible outcome

Ω **=** {LLL, LLR, LRL, LRR, RLL, RLR, RRL, RRR}

Thus, there are eight simple events:

$E_1 = \{LLL\}$, $E_2 = \{LLR\}$, $E_3 = \{LRL\}$, $E_4 = \{LRR\}$, $E_5 = \{RLL\}$, $E_6 = \{RLR\}$, $E_7 = \{RRL\}$, $E_8 = \{RRR\}$

Example #7

Consider an experiment in which each of three vehicles taking a particular highway exist turns left (L) or right (R) at the end of the off-ramp.

b. Describe the following events:

Let A be the event that exactly one of the three vehicles turns right.

A = {RLL, LRL, LLR}

Let B be the event that at most one of the vehicles turns right.

B = {LLL, RLL, LRL, LLR}

Let C be the event that all three vehicles turn in the same direction.

C = {LLL, RRR}

Example #7

Consider an experiment in which each of three vehicles taking a particular highway exist turns left (L) or right (R) at the end of the off-ramp.

Let A be the event that exactly one of the three vehicles turns right. A = {RLL, LRL, LLR}
Let B be the event that at most one of the vehicles turns right. B = {LLL, RLL, LRL, LLR}
Let C be the event that all three vehicles turn in the same direction. C = {LLL, RRR}

Suppose that when the experiment is performed, the outcome is LLL.
Then the simple event $E_1 = \{LLL\}$ has occurred so also events B and C (but not A).

More about Sample Spaces

- **Discrete** sample space:
  - Finite sample space: Set with finite (countable) number of elements

  E.g., Toss a fair coin twice : Ω = {HH, HT, TH, TT}

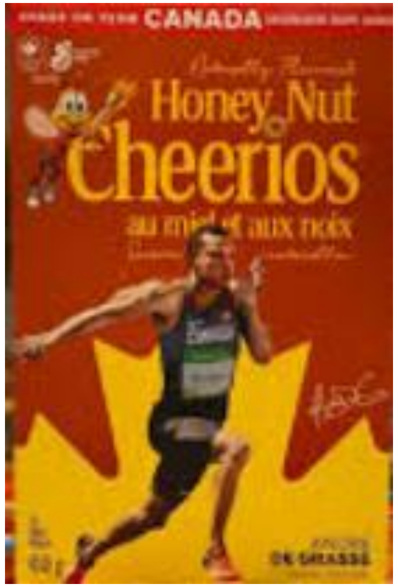
  - Countably infinite sample space: Set with countably many points

  E.g., # of emails in a day: $\Omega = \{x \mid x \in \mathbb{Z}\ ,\ x \geq 0\}$ (non-negative integers)

- **Continuous** sample space: Uncountable set
- E.g., Time it takes to solve a  Rubrik's Cube: $\Omega = \{x \mid x \in \mathbb{R},\ x > 0\}$ (positive real numbers)

[Fastest 3x3x3 Cube Solve EVER! - Guinness World Records (youtube.com)]

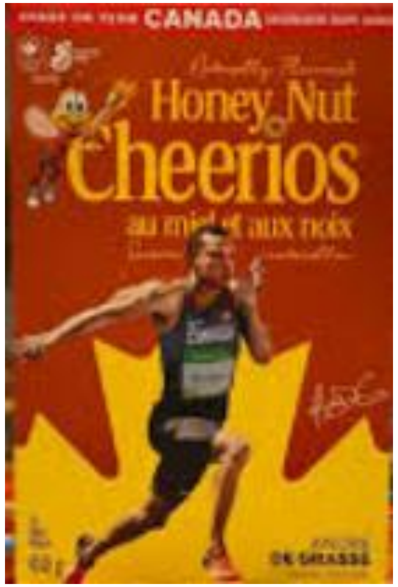
Example of Infinite Sample Space



Suppose that 10% of all cereal boxes of a certain bran contain a particular prize.
Denote a box as a success S if it has the prize and a failure F otherwise.
Now consider opening box after box until finding one that has the desired prize.
Although, it may not be very likely, a possible of outcome of this experiment is that the first 10 (or 100 or 1000 or …) are F's and the next one is an S.
That is for any positive integer n, we may have to examine n boxes before seeing the first S.

The sample space is $\Omega = \{S, FS, FFS, FFFS, \ldots\}$,
which contains an infinite number of possible outcomes.
In other words, there are infinite number of simple events.

Example of Infinite Sample Space



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That is for any positive integer n, we may have to examine n boxes before seeing the first S.

The sample space is Ω = {S, FS, FFS, FFFS, …}, which contains an infinite number of possible outcomes.
In other words, there are infinite number of simple events.
Some examples of compound events:

A = {S, FS, FFS} = the event that at most three cereal boxes are opened.
B = {S, FFS, FFFFS} = the event exactly one, three, or five boxes are opened.
C = {FS, FFFS, FFFFFS, …} = the event that an even number of boxes are opened.

Connections to Set Theory

- Since events are, by their very definition, sets, it will be useful for us to review some basic set theory.

- Sets are (unordered) collections of elements.

  E.g., set $A=\{a1,a2,a3\}$ has 3 elements

- Sets can be defined by listing or describing their elements.

  E.g., Let event A denote "roll 3 or less" $A = \{1, 2, 3\} = \{n \in \mathbb{N} : 1 \leq n \leq 3\}$

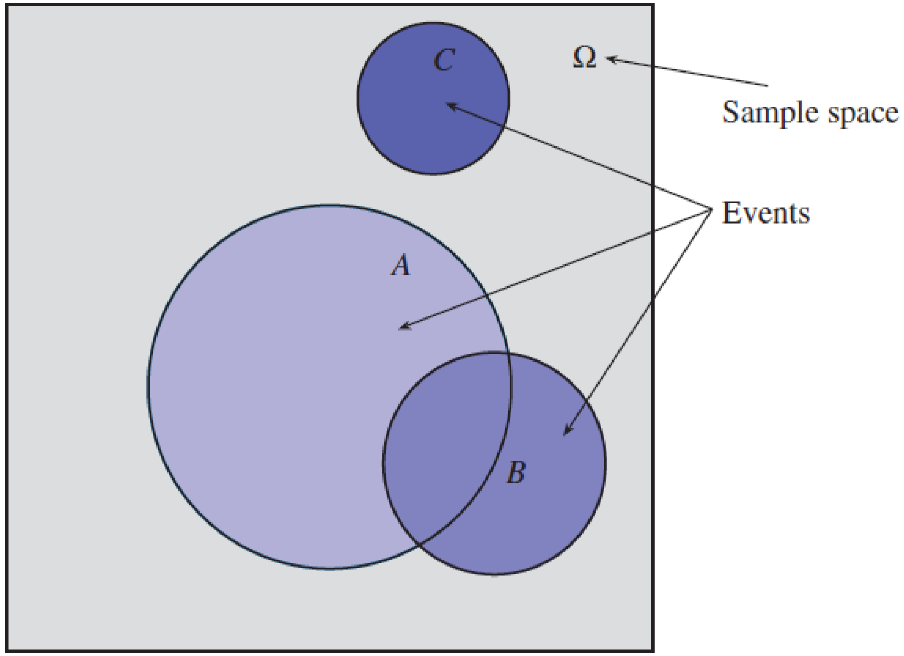
- Universal set Ω contains all possible elements.

- Null or empty set(Ø) contains no elements

- In probability:
  - Sample space ~ Universal set
  - Outcomes ~ Elements
  - Events ~ Sets

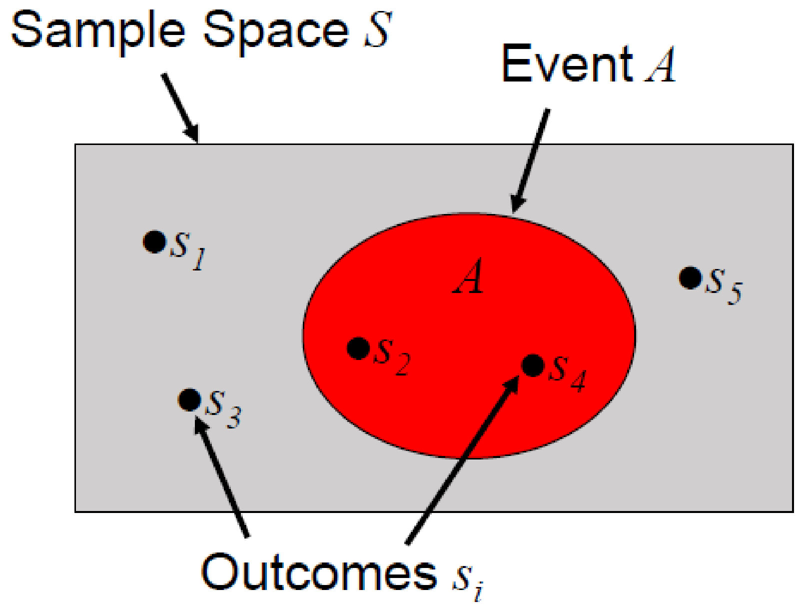
Venn Diagram

A Venn diagram is a useful tool for working with events and subsets. A rectangular box denotes the sample space Ω, and circles are used to denote events.

