The work of Landesman-Litt

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1 Introduction

Let $X/\mathbb{C}$ be a smooth, connected variety.

Definition 1.1. *Let A be a commutative ring. An A-local system on X is a locally constant sheaf of finite free A-modules on X^{an}.*

The following is well-known: let $\bar{x} \in X$. An A-local system L of rank N on X is then equivalent to a homomorphism: $\rho\colon \pi_1(X^{\mathrm{an}}, \bar{x}) \to \mathrm{GL}_N(A)$; this is colloquially called the *monodromy representation*. We say two local systems ρ, ρ' are isomorphic if there exists an invertible matrix M such that $\rho = \mathrm{Ad}_M \circ \rho'$, i.e., if the two representations are conjugate by M.

Definition 1.2. *Let $X/\mathbb{C}$ be a smooth, connected variety. Let L be a $\mathbb{C}$-local system on X. We say that L is* of geometric origin *(or geometric, or motivic) if threre exists an open dense $U \subset X$, a smooth projective morphism $f\colon \mathcal{Y} \to U$, and an integer $i \geq 0$ such that $L|_U$ is a subquotient of:*

$$R^i f_*(\mathbb{C})$$

.

Let L be a local system of geometric origin. Then the following properties hold. (The first three are due to Deligne, the fourth is due independently to Brieskorn, Borel, Grothendieck, and Landman.)

- L is semi-simple. Therefore, in the definition, we may replace the word "sub-quotient" by the word "summand".
- L is integral, i.e., there exists a number field K such that the associated representation factors:

  $$\rho\colon \pi_1(X) \to GL_N(\mathcal{O}_K) \to GL_N(\mathbb{C}).$$

  In particular, the traces of $\gamma \in \pi_1(X)$ on L are algebraic integers.
- L underlies a complex polarized variation of Hodge structures (CPVHS).
- L has quasi-unipotent monodromy around ∞.
- $\det L$ has finite order.

There are of course many more properties that local systems of geometric origin enjoy. The purpose of this seminar is roughly to understand how common/rare local systems of geometric origin are.

2 Some arithmetic motivation

There are analogous notions of "local systems of geometric origin" for varieties over any field. In the case of a base field being finite, we have the following striking result, which we state somewhat imprecisely below.

Theorem 2.1. *Let $U/\mathbb{F}_q$ be a smooth curve over a finite field.*

- *Let ρ be an irreducible $\overline{\mathbb{Q}}_\ell$-local system on U with finite order determinant. Then ρ is of geometric origin.*
- *Let $\bar{\varphi}$ be an $\mathbb{F}_{\ell^r}$ local system on U, where ℓ is prime to p. Suppose that*

$$\bar{\varphi}^{geo} := \bar{\varphi}|_{\pi_1(U_{\bar{\mathbb{F}}_q})}$$

is absolutely irreducible, i.e., that $\bar{\varphi}$ is geometrically absolutely irreducible. Then in the rigid generic fiber of the deformation space Def($\bar{\varphi}^{geo}$) of $\bar{\varphi}^{geo}$, the set of those representations of geometric origin is Zariski dense.

The proof of this theorem involves Gaitsgory's solution [G07] to a conjecture of de Jong [dJ01] as well as the *proof* (and not merely the statement) of the Langlands' conjectures for function fields due to Lafforgue. We have the following striking corollary.

Corollary 2.2. *Let $U/\bar{\mathbb{F}}_p$ be a smooth, connected, hyperbolic curve. Let $r \geq 1$. Then there are infinitely many irreducible local systems of rank r with finite order determinant on U that have geometric origin. This holds even when we bound the ramification at ∞ and fix the profile of the local monodromy around the cusps.*

(As a cute example, every affine hyperbolic curve over a finite field of odd characteristic admits a modular embedding, and in particular a local system of rank 2 of geometric origin.)[1]

Remark 2.3. *Here are two interesting applications of the aforementioned density in Theorem 2.1.*

