

## MATH 410: NOTES

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### CONTENTS

1. Groups	2
1.1. Group Actions and Orbit-Stablizer Theorem	2
1.2. Centralizers and Normalizers	3
1.3. The Symmetric Group	3
1.4. Dihedral Group	4
2. Cosets	4
2.1. Normal Subgroups	5
3. Conjugacy Classes	5
4. Sylow Theorems	6
4.1. Counting to Force Normalization of a Sylow $p$ -subgroup	6
5. Semidirect Products	6
5.1. Inner Direct Product	7
6. Rings	8
6.1. Euclidean Domains	8
6.2. UFD	9
6.3. Polynomial Rings	10
6.4. Irreducibility of Polynomials	11
7. Assortment of Exercises since last Midterm	12
7.1. 8.1: Euclidean Domains	12
7.2. 8.2 PID	13
7.3. 8.3 UFD	13
7.4. Polynomial Rings that are UFDS	13
7.5. 9.4: Irreducibility Criteria	14
8. First Midterm:	15
9. Ring Theory Exercises and Notes done long ago	17

## 1. GROUPS

### 1.1. Group Actions and Orbit-Stabilizer Theorem.

**Definition 1.1.** Let  $G$  be a group and  $A$  be a set. Then a (left) **group action** is a map  $\alpha: G \times A \rightarrow A$  such that the following two axioms are satisfied for all  $g, h \in G$  and  $a \in A$ :

- $\alpha(e, x) = x$ , and
- $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ .

The usual convention for the notation of a group action is that we will denote it simply using a dot when the context is clear, that is, if we let  $G$  act on a set  $A$ , then  $e \cdot x = x$  and  $g \cdot (h \cdot x) = (gh) \cdot x$  is the way we denote the two needed axioms above.

**Example 1.1.** The standard example is that if we have  $G = S_n$  (or a  $G$  be a subgroup of  $S_n$  such as  $A_n$ ) and  $A = \{1, 2, \dots, n\}$ . Then  $G$  acts on  $A$  by the formula  $g \cdot x = g(x)$ ; that is, we're taking a bijective map  $g: G \rightarrow G$  and  $\alpha(g, x) = g(x) \in X$ . The first axiom is  $e \cdot x = e(x) = x$  is clear as  $e: G \rightarrow G$  is the trivial mapping, and the second axiom is simple: given two bijective maps  $g, h: G \rightarrow G$ , then  $g \cdot (h \cdot x) = g \cdot (h(x)) = g(h(x)) = (g \circ h)(x) = (gh) \cdot x$ . Furthermore, to make things clear, let  $A = \{1, 2, 3\}$ . Then if we have the cycle  $(1 \ 2 \ 3)$ , then  $(1 \ 2 \ 3) \cdot x = (1 \ 2 \ 3)(x)$  and so checking all values in  $A = \{1, 2, 3\}$ , then  $(1 \ 2 \ 3)(1) = 2$ ,  $(1 \ 2 \ 3)(2) = 3$ ,  $(1 \ 2 \ 3)(3) = 1$ . For great measure consider the composition of two cycles  $(1 \ 2 \ 3)(1 \ 3 \ 2)$  in  $S_3$ . Then we look towards checking the second axiom:  $(1 \ 2 \ 3) \cdot (1 \ 3 \ 2) \cdot x = (1 \ 2 \ 3) \cdot (1 \ 3 \ 2)(x)$ , and checking all values of  $x \in A$ , then  $(1 \ 3 \ 2)(1) = 3$ ,  $(1 \ 3 \ 2)(2) = 1$ , and  $(1 \ 3 \ 2)(3) = 2$ , and so  $(1 \ 2 \ 3)((1 \ 3 \ 2)(1)) = (1 \ 2 \ 3)(3) = 1$ ,  $(1 \ 2 \ 3)((1 \ 3 \ 2)(2)) = (1 \ 2 \ 3)(1) = 2$ , and  $(1 \ 2 \ 3)((1 \ 3 \ 2)(3)) = (1 \ 2 \ 3)(2) = 3$ . Now that we've checked this, we should have that this condides with the other side of this axiom: composing  $(1 \ 2 \ 3)(1 \ 3 \ 2)$  gives us  $(1)(2)(3) = e$  the identity permutation. Thus  $(1)(2)(3) \cdot x = (1)(2)(3)(x)$  gives us  $(1)(2)(3)(1) = 1$ ,  $(1)(2)(3)(2) = 2$ , and  $(1)(2)(3)(3) = 3$ . Thus the permutations agree on all values of  $x \in A$ . The second axiom is satisfied for these two cycles, but we've already done the general case in the first few sentences.

**Definition 1.2.** Let  $G$  be a group acting on a set  $X$ .

- A **fixed point** of an element  $g \in G$  is an element  $x \in X$  such that  $g \cdot x = x$ .
- The **stabilizer**  $G_x$  of a point  $x \in X$  is the set of elements  $g \in G$  such that  $x$  is a fixed point of  $g$ , i.e.  $G_x = \{g \in G: g \cdot x = x\}$ .
- The **orbit** of an element  $x \in X$  is the set of elements  $y \in X$  such that  $g \cdot x = y$  for some  $g \in G$ , and the orbit of  $x$  is denoted by  $G \cdot x = \{g \cdot x: g \in G\}$ .

**Remark 1.1.** We say that a group action is *transitive* if and only if it has exactly one orbit, that is, if there exists  $x \in X$  with  $G \cdot x = X$ . This is true if and only if  $G \cdot x = X$  for all  $x \in X$ . Or, rather, a group action is transitive if for any two  $x, y \in X$  there exists a  $g \in G$  such that  $g \cdot x = y$ . The set of all orbits of  $X$  under the group action of  $G$  is written as  $X/G$ , and is called the *quotient* of the action. A  $G$ -invariant element is an element  $x \in X$  such that  $g \cdot x = x$  for all  $g \in G$ . The set of all such  $x$  is denoted  $X_G$  and is called the  $G$ -invariants of  $X$ . When  $X$  is a  $G$ -module, i.e.  $X$  is an additive abelian group and  $G$  is acting on it with the additional action axiom that  $g \cdot (a + b) = g \cdot a + g \cdot b$  with  $a, b \in X$ , then  $X^G$  is the zeroth cohomology group of  $G$  with coefficients in  $X$ , that is,  $H^0(G; X) = X^G$ .

For each  $x \in X$ , the **stabilizer** of  $x$  by  $G$  is defined as the set  $G_x = \{g \in G: g \cdot x = x\}$ , and another common notation for this set is  $\text{stab}(x)$ . Sometimes this set is defined as being the "stabilizer subgroup" of  $G$ . The reason should be clear: Clearly  $G_x \subseteq G$  and  $G_x \neq \emptyset$  as  $e \in G$  and  $e \cdot x = x$  by the action axioms, and so  $e \in G_x$ . Now take  $a, b \in G_x$ , then we need to show  $ab^{-1} \in G_x$  (recall this is the subgroup test). So, as  $a \cdot x = x$  and  $b \cdot x = x$ , and as  $e \cdot x = x$ , then  $e \cdot x = bb^{-1} \cdot x = b^{-1} \cdot b \cdot x = b^{-1} \cdot x = x$ ; thus  $b^{-1} \in G_x$ . Then  $ab^{-1} \cdot x = a \cdot (b^{-1} \cdot x) = a \cdot x = x$ . Therefore  $G_x \leq G$ . This is typically not a normal subgroup, however.

The action of  $G$  on  $X$  is *free* if and only if all stabilizers are trivial; that is, for all  $x \in X$ , we have that  $G_x = \{e_G\}$ . An action is called *faithful* if we have that the intersection of all stabilizers is trivial  $\{e_G\}$ . Thus, if we let  $E = \{G_x\}_{x \in X}$  denote the set of stabilizers for all  $x \in X$ , then the action of  $G$  on  $X$  is free if and only if  $\bigcap E = \{e_G\}$ ; the action of  $G$  on  $X$  is called faithful if  $\bigcap E = \{e_G\}$ . Furthermore, we can reformulate this as saying an action is faithful if for two distinct elements  $g \neq h$  in  $G$  there exists an  $x \in X$  such that  $g \cdot x \neq h \cdot x$ ; contrapositively, if for all  $x \in X$ , we have  $g \cdot x = h \cdot x$ , then  $g = h$ . Moreover, an action is called free if for any two distinct elements  $g \neq h$  in  $G$  and all  $x \in X$  we have  $g \cdot x \neq h \cdot x$ ; contrapositively, if we have  $g \cdot x = h \cdot x$  for some  $x \in X$ , then  $g = h$  in  $G$ . We can see how related these two notions are to one another. In fact, as you should've seen, a free action is faithful.

**Proposition 1.1.** A group  $G$  acts faithfully on  $X$  if and only if the homomorphism  $G \rightarrow S_X$  has a trivial kernel.

*Proof.* Let  $G$  be a group acting on  $X$  and let the action be faithful. For all  $g \in G$ , define  $\sigma_g: X \rightarrow X$  where  $\sigma_g: x \mapsto g \cdot x$ ; this map is a bijection from  $X \rightarrow X$ . Now define the map  $\varphi: G \rightarrow S_X$  by  $\varphi: g \mapsto \sigma_g$ . Take  $g, h \in G$ . Then  $\varphi(gh) = \sigma_{gh}$ , and so take some  $x \in X$ . So  $\sigma_{gh}(x) = gh \cdot x = g \cdot (h \cdot x) = \sigma_g(\sigma_h(x)) = (\sigma_g \sigma_h)(x)$ ; thus  $\varphi(gh) = \sigma_g \sigma_h = \varphi(g)\varphi(h)$  and  $\varphi$  is a homomorphism. Now consider the kernel of this maps, i.e.  $\ker(\varphi) = \{z \in G: \varphi(z) = \sigma_z = \text{id}_X\}$ . If we take the trivial  $e \in G$  then  $\varphi(e) = \sigma_e$ , and so on every element  $x \in X$ ,

$\sigma_e(x) = e \cdot x = x$  by the first action axiom, and we see that  $\sigma_e = \text{id}_X$ ; therefore  $e_G \in \ker(\varphi)$ . Suppose that we have some other element  $q \in \ker(\varphi)$  distinct from  $e_G$ . Then we have that  $\sigma_q = \sigma_e$ , i.e. for all  $x \in X$ ,  $\sigma_q(x) = q \cdot x = e \cdot x = \sigma_e(x)$ , but as the action is faithful then  $q \cdot x = e \cdot x$  implies that  $q = e$ . Therefore the kernel is trivial.

For the opposite direction, suppose that the homomorphism we define in the first part of the proof has trivial kernel. As the homomorphism has trivial kernel, then we have that  $\varphi: G \rightarrow S_X$  is injective. Suppose that we have  $\varphi(g) = \varphi(h)$  for some  $g, h \in G$ . Then  $\sigma_g = \sigma_h$ , for all  $x \in X$ ,  $\sigma_g(x) = \sigma_h(x)$  so  $g \cdot x = h \cdot x$  and as  $\varphi$  is injective, we have  $g = h$ . Therefore the action is faithful.  $\square$

**Theorem 1.1 (Orbit-Stabilizer).** Let  $G$  be a group which acts on a finite set  $X$  and fix  $x \in X$ . Then  $|G \cdot x| = [G : G_x]$ .

**1.2. Centralizers and Normalizers.** Let  $G$  be a group and let  $A \subseteq G$ .

**Definition 1.3.** Define  $C_G(A) = \{g \in G : gag^{-1} = a \text{ for all } a \in A\}$ . This subset of  $G$  is called the **centralizer** of  $A$  in  $G$ . Since  $gag^{-1} = a$  if and only if  $ga = ag$ ,  $C_G(A)$  is the set of elements of  $G$  which commute with every element of  $A$ .