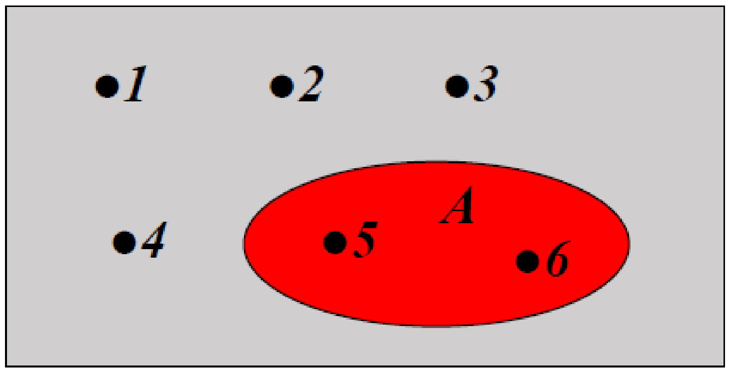


Venn Diagram

- Visual representation of sets
  - Sample space is rectangle
  - Outcomes are point in rectangle
  - Events are collections of points, i.e., subsets of rectangle



- E.g., Rolling 6-sided die

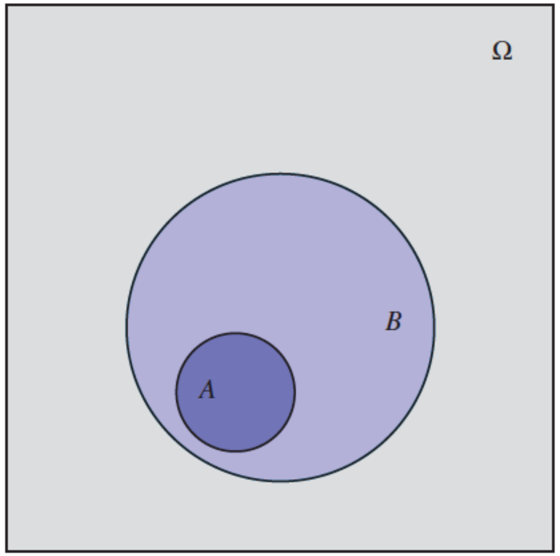


Operations Between Events

- For the study of events associated with a certain experiment, and the assignment of probabilities to these events later on, it is essential to consider various relations among the events of a sample space, as well as operations among them.
- Each event is mathematically represented by a set (a subset of Ω); thus, it is no surprise that the relations and operations we consider are borrowed from mathematical set theory.

Related Events

- Assume that A and B are events on the same sample space Ω.
- If every element (sample point) of A is also a member of B, then we use the standard notation for subsets and write $A \subseteq B$ (A is a subset of B).
- In words, this means that whenever A occurs, B occurs too.



Example

For instance, in a single throw of a six-sided die consider the events:

A: the outcome of the die is 4

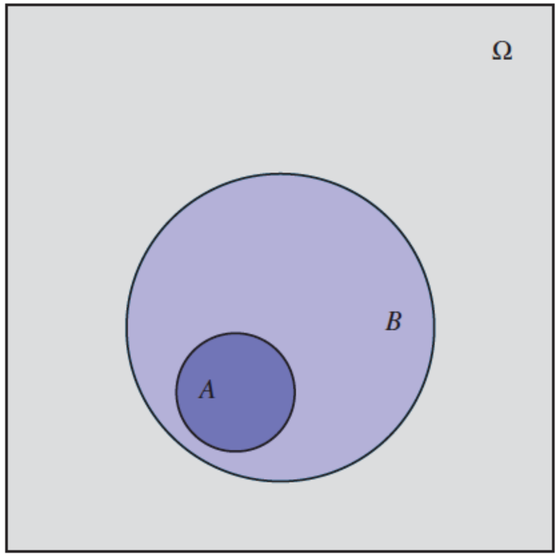
B: the outcome of the die is an even integer

Then, expressing *A* and *B* as sets, we have

$A = \{4\}$

$B = \{2, 4, 6\}$

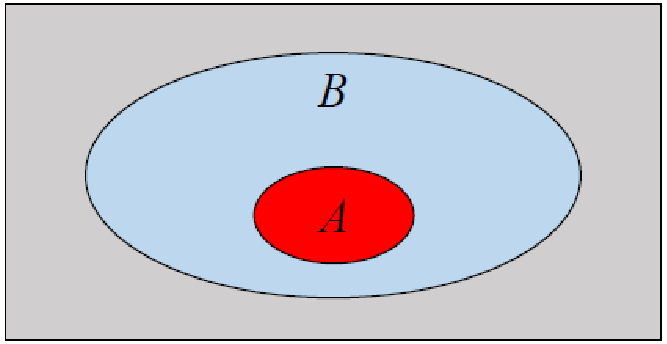
and it is clear that $A \subseteq B$.



On the other hand, if we know that the outcome is 4 (*A* has occurred), then necessarily the outcome is even, so that *B* occurs, too.

Equivalent Events

- If $A \subseteq B$ and $B \subseteq A$, then obviously A occurs iff (if and only if) B occurs.
- Definition: Two events A and B, defined on a sample space Ω, are called equivalent if when A appears, then B appears, and vice versa. In this case, we shall write $A = B$.



Example

Consider the experiment of throwing a six-sided die. Consider the following events:
$A = \{4\}$
B: the outcome of the die is 5

Suppose now that Nick, who likes gambling, throws a die and wins a bet if the outcome of the die is *either* 4 *or* 5.
Then, the event C: Nick wins the bet occurs if and only if at least one of the events A and B occur.

The event C is the same as the event “at least one of A and B occur”
(these two events are equivalent), and so, using set notation, we can write $C = \{4, 5\}$.

We thus see that, expressed as a set, C coincides with the union of the two sets A and B.

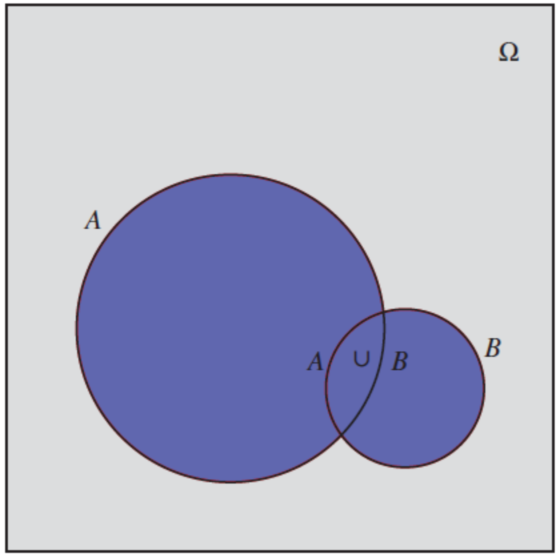
Combined Events

- Events can be combined together to create new events using the connectives “or” “and” and “not”
- These correspond to the set operations union, intersection, and complement.

Union of Events

For sets A, $B \subseteq \Omega$, the ***union A*** $\cup$ ***B*** is the **set of all elements of Ω that are in either *A* or *B* or both**.

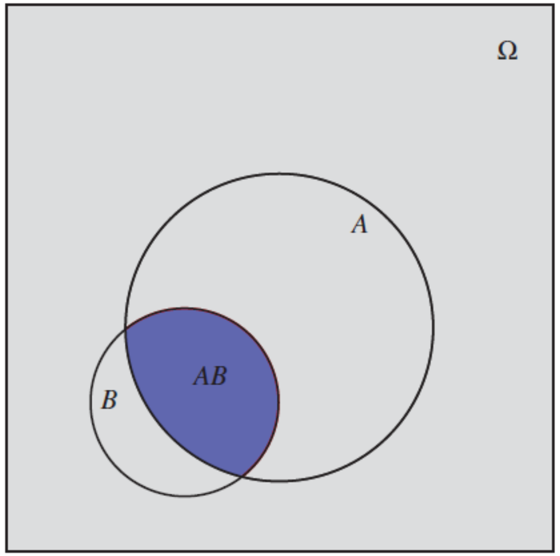
‘Event $A \cup B$ occurs’ means ‘either A or B occurs or both occur’.



Intersection of Events

The ***intersection AB*** is the **set of all elements of Ω that are in both *A* and *B***. (Another common notation for the intersection of two events is ***A* ∩ *B***.)

‘Event $A \cap B$ occurs’ means ‘both A and B occur’.



Example

Roll a six-faced dice and observe the score on the uppermost face.

Here S = (1, 2, 3, 4, 5, 6), which is composed of six elementary events.

Suppose A is the event that *an even number is observed*.

This event consists of the set of outcomes 2, 4 and 6, i.e. $A = \{x : x \text{ is even}\} = \{2, 4, 6\}$

Suppose B is the event that *a number larger than 3 is observed*.

This event consists of the outcomes 4, 5 and 6, i.e. $B = \{x : x \text{ is greater than } 3\} = \{4, 5, 6\}$

The event $A \cup B = \{x : x \text{ is an even number or a number larger than } 3\} = \{2, 4, 5, 6\}$.

The event $AB = \{x : x \text{ is an even number and a number larger than } 3\} = \{4, 6\}$.

Union/Intersection Properties

- Commutative: $A \cap B = B \cap A, \quad A \cup B = B \cup A$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C,$

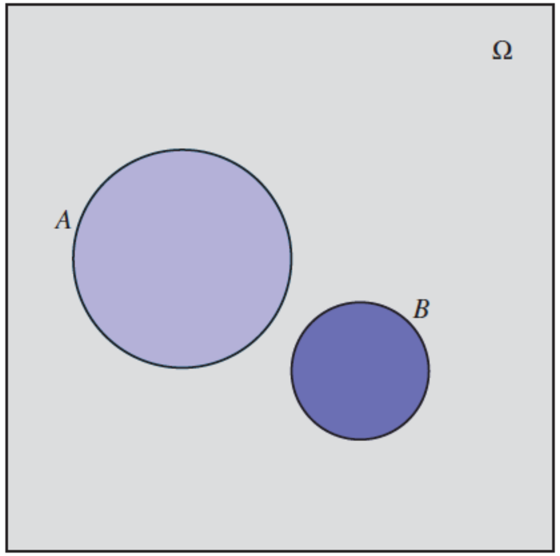
  $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$

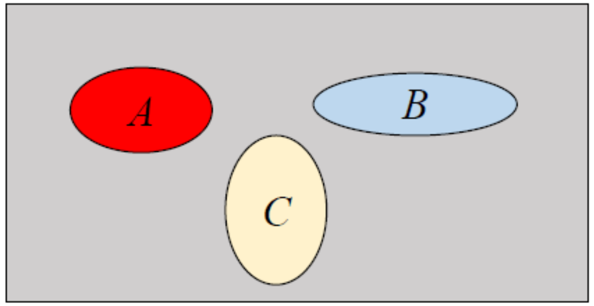
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - Apply for arbitrary collections of sets (not just pairs)
- Note: $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$ and similarly for $\cap$

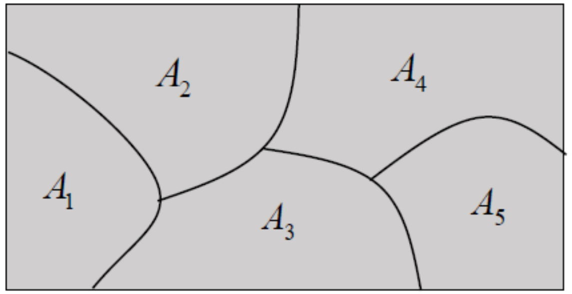
Disjoint Events

Two or more events are ***disjoint*** or ***mutually exclusive***, if they have no elements in common.

