Global gole: $\rho : \pi_1(\Sigma_{g,n}) \longrightarrow GL_r(\mathbb{C})$,

ρ is MCG-finite $\left(\text{that is, } |MCG_{g,n} \cdot [\rho]| < \infty \quad [\rho] \in M_B^r(\Sigma_{g,n})\right)$

If $r < \sqrt{g+1}$, then $|\mathrm{Im}\,\rho| < \infty$

Set-up:

$\mathcal{C}/\mathcal{M}$ - versal family of (g,n)-curves

$$\begin{tikzcd}
\mathcal{C}^\circ \arrow[r, "j"] \arrow[dr, "\pi^\circ"'] & \mathcal{C} \arrow[d, "\pi"] & \mathcal{D}_1 \arrow[l] & \cdots & \mathcal{D}_n \arrow[lll, bend right] \\
& \mathcal{M} \arrow[ur, "s_1 \cdots"] \arrow[urrr, "s_n"'] & & &
\end{tikzcd}$$

versal: $c : \mathcal{M} \longrightarrow \mathcal{M}_{g,n}$ is dominat étale

$s_1, \dots, s_n : \mathcal{M} \longrightarrow \mathcal{C}$ non-intersecting sections ("punctures")

$\mathcal{D}_i := s_i(\mathcal{M})$, $\mathcal{D} := \bigcup \mathcal{D}_i$

$\mathcal{C}^\circ := \mathcal{C} \setminus \mathcal{D}$ family of quasi-projective curves

$\mathbb{V} \longrightarrow \mathcal{C}^\circ$ unitary semisimple local system, $\mathrm{rk}\,\mathbb{V} = r$

Theorem (1.7.1): $R^1\pi^\circ_* \mathbb{V}$ has no sub-local systems of low rank:

$\mathbb{L} \subset R^1\pi^\circ_*\mathbb{V}$, $\mathbb{L} \neq 0 \Rightarrow \mathrm{rk}\,\mathbb{L} \geq 2g - 2r$

Remarks:

1) Corollary: if $r<g$ then $H^0(\mathcal{M}, R^1\pi_*^\circ \mathbb{V})=0$ (otherwise there is a rk 1 subsystem generated by an invariant vector, but $\mathrm{rk}\,\mathbb{L}\geq 2$)

2) What if $\mathbb{V}=\underline{\mathbb{C}}_{C^\circ}$?

$$MCG_{g,n} \xrightarrow{\sigma} Sp_{2g}(\mathbb{Z}) \circlearrowright H^1(\Sigma_g, \mathbb{C})$$

$$\sigma(T_\alpha)\cdot[\beta] = [\beta] + \langle\alpha,\beta\rangle[\alpha] \quad (\text{"Picard-Lefschetz"})$$

$\hookrightarrow$ Dehn twist around α

$\Rightarrow \sigma$ is surjective $\Rightarrow H^0(\sigma)=0$

3) Assume $\mathbb{V}$ is fiberwise irreducible. Apply to $\mathrm{ad}\,\mathbb{V} = \mathrm{End}(\mathbb{V})/\mathbb{C}\cdot\mathrm{id}_{\mathbb{V}}$.

$\mathrm{rk}\,\mathrm{ad}\,\mathbb{V} = r^2-1<g$ if $r<\sqrt{g+1}$

- By 1): $H^0(\mathcal{M}, R^1\pi_*^\circ\,\mathrm{ad}\,\mathbb{V})=0$
- By Schur lemma: $\pi_*^\circ\,\mathrm{ad}\,\mathbb{V}=0 \Rightarrow H^1(\mathcal{M}, \pi_*^\circ\,\mathrm{ad}\,\mathbb{V})=0$
- By Leray sp. seq.: $H^1(C^\circ, \mathrm{ad}\,\mathbb{V})=0$

(rigidity result!!!)

1. VHS on $R^1\pi_*^\circ\mathbb{V}$

If $\mathbb{V}$ is constant, one may study $R^\bullet \pi_* \mathbb{V}$ using Hodge theory. What if $\mathbb{V}$ is locally constant?

Assume $\mathbb{V}$ is real ($\exists \mathbb{V}_{\mathbb{R}} \in Loc_{\mathbb{R}}(\mathcal{C}^\circ)$: $\mathbb{V} = \mathbb{V}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$)

[N.B.: $\mathbb{V}$ is unitary $\Rightarrow$ $\mathbb{V} \oplus \mathbb{V}^\vee$ is real]
$\mathcal{H}^{i,j}_{\mathbb{V}} = \mathbb{V} \cap \mathcal{H}^{i,j}_{(\mathbb{V} \oplus \mathbb{V}^\vee)}$

Fact: $R^1 \pi^\circ_* \mathbb{V}$ underlies an admissible polarisable $\mathbb{R}$-MVHS.

