

Introduction to geometric local systems 2

Last time, we briefly discussed de Jong's conjecture, resolved by Gaitsgory. Combined w/ the work of L. Lafforgue, we have the following upshot.

Thm Let $U/\bar{\mathbb{A}}_q$ be hyperbolic.

Let $\mathcal{J} := \left\{ L \text{ Q}_p\text{-local systems} \right. \atop \left. \text{on } U/\bar{\mathbb{A}}_q \text{ w/ fixed} \right. \atop \left. \text{rank, bounded ramification} @ \right. \atop \left. \infty, \text{fixed local monodromy} \right. \atop \left. \text{geo} \right\}$

Then

- ① $\Delta^{\text{geo}} \rightarrow \text{infinit}$
- ② \sim " Δ^{geo} " \rightarrow Zariski dense in Δ "

Rmk: This theorem is crazy!!

For instance, we have the exmpl.

Example let $\lambda \in \overline{\mathbb{F}_p}$. and set.

$$U := \mathbb{P}^1 \setminus \{0, 1, \infty, \lambda\}.$$

Then \mathcal{Y}_{an} may (isogeny classes of)

Simple abelian schemes

$A_{\mathcal{U}}$



\mathcal{U}

of $G_{\mathbb{Q}_2}$ -type and semi-stable reduction.
(skip?)

+
Moreover, there is a sense in
which this α , appropriately
weighted, is independent of λ

Q: Direct proof / construction?

They are all of Hilbert
modular type.

Rmk Example true for any
affine hyperbolic curve.

Q: Why do we care about
the theorem?!

A:  Using JJ's conj., Drinfeld
proved thus semi-simplicity + Hard Lefschetz
for perverse sheaves in char 0.

 Using JJ + compactness + ...

JJ-Esnault proved that

$$\pi_! : X \rightarrow Y \quad (\text{quasi-proj, normal})$$

\mathcal{L}_Y is s.s. \mathbb{C} -local system

Then $\pi^{\mathcal{A}L_Y} =: L_X$ is

semi-simple on X .

Q: Is Cor dJG true for other
base fields? E.g.

Question: let X/\mathbb{C} be smooth.

Let $\text{Char}^B(X)$ be a character
variety (moduli of $\pi_1(X) \rightarrow \text{GL}_N$)

Is the set of pts of geo
origin dense???

L-L prove that such a statement
is false in general for low rank.

Notation

C s.m. proj. curve/c
gens x_1, \dots, x_N distinct pts of C .
 $V := C \setminus \{x_1, \dots, x_N\}$

Thm L-L let (C, x_1, \dots, x_N)

be analytically very general in $M_{g,n}$.

Let L be a local system
of geo origin on U w/
 ∞ -monday. Then

$$\text{rank } L \geq 2\sqrt{g+1}$$

Slogans: "very general case admits
NO low rank local systems &
geo origin" =

In fact, they prove that if

L is an Ω_K -local system on U ,

s.t. $\forall \iota: K \hookrightarrow L$,

L underlies a PVHS,

then : $\text{rank}(L) \leq 2\sqrt{g+1}$
 $\Rightarrow L$ has finite monodromy

Note: such a statement requires the # field K : every curve (at even Euler char) has a "natural" rank 2 \mathbb{R} -local system underlying an \mathbb{R} VHS:

$$\rho: \pi_1(C) \longrightarrow \mathrm{SL}_2(\mathbb{R})$$

$$f_2 \mathcal{G}_{\mathrm{PSL}_2(\mathbb{R})}$$

$$\downarrow \\ C$$

NOTE: Any statement like Thm LL1 requires a rank bound by Kodai-Parsadan trick.

Motivation

M_h not proper.

Do there exist complete curves
in M_h ?

Answer $h > 2$, YES:

$$M_h \hookrightarrow \underbrace{d_h \hookrightarrow d_{h+1}}_{\text{boundary has codim } h}$$

take general intersection of ample
divisors.

