MAC2311.03 – Test 2 Name: ______________________

Show your work and justify your answers. Circle your final answer when applicable.

1. (5 points) Suppose $f'(2) = 1$, $g(1) = 2$, and $g'(1) = 3$. Find $h'(1)$ if $h(x) = f(g(x))$.

$h'(x) = f'(g(x))g'(x)$ by the chain rule. So, $h'(1) = f'(g(1))g'(1) = f'(2) \cdot 3 = 1 \cdot 3 = 3$

2. (5 points) Compute the limit: $\lim_{x\to 0} \frac{\sin 3x}{x}$

Recall that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. So, $\lim_{x\to 0} \frac{\sin 3x}{x} = \lim_{x\to 0} 3 \cdot \frac{\sin 3x}{3x} = 3 \cdot \lim_{x\to 0} \cdot \frac{\sin 3x}{3x} = 3 \cdot 1 = 3$ by letting u = 3x in the final limit.

3. (5 points) Find $f'(x)$ if $f(x) = 2x^3 - 6\sqrt{x^3}$

Write $f(x) = 2x^3 - 6x^{3/2}$ so that $f'(x) = 6x^2 - 9x^{1/2}$

4. (5 points) Find $f'(x)$ if $f(x) = \cos(e^{-x} + e^{-2})$

$$f'(x) = -\sin\left(e^{-x} + e^{-2}\right) \frac{d}{dx}\left(e^{-x} + e^{-2}\right) = -\sin\left(e^{-x} + e^{-2}\right)\left(-e^{-x}\right) = e^{-x} \sin\left(e^{-x} + e^{-2}\right)$$

5. (5 points) Find $f'(x)$ if $f(x) = \log_2(2x)$

$$f'(x) = \frac{1}{\ln(2) \cdot (2x)} \cdot \frac{d}{dx}(2x) = \frac{1}{\ln(2) \cdot x}$$

6. (5 points) Find y' if $y = \tan^{-1}(2x)$. HINT: If you can't remember the derivative of $\tan^{-1}(x)$, it may be helpful to recall that $y = \tan^{-1}(x) \Leftrightarrow x = \tan(y)$ and $\frac{d}{dx} \tan(y) = (1 + \tan^2(y))y'$

$$y' = \frac{1}{1 + (2x)^2} \frac{d}{dx}(2x) = \frac{2}{1 + 4x^2}$$

7. (10 points) Consider the function:

$$f(x) = \cos^2(x) + \sin^2(x)$$

a.) Find $f'(x)$. Do not use the fact that $\cos^2(x) + \sin^2(x) = 1$.

$f'(x) = -2\cos(x)\sin(x) + 2\sin(x)\cos(x) = 0$

b.) Does your answer from part a.) match what'd you expect, using that $\cos^2(x) + \sin^2(x) = 1$? Briefly explain.

Yes. $\cos^2(x) + \sin^2(x) = 1$ implies $f(x) = 1$, which of course has derivative $f'(x) = 0$.

8. (10 points) Find $f'(x)$ if $f(x) = \dfrac{1+3x}{5-4x}$.

$$\begin{aligned} f'(x) &= \frac{(5-4x)3 - (1+3x)(-4)}{(5-4x)^2} \\ &= \frac{15 - 12x + 4 + 12x}{(5-4x)^2} \\ &= \frac{19}{(5-4x)^2} \end{aligned}$$

9. (10 points) Find $f'(x)$ if $f(x) = \dfrac{e^x}{1+e^x}$

$$\begin{aligned} f'(x) &= \frac{(1+e^x)e^x - e^x \cdot e^x}{(1+e^x)^2} \\ &= \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2} \\ &= \frac{e^x}{(1+e^x)^2} \end{aligned}$$

10. (10 points) A ball rolls down a hill. The height of the ball at any time, t, is given by $y(t) = \frac{1}{1+t^2}$.

a.) Find the function describing the ball's velocity at any time, t.

The function describing the ball's velocity is $y'(t)$. Note that $y(t) = (1+t^2)^{-1}$. Then,

$$y'(t) = -(1+t^2)^{-2}\frac{d}{dt}(t^2) = -\frac{2t}{(1+t^2)^2}$$

b.) What is the ball's velocity at $t = 0$? What is the ball's velocity at $t = 1$?

$y'(0) = 0$ and $y'(1) = -1/2$

11. (10 points) Use logarithmic differentiation to find y' if $y = \frac{(x-2)^{28}(x+1)^4}{\sqrt{x^2-3}}$

$$\begin{aligned}
\ln(y) &= 28\ln(x-2) + 4\ln(x+1) - \frac{1}{2}\ln(x^2-3) \\
\frac{y'}{y} &= 28\frac{1}{x-2} + 4\frac{1}{x+1} - \frac{1}{2}\frac{1}{x^2-3}\frac{d}{dx}\left(x^2\right) \\
y' &= y\left(\frac{28}{x-2} + \frac{4}{x+1} - \frac{x}{x^2-3}\right) \\
y' &= \frac{(x-2)^{28}(x+1)^4}{\sqrt{x^2-3}}\left(\frac{28}{x-2} + \frac{4}{x+1} - \frac{x}{x^2-3}\right)
\end{aligned}$$

12. (10 points) Consider the equation,

$$\ln(y+x) = e^{2x}$$

a.) Find y'.

$$\begin{aligned} \frac{y'+1}{y+x} &= 2e^{2x} \\ y'+1 &= 2(y+x)e^{2x} \\ y' &= -1 + 2(y+x)e^{2x} \end{aligned}$$

b.) Find y'' in terms of y and x only.

$$\begin{aligned} y'' &= 2(y'+1)e^{2x} + 2(y+x)2e^{2x} \\ y'' &= 2(2(y+x)e^{2x})e^{2x} + 4(y+x)e^{2x} \\ y'' &= 4(y+x)e^{2x}e^{2x} + 4(y+x)e^{2x} \\ y'' &= 4(y+x)e^{2x}\left(e^{2x}+1\right) \end{aligned}$$

13. (10 points) Let $f(x) = \sin(x)$. Find $f^{(4)}(x)$. In other words, find the fourth derivative.

$$\begin{aligned} f'(x) &= \cos(x) \\ f''(x) &= -\sin(x) \\ f^{(3)}(x) &= -\cos(x) \\ f^{(4)}(x) &= \sin(x) \end{aligned}$$

One can interpret the derivative of $\sin(x)$ as a 90 degree phase shift. Thus the fourth derivative is a 360 degree phase shift, which puts us back where we started.