Weight filtration: $R^1 \pi_* j_* \mathbb{V} \hookrightarrow R^1 \pi^\circ_* \mathbb{V}$
$\quad\quad W^1 = R^1 \pi_* j_* \mathbb{V}, \quad W^2 = R^1 \pi^\circ_* \mathbb{V}$

Hodge filtration:

Let $(\mathcal{E}, \nabla)$ be Deligne canonical extension of $\mathbb{V} \otimes \mathcal{O}_{\mathcal{C}^\circ}$ on $(\mathcal{C}, \mathcal{D})$.

De Rham complex:

$$[0 \to j_* \mathbb{V} \to \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1_{\mathcal{C}}(\log \mathcal{D})] = DR^\bullet(\mathcal{E})$$

Fact (Hodge-to-de Rham degeneration):

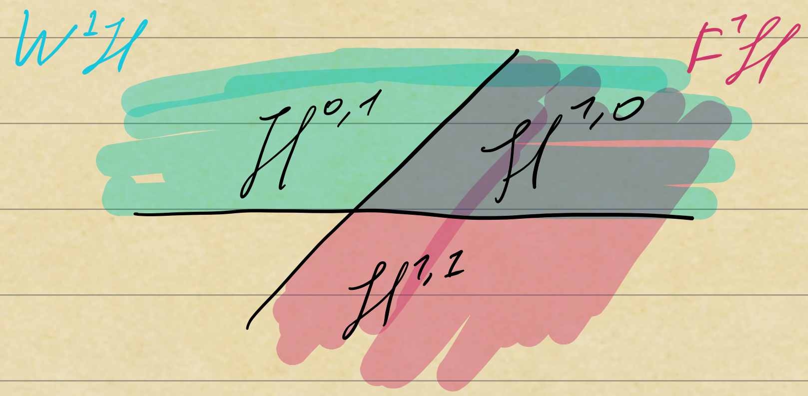
$$\mathbb{R}^1\pi_*^\circ \mathbb{V} = \mathbb{R}^1\pi_*(DR^\bullet(\mathcal{E}))$$

filtration bête $\rightsquigarrow$ Hodge filtration

More explicitly: $\mathcal{H} := R^1\pi_*^\circ \mathbb{V} \otimes \mathcal{O}_M$

$$F^1\mathcal{H} = \mathrm{im}\left(\pi_*(\mathcal{E}\otimes\Omega^1_C(\log \mathcal{D})) \longrightarrow R^1\pi_*^\circ \mathbb{V}\right)$$

$$\mathcal{H}/F^1\mathcal{H} \simeq R^1\pi_*\mathcal{E}$$



2. Period map:

Let $\mathcal{B} \subset M$ be a ball.

Locally we have a period map

$$P : \mathcal{B} \longrightarrow Gr(f^1; h)$$
$$b \longmapsto F^1\mathcal{H} \subseteq \mathcal{H}$$

$\mathrm{rk}\, F^1\mathcal{H} = f^1$, $\mathrm{rk}\, \mathcal{H} = h$

$$dP : T\mathcal{B} \longrightarrow TGr(f^1; h) = \mathrm{Hom}(F^1\mathcal{H}, \mathcal{H}/F^1\mathcal{H})$$

Fix a point $b \in \mathcal{B}$. Let $C := \mathcal{C}_b$

$$(C^\circ = \mathcal{C}^\circ_b,\ D = \mathcal{D}_b,\ E = \mathcal{E}|_C)$$

Our aim is to understand

$$dP_b : T_b\mathcal{B} \longrightarrow (F^1\mathcal{H})^\vee \otimes (\mathcal{H}/F^1\mathcal{H})$$

Dually: $F^1\mathcal{H}\otimes(\mathcal{H}/F^1\mathcal{H})^\vee \xrightarrow{dP^\vee} \Omega^1_b\mathcal{B}$

$c: \mathcal{M} \longrightarrow \mathcal{M}_{g,n}$ is dominant étale $\Rightarrow$
$c^*: \Omega^1_b\mathcal{B} \xrightarrow{\sim} H^0(C, \omega_C^{\otimes 2}(D)) = \Omega^1_{[C]}\mathcal{M}_{g,n}$

$$\begin{tikzcd}
F^1\mathcal{H}_b \otimes (\mathcal{H}/F^1\mathcal{H})^\vee_b \arrow[r] \arrow[d, equal] & \Omega^1_b\mathcal{B} \arrow[d, equal] \\
H^0(C, E\otimes\Omega^1_C(\log D))\otimes H^1(C,E)^\vee \arrow[r, "dP^\vee"] \arrow[d, equal, "\text{Serre duality}"] & H^0(C, \omega_C^{\otimes 2}(D)) \arrow[d, equal] \\
H^0(C, E\otimes\Omega^1_C(\log D))\otimes H^0(C, E^\vee\otimes\Omega^1_C(\log D)) \arrow[r] & H^0(C, \omega_C^{\otimes 2}(D))
\end{tikzcd}$$