Mathematicians write this compatibility as $A \cap B = \emptyset$, where $\emptyset$ denotes the event consisting of no outcomes whatsoever (the "null" or "empty" event).







Two or more events form a ***partition*** if they are disjoint *and* their union is the sample space.

Example # 1

A small city has three automobile dealerships: a GM dealer selling Chevrolets and Buicks, a Ford dealership selling Fords and Lincolns, and a Toyota dealer with Toyota and Lexus vehicles. If an experiment consist of observing the brand of the next car sold, the event A = {Chevrolet, Buick} and

B = {Ford, Lincoln} are mutually exclusive because the next car sold cannot be both a GM product and a Ford product.

Example #2

Roll a six-faced dice and observe the score on the uppermost face.

Here S = (1, 2, 3, 4, 5, 6), which is composed of six elementary events.

Suppose A is the event that *an even number is observed*.

This event consists of the set of outcomes 2, 4 and 6, i.e. $A = \{x : x \text{ is even}\} = \{2, 4, 6\}$

Suppose B is the event that *an odd number is observed*.

This event consists of the outcomes 1, 3 and 5, i.e. $B = \{x : x \text{ is odd}\} = \{1, 3, 5\}$

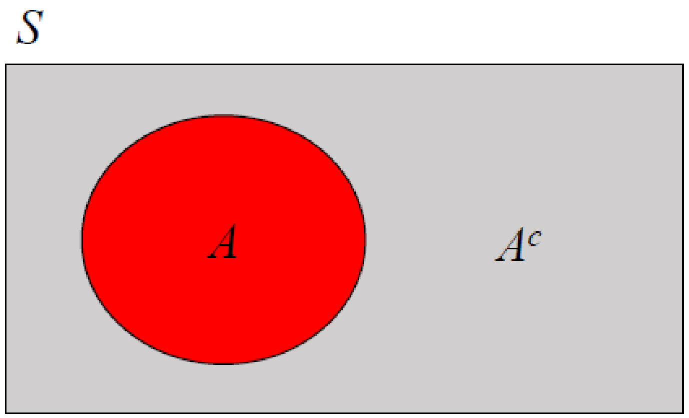
The event $A \cup B = \{x : x \text{ is an even number or an odd number}\} = \{1, 2, 3, 4, 5, 6\}$

$A \cup B = S$

The event $AB = \{x : x \text{ is an even number and an odd number}\} = \emptyset$

Complement of an Event

- Let *A* be an event on a sample space.
- The **complement**, or the **complementary event of *A***, is the **event which occurs if and only if *A* does not occur**.
- An equivalent definition is that the complement of *A* contains exactly those elements of the sample space (the set of all elements of Ω) which are not elements of *A*.
- For the complement of an event *A*, there are at least three different symbols which are used quite commonly, and they are A^c or $\bar{A}$ or A'



Also:

$A \cap A^c = \varnothing$

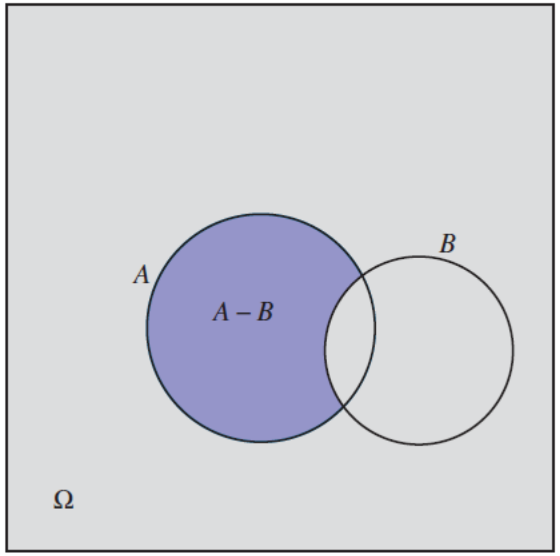
$A \cup A^c = S$

$S^c = \varnothing$

Difference Between Two Events

The **difference of an event *B* from an event *A*** refers to the **event which occurs when *A* occurs but *B* does not.**

The symbol we use for the difference of B from A is $A - B$ (or AB' or A\B).



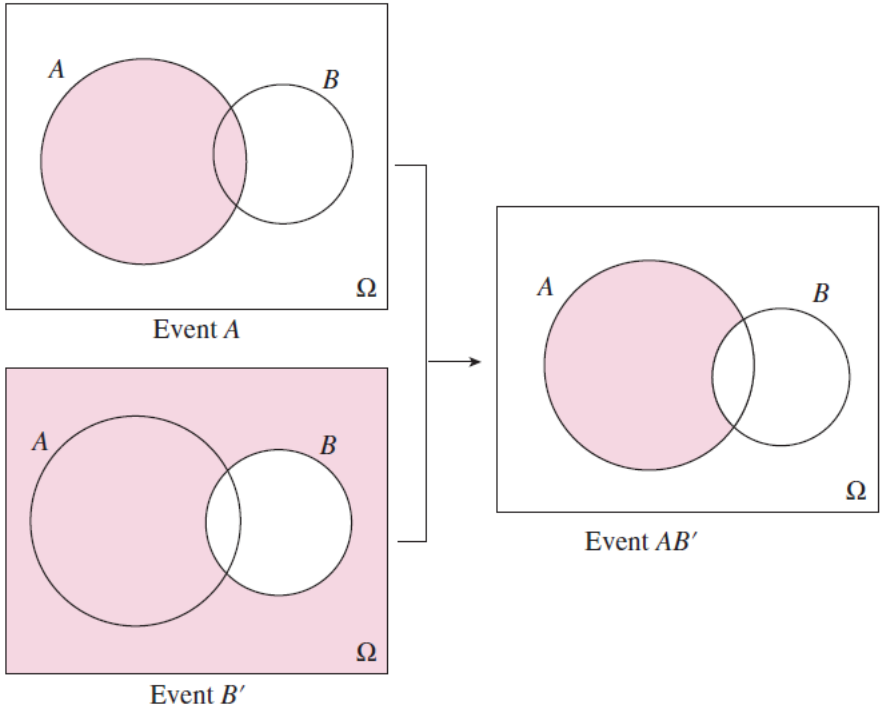
For example:

A = All STA107 students majoring in Art and Art History

B = All STA107 students majoring in Computer Science

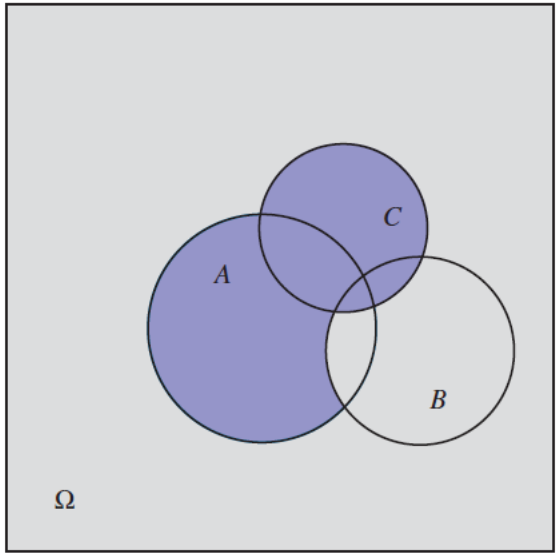
A – B = All STA107 students **only** majoring in Art and Art History

Difference Between Two Events



Example with Three Events

Let A, B,C be three events on a sample space and consider the event C: either A occurs but not B, or else C occurs.



$(A - B) \cup C$

Set Properties

The following properties hold true for the operations among events:

1. $A \cup A = A$, $AA = A$
2. $A \cup \emptyset = A$, $A\emptyset = \emptyset$
3. $A \cup \Omega = \Omega$, $A\Omega = A$
4. $A \cup B = B \cup A$, $AB = BA$
5. $A \cup (B \cup C) = (A \cup B) \cup C$, $A(BC) = (AB)C$
6. $A \cup (BC) = (A \cup B)(A \cup C)$, $A(B \cup C) = (AB) \cup (AC)$
7. $A \cup A' = \Omega$, $AA' = \emptyset$
8. $(A')' = A$
9. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
10. If $A \subseteq B$, then $B' \subseteq A'$ and vice versa.
11. If $A \subseteq B$, then $AB = A$ and $A \cup B = B$.