- *Drinfeld has shown a hard Lefschetz theorem and a semi-simplicity theorem for direct images of perverse sheaves in characteristic 0 by reduction to characteristic p [Dri01]. This was later proven with analytic techniques.*
- *de Jong-Esnault provide an arithmetic proof of the following theorem of Mochizuki: if $f\colon X \to Y$ is an algebraic map of complex, quasi-projective varieties, and L_Y is a semi-simple $\mathbb{C}$ local system on Y, then $L_X := f^*L_Y$ is a semi-simple $\mathbb{C}$ local system on X [dJE22]. Before this recent work, the only known proof of this involved the construction of a tame harmonic metric.*

As is well known, varieties over $\bar{\mathbb{F}}_p$ tend to be "special". For instance, every abelian variety over $\bar{\mathbb{F}}_p$ has complex multiplication, while most abelian varieties over $\mathbb{C}$ do not have CM. Similarly, most affine curves over $\mathbb{C}$ do not admit modular embeddings. (Indeed, both CM abelian varieties and curves that admit modular embeddings are defined over $\bar{\mathbb{Q}}$.) It is therefore natural to wonder whether some analog of Theorem 2.1 holds over more general base fields k.

The first part of Theorem 2.1 does not hold over $\mathbb{C}$: it is easy to write down local systems on curves that are not of geometric origin. (For several new obstructions, we refer the reader to [L18].) However, the question of "how many" local systems of geometric origins there are on a given curve U remains.

Question 2.4. *Let $U/\mathbb{C}$ be a smooth, connected curve and fix $r \geq 2$. Are there are infinitely many $\mathbb{C}$-local systems of rank r on U of geometric origin? Are they perhaps even Zariski dense in the appropriate representation variety?*

This question was indeed recently posed by Esnault-Kerz, with several cohomological applications in mind [EK21a]. (See also [EK20b].)

3 Main Results

The recent work of Landesman-Litt shows that the answer to Question 2.4 is "no", first for an analytically very general curve over $\mathbb{C}$, and then using this, for a "generic" curve. We emphasize that Question 2.4 could still hold for generic curves and *high rank* local systems.

[1] The proof of this is based on the canonical lift of ordinary curves of Mochizuki, reinterpreted by Lan-Sheng-Zuo in terms of p-adic nonabelian Hodge theory.

Definition 3.1. *A pair $(C, x_1, \ldots, x_n)$, where C is a smooth projective curve of genus g and the x_i are distinct points of $\mathbb{C}$, is* hyperbolic *if $2g - 2 + n > 0$.*

Theorem 3.2. *[LL22b, Corollary 1.2.7] Let $(C, x_1, \ldots, x_n)$ be an analytically very general hyperbolic n-pointed curve, where C is a smooth projective with genus g. Let U be the open complement. Let L be a local system on U of geometric origin. If L has infinite monodromy, then the rank of L must be at least $2\sqrt{g+1}$.*

In other words, for every (g, n), with $2g - 2 + n > 0$, there exists an n-pointed genus g curve C that has *no* low rank local systems of geometric origin that have infinite monodromy. In fact, Theorem 3.2 holds in somewhat greater generality; they show the result for those representations $\rho\colon \pi_1(U) \to GL_N(\mathcal{O}_K)$ such that for every $\iota\colon K \hookrightarrow \mathbb{C}$, the induced complex representation $\iota \circ \rho$ underlies a $\mathbb{C}$-PVHS.

We now setup the second main result we will discuss. Let $\Sigma_{g,n}$ denote the (orientable) topological space underlying an n-punctured genus g Riemann surface. We set the mapping class group $\mathrm{MCG}_{g,n}$ to be the quotient:

$$\mathrm{Homeo}^{\mathrm{oriented}}(\Sigma_{g,n})/(\text{isotopies preserving the punctures}).$$

Fixing a base point $x \in \Sigma_{g,n}$, there is a natural *outer* action of $\mathrm{MCG}_{g,n}$ on $\pi_1(\Sigma_{g,n}, x)$ and hence there is an induced action of $\mathrm{MCG}_{g,n}$ on the character variety of $\pi_1(\Sigma_{g,n}, x)$. (By this, we mean the moduli space parametrizing *conjugacy classes* of homomorphisms $\pi_1(\Sigma_{g,n}, x) \to GL_N(\mathbb{C})$.)[2]