[Formal check: $dP^\vee$ coincides with the composition

$$\begin{tikzcd}
H^0(C, E\otimes\omega_C(\log D))\otimes H^0(C, E^\vee\otimes\omega_C(\log D)) \arrow[r, "\otimes"] \arrow[dr] & H^0(C, E\otimes E^\vee\otimes\omega_C^{\otimes 2}(\log D)) \arrow[d, "\mathrm{tr}\otimes\mathrm{id}"] \\
 & H^0(C, \omega_C^{\otimes 2}(\log D))
\end{tikzcd}$$

((A well-known particular case:
$E = \mathcal{O}_C$, $n=0$:

$$\begin{tikzcd}
H^{1,0}(C)\otimes H^{0,1}(C)^\vee \arrow[r] \arrow[d, equal] & H^0(C, K_C^{\otimes 2}) \arrow[d, equal] \\
H^0(C,\omega_C)\otimes H^0(C,\omega_C) \arrow[r, "\otimes"] & H^0(C, K_C^{\otimes 2})
\end{tikzcd}$$
)

The period map differential is the same as the map defined by the Gauß-Manin connection:

$$\begin{tikzcd} F^1\mathcal{H} \arrow[r, hook] \arrow[rrr, bend right, "\overline{\nabla}"] & \mathcal{H} \arrow[r, "\nabla_{GM}"] & \mathcal{H}\otimes\Omega^1_M \arrow[r] & (\mathcal{H}/F^1\mathcal{H})\otimes\Omega^1_M \end{tikzcd}$$

(By Griffiths transversality this is a linear map!)

Polarisation: One can think about this map as a pairing

$$B_E : H^0(C, E\otimes\omega_C(D))\otimes H^0(C, E^\vee\otimes\omega_C(D)) \longrightarrow H^0(C, \omega_C^{\otimes 2}(D))$$

More precisely:

$$\overline{\nabla} : v \longmapsto B_E(v, -).$$

Idea of the proof of Thm 1.7.1:

$\forall v$ we get a map $\overline{\nabla}(v) : E \to (\mathcal{H}/F^1\mathcal{H})\otimes\Omega^1_b M$

We will show:

- If rk $\mathrm{Im}(\overline{\nabla}(v))$ on H^0 is big, then rk E is big
- If $\exists\, \mathbb{U} \subset E$ of small rank, then $\exists v$: $\mathrm{Im}(\overline{\nabla}(v))$ is of

3. Bound on $\operatorname{rk} \operatorname{Ker}(\overline{\nabla}(v))$

small rank

Prop. 6.3.6 from 2202.00039 ("non-GGG lemma"):

C - smooth proper curve of genus g
$D \subset C$ - effective reduced divisor
$E \longrightarrow C \setminus D$ vector bundle
$E_\star$ parabolic structure, semistable
$\widehat{E}_\star$ - coparabolically stable
$\mathcal{U} \subset \widehat{E}_0$ subbundle
$c := \operatorname{rk} \widehat{E}_0 - \operatorname{rk} \mathcal{U}$
$\delta := h^0(C, \widehat{E}_0) - h^0(C, \mathcal{U})$
If $\mu_\star(E_\star) > 2g-2+n$, then $\operatorname{rk} E > gc - \delta$
If $\mu_\star(E_\star) = 2g-2+n$, then $\operatorname{rk} E \geq gc - \delta$.

Proposition (4.2.3.):
$E_\star$ is a parabolic semistable on (C, D)
$\operatorname{degpar}(E_\star)$ $\quad 0 \neq v \in H^0(C, E \otimes \omega_C(D))$

Suppose $f_v := B_E(v, -)$, $f_v : E^\vee \otimes \omega_C(D) \longrightarrow \omega_C^{\otimes 2}(D)$

$H^0(C, f_v) : H^0(C, E^\vee \otimes \omega_C(D)) \longrightarrow H^0(C, \omega_C^{\otimes 2}(D))$
has rank r. Then $\operatorname{rk}(E) \geq g - r$.