De Morgan's Law

Suppose A and B are events on the same sample space Ω. Then, the following identities hold:

$$(A \cup B)^c = A^c \cap B^c$$

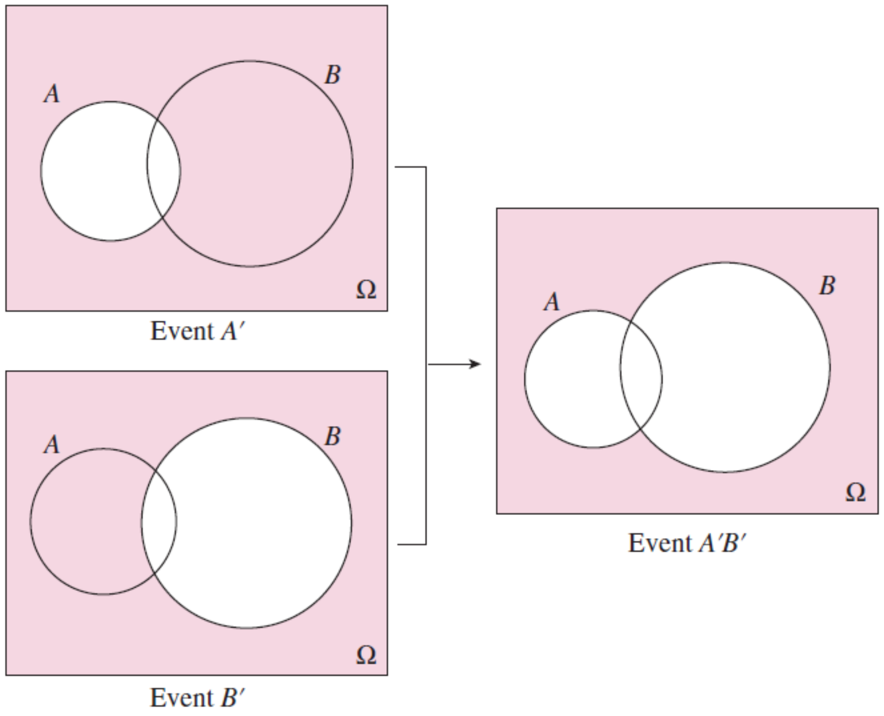
The complement of a union is

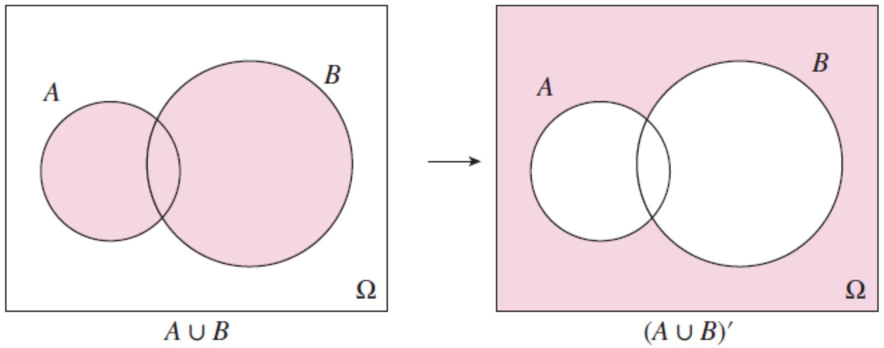
an intersection of complements

$$(A \cap B)^c = A^c \cup B^c$$

The complement of an intersection is

a union of complement





These identities are known as De Morgan identities or De Morgan formulas.

De Morgan’s Law

Generally: $\left(\bigcap_{i=1}^{n} A_i\right)^c = A_1^c \cup A_2^c \cup \cdots \cup A_n^c$

$$\left(\bigcup_{i=1}^{n} A_i\right)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

TABLE 1.2. Events and sets.

Description	Set notation
Either A or B or both occur	$A \cup B$
A and B	AB
Not A	A^c
A implies B; A is a subset of B	$A \subseteq B$
A but not B	AB^c
Neither A nor B	$A^c B^c$
At least one of the two events occurs	$A \cup B$
At most one of the two events occurs	$(AB)^c = A^c \cup B^c$

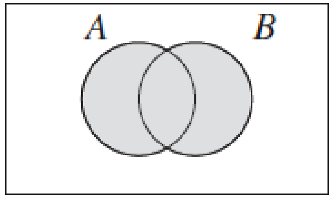
Note that in probability word problems, descriptive phrases are typically used rather than set notation.

Source:

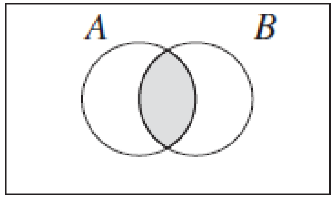
Probability with Applications AND R. Second Ed.(2021). Amy S. Wagaman and Robert P. Dobrow

Chapter 1, page 8.

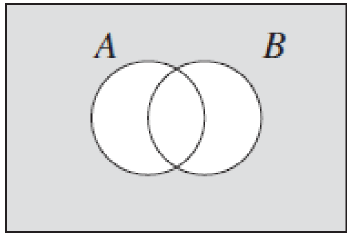
Venn diagrams for the most common combined events obtained from two events *A* and *B*.



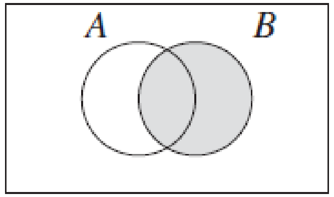
A or *B* occur



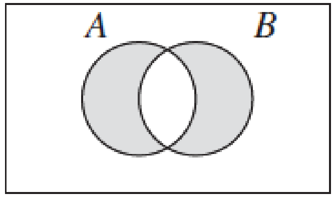
A and *B* occur



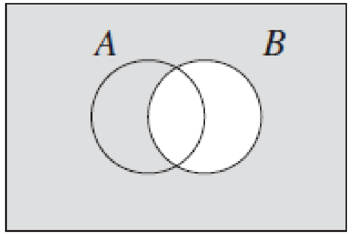
Neither *A* nor *B* occurs



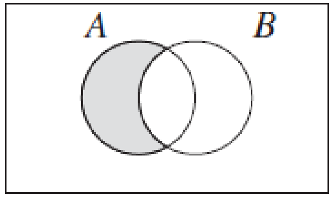
B occurs



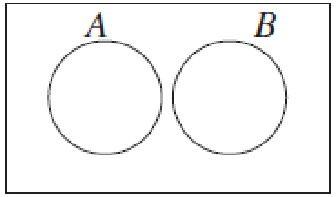
Exactly one of *A* or *B* occurs



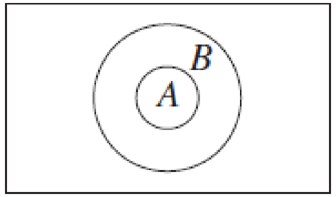
B does not occur



Only *A* occurs



A and *B* mutually exclusive



A implies *B*

Experiment: Rolling Two Six-sided Dice

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.

The sample space Ω is given by

$$\begin{pmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{pmatrix}$$

There are 36 possible outcomes.
6^2 = 36

Let event E be "the sum of the two dice is 6". E = {(1, 5),(2, 4),(3, 3),(4, 2),(5, 1)}

Let event F be "At least one of the dice is a 2." F = {(2, 1),(2, 2),(2, 3),(2, 4),(2, 5),(2, 6),(1, 2),(3, 2),(4, 2),(5, 2),(6, 2)}

Find the following events:

a. $E \cap F$
b. $E \cup F$
c. E'

Experiment: Rolling Two Six-sided Dice

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.

Let event E be “the sum of the two dice is 6”. E = {(1, 5),(2, 4),(3, 3),(4, 2),(5, 1)}

Let event F be “At least one of the dice is a 2.” F = {(2, 1),(2, 2),(2, 3),(2, 4),(2, 5),(2,6),(1, 2),(3, 2),(4, 2),(5, 2),(6, 2)}

$E \cap F$ = {(2, 4),(4, 2)}

$E \cup F$ = {(1, 5),(2, 4),(3, 3),(4, 2),(5, 1),(2, 1),(2, 2),(2, 3),(2, 5),(2, 6),(1, 2),(3, 2),(5, 2),(6, 2)}

Experiment: Rolling Two Six-sided Dice

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.

The sample space Ω is given by

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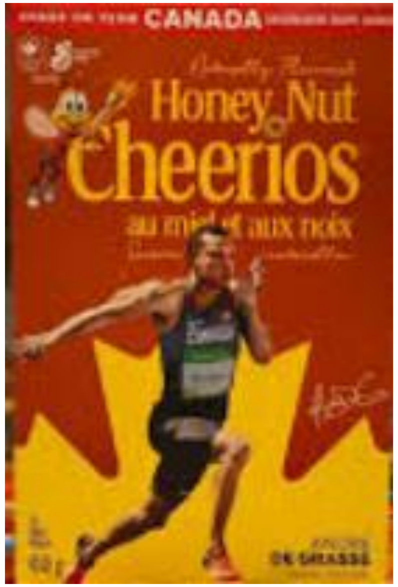
There are 36 possible outcomes.
$6^2 = 36$

Let event E be “the sum of the two dice is 6”. E = {(1, 5),(2, 4),(3, 3),(4, 2),(5, 1)}

$E' =$

$$\begin{pmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{pmatrix}$$

Experiment: Cereal Box Prize



In the cereal box prize experiment, recall events A, B, and C as follows

A = {S, FS, FFS} = the event that at most three cereal boxes are opened.

B = {S, FFS, FFFFS} = the event exactly one, three, or five boxes are opened.

C = {FS, FFFS, FFFFFS, …} = the event that an even number of boxes are opened.

Find the following events:

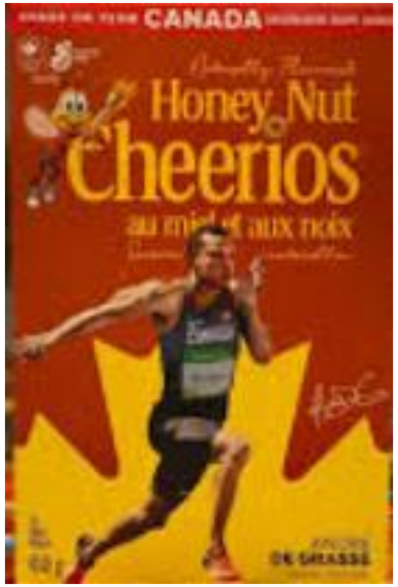
a. $A \cap B$

b. $A \cup B$

c. A'

d. C'

Experiment: Cereal Box Prize



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B = {S, FFS, FFFFS} = the event exactly one, three, or five boxes are opened.