Definition 3.3. *A representation $\rho\colon \pi_1(\Sigma_{g,n}, x) \to GL_N(\mathbb{C})$ is said to be* MCG-finite*, or a* canonical representation*, if the orbit of $[\rho]$ under $MCG_{g,n}$ is a finite set.*

MCG-finite representations are also called canonical because they morally extend to local systems on the universal punctured curve over $\mathscr{M}_{g,n}$. (This is not quite true on the nose, but it is close to being true: the group $\mathrm{MCG}_{g,n}$ is close to being the fundamental group of $\mathscr{M}_{g,n}$.) From now on, we suppress the basement. The main theorem is now the following

Theorem 3.4. *[LL22c, Theorem 1.2.1] Let $\rho\colon \pi_1(\Sigma_{g,n}) \to GL_r(\mathbb{C})$ be an MCG-finite representation. If $r < \sqrt{g+1}$, then ρ has finite image.*

Theorem 3.4 may be phrased purely group theoretically. However, the proof relies on deep arithmetic work (ultimately relying on the Langlands correspondence for function fields) as well as deep analytic work (using non-abelian Hodge theory for non-compact curves).

Here is some motivation For 3.4. Given a representation, $\rho\colon \pi_1(\Sigma_{g,n}) \to GL_r(\mathbb{C})$, one can take the "monodromy of monodromy", i.e., the monodromy of the isomonodromic deformation. As is explained in [LL22d, Slogan 1.8], the general expectation is that monodromy groups should be big. Theorem 3.4 confirms some weak version of this for low rank. As is explained in [LL22d, p. 3], this motivation can also be used as motivation for the general Putnam-Wieland conjecture.

We emphasize, in both Theorems 3.2 and 3.4, the "low-rank" condition is absolutely necessary; using the Kodaira-Parshin trick, one can construct very high rank local systems over the generic curve; such local systems will yield MCG-finite representations.

We will now state an arithmetic corollary of Theorem 3.4. We first require a definition.

Definition 3.5. *Let X/K be a smooth, geometrically connected variety over a finitely generated field of characteristic 0. A continuous homomorphism:*

$$\rho\colon \pi_1(X_{\bar{K}}) \to GL_N(\overline{\mathbb{Q}}_\ell)$$

is said to be arithmetic *if there exists a finite extension K'/K and a representation $\rho'\colon \pi_1(X_{K'}) \to GL_N(\overline{\mathbb{Q}}_\ell)$ extending ρ.*

By a standard spreading-out argument, local systems of geometric origin are arithmetic. It has been conjectured that irreducible arithmetic local systems with algebraic determinant are of geometric origin, up to a (possibly fractional) Tate twist. (Sometimes, this is referred to as a relative Fontaine-Mazur conjecture.)

[2]Note that there is also an honest action $\mathrm{MCG}_{g,n+1}$ on $\pi_1(\Sigma_{g,n}, x)$.

Theorem 3.6. *[LL22c, Theorem 8.1.2] Let K be a finitely generated field of characteristic 0 and $(C, x_1, \ldots, x_n)$ an n-pointed hyperbolic curve over K of genus g with open complement U. Let $(C, x_1, \ldots, x_n)$ be general, i.e., such that the induced moduli map Spec$(K) \to \mathscr{M}_{g,n}$ is dominant. Suppose ρ is a continuous representation:*

$$\rho\colon \pi_1(U_{\bar{K}}) \to GL_N(\overline{\mathbb{Q}}_\ell)$$

If $N < \sqrt{g+1}$ and ρ is arithmetic, then ρ has finite monodromy.

In particular, Theorem 3.4 implies that if U/K is a generic curve (in the sense that the map to $\mathscr{M}_{g,n}$ is dominant), then there are *no* low-rank local systems of geometric origin on $U_{\bar{K}}$ with infinite monodromy.