We can clearly see that  $C_G(A) \subseteq G$  and  $C_G(A) \neq \emptyset$  as  $ea e^{-1} = ea = a$  so  $e \in C_G(A)$ . Now take  $x, y \in C_G(A)$ . Then  $x, y \in G$  and  $xax^{-1} = a = yay^{-1}$ . Lastly let's consider  $xy^{-1} : (xy)a(xy)^{-1} = (xy)a(y^{-1}x^{-1}) = x(yay^{-1})x^{-1} = xax^{-1} = a$ , and so  $xy^{-1} \in C_G(A)$ . Therefore the centralizer of  $A$  in  $G$  is a subgroup of  $G$ , i.e.  $C_G(A) \leq G$ .

Now if  $G$  is abelian, then we clearly see that  $ga = ag$  for all  $g \in G$  and  $a \in A$ , so  $C_G(A) = G$ .

**Definition 1.4.** Define  $Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}$ , the set of elements commuting with all the elements of  $G$ . This subset of  $G$  is called the **center** of  $G$ .

We can make the clear observation that  $C_G(G) = Z(G)$  and so we also have that  $Z(G) \leq G$ . If we do this directly, however, in good spirit,  $Z(G)$  is trivially nonempty and obviously a subset of  $G$ . If we take  $x, y \in Z(G)$ . Then  $yt = ty$  and  $xs = sx$  for all  $s, t$ . Now  $et = te$ , that is,  $y^{-1}yt = y^{-1}ty = ty y^{-1}$ , multiplying by  $y$  we get  $y^{-1}ty^2 = ty$  so  $y^{-1}ty = t$  which gives  $y^{-1}t = t^{-1}$ ; thus  $y^{-1} \in Z(G)$ . Lastly, take some  $l \in G$ , then  $xy^{-1}l = xly^{-1} = lxy^{-1}$ . Therefore  $Z(G)$  is a subgroup of  $G$ .

**Definition 1.5.** Define  $gAg^{-1} = \{gag^{-1} : a \in A\}$ . Define the **normalizer** of  $A$  in  $G$  to be the set  $N_G(A) = \{g \in G : gAg^{-1} = A\}$ .

The reason for defining the normalizer in this manner is possibly the main cannon of confusion for this definition: for the set's needed condition  $gAg^{-1} = A$ , we're saying that  $gag^{-1} = a'$  for some  $a, a' \in A$  and we don't necessarily have that  $a = a'$ . Note that in the case where they are equivalent however, then we would have this element  $g \in G$  that makes  $gag^{-1} = a$  a true statement so  $g \in C_G(A)$ . Thus we see can predict, with good vision, that  $C_G(A) \leq N_G(A)$ . Furthermore, we have that  $N_G(A) \leq G$ .

**1.3. The Symmetric Group.** Let  $X$  be any set. Then we define the **symmetric group** to be the set  $\mathfrak{S}_X = \{\varphi : X \rightarrow X : \varphi \text{ is a bijection}\}$ . You can easily verify that this is in fact a group with the group operation being the usual composition of functions. Typically, we deal simply with the set  $X = \{1, 2, 3, \dots, n\}$ . We assume familiarity with cycle notation.

**Proposition 1.2.** If the pair of cycles  $\alpha = (a_1 a_2 \dots a_m)$  and  $\beta = (b_1 b_2 \dots b_n)$  have no common entries, i.e. they're disjoint, then  $\alpha\beta = \beta\alpha$ .

*Proof.* Let  $a_i$  be in  $\{a_1, a_2, \dots, a_m\}$ . Then  $\alpha\beta(a_i) = \alpha(\beta(a_i)) = \alpha(a_i) = a_{i+1}$ , and  $\beta\alpha(a_i) = \beta(\alpha(a_i)) = \beta(a_{i+1}) = a_{i+1}$ . Similarly, let  $b_i$  be in  $\{b_1, b_2, \dots, b_n\}$ . Then  $\alpha\beta(b_i) = \alpha(\beta(b_i)) = \alpha(b_{i+1}) = b_{i+1}$ , and  $\beta\alpha(b_i) = \beta(\alpha(b_i)) = \beta(b_i) = b_{i+1}$ .  $\square$

**1.3.1. Transpositions and the Alternating Group.** First we need to recall that any element of  $\mathfrak{S}_n$  can be written as a product of disjoint cycles in a *unique* way. However, if we don't write the cycle as a product of disjoint cycles, then the representation of the cycle can take many forms: for example,  $\sigma = (1 \ 2 \ 3)$  in  $\mathfrak{S}_3$  may be written as  $(1 \ 3)(1 \ 2)$ ,  $(1 \ 2)(1 \ 3)(1 \ 2)(1 \ 3)$ , and there are more representations (in fact, infinitely many ways!). And so, if we allow for the cycle to not be disjoint, then we're destroying the uniqueness of a representation of a permutation as a product of cycles.

**Remark 1.2.** Once again, recall that a 2-cycle is called a transposition, and for any  $\mathfrak{S}_n$ , we may write any  $\sigma \in \mathfrak{S}_n$  as a product of two cycles (however not in a unique way). For example, we have  $(1 \ 2 \ 3) = (1 \ 3)(1 \ 2)$  for  $\mathfrak{S}_3$ . In general, if  $\sigma = (a_1 a_2 \dots a_m) \in \mathfrak{S}_n$ , then we can write this as a product of transpositions:  $\sigma = (a_1 a_m)(a_1 a_{m-1})(a_1 a_{m-2}) \dots (a_1 a_2)$ . And so, we can say that  $\mathfrak{S}_n$  is finitely generated if  $n < \infty$  by the given set  $\langle T \rangle = \langle \{(i \ j) : 1 \leq i < j \leq n\} \rangle = \mathfrak{S}_n$ .

**1.4. Dihedral Group.** In general, the **dihedral group** is the group of symmetries of a regular polygon, where the elements in the groups are representative of rotations and reflections. The group is defined as  $D_{2n} = \langle r, s : r^n = s^2 = 1, rs = sr^{-1} \rangle$ . We use the convention of denoting the dihedral group as  $D_{2n}$  rather than  $D_n$  as  $|D_{2n}| = 2n$ , as opposed to using  $D_n$  to denote the  $n$ -gon (e.g. for the group of symmetries of a triangle,  $D_{2(3)} = D_6$  rather than  $D_3$ ). We can note that, taking the geometric point of view, that  $s \neq r^i$  for any  $i$ , as if we have a regular polygon (e.g. take a square) then if we rotate it as many times as we please, then it wouldn't make any sense for it to be equivalent to a rotation along any axis (label and work it with a square to help visual aid). Furthermore, we also have that  $sr^i \neq sr^j$ , for all  $0 \leq i, j \leq n-1$  with  $i \neq j$ , so a presentation of  $D_{2n}$  in terms of looking at its elements,

$$D_{2n} = \{1, r, r^2, \dots, r^{n-1}, s, sr, sr^2, \dots, sr^{n-1}\}.$$

**Remark 1.3.** As we've seen in the definition of  $D_{2n}$  we have that  $rs = sr^{-1}$  for all  $s, r$ , but we in fact have  $r^i s = sr^{-i}$  for all  $i \geq 1$ . We proceed by induction: the base case is trivial, as we have it by definition. Let  $r^k s = sr^{-k}$  for all  $k \geq 1$ . Then  $r(r^k s) = (r r^k s) = r^{k+1} s = r(sr^{-k}) = (rs)r^{-k} = (sr^{-1})r^{-k} = s(r^{-1}r^{-k}) = s^{-1(k+1)}$ , and we're done.

In general, all the elements of  $D_{2n}$  have a (unique) representation in the form  $s^k r^i$ ,  $k = 0$  or  $1$  and  $0 \leq i \leq n-1$ . If we take  $x, y \in D_{2n}$ , then  $x = s^k r^i$  and  $y = s^l r^j$  and for now we make no assumptions about the exponents on  $s$ . Then  $xy = (s^k r^i)(s^l r^j) = s^k (r^i s^l) r^j = s^k (s^l r^{-i}) r^j = s^{k+l} r^{j-i}$ . Thus this works, and it is clear that any assumptions upon the exponents  $l$  and  $k$  would've not changed the outcome.

We can note here that  $D_2$  is isomorphic to  $Z_2$ , which makes  $D_2$  abelian, and that  $D_4$  is isomorphic to  $K_4$ , the Klein four-group. However,  $D_2$  and  $D_4$  are the exception as no other dihedral group is abelian beyond these two examples. Moreover, we have that  $D_{2n}$  is a subgroup of the symmetric group  $S_n$  for  $n \geq 3$ .

## 2. COSETS

Let  $G$  be a group and let  $H \leq G$ . Then we define a (left) *coset* to be the set  $gH = \{gh : h \in H\}$  for each  $g \in G$ . The element  $g$  of the coset  $gH$  is said to be a *representative* of the coset. Moreover, the collection of left cosets is denoted  $G/H$ . That is to say,  $G/H = \{gH : g \in G\}$ . The common short-hand notation is to denote a representative in  $G/H$  by  $\bar{g}$ , however, we will omit that largely here as it is perhaps clearer to not use it. To provide some interest, we should also note here that we can say, equivalently, that a left coset is an equivalence class of  $G/\sim$  where  $g_1 \sim g_2$  if and only if  $g_1 = g_2 h$  for some  $h \in H$ . So, two cosets equivalent if  $g_1 H = g_2 H$  as  $g_1 h = g_2 l$  for some  $h, l \in H$ , so  $g_1 = g_2 (l h^{-1}) = g_2 h_2$  where  $h_2 = l h^{-1} \in H$ .

**Proposition 2.1.** Let  $G$  be a group and let  $H \leq G$ . The left cosets of  $H$  in  $G$  form a partition of  $G$ .

*Proof.* Let  $g \in G$ . Since  $1 \in H$ , it follows that  $g \cdot 1 = g \in gH$ , so every element of  $G$  is in some coset of  $H$  and so the union of all cosets is all of  $G$ . By contrapositive, assume that we have  $g_1 H$  and  $g_2 H$  and  $g_1 H \cap g_2 H \neq \emptyset$ . Then we have at least one element in both cosets, so let  $x \in g_1 H \cap g_2 H$ . Then we may write  $x$  as  $x = g_1 h_1$  and  $x = g_2 h_2$ , which means that  $g_1 h_1 = g_2 h_2$ , and so  $g_1 = g_2 h_2 h_1^{-1}$ ; each element of  $g_1 H$  has the form  $g_1 h$  for some  $h \in H$ , and  $g_1 h = (g_2 h_2 h_1^{-1}) h = g_2 (h_2 h_1^{-1} h) \in g_2 H$ . Thus we see that  $g_1 H \subseteq g_2 H$ , and the other inclusion follows by a similar argument. Therefore  $g_1 H = g_2 H$ .  $\square$

**Proposition 2.2.** Let  $H \leq G$ . Then any two cosets have the same order.

*Proof.* Take  $g, k \in G$ . Then let  $\varphi: gH \rightarrow kH$  by  $\varphi: gh \mapsto kh$ . This map is well defined as if we let  $gh = gh'$ , then  $g^{-1}gh = h' = h$ . We can also define the map  $\psi: kH \rightarrow gH$  by  $\psi: kh \mapsto gh$ , and this map is indeed well defined. Now take  $gh \in gH$ , then  $(\psi\varphi)(gh) = \psi(\varphi(gh)) = \psi(kh) = (gh) = \text{id}(gh)$ , and similarly, if we take  $kh \in kH$  then  $\varphi(\psi(kh)) = \varphi(gh) = kh = \text{id}(kh)$ . Therefore we have a bijection between the two cosets and thus their cardinalities are the same.  $\square$

**Corollary 2.1.** Any left coset has the same order as  $H$ .

**Definition 2.1.** If  $H \leq G$  then the **index** of  $H$  in  $G$ , written  $[G : H]$ , is the number of left (or right) cosets of  $H$  in  $G$ . That is,  $[G : H] = |\{gH : g \in G\}|$

**Remark 2.1.** If  $G$  is finite, then  $[G : H] = |G|/|H|$ . Moreover, if  $K \leq H \leq G$  then  $[G : K] = [G : H][H : K]$  as we have

$$[G : H][H : K] = \frac{|G|}{|H|} \cdot \frac{|H|}{|K|} = \frac{|G|}{|K|} = [G : K].$$

We will note here, without proof, that this is also true when considering infinite groups.

