

Introduction to P-curvature.

Recall setup from the end of last lecture:

- S/k smooth, $\text{char}(k) = p$.

$$\begin{array}{ccccc}
 S & \xrightarrow{\quad} & S' & \xrightarrow{\pi} & S \\
 \downarrow F_{S/k} & \nearrow & \downarrow & & \downarrow \\
 \text{Spec}(k) & \xrightarrow{\quad} & \text{Spec}(k) & \xrightarrow{\quad} & \text{Spec}(k)
 \end{array}$$

$F_{\text{rb}, k}$

Observation: for any $E \in \text{coh}(S')$, \exists

a "canonical" connection on $F_{S/k}^*(E) \in \text{coh}(S)$

$$\text{Indeed, } F_{S/k}^*(E) \cong F_{S/k}^{-1}(E) \otimes \begin{cases} \mathcal{O}_S \\ F_{S/k}^{-1}(\mathcal{O}_{S'}) \end{cases}$$

$$\nabla^{\text{can}}(s \otimes f) := s \otimes df$$

Why is this well-defined?

Let $\gamma \in \Omega_S$. Then

$$\nabla^{\text{can}}(\gamma s \otimes f) := \gamma s \otimes df$$

???

$$\nabla^{\text{can}}(s \otimes F_{S/k}^*(\gamma)f) := s \otimes d(F_{S/k}^*(\gamma)f)$$

???

$$s \otimes df \checkmark$$

- Rmk In char 0, the following are true.
- $(E, \nabla) \in \text{MIC}(S/k) \Rightarrow E$ locally free
 - S/k proper, $\Rightarrow c_1(E) = 0$
 - $(E, \nabla) \in \text{MIC}(S/k)$

Both are false in char p .

Upshot If S/k is smooth and $\text{char}(k) = p$, then \exists a "canonical" subcategory of $\text{MIC}(S/k)$.

Questions

- How do we describe this subcategory "intrinsically"? (i.e., when is an object of $\text{MIC}(S/k)$ a Frobenius pullback?)
- What are properties of this abelian subcategory? E.g., is it thick?

To answer these questions, we return to the ring of crystalline differential operators.

The Ring D_S

Learned from
M. Gröchenig

Setup • S/k smooth,

- If $\mathcal{Q} \subseteq k$, say has "char ∞ "

Def The ring D_S of crystalline differential operators is the quasi-coherent sheaf of D_S algs, described as the following quotient:

$$\bigoplus_{i \geq 0} (T_S^{\otimes i}) / \mathfrak{J}$$

$T_S \otimes_k T_S \otimes \dots \otimes T_S'$

where \mathfrak{J} is the 2-sided ideal generated by relations:

- $\partial \otimes f - f \otimes \partial - (\partial f)$
- $\partial_1 \otimes \partial_2 - \partial_2 \otimes \partial_1 - [\partial_1, \partial_2]$

Rmk

- D_S is "universal enveloping algebra" of the Lie algebroid $T\mathcal{S}|_k$
- D_S is a filtered algebra:
$$D_S^{\leq d} := \text{Im} \left(\bigoplus_{i \leq d} T_S^{\otimes i} \rightarrow D_S \right)$$
- $\text{MIC}_S =$ " O_S -coherent D_S modules"

slogan: a flat connection is equivalent to the data of a D_S -action

Ex let $S = A'_k = \text{Spec}(k[t])$.

Then D_S is generated, as an algebra $| k[+]$, by the symbols $t, \frac{\partial}{\partial t}$

w/ the relation $[\frac{\partial}{\partial t}, t] = 1$.

"Weyl alg": $k\langle t, \partial \rangle$. Note when $\text{char}(k) = p$, $\mathcal{Z}(k\langle t, \partial \rangle) = k[t^p, \partial^p]$.

$$\underline{\text{Obs}} \quad D_S \rightarrow \underline{\text{End}}_k(\Omega_S)$$

Q: When is this faithful?

