MAC2311.03 – Test 4 Name: ______________________________

Show your work and justify your answers. Circle your final answer when applicable.

1. (10 points) Determine $f(x)$ if $f'(x) = \sin(x) + x$ and $f(0) = 0$.

We first have via antidifferentiation that $f(x) = -\cos(x)+\frac{x^2}{2}+C$ for some constant C. Next, $0 = f(0) = -1+0+C$, so that $C = 1$, and $f(x) = -\cos(x) + \frac{x^2}{2} + 1$

2. (10 points) The function, $f(x) = \ln(x) - e^{-x}$, has a root on the interval $[1, 2]$. Find the second value of Newton's method, x_2, starting from the initial guess, $x_1 = 1$.

According to Newton's method, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. So, we need to find f' first. $f'(x) = \frac{1}{x} + e^{-x}$. Next, $f(x_1) = f(1) = \ln(1) - e^{-1} = 0 - \frac{1}{e} = -\frac{1}{e}$, and $f'(x_1) = \frac{1}{1} + \frac{1}{e}$. Finally,

$$\begin{aligned} x_2 &= 1 - \frac{-\frac{1}{e}}{1+\frac{1}{e}} \\ &= 1 + \frac{1}{e+1} \\ &= \frac{e+2}{e+1} \approx 1.27 \end{aligned}$$

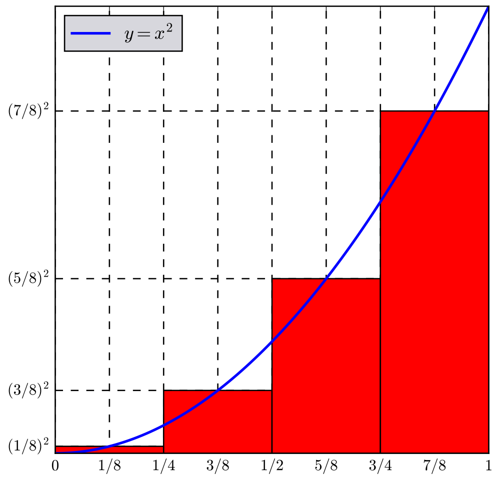
Note that $f(x_1) \approx -0.37$ and $f(x_2) \approx -0.04$ so Newton's method appears to be progressing toward a root.

3. (10 points) Find $g'(x)$ if $g(x) = \displaystyle\int_0^{\cos(x)} \cos(t)dt$.

First, note that if $f(x) = \displaystyle\int_0^{x} \cos(t)dt$, then $g(x) = f(\cos(x))$ so that $g'(x) = f'(\cos(x))\frac{d}{dx}\cos(x) = -f'(\cos(x))\sin(x)$. By the 1st FTC, $f'(x) = \cos(x)$, so $f'(\cos(x)) = \cos(\cos(x))$. Putting everything together, $g'(x) = -\cos(\cos(x))\sin(x)$.

4. (15 points) Consider the integral, $\int_0^1 x^2 dx$.

a.) Approximate the integral, using the **mid-point rule** and 4 subintervals.



See the figure to the left. We have then that:

$$
\begin{aligned}
\int_0^1 x^2 dx &\approx \frac{1}{4}\left(\frac{1}{8}\right)^2 \frac{1}{4}\left(\frac{3}{8}\right)^2 + \frac{1}{4}\left(\frac{5}{8}\right)^2 + \frac{1}{4}\left(\frac{7}{8}\right)^2 \\
&= \frac{1}{4}\left(\frac{1+9+25+49}{64}\right) \\
&= \frac{1}{4}\cdot\frac{84}{64} = \frac{21}{64}
\end{aligned}
$$

b.) Write the integral approximation, using the **right end-point rule** using a general number of subintervals, n. Simply write the sum. You do not need to evaluate the limit of the sum as $n \to \infty$.

Using the right end-point rule with n subintervals, $\Delta x = \frac{1-0}{n} = \frac{1}{n}$ and $x_i = 0 + i \cdot \Delta x = \frac{i}{n}$. Therefore,

$$\int_0^1 x^2 dx \approx \Delta x \sum_{i=1}^{n} (x_i)^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2$$

5. (10 points) A particle's velocity in cm/s is given by the equation, $v(t) = \sin(t)$. Find the total distance travelled by the particle from 0 seconds to 2π seconds.