## 2.1. Normal Subgroups.

**Definition 2.2.** If  $G$  is a group and  $H \leq G$ , and for all  $g \in G$  we have  $gH = Hg$ , then we say that  $H$  is a **normal subgroup** of  $G$ , and write  $H \triangleleft G$ .

**Lemma 2.1.**

- (i) Every subgroup of an abelian group  $G$  is normal.
- (ii) A subgroup  $H$  is normal in  $G$  if and only if for all  $g \in G$  and  $h \in H$ , then  $ghg^{-1} \in H$

*Proof.* Suppose that  $G$  is an abelian group and  $H \leq G$ . Then if we take  $g \in G$  and  $gh \in gH$ , then  $gh = hg \in Hg$ . So the equality of sets is obvious. Now for (ii): If we have a normal subgroup  $H$  of  $G$ , then  $gH = Hg$ , so if we take some  $gh \in gH$ , then  $gh = h'g \in Hg$ , so  $ghg^{-1} = h' \in H$ . For the opposite direction, take  $g \in G$  and  $h \in H$  and suppose  $ghg^{-1} = h' \in H$ . Then if  $gh = h'g$ , so  $gh \in Hg$  and  $h'g \in gH$ , so we can see a containment of sets, i.e.  $gH = Hg$  as we chose everything arbitrarily.  $\square$

**Proposition 2.3.** Let  $\varphi: G \rightarrow H$  be a group homomorphism.

- (i)  $\ker \varphi \triangleleft G$ .
- (ii) If  $K \triangleleft H$ , then  $\varphi^{-1}(K) \triangleleft G$ .

*Proof.* For (i):  $\ker \varphi$  is trivially nonempty, and so take  $x, y \in \ker \varphi$ . So  $\varphi(1) = 1 = \varphi(yy^{-1}) = \varphi(y)\varphi(y^{-1}) = \varphi(y^{-1})$ ; thus  $y^{-1} \in \ker \varphi$ . Now  $\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1}) = 1$ ; thus  $xy^{-1} \in \ker \varphi$  and  $\ker \varphi \leq G$ . Take  $g \in G$  and  $h \in \ker \varphi$ . Then  $\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) = \varphi(g)\varphi(g^{-1}) = \varphi(gg^{-1}) = 1$ . Therefore  $ghg^{-1} \in \ker \varphi$  and  $\ker \varphi \triangleleft G$ . For (ii): Let  $K \triangleleft H$ . Then recall that the preimage is defined as  $\varphi^{-1}(K) = \{g \in G: \varphi(g) \in K\}$ . This set is nonempty as  $\varphi(1) = 1 \in K$ . Take  $x, y \in \varphi^{-1}(K)$ . Now  $\varphi(1) = 1 = \varphi(yy^{-1}) = \varphi(y)\varphi(y^{-1}) \in K$ . Clearly we will have that  $xy^{-1} \in K$ , so  $K \leq G$ . Now take  $h \in \varphi^{-1}(K)$  and  $g \in G$ . Then  $\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) \in \varphi(g)K\varphi(g^{-1}) \subseteq K$ , as  $K$  is normal. Thus  $ghg^{-1} \in \varphi^{-1}(K)$  and  $\varphi^{-1}(K) \triangleleft G$ .  $\square$

**Proposition 2.4.** If  $H \leq G$  and  $[G: H] = p$  for some prime  $p$ , then  $H \triangleleft G$ .

## 3. CONJUGACY CLASSES

In a group  $G$ , two elements  $h$  and  $\ell$  are called *conjugate* when  $h = g\ell g^{-1}$  for some  $g \in G$ . Recall that for  $H \leq G$ , the *conjugate* subgroup of  $H$  by a fixed  $g \in G$  is

$$gHg^{-1} = \{ghg^{-1}: h \in H\}.$$

If we fix some element  $\ell \in G$ , we may seek to know which, and how many, elements in  $G$  can be written as  $g\ell g^{-1}$  for some  $g \in G$ . A trivial observation is that we may write the  $\ell$  itself as a conjugate of  $\ell$ , that is,  $\ell(\ell)\ell^{-1} = \ell(1) = \ell$ . We wish to prescribe a name to all these elements that can be written in terms of conjugacy by a fixed element (in our case,  $\ell$ ): we denote the set of all such elements in  $G$  by  $\text{cl}_G(\ell)$  and it is called the *conjugacy class* of  $\ell$ . Formally, for an element  $q \in G$ , its *conjugacy class* is the set of all elements to it, i.e.  $\text{cl}_G(x) = \{gxg^{-1}: g \in G\}$ .

**Remark 3.1.** In any group,  $\text{cl}_G(1) = \{1\}$ , as  $g1g^{-1} = gg^{-1} = 1$  for all  $g \in G$ . Moreover, if  $x \in Z(G)$ , then  $gxg^{-1} = x$  for all  $g \in G$ , so  $\text{cl}_G(x) = \{x\}$ . The converse is true here as well: if a conjugacy class has size 1, then it is in the center of the group. Moreover, if  $G$  is abelian then every element is its own conjugacy class.

**Proposition 3.1.** Conjugacy is an equivalence relation.

*Proof.* In this instance, two elements  $x \sim y$  if and only if there is a  $g \in G$  such that  $x = gyg^{-1}$ . Now  $x = 1x1^{-1} = x$ , so  $x \sim x$ . Now let  $x \sim y$  such that  $x = gyg^{-1}$ , so then  $xg = gy$  and  $g^{-1}xg = y$ ; thus  $y \sim x$ . Lastly, let  $x \sim y$  and  $y \sim z$  with  $x = gyg^{-1}$  and  $y = hzh^{-1}$ . Then  $x = g(hzh^{-1})g^{-1} = (gh)z(gh)^{-1}$ , and so  $x \sim z$ .  $\square$

**Remark 3.2.** As conjugacy is an equivalence relation, then it forms a partition of  $G$ . Thus the size of  $G$  is the sum of all conjugacy classes, that is,  $|G| = \sum_{\ell \in G} |\text{cl}_G(\ell)|$ .

**Definition 3.1.** For a group  $G$  and  $g \in G$ , its **centralizer**  $Z(g)$  is the set of elements of  $G$  commuting with  $g$ ; that is,

$$Z(g) = \{x \in G: xg = gx\}$$

A clear observation that follows from this definition is that if we intersect all centralizers of fixed elements in  $G$ , then we get the whole centralizer of the group, i.e.  $\bigcap_{g \in G} Z(g) = Z(G)$ .

**Theorem 3.1.** For each  $g \in G$ , its conjugacy class has the same size as the index of its centralizer:

$$|\text{cl}_G(g)| = [G: Z(g)].$$

**Corollary 3.1.** For any finite group  $G$ ,

$$|G| = |Z(G)| + \sum |\text{cl}_G(x_i)|,$$

where the sum is taken over distinct conjugacy classes of size greater than 1.

*Proof.* As we showed earlier, conjugacy classes partition  $G$ , so  $|G| = \sum_{\ell \in G} |\text{cl}_G(\ell)|$ , and by Theorem 4.1., we have  $|G| = \sum_{\ell \in G} [G : Z(g)] = \sum_{\ell \in G} |G|/|Z(g)|$ . If we take some  $w \in G$ , and  $|\text{cl}_G(w)| = 1$ , then we must have that the element in  $\text{cl}_G(w)$  is  $w$  itself: as every conjugacy class itself is in it as  $w = 1w1^{-1} = w$ , so this is necessary. Moreover, if  $\text{cl}_G(w) = \{w\}$ , then we have that  $w = gwg^{-1}$ , so  $wg = gw$ , i.e.  $w \in Z(G)$ . So we can collect all these conjugacy classes with size one (all those  $w_i \in G$  such that  $Z(w_i) = G$ ) into a single term to get

$$|G| = |Z(G)| + \sum |\text{cl}_G(x_i)|$$

where the sum is carried out only over those conjugacy classes with more than one element.  $\square$

**Theorem 3.2.** If  $G$  is a finite group then each conjugacy class in  $G$  has size dividing  $|G|$ .

Note here that this is not an immediate consequence of Lagrange, as conjugacy classes are not subgroup. A simple way to see this is that the only conjugacy class containing the identity is the identity itself. As, if you had  $1 \in \text{cl}_G(x)$  for some  $x \neq 1 \in G$ , then  $1 = gxg^{-1}$  for some  $g \in G$ , so  $1g = g = xg$ , so  $x = 1$ , which is a contradiction and we must have  $1 \notin \text{cl}_G(x)$ .

#### 4. SYLOW THEOREMS

**Definition 4.1.** Let  $G$  be group and  $p$  be prime.

- A group of order  $p^\alpha$  for some  $\alpha \geq 1$  is called a  $p$ -group. Subgroups of  $G$  which are  $p$  groups are called  $p$ -subgroups.
- If  $G$  is a group of order  $p^\alpha m$ , where  $p \nmid m$ , then a subgroup of order  $p^\alpha$  is called a **Sylow  $p$ -subgroup** of  $G$ .
- The set of Sylow  $p$ -subgroups of  $G$  will be denoted by  $\text{Syl}_p(G)$  and the number of Sylow  $p$ -subgroups of  $G$  will be denoted by  $n_p(G)$  (or just  $n_p$  when clear).

**Theorem 4.1** (Sylow's Theorem). Let  $G$  be a group of order  $p^\alpha m$ , where  $p$  is a prime not dividing  $m$

- Sylow  $p$ -subgroups of  $G$  exist, i.e.  $\text{Syl}_p(G) \neq \emptyset$ .
- If  $P$  is a Sylow  $p$ -subgroup of  $G$  and  $Q$  is any  $p$ -subgroup of  $G$ , then there exists  $g \in G$  such that  $Q \leq gPg^{-1}$ , i.e.  $Q$  is contained in some conjugate of  $P$ . In particular, any two Sylow  $p$ -subgroups of  $G$  are conjugate in  $G$ .
- The number of Sylow  $p$ -subgroups of  $G$  is given by considering  $n_p \equiv 1 \pmod{p}$  and the fact that  $n_p \mid m$ .

**4.1. Counting to Force Normalization of a Sylow  $p$ -subgroup.** Let  $G$  be a group and  $|G| = n$ , let  $p$  be prime dividing  $n$  and let  $P \in \text{Syl}_p(G)$ . If  $|P| = p$ , then every nonidentity element of  $P$  has order  $p$  (we use this a lot in semidirect products) and every element of  $G$  of order  $p$  is in some conjugate of  $P$ . By Lagrange, distinct conjugates of  $P$  intersect in the identity, and so in this case the number of elements of  $G$  of order  $p$  is  $n_p(p-1)$ .

If Sylow  $p$ -subgroups for different primes have different orders (cf. Exercise 6.1 that follows this section) and we assume none of these is normal, we can sometimes show that the number of elements of prime order is  $> |G|$ . This contradiction forces one of the  $n_p$  to be 1 and thus giving us a normal Sylow  $p$ -subgroup in  $G$ .

**Example 4.1.** Let  $|G| = 105 = 3 \cdot 5 \cdot 7$ . Then use many choices for the  $n_3, n_5$ , and  $n_7$ , but if we assume  $n_3 = 7, n_5 = 21$ , and  $n_7 = 15$  (for their respective max):

$$\begin{aligned} \text{the number of elements of order 3 is } 7 \cdot 2 &= 14 \\ \text{the number of elements of order 5 is } 21 \cdot 4 &= 84 \\ \text{the number of elements of order 7 is } 15 \cdot 6 &= 90 \end{aligned}$$

which implies that the number of elements of prime order is  $90 + 84 + 14 = 188$ , but this greater than 105, so we have a contradiction.