Proof: Let $n = \deg D$

$\mathcal{U} = \ker f_v = v^{\perp}$ has corank 1, so
$c = 1$

Since $0 \to H^0(C, \mathcal{U}) \to H^0(C, E^{\vee} \otimes \omega_C(D)) \to \mathrm{Im}\, H^0(f_v) \to 0$
$\delta = r$

$\overbrace{(E_*)^{\vee} \otimes \omega_C(D)}$ is Serre-dual to E_*

Since $\mu_*(E_*) = 0$, $\mu_*(\overbrace{(E_*)^{\vee} \otimes \omega_C(D)}) = 2g - 2 + n$

But one can check:

$\overbrace{(E_*)^{\vee} \otimes \omega_C(D)}_0 = E^{\vee} \otimes \omega_C(D)$ and

By non-GBB lemma $\mathrm{rk}\, E \geq g - r$

4. VHS on sub-local systems:

Let $\mathbb{L} \subset R^1 \pi_*^{\circ} \mathbb{V}$.

Define $\overline{\mathbb{L}} := \begin{cases} \mathbb{L}, & \text{if } \mathbb{L} \text{ has } \mathbb{R}\text{-structure} \\ \mathbb{L} \oplus \overline{\mathbb{L}}, & \text{otherwise.} \end{cases}$

Clearly, $\widetilde{R^1 \pi_* \mathbb{V}} = R^1 \pi_* \widetilde{\mathbb{V}} =: \mathbb{W}$

$\mathbb{L} \subset \mathbb{W}_{\mathbb{C}}$. There exists i, s.t.

$\mathbb{L} \to gr^i_W \mathbb{W}_{\mathbb{C}}$ is non-zero

By Deligne semi-simplicity:

$$gr^i_W \mathbb{W}_{\mathbb{C}} = \bigoplus_j \mathbb{V}_j \otimes W_j, \text{ where}$$

$\mathbb{V}_j$ are irr. PVHS on $\mathcal{M}$, W_j are Hodge str.-s

Enough to assume that $\mathbb{L}$ is irreducible (if $\mathbb{L}' \subseteq \mathbb{L}$ is irr., rank bound on $\mathbb{L}' \Rightarrow$ rank bound on $\mathbb{L}$)

Therefore, $\exists j$: $\mathbb{L} \simeq \mathbb{V}_j$. Thus $\mathbb{L}$ underlies a pure VHS!

The morphism $\mathbb{L} \hookrightarrow \mathbb{W}$ is not necessary a ~~morphism~~ morphism of VHS, but:

pure VHS

admissible VMHS

$$Q := \mathrm{Hom}_{\mathbb{R}\text{-VHS}}(\overline{\mathbb{L}}, \mathbb{W}) = H^0(\mathcal{M}, \underbrace{\overline{\mathbb{L}}^\vee \otimes \mathbb{W}}_{\text{admissible VMHS}})$$

By fixed part theorem for admissible VMHS the space Q is endowed with $\mathbb{R}$-MHS and the natural map

$z: Q \otimes \overline{\mathbb{L}} \longrightarrow \mathbb{W}$ is a non-zero morphism of $\mathbb{R}$-VMHS. This is not true in general! However, it is true in this particular case, see ⊗ in

Let $Q \otimes \mathcal{L} := (Q \otimes \overline{\mathbb{L}}) \otimes \mathcal{O}_{\mathcal{M}}$.

If $v \in F^1(Q\otimes L)$ we have   the end of these notes

$$\begin{tikzcd} (Q\otimes L)/F^1(Q\otimes L) \arrow[r, "f_v = \overline{\nabla}_L(v)"] \arrow[d, "z \bmod F^1"] & T_b^* B \arrow[d, equal] \\ \mathcal{H}/F^1\mathcal{H} \arrow[r, "f_{z(v)} = \overline{\nabla}_{\mathcal{H}}(z(v))"] & T_b^* B \end{tikzcd}$$

Since z is a morphism of VHS

5. Rank bound for the differential of period map on $\mathbb{L}$

Lemma (5.1.1): Up to replacing $\mathbb{V} \rightsquigarrow \overline{\mathbb{V}}$, $\mathbb{L} \rightsquigarrow \overline{\mathbb{L}}$

there exists $v \in F^{\#}\mathcal{H}_m \cap \mathbb{L}$, s.t. $\operatorname{rk} \overline{\nabla}_m(v) \leq \frac{\operatorname{rk}\mathbb{L}}{2}$.

