

# Introduction to P-curvature.

Recall setup from the end of last lecture:

- $S/k$  smooth,  $\text{char}(k) = p$ .

$$\begin{array}{ccccc}
 S & \xrightarrow{\quad} & S' & \xrightarrow{\pi} & S \\
 \downarrow F_{S/k} & \nearrow & \downarrow & & \downarrow \\
 \text{Spec}(k) & \xrightarrow{\quad} & \text{Spec}(k) & \xrightarrow{\quad} & \text{Spec}(k)
 \end{array}$$

$F_{\text{rb}, k}$

Observation: for any  $E \in \text{coh}(S')$ ,  $\exists$

a "canonical" connection on  $F_{S/k}^*(E) \in \text{coh}(S)$

$$\text{Indeed, } F_{S/k}^*(E) \cong F_{S/k}^{-1}(E) \otimes \begin{cases} \mathcal{O}_S \\ F_{S/k}^{-1}(\mathcal{O}_{S'}) \end{cases}$$

$$\nabla^{\text{can}}(s \otimes f) := s \otimes df$$

Why is this well-defined?

Let  $\gamma \in \Omega_S$ . Then

$$\nabla^{\text{can}}(\gamma s \otimes f) := \gamma s \otimes df$$

???

$$\nabla^{\text{can}}(s \otimes F_{S/k}^*(\gamma)f) := s \otimes d(F_{S/k}^*(\gamma)f)$$

???

$$s \otimes df \checkmark$$

- Rmk In char 0, the following are true.
- $(E, \nabla) \in \text{MIC}(S/k) \Rightarrow E$  locally free
  - $S/k$  proper,  $\Rightarrow c_1(E) = 0$
  - $(E, \nabla) \in \text{MIC}(S/k)$

Both are false in char  $p$ .

Upshot If  $S/k$  is smooth and  $\text{char}(k) = p$ , then  $\exists$  a "canonical" subcategory of  $\text{MIC}(S/k)$ .

### Questions

- How do we describe this subcategory "intrinsically"? (i.e., when is an object of  $\text{MIC}(S/k)$  a Frobenius pullback?)
- What are properties of this abelian subcategory? E.g., is it thick?

To answer these questions, we return to the ring of crystalline differential operators.

# The Ring $D_S$

Learned from  
M. Gröchenig

Setup •  $S/k$  smooth,

- If  $\mathcal{Q} \subseteq k$ , say has "char  $\infty$ "

Def The ring  $D_S$  of crystalline differential operators is the quasi-coherent sheaf of  $D_S$  algs, described as the following quotient:

$$\bigoplus_{i \geq 0} (T_S^{\otimes i}) / \mathfrak{J}$$

$T_S \otimes_k T_S \otimes \dots \otimes T_S'$

where  $\mathfrak{J}$  is the 2-sided ideal generated by relations:

- $\partial \otimes f - f \otimes \partial - (\partial f)$
- $\partial_1 \otimes \partial_2 - \partial_2 \otimes \partial_1 - [\partial_1, \partial_2]$

Rmk

- $D_S$  is "universal enveloping algebra" of the Lie algebroid  $T\mathcal{S}|_k$
- $D_S$  is a filtered algebra:
$$D_S^{\leq d} := \text{Im} \left( \bigoplus_{i \leq d} T_S^{\otimes i} \rightarrow D_S \right)$$
- $\text{MIC}_S =$  "  $O_S$ -coherent  $D_S$  modules"

slogan: a flat connection is equivalent to the data of a  $D_S$ -action

Ex let  $S = A'_k = \text{Spec}(k[t])$ .

Then  $D_S$  is generated, as an algebra  $| k[+]$ , by the symbols  $t, \frac{\partial}{\partial t}$

w/ the relation  $[\frac{\partial}{\partial t}, t] = 1$ .

"Weyl alg":  $k\langle t, \partial \rangle$ . Note when  $\text{char}(k) = p$ ,  $\mathcal{Z}(k\langle t, \partial \rangle) = k[t^p, \partial^p]$ .

$$\underline{\text{Obs}} \quad D_S \rightarrow \underline{\text{End}}_k(\Omega_S)$$

Q: When is this faithful?