#### 5. SEMIDIRECT PRODUCTS

**Exercise 5.1.** Classify the structure of a group of order 20.

*Proof.* Let  $G$  be a group of order 20. Then  $|G| = 20 = 2^2(5)$ , and so by Sylow's Theorem, we have  $n_5 \mid 4$  and  $n_5 \equiv 1 \pmod{5}$ , which implies that  $n_5 = 1$ , however,  $n_2 = 1$  or  $n_2 = 5$ . Let  $K$  be the unique, normal 5-Sylow and let  $H$  be a 2-Sylow. Then the orders of  $H$  and  $K$  are relatively prime as  $\gcd(5,4) = 1$  and so their intersection must be trivial, i.e.  $H \cap K = \{1\}$ . Moreover,  $HK \leq G$ , and so  $|HK| = (|H||K|)/(H \cap K) = 20/1 = 20$ , and so by the Recognition Theorem, we have that  $G \simeq K \rtimes_\varphi H$ , where  $\varphi: H \rightarrow \text{Aut}(K)$ . As  $|K| = 5$ , we have that  $K \simeq Z_5$ , and as  $|H| = 4$ , then  $H \simeq Z_2 \times Z_2$  or  $H \simeq Z_4$ .

For the case of  $H \simeq Z_4$ , we have that  $\varphi: Z_4 \rightarrow \text{Aut}(Z_5)$ , but  $\text{Aut}(Z_5) \simeq Z_4$ , so  $\varphi: Z_4 \rightarrow Z_4$ . Let  $x$  denote the generator of  $Z_4$  and  $y$  denote the generator of the codomain  $Z_4$ . Now we need for the order of  $y$  to satisfy specific properties, in particular,  $|y^i| = 4/(\gcd(4, i))$  and we need for  $|y^i|$  to divide 4. Testing for values of  $i$  up to 4, we have that  $i \in \{1, 4\}$  are all choices that work. Now we have four mappings:

$$\begin{aligned}\psi_1: Z_4 &\rightarrow Z_4, \psi_1: x \mapsto y^4 = 1 \\ \psi_2: Z_4 &\rightarrow Z_4, \psi_2: x \mapsto y \\ \psi_3: Z_4 &\rightarrow Z_4, \psi_3: x \mapsto y^2 \\ \psi_4: Z_4 &\rightarrow Z_4, \psi_4: x \mapsto y^3\end{aligned}$$

The map  $\psi_1$  produces the group  $Z_5 \times Z_4$ . We need to consider the other maps and whether or not they are isomorphic to one another. We do this by recognizing that  $\psi_2$  and  $\psi_4$  both have order 4, as when we map  $\psi_2(x^4) = y^4 = 1$  and  $\psi_3(x^4) = (y^3)^4 = y^{12} = 1$ . This implies that  $Z_5 \rtimes_{\psi_1} Z_4 \simeq Z_5 \rtimes_{\psi_3} Z_4$ . Lastly, we have  $Z_5 \rtimes_{\psi_3} Z_4$ .

For the case of  $H \simeq Z_2 \times Z_2$ , we have  $\varphi: Z_2 \times Z_2 \rightarrow Z_4$ . We note here that  $Z_2 \times Z_2 = \langle a, b: a^2 = b^2 = (ab)^2 = 1 \rangle$ . Once again, let  $y$  denote the generator of  $Z_4$ . Now we, similarly, need  $\varphi$  to satisfy certain conditions:  $|y^j| = 4/(\gcd(4, j))$  and  $y^j \mid 2$ . Thus the only satisfactory values of  $j$  for which this works for are  $j = 2$  and  $4$ . So we will mappings:

$$\begin{aligned}\alpha_1: Z_2 \times Z_2 &\rightarrow Z_4, \alpha_1: a, b \mapsto y^2, 1 \\ \alpha_2: Z_2 \times Z_2 &\rightarrow Z_4, \alpha_2: a, b \mapsto 1, y^2 \\ \alpha_3: Z_2 \times Z_2 &\rightarrow Z_4, \alpha_3: a, b \mapsto y^2, y^2 \\ \alpha_4: Z_2 \times Z_2 &\rightarrow Z_4, \alpha_4: a, b \mapsto 1, 1\end{aligned}$$

We can immediately notice that  $\alpha_4$  gives rise to  $Z_4 \times Z_2 \times Z_2$ . Moreover, we can see that we will have isomorphisms between  $\alpha_1$  and  $\alpha_2$ , and so  $Z_4 \rtimes_{\alpha_1} Z_2 \times Z_2 \simeq Z_4 \rtimes_{\alpha_2} Z_2 \times Z_2$ . The nontrivial case that we actually hence have is  $Z_4 \rtimes_{\alpha_3} Z_2 \times Z_2$ .  $\square$

**Example 5.1.** Classify the structure of a group of order 28.

*Proof.* Let  $G$  be a group of order 28. Then  $|G| = 28 = 14 \cdot 2 = 7 \cdot 2^2$ , and so  $n_7 \mid 4$  and  $n_7 \equiv 1 \pmod{7}$  which means that  $n_7 = 1$ . However,  $n_2 = 7$  or  $1$ . Let  $K$  denote the unique, normal 7-Sylow and let  $H$  be a 2-Sylow. Then  $|K| = 7$  so  $K \simeq Z_7$  and  $|H| = 4$  so either  $H \simeq Z_4$  or  $H \simeq Z_2 \times Z_2$ . Furthermore, by the same reasoning as the last exercise, we have  $G \simeq K \rtimes_{\varphi} H$ , where  $\varphi: H \rightarrow \text{Aut}(K)$ , or, rather,  $\varphi: H \rightarrow Z_6$ .

If  $H \simeq Z_4$ , then we will consider mappings  $\psi: Z_4 \rightarrow Z_6$ . Let  $Z_4 = \langle x \rangle$  and  $Z_6 = \langle y \rangle$ . Then we need that  $|y^i| = 6/(\gcd(6, i))$  to divide 4. So the only choices of  $i$  that satisfy this are  $i = 3$  and  $6$ . Thus we have the maps

$$\begin{aligned}\psi_1: Z_4 &\rightarrow Z_6, \psi_1: x \mapsto y^6 = 1 \\ \psi_2: Z_4 &\rightarrow Z_6, \psi_2: x \mapsto y^3\end{aligned}$$

The first mapping gives rise to  $Z_6 \rtimes_{\psi_1} Z_4 \simeq Z_6 \times Z_4$ , and the second mapping simply produces  $Z_6 \rtimes_{\psi_2} Z_4$ .

If  $H \simeq Z_2 \times Z_2$ , then we consider mappings  $\alpha: Z_2 \times Z_2 \rightarrow Z_6$ . Once again, we need to find satisfactory conditions for  $j$  respective to  $|y^j| = 6/\gcd(6, j)$  such that this divides 2. In this instance, the only satisfactory choice are  $j = 6$  and  $3$ . Thus we have mappings

$$\begin{aligned}\alpha_1: Z_2 \times Z_2 &\rightarrow Z_6, \alpha_1: a, b \mapsto 1, 1 \\ \alpha_2: Z_2 \times Z_2 &\rightarrow Z_6, \alpha_2: a, b \mapsto y^3, y^3 \\ \alpha_3: Z_2 \times Z_2 &\rightarrow Z_6, \alpha_3: a, b \mapsto 1, y^3 \\ \alpha_4: Z_2 \times Z_2 &\rightarrow Z_6, \alpha_4: a, b \mapsto 1, y^3\end{aligned}$$

The last two mappings can be seen to induce the same group, i.e.  $Z_6 \rtimes_{\alpha_3} Z_2 \times Z_2 \simeq Z_6 \rtimes_{\alpha_4} Z_2 \times Z_2$ , and we of course have the trivial mapping given by  $\alpha_1$ , so we also have  $Z_6 \times Z_2 \times Z_2$ . The other group we have is  $K_6 \rtimes_{\alpha_2} Z_2 \times Z_2$ .  $\square$

**5.1. Inner Direct Product.** Suppose that  $G$  is a group and  $H \triangleleft G$ ,  $K \leq G$ , and  $H \cap K = \{1\}$ . We seen in the past that  $HK = \{hk: h \in H, k \in K\}$  does indeed form a subgroup of  $G$ , i.e.  $HK \leq G$ . Now, since  $H \cap K = \{1\}$ , then for every  $x \in HK$ , we have a unique pair  $(h, k)$  such that  $x = hk$ . For now, Now, if we take some  $g_1, g_2 \in HK$  such that  $g_1 = h_1 k_1$  and  $g_2 = h_2 k_2$  for some  $h_1, h_2 \in H$  and  $k_1, k_2 \in K$ , then  $g_1 g_2 = (h_1 k_1)(h_2 k_2) = h_1 k_1 h_2 k_2 = h_1 k_1 h_2 k_1^{-1} k_1 k_2 = h_1 (k_1 h_2 k_1^{-1}) k_1 k_2 = h_1 \varphi_{k_1}(h_2) k_3$ , where  $\varphi: K \rightarrow \text{Aut}(H)$  by  $\varphi: k \mapsto \varphi_k$  and  $\varphi$  takes some element  $x \in H$  to  $kxk^{-1}$ , i.e.  $\varphi_k: x \mapsto kxk^{-1}$ . Therefore we have that the maps coincide.

We can take note of that the fact that  $g_1 g_2$  is completely determined by the group homomorphism:

$$\begin{aligned}\varphi: K &\longrightarrow \text{Inn}(H) \leq \text{Aut}(H) \\ k &\longmapsto \varphi_k\end{aligned}$$

Moreover, given two groups  $H$  and  $K$ , we want to define possible group structures on  $H \times K = \{(h, k) : h \in H, k \in K\}$ .

**Theorem 5.1.** Let  $H$  and  $K$  be two groups. Let  $\psi: K \longrightarrow \text{Inn}(H)$  be a group homomorphism. Equip the set  $H \times K$  by the following group operation: for  $(h_1, k_1), (h_2, k_2) \in H \times K$ ,

$$(h_1, k_1) \cdot (h_2, k_2) = (h_1 \psi(k_1)(h_2), k_1 k_2).$$

Then  $(H \times K, \cdot)$  is a group, and is denoted by  $H \rtimes_{\psi} K$ , called the **semi-direct product** of  $H$  and  $K$  with respect to  $\psi$ .

*Proof.* As  $H$  and  $K$  are both groups and are both nonempty as needed, then  $H \times K$  is nonempty. Now, suppose we have  $(h_1, k_1), (h_2, k_2) \in H \rtimes_{\psi} K$ . Then  $(h_1, k_1) \cdot (h_2, k_2) = (h_1 \psi(k_1)(h_2), k_1 k_2)$ , and since  $\psi(k_1)(h_2)$  produces an element in  $H$  as  $\psi$  is itself an automorphism, we have that  $(h_1, k_1) \cdot (h_2, k_2) = (h_2 h, k_1 k_2) \in H \times K$ , where  $h = \psi(k_1)(h_2)$ . Thus we have closure. Now, the identity element of  $H \rtimes_{\psi} K$  is simply  $(1, 1)$ . For inverses, we're going to define that for some  $(h, k) \in H \rtimes_{\psi} K$ , its inverse is  $(h, k)^{-1} = (\psi(k^{-1})(h^{-1}), k^{-1})$ ; so,  $(h, k) \cdot (h, k)^{-1} = (h \psi(k)(\psi(k^{-1})(h^{-1})), k k^{-1})$ . The left side of this is a cluster fuck but its not so bad to actually compute: firstly  $\psi(k^{-1})(h^{-1}) = k^{-1} h^{-1} k$ , and so  $\psi(k)(k^{-1} h^{-1} k) = k(k^{-1} h^{-1} k)k^{-1} = h^{-1}$ ; thus,  $(h, k) \cdot (h, k)^{-1} = (h(h^{-1}), k k^{-1}) = (1, 1)$ . Lastly we need to check associativity:

$$\begin{aligned}((h, k) \cdot (x, y)) \cdot (l, t) &= (h \psi(k)(x), ky) \cdot (l, t) \\ &= (h k x k^{-1}, ky) \cdot (l, t) \\ &= (h k x k^{-1} \psi(ky)(l), kyt) \\ &= (h k x k^{-1} (ky) l (ky)^{-1}, k(yt)) \\ &= (h k x y l y^{-1} k^{-1}, k(yt)) \\ &= (h \psi(k)(x y l y^{-1}), k(yt)) \\ &= (h \psi(k)(x \psi(y)(l)), k(yt)) \\ &= (h, k) \cdot (x \psi(y)(l), yt) \\ &= (h, k) \cdot ((x, y) \cdot (l, t));\end{aligned}$$

we should note here that we were using the fact that  $H$  and  $K$  are groups to do a lot of the work in their respective slots. Therefore  $H \rtimes_{\psi} K$  is a group.  $\square$

## 6. RINGS

### 6.1. Euclidean Domains.

**Definition 6.1.** Any function  $N: R \rightarrow \mathbb{Z}_{\geq 0}$  with  $N(0) = 0$  is called a **norm** on the integral domain  $R$ . If  $N(a) > 0$  for all  $a \neq 0$  define  $N$  to be a positive norm.