Proof: As we explained, $\mathbb{L}$ underlies a pure VHS and there is a map

$z: Q \otimes \mathbb{L} \longrightarrow R^1\overline{u}_*^{\circ}\mathbb{V} = \mathbb{W}$

We have a diagram

$$F^1\mathcal{H} \xrightarrow{\overline{\nabla}_W} \mathcal{H}/F^1\mathcal{H} \otimes \Omega^1_{\mu}$$

$$F'(Q\otimes L) \xrightarrow{\overline{\nabla}_{Q\otimes L}} (Q\otimes L)/F'(Q\otimes L) \otimes \Omega^1_M$$

By duality:

$$\begin{tikzcd}
\left(\mathcal{H}/F'\mathcal{H}\right)^\vee \arrow[rr, "\overline{\nabla}_{\mathbb{W}}(z(v))"] \arrow[d] & & \Omega^1_M \\
\left((Q\otimes L)/F'(Q\otimes L)\right)^\vee \arrow[rr, "\overline{\nabla}_{Q\otimes \mathbb{L}}(v)"] & & \Omega^1_M \arrow[u]
\end{tikzcd}$$

$\mathrm{rk}\,\overline{\nabla}_{\mathbb{W}}(z(v)) \leq \mathrm{rk}\,\overline{\nabla}_{Q\otimes\mathbb{L}}(v)$; since Q is constant $\mathrm{rk}\,\overline{\nabla}_{Q\otimes\mathbb{L}}(v) = \mathrm{rk}\,\overline{\nabla}_{\mathbb{L}}(v)$

$\mathbb{W} = \mathbb{W}^{1,0} \oplus \mathbb{W}^{0,1} \oplus \mathbb{W}^{1,1}$

We have a morphism of MVHS

$$Q\otimes\tilde{\mathbb{L}} \longrightarrow \mathbb{W}.$$

$\mathbb{L}$ is pure, so there are two possibilites (maybe up to twist)

1) $\mathbb{L} = \mathbb{L}^{0,0}$ and $Q = Q^{1,0}\oplus Q^{0,1}\oplus Q^{1,1}$

But then $\mathbb{L}$ does not vary at all!

(and Q is a constant variation

$\forall v \in F'(Q\otimes L)$ the map

$$\overline{\nabla}(v) : \left((Q\otimes L/ \quad\right)^\vee \longrightarrow H^0(C, F_{\ldots}(D))$$

$\overline{\nabla}_{\mathbb{L}}(v) : \left(Q\otimes\mathbb{L}/F'(Q\otimes\mathbb{L})\right) \to H(C, L\otimes\omega_C(D))$

vanishes. Therefore, $\operatorname{rk}\overline{\nabla}_{\mathbb{W}}(z(v)) \leq \operatorname{rk}\overline{\nabla}_{Q\otimes\mathbb{L}}(v) = 0$

((We can always find such v that $z(v) \neq 0$)

2) $\mathbb{L} = \mathbb{L}^{1,0} \oplus \mathbb{L}^{0,1}$, $Q = Q^{0,0}$

Up to replacing $\mathbb{L}$ with $\overline{\mathbb{L}}$ we may assume

$$\dim \mathbb{L}^{1,0}_m \geq \dim \mathbb{L}^{0,1}_m$$

N.B.: It is not necessary that $\dim \mathbb{L}^{1,0} = \dim \mathbb{L}^{0,1}$, since $\mathbb{L} \subset \overline{\mathbb{L}}$ might be not real

Then $\operatorname{rk}\overline{\nabla}_{Q\otimes\mathbb{L}}(v) = \operatorname{rk}\overline{\nabla}_{\mathbb{L}}(v) \leq \dim(\mathbb{L}_m/F'\mathbb{L}_m) =$
$= \dim \mathbb{L}^{0,1}_m \leq \frac{\operatorname{rk}\mathbb{L}}{2}$

- It is important that since Q is constant, $\operatorname{rk}\overline{\nabla}_{Q\otimes\mathbb{L}} = \operatorname{rk}\overline{\nabla}_{\mathbb{L}}$!

6. Proof of the Theorem

Proof of Thm 1.7.1:

From Lemma we know that there exists $v \in F^1\mathcal{H}$, such that $\mathrm{rk}\, \overline{\nabla}_{\mathcal{H},m}(v) \leq \frac{\mathrm{rk}\, \mathbb{U}}{2}$.

From the identification $\overline{\nabla} = B_E(v,-)$ we get $\mathrm{rk}\, B_E(v,-) \leq \frac{\mathrm{rk}\, \mathbb{U}}{2}$, and by Prop. 4.2.3.

$$\mathrm{rk}\, E > g - \frac{\mathrm{rk}\, \mathbb{U}}{2}$$
$$\Updownarrow$$
$$2r - 2g > -\mathrm{rk}\, \mathbb{U}$$
$$\Updownarrow$$
$$\mathrm{rk}\, \mathbb{U} > 2g - 2r. \qquad \blacksquare$$

We also would like to deduce the following vanishing result:

Theorem (5.2.1): Assume now that $\mathbb{V}$ is unitary only in one point, i.e. $\exists m \in \mathcal{M}$: $\mathbb{V}|_{C_m^0}$ is unitary. and $\mathrm{rk}\, \mathbb{V} < g$.

