Crystals II

Let $S|k$ be smooth, w/ $\mathrm{char}(k) = 0$.

Let $\mathcal{E}$ be a crystal on S_{inf}

Claim: $\mathcal{E}$ gives rise to an object $(M, \nabla) \in MIC(S|k)$

Pf Recall:

$$S \times S \supseteq S(1) = (S, \mathcal{O}_{S\times S}/\mathcal{J}^2)$$

$$\begin{tikzcd} & S(1) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ S & & S \end{tikzcd}$$

$$\begin{tikzcd} S(1) \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_2"'] & S \arrow[l, "\Delta"' description] \end{tikzcd}$$

Set $M := \mathcal{E}(\underset{\substack{\uparrow \\ S_{inf}}}{S \hookrightarrow S})$

Then

$$\underset{\substack{\wr \\ \text{by virtue of being a crystal.}}}{P_1^* M \xrightarrow{\sim}} \mathcal{E}(S \hookrightarrow S(1)) \xleftarrow{\sim} P_2^* M$$

$\rightsquigarrow$ $\exists$ a natural isomorphism

$$\varphi_{12} : P_1^* M \longrightarrow P_2^* M$$

$\rightsquigarrow$ there is a natural connection ∇ on M.

Again using the crystal property, one checks ∇ is integrable.

Claim: $(M, \nabla) \in MIC(S/k)$ gives rise to a crystal $\mathcal{E}$ on S_{inf}.

Pf Let $(\mathcal{U} \hookrightarrow T)$ be an affine object in Sinf, corresponding to the ring map $A \twoheadleftarrow B$

<u>Subclaim:</u>

$\exists$ a commutative diagram

$$\star \quad \begin{tikzcd} A & B \arrow[l, two heads] \\ P := k[T_1,\dots,T_N] \arrow[u, two heads] \arrow[r, hook] & k[T_1,\dots,T_N]^{\wedge}_{J} =: D \arrow[u, "\varphi"'] \end{tikzcd}$$

$\uparrow$
$\ker = J$

$$\mathcal{E}(\mathcal{U} \hookrightarrow T) := M \otimes_{D,\varphi} B$$

Idea: " j_* extension, then restrict"

"trivial along vertical directions"

<u>Question</u>: why is this well-defined?!

I.e., given two maps φ_1, φ_2 as in $\star$, how do the outputs relate?

subclaim:

$\exists$ a natural B-isomorphism

$$\varphi_{12} : M \otimes_{D,\varphi_1} B \longrightarrow M \otimes_{D,\varphi_2} B,$$

induced from ∇:

$$m \otimes 1$$

$$\downarrow$$

in I

ξ

$$\sum_{\vec{v} \in \mathbb{Z}^N_{\geq 0}} \left(\left(\prod_{i=1}^{N} \left(\nabla_{\partial_{T_i}}\right)^{v_i} \right) m \otimes \prod \frac{\left(\varphi_1(T_i) - \varphi_2(T_i)\right)^{v_i}}{(v_i)!} \right)$$

Q: Why does this sum "converge"?

A: $A \twoheadleftarrow B$ is a nilimmersion,

hence the sum is finite.

Q: What does the symbol $\nabla^{M}_{\frac{\partial}{\partial T_i}}$ mean?

$\Leftrightarrow \theta_i : M \to M$

$$\begin{array}{c} A = k[T_1, \dots, T_N]/J \\ \uparrow\!\!\uparrow \\ P = k[T_1, \dots, T_N] \end{array}$$

$$0 \to T_U \to T_{\mathbb{A}^N}|_U \to N_{U \subseteq \mathbb{A}^n} \to 0$$

replacing U by a cover, this sequence splits $\rightsquigarrow$ there is a (non-canonical) notion of $\frac{\partial}{\partial T_i} \in T_U$. However,

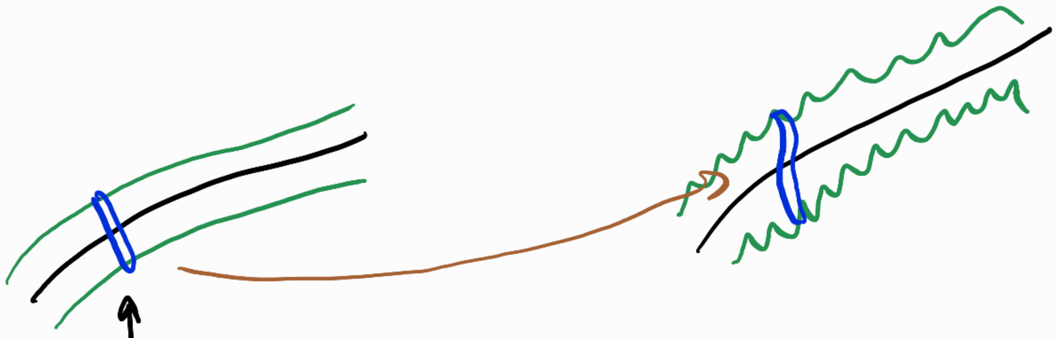
$\theta_i = \nabla_{\frac{\partial}{\partial T_i}}$ depends only on $P \twoheadrightarrow A$.