**Definition 6.2.** The integral domain  $R$  is said to be a **Euclidean Domain** if there is a norm  $N$  on  $R$  such that for any two elements  $a, b \in R$  with  $b \neq 0$  there exists elements  $q, r \in R$  with

$$a = qb + r \text{ with } r = 0 \text{ or } N(r) < N(b).$$

The element  $q$  is called the **quotient** and the element  $r$  is the **remainder** of the division.

**Remark 6.1.** Recall that if  $D$  is a square free integer, then we can produce the field  $\mathbb{Q}(\sqrt{D}) = \{a + b\sqrt{D} : a, b \in \mathbb{Q}\}$ . We may observe that  $\mathbb{Q}(\sqrt{D}) \subseteq \mathbb{C}$ , and so we can show that  $\mathbb{Q}(\sqrt{D})$  is subring of  $\mathbb{C}$ , however, we will omit the proof here as it is easy. In fact, this shows that  $\mathbb{Q}(\sqrt{D})$  is an integral domain as any subring of an field is an integral domain. The reason for picking  $D$  to be square free ( $D$  has prime factorization with exponents at most one, i.e.  $D = p_1 p_2 \cdots p_n$  with all  $p_i$  prime and  $1 \leq i \leq n$ ) is so that every element in  $\mathbb{Q}(\sqrt{D})$  can be written uniquely in the form  $a + b\sqrt{D}$ . Moreover, this further gives us that if  $a, b \neq 0 \in \mathbb{Q}$ , then  $a^2 - Db^2 \neq 0$ , and furthermore, as  $(a + b\sqrt{D})(a - b\sqrt{D}) = a^2 - Db^2$ , then we can recognize that  $(a - b\sqrt{D})/(a^2 - Db^2)$  is the multiplicative inverse of  $a + b\sqrt{D}$  in  $\mathbb{Q}(\sqrt{D})$  if  $a + b\sqrt{D} \neq 0$ . Thus we have that  $\mathbb{Q}(\sqrt{D})$  is a field (called a *quadratic field*).

Continuing our discussion, we can immediately recognize that  $\mathbb{Z}[\sqrt{D}]$  is a subring of  $\mathbb{Q}(\sqrt{D})$ . Now if  $D \equiv 4 \pmod{4}$  then we can product the slightly larger subset

$$\mathbb{Z} \left[ \frac{1 + \sqrt{D}}{2} \right] = \left\{ a + b \frac{1 + \sqrt{D}}{2} : a, b \in \mathbb{Z} \right\}.$$



This, once again, forms a subring of  $\mathbf{Q}\sqrt{D}$ . We define  $\mathcal{O}_{(\mathbf{Q}\sqrt{D})} = \mathbf{Z}[\omega] = \{a + b\omega : a, b \in \mathbf{Z}\}$ , where  $\omega = \sqrt{D}$  if  $D \equiv 2, 3 \pmod{4}$  or  $\omega = (1 + \sqrt{D})/2$  if  $D \equiv 1 \pmod{4}$ . We call  $\mathcal{O}_{(\mathbf{Q}\sqrt{D})}$  the **ring of integers** in the quadratic field  $\mathbf{Q}\sqrt{D}$ .

Define the *field norm*  $N: \mathbf{Q}(\sqrt{D}) \rightarrow \mathbf{Q}$  by  $N(a + b\sqrt{D}) = (a + b\sqrt{D})(a - b\sqrt{D}) = a^2 - Db^2 \in \mathbf{Q}$ , and as  $a, b, D \in \mathbf{Z}$ , we also have that  $N(a + b(\sqrt{D})) \in \mathbf{Z}$ , and so we can begin to talk about the Euclidean aspect to this all. We do this by considering  $\mathcal{O}_{(\mathbf{Q}\sqrt{D})}$  as they are integral domains with a norm defined by taking the absolute value of the field norm, i.e.  $|N(a + b\sqrt{D})|$ , so as to satisfy that  $N: \mathbf{R} \rightarrow \mathbf{Z}_{\geq 0}$ . In general, however,  $\mathcal{O}_{(\mathbf{Q}\sqrt{D})}$  isn't Euclidean; in fact,  $\mathcal{O}_{(\mathbf{Q}\sqrt{D})}$  is only Euclidean, using the norm we just defined, for finitely many choices of  $D$ , and in, general, for any possible norm on  $\mathcal{O}_{(\mathbf{Q}\sqrt{D})}$ , and  $D < 0$ , there are once again finitely many choices of  $D$  in which it is Euclidean, and it is an open problem to classify when  $\mathcal{O}_{(\mathbf{Q}\sqrt{D})}$  is Euclidean, with whatever norm, for  $D > 0$ .

**Proposition 6.1.**  $\mathcal{O}_{(\mathbf{Q}\sqrt{-1})} = \mathbf{Z}[\sqrt{-1}] = \mathbf{Z}[i]$  is a Euclidean domain.

*Proof.* We define the norm  $N: \mathbf{Z}[i] \rightarrow \mathbf{Z}_{\geq 0}$  to be  $N(a + bi) = (a + bi)(a - bi) = a^2 + b^2$ . Now take  $\alpha = a + bi, \beta = c + di \in \mathbf{Z}[i], \beta \neq 0$ . Then in the field  $\mathbf{Q}(i)$  we have that

$$\frac{\alpha}{\beta} = \frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{c^2 + d^2} = \frac{ac - adi + bci + bd}{c^2 + d^2} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{(bc - ad)}{c^2 + d^2}i.$$

Now, we define in the clear manner,  $\alpha/\beta = r + si$ , and clearly  $r, s \in \mathbf{Q}$ . Let  $p$  be an integer closest to the rational number  $r$  and let  $q$  be an integer closest to the rational number  $s$ , so that both  $|r - p|$  and  $|s - q|$  are at most  $1/2$ . (It may help to think of the rational numberline here.) The division algorithm follows once we have that

$$\alpha = (p + qi)\beta + \gamma \text{ for some } \gamma \in \mathbf{Z}[i] \text{ with } N(\gamma) \leq \frac{1}{2}N(\beta).$$

Let  $\theta = (r - p) + (s - q)i$  and set  $\gamma = \beta\theta$ . Then  $\gamma = \alpha - (p + qi)\beta$ , so that  $\gamma \in \mathbf{Z}[i]$  is a Gaussian integer and  $\alpha = (p + qi)\beta + \gamma$ . Since  $N(\theta) = (r - p)^2 + (s - q)^2$  is at most  $1/4 + 1/4 = 1/2$ , and as  $N$  is multiplicative, we have  $N(\gamma) = N(\beta)N(\theta)$  so  $N(\gamma) \leq \frac{1}{2}N(\beta)$ .  $\square$

**Proposition 6.2.** Every ideal in a Euclidean domain is principal. More precisely, if  $I$  is any nonzero ideal in the Euclidean domain  $R$  then  $I = (d)$ , where  $d$  is any nonzero element of  $I$  of minimum norm.

*Proof.* Let  $A$  be a Euclidean domain. Then let  $J$  be an ideal of  $A$ . If  $J$  is the zero ideal, then  $J = (0)$  and we are done. Let  $m$  be the nonzero, minimum element of  $J$  with respect to the norm, say,  $N$  endowed on  $A$ . We may note here that the set  $\{N(a) : a \in J\}$  indeed has a minimum element by the Well Ordering of  $\mathbf{Z}$ ; recall that  $N(a) \in \mathbf{Z}_{\geq 0}$  as  $N: A \rightarrow \mathbf{Z}_{\geq 0}$ . So as  $J$  is an ideal then  $mx \in J$  for  $x \in A$ ; thus  $(m) \subseteq J$ . Let  $\ell \in J$ , and so as  $A$  is Euclidean, we may write  $\ell = mq + r$  for some  $q, r \in A$  and  $N(r) < N(m)$ . We may rewrite this as  $\ell - mq = r$ , and as  $J$  is an ideal, then  $\ell - mq \in J$ , but then  $N(r) < N(m)$  is smaller so this contradicts the minimality of  $N(m)$ . We must have that  $\ell = mq + 0 = mq$ , and so every element in  $J$  can be written as  $mt$  for some  $t \in A$ . Thus  $J \subseteq (m)$ , which means  $J = (m)$ , and so  $A$  is a PID.  $\square$

## 6.2. UFD.

**Definition 6.3.** Let  $R$  be an integral domain.

- Suppose  $r \in R$  is nonzero and is not a unit. Then  $r$  is called **irreducible** in  $R$  if whenever  $r = ab$  with  $a, b \in R$ , at least one of  $a$  or  $b$  must be a unit in  $R$ . Otherwise,  $r$  is said to be **reducible**.
- The nonzero element  $p \in R$  is called **prime** in  $R$  if the ideal  $(p)$  generated by  $p$  is a prime ideal. In other words, a nonzero element  $p$  is called prime if it is not a unit and whenever  $p \mid ab$ , then either  $p \mid a$  or  $p \mid b$ .
- Two elements  $a$  and  $b$  of  $R$  differing by a unit are said to be **associate** in  $R$  (i.e.  $a = ub$  for some unit  $u$  in  $R$ ).

We can note here that the reason we define these things in this manner is that we take the ring  $\mathbf{Z}$  our motivation. For example, the condition of *primeness* comes from Euclid's lemma in  $\mathbf{Z}$ . And the notion of irreducibility is what is associated to a prime in  $\mathbf{Z}$ . The prime  $p = 11$  can only be written in the form  $11 = 11 \cdot 1$ , and so we need a unit, i.e.  $1 \in R$  in order to write  $11$  as a product of more than one integer.

**Proposition 6.3.** In an integral domain a prime element is always irreducible.

*Proof.* Suppose  $R$  is an integral domain and let  $p \in R$  be prime. If we let  $p$  be reducible, then we can write  $p = ab$  where  $a, b \in R - R^\times$ . This means that  $ab \in (p)$  and so either  $a \in (p)$  or  $b \in (p)$ . If we let  $a \in (p)$ , then  $a = pt$  for some  $t \in R$ , and so  $p = ab = (pt)b$ , which means that  $1 = bt$  and this is a contradiction as we assume  $b \notin R^\times$ . Similarly, if we let  $b \in (p)$ , then  $b = ps$  for some  $s \in R$ . Thus,  $p = ab = a(ps)$  which implies that  $a \in R^\times$ ; a contradiction. Therefore we must have that  $p$  is irreducible.  $\square$

**Proposition 6.4.** In a PID a nonzero element is prime if and only if it is irreducible.

*Proof.* Let  $A$  be a PID. As  $A$  is a PID then it is also integral domain, by definition, therefore we're done for the forward direction. Now, let  $p$  be irreducible. If  $M$  is an ideal containing  $(p)$  then  $M = (m)$  is principal, i.e.  $(p) \subseteq (m)$ . So  $p = mx$  for some  $x \in A$ , but as  $p$  is irreducible we must have that either  $m$  is a unit or  $x$  is a unit. If  $m \in A^\times$ , then  $mv = 1$  for some  $v \in R^\times$ , but this then implies that  $1 \in (m)$ , and so  $(m) = A$ . Moreover, if  $x \in R^\times$  then  $xl = 1$  for some  $l \in R^\times$ , so  $p = p \cdot 1 = p(xl) = (mx) = plx$ , and so  $m = pl$ ; thus  $m \in (p)$  so  $(m) \subseteq (p)$ , which means  $(m) = (p)$ . Thus  $(p)$  is maximal, and so as  $A$  is a PID, we have that  $(p)$  is a prime ideal.  $\square$

**Definition 6.4.** A **Unique Factorization Domain (UFD)** is an integral domain  $R$  in which every nonzero element  $r \in R$  which is not a unit has the following properties:

- (i)  $r$  can be written as a finite product of irreducibles  $p_i$  of  $R$  (not necessarily distinct):  $r = p_1 p_2 \cdots p_n$ , and
- (ii) the decomposition in (i) is *unique up to associates*: namely, if  $r = q_1 q_2 \cdots q_m$  is another factorization of  $r$  into irreducibles, then  $m = n$  and there is some renumbering of the factors so that  $p_i$  is associate to  $q_i$ , that is,  $p_i = q_i \ell$  where  $\ell \in R^\times$ , for  $i = 1, 2, \dots, n$ .