Then $H^0(\mathcal{M}, R^1\pi_*^0 \mathbb{V}) = 0$.

Preliminaries from representation theory:

(2.1.3.)

Prop.: Let $\mathcal{C} \xrightarrow{\pi} \mathcal{M}$ be a versal family of (g,n)-curves, s_i

$\rho : \pi_1(\mathcal{C}^\circ) \to GL_r(\mathbb{C})$ is a representation.

Then $\rho|_{C_m^\circ}$ is MCG-finite $\forall m \in \mathcal{M}$.

$$1 \to PMod_{g,n} \to MCG_{g,n} \to \mathfrak{S}_n \to 1$$

$$\begin{array}{c} \| \\ \pi_1(\mathcal{M}_{g,n}) \end{array}$$

(action on punctures)

and

$$1 \to \pi_1(C_m^\circ) \to \pi_1(\mathcal{C}^\circ) \to \pi_1(\mathcal{M}) \to 1$$

(homotopy exact sequence)

$$1 \to \pi_1(C_m^\circ) \to \pi_1(\mathcal{M}_{g,n+1}) = PMod_{g,n+1} \to \pi_1(\mathcal{M}_{g,n}) = PMod_{g,n} \to 1$$

(Birman exact sequence)

Fact: If $\mathcal{M} \to \mathcal{M}_{g,n}$ is dominant étale, $\mathrm{im}(\pi_1(\mathcal{M}) \to \pi_1(\mathcal{M}_{g,n}))$ has finite index in $\pi_1(\mathcal{M}_{g,n}) = PMod_{g,n}$.

Now, we have an action $\pi_1(\mathcal{M}_{g,n}) \to Out(\pi_1(C_m^\circ))$

Since the action $\pi_1(\mathcal{M}) \to \pi_1(C_m^\circ)$ preserves ρ, its image in $\pi_1(\mathcal{M}_{g,n})$ preserves ρ, but it is a finite index subgroup $\boxtimes$

(2.5.1)
Lemma: Let G be a group, $H \trianglelefteq G$ - normal
$\rho : G \to GL_n(\mathbb{C})$ a representation,
- $\det \rho$ is finite
- $\rho|_H$ is irreducible.

If $\rho|_H$ is unitary, then ρ is unitary.

Proof: Let h be a Hermitian form on $\rho|_H$.

Well if V is underlying vector space,
$h : V \xrightarrow{\sim} \overline{V}^\vee$.
By Schur lemma such h is unique up to scaling (V is an irreducible H-module),
$\forall g \in G$ $\quad h^g : (v, w) \longmapsto h(\rho(g)v, \rho(g)w)$ is another Hermitian form, thus $h^g = \chi(g) \cdot h$, where $\chi : G \to \mathbb{C}^\times$ is a character. But $\det \rho$ is finite, hence $\chi \in \mathrm{Hom}(G, U(1))$ and ρ is unitary $\boxtimes$

Decomposition of a local system unitary in one point.

Lemma (2.5.2.): Let $\mathbb{V} \to C^\circ$ be a local system,
assume $\mathbb{V}|_{C^\circ_m}$ is unitary.

There is a dominant étale base change

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d, "\pi'"] & \mathcal{C} \arrow[d, "\pi"] \\ \mathcal{M}' \arrow[r] \arrow[u, bend left, "s_i'"] & \mathcal{M} \arrow[u, bend left, "s_i"] \end{tikzcd}$$

with $\pi'^{0}: \mathcal{C}'^{0} \to \mathcal{M}'$ the associated family of punctured curves, s.t.

$$\mathbb{V}\big|_{\mathcal{C}'^{0}} = \bigoplus_{i=1}^{s} \mathbb{U}_i \otimes (\pi'^{0})^* \mathbb{W}_i,$$

where $\mathbb{W}_i \in Loc(\mathcal{M}')$, $\mathbb{U}_i \in Loc_{U_r}(\mathcal{C}'^{0})$.

Moreover, $\bullet\ \forall m \in \mathcal{M}'$ $\mathbb{U}_i\big|_{C_i'^{0}}$ is irreducible,

$\bullet$ $\mathbb{U}_i \not\simeq \mathbb{U}_j$ for $i \neq j$, and $\mathbb{W}_i \simeq \pi_*'^{0} Hom(\mathbb{U}_i, \mathbb{V}|_{\mathcal{C}'^{0}})$.