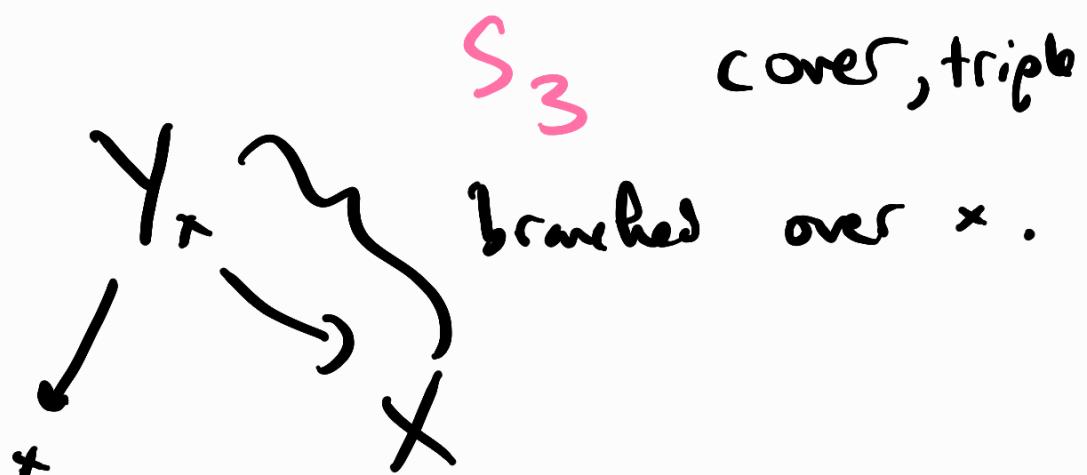
Explicit construction?

$$\begin{matrix} X & \longrightarrow & M_h \\ (\Rightarrow) \exists \text{ curve } & & \downarrow \\ & y & \text{ of genus } h. \\ & x & \end{matrix}$$

For every $p \in X$, want a curve y_p .

How on earth are we going to
construct such a thing?

IDEA: MAKE IT RELATED TO
 $X!!$



Q: Does this give me my
family Y ?



A: NO! triple cover not uniquely
determined!

\exists 3 choices
2 genus(X)

choices

~)

$$\mathcal{X} \longrightarrow \mathcal{C} = M_{g,1}$$

Relative Curve,
fibers not connected!
 $\sim 3^{2g}$ components

rel genus $\sim 6g$

~) \exists geometric local system

over general curve of rank
 $O(g \cdot 3^{2g})$

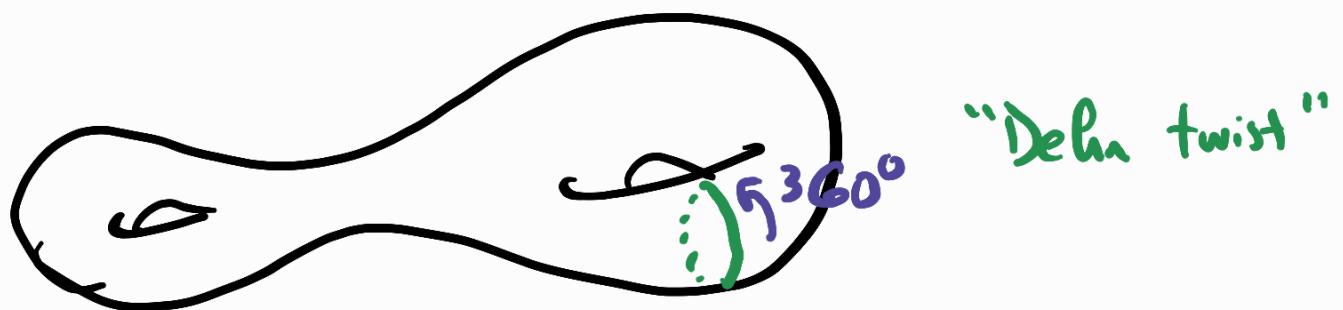
To state next result, need to recall the mapping class group.

Σ_g := compact orientable top. surface of genus g.

$$\Sigma_{g,n} := \Sigma_g \setminus \{x_1, \dots, x_n\}$$

$$MCG_{g,n} := \overbrace{\text{Homeo}^+(\Sigma_g, \{x_1, \dots, x_n\})}^{\text{oriented homeomorphisms } \Sigma_g \rightarrow \Sigma_g \text{ preserving } x_1, \dots, x_n} / \text{isotopy}$$

$$= \pi_0(\text{Homeo}^+(\Sigma_g, \{x_1, \dots, x_n\}))$$



skip

$$1 \rightarrow \text{Torelli} \rightarrow MCG_g \rightarrow Sp_{2g}(\mathbb{Z}) \rightarrow 1$$

The (Dehn)

MCG_g generated by sink
set of Dehn twists around
simple closed curves.