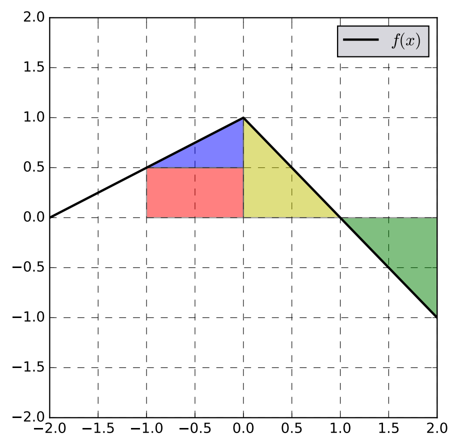
Total distance travelled $= \int_0^{2\pi} |v(t)|dt = \int_0^{2\pi} |\sin(t)|dt$. Because $\sin(t) \geq 0$ for $0 \leq t \leq \pi$ and $\sin(t) \leq 0$ for $\pi \leq t \leq 2\pi$, we then have that

$$
\begin{aligned}
\int_0^{2\pi} |\sin(t)|dt &= \int_0^{\pi} \sin(t)dt + \int_{\pi}^{2\pi} (-\sin(t))dt \\
&= -\cos(t)\Big|_0^{\pi} + \cos(t)\Big|_{\pi}^{2\pi} \\
&= [-\cos(\pi) - (-\cos(0))] + [\cos(2\pi) - \cos(\pi)]\Big|_{\pi}^{2\pi} \\
&= 4 \text{ cm}
\end{aligned}
$$

6. (10 points) The **arithmetic mean** between two numbers x and y is $\dfrac{x+y}{2}$. The **harmonic mean** between x and y is given by $\dfrac{2xy}{x+y}$. Find the numbers, x and y, whose harmonic mean is maximum when their arithmetic mean is equal to 1.

We want to maximize the quantity, $\dfrac{2xy}{x+y}$, subject to the constraint, $\dfrac{x+y}{2} = 1$. Solving for y in the constraint equation yields $y = 2-x$. We subsitute y into the quantity to be maximized, so we have $H(x) = \dfrac{2x(2-x)}{x+(2-x)} = x(2-x)$, which is a single-variable function that we can maximize with our calculus techniques. It follows that $H'(x) = 2-2x$ and $H'(x) = 0 \rightarrow x = 1$. This is a global maximum because $H(x)$ is a downward-facing parabola. When $x = 1$, $y = 2-1 = 1$. Thus the harmonic mean is maximized when x and y are both equal to 1.

7. (10 points)



Use the graph above to find $\displaystyle\int_{-1}^{2} f(x)dx$

The integral is the net area under the curve. See the color-coded areas above. The integral is thus:

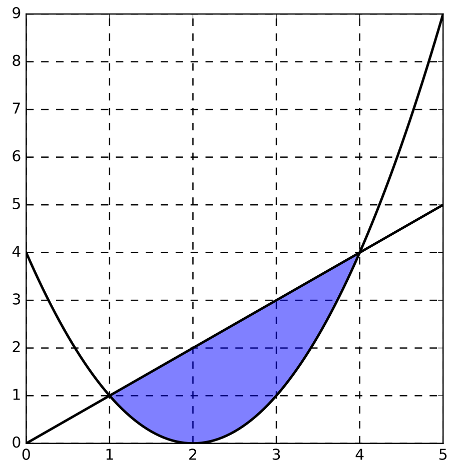
$$\int_{-1}^{2} f(x)dx = \text{A} + \text{A} + \text{A} - \text{A} = \tfrac{1}{2}\cdot 1\cdot \tfrac{1}{2} + 1\cdot \tfrac{1}{2} + \tfrac{1}{2}\cdot 1\cdot 1 - \tfrac{1}{2}\cdot 1\cdot 1 = \tfrac{3}{4}$$

8. (10 points) Evaluate $\displaystyle\int_e^{e^3} \frac{dx}{x\sqrt{\ln(x)}}$

Let $u = \ln(x)$. Then $du = \frac{1}{x}dx$, $u(e) = 1$, and $u(e^3) = 3$. Thus,

$$\int_e^{e^3} \frac{dx}{x\sqrt{\ln(x)}} = \int_1^3 \frac{1}{\sqrt{u}}du = \left.\frac{u^{1/2}}{1/2}\right|_1^3 = 2(\sqrt{3} - 1)$$

9. (15 points) Consider the graphs of $y = (x-2)^2$ and $y = x$. Sketch these two graphs, and find the area enclosed between them. The enclosed area refers to the area that is both between the two graphs and between the minimum and maximum x-values of the points of intersection.



The points of intersection between the graphs occur at $x = 1$ and $x = 4$, which is found by setting $(x-2)^2 = x$ and solving for x. The area between the graphs is thus:

$$\int_1^4 x - (x-2)^2 dx = \int_1^4 x - \int_1^4 (x-2)^2 dx = I_1 - I_2$$

Now,

$$I_1 = \left.\frac{x^2}{2}\right|_1^4 = \frac{4^2}{2} - \frac{1^2}{2} = \frac{15}{2}$$

On the other hand, we letting $u = x - 2$ in I_2 we find:

$$I_2 = \int_{-1}^2 u^2 du = \left.\frac{u^3}{3}\right|_{-1}^2 = \frac{2^3}{3} - \frac{(-1)^3}{3} = \frac{9}{3} = 3$$

Thus, $\int_1^4 x - (x-2)^2 dx = I_1 - I_2 = \frac{15}{2} - 3 = \frac{9}{2}$.