Introduction to local systems of <u>geometric</u> origin

<u>Notation</u>:

- X/k means sm., q-proj variety over perfect field k

$$\left\{ k = \begin{array}{l} \cdot\, \mathbb{F}_q \\ \cdot\, \#\text{field} \\ \cdot\, \mathbb{C} \end{array} \right\}$$

- L stands for a ($\overline{\mathbb{Q}}_\ell$ or $\mathbb{C}$) local system on X

Let S be a $\mathbb{C}$-manifold

<u>Def</u> Let A be a comm. ring. An A-local system (on S) is a locally constant sheaf of finite free A-modules on S.

There is a "classical" correspondence:

hol v.b flat conn.

$$\begin{tikzcd}[column sep=small] & \left\{ \mathbb{C}\text{-local systems on } S \right\} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \left\{ (V, \nabla) \text{ on } S \right\} \arrow[rr, hookleftarrow, leftrightarrow] & & \left\{ \begin{array}{c} \text{fin. dim rep} \\ \text{of } \pi_1(S) \end{array} \right\} \end{tikzcd}$$

$$\begin{tikzcd} & L \arrow[dl] \arrow[dr] & \\ (V, \nabla) \arrow[rr, leftrightarrow] & & \rho \end{tikzcd}$$

<u>Examples</u>

- $S = \mathbb{C}^\times$, $\pi_1(S) = \mathbb{Z}$

  $\rightsquigarrow$ rank n local system on S "is" $M \in GL_n(\mathbb{C})$

- $X | \mathbb{C}$ be <u>hyperbolic</u> curve

  $g - 1 + r$

Exercise: show that $\pi_1(X^{an})$ has a finite index free subgroup.

$\rightsquigarrow$ zillions of local systems on X^{an}.

- $\Gamma(2) := \{ M \in SL_2(\mathbb{Z}) \mid M \equiv I_2 \pmod 2 \}$

$$SL_2(\mathbb{Z}) \curvearrowright \mathbb{H}$$

$$\rightsquigarrow \left[\mathbb{H} / \Gamma(2) \right]$$

$$\wr\!\!\mid$$

$$Y(2) = \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

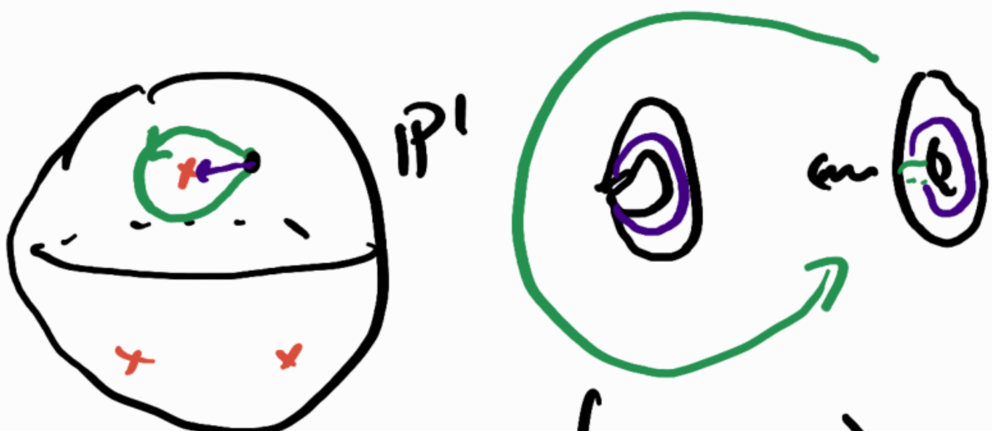
"moduli of elliptic curves w/ full level 2 structure"

$\exists$ "natural" local system on $Y(2)$

$$\begin{array}{c}\mathcal{E}\\ \big\downarrow \pi \\ Y(2)\end{array} \quad \hookrightarrow \quad z^2 = x(x-1)(x-\lambda)$$

λ-coordinate

$\rightsquigarrow$ $R^1\pi_*\mathbb{Z}$ is a rank 2 $\mathbb{Z}$-local system on $Y(2)$



$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

rep? $\pi_1(Y(2)) = \Gamma(2)$

$$\Gamma(2) \hookrightarrow SL_2(\mathbb{Z})$$

Def Let $B/\mathbb{C}$, let L be a $\mathbb{C}$-local system on B^{an}.