Lemma 1

$$\text{The map } D_S \xrightarrow{\leq \text{char}(k)-1} \underline{\text{End}}_k(\Omega_S)$$

is faithful

Pf

- First, consider the case that

$$S = \mathbb{A}_{\mathbb{K}}^n = \text{Spec } \mathbb{K}[t_1, \dots, t_n].$$

Then we claim:

$$\text{Let } s \in D_{\mathbb{A}^n}^{\leq d}, w \quad d \leq \text{char}(k)-1.$$

Then  $\exists f \in \mathbb{K}[t_1, \dots, t_n]$  of  $\deg \leq d$   
s.t.  $sf \neq 0$ .

Pf: Induction:  $s = s_{\leq d} + s_d$ ,

where  $s_{\leq d}$  is in  $D_S$

$$\text{and } \delta_d = \sum_{|\mathbf{j}|=d} a_{\mathbf{j}} \partial^{\mathbf{j}}$$

$$\frac{\partial^d}{\partial t_1^{j_1} \partial t_2^{j_2} \cdots \partial t_n^{j_n}}$$

If  $\delta_{d-1} \neq 0$ , then done by induction.

Else,  $\delta = \delta_d$ . Suppose

$$k[t_1, \dots, t_n] \ni a_j = a_{j_1, \dots, j_n} \neq 0.$$

Then  $f = \prod_{i=1}^n t_i^{j_i}$  is of deg  $d$ ,

$$\text{and } \delta f = \delta_d f = j_1! \cdots j_n! a_j$$

$$\neq 0 \quad \square$$

In general, use étale descent to reduce to this case.

$\square$

$$D_S \longrightarrow \text{gr } D_S$$

$\hookrightarrow$  "Symbol( $\delta$ )"

Thm (PBW)

There is an iso, natural in pullback  
along étale map,

$$\text{gr}(D_S) \simeq \text{Sym } T_S = \pi_* \underbrace{\Omega_{T^*S}},$$

$$\pi: T^*S \rightarrow S$$

Idea of Pf

"Étale coordinates"  $\Rightarrow$  reduce to (open  
subset of)  $A^n$

$$\delta \in D_{A^n}^{\leq 2}$$

$$\delta = \sum_{i=0}^2 \left( \sum_{|J|=i} a_J \partial^J \right),$$

w/  $a_J \in k[t_1, \dots, t_n]$

$$\cdot \delta \mapsto \sum_{|J|=d} a_J y^J \in \text{Sym}^d(T_S)$$

where " $y_i = \partial_i$ " is the corresponding vector field, gives a map

$$D_{A^n}^d / D_{A^n}^{d-1} \rightarrow \text{Sym}^d(T_{A^n})$$

$$\sim D_{A^n} \longrightarrow \text{Sym}^d(T_{A^n})$$

=

Recall that  $T^* S$  is canonically "symplectic": there exists a closed, non-degen 2-form  $\omega$ :

$$\omega = d\gamma, \text{ where } \gamma \text{ is the}$$

tautological 1-form on  $T^*S$ :

$$p : T(T^*S) \rightarrow T^*S$$

$\alpha_s$

Given  $\xi_{\alpha_s} \in T_{\alpha_s}(T^*S)$

$\eta$  is determined by the tautological

$$\text{map: } T_{\alpha_s}(T^*_s S) \rightarrow k$$

$$\xi_{\alpha_s} \longmapsto \alpha_s(p_* \xi_{\alpha_s})$$

=

Given (local) function  $f \in \Omega_s$ ,

$\omega$  induces a "Hamiltonian vector field"

$\xi_f$ :

$$\omega : T^*S \xrightarrow{\sim} TS \quad (\text{non deg})$$

$$df \mapsto X_f$$

Then  $\omega$  induces a Poisson bracket

on  $\pi \times \Omega_{T^*S}$

- bilinear
- skewsymmetric
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
- Leibniz:  $\{fg, h\} = f\{g, h\} - g\{f, h\}$

$$\cancel{\{f, g\}_\omega := \omega(X_f, X_g)}$$

"Example"

$$S = /A^*_k = \text{Spec}(k[t_1, \dots, t_n])$$

$$T^*S = \text{Spec}(k[t_1, \dots, t_n, y_1, \dots, y_n])$$

$$"y_i = \frac{\partial}{\partial t_i}"$$

$$\cdot \omega = \sum dt_i \wedge dy_i$$

$$\cdot \{f, g\} = \sum \frac{\partial f}{\partial t_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial t_i}$$

