Crystals & Isocrystals

Alternative perspectives on flat connections

Setup $S|k$ smooth, k perfect.

I. Via $S \times S$

Def $S(1) := \mathcal{O}_{S\times S}/\mathcal{J}^2$, where $\mathcal{J}$ is the ideal of $\Delta(S) \subseteq S \times S$.

Note that there are maps

$$\begin{tikzcd} & S(1) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ S & & S \end{tikzcd}$$

Claim Let $M \in QCoh(S)$.

① A connection ∇ on M is the same as the datum of an isomorphism

$$\varphi_{12} : p_1^* M \longrightarrow p_2^* M \quad / S(1)$$

② A connection ∇ on M is flat $\Longleftrightarrow$ φ satisfies a cocycle condition of the form $\boxed{\varphi_{23} \circ \varphi_{12} = \varphi_{13}}$

occuring on the scheme $(S, \mathcal{O}_{S\times S\times S}/J^2$, where J is the ideal of the Δ.)

③ $\mathrm{char}(k) = 0$ & ∇ is flat

$\Rightarrow$ the map φ_{12} may be extended over $\boxed{\widehat{S\times S}_{/\Delta}}$

formal completion of $S\times S$ along $\Delta(S)$

[More concretely, $\forall n$, φ_{12} lifts to $\tilde{\varphi}_{12} : p_1^* M \to p_2^* M$ over $(S, \mathcal{O}_{S\times S}/J^n)$]

Rmk ① "follows" from the fact that

$$\Omega^1_{S/k} \simeq \mathcal{J}/\mathcal{J}^2$$

③ • Will use a Taylor formula and is NOT TRUE in char p!

• What is true is that φ_{12} lifts $\tilde{\varphi}_{12}$ over $(S, \mathcal{O}_{S\times S}/\mathcal{J}^{p-1})$.

If the p-curvature vanishes, it in fact lifts to $(S, \mathcal{O}_{S\times S}/\mathcal{J}^{p})$

• When is ③ true in char p?

when M is "infinitely Frobenius divisible" $\iff$ M admits an action of Grothendieck's ring of differential operators.

("M is a stratified sheaf")

(II) Via $\mathrm{S}_{\inf}$

no relation to S

Def $S_{\inf}$ is the following site:

- objects are $\left(\begin{tikzcd} U \arrow[r, hook] \arrow[dr] & T \arrow[d] \\ & k \end{tikzcd}\right)$ where $U \subseteq S$ open subset, and $U \hookrightarrow T$ is a nil immersion
- morphisms are commuting squares
- covers are determined by the "Zariski topology on T":

$$\coprod (U_i \to T_i) \longrightarrow (U \to T)$$

is a cover $\iff$ $\coprod T_i \to T$ is a cover.

The site $S_{\inf}$ has a natural sheaf of rings:

$$\mathcal{O}_{\inf}(U \hookrightarrow T) := \mathcal{O}_T(T)$$

Fact ① Good sheaf thy on S_{inf}
$\rightsquigarrow$ good sheaf cohomology theory [not obvious! no final object!!]

② $\mathbb{Q} \subseteq k \implies$

$$\boxed{H^i(S_{inf}, \mathcal{O}_{inf}) = H^i_{dR}(S/k)}$$

Def A crystal (of coherent $\mathcal{O}_{inf}$-modules) on S_{inf} is

a sheaf $\mathcal{F}$ of coherent $\mathcal{O}_T$-modules on S_{inf}, s.t.

$\bigstar$ $\forall f : (U' \hookrightarrow T') \to (U \hookrightarrow T)$ in S_{inf},

the natural map

$$c_f : f^* \mathcal{F}(U \hookrightarrow T) \to \mathcal{F}(U' \hookrightarrow T')$$

(from the definition of a presheaf)

is an isomorphism.

Fact If $k \subseteq \mathbb{Q}$, then crystals on S_{inf} are equivalent to $\text{MIC}(S/k)$

SLOGAN: A flat connection is the data of "canonical parallel transport for every infinitesimal thickening".

III. Crystalline Site (and Variants)

Def Let A be a commutative ring, and let $I \subseteq A$ be an ideal. A divided power structure on I is a collection of maps:

$$\left(\gamma_n : I \longrightarrow I\right)_{n \geq 1} \text{ w/}$$

the following properties

① $\gamma_1(x) = x$ $\quad$ [set $\gamma_0(x) = 1$]

② $\gamma_n(x)\,\gamma_m(x) = \dfrac{(n+m)!}{n!\,m!}\,\gamma_{n+m}(x)$

③ $\gamma_n(ax) = a^n\,\gamma_n(x)$

④ $\gamma_n(x+y) = \displaystyle\sum_{i=0}^{n} \gamma_i(x)\,\gamma_{n-i}(x)$

⑤ $\gamma_n(\gamma_m(x)) = \dfrac{(nm)!}{n\,(m!)^n}\,\gamma_{nm}(x)$

Rmk • These abstract the properties of the function $x \longmapsto \dfrac{x^n}{n!}$

• Both blue circled expressions are integers.