C = {FS, FFFS, FFFFFS, …} = the event that an even number of boxes are opened.

$A \cap B$ = {S, FFS}

$A \cup B$ = {S, FS, FFS, FFFFS}

A' = {FFFS, FFFFS, FFFFFS …}

C' = {S, FFS, FFFFS, ….} = the event that an odd number of boxes are opened.

Three Events

For any three events A, B, and C, the event $A \cap B \cap C$ is set of outcomes contained in all three events, whereas $A \cup B \cup C$ is a set of outcomes contained in at least one of the three events.

Note:

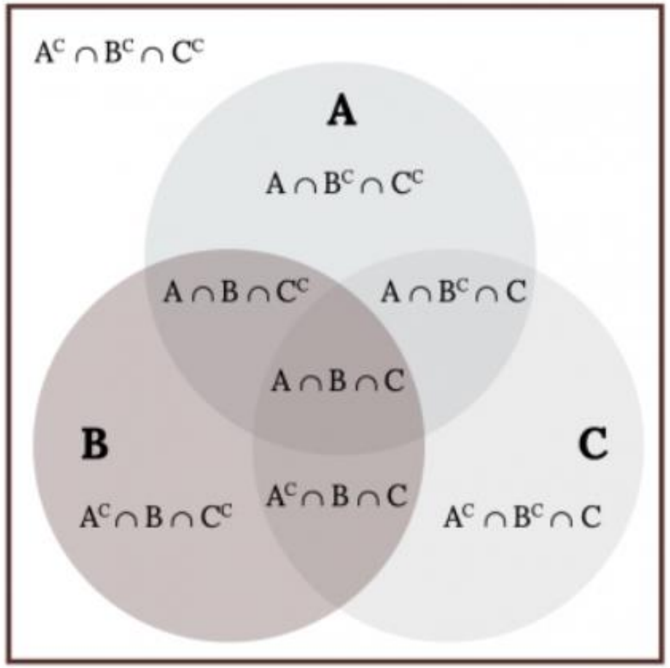
A collection of several events is said to be mutually exclusive

(or pairwise disjoint) if no two events have any outcomes in common.

De Morgan's Law extends similarly; for example,

$(A \cup B \cup C)' = A' \cap B' \cap C'$ and

$(A \cap B \cap C)' = A' \cup B \cup C'$



In words, the complement of an at least one event occurs is the intersections of the complements of the events occur, and the complement of all events occur is at least one event does not occur.

Probability Axioms

- ***Probability function/measure*** is function that takes events and assigns probability values to them
  - Denoted by $P(\cdot)$, where $P(A)$ is probability value of specific event A

- ***Probability Axioms***:
  - Assume that Ω is a sample space for a chance experiment.
  - Assume also that to each event A in Ω, there corresponds a real number, $P(A)$.
  - If the function $P(\cdot)$ satisfies the following three axioms, then P will be called a probability on the sample space Ω, while the number $P(A)$ will be referred to as the probability of the event A:

    1. $P(A) \geq 0$ for any event A defined in the sample space Ω;

    2. $P(\Omega) = 1$;

    3. If $A1$, $A2$,… is a sequence of events in Ω which are pairwise disjoint

    (that is, $A_i A_j = \emptyset$ for any $i \neq j$), then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

- ***Probability model*** consist of triple $(\underbrace{S,}_{\text{sample space}}\ \underbrace{\{A,B,\ldots\}}_{\text{set of events}},\ \underbrace{P}_{\text{probability function}})$

More About the Probability Axioms

- Axiom 1 reflects the intuitive notion that the chance of *A* occurring should be nonnegative.
- The sample space is by definition the event that must occur when the experiment is performed (contains all possible outcomes), so Axiom 2 says that the maximum possible probability of 1 is assigned to Ω.
- The third axiom formalizes the idea that if we wish the probability that at least one of a number of events will occur and no two of the events can occur simultaneously, then the chance of at least one occurring is the sum of the chances of the individual events.
- Wonder why the third axiom contains no reference to a *finite* collection of disjoint events?
- It is because the corresponding property for a finite collection can be derived from our three axioms.
- Similarly, the statement $0 \leq P(A) \leq 1$ (i.e. all probabilities are between 0 and 1 inclusive) can be proved using the axioms.
- We want our axiom list to be as short as possible and not contain any property that can be derived from others on the list.

Equally Likely Outcome

- The simplest probability model for a finite sample space is that all outcomes are equally likely.
- If Ω has k elements, then the probability of each outcome is $1/k$, as probabilities sum to 1.
- That is, $P(\omega) = 1/k$, $\forall\ \omega\ in\ \Omega$
- Computing probabilities for equally likely outcomes takes a fairly simple form.
- Suppose A is an event with s elements, with $s \leq k$.
- As P(A) is the sum of all the outcomes contained in A,

$$P(A) = \sum_{\omega \epsilon A} P(\omega) = \sum_{\omega \epsilon A} \frac{1}{k} = \frac{s}{k} = \frac{Number\ of\ elements\ of\ A}{Number\ of\ elements\ of\ \Omega} = \frac{|A|}{|\Omega|}$$

In other words, probability with equally likely outcomes reduces to counting elements in A and Ω.

Example #1

A palindrome is a word that reads the same forward and backward.

Examples include mom, civic, rotator, and wow.

Pick a three-letter “word” at random from D, O, or, G. What is the probability that the resulting word is a palindrome? (Words in this context do not need to be real words in English, e.g., OGO is a palindrome.)

Example #1

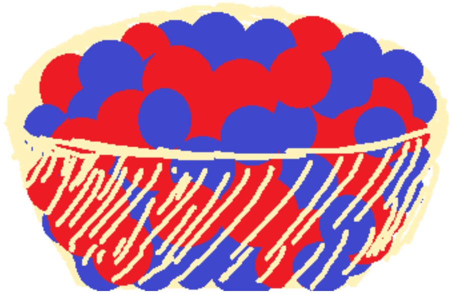
Pick a three-letter “word” at random from D, O, or, G. What is the probability that the resulting word is a palindrome? (Words in this context do not need to be real words in English, e.g., OGO is a palindrome.)

There are 3^3 = 27 possible words (three possibilities for each of the three letters).

List and count the palindromes: DDD, OOO, GGG, DOD, DGD, ODO, OGO, GDG, and GOG.

The probability of getting a palindrome is: $\frac{9}{27} = \frac{1}{3}$

Example #2



A bowl has r red balls and b blue balls. A ball is drawn randomly from the bowl.

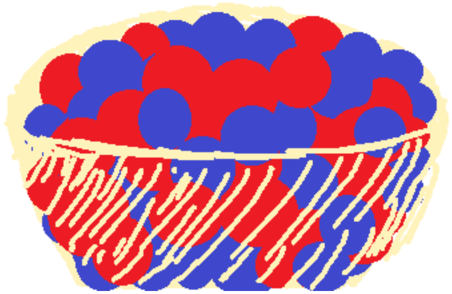
What is the probability of selecting a red ball?

The sample space consists of $r + b$ balls.

The event A = {Red balls} has r elements.

Therefore, $\mathrm{P(A)} = \frac{r}{(r+b)}$

Example #2



A bowl has r red balls and b blue balls. A ball is drawn randomly from the bowl.

What is the probability of selecting a red ball?

The sample space consists of $r + b$ balls.

The event A = {Red balls} has r elements.

Therefore, $\mathrm{P(A)} = \frac{r}{(r+b)}$

More about Equally Likely Outcome

- A model for equally likely outcomes assumes a finite sample space.
- Interestingly, it is impossible to have a probability model of equally likely outcomes on an infinite sample space. Why?
- Suppose $\Omega = \{\omega_1, \omega_2, \ldots\}$ and $P(\omega_i) = c \ \forall \ i,$ where c is a nonzero constant. Then summing the probabilities gives

$$\sum_{i=1}^{\infty} P(\omega_i) = \sum_{i=1}^{\infty} c = \infty \neq 1$$

Theoretical Probability

❖ Sometimes can argue (in a mathematical model not from observation) what probabilities should be.

For example, roll a die:

- in theory it is a perfect cube, so each of the 6 faces equally likely to occur.
  - e.g., Probability(6 occur) = 1/6.

❖ More generally, any time you have equally likely outcomes:

Probability of event A is:

$$P(A) = \frac{Number\ of\ Outocmes\ in\ A}{Number\ of\ Outcomes\ in\ S} = \frac{|A|}{|S|}$$

In the above example:

- All possible outcomes are S = {1, 2, 3, 4, 5, 6}
- Let A be the event the outcome 6 is occurred when rolling a die: A = {6}
- P(A) = 1/6

Experiment: Rolling Two Six-sided Dice

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.

The sample space S is given by

$$\begin{pmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{pmatrix}$$

There are 36 possible outcomes. $6^2 = 36$

By the symmetry of the dice, we expect all 36 possible outcomes to be equally likely.

So, the probability of each outcome is 1/36.

Example #1

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.

The sample space S is given by

$$\begin{pmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{pmatrix}$$

Let E be the event that “the sum of the two dice is 6”.
E = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}
Find the probability that “the sum of the two dice is 6”.

The number of elements in event E is: n(E) = 5
The number of elements in the sample space S: n(S) = 36
P(E) = $\frac{n(E)}{n(S)} = \frac{5}{36}$ = 0.139

Example #2

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.

The sample space S is given by

$$\begin{pmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{pmatrix}$$

Let F be the event such that "At least one of the dice is a 2."
F = {(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(1,2),(3,2),(4,2),(5,2),(6,2)}
Find the probability that "at least one of the dice is a 2."

The number of elements in event F is: n(F) = 11
The number of elements in the sample space S: n(S) = 36
P(F) = $\frac{n(F)}{n(S)} = \frac{11}{36}$ = 0.306

The Law of Large Numbers & Empirical Probability

The Law of Large Numbers (LLN) says that as we repeat a random process over and over,
the proportion of times (fraction of times) that an event occurs does settle down to one number.