Lemma 1

$$\text{The map } D_S \xrightarrow{\leq \text{char}(k)-1} \underline{\text{End}}_k(\Omega_S)$$

is faithful

Pf

- First, consider the case that

$$S = \mathbb{A}_{\mathbb{K}}^n = \text{Spec } \mathbb{K}[t_1, \dots, t_n].$$

Then we claim:

$$\text{Let } s \in D_{\mathbb{A}^n}^{\leq d}, w \quad d \leq \text{char}(k)-1.$$

Then $\exists f \in \mathbb{K}[t_1, \dots, t_n]$ of $\deg \leq d$
s.t. $sf \neq 0$.

Pf: Induction: $s = s_{\leq d} + s_d$,

where $s_{\leq d}$ is in D_S

$$\text{and } \delta_d = \sum_{|\mathbf{j}|=d} a_{\mathbf{j}} \partial^{\mathbf{j}}$$

$$\frac{\partial^d}{\partial t_1^{j_1} \partial t_2^{j_2} \cdots \partial t_n^{j_n}}$$

If $\delta_{d-1} \neq 0$, then done by induction.

Else, $\delta = \delta_d$. Suppose

$$k[t_1, \dots, t_n] \ni a_j = a_{j_1, \dots, j_n} \neq 0.$$

Then $f = \prod_{i=1}^n t_i^{j_i}$ is of deg d ,

$$\text{and } \delta f = \delta_d f = j_1! \cdots j_n! a_j$$

$$\neq 0 \quad \square$$

In general, use étale descent to reduce to this case.

\square

$$D_S \longrightarrow \text{gr } D_S$$

\hookrightarrow "Symbol(δ)"

Thm (PBW)

There is an iso, natural in pullback
along étale map,

$$\text{gr}(D_S) \simeq \text{Sym } T_S = \pi_* \underbrace{\Omega_{T^*S}},$$

$$\pi: T^*S \rightarrow S$$

Idea of Pf

"Étale coordinates" \Rightarrow reduce to (open
subset of) A^n

$$\delta \in D_{A^n}^{\leq 2}$$

$$\delta = \sum_{i=0}^2 \left(\sum_{|J|=i} a_J \partial^J \right),$$

w/ $a_J \in k[t_1, \dots, t_n]$

$$\cdot \delta \mapsto \sum_{|J|=d} a_J y^J \in \text{Sym}^d(T_S)$$

where " $y_i = \partial_i$ " is the corresponding vector field, gives a map

$$D_{A^n}^d / D_{A^n}^{d-1} \rightarrow \text{Sym}^d(T_{A^n})$$

$$\sim D_{A^n} \longrightarrow \text{Sym}^d(T_{A^n})$$

=

Recall that $T^* S$ is canonically "symplectic": there exists a closed, non-degen 2-form ω :

$$\omega = d\gamma, \text{ where } \gamma \text{ is the}$$

tautological 1-form on T^*S :

$$p : T(T^*S) \rightarrow T^*S$$

α_s

Given $\xi_{\alpha_s} \in T_{\alpha_s}(T^*S)$

η is determined by the tautological

$$\text{map: } T_{\alpha_s}(T^*_s S) \rightarrow k$$

$$\xi_{\alpha_s} \longmapsto \alpha_s(p_* \xi_{\alpha_s})$$

=

Given (local) function $f \in \Omega_s$,

ω induces a "Hamiltonian vector field"

ξ_f :

$$\omega : T^*S \xrightarrow{\sim} TS \quad (\text{non deg})$$

$$df \mapsto X_f$$

Then ω induces a Poisson bracket

on $\pi \times \Omega_{T^*S}$

- bilinear
- skewsymmetric
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
- Leibniz: $\{fg, h\} = f\{g, h\} - g\{f, h\}$

$$\cancel{\{f, g\}_\omega := \omega(X_f, X_g)}$$

"Example"

$$S = /A^*_k = \text{Spec}(k[t_1, \dots, t_n])$$

$$T^*S = \text{Spec}(k[t_1, \dots, t_n, y_1, \dots, y_n])$$

$$"y_i = \frac{\partial}{\partial t_i}"$$

$$\cdot \omega = \sum dt_i \wedge dy_i$$

$$\cdot \{f, g\} = \sum \frac{\partial f}{\partial t_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial t_i}$$