Subrmk

- $$\frac{(\varphi_1(T_i)-\varphi_2(T_i))^k}{k!} = \gamma_k(\varphi_1(T_i)-\varphi_2(T_i)) =: (\varphi_1(T_i)-\varphi_2(T_i))^{[k]}$$

- In crystalline setting, sum will converge either if ∇ is topologically quasi-nilpotent ($\nabla \circ \dots \nabla$ is divisible by p) OR if $\gamma_k \to 0$ as $k \to \infty$. Former corresponds to crystalline site, latter to nilpotent crystalline site. (Note that $p \mid \varphi_1 - \varphi_2$)

<u>Picture</u>



$M \otimes_{D,\nabla} B$ constant along vertical slices

Now, one can play an analogous game in the crystalline site.

Let $S|k$ smooth | k perfect of char p.

We work w/ the sites

$$\mathrm{CRYS}(S/W(k))$$

(resp.) $$\mathrm{NCRYS}(S/W(k))$$

Let $\hat{S}/W(k)$ be a p-adic formal scheme w/ an iso

$$\hat{S}\Big|_k \simeq S$$

__Fact__

- Crystals on $NCRYS(S/W(k))$ $\Longleftrightarrow$ formal flat connections on $\hat{S}$
- Crystals on $CRYS(S/W(k))$ $\Longleftrightarrow$ formal flat connections on $\hat{S}$ that are top. quasi-nilpotent.

What does this look like?

$S = \mathrm{Spec}(\mathbb{F}_p[T])$ $\quad \hat{S} = \mathrm{Spf}(\mathbb{Z}_p\langle T\rangle)$

$$\mathbb{Z}_p\langle T\rangle = \left\{ \sum_{i\geq 0} a_i T^i \, , \, \lim_{i\to\infty} a_i = 0 \right\}$$

"functions" on unit disk."

$$\Omega^1_{\hat S/\mathbb{Z}_p} \hookrightarrow \mathbb{Z}_p\langle T\rangle\{dT\}$$

a formal flat connection:

p-adic completion

$$\hat\nabla : \hat M \longrightarrow \hat M \hat\otimes \Omega^1_{\hat S/\mathbb{Z}_p},$$

where $\hat M$ is a p-adically complete + separated quasi-coherent sheaf on $\mathbb{Z}_p\langle T\rangle$.

Suppose $\hat S/W(k)$ is projective,

i.e., that $\hat S$ is the p-adic completion of some $\tilde S/W(k)$.

Although $\hat M$ is algebraic, there is $\boxed{\text{No}}$ a priori reason that ∇ is algebraic, i.e., comes from a connection

$$\nabla : \tilde M \longrightarrow \tilde M \otimes \Omega^1_{\tilde S/\mathbb{Z}_p}$$

Last time, we briefly discussed functoriality of the topos.

$$f : S \to T \rightsquigarrow$$

$$\begin{array}{ccc} \mathrm{Sheaves}(\mathrm{CRYS}(T/W)) & \to & \mathrm{Sheaves}(\mathrm{CRYS}(S/W)) \\ \cup\!\mid & & \cup\!\mid \\ \mathrm{Crystals}(\mathrm{CRYS}(T/W)) & \to & \mathrm{Crystals}(\mathrm{CRYS}(S/W)) \end{array}$$

What about when $T = S$, $f = \mathrm{Frob}_S$?

Suppose S has a formal lift $\hat{S}/W$ s.t. Frob_S lifts to $\widehat{\mathrm{Frob}_S}$.

Then the pulled back formal connection is the "evaluation of the crystal on $\hat{S}$".

This glues! (See Esnault's lecture notes, §8.

the argument is the same Taylor formula: assume there are two lifts of Frob_S and canonically construct a map btwn the corresponding pullbacks.)

Example of earlier

Let $E / \mathbb{F}_p$ be an elliptic curve

Let $\hat{E} / \mathbb{Z}_p$ be a lift w/out CM

(e.g., w/ transcendental j-invariant.)

Then

$$
\begin{array}{ccc}
\text{Crystals}(\mathrm{NCRYS}(E/\mathbb{Z}_p)) & \longleftrightarrow & \text{Formal flat connections on } \hat{E} \\
\cup & & \cup \\
\text{rank 1 crystals} & \longleftrightarrow & \text{rank 1 formal flat connections on } \hat{E}
\end{array}
$$

The RHS has a Frobenius action,

even though $\{\text{rank } 1 \text{ flat connections on } \tilde{E}/\mathbb{Z}_p\}$

DOES NOT!!