- We call this number probability of the event.
- This LLN is based on two assumptions:

1. The random phenomenon we are studying must not change (that is, the outcomes must have the same probability for each trial).
2. The events must be independent (that is, the outcome of one trial does not affect the outcomes of the others).

- So, LLN says that as the number of independent trials increases, the long run relative frequency of repeated events get closer and closer to a single value.

- Because this definition is based on repeatedly observing the event's outcome; this definition of probability is often called empirical probability.

❖ Empirical Probability: For any Event A, P(A) = $\frac{\#\ times\ A\ occurs}{total\ \#\ of\ trails}$ in the long run

Simulation

- The goal of probability and statistics is to understand the real world.
- Statistical experiments in the real world are usually slow and often expensive.
- Instead of running real world experiments, it is easier to model these experiments and then use a computer to imitate the results.
- This process goes by several names, such as *stochastic simulation*, *Monte Carlo simulation or probability simulation.*
- Since this is the only type of simulation we discuss in this course, we simply call it *simulation*.
- The goal of simulation is usually to estimate the probability of an event.
- The foundational mathematical theorem which justifies using simulations to estimate probabilities as described in this lecture and throughout the course is the Law of Large Numbers.

Simulation with sample in R

For an experiment with a finite sample space $S=\{x_1,x_2,\ldots,x_n\}$, the R command **sample()** can simulate one or many trials of the experiment. Essentially, sample treats S as a bag of outcomes, reaches into the bag, and picks one.

The syntax of sample is **sample(x, size, replace = FALSE, prob = NULL)**
where the parameters are:

X
The vector of elements from which you are sampling.

Size
The number of samples you wish to take.

Replace
Whether you are sampling with replacement or not. Sampling without replacement means that sample will not pick the same value twice, and this is the default behavior. Pass replace = TRUE to sample if you wish to sample with replacement.

prob
A vector of probabilities or weights associated with x. It should be a vector of nonnegative numbers of the same length as x. If the sum of prob is not 1, it will be normalized. If this value is not provided, then each element of x is considered to be equally likely.

Examples of Simulation with sample function in R

To get a random number from 1 to 10, with size 1:

```
> sample(x = 1:10, size = 1)
[1] 5
> sample(x = 1:10, size = 1)
[1] 7
> sample(x = 1:10, size = 1)
[1] 7
> sample(x = 1:10, size = 1)
[1] 6
```

Examples of Simulation with sample function in R

To get a random number from 1 to 10, with different sizes:

```
> # Use set.seed() before sampling.
> # The set.seed() function in R will create reproducible results.
> set.seed(107)
> sample(x = 1:10, size = 1)
[1] 2
> sample(x = 1:10, size = 5)
[1]  3  6  2 10  5
> sample(x = 1:10, size = 15)
Error in sample.int(length(x), size, replace, prob) :
  cannot take a sample larger than the population when 'replace = FALSE'
> sample(x = 1:10, size = 5, replace = TRUE)
[1] 3 3 6 8 6
```

Using Simulation to Estimate Probabilities
Die Experiment

```
> # The set.seed() function in R will create reproducible results.
> set.seed(107)
> die_face_values <- c(1, 2, 3, 4, 5, 6)
> die_face_values_probs <- c(1/6, 1/6, 1/6, 1/6, 1/6, 1/6)
> simulate_data <- sample(x = die_face_values, size = 30, prob = die_face_values_probs, replace = TRUE)
> library(mosaic)
> tally(simulate_data, margin = TRUE)
X
    1     2     3     4     5     6 Total
    2     6     5     4     7     6    30
> tally(simulate_data, format = "proportion", margin = TRUE)
X
         1          2          3          4          5          6      Total
0.06666667 0.20000000 0.16666667 0.13333333 0.23333333 0.20000000 1.00000000
```

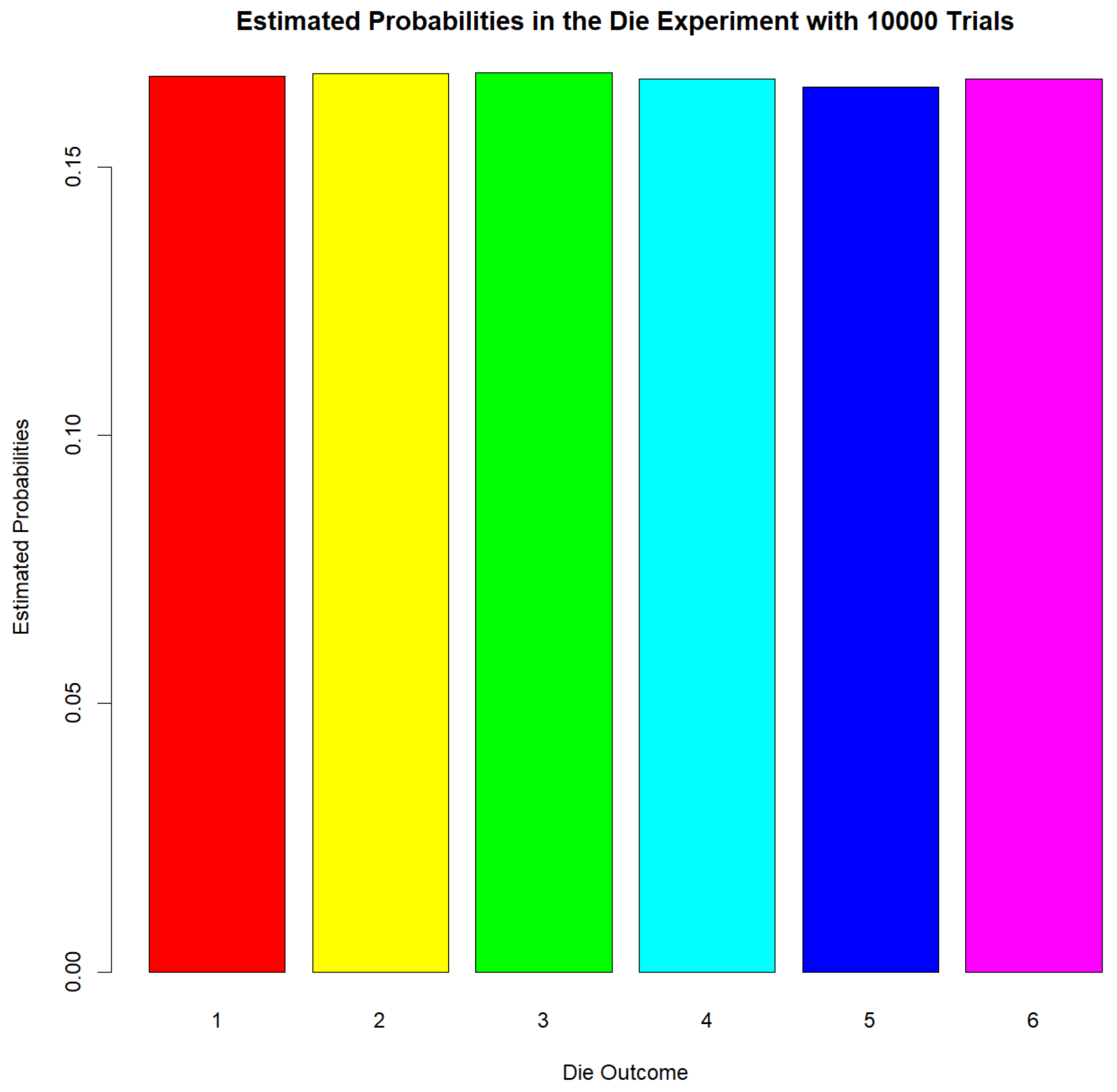
Using Simulation to Estimate Probabilities
Die Experiment

```
> # Increase the size of the sample
> sim_data <- sample(x = die_face_values, size = 10000, prob = die_face_values_probs, replace = TRUE)
> tally(sim_data, margin = TRUE)
X
    1     2     3     4     5     6 Total
 1669  1675  1677  1665  1649  1665 10000
> tally(sim_data, format = "proportion", margin = TRUE)
X
     1      2      3      4      5      6  Total
0.1669 0.1675 0.1677 0.1665 0.1649 0.1665 1.0000
```

Observe that a large sample reproduces the original probabilities with reasonable accuracy.

Visualizing the Estimate Probabilities in the Die Experiment with 10000 Trials

```
> # Visualize the results of the random experiment by constructing a bar plot
> barplot(tally(sim_data, format = "proportion"), col = rainbow(6),
+         main = "Estimated Probabilities in the Die Experiment with 10000 Trials",
+         xlab = "Die Outcome",
+         ylab = "Estimated Probabilities")
```



