# Computational Data Assimilation: Project 1

September 23, 2022

# 1 Resources

The following resources may be useful while working on the project, beyond the class notes:

1. The syllabus contains pointers to three data assimilation textbooks, by Asch et al., Law et al., and Reich et al. The Asch et al. book is written at an introductory level and is particularly well accessible.

2. The OTP Matlab package available from URL
`https://github.com/ComputationalScienceLaboratory/ODE-Test-Problems`.
It contains implementations of the Lorenz models, among others. Start with the readme file.

3. The DATools Matlab package available from
`https://github.com/ComputationalScienceLaboratory/DATools`.
It contains implementations of all the data assimilation algorithms we will discuss in class, including EnKF and EnSRF. Start with the readme file.

4. The MATLODE Matlab package available from
`https://github.com/ComputationalScienceLaboratory/MATLODE`.
It contains implementations of many time stepping algorithms and direct and adjoint sensitivity analyses.

## 2 The Lorenz-96 model

The Lorenz-96 model is described by the following system of ordinary differential equations [2]:

$$\frac{dx_i}{dt} = \begin{cases} (x_2 - x_{\mathrm{N_{state}}}) \cdot x_{\mathrm{N_{state}}} - x_1 + \varphi & \text{for } i = 1, \\ (x_{i+1} - x_{i-2}) \cdot x_{i-1} - x_i + \varphi & \text{for } 2 \le i \le \mathrm{N_{state}} - 1, \\ (x_1 - x_{\mathrm{N_{state}}-2}) \cdot x_{\mathrm{N_{state}}-1} - x_{\mathrm{N_{state}}} + \varphi & \text{for } i = \mathrm{N_{state}}, \end{cases} \tag{1}$$

which mimics fundamental properties of atmospheric dynamics. Typically $\mathrm{N_{state}} = 40$. This model exhibits extended chaos with an external forcing value $\varphi = 8.0$, when the solution is in the form of moving waves. For this reason, the model is adequate to perform basic studies of predictability. One time unit of the Lorenz-96 model (1) corresponds to 1.5 days of the atmosphere.

The observation operator for this problem is linear and selects only the odd-numbered states:

$$\mathbf{H} \in \mathbb{R}^{20\times 40}, \qquad \mathbf{H} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{40} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ \vdots \\ x_{39} \end{bmatrix}.$$

# 3 Question 1

1. Implement the Lorenz-96 model (1) and solve it using a Matlab ode solver (ode45 or one integrator selected from MATLODE). Start with an initial condition where the states are drawn from a random distribution (e.g., $\mathbf{x}_0 \sim \mathcal{N}(0, 16)$), then integrate for 5 time units. Save the result. This is our $\mathbf{x}_{-1}^{\text{true}}$. Propagate for $\Delta t$ time units, and save. This is our reference state $\mathbf{x}_0^{\text{true}}$. (You can use $\Delta t = 0.1$.)

2. Compute $\mathrm{N}_{\text{ens}}$ normal random perturbations $\varepsilon_i^{\text{b}} \sim \mathcal{N}(0, \sigma^2\, \mathbf{I}_{\mathrm{N}_{\text{state}}})$ with $\sigma = 0.2$. Add to $\mathbf{x}_{-1}^{\text{true}}$, and integrate the system starting from the perturbed initial conditions for $\Delta t$ time units. Save the resulting $\mathrm{N}_{\text{ens}}$ states $\mathbf{x}$. The members of this ensemble represent our set of background states $\mathbf{X}_0^{\text{b}}$.

3. Using a large number of ensemble members compute the ensemble covariance $\mathbf{P}_0^{\text{b}}$. Check its condition number. If it is large build a modified background covariance using $(1-\alpha)\mathbf{P}_0^{\text{b}} + \alpha \mathbf{I}_{\mathrm{N}_{\text{state}}}$ with a small $\alpha$ that leads to a reasonably small condition number of the resulting covariance. This is our $\mathbf{B}_0$.

4. Obtain $\mathbf{x}_0^{\text{b}}$ by selecting one of the states $\mathbf{X}_0^{\text{b}}$, e.g., the one with the smallest $\left\|\mathbf{X}_0^{\text{b}} - \mathbf{x}_0^{\text{true}}\right\|$.

5. Run the reference trajectory starting from $\mathbf{x}_0^{\text{true}}$ over the interval $[0, T]$. Save the solution every $\Delta t$ time units. At each $t_i = i\,\Delta t$ generate $\mathrm{N}_{\text{ens}}$ synthetic observations by adding to the reference solution $\mathbf{H}\,\mathbf{x}_i$ normal random noise drawn from $\mathcal{N}(0, \mathbf{R})$, $\mathbf{R} = r^2\, \mathbf{I}_{\mathrm{N}_{\text{obs}}}$ with $r = 0.025$.

## 4 Question 2

1. Consider the data assimilation problem posed over the interval $[0, T]$, starting with initial background states from $\mathcal{N}(\mathbf{x}_0^{\mathrm{b}}, \mathbf{B}_0)$. Observations are available at times $t_i = i\,\Delta t$, $i = 0, \ldots, N$, $t_N = T$. Use your synthetic observations, observation error covariance, and observation operator from above.

2. Solve the data assimilation problem using the standard (perturbed observations) version of EnKF. You will need to implement the filter, run it, and assess the quality of the results.

3. Solve the data assimilation problem using a square root filter (EnSRF), e.g., ETKF. You will need to implement the filter, run it, and assess the quality of the results.

4. For each case plot:

   - The time evolution of the error in the analysis ensemble mean, compared to the reference trajectory.
   - The rank histogram for one of the observations.

5. Apply covariance localization. Use a Gaspari-Cohn function with a decorrelation parameter $L = 5$. Repeat the EnKF and ETKF numerical experiments, and compare the results against the analysis obtained without localization.

6. In addition to localization, apply multiplicative covariance inflation with a parameter $\alpha = 1.02$. Again, repeat the EnKF and ETKF numerical experiments, and compare the results against the analysis obtained before.

# 5 What to submit

1. A well articulated report detailing your work and the results you obtained, and
2. The well-organized, structured, and commented code you developed for each of the steps.

# References

[1] Edward N. Lorenz. Deterministic nonperiodic flow. *Journal of Atmospheric Sciences*, 20(2):130 – 141, 1963.

[2] E.N. Lorenz. Predictability: a problem partly solved. In *Seminar on Predictability, 4-8 September 1995*, volume 1, pages 1–18, Shinfield Park, Reading, 1995. ECMWF, ECMWF.