MAC2311.03 – Test 3 Name: ________________________

Show your work and justify your answers. Circle your final answer when applicable.

1. (10 points) Provide an example of a simple function, f, that is differentiable on an interval and where $f'(c) = 0$ for some c on the interval, but neither a minimum nor maximum exists at $x = c$.

The function $f(x) = x^3$ on any interval that contains $x = 0$ is such a function since $f'(0) = 0$, but no max or min exits at $x = 0$.

2. (10 points) A ball is thrown upward, and its height at a time, t, is given by $f(t) = \sqrt{t}$ for t on $[0, 9]$. What does the mean value theorem assert about the velocity of the ball on this interval?

The mean value theorem state that there exists at least one point, say $t = u$, such that $0 < u < 9$ and $f'(u) = \frac{f(9)-f(0)}{9-0} = \frac{1}{3}$. In other words, the mean value theorem says there's a point in time between 0 and 9 where the velocity is equal to $1/3$.

3. (10 points) Air is pumped into a spherical balloon at a rate of 16 cm^3/s. Recall that the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$ where r is the radius of the sphere. At what rate is the radius of the balloon increasing when the radius is equal to 2 cm? Be sure to label your answer with the correct units.

First, we have that $\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt}$. We are told that $\frac{dV}{dt} = 16$ for all t so this means $16 = 4\pi r^2 \frac{dr}{dt}$. Furthermore, we are asked what the rate of change of r is when r is equal to 2. So, we have $16 = 4\pi 2^2 \frac{dr}{dt}$. Solving for derivative of r, we find $\frac{dr}{dt} = \frac{1}{\pi}$ cm/s.

4. (10 points) Let $f(x) = 2x^3 - 3x^2$ on $(-\infty, \infty)$.

a.) Determine intervals where f is positive and negative, noting roots of f.

$f(x) = x^2(2x-3) = 0 \to$ roots are $x = 0, \frac{3}{2}$

interval	sign($f(x)$)
$(-\infty, 0)$	-
$(0, \frac{3}{2})$	-
$(\frac{3}{2}, \infty)$	$+$

b.) Determine intervals where f is increasing and decreasing, noting local extrema of f and where they occur.

$f'(x) = 6x^2 - 6x = 6x(x-1) = 0 \to$ critical numbers are $x = 0, 1$

interval	sign($f'(x)$)	increasing / decreasing
$(-\infty, 0)$	$+$	increasing
$(0, 1)$	-	decreasing
$(1, \infty)$	$+$	increasing

From the table, there is a local max at $x = 0$ of $f(0) = 0$ and a local min at $x = 1$ of $f(1) = -1$.

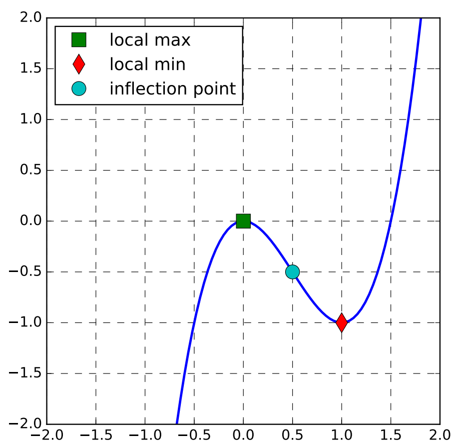
c.) Determine intervals where f is concave up and concave down, noting inflection points of f.

$f''(x) = 12x - 6 = 0 \to x = \frac{1}{2}$

interval	sign($f''(x)$)	concave down / up
$(-\infty, \frac{1}{2})$	-	down
$(\frac{1}{2}, \infty)$	$+$	up

Because concavity changes at $x = \frac{1}{2}$, there is an inflection point there.

d.) Sketch the graph of f, indicating the information you've found in the previous parts.



5. (10 points) Let $f(x) = x + \sqrt{2}\cos(x)$ on $(0, \pi)$. Find all critical numbers in this interval. Determine if f has a local minimum or local maximum at each critical number, using the second derivative test. You do not need to provide the value of the function at the critical numbers.

First, $f'(x) = 1 - \sqrt{2}\sin(x) = 0 \to$ critical numbers are $x = \frac{\pi}{4}, \frac{3\pi}{4}$ since $0 < x < \pi$. The second derivative is $f''(x) = -\sqrt{2}\cos(x)$ so $f''(\pi/4) = -1$ implies a local max at $x = \pi/4$, while $f''(3\pi/4) = 1$ implies a local min at $x = 3\pi/4$.