Using Simulation to Estimate Probabilities

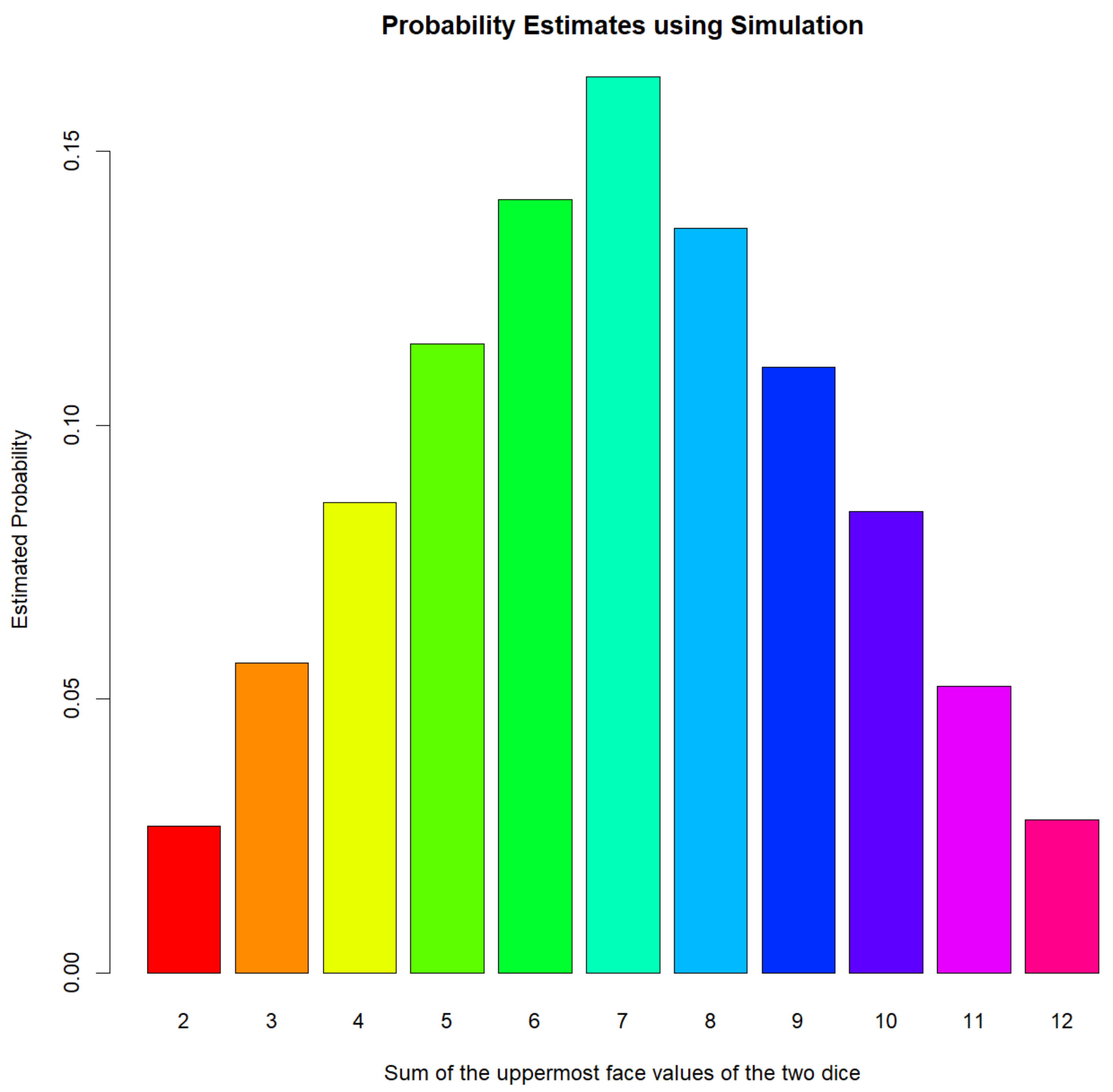
The goal of simulation is usually to estimate the probability of an event. This is a three-step process:

- Simulate the experiment many times to produce a vector of outcomes.
- Test if the outcomes are in the event to produce a vector of TRUE/FALSE.
- Compute the mean of the TRUE/FALSE vector to compute the probability estimate.

Using Simulation to Estimate Probabilities
Pair of Dice Experiment

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are added. Simulate this experiment by performing 10,000 rolls of each die with sample and then adding the two dice:

```
> # The set.seed() function in R will create reproducible results.
> set.seed(107)
> die1 <- sample(x = 1:6, size = 10000, replace = TRUE)
> die2 <- sample(x = 1:6, size = 10000, replace = TRUE)
> sumDice <- die1 + die2
> # Let's take a look at the simulated data:
> head(die1)
[1] 3 2 3 6 2 3
> head(die2)
[1] 2 3 2 5 2 3
> head(sumDice)
[1]  5  5  5 11  4  6
> # Let E be the event "the sum of the dice is 6"
> eventE <- sumDice == 6
> head(eventE)
[1] FALSE FALSE FALSE FALSE FALSE  TRUE
> # Display the results of the simulated data.
> library(mosaic)
> tally(sumDice, format = "proportion")
X
     2      3      4      5      6      7      8      9     10     11     12 
0.0268 0.0565 0.0858 0.1148 0.1412 0.1637 0.1360 0.1106 0.0843 0.0523 0.0280 
> barplot(tally(sumDice, format = "proportion"),
+         xlab = "Sum of the two dice",
+         ylab = "Estimated Probability",
+         main = "Probability Estimates using Simulation",
+         col = rainbow(11))
```



Probability Estimates using Simulation
Estimated Probability
0.00
0.05
0.10
0.15
2
3
4
5
6
7
8
9
10
11
12
Sum of the uppermost face values of the two dice

Using Simulation to Estimate Probabilities
Pair of Dice Experiment

```
> # From theory P(E) = 5/36 which is approx 0.139
> # Using the function mean() in R
> # we find out what percentage of the time our events occurred in the simulation,
> # which estimates the correct probability:
> # P(E)
> mean(eventE)
[1] 0.1412
```

Using replicate in R to Repeat Experiments

- The size argument to sample allowed us to perform many repetitions of an experiment.

- For more complicated statistical experiments, we use the R function replicate, which can take a single R expression and repeat it many times.

- The function replicate is an example of an *implicit loop* in R.

- Suppose that expr is one or more R commands, the last of which returns a single value.

The call
replicate(n, expr)
repeats the expression stored in expr n times and stores the resulting values as a vector.

Using replicate in R to Repeat Experiments

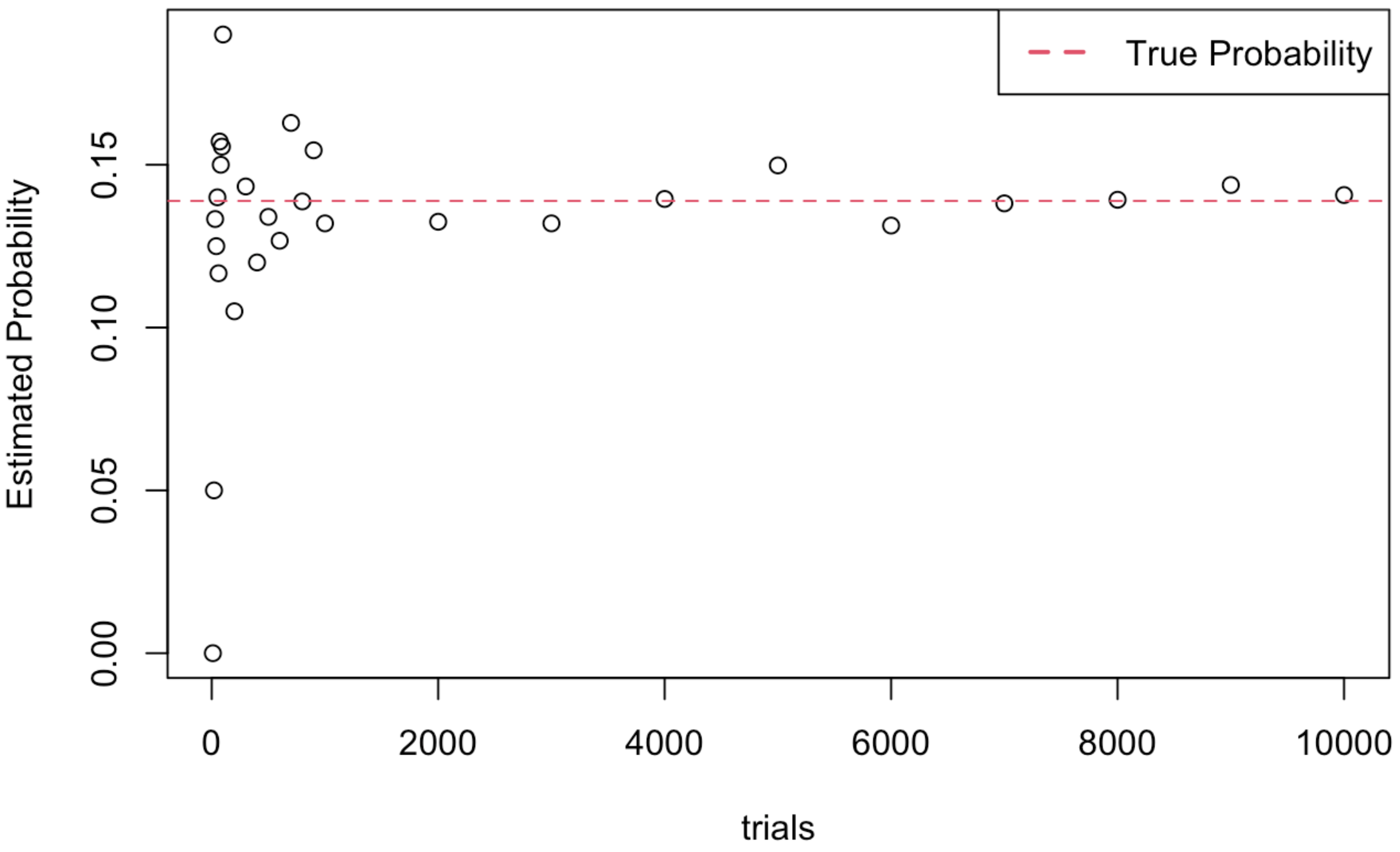
Suppose two dice are rolled.

- Let E denote the event that the sum of the uppermost face values of the pair of dice is six.
- By counting, we know that P(E) = 5/36 = 0.139
- Let's run the code below several times and perform many repetitions of the experiment.

```
> # replicate to repeat experiments
> eventE <- replicate(10, {
+   die_roll <- sample(1:6, 2, TRUE)
+   sum(die_roll) == 6
+ })
> mean(eventE)
[1] 0.2
> mean(replicate(200, {
+   sum(sample(1:6, 2, T)) == 6
+ }))
[1] 0.145
> mean(replicate(10000, {
+   sum(sample(1:6, 2, T)) == 6
+ }))
[1] 0.1367
```

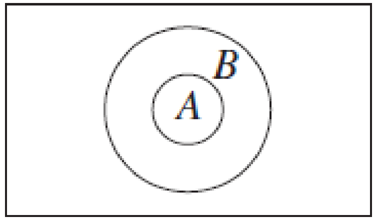
When we perform 10000 trials to estimate the probability, we observe can get closer to the theoretical probability, on average.