Lemma 3

$$\text{let } s_1, t \in D_s^{\leq d_1}, s_2 \in D_s^{\leq d_2} \\ \leq d_1 + d_2 - 1$$

$$\text{Then } [s_1, s_2] \in D_s^{d_1 + d_2 - 2}$$

$$[s_1, s_2]_{\text{mod }} D_s \\ = \{s_1 \text{ mod } D^{\leq d_1 - 1}, s_2 \text{ mod } D^{\leq d_2 - 1}\}$$

$$\text{If } [s_1, s_2] \notin D_s^{d_1 + d_2 - 2}, \text{ then}$$

$$\sigma[s_1, s_2] = \{ \sigma(s_1), \sigma(s_2) \}$$

Idea of Pf: coordinates

Lemma 3

Let  $f \in \pi_* \Omega_{T^* S} = \text{Sym}^\alpha T_S$   
 be a local section s.t.  $\{f, -\} = 0$   
 (i.e.,  $f$  is in the Poisson center).

Then  $\exists g \in \pi_* \Omega_{T^* S'}$  n Frobenius twist  
 s.t.  $f = F_{S/k}^* g$ .

Pf (coordinates):  $S = /A^1_K$ ,

$$T^*/A^1_K = \text{Spec}(k[t_1, \dots, t_n, y_1, \dots, y_n])$$

Suppose  $f \in k[t_1, \dots, t_n, y_1, \dots, y_n]$  has  
 at least one exponent that is not a  
 $p^{th}$  power. Then  $\exists i$  s.t.

$$\frac{\partial f}{\partial t_i} \quad \underline{\text{OR}} \quad \frac{\partial f}{\partial y_i} \quad \text{is non-zero.}$$

$$\text{If } \frac{\partial f}{\partial t_i} \neq 0, \quad \{f, y_i\} \neq 0$$

$$\frac{\partial f}{\partial t_i} \frac{\partial y_i}{\partial y_j} - \frac{\partial y_i}{\partial t_i} \frac{\partial f}{\partial y_j}$$

$$\text{If } \frac{\partial f}{\partial y_i} \neq 0, \quad \{f, t_i\} \neq 0$$

Hence:  $\{f, -y_i\} = 0 \Rightarrow$  all  
 exponents are multiples of  $p.$

### Birth of $p$ -curvature

Notation: let  $\partial$  be a local section of  $T_S.$

Then  $\underbrace{\partial \circ \dots \circ \partial}_P$  is again a  $\underbrace{\text{derivation}}_{\text{by binomial theorem}}$

$$\partial_S \rightarrow \partial_S$$

$\rightsquigarrow$  corresponds to a vector field,  
 $\partial^{[p]}$

$$\text{Def} \circ \iota : T_S \rightarrow D_S^{\leq p}$$

$$\partial \mapsto \underbrace{\partial}_\text{order p}^p - \underbrace{\partial}_\text{order 1}^{[p]}$$

, If  $(E, \nabla) \in \text{MIC}(S/k)$ ,

$$\sim \cap E$$

$$D_S$$

$\psi_\nabla(\partial) :=$  action of  $(\partial)$  on  $E$

is the  $p$ -curvature.

A priori,  $\iota$  is just a map of sheaves.

Example  $S = \mathbb{A}_{F_2}^1 = \text{Spec}(F_2[t])$

$$\partial = t \frac{\partial}{\partial t} \quad \partial^{[a]} : \mathcal{O}_S \rightarrow \mathcal{O}_S$$

$$f \mapsto t \frac{\partial}{\partial t} (t \frac{\partial f}{\partial t})$$

$$= t \frac{\partial f}{\partial t}$$

$$\begin{aligned}\partial^2 &= t \frac{\partial}{\partial t} \left( t \frac{\partial}{\partial t} \right) = t \left( \left( 1 + t \cdot \frac{\partial}{\partial t} \right) \frac{\partial}{\partial t} \right) \\ &= t \frac{\partial}{\partial t} + t^2 \frac{\partial^2}{\partial t^2}\end{aligned}$$

$$((\partial)) = t^2 \frac{\partial^2}{\partial t^2} \in D_{A' F_2}^{\leq 2}$$

Lemma 4  $\iota: T_S \rightarrow D_S^{\leq p}$  is

$p$ -linear:

$$\iota(f \partial_1 + \partial_2) = f^p \iota(\partial_1) + \iota(\partial_2)$$

Pf

Consider the symbols

$$\sigma(\iota(f \partial_1 + \partial_2)) = (f \partial_1 + \partial_2)^p$$

$$\sigma(f^p \iota(\partial_1) + \iota(\partial_2)) = f^p \partial_1^p + \partial_2^p$$

(by RHS is in a  
commutative ring!)