Ohs: MCG does not "act"

on $\pi_1(\Sigma_{g,n}, \times)$

However, it "acts by outer
automorphisms":

$MCG_{g,n} \rightarrow \text{Out}(\pi_1(\Sigma_{g,n}, \times))$

Hence $MCG_{g,n}$ does NOT

act on $\text{Hom}(\pi_1(\Sigma_{g,n}, \times), GL_n(\mathbb{Q}))$

But, it does act on

$\text{Char}(\Sigma_{g,n}) :=$

$\text{Hom}(\pi_1(\Sigma_{g,n}), \text{GL}_n(\mathbb{C}))$

conj.

(in general,

$$\begin{array}{ccc} \text{Aut}(G) & \xrightarrow{\sim} & \text{Hom}(G, H) \\ \downarrow & & \downarrow \\ \text{Out}(G) & \xrightarrow{\sim} & \text{Hom}(G, H)/_{\substack{\text{conj.} \\ H}} \end{array}$$

Let $\tilde{g} \in \text{Aut}(G)$ $\overset{g^{-1}-g}{\text{Ad}_g \circ \tilde{g}}$

$$\tilde{g} \cdot f(x) := f(\tilde{g}^{-1}(x))$$

$$\text{Ad}_g \circ \tilde{g} \cdot f(x) := f(g \tilde{g}^{-1}(x) g^{-1})$$

Def A rep

$$p: \pi_1(\Sigma_{g,n}) \rightarrow \mathrm{GL}_n(\mathbb{C})$$

is MCG-link (or canonical)

If $\mathrm{MCG}_{g,n}$ orbit of p is finite.

Idea: MCG link representations
morally correspond to (log) flat
connections on the universal curve
 $U := \mathbb{C} \setminus \{x_1, \dots, x_N\}$
 \downarrow
 $M_{g,n}$
"alg isomonodromic deformation"

Reason: Let $\overline{\mathcal{T}}_g$ be Teichmüller space
universal cover,
 $\pi_1(M_g) \rightarrow$ almost M_g

MCG_g

$$\begin{array}{ccc} \mathcal{D}^{\text{univ}} & \longrightarrow & \overline{\mathcal{T}}_g \\ \downarrow & & \downarrow \\ \mathcal{E}^{\text{univ}} & \xrightarrow{\quad} & M_g \end{array}$$

universal

Given a top \mathbb{C} -local system L or
 Σ_g , L canonically extends
isomonodromically to a local
system L on $\mathcal{D}^{\text{univ}}$
(\leadsto relative flat connection.)

We say \mathcal{L} admits a (versal) alg
isomonodromic deformation if
 \mathcal{L} on \mathbb{P}^{univ} descends to

$\mathcal{L}^{\text{univ}}$ (or, more precisely,
to a scheme $\mathcal{C}' \rightarrow \mathcal{C}^{\text{univ}}$
where étale & dominant)

Idea: the orbit of $[P]$
under $MCG_{g,n}$ is "monodromy
of monodromy"
Philosophy: this should generally
be big.