6. (10 points) Compute the limit: $\displaystyle\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[3]{x}}$

l'Hopital's rule applies because the limit is of the form, ∞/∞. Thus,

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{3}x^{-\frac{2}{3}}} = \frac{3}{x^{\frac{1}{3}}} = 0$$

7. (10 points) Compute the limit: $\displaystyle\lim_{x\to 0^+} \left[\cot(x) - \frac{1}{x}\right]$

l'Hopital's rule currently doesn't apply because the limit is of the form $\infty - \infty$. However, by writing $\cot$ in terms of $\sin$ and $\cos$, and finding a common denominator, we find we can apply l'Hopital's rule:

$$\begin{aligned}
\lim_{x\to 0^+} \left[\cot(x) - \frac{1}{x}\right] &= \lim_{x\to 0^+} \frac{x\cos(x) - \sin(x)}{x\sin(x)} && \text{(form 0/0)} \\
&= \lim_{x\to 0^+} \frac{\cos(x) - x\sin(x) - \cos(x)}{\sin(x) + x\cos(x)} && \text{(also form 0/0)} \\
&= \lim_{x\to 0^+} \frac{-\sin(x) - x\cos(x)}{\cos(x) + \cos(x) - x\sin(x)} = \frac{0}{2} = 0
\end{aligned}$$

8. (10 points) Make an argument for the value of $\displaystyle\lim_{x\to\infty} \frac{x^{100}}{e^x}$ without differentiating 100 times.

The limit is 0. l'Hopital's rule clearly applies as is because we have the form ∞/∞. Furthermore, each time we differentiate using l'Hopital's rule, we also have the form ∞/∞, but the powers of x on top decrease each time, while the bottom remains the same. This happens until we differentiate for the 100th time where we are just left with a constant on top divided by the exponential, which goes to zero.

9. (10 points) Recall that $\dfrac{d}{dx}\tanh^{-1}(x) = \dfrac{1}{1-x^2}$. Also note that $\lim_{x\to 1^-}\tanh^{-1}(x) = \infty$. Compute the limit:

$$\lim_{x\to 1^-}(x-1)\tanh^{-1}(x)$$

We have the form $0\cdot\infty$, but by writing $(x-1)\tanh^{-1}(x) = \dfrac{\tanh^{-1}(x)}{\frac{1}{x-1}}$, we have a limit of the form ∞/∞. Using the provided derivative, we have:

$$\begin{aligned}
\lim_{x\to 1^-}\frac{\tanh^{-1}(x)}{\frac{1}{x-1}} &= \lim_{x\to 1^-}\frac{\frac{1}{1-x^2}}{\frac{-1}{(x-1)^2}} \\
&= \lim_{x\to 1^-}\frac{-(x-1)^2}{1-x^2} \\
&= \lim_{x\to 1^-}\frac{-(x-1)^2}{(1-x)(1+x)} = \lim_{x\to 1^-}\frac{x-1}{1+x} = 0
\end{aligned}$$

10. (10 points) Suppose you are measuring the amount of some substance, and at two different points in time, you take the measurements, m_1 and m_2, where $m_1 \neq m_2$ due to noise. In place of these two measurements, you would like to use a single constant value, c. To choose an appropriate value for c, you decide that it should minimize the following measurement of error:

$$E(c) = (m_1 - c)^2 + (m_2 - c)^2$$

a.) Find all critical numbers of $E(c)$.

$E'(c) = -2(m_1 - c) - 2(m_2 - c)$ and $E'(c) = 0 \to c = \dfrac{m_1 + m_2}{2}$.

b.) Use the second derivative test to determine whether or not a local minimum or local maximum occurs at the critical points.

$E''(c) = 2 + 2 = 4 > 0$. So, $E(c)$ is concave up everywhere, and in particular it is concave up at the critical point, indicating that a local min occurs at the critical point.

Note that the critical point is exactly the sample mean (or average) of the two measurements. So, the sample mean minimizes $E(c)$. $E(c)$ is sometimes called the "sum of squared errors". This result generalizes to n samples, $m_1, m_2, \ldots, m_n$, where the sum of squared is minimized by the average of the measurements.