$$\Rightarrow \iota(f \partial_1 + \partial_2) - f^p \iota(\partial_1) - \iota(\partial_2) \in D_S^{\leq p-1}$$

Moreover, for any  $g \in \mathcal{O}_S$ ,

$$\begin{aligned} ((f\partial_1 + \partial_2)(g)) &= (f^p L(\partial_1) + L(\partial_2)) g \\ &= 0! \end{aligned}$$

[Lemma 1  $\Rightarrow \square$ ]

$\sim_L$  induces  $\mathcal{O}_S$ -linear map  $Frob_S^* T_S \rightarrow D_S$ ,

which induces:

- an  $\mathcal{O}_S$ -linear map

$$L: F_{S/k}^* T_{S'} \rightarrow D_S,$$

- an  $\mathcal{O}_{S'}$ -linear map

$$L: T_{S'} \rightarrow (F_{S/k})_* D_S$$

Example

$$S = \text{Spec}(k[t])$$

free rank 1 module

$$\begin{aligned} L: \overbrace{k[t] \cdot y}^{\text{free rank 1 module}} &\longrightarrow (F_{S/k})_a \quad k\langle t, y \rangle \\ f \cdot y &\longmapsto f^p y^p - (\underbrace{f \partial_0 \circ \dots \circ f \partial_0}_{p \text{ times}}) \end{aligned}$$

Lemma 5 : Let  $\partial'$  be a local section  
of  $T_S$ . Then

$$((\partial')) \in Z((F_{S/k})_*, D_S)$$

Pf Let  $\partial \in T_S$

$$[\overset{\text{deg } p}{\underbrace{((\partial))}}, \partial] \bmod D^{\leq p-1}$$

$$= \left\{ ((\partial) \bmod D^{\leq p-1}), \partial \right\}$$

$$= \left\{ (\sigma(\partial))^p, \partial \right\}$$

$$(\sigma((\partial))) = \sigma(\partial^p) = \sigma(\partial)^p$$

b/c  $\sigma$  has values in  
a commutative ring.)

$$= \emptyset$$

As this is true for all local sections  
 $\partial$  of  $T_S$ , we deduce:

$$\Rightarrow L(\partial) \in Z(F_{S/k} \rtimes D_S)$$

□

~)  $L$  induces a map

$$\tilde{\iota}: \text{Sym}^* T_S \xrightarrow{\sim} Z(F_{S/k} \rtimes D_S)$$

Lemma 6  $\tilde{\iota}$  is an isomorphism.

$$\text{Sym}^* T_S \rightarrow Z(F_{S/k} \rtimes D_S)$$

$$T_S \xrightarrow{F_{T_S/k}} (T_S)' \quad \text{as } \sigma = "g \circ F.e" \text{ kills lower order terms.}$$

- ~)  $\tilde{\iota}$  is injective ( $T_S$  is reduced)
- image is the Poisson center of  $(F_{S/k})_* \text{Sym}^* T_S$  (Lemma 3)

• Lemma 2 :  $\text{Im}(\sigma) \subseteq \text{Poisson center}$

$$\text{of } (\mathcal{F}_{S/k})_* \text{Sym}^* T_S$$

$$\Rightarrow \text{Im}(\sigma) = \text{Poisson center}$$

$\Rightarrow$   $\sigma$  isomorphism

$\Rightarrow$   $\tilde{\sigma}$  isomorphism!

□.

Lemmas 485 elegantly reprove  
 "classical" facts about the  $p$ -curvature.

Let  $S/k$  be a smooth var.,  $\text{char}(k) = p$ .

Theorem 7

$$(E, \nabla) \in \text{MIC}(S/k)$$

$$\sim \psi_\nabla : \mathcal{F}_{S/k}^* T_S \rightarrow \text{End}_{\mathcal{O}_S}(E) \subseteq \text{End}_k(E)$$

Pf lemma 5:  $\{(\sigma), f\} = 0$

Thm 8  $\psi_\sigma$  induces a map:

$$\psi: E \rightarrow E \otimes F_{S/k}^* \Omega^1_{S'/k}$$

Then this map  $\rightarrow$  flat (here,

$F_{S/k}^* \Omega^1_{S'/k}$  is equipped w/ the canonical connection.)

Pf Lemma 5:  $[c(\partial), \partial'] = 0 \cdot \square$

To see the utility of Lem 6, we need:

Lemma 6) The ring  $F_{S/k} \rtimes D_S$   
is an Azumaya algebra over its  
center  $\text{Sym}^2 T_S \xrightarrow{\sim} Z(F_{S/k} \rtimes D_S)$   
 $\uparrow$  derived from the p-curvature.