Lemma 3

Let $s_1, t \in D_s^{\leq d_1}, s_2 \in D_s^{\leq d_2}$
 $\leq d_1 + d_2 - 1$

Then $[s_1, s_2] \in D_s$

Symbol $\sigma[s_1, s_2]$
 $= \{\sigma(s_1), \sigma(s_2)\}$

here, $\sigma[s_1, s_2]$ means: $d_1 + d_2 - 2$
 $[s_1, s_2] \bmod D$

Idea of P: Coordinates

Lemma 3

Let $f \in \pi_* \Omega_{T^* S} = \text{Sym}^\alpha T_S$
 be a local section s.t. $\{f, -\} = 0$
 (i.e., f is in the Poisson center).

Then $\exists g \in \pi_* \Omega_{T^* S'}$ n Frobenius twist
 s.t. $f = F_{S/k}^* g$.

Pf (coordinates): $S = /A^1_K$,

$$T^*/A^1_K = \text{Spec}(k[t_1, \dots, t_n, y_1, \dots, y_n])$$

Suppose $f \in k[t_1, \dots, t_n, y_1, \dots, y_n]$ has
 at least one exponent that is not a
 p^{th} power. Then $\exists i$ s.t.

$$\frac{\partial f}{\partial t_i} \quad \underline{\text{OR}} \quad \frac{\partial f}{\partial y_i} \quad \text{is non-zero.}$$

$$\text{If } \frac{\partial f}{\partial t_i} \neq 0, \quad \{f, y_i\} \neq 0$$

$$\frac{\partial f}{\partial t_i} \frac{\partial y_i}{\partial y_j} - \frac{\partial y_i}{\partial t_i} \frac{\partial f}{\partial y_j}$$

$$\text{If } \frac{\partial f}{\partial y_i} \neq 0, \quad \{f, t_i\} \neq 0$$

Hence: $\{f, -y_i\} = 0 \Rightarrow$ all
 exponents are multiples of $p.$

Birth of p -curvature

Notation: let ∂ be a local section of $T_S.$

Then $\underbrace{\partial \circ \dots \circ \partial}_P$ is again a $\underbrace{\text{derivation}}_{\text{by binomial theorem}}$

$$\partial_S \rightarrow \partial_S$$

\rightsquigarrow corresponds to a vector field,
 $\partial^{[p]}$

$$\text{Def} \circ \iota : T_S \rightarrow D_S^{\leq p}$$

$$\partial \mapsto \underbrace{\partial}_\text{order p}^p - \underbrace{\partial}_\text{order 1}^{[p]}$$

, If $(E, \nabla) \in \text{MIC}(S/k)$,

$$\sim \cap E$$

$$D_S$$

$\psi_{\nabla}(\partial) := \text{action of } ((\partial)) \text{ on } E$

is the p -curvature.

A priori, ι is just a map of sheaves.

Example $S = \mathbb{A}_{F_2}^1 = \text{Spec}(F_2[t])$

$$\partial = t \frac{\partial}{\partial t} \quad \partial^{[a]} : \mathcal{O}_S \rightarrow \mathcal{O}_S$$

$$f \mapsto t \frac{\partial}{\partial t} (t \frac{\partial f}{\partial t})$$

$$= t \frac{\partial f}{\partial t}$$

$$\begin{aligned}\partial^2 &= t \frac{\partial}{\partial t} \left(t \frac{\partial}{\partial t} \right) = t \left(\left(1 + t \cdot \frac{\partial}{\partial t} \right) \frac{\partial}{\partial t} \right) \\ &= t \frac{\partial}{\partial t} + t^2 \frac{\partial^2}{\partial t^2}\end{aligned}$$

$$((\partial)) = t^2 \frac{\partial^2}{\partial t^2} \in D_{A' F_2}^{\leq 2}$$

Lemma 4 $\iota: T_S \rightarrow D_S^{\leq p}$ is

p -linear:

$$\iota(f \partial_1 + \partial_2) = f^p \iota(\partial_1) + \iota(\partial_2)$$

Pf

Consider the symbols

$$\sigma(\iota(f \partial_1 + \partial_2)) = (f \partial_1 + \partial_2)^p$$

$$\sigma(f^p \iota(\partial_1) + \iota(\partial_2)) = f^p \partial_1^p + \partial_2^p$$