We say L is of geometric origin if

- $\exists\, \mathcal{U} \subseteq B$ open dense
- $\begin{array}{c} \mathcal{Y} \\ \downarrow \pi \\ \mathcal{U} \end{array}$ sm. proj.
- $i \geq 0$

s.t. $L|_{\mathcal{U}^{an}}$ is a subquotient of $R^i\pi_*\mathbb{C}$.

Properties of Geometric local systems $\underline{\hphantom{xxx}}$ (over $X/\mathbb{C}$)

- $\mathbb{L}$ is semi-simple
- $\mathbb{L}$ is "integral": $\exists$ # subfield $F \subseteq \mathbb{C}$ s.t.

$$\begin{tikzcd} \rho: \pi_1(X) \arrow[r] \arrow[dr] & GL_n(\mathbb{C}) \\ & GL_n(\mathcal{O}_F) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

$\Rightarrow$ (traces $\in \overline{\mathbb{Z}}$)

- $\mathbb{L}$ underlies $\mathbb{C}$-PVHS

(for every $\iota \in \mathrm{Aut}(\mathbb{C})$,

$\rightsquigarrow \mathbb{L}_\iota \leftrightarrow {}^\iota\rho: \pi_1(X) \to GL_n(\mathbb{C}) \xrightarrow{\iota} GL_n(\mathbb{C})$

also underlies $\mathbb{C}$-PVHS)

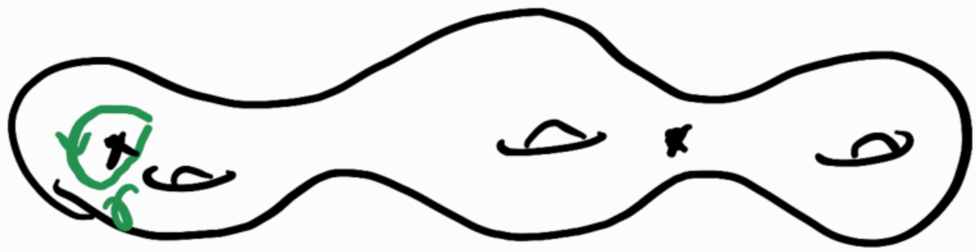
(i.e. $\mathbb{L}$ is summand of $\mathbb{Z}$-PVHS)

• L has quasi-unipotent monodromy @ ∞.

WLOG X is a curve:

$\cap$

$\overline{X}$



$L \rightsquigarrow \rho$

γ is not "uniquely defined" in $\pi_1(X)$, BUT $[\gamma]$ is well defined.

$\Rightarrow \rho(\gamma)$ not well defined but $[\rho(\gamma)]$ is.

eigenvalues of $\rho(\gamma)$ are in μ_∞. (Many proofs!)

- L has finite order determinant.
  ($\Leftarrow$ det trivial on a finite cover)

- (V, ∇) has vanishing p-curvature for $p >> 0$
  ($\Leftarrow$ underlies crystal over every p-adic completion + F-structure for all $p >> 0$)

- more: - arithmetic.
  - Simpson has some bi-algebraicity conj.,

Conj (Simpson)

Let $X/\mathbb{C}$, let L be a $\mathbb{C}$-local system on X^{an} s.t.

1. irr.
2. quasi-unipotent monodromy @ ∞
3. triv. det
4. rigid

$\leadsto$ Then L is of geometric origin

① ✓

② ✓

③ ✓

④ Assume for simplicity that X is proj.

$\exists$ f.t. moduli space$/\mathbb{Q}$,

$\mathcal{M}_1$ parametrizing rank n local systems (w/ triv det) on X^{an}.

$\Longleftrightarrow \{\pi_1(X^{an}) \to SL_n(\mathbb{C})\}/\sim$

$$\rho \longmapsto [\rho] \in \mathcal{M}(\mathbb{C})$$

L is rigid $\Longleftrightarrow$ $[\rho]$ is an isolated pt of $\mathcal{M}(\mathbb{C})$.

Evidence

Simpson
- $\rho \sim \pi_1(X) \to SL_n(K)$, K # field
- X proj: L underlies a $\mathbb{Q}$VHS.

Esnault-Gröchenig
- L is coh rigid
  $\rightsquigarrow \rho \sim \pi_1(X) \to SL_n(\mathcal{O}_K)$
- X proj: (V, ∇) has nilpotent p-curvature for all $p \gg 0$, underlies F^δ structure for all $p \gg 0$

$\underbrace{\hphantom{F^\delta \text{ structure for all } p \gg 0}}$

also when X not. nec. proj, w/ L strongly coh. rigid

Question (Simpson)

Let $X/\mathbb{C}$ be proj of dim ≥ 2.
Let $D \subseteq X$ be smooth ample divisor
Let $\mathbb{L}$ be a local system on X^{an}, s.t. $\mathbb{L}|_{D^{an}}$ is motivic.