<u>Proof</u>: Since $\mathbb{V}|_{C_m^0}$ is unitary, it is semi-simple:

$$\mathbb{V}\big|_{C_m^0} \simeq \bigoplus_{i=1}^{s} \mathbb{S}_i^{\oplus k_i} \quad \text{- decomposition to irr. summands}$$

Let ρ be the monodromy of $\mathbb{V}$ and $\rho = \bigoplus_{i=1}^{s} \rho_i^{\oplus k_i}$.
By Proposition 2.1.3. ρ is MCG-finite. But then each ρ_i is MCG-finite. Indeed $\forall \gamma \in MCG\ \exists t$: $\gamma^t \cdot [\rho] \simeq [\rho]$, hence $\gamma^t[\rho_i]$ is a subquotient of $[\rho_j]$, $\gamma^t[\rho_i] \simeq [\rho_j]$. But the set of indices is finite.

<u>Claim</u>: $\exists\, \mathcal{M}' \to \mathcal{M}$ and $\rho_i': \pi_1(\mathcal{C}^0) \to GL_r(\mathbb{C})$ s.t.

$\forall i$ $\rho_i'|_{\mathcal{C}^0_{m'}}$ is identified with ρ_i

We would like to extend each ρ_i to a global representation $\rho_i' : \pi_1(\mathcal{C}^0) \longrightarrow GL_r(\mathbb{C})$.

<u>Step 1</u>: Since ρ_i is MCG-finite, after finite étale base change $MCG \cdot [\rho_i] = \{[\rho_i]\}$ (it is fixed)

But $\mathrm{Aut}([\rho_i]) = \mathbb{C}^\times$, so it only extends to a projective rep. $\tilde{\rho}_i : \pi_1(\tilde{\mathcal{C}}^0) \longrightarrow PGL_r(\mathbb{C})$

<u>Step 2.</u>: $\det \rho_i$ is finite $\Rightarrow$ extends to ρ_i' after another covering.

Let $\mathbb{U}_i$ be the corr. loc. sys. By Lemma 2.6.1 they're unitary. There is a map $\bigoplus_{i=1}^{s} (\pi'^0)^* \mathbb{W}_i \otimes \mathbb{U}_i \to \mathbb{V}$, $\mathbb{W}_i$ as above.

It is fiberwise an isomorphism $\Rightarrow$ an iso $\boxtimes$

<u>Proof of Theorem 5.2.1.</u>:

If $M' \longrightarrow M$ is dominant,

$$H^0(M, R^1\pi_*^0 \mathbb{V}) \hookrightarrow H(M', R^1\pi_*^0 \mathbb{V}|_{M'}) \text{ is injective.}$$

Hence we can assume:

$$\mathbb{V} = \bigoplus_{i=1}^{s} \mathbb{U}_i \otimes (\pi^0)^* \mathbb{W}_i \quad , \quad \mathbb{U}_i \text{ are unitary}$$

$\mathbb{W}_i$ are in $Loc(M)$

Enough to show:

$$H^0(M, R^1\pi^\circ_*(\mathbb{U}\otimes(\pi^\circ)^*\mathbb{W})) = H^0(M, R^1\pi^\circ_*\mathbb{U}\otimes\mathbb{W}) = 0$$
if $\mathrm{rk}(\mathbb{U}\otimes\mathbb{W}) < g$.

Let $0 \neq \alpha \in H^0(M, R^1\pi^\circ_*\mathbb{U}\otimes\mathbb{W})$. It can be regarded as:

$$\alpha : \mathbb{W}^\vee \longrightarrow R^1\pi^\circ_*\mathbb{U}.$$

If $\mathbb{L} = \mathrm{Im}\,\alpha$, by Theorem 1.7.1: $\mathrm{rk}\,\mathbb{L} \geq 2g - 2\,\mathrm{rk}\,\mathbb{U}$, hence $\mathrm{rk}\,\mathbb{W} \geq \mathrm{rk}\,\mathbb{L} > 2g - 2\,\mathrm{rk}\,\mathbb{U}$.