"Explanation" of Lemma 6' for  $A'_k$

$$\text{rank } P^2 \left\{ \begin{array}{l} k[t, \partial] \\ \text{UI} \\ k[t^P, y^P] \end{array} \right. \quad \begin{array}{l} w \cdot \partial = y^P \\ \cdot [\partial, \cdot] = 1 \end{array}$$

We claim this is an Azumaya alg.

$\Leftrightarrow$  it splits after a flat cover

C thm of Grothendieck

$$k[t, \partial] \longrightarrow k[t, \partial] \otimes_{k[t^P, y^P]} k[s, y^P]$$

UI

$$k[t^P, y^P] \longrightarrow k[s, y^P]$$

Key Claim:

$$k[s, y^P] \otimes_{k[t^P, y^P]} k[t, \partial] \xrightarrow{\sim} M_{P \times P} (k[t, y^P])$$

$$k[s, y^P] \otimes_{k[t^P, y^P]} k[t, \partial] \rightarrow E_{1,1}$$

mult on left,  
s acts as t.

mult on right

$k[t, \partial]$

$k[t^P, y^P]$

as a module

Rank In fact, the Azumaya  $\alpha^{\text{lg}}$   
 $(F_{S/k})_S \otimes D_S$  over  $T^* S'$

splits over the zero section.

Concretely:  $S = M^1_{/\kappa}$

Restatement  $k\langle t, \partial \rangle / \left( \partial^p = 0, [\partial, t] = 1 \right)$   
 $\cong k[t^\circ]$

isomorphic to  $M_{p \times p}(k[t^\circ])$

Abstract pf:  $(F_{S/k})_S \otimes D_S$  is Azumaya

$T^* S'$  of rank  $p^{2n}$ . On the other hand,  $(F_{S/k})_S \otimes D_S$  has rank  $p^n$  over  $S'$  and has an action of  $(F_{S/k})_S \otimes D_S$  (from  $\alpha$ )

$\Rightarrow (F_{S/k})_S \otimes D_S \mid_{\text{zero section}} \xrightarrow{\text{has action}}$   
 on rank  $p^n$  vector bundle

$\Rightarrow (F_{S/k})_S \otimes D_S$  splits.

In other words,

$$(F_{S|k})_* D_S \xrightarrow{N} \text{End}(F_{S|k} \otimes_s)$$

zero section  
 $s' \hookrightarrow T^* s'$

In coordinates,

$$k\langle t, \partial \rangle / \begin{matrix} \partial^p = 0 \\ [\partial, t] = 1 \end{matrix} \xrightarrow{N} \text{End}_{k[t^p]} k[1]$$

Thm (Cartier Descent)

$$\text{Coh}(S'/k) \longrightarrow \text{MIC}(S/k)$$

$$M \longmapsto (F_{S/k}^* M, \nabla^{(n)})$$

has essential image precisely those

$$(E, \nabla) \text{ s.t. } \Psi_\nabla = 0.$$

"Pf"  $(E, \nabla) \rightsquigarrow E$  is a  $D_S$ -module

Monta theory:

$F_{S/k}^* D_S$  is Azumaya over  $\mathcal{O}_{T^* S'}$

$\Rightarrow$   
Morita:  $\text{Coh}(S, D_S) \longrightarrow \text{Coh}(T^* S', F_{S/k}^* D_S)$   
equivalence

$$\pi_2(\text{Monta}(E)) \simeq F_{S/k}^* E$$

$$\text{in } \text{Coh}(S')$$

- $\psi_\triangleright$  induces an action of  
 $Z(F_{S/k} \star D_S) \cong \bar{\pi}_*^1 \mathcal{O}_{T^* S'}$   
on  $F_{S/k} \star E$ .  
 $\psi_\triangleright \equiv 0 \Leftrightarrow$  Morita ( $E$ ) supported  
on the zero-section  
 $S' \hookrightarrow T^* S'$   
 $(\zeta(\alpha) = 0 \wedge \alpha \in T_S)$
- $F_{S/k} \star D_S$  split over zero-  
section  $S' \hookrightarrow T^* S'$   
(induced by  $(0, \zeta)$ ).

May use these two facts to  
construct  $E'|_{S'}$  s.t.  
 $E'^{\nabla=0} \xrightarrow{\sim} E'$