We further add that the technique of [LL22c] provide techniques for showing that residual local systems (i.e., representations into $GL_N(\mathbb{F}_\ell)$) over generic curves are not of geometric origin.

We will discuss the proofs of Theorems 3.2, 3.4, and 3.6 under the further assumption that the local systems have *unipotent monodromy at* ∞. The proof of Theorem 3.2 in this case has been written up in [LL22a] and avoids working with parabolic structures. For Theorems 3.4 and 3.6, we will also assume that the representations are semi-simple.

4 Outline

1. Introduction (Raju).

2. **Goal**: Introduce the following objects carefully: the Atiyah bundle (of a filtered vector bundle), isomonodromic deformation (of a log flat connection).

   All references will be to [LL22a]. Give Definitions 2.1.1, 2.1.2, and 2.1.5. State Remarks 2.1.3 and 2.1.6. State Proposition 2.1.7 (and provide some intuition) and prove Proposition 2.1.8.

   State Lemma 2.2.2 (no proof necessary), explain Definitions 2.2.3, 2.2.4.

   State Proposition 2.3.5. Time permitting, state and give an indication of the proof of Proposition 2.3.8.

3. State and prove the non-GGG Lemma, [LL22b, Proposition 6.3.1] (or [LL22a, Proposition 5.1.3]).

   All references will be to [LL22a]. State and prove Proposition 5.1.3, a.k.a. the "not GGG lemma".

4. **Goal**: Prove Theorem 1.3.4 of [LL22a].

   Prove Theorem 1.3.4 (contained in section 5.2) (the summary in section 1.5 might be helpful).

5. **Goal**: Explain $\mathbb{C}$-PVHS, associated graded Higgs bundle, and attendant positivity properties. Prove Theorem 1.2.4.

   All references will be to [LL22a]. State Definitions 3.1.1 - 3.1.4. State Propositions 3.1.5 (or maybe I'll just print out copies as a handout). State 3.1.6 and "prove" Corollary 3.1.9.

   Prove Theorems 1.2.8, 1.2.4, and 1.2.6 (in Section 6).

6. **Goal**: Define the notion of a canonical representation (a.k.a. MCG-finiteness). Explain the relation with local systems on versal families. State the main theorem.

   All references to [LL22c]. Define MCG$_{g,n}$ and state Lemma 2.0.1. Explain Proposition 2.1.3. State Corollary 2.4.5 and explain the idea of the proof.

   State Theorem 1.2.1 and Corollary 1.3.1.

7. **Goal**: Explain the cohomology rank bound for a unitary local system on a versal family of curves.

   Prove [LL22c, Theorem 1.7.1] in the setting where the parabolic structure is trivial, using the non-GGG lemma [LL22b, Proposition 6.3.1]. Explain the idea behind [LL22c, Lemma 5.1.1]. Prove [LL22c, Lemma 5.2.1].

8. **Goal:** Prove (cohomological) rigidity of unitary canonical representations. Using the results of Esnault-Gröchenig and Klevdal-Patrikis on integrality, deduce integrality for unitary canonical representations. Deduce the main result in the case that the representation is unitary.

   All references to [LL22c]. Give Definition 7.1.1 (with caveat of Remark 7.1.2 for curves) and 7.1.4. State the implication Lemma 7.1.3. Prove 7.2.1. State 7.3.2 and sketch the proof of 7.3.3. State 7.3.4. State Lemma 3.3.2 and use it to prove 7.4.1.

9. **Goal:** Prove the main theorem in the case of semi-simple local systems. Sketch applications to arithmetic local systems.

   All references to [LL22c]. Prove Lemma 7.5.1. This might require explaining Theorem 5.2.1 (if there wasn't time earlier) and explaining enough of the theory of deformations of representations in section 2.2.

   State Theorem 8.12 and sketch the proof. In particular, explain how we move from arithmetic local systems over a generic curve to MCG-finite representations.

10. **Goal:** Explain applications to the Putnam-Wieland conjecture.

References

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