(by RHS is in a
commutative ring!)

$$\Rightarrow \iota(f \partial_1 + \partial_2) - f^p \iota(\partial_1) - \iota(\partial_2) \in D_S^{\leq p-1}$$

Moreover, for any $g \in \mathcal{O}_S$,

$$\begin{aligned} L(f\partial_1 + \partial_2)(g) &= (f^p L(\partial_1) + L(\partial_2)) g \\ &= 0! \end{aligned}$$

[Lemma 1 \Rightarrow \square]

$\rightsquigarrow L$ induces \mathcal{O}_S -linear map $Frob_S^* T_S \rightarrow D_S$,

which induces:

- an \mathcal{O}_S -linear map

$$L: F_{S/k}^* T_{S'} \rightarrow D_S,$$

- an $\mathcal{O}_{S'}$ -linear map

$$L: T_{S'} \rightarrow (F_{S/k})_* D_S$$

Lemma 5 : Let σ be a local section
of T_S . Then

$$(\sigma) \in Z((F_{S/k})_*, D_S)$$

Pf let $s \in D_S$.

$$\sigma([(\sigma), s])$$

$$= \{ \sigma((\sigma)), \sigma(s) \}$$

$$= \{ (\sigma(\sigma))^p, \sigma(s) \}$$

$$(\sigma((\sigma))) = \sigma(\sigma^p) = \sigma(\sigma)^p$$

b/c σ has values in
a commutative ring.)

$$= 0$$

$$\Rightarrow [(\sigma), s] = 0$$

As this is true for all s ,

$$\Rightarrow L(\partial) \in Z(F_{S/k} \rtimes D_S)$$

□

~) L induces a map

$$\tilde{\iota}: \text{Sym}^* T_S \xrightarrow{\sim} Z(F_{S/k} \rtimes D_S)$$

Lemma 6 $\tilde{\iota}$ is an isomorphism.

$$\text{Sym}^* T_S \rightarrow Z(F_{S/k} \rtimes D_S)$$

A commutative diagram with four nodes:

- Top-left: $\text{Sym}^* T_S$
- Top-right: $Z(F_{S/k} \rtimes D_S)$
- Bottom-left: $F_{T_S/k}^*$
- Bottom-right: $F_{S/k} \rtimes \text{Sym}^* T_S$

 Arrows:

- Vertical arrow from top-left to top-right.
- Vertical arrow from bottom-left to bottom-right.
- Horizontal arrow from top-left to bottom-right.
- Diagonal arrow from top-left to bottom-right, passing through the bottom-left node.
- Downward arrow labeled σ from top-right to bottom-right.

$$T_S \xrightarrow{F_{T_S/k}} (T_S)'$$

~) $\tilde{\iota}$ is injective,

- image is the Poisson Center of $\text{Sym}^* T_S$ (Lemma 3)

• Lemma 2 :

$$\text{Poisson center} \left(\text{Sym}^* T_S \right) \\ = \mathcal{G}(\mathcal{Z}(D_S))$$

$\Rightarrow \tilde{\iota}$ is an isomorphism.

□.

Lemmas 485 elegantly reprove
"classical" facts about the p-curvature.

Let S/k be a smooth var., $\text{char}(k) = p$.

Theorem 7

$$(E, \sigma) \in \text{MIC}(S/k)$$

$$\sim \psi_\sigma : F_{S/k}^* T_{S'} \rightarrow \text{End}_{\mathcal{O}_S}(E) \subseteq \text{End}_k(E)$$

Pf Lemma 5: $\{(\sigma), f\} = 0$

Thm 8 ψ_σ induces a map:

$$\psi: E \rightarrow E \otimes F_{S/k}^* \Omega^1_{S'/k}$$

Then this map \rightarrow flat (here,

$F_{S/k}^* \Omega^1_{S'/k}$ is equipped w/ the canonical connection.)

Pf Lemma 5: $[c(\partial), \partial'] = 0 \cdot \square$

To see the utility of Lem 6, we need:

Lemma 6) The ring $F_{S/k} \rtimes D_S$
is an Azumaya algebra over its
center $\text{Sym}^2 T_S \xrightarrow{\sim} Z(F_{S/k} \rtimes D_S)$
 \uparrow derived from the p-curvature.

