MAD2311.03 – Test 1 Name: ______________________

Show your work and justify your answers. Circle your final answer when applicable.

1. (2 points) Circle $\boxed{\textbf{True}}$ **or False** – If f is continuous function at $x = a$, then $\lim_{x \to a} f(x) = f(a)$

2. (2 points) Circle the correct choice:

   a.) According to the intermediate value theorem, if f is a continuous function on $[a, b]$, then for every value y between $f(a)$ and $f(b)$, there is a value x between a and b such that $f(x) = 0$.

   b.) According to the intermediate value theorem, if f is a continuous function on $[a, b]$, then for every value y between $f(a)$ and $f(b)$, there is a value x between a and b such that $f(x) = x$.

   c.) According to the intermediate value theorem, if f is a continuous function on $[a, b]$, then for every value y between $f(a)$ and $f(b)$, there is a value x between a and b such that $f(x) = y$.

3. (2 points) If f is differentiable at $x = a$, then $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ or $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

4. (10 points) Let $f(x) = 2^{-x} - \log_2(x)$.

   a.) Is $f(x)$ continuous on $[1, 2]$? Why or why not? Yes, f is the sum of two continuous functions on this interval.

   b.) If your previous answer was "yes", show that $f(x)$ has a root on this interval using the intermediate value theorem.
   $f(1) = 1/2$ and $f(2) = -3/4$. By the I.V.T., there exists x with $f(x) = 0$ since $-3/4 < 0 < 1/2$.

5. (10 points) Let $f(x) = \sqrt{x}$. Note that $f'(x) = \dfrac{1}{2\sqrt{x}}$. Determine the equation of the tangent line to $f(x)$ at $x = 3$.
The line is give by $y = mx + b$. We know $m = f'(3) = \frac{1}{2\sqrt{3}}$. We also know that the line passes through the point, $(3, f(3))$ or $(3, \sqrt{3})$. So, $\sqrt{3} = \frac{1}{2\sqrt{3}} \cdot 3 + b$. This yields $b = \sqrt{3} - \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$. The tangent line is $y = \frac{1}{2\sqrt{3}} x + \frac{\sqrt{3}}{2}$

6. (12 points) You drop a penny from the top of the parking garage to the ground and record its distance away from you at various moments in time:

t (sec)	1	1.001	1.01
$f(t)$ (feet)	16	16.032016	16.3216

a.) Compute the average velocity of the penny (in feet per second) from 1 second to 1.01 seconds.

$$\frac{16.3216 - 16}{1.01 - 1} = \frac{0.3216}{0.01} = 32.16 \text{ ft/sec}$$

b.) Compute the average velocity of the penny (in feet per second) from 1 second to 1.001 seconds.

$$\frac{16.032016 - 16}{1.001 - 1} = \frac{0.032016}{0.001} = 32.016 \text{ ft/sec}$$

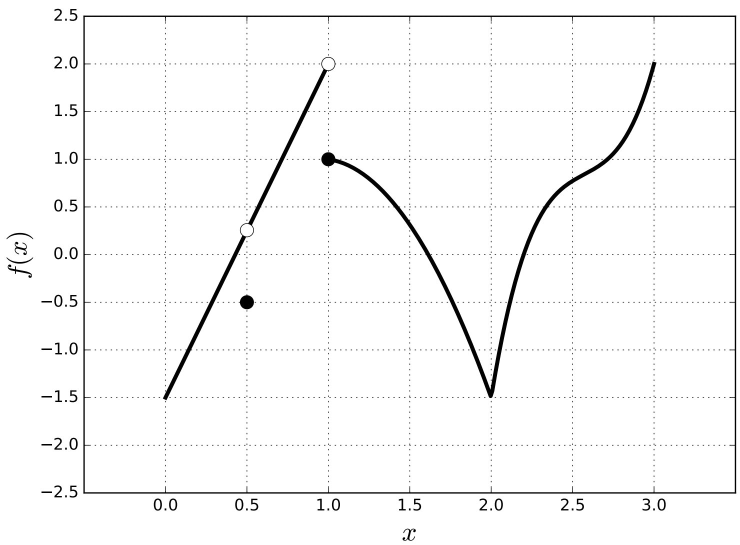
c.) Using your previous two answers, estimate to the nearest integer the instantaneous velocity of the penny at 1 second.
32 ft/sec

d.) Suppose $f(t) = 16t^2$. Compute the instantaneous velocity at $t = 1$ by computing the derivative from the definition. Compare to your previous answer.