Then is $\mathbb{L}$ motivic?

(in fact, suggestion was that the motive over D should extend, after an isogeny, to all of X)

Exercise: prove it when $\mathbb{L}|_D$ comes from a family of AVs!
(corollary of work of Simpson)

Arithmetic

If X/k general $\rightsquigarrow$ $\pi_1^{\acute{e}t}(X)$, profinite group.

Def A $\mathbb{Q}_\ell$-local system on X is a conti hom

$$\rho: \pi_1^{\acute{e}t}(X) \to GL_n(\mathbb{Q}_\ell)$$

Conj (Deligne) (from Weil II)
Let $X/\mathbb{F}_q$, let L be a $\overline{\mathbb{Q}}_\ell$ local system on X s.t.

1. irr.
2. N/A
3. triv. det
4. N/A

Then L is of geom. origin (up to a Tate twist)

Question: why fewer hypotheses?

Note ② "Automatic" by Grothendieck's quasi-unipotent monodromy theorem

④ Morally "automatic" after finiteness results of Deligne, Drinfeld (~2010)

Rmk

① Conj. D is known when $\dim X = 1$ due to proof (not just statement) of Langlands, due to L. Lafforgue.

"Why"

$$L \longleftrightarrow \overline{\mathbb{Q}}_\ell \text{ cusp form}$$

$$\wedge$$

$$C^\infty\left(G(\hat{\mathcal{O}}) \backslash G(\mathbb{A}_{\mathbb{F}_q(X)}) / G(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell\right)$$

has no topology

$$\overline{\mathbb{Q}}_\ell \xrightarrow{\iota} \overline{\mathbb{Q}}_{\ell'} \qquad \downarrow s$$

$$\text{"}\iota L\text{"} = \underline{L'} \longleftrightarrow C^\infty(\quad , \overline{\mathbb{Q}}_{\ell'})$$

cont rep of $\pi_1(X) \to GL_n(\overline{\mathbb{Q}}_{\ell'})$

"compatible"

local-global compatible of Langlands

$\iff$ Frobenius eigenvalues match.

Rmk A conj. of dJ, proved by Gaitsgory for $\ell > 2$, implies the following:

Cor (dJG) Let $U/\overline{\mathbb{F}}_q$ be hyperbolic.

Let $\mathcal{S} := \{ L$ $\overline{\mathbb{Q}}_\ell$ - local systems on $U/\overline{\mathbb{F}}_q$ w/ fixed rank, bounded ramification @ ∞, fixed local monodromy $\}$

$U|$

$\mathcal{S}^{geo}$

Then ① $\mathcal{S}^{geo}$ is infinite

② ~ "$\mathcal{S}^{geo}$ is Zariski dense in $\mathcal{S}$"

Rmk • Using dJ's conj., Drinfeld proved semi-simplicity + Hard Lefschetz thms for perverse sheaves in char 0.

• Using dJ + companions + ...

dJ-Esnault proved that

$\pi : X \to Y$ (quasi proj. normal)

L_Y is s.s. $\mathbb{C}$-local system

$\Rightarrow \pi^* L_Y =: L_X$ is s.s.

Rmk Recently, several authors have generalized Conj D to include the finitely generated /Q (Litt, Petrov)

This may be thought of as a relative Fontaine-Mazur conjecture.

maybe skip

Q: Is CorDJG true for other base fields? E.g.

Question: Let $X/\mathbb{C}$ be smooth. Let $\mathrm{Char}^{\mathcal{B}}(X)$ be a character variety (moduli of $\pi_1(X) \to GL_n$)

Is the set of pts of geo origin dense?!?

L-L prove that such a statement is false in general for low rank.

Notation C sm. proj. curve/$\mathbb{C}$ genus g

$x_1, \ldots, x_N$ distinct pts of C.

$U := C \setminus \{x_1, \ldots, x_N\}$

Thm (LL'1) Let $(C, x_1, \ldots, x_N)$ be analytically very general in $\mathcal{M}_{g,n}$. Let $\mathbb{L}$ be a local system of geo origin on U w/ ∞-monodromy. Then

$$\operatorname{rank} \mathbb{L} \geq 2\sqrt{g+1}$$

Slogans: "very general curve admits

NO low rank local systems of

geo origin"