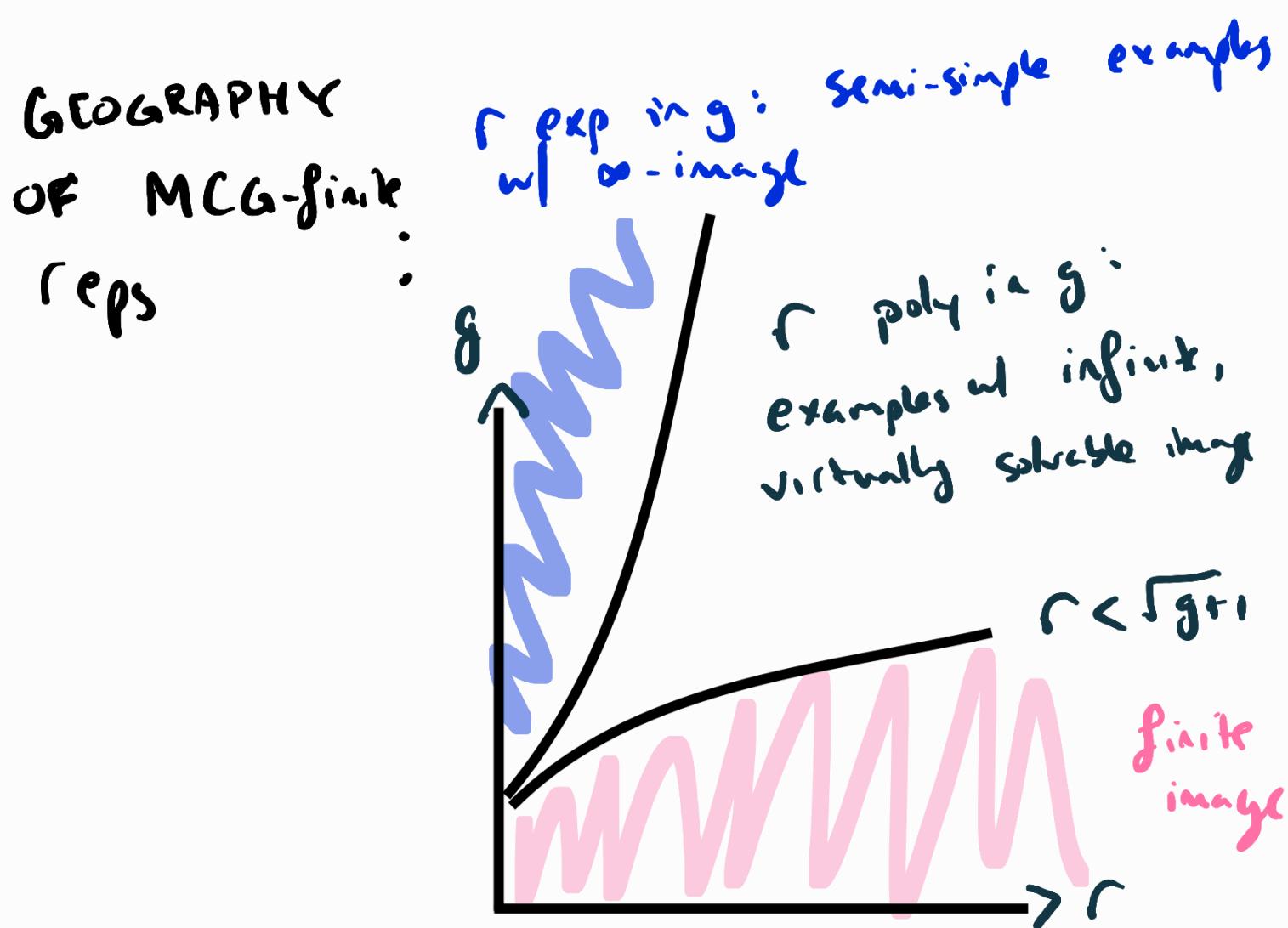
Thm LL2 $p: \pi_1(\Sigma_{g,n}) \rightarrow GL_N(\mathbb{C})$

be MCG-finite. If p has

∞ -monodromy, then

$$N > \sqrt{g+1}$$

Slogans: canonical representations
have large rank.



Cor of Thm LL2

$$\pi_1(\sum_{g,1}) = F_{2g}$$

Let $\rho: F_{2g} \rightarrow GL_r(\mathbb{C})$

be a rep such that

$\text{Out}(F_{2g}).[\rho]$ is a finite set

(i.e., $[\rho]$ has finite orbit under

$\text{Out}(F_{2g})$.)

Then if r is small (i.e., $r < \sqrt{g+1}$), then ρ is

finite.

Note: can describe generators
of $\text{Out}(F_n)$, see Remark
1.6.3 of [LL2ac].

Then L13: $(C, x_1, \dots, x_n) / K$ ^{wsg.}

be "general"; i.e.,



Let $\rho: \pi_1(U_{\bar{K}}) \rightarrow \text{GL}_n(\bar{\mathbb{Q}}_\ell)$.