"Explanation" of Lemma 6' for A'_k

$$\text{rank } p^2 \left\{ \begin{array}{l} k[t, \partial] \\ \text{UI} \\ k[t^p, y^p] \end{array} \right. \quad \begin{array}{l} w \cdot \partial = y^p \\ \cdot [\partial, \cdot] = 1 \end{array}$$

We claim this is an Azumaya alg.

\Leftrightarrow it after a flat cover

$$\begin{array}{ccc} k[t, \partial] & \longrightarrow & k[t, \partial] \otimes_{k[t^p, y^p]} k[s, y^p] \\ \text{UI} & & \uparrow \\ k[t^p, y^p] & \longrightarrow & k[s, y^p] \end{array}$$

Key Claim:

$$k[t, \partial] \otimes_{k[t^p, y^p]} k[s, y^p] \xrightarrow[p \times p]{\sim} M(k[t, y^p])$$

$$k[s, y^p] \otimes_{k[t^p, y^p]} k[t, \partial] \rightarrow E_1$$

mult on left,
s acts as t.

↑ ↑

mult on right

k[t, \partial]

↑

as a module

Rank In fact, the Azumaya α^{lg}
 $(F_{S/k})_a D_S$ over $T^a S'$

splits over the zero section.

Concretely: $S = M^1_{\mathbb{K}}$

Restatement $k[t, \alpha] / \left(\alpha^p = 0, [\alpha, t] = 1 \right)$
 U_1
 $k[t^\circ]$

isomorphic to $M_{p \times p}(k[t^\circ])$

Abstract pf: $(F_{S/k})_a D_S$ is Azumaya/
 $T^a S'$ of rank p^{2n} . On the other
 hand, $(F_{S/k})_a$ has rank $p^n \mid s'$,
 and has an action of $(F_{S/k})_a D_S$ (from α)

$\Rightarrow (F_{S/k})_a D_S \mid$ zero section $\xrightarrow{\text{has action}}$
 on rank p^n vector bundle

$\Rightarrow (F_{S/k})_a D_S$ splits.

In other words,

$$(F_{S|k})_* D_S \xrightarrow{N} \text{End}(F_{S|k} \otimes_s)$$

zero section
 $s' \hookrightarrow T^* s'$

In coordinates,

$$k\langle t, \partial \rangle / \begin{matrix} \partial^p = 0 \\ [\partial, t] = 1 \end{matrix} \xrightarrow{N} \text{End}_{k[t^p]} k[1]$$

Thm (Cartier Descent)

$$\text{Coh}(S'/k) \longrightarrow \text{MIC}(S/k)$$

$$M \longmapsto (F_{S/k}^* M, \nabla^{(n)})$$

has essential image precisely those

$$(E, \nabla) \text{ s.t. } \Psi_\nabla = 0.$$

"Pf" $(E, \nabla) \rightsquigarrow E$ is a D_S -module

Monta theory:

$F_{S/k}^* D_S$ is Azumaya over $\mathcal{O}_{T^* S'}$

\Rightarrow
Morita: $\text{Coh}(S, D_S) \longrightarrow \text{Coh}(T^* S', F_{S/k}^* D_S)$
equivalence

$$\pi_2(\text{Monta}(E)) \simeq F_{S/k}^* E$$

$$\text{in } \text{Coh}(S')$$

- ψ_\triangleright induces an action of
 $Z(F_{S/k} \star D_S) \cong \bar{\pi}_*^1 \mathcal{O}_{T^* S'}$
on $F_{S/k} \star E$.
 $\psi_\triangleright \equiv 0 \Leftrightarrow$ Morita (E) supported
on the zero-section
 $S' \hookrightarrow T^* S'$
 $(\zeta(\alpha) = 0 \wedge \alpha \in T_S)$
- $F_{S/k} \star D_S$ split over zero-
section $S' \hookrightarrow T^* S'$
(induced by $(0, \iota)$).

May use these two facts to
construct $E'|_{S'}$ s.t.
 $E'^{\nabla=0} \xrightarrow{\sim} E'$