This figure shows the plot of the estimated probability of E for simulations ranging from 10 trials to 10,000 trials



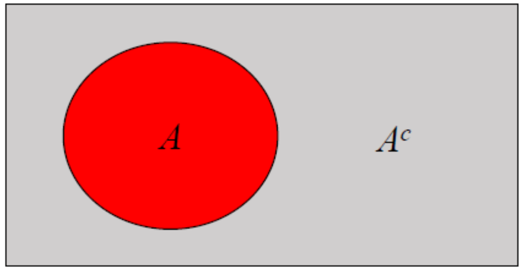
Properties of Probability

1. If A implies B, that is if A ⊆ B, then P(A) ≤ P(B)



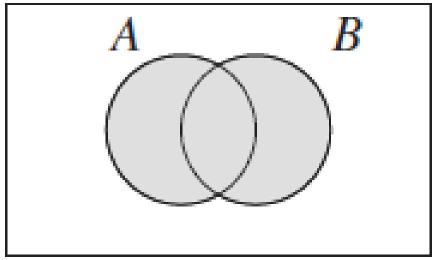
A implies *B*

2. P(A does not occur) = $P(A^C)$ = 1 – P(A)



3. For all events A and B,

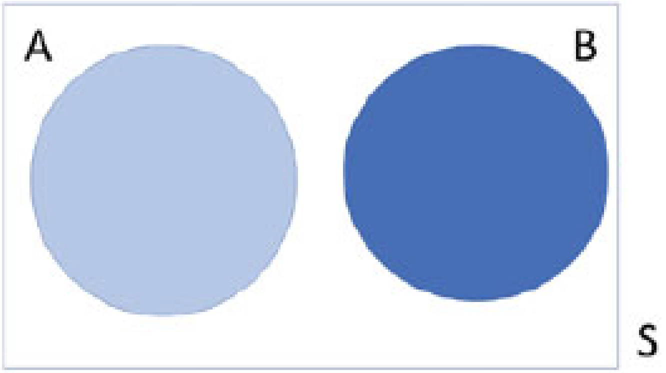
$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(AB)$$



A or *B* occur

Addition Rule for Mutually Exclusive Events

If A and B are mutually exclusive events ($AB = \emptyset$), then



$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$$

Consider two events A, B, on the same sample space S.

$A = (A \cap B^C) \cup (A \cap B)$

Events $(A \cap B^C)$ and $(A \cap B)$ are mutually exclusive.

$P(A) = P(A \cap B^C) + P(A \cap B)$

$P(A \cap B^C) = P(A) - P(A \cap B)$

And, $P(A^C \cap B) = P(B) - P(A \cap B)$

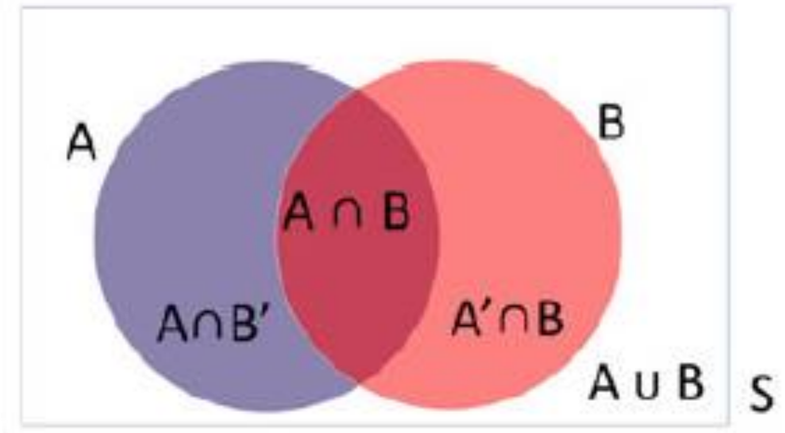
$A \cup B = (A \cap B^C) \cup (A \cap B) \cup (A^C \cap B)$

All three events are mutually exclusive events.

$P(A \cup B) = P(A \cap B^C) + P(A \cap B) + P(A^C \cap B)$

$= P(A) - P(A \cap B) + P(A \cap B) + P(B) – P(A \cap B)$

$= P(A) + P(B) – P(A \cap B)$ ❤



Proof We can write $A \cup B = (A \cap B') \cup (A \cap B) \cup (A' \cap B)$, see the right panel of Fig. 3.3. All three of these are mutually exclusive events. From (3.1) we have,

$$P(A \cap B') = P(A) - P(A \cap B),\ P(A' \cap B) = P(B) - P(A \cap B).$$

Hence, using Axiom (A3),

$$\begin{aligned} P(A \cup B) &= P(A \cap B') + P(A \cap B) + P(A' \cap B) \\ &= P(A) - P(A \cap B) + P(A \cap B) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - P(A \cap B). \end{aligned}$$

Example #1

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.

The sample space Ω is given by

$$\begin{pmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{pmatrix}$$

There are 36 possible outcomes.
6^2 = 36

Let event E be “the sum of the two dice is 6”. E = {(1, 5),(2, 4),(3, 3),(4, 2),(5, 1)}

Let event F be “At least one of the dice is a 2.” F = {(2, 1),(2, 2),(2, 3),(2, 4),(2, 5),(2, 6),(1, 2),(3, 2),(4, 2),(5, 2),(6, 2)}

a. Find $\mathrm{P}(\mathrm{E} \cup \mathrm{F})$.

b. Find $\mathrm{P}(\mathrm{E}')$.

Example #1

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.

The sample space S is given by

$$\begin{pmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{pmatrix}$$

E = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}

P(E) = $\frac{n(E)}{n(S)} = \frac{5}{36}$ = 0.139

F = {(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(1,2),(3,2),(4,2),(5,2),(6,2)}

P(E) = $\frac{n(F)}{n(S)} = \frac{11}{36}$ = 0.306

E ∩ F = {(2, 4),(4, 2)}

P(E ∩ F) = $\frac{2}{36}$

P(E or F) = P(E ∪ F) = P(E) + P(F) – P(E ∩ F) = $\frac{5}{36} + \frac{11}{36} - \frac{2}{36} = \frac{14}{36}$

Example #1

Suppose that two six-sided dice are rolled and the numbers appearing on the dice are observed.

The sample space Ω is given by

$$\begin{pmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{pmatrix}$$

There are 36 possible outcomes.
$6^2 = 36$

Let event E be “the sum of the two dice is 6”. E = {(1, 5),(2, 4),(3, 3),(4, 2),(5, 1)}

$E' =$

$$\begin{pmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{pmatrix}$$

$P(E') = 1 - P(E) = 1 - \frac{5}{36} = \frac{31}{36}$

Example #2

A retail establishment accepts either the American Express or the VISA credit card.

A total of 24% of its customers carry an American Express card,

61% carry a VISA card, and 11% carry both.

What *percentage* of its customers, carry a card that the establishment will accept?

Example #2

A retail establishment accepts either the American Express or the VISA credit card.

A total of 24% of its customers carry an American Express card,

61% carry a VISA card, and 11% carry both.

What *percentage* of its customers, carry a card that the establishment will accept?

Let A be the event carrying American Express card; $P(A) = 0.24$

Let B be the event carrying Visa card; $P(B) = 0.61$

Let AB be the event carrying both cards; $P(AB) = 0.11$

Let $A \cup B$ be the event carrying either American Express or Visa or both cards (at least one type of card);

$P(A \cup B) = ?$

$P(A \cup B) = P(A) + P(B) - P(AB) = 0.24 + 0.61 - 0.11 = 0.74$

74% of customers carry a card that the establishment will accept.

Example #3

A survey of magazine subscribers showed that 45.8% rented a car during the past 12 months for business reasons, 54% rented a car during the past 12 months for personal reasons, and 30% rented a car during the past 12 months for both business and personal reasons.

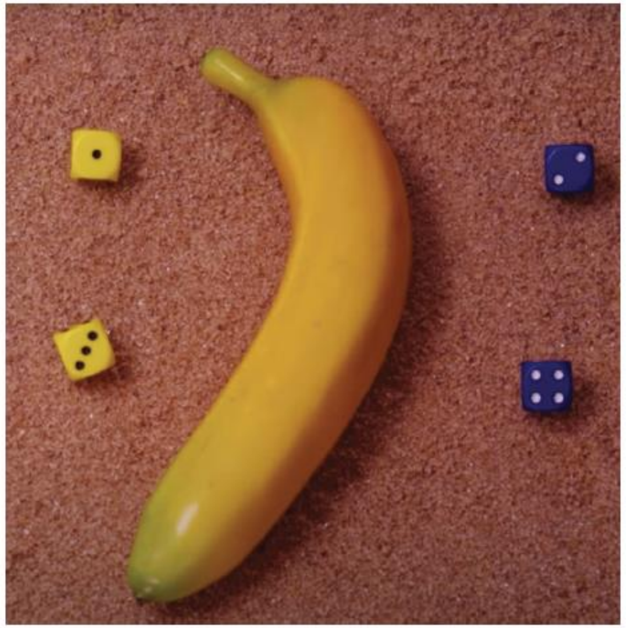
a. What is the probability that a subscriber rented a car during the past 12 months for business or personal reasons?

b. What is the probability that a subscriber did not rent a car during the past 12 months for either business or personal reasons?

Illustrate events with Venn Diagram (or with contingency table)

Game: The Last Banana

Two people are on a deserted island, and they decide to play a game to see who will get the last banana to eat. They have two dice with them on the island. The game is as follows: Both players roll their die. If a one, two, three, or four is the highest number rolled, then player A wins. If instead a five or a six is the highest number rolled, then player B wins.



Which player would you rather be?

Goal:

- Use simulations to estimate probability and make decisions.
- Compute the theoretical probabilities using formulas and probability rules.

[The last banana: A thought experiment in probability - Leonardo Barichello]