**Proposition 6.5.** In a UFD, a nonzero element is prime if and only if it is irreducible.

*Proof.* By the Proposition 5.3., we have the forward direction already done. Let  $A$  be a UFD and let  $\ell$  be irreducible in  $A$  and assume that  $\ell \mid ab$  for some  $a, b \in A$ . Then we may write  $ab = \ell k$  for some  $k \in A$ . Writing  $a$  and  $b$  as a product of irreducibles, then we have two characterizations of  $ab$  and so we must have that the irreducible  $\ell$  must be associate to one of the irreducibles in either  $a$  or  $b$ . WLOG, let  $\ell$  be associate to one of the irreducibles in the factorization of  $a$ , and so we may write  $a = (u\ell)p_1 p_2 \cdots p_n$  for  $u \in A^\times$  and some (possibly empty set of) irreducibles  $p_1, \dots, p_n$ . Thus  $a = \ell(u p_1 p_2 \cdots p_n)$  and so  $\ell \mid a$ .  $\square$

**6.3. Polynomial Rings.** Recall that the polynomial ring  $R[x]$  in the indeterminate  $x$  with coefficients from  $R$  is the set of all formal sums  $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$  with  $n \geq 0$  and each  $a_i \in R$ . Addition of polynomials is done "componentwise":

$$\sum_{i=0}^n a_i x^i + \sum_{i=0}^n b_i x^i = \sum_{i=0}^n (a_i + b_i) x^i.$$

Multiplication may at first seem odd without actually computing what multiplication might look like for some two polynomials: take  $f, g \in R[x]$  with  $f = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$  and  $g = b_m x^m + a_{m-1} x^{m-1} + \cdots + b_1 x + b_0$ . Then

$$fg = (a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n)(b_0 + b_1 x + \cdots + b_{m-1} x^{m-1} + b_m x^m) \quad (1)$$

$$= (a_0 b_0 + a_0 b_1 x + \cdots + a_0 b_{m-1} x^{m-1} + a_0 b_m x^m) + a_1 b_0 x + a_1 b_1 x^2 + \cdots + a_1 b_{m-1} x^m + a_1 b_m x^{m+1} + \cdots \quad (2)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1)x^2 + \cdots \quad (3)$$

This continues going on, however, this multiplication can be described by

$$fg = \left( \sum_{i=0}^n a_i x^i \right) \left( \sum_{i=0}^m b_i x^i \right) = \sum_{k=0}^{n+m} \left( \sum_{i=0}^k a_i b_{k-i} \right) x^k.$$

**Proposition 6.6.** Let  $R$  be an integral domain. Then

- (i)  $\deg(p(x)q(x)) = \deg(p(x)) + \deg(q(x))$  if  $p(x), q(x) \in R[x]$  and both polynomials are nonzero,
- (ii)  $(R[x])^\times = R^\times$ , and
- (iii)  $R[x]$  is an integral domain.

*Proof.* (i) This is obvious.

(ii) If  $p(x), q(x) \in R[x]$  such that  $p(x)q(x) = 1$ , then we need for the degrees of the polynomials to match up, i.e.  $\deg(p(x)q(x)) = \deg(p(x)) + \deg(q(x)) = \deg(1) = 0$ . Thus we must have that  $p(x)$  and  $q(x)$  are both constant polynomials. Lastly, let  $p(x) = a$  and  $q(x) = b$ , and so  $ab = 1$ ; thus  $p(x) = a \in R^\times$  and same goes for  $b = q(x)$ . For the other direction, if we have  $st = 1$  for  $s, t \in R^\times$ , then this is trivially in the units of  $R[x]$  as  $s, t$  are "polynomials" of degree 0.

(iii) Suppose we have  $f(x), g(x) \neq 0 \in R[x]$  such that  $f(x)g(x) = 0$ . Then we write out  $f = a_n x^n + \cdots + a_1 x + a_0$  and  $g = b_m x^m + \cdots + b_1 x + b_0$ , and if we have that  $fg = 0$ , then each polynomial coefficient must be 0, and in particular, for the term  $a_n b_m x^{n+m}$  in the product, we have that  $a_n b_m = 0$ , so as  $R$  is an integral domain, we either have that  $a_n = 0$  or  $b_m = 0$ . WLOG, if we have  $a_n = 0$ , then we have that the original degree of  $f$  is not  $n$ , which is a contradiction. Therefore one of our original polynomials must be zero and thus  $R[x]$  is an integral domain.  $\square$

If we consider localization, i.e. the quotient field of  $R[x]$ , then the quotient field, say,  $K$  of  $R[x]$  consists of all quotients  $p(x)/k(x)$  where  $k(x)$  is not the zero polynomial. The quotient field  $K$  is called the field of rational functions in  $x$  with coefficients in  $R$ .

**Proposition 6.7.** Let  $I$  be an ideal of the ring  $R$  and let  $(I) = I[x]$  denote the ideal of  $R[x]$  generated by  $I$  (the set of polynomials with coefficients in  $I$ ). Then

$$R[x]/(I) \simeq (R/I)[x].$$

In particular, if  $I$  is a prime ideal of  $R$  then  $(I)$  is a prime ideal of  $R[x]$ .

*Proof.* Define the map  $\varphi: R[x] \rightarrow (R/I)[x]$  by taking a polynomial  $\varphi(f) = \varphi(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) = (a_n + I)x^n + (a_{n-1} + I)x^{n-1} + \cdots + (a_1 + I)x + (a_0 + I)$ . This map can be easily realized to be a homomorphism (think of terms simply as  $\overline{a_i x^n}$ ) and surjective. Now if we consider the kernel of the map, then we have

$$\ker(\varphi) = \{q \in R[x] : \varphi(q) = I\}.$$

A polynomial in  $R[x]$  that's in the kernel of  $\varphi$  can clearly be seen to need coefficients in  $I$ , and so  $\ker(\varphi) = I[x]$ . As the map is surjective as a homomorphism, then  $R[x]/I[x] \simeq (R/I)[x]$ .  $\square$

**Example 6.1.** If we let  $R = \mathbf{Z}$  and take the ideal  $n\mathbf{Z} = (n)$  of  $\mathbf{Z}$ . Then the above proposition gets us that  $\mathbf{Z}[x]/n\mathbf{Z}[x] \simeq (\mathbf{Z}/n\mathbf{Z})[x]$ . Moreover as  $\mathbf{Z}/p\mathbf{Z} \simeq \mathbf{F}_p$  is a field then  $\mathbf{Z}/p\mathbf{Z}[x]$  is an integral domain (as well as a Euclidean domain). The last remark could've been also realized another way:  $p\mathbf{Z} = (p)$  is a prime ideal of  $\mathbf{Z}$  and so  $\mathbf{Z}/p\mathbf{Z}$  is an integral domain and by Proposition 5.6. (iii), we have that  $(\mathbf{Z}/p\mathbf{Z})[x]$  is an integral domain.

### 6.3.1. Polynomial Rings Over Fields.

**Theorem 6.1.** Let  $F$  be a field. The polynomial ring  $F[x]$  is a Euclidean domain. Specifically, if  $a(x)$  and  $b(x)$  are two polynomials in  $F[x]$  with  $b(x)$  nonzero, then there are *unique*  $q(x)$  and  $r(x)$  in  $F[x]$  such that

$$a(x) = q(x)b(x) + r(x) \text{ with } r(x) = 0 \text{ or } \deg(r(x)) < \deg(b(x))$$

**Corollary 6.1.** If  $F$  is field, then  $F[x]$  is a PID and UFD.

**Remark 6.2.** Recall that  $F$  is a field if and only if  $F[x]$  is a PID. Furthermore, if  $R$  is any commutative ring such that  $R[x]$  is a Euclidean domain, then  $R$  is a field.

6.3.2. *Polynomial Rings that are UFD.* Recall that if we have an integral domain  $R$  then we can embed it into its field of fraction, say,  $F$ , so that  $R[x] \subseteq F[x]$  is a subring and  $F[x]$  is a Euclidean domain (and hence a PID and UFD). A motivation towards using  $F[x]$  rather than  $R[x]$  is that, well, we have more ready machinery in  $F[x]$ , and as  $R[x] \subseteq F[x]$  then we can approach some questions about  $R[x]$  in  $F[x]$ . For example, if we have  $p(x) \in R[x]$ . Then  $p(x) \in F[x]$  and as  $F[x]$  is a UFD we can factor  $p(x)$  uniquely into a product of irreducibles in  $F[x]$ . In general  $R[x]$  is not a UFD, and so if  $R$  is an integral domain that isn't a UFD, then  $R[x]$  cannot be a UFD as well.

**Proposition 6.8** (Gauss' Lemma). Let  $R$  be a UFD with field of fractions  $F$  and let  $p(x) \in R[x]$ . If  $p(x)$  is reducible in  $F[x]$  then  $p(x)$  is reducible in  $R[x]$ . More precisely, if  $p(x) = A(x)B(x)$  for some nonconstant polynomials  $A(x), B(x) \in F[x]$ , then there are nonzero elements  $r, s \in F$  such that  $rA(x) = a(x)$  and  $sB(x) = b(x)$  both lie in  $R[x]$  and  $p(x) = a(x)b(x)$  is a factorization in  $R[x]$ .

**Corollary 6.2.** Let  $R$  be a UFD, let  $F$  be its field of fractions and let  $p(x) \in R[x]$ . Suppose the greatest common divisor of the coefficients of  $p(x)$  is 1. Then  $p(x)$  is irreducible in  $R[x]$  if and only if it is irreducible in  $F[x]$ . In particular, if  $p(x)$  is a monic polynomial that is irreducible in  $R[x]$ , then  $p(x)$  is irreducible in  $F[x]$ .

*Proof.* By Gauss' Lemma, we have that if  $p(x)$  is irreducible in  $R[x]$ , then  $p(x)$  is irreducible in  $F[x]$  (contrapositive). Now for the other direction, irreducible in  $F[x] \implies$  irreducible in  $R[x]$ , we do this again by contrapositive: suppose  $p(x)$  is reducible in  $R[x]$ , and so let  $p(x) = a(x)b(x)$ , where  $a(x), b(x) \in R[x] - R^\times$ , as the gcd of all the coefficients is one. This factorization of  $p(x)$  is once again reducible in  $F[x]$  (note here that no information of reducibility is lost when transferring this over to the quotient field as we can write  $p(x) = \frac{a(x)b(x)}{1}$  and  $1 \in R[x]$ , so  $p(x)$  remains reducible).  $\square$

**Theorem 6.2.**  $R$  is a UFD if and only if  $R[x]$  is a UFD.

**Corollary 6.3.** If  $R$  is a UFD, then a polynomial ring in an arbitrary number variables with coefficients in  $R$  is also a UFD.

### 6.4. Irreducibility of Polynomials.

**Proposition 6.9.** Let  $F$  be a field and let  $p(x) \in F[x]$ . Then  $p(x)$  has a factor of degree one if and only if  $p(x)$  has a root in  $F$ .