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{16(1+h)^2 - 16}{h} \\ &= 16 \cdot \lim_{h\to 0} \frac{1 + 2h + h^2 - 1}{h} \\ &= 16 \cdot \lim_{h\to 0} 2 + h = 32 \end{aligned}$$

This matches the previous answer.

7. (12 points) Consider the function, $f(x)$, whose graph is shown below:



a.) $\lim_{x\to 0.5} f(x) =$0.25

b.) $\lim_{x\to 1^-} f(x) =$2

c.) $\lim_{x\to 1^+} f(x) =$1

d.) Does $\lim_{x\to 1} f(x)$ exist? Why or why not? No, the left and right limits must agree in order for the limit to exist at a point.

e.) Where is $f(x)$ continuous? $(0, 0.5) \cup (0.5, 1) \cup (1, 3)$

f.) Where is $f(x)$ differentiable? $(0, 0.5) \cup (0.5)1) \cup (1, 2) \cup (2, 3)$

8. (15 points) Determine the following limits. Do not skip steps.

a.) $\lim_{x\to 3} \ln(1+\sin(\pi x))$

$$\begin{aligned}\lim_{x\to 3} \ln(1+\sin(\pi x)) &= \ln\left(\lim_{x\to 3}(1+\sin(\pi x))\right) \\ &= \ln\left(1+\lim_{x\to 3}\sin(\pi x)\right) \\ &= \ln\left(1+\sin(\lim_{x\to 3}(\pi x))\right) \\ &= \ln(1+\sin(\pi\cdot 3))) = \ln(1+0) = 0\end{aligned}$$

b.) $\lim_{x\to\infty} \dfrac{4x^3-x}{2x^2-3x^3+1}$

$$\begin{aligned}\lim_{x\to\infty} \frac{4x^3-x}{2x^2-3x^3+1} &= \lim_{x\to\infty} \frac{x^3\left(4-\frac{1}{x^2}\right)}{x^3\left(\frac{2}{x}-3+\frac{1}{x^3}\right)} \\ &= \lim_{x\to\infty} \frac{4-\frac{1}{x^2}}{\frac{2}{x}-3+\frac{1}{x^3}} \\ &= \frac{\lim_{x\to\infty}\left(4-\frac{1}{x^2}\right)}{\lim_{x\to\infty}\left(\frac{2}{x}-3+\frac{1}{x^3}\right)} \\ &= \frac{\lim_{x\to\infty}4-\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty}\frac{2}{x}-\lim_{x\to\infty}3+\lim_{x\to\infty}\frac{1}{x^3}} \\ &= \frac{4-0}{0-3+0} = -\frac{4}{3}\end{aligned}$$

c.) $\lim_{t\to 0}\left(\dfrac{1}{t}-\dfrac{1}{t^2+t}\right)$

$$\begin{aligned}\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right) &= \lim_{t\to 0}\left(\frac{t+1}{t(t+1)}-\frac{1}{t(t+1)}\right) \\ &= \lim_{t\to 0}\frac{t}{t(t+1)} \\ &= \lim_{t\to 0}\frac{1}{t+1} \\ &= \frac{\lim_{t\to 0}1}{\lim_{t\to 0}(t+1)} = \frac{1}{0+1} = 1\end{aligned}$$

9. (10 points) Suppose $3x-1 \le f(x) \le x^2-3x+8$ near $x=3$. Find $\lim_{x\to 3} f(x)$.

$3\cdot 3-1=8$ and $3^2-3\cdot 3+8=8$ so $\lim_{x\to 3} f(x) = 8$ by the squeeze theorem.

10. (10 points) Let $f(x) = \begin{cases} cx^2+2x & x<2 \\ x^3+cx & x\ge 2 \end{cases}$. Determine the value c that makes $f(x)$ continuous on $(-\infty,\infty)$.

If $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$, then $f(x)$ will be continuous at $x=2$. Since is a polynomial on either side of $x=2$, this would make $f(x)$ continuous everywhere. So, we need $cx^2+2x = x^3+cx$ at $x=2$ or $c2^2+2\cdot 2 = 2^3+c\cdot 2$. This says $c=2$.