<u>Def</u> Let $S|k$ be smooth over a perfect k of char $p>0$.

S_{crys} is the following site:

- Objects: $(U \hookrightarrow T, (\gamma)_m)$ where $U \subseteq S$, $U \hookrightarrow T$ nilimmersion, $(\gamma)_m$ PD structure on the ideal sheaf
- morphisms: commute w/ γ
- coverings: Zariski on T.

S_{Ncrys} is the following full subcat of S_{crys}, where $(\gamma)_m$ must be nilpotent: $\forall\ (U \hookrightarrow T, (\gamma))$, $\exists\ M$ s.t.

$$\gamma_m = 0 \text{ for } m > M.$$

Examples

- $S = \operatorname{Spec} \mathbb{F}_p$.

  $U = S$, $T = \operatorname{Spec}(\mathbb{Z}/p^n\mathbb{Z})$,

  $$\gamma : (p) \longrightarrow (p)$$

  $$\gamma_m \quad x \longmapsto \frac{x^m}{m!}$$

  is in S_{crys}.

  if $p > 2$, also in S_{Ncrys}.

- $U = S$, $T = \operatorname{Spec}(\mathbb{F}_p[t]/t^{p-1})$

  $$\gamma : (t) \longrightarrow (t)$$

  $\gamma_p(t) = t$ induces an object of S_{crys} <u>NOT</u> in S_{Ncrys}.

  $\left(\gamma_m(t) = \frac{t^m}{m!} \text{ for } m < p \right)$

- Any square-zero thickening has canonical divided powers.

=

Let $\tilde{S}/W(k)$ be a p-adic formal lift of S/k. (Always exists if S smooth & affine.) Then

flat

$$\left\{\begin{array}{c}\text{crystals on} \\ S_{crys}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}(\tilde{M}, \tilde{\nabla}) \text{ on} \\ \tilde{S}, \text{ s.t.}\end{array}\right\}$$

∇ is top. quasi-nilpotent

" $\nabla_{\partial_1} \circ \cdots \circ \nabla_{\partial_n} \to 0$ " as $n \to \infty$

$\Updownarrow$

$(\tilde{M}, \tilde{\nabla})$ mod p has nilpotent p-curvature.

$$\left\{\begin{matrix}\text{crystals on}\\ S_{N\,crys}\end{matrix}\right\} \hookleftarrow\!\!\rightarrow \underbrace{\left\{\begin{matrix}(\tilde{M}, \nabla) \text{ on}\\ \tilde{S}\end{matrix}\right\}}_{\text{No nilpotence condition.}}$$

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Now, by "functoriality of the crystalline topos", for any sheaf $\mathcal{F}$ on either S_{crys} or $S_{N\,crys}$, $\exists$ "$\mathrm{Frob}_S^* \mathcal{F}$". If $\mathcal{F}$ is a crystal, so is $\mathrm{Frob}_S^* \mathcal{F}$.

<u>Rmk</u> This is remarkable and surprising! Let $C/\mathbb{F}_p$ be a proper hyperbolic curve. Let $\tilde{C}/\mathbb{Z}_p$ be

a lift. Then

$$\left\{ \begin{array}{c} \text{crystals on} \\ C_{crys} \end{array} \right\} \longleftrightarrow \underbrace{\left\{ \begin{array}{c} (\tilde{M}, \tilde{\nabla}) \text{ on} \\ \tilde{C} \quad + \text{ top.} \\ \text{quasi-nilpotent} \end{array} \right\}}$$

has a "Frobenius pullback" even when $\tilde{C}$ has no global Frobenius

Even more surprising for crystals on $C_{N crys}$,

<u>Idea</u>: For an affine $U \subseteq C$, the p-adic formal schem $\tilde{U}$ has a (non-canonical) Frobenius lift.

prove the resulting Frobenius pullback is "independent of choice of $\widetilde{\mathrm{Frob}}$"

by Taylor formula $\Rightarrow$ glues.

See Esnault's lecture notes, section 8.

Def S/k, k perfect of char p

- An F-crystal is a pair (M, F), where

  M is a crystal on S_{cry},

  $F : \mathrm{Frob}_S^* M \longrightarrow M$ is an isogeny

- The Tate F-crystal, $\mathbb{Z}_p(-1)$, is the pair

$$(\mathcal{O}_{crys}, p)$$

- The category $F\text{-}Isoc(S)$ is obtained from $F\text{-}Crys(S)$ by
  - $\otimes$ Hom spaces w/ $\mathbb{Q}_p$
  - $\otimes$ invert $\mathbb{Z}_p(-1)$.

Slogan F-Isocrystals are the (rational) output of crystalline cohomology (resp. rigid cohomology).

$\rightsquigarrow$ they are "p-adic cousins" of lisse $\mathbb{Q}_\ell$ - sheaves.

<u>Thm</u> (Dwork's trick)

Let (M,F) be an F-crystal on $S_{N\text{crys}}$. Then (M,F) canonically extends to an F-crystal on S_{crys}

Idea of Pf: Frobenius structure forces $\tilde{\nabla}$ to be topologically quasi-nilpotent!