*Proof.* Suppose that  $p(x)$  has a factor of degree one, i.e.  $p(x)$  has a linear factor. Now as were working with a field  $F$ , we may assume the factor is monic, and so the linear factor is of the form  $(x - \alpha)$  with  $\alpha \in F$ . But clearly if  $\alpha = x$ , then we get that  $p(\alpha) = 0$ , i.e.  $\alpha$  is a root in  $F$ . Now suppose we have that  $p(\ell) = 0$  for some  $\ell \in F$ . Then we may write  $p(x) = q(x)(x - \ell) + r$  and  $r$  is constant as we needed  $\deg(x - \ell) = 1 > \deg(r)$ . So then  $p(\ell) = q(\ell)(\ell - \ell) + r = 0 + r = r$ , and so then  $r = 0$  and  $(x - \ell)$  is a linear factor of  $p(x)$ .  $\square$

**Proposition 6.10.** A polynomial of degree two or three over a field  $F$  is irreducible if and only if it has no roots in  $F$ .

*Proof.* Let  $f \in F[x]$  with  $f$  having degree two or three. If  $f$  has a root  $\alpha$  in  $F$ , then  $f$  is divisible by  $(x - \alpha)$ , and so  $f$  is reducible. Now, for the other direction, if  $f$  is reducible in  $F[x]$ , then it has a factor of degree 1, which is of the form  $ax + b$ . Then  $-b/a$  is a root of  $f$ .  $\square$

**Proposition 6.11** (Rational Root Theorem). Let  $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$  be a polynomial of degree  $n$  with integer coefficients. If  $r/s \in \mathbf{Q}$  is in lowest terms (i.e.  $r$  and  $s$  are relatively prime integers) and  $r/s$  is a root of  $p(x)$ , then  $r$  divides the constant term and  $s$  divides the leading coefficient of  $p(x)$ :  $r \mid a_0$  and  $s \mid a_n$ . In particular, if  $p(x)$  is a monic polynomial with integer coefficients and  $p(d) \neq 0$  for all integers  $d$  dividing the constant term of  $p(x)$ , then  $p(x)$  has no roots in  $\mathbf{Q}$ .

*Proof.* Suppose that  $r/s \in \mathbf{Q}$  and  $\gcd(r, s) = 1$  with  $p(r/s) = 0$ . Then  $p(r/s) = 0 = a_n (r/s)^n + a_{n-1} (r/s)^{n-1} + \cdots + a_1 (r/s) + a_0$ , and so  $0 = a_n r^n + a_{n-1} r^{n-1} s + \cdots + a_0 s^n$ , which can be rewritten as  $a_n r^n = s(-a_{n-1} r^{n-1} - \cdots - a_0 s^{n-1})$ . Hence  $s \mid a_n r^n$ . So, as  $\gcd(r, s) = 1$ , we must have that  $s \mid a_n$ . Solving the equation for  $a_0 s^n$  would've given us that  $r \mid a_0$ .  $\square$

**Corollary 6.4.** If  $f(x) = p_n x^n + p_{n-1} x^{n-1} + \cdots + p_1 x + p_0$  is a monic polynomial, then the rational roots of  $f(x)$  must be integers

*Proof.* By the rational root theorem, if  $r = a/b$  is a root of  $f(x)$ , then  $b \mid p_n = 1$  so  $b = 1$ , and so then  $a/b = a/1 = a \in \mathbf{Z}$ .  $\square$

**Example 6.2.** For any prime  $p$  the polynomials  $x^2 - p$  and  $x^3 - p$  are irreducible in  $\mathbf{Q}[x]$ , as they are degree less than or equal to three and it's obvious that they have no rational roots. The fact that  $x^2 - p$  is irreducible in  $\mathbf{Q}[x]$  provides with the fact that  $\sqrt{2}$  is irrational for the case of  $p = 2$ . However, using the previous proposition, and supposing we had a hypothetical root in  $r/s \in \mathbf{Q}$ , then we would need for  $r \mid p$  and  $s \mid 1$ , and so this would mean that we need have the choice between the roots  $\pm 1$  or  $\pm p$ , but in either case case, when substituted into either equation, we don't produce a zero. (Looking back at this it would've been easier to simply observe that as the quadratic and cubic are monic and have integer coefficients, then we would only need to look at the divisors of  $-p$ , which are  $\pm 1$  and  $\pm p$ , and neither provide zeros for either polynomial. Thus they are irreducible.)

**Proposition 6.12.** Let  $I$  be a proper ideal in the integral domain  $R$  and let  $p(x)$  be a nonconstant monic polynomial in  $R[x]$ . If the image of  $p(x)$  in  $(R/I)[x]$  cannot be factored in  $(R/I)[x]$  into two polynomials of smaller degree, then  $p(x)$  is irreducible in  $R[x]$ .

*Proof.* Suppose  $p(x)$  cannot be factored in  $(R/I)[x]$  but that  $p(x)$  is reducible in  $R[x]$ , in aim of contradiction. Then there are monic polynomials  $a(x)$  and  $b(x)$  in  $R[x]$  such that  $p(x) = a(x)b(x)$ . By the isomorphism between  $R[x]/I[x] \simeq (R/I)[x]$ , reducing the coefficients modulo  $I$ , gives a factorization in  $(R/I)[x]$  with nonconstant factors, which is a contradiction. Thus  $p(x)$  is irreducible.  $\square$

**Proposition 6.13** (Eisenstein's Criterion). Let  $P$  be a prime ideal of the integral domain  $R$  and let  $f(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$  be a polynomial in  $R[x]$  (here  $n \geq 1$ ). Suppose  $a_{n-1}, \dots, a_1, a_0$  are all elements of  $P$  and suppose  $a_0$  is not an element of  $P^2$ . Then  $f(x)$  is irreducible in  $R[x]$

## 7. ASSORTMENT OF EXERCISES SINCE LAST MIDTERM

### 7.1. 8.1: Euclidean Domains.

**Exercise 7.1** (3.). Let  $R$  be a Euclidean domain. Let  $m$  be the minimum integer in the set of norms of nonzero elements of  $R$ . Prove that every nonzero element of  $R$  of norm  $m$  is a unit. Deduce that a nonzero element of norm zero (if such an element exists) is a unit.

*Proof.* If we let  $S$  denote the set of all norms of nonzero elements of  $R$ , then the elements in  $S$  belong to  $\mathbf{Z}_{>0}$ , and so using the well-ordering principle, we have a minimum element, which we will call  $m$ ; that is,  $m < N(a)$  of  $m$ . Let  $x \neq 0 \in R$ , and  $N(x) = m$ . Then, as  $R$  is Euclidean, we can write  $1 = ax + r$  where  $a, r \in R$  and  $N(r) < N(x) = m$ , which forces  $r = 0$  because of the minimality of  $m$ . Thus we have that  $ax = 1$  and  $x$  is a unit.

Suppose  $\ell \neq 0 \in R$  such that  $N(\ell) = 0$ . Then, in a similar manner, we can write  $1 = b\ell + r$  where  $b, r \in R$  and  $N(r) < N(\ell) = 0$ , which means that  $r = 0$ , and  $\ell$  is hence a unit.  $\square$

**Exercise 7.2** (10.). Prove that the quotient ring  $\mathbf{Z}[i]/I$  is finite for any nonzero ideal  $I$  of  $\mathbf{Z}[i]$ .

*Proof.* By Proposition 1 in D&F in 8.1., we have that  $I$  is generated by an element in  $I$  with minimum norm in  $I$ , say,  $\alpha$ , and so  $I = (\alpha)$ . An element in  $\mathbf{Z}[i]/I$  is written as  $\ell + (\alpha)$  for some  $\ell \in \mathbf{Z}[i]$ . Now let  $x = \alpha q + r$  for some  $q, r \in \mathbf{Z}[i]$  and  $N(r) < N(\alpha)$ . Then this is exactly a representation of  $\mathbf{Z}[i]/I$ , and so  $x = r + (\alpha)$ , and as we have  $N(r) < N(\alpha)$ , and  $N(\alpha)$  is just some integer, then there is only going to be finitely many choices of  $r$ , as  $\mathbf{Z}[i]$  is Euclidean, and then we will have a finite list of different representations of  $x$ , i.e.  $r_1 + (\alpha), r_2 + (\alpha), \dots, r_k + (\alpha)$  for  $N(r_i) < N(\alpha)$  for  $1 \leq i \leq k$ . Thus as we have finitely many representations of elements in  $\mathbf{Z}[i]/I$ , then we have that  $\mathbf{Z}[i]/I$  is finite.  $\square$

## 7.2. 8.2 PID.

**Exercise 7.3 (3.).** Prove that a quotient of a PID by a prime ideal is again a PID.

*Proof.* Suppose that  $A$  is a PID, and let  $(p)$  be a prime ideal of  $A$ . Then  $A/(p)$  is a field as  $(p)$  is maximal, and so we have that the only ideals of  $A/(p)$  are  $(0)$  and  $(1) = R$ . Thus  $A/(p)$  is a PID.  $\square$

**Exercise 7.4 (5.).** Let  $R$  be the quadratic integer ring  $\mathbf{Z}[\sqrt{-5}]$ . Define the ideals  $I_2 = (2, 1 + \sqrt{-5})$ ,  $I_3 = (3, 2 + \sqrt{-5})$ , and  $I_1 = (3, 2 - \sqrt{-5})$ .

*Proof.* We will only show one of the ideals is not principal for (a), as the rest follow pretty easily using the same structure as proving for it one of the ideals: Suppose that  $I_2$  is a principal ideal, that is, let  $I_2 = (2, 1 + \sqrt{-5}) = (\alpha)$ . Then we have that  $2 = \alpha x$  and  $1 + \sqrt{-5} = \alpha y$  for some  $x, y \in \mathbf{Z}[\sqrt{-5}]$ . So then  $N(2) = 4 = N(\alpha)N(x)$  and  $N(1 + \sqrt{-5}) = 1^2 + 5 = 6 = N(\alpha)N(y)$ . Thus we have that  $N(\alpha) = 1$  or  $2$ , as  $N(\alpha) \mid 4$  and  $N(\alpha) \mid 6$ . When  $N(\alpha) = 1$ , then  $\alpha$  is a unit, so  $(\alpha) = R$ , and so we can write  $1 = 2s + (1 + \sqrt{-5})t$  for some  $s, t \in \mathbf{Z}[\sqrt{-5}]$ . Moreover,  $(1 + \sqrt{-5}) = 2s(1 + \sqrt{-5}) + 6t = 2(s + s(1 + \sqrt{-5}) + 3t)$ , and so the LHS and RHS don't align as the RHS has coefficient divisible by 2, but the LHS does not. Thus we cannot have  $N(\alpha) = 1$ . If  $N(\alpha) = 2$ , then  $\alpha^2 + 5b^2 = 2$ , which clearly has no integral solutions. Thus we cannot have  $N(\alpha) = 2$ , either. So then we have that  $(\alpha) \neq I_2$ , and so  $I_2$  cannot be principal.  $\square$

## 7.3. 8.3 UFD.

**Exercise 7.5 (6.).**

- (a) Prove that the quotient ring  $\mathbf{Z}[i]/(1 + i)$  is a field of order 2.
- (b) Let  $q \in \mathbf{Z}$  be a prime with  $q \equiv 3 \pmod{4}$ . Prove that the quotient ring  $\mathbf{Z}[i]/(q)$  is a field with  $q^2$  elements.

*Proof.* For (a): We have shown in an above exercise that  $\mathbf{Z}[i]/I$  is finite for any nonzero ideal  $I$  of  $\mathbf{Z}[i]$ . We simply need to determine the possible cosets of  $\mathbf{Z}[i]/(1 + i)$ . Firstly  $N(1 + i) = 1^2 + (-1)^2 = 2$ , and so as this is prime we have that  $(1 + i)$  is irreducible. Now as it is prime it is also maximal and thus we have a field  $\mathbf{Z}[i]/(1 + i)$ . A element in  $x \in \mathbf{Z}[i]/(1 + i)$  is of the form  $x = a + bi + (1 + i)$  for some  $a, b \in \mathbf{Z}$ . Since  $\mathbf{Z}[i]$  is Euclidean, write  $x = (1 + i)q + r$  for some  $q, r \in \mathbf{Z}[i]$  such that  $N(r) < N(1 + i) = 2$ . The only values that  $N(r)$  can be are 0, 1, and  $i$ , and so we have to we have, essentially, three elements of  $\mathbf{Z}[i]/(1 + i)$ :  $0 + (1 + i) = (i + 1)$ ,  $1 + (1 + i)$ , and  $i + (1 + i)$ . But we should not here that  $1 - i = -i(1 + i) \in (1 + i)$  and so we thus have two coset representations. Thus  $\mathbf{Z}[i]/(1 + i)$  is a field of order 2.