Thus: $\mathrm{rk}\,\mathbb{W} + 2\,\mathrm{rk}\,\mathbb{U} \geq 2g$. We are interested in $\mathrm{rk}(\mathbb{U}\otimes(\pi^\circ)^*\mathbb{W}) = \mathrm{rk}\,\mathbb{U}\cdot\mathrm{rk}\,\mathbb{W}$

Exercise: If $w, u \in \mathbb{N}^\times$, $w + 2u \geq 2g$, then $w \cdot u \geq g$

$$\left(w\cdot u \geq u, \quad u \geq \left\lceil \tfrac{2g-1}{2} \right\rceil \geq g\right)$$

$$wu \geq u(2g - 2u) = 2u(g-1) \geq 2g - 2 \geq g, \text{ since } g \geq 2.$$

Appendix: Let $\mathbb{V}$ be a real variation of Hodge structures, $\mathbb{L} \subset \mathbb{V}_{\mathbb{C}}$ a local system. Assume that $\mathbb{L}$ is irreducible. How far is $\mathbb{L}$ from being a VHS?

1) Assume $\mathbb{V}$ is pure. By Deligne, it splits

$$\mathbb{V} = \bigoplus_i \mathbb{S}_i \otimes V_i,$$ where $\mathbb{S}_i$ are simple VHS, V_i are Hodge structures.

$\exists i : \mathbb{L} \simeq \mathbb{S}_i$. This defines a VHS-str. on $\mathbb{L}$.
However, the natural map $\mathbb{L} \hookrightarrow \mathbb{V}$ is not a morphism of VHS. Consider, e.g. the case when

$\mathbb{L} = \mathbb{S}_i \otimes v$, $v \in V_i$ - is not a Hodge vector (not of type $(0,0)$).

If one put $Q := \mathrm{Hom}_{\text{Loc.Sys.}}(\mathbb{L}, \mathbb{V}) = H^0(X, \mathbb{L}^\vee \otimes \mathbb{V})$, this carries a Hodge structure (by fixed part theorem) and the natural morphism

$$Q \otimes \mathbb{L} \to \mathbb{W}$$

is a morphism of VHS. (in fact $Q = V_i$ as VHS)

2) Assume $\mathbb{V}$ is mixed. It is true that:

1) $\exists j$ s.t. the projection $\mathbb{L} \to gr^j_W \mathbb{V}$ is non-zero.

2) There exists a decomposition

$$gr^j_W \mathbb{V} = \bigoplus \mathbb{S}_i \otimes V_i$$

and $\mathbb{L} \simeq \mathbb{S}_i$ for some i.

This defines a VHS-str. on $\mathbb{L}$

3) If $\mathbb{W}$ is admissible, then

$$Q := \mathrm{Hom}_{\text{Loc.Sys.}}(\mathbb{L}, \mathbb{W}) = H^0(X, \mathbb{L}^\vee \otimes \mathbb{W})$$ carries a canonical Hodge structure

BUT the morphism $Q \otimes \mathbb{L} \to \mathbb{W}$ is not a morphism of VHS.

Here is the reason: let V be a mixed Hodge structure s.t. the weight filtration has two steps, but do not split

$$\left(\begin{array}{l} V \neq gr^0_W V \oplus gr^1_W V \text{ as Hodge structures} \\ \text{but} \simeq \text{ as vector spaces} \end{array} \right)$$

Let X be a compact complex manifold and $\mathbb{W} := V \otimes \mathbb{C}_X$ a constant VHS.

Then $\mathbb{V} \simeq gr^0_W \mathbb{V} \oplus gr^1_W \mathbb{V}$ in category local systems but if $\mathbb{L} = gr^1_W \mathbb{V}$, the natural morphism

$$Q \otimes \mathbb{L} \to \mathbb{V}$$ does not have to be a morphism of MVHS.

However, there is a situation when it is still true. For instances if both W° and F° have two steps (this is the case in Landesman-Litt's paper).

In this case $\mathcal{H} := \mathbb{V} \otimes \mathcal{O}_X$ splits as

$$\mathcal{H} = \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1} \oplus \mathcal{H}^{1,1}$$

(in cat. of vector bundles)

and $\mathcal{H}^{1,0} = W_1 \cap F'$
$\mathcal{H}^{0,1} = W_1 \cap \bar{F}'$
$\mathcal{H}^{1,1} = F' \cap \bar{F}'$.

Thus, if $\mathbb{L} \subset \mathbb{V}$ a local sub-system:

1) if $\mathbb{L} \subset W_1\mathbb{V}$, then $\mathbb{L}$ is a local subsystem inside pure sub-VHS and the considerations as above apply.

2) otherwise, $\mathbb{L} \simeq gr^1_W \mathbb{V}$, that can be lifted to $\mathbb{V}$ as $gr^1_W\mathbb{V} \xrightarrow{\sim} F'\mathbb{V} \cap \bar{F}'\mathbb{V}$. This is a lifting in category of Hodge structures, so $\mathbb{L}$ is isomorphic to a sub-VHS in $\mathbb{V}$.