Suppose

① ρ is of geo origin

② ρ has no-image

Then $\text{rank}(L) \geq \sqrt{g+1}$

OPTIONAL

Putnam-Wieland

Let H be a finite gp,

$$\Sigma_{g', n'} \rightarrow \Sigma_{g, n} \text{ an}$$

unbranched H cover.

$$MCG_{g, n+1} \curvearrowright \pi_1(\Sigma_{g, n}, *)$$

$\downarrow \phi$
 H

Let $\Gamma \subseteq MCG_{g, n+1}$ be stabilizer
of $\ker(\phi)$.

$$\rightsquigarrow \Gamma \curvearrowright H_1(\Sigma_{g', n'}, \mathbb{C})$$

$$\rightsquigarrow \Gamma \curvearrowright H_1(\Sigma_g, \mathbb{C})$$

"fill in punctures"

For any $\rho: H \rightarrow GL_n(\mathbb{C})$,

$H_1(\sum_{g',n'}, \mathbb{C})^\rho$ is isotypic component (H acts on $\sum_{g',n'}$)

Def

Fix $g \geq 2$, $n \geq 0$, and H as above.

We say $PW_{g,n}^H$ holds if:

\sqrt{H} -corel $\sum_{g',n'} \rightarrow \sum_{g,n}$

$\Gamma \curvearrowright H_1(\sum_{g',n'}, \mathbb{C})$

has no vectors of finite orbit.

Conj (PW) $\forall g \geq 2, n \geq 0, H$

$PW_{g,n}^H$ holds.

Cir of Thm LL2

For fixed g, n , $PW_{g,n}^H$ holds
for any group H w/ following
property:

- all irreps ρ of H
have rank $< g$.

Sketch of Pf of Thurlli

- (C, x_1, \dots, x_N) , (E, J) on
curve . rank $E < 2\sqrt{g+1}$
 - reg. sing. nilp. res.

Suppose isomorophism of
to any general nearby n-p'ted
curve underlies CPVHS

Then we want to prove
 (E, J) has unitary monodromy
(i.e. $KS = 0$)

- For η_j such as above,
we will show that

★ If (\mathcal{E}, \square) is the isomonodromic def, then \mathcal{E} is semi-stable or the (analytically) very general case.

If ★ holds, then:

- Hodge filtration consists of one piece.
- \Rightarrow ∃ definite Hermitian form (polarization) preserved by monodromy
- \Rightarrow monodromy is unitary

Therefore, reduced to proving .

This follows from

Theorem 1.3.4 (LL22a),
which morally says the following:

Let (E, ∇) be a flat connection
on an analytically very general
curve (C, x_1, \dots, x_n) . Then E
^(log)
is semi-stable.

Pf : Def thy of $(E, \nabla, \text{Fil}_{\text{HN}})$
+ Clifford theory for vector
bundles

Key: "non-GGG" Lemma,
(5.1.3, 5.1.4 of [LL22a])

To prove Thm II.2

- cohomology vanishing for unitary local systems on versal families of curve
($\mathbb{R}^1\bar{\pi}_\alpha$ (unitary) underlies PVHS)
- MCG-finite \Rightarrow strongly (wh. rigid or $\overset{\circ}{\downarrow}$)
 \Rightarrow integral
 $E \cdot G, K \cdot P$
(these will imply the result for unitary local systems)

- In general, deform \rightarrow PVHS.
over a fiber, will have
unitary monodromy. USE
above techniques to show
that MCG-finite reps don't
have non-MCG-finite
deformations.

Talks

- ① Intro
- ② Atiyah bundles + Isomonodromic deformation
- ③ "non-Galois lemma"
very important!

(4) Prove Thm 1.3.4 of [L22a]:
analysis of F_{dHN} of
isomonodromic deformation.

(4) (PRHS, Higgs, positivity.)
. Explain proof of main
thm of [L22a]

(5) Basics on $M(G_{\mathbb{R}}, r)$,
canonical reps. Explain
equivalence w/ local systems
on versal families.

(6) Use non-Galois to
prove cohomology rank
bound for a unitary local

system over a versal family.

⑦ (coho) rigidity (using
non-b(G)). Deduce
main thm for
unitary M(G-finite) reps.

⑧ thm for semi-simple
reps, arithmetic applications

⑨ Putnam-Wieland