For (b): As  $q$  is prime and  $q \equiv 3 \pmod{4}$ , we have that this prime is irreducible and  $(q)$  is prime. Thus we once again have that  $\mathbf{Z}[i]/(q)$  is a field. Now as  $\mathbf{Z}[i]$  is Euclidean, we can write  $x = ql + r$  for some  $l, r \in \mathbf{Z}[i]$  such that  $N(r) < N(q) = q^2$ . We claim that all elements  $N(r)$  determined by being less than  $N(q) = q^2$  give rise to distinct cosets in  $\mathbf{Z}[i]/(q)$ . Suppose that we have two cosets  $a_1 + b_1 i + (p) = a_2 + b_2 i + (p)$ , then we have  $(a_1 - a_2) + (b_1 - b_2)i \in (p)$ , which implies that  $a_1 \equiv a_2$  and  $b_1 \equiv b_2$  as  $p \mid a_1 - a_2$  and  $p \mid b_1 - b_2$ . Thus we have that the cosets are distinct and so we have a field  $\mathbf{Z}[i]/(q)$  of order  $q^2$ .  $\square$

## 7.4. Polynomial Rings that are UFDs.

**Exercise 7.6 (1.).** Let  $R$  be an integral domain with quotient field  $F$  and let  $p(x)$  be a monic polynomial in  $R[x]$ . Assume that  $p(x) = a(x)b(x)$  where  $a(x)$  and  $b(x)$  are monic polynomials in  $F[x]$  of smaller degree than  $p(x)$ . Prove that if  $a(x) \notin R[x]$  then  $R$  is not a UFD. Deduce that  $\mathbf{Z}[\sqrt{2}]$  is not a UFD.

*Proof.* Let  $p(x)$  be a monic polynomial in  $R[x]$ . Moreover, let  $p(x) = a(x)b(x)$  where  $a(x)$  and  $b(x)$  are monic polynomials in  $F[x]$  of smaller degree than  $p(x)$ . Now suppose that  $a(x) \notin R[x]$ . In aim of contradiction, assume that  $R[x]$  is a UFD. By Gauss, as  $p(x)$  is reducible in  $F[x]$ , then it is reducible in  $R[x]$  as well, and specifically: there are some  $r, s \in F$  such that  $ra(x), sb(x) \in R[x]$  and  $p(x) = (rs)a(x)b(x)$ . Since  $p(x), a(x)$ , and  $b(x)$  are all monic polynomials then we must have that  $rs = 1$  and so  $r, s \in R$  and both are units. As  $r$  is a unit in  $R$ , then  $a(x) = rr^{-1}a(x) \in R[x]$ .  $\square$

**Exercise 7.7 (4.).** Let  $R = \mathbf{Z} + x\mathbf{Q}[x] \subset \mathbf{Q}[x]$  be the set of polynomials in  $x$  with rational coefficients whose constant term is an integer.

- (a) Prove  $R$  is an integral domain and its units are  $\pm 1$ .
- (b) Show that the irreducibles in  $R$  are  $\pm p$  where  $p$  is a prime in  $\mathbf{Z}$  and polynomials  $f(x)$  that are irreducible in  $\mathbf{Q}[x]$  and have constant term  $\pm 1$ . Prove that these irreducibles are prime in  $R$ .

- (c) Show that  $x$  cannot be written as the product of irreducibles in  $R$  (in particular,  $x$  is not irreducible) and conclude that  $R$  is not a UFD.
- (d) Show that  $x$  is not prime in  $R$  and describe the quotient ring  $R/(x)$ .

*Proof.* For (a): Note that we aren't given initially that  $R$  is a ring. However, it is easy to see that  $R$  is closed under subtraction and multiplication and it is also trivially nonempty, and so  $R$  is a subring of  $\mathbf{Q}[x]$  and so it is an integral domain. Suppose that we have  $(n + xf(x))(m + xg(x)) = 1$  for some  $n, m \in \mathbf{Z}$  and  $f, g \in \mathbf{Q}[x]$ . Looking at this through the scope of being a subring of  $\mathbf{Q}[x]$  the degree norm is multiplicative and so  $N((n + xf(x))(m + xg(x))) = N(n + xf(x)) + N(m + xg(x)) = N(1) = 0$ . And so we must have constants on the LHS that are integers, and so the only possible units are those of  $\mathbf{Z}$ ; that is,  $\pm 1$ .

For (b): We will show that  $\pm 1$  satisfies irreducibility: Suppose  $f(X)$  is of the form  $\pm p$  where  $p$  is prime in  $\mathbf{Z}$ . If we had that  $f(X)$  can be written as a product  $f(X) = a(X)b(X)$  for some  $a(X), b(X) \in R$ , then the degree terms would need to add to zero, as  $\pm p$  is constant. And so  $f(X)$  gives rise to a product of integers. As primes in  $\mathbf{Z}$  are irreducible, then we have that either  $a(X)$  or  $b(X)$  is a unit, and so we can conclude that  $f(X)$  is irreducible. Other part is hard...

For (c): We have that  $\frac{1}{n}x$  and  $n$  for  $n \neq 0 \in \mathbf{Z}$  are non unit elements in  $R$  that give a product of  $x$ , and so  $x$  is not irreducible in  $R$ . Now suppose that we could write  $x$  as a product of irreducibles, that is,  $x = p_1(x)p_2(x) \cdots p_n(x)$  for irreducible elements  $p_i(x)$  in  $R$ , but  $N(x) = 1 = N(p_1(x)p_2(x) \cdots p_n(x)) = N(p_1(x)) + N(p_2(x)) + \cdots + N(p_n(x))$  and so we must have at most one polynomial with degree 1 on the RHS and all others must be of degree 0. WLOG, let  $\deg(p_1(x)) = 1$ . So then we can write  $p_1(x) = ax + b$  for  $a \in \mathbf{Q}$  and  $b \in \pm 1$ . For the other  $p_i(x)$  terms with  $i > 1$ ,  $p_i(x) = p_i$  as an irreducible in  $\mathbf{Z}$  (using part (b)). However this cannot happen as we would need the product  $p_2(x)p_3(x) \cdots p_n(x) = p_1p_2p_3 \cdots p_n$  to multiply to a polynomial with 0 constant term, and so  $x$  is not a product of irreducibles of  $R$ . Thus we have that  $R$  is not a UFD.  $\square$

For (d): Consider once again  $\frac{1}{n}x$  and  $n$ , which are both in  $R$ , but neither of whom are in  $(x)$ . But  $(x/n) \cdot n = x \in (x)$ . Thus  $(x)$  is not a prime ideal. We know immediately that  $R/(x)$  cannot be an integral domain.

**Exercise 7.8 (2).** Prove that if  $f(x), g(x)$  are polynomials with rational coefficients whose product  $f(x)g(x)$  has integer coefficients, then the product of any coefficient of  $g(x)$  with any coefficient of  $f(x)$  is an integer.

*Proof.* Suppose that  $f(x), g(x) \in \mathbf{Q}[x]$  and  $f(x)g(x) \in \mathbf{Z}[x]$ . By Gauss' lemma, there exists  $r, s \in \mathbf{Q}$  such that  $rf(x), sg(x) \in \mathbf{Z}[x]$ , so then we have that  $(rf(x))(sg(x)) = (rs)f(x)g(x) = f(x)g(x)$ , and so we have that  $s = r^{-1}$ . Thus for the coefficients of  $f(x)$  and  $g(x)$ , say, we have coefficients  $a_i$  of  $f(x)$  and coefficients  $b_j$  for  $g(x)$ , then  $ra_i, r^{-1}b_j \in \mathbf{Z}$  and so  $ra_i r^{-1}b_j = a_i b_j \in \mathbf{Z}$ .  $\square$

## 7.5. 9.4: Irreducibility Criteria.

**Exercise 7.9 (1).**

- (a)  $x^2 + x + 1$  in  $\mathbf{Z}/2\mathbf{Z}[x]$ .  
 (b)  $x^3 + x + 1$  in  $\mathbf{Z}/3\mathbf{Z}[x]$ .  
 (c)  $x^4 + 1$  in  $\mathbf{Z}/5\mathbf{Z}[x]$ .  
 (d)  $x^4 + 10x^2 + 1$  in  $\mathbf{Z}[x]$ .

*Proof.* For (a): it suffices to check all cases:  $0^2 + 0 + 1 = 1$  and  $1^2 + 1 + 1 = 3 \neq 0$ , and so by Proposition 10. (308) in D&G, this polynomial is irreducible.

For (b): once again, it suffices to check all cases:  $0^3 + 0 + 1 = 1$ ,  $1^3 + 1 + 1 = 0$ , and  $3^2 + 3 + 1 = 1$ , and so this polynomial is reducible and can be factored as  $(x - 1)(x^2 + x + 2) = x^3 + x - 2$  and  $1 \equiv -2 \pmod{3}$  so  $x^3 + x - 2 \equiv_3 x^3 + x + 1$ . Thus  $x^3 + x + 1 = (x - 1)(x^2 + x + 2)$ .

For (c): Observe that  $1 \equiv -4 \pmod{5}$ , and so  $x^4 + 1 = x^4 - 4 = (x^2 + 2)(x^2 - 2)$ ; thus  $x^4 + 1$  is reducible in  $\mathbf{Z}/5\mathbf{Z}[x]$ .

For (d): Let  $p = x^4 + 10x^2 + 1$ . We're going to take  $p \in \mathbf{Q}[x]$ . By the rational root test and the corollary that follows, if we have a root of  $p$ , then it must be an integer. But clearly  $p$  doesn't have integral solutions. Thus by Proposition 6.9., we don't have a linear factor. Now suppose that  $p$  is reducible, that is,  $p = x^4 + 10x^2 + 1 = (x^2 + ax + b)(x^2 + cx + d)$ , then expanding this out gives us:  $p = x^4 + x^3c + x^2d + ax^3 + ax^2c + adx + x^2b + bcx + bd = x^4 + (a + c)x^3 + (b + d + ac)x^2 + (ad + bc)x + bd$ . This then implies the following list conditions, by matching coefficients:  $(a + c) = 0, b + d + ac = 10, ad + bc = 0, bd = 1$ . So, in particular,  $a = -c$  and  $b, d = \pm 1$ , which implies that, using  $b + d + ac = 10$ , we get either  $2 - a^2 = 10$  or  $-2 - a^2 = 10$ , so then  $a^2 = -8$  or  $a^2 = -12$ , and in either case this doesn't work as  $a \in \mathbf{Z}$ . Thus  $p$  is irreducible.  $\square$

**Exercise 7.10 (2).** Prove that the following polynomials are irreducible in  $\mathbf{Z}[x]$ .

- (a)  $x^4 - 4x^3 + 6$   
 (b)  $x^6 + 30x^5 - 15x^3 + 6x - 120$   
 (c)  $x^4 + 4x^3 + 6x^2 + 2x + 1$  [Substitute  $x - 1$  for  $x$ .]

(d)  $\frac{(x+2)^p-2^p}{x}$ , where  $p$  is an odd prime.

*Proof.* For (a): We see that all coefficients besides the leading term have 2 as a factor and so applying Eisenstein, and seeing that  $6 \notin (4)$ , we have that this polynomial is irreducible.

For (b): Similarly, we have that the coefficients all have a factor of three but  $120 \notin (9)$ , and by Eisenstein this polynomial is irreducible.

For (c): Let  $f(x) = x^4 + 4x^3 + 6x^2 + 2x + 1$ , and so  $G(x) = f(x-1) = (x-1)^4 + 4(x-1)^3 + 6(x-1)^2 + 2(x-1) + 1 = x^4 - 2x + 2$  (did this with a calculator), and so we can apply Eisenstein again with the prime ideal  $(2)$  and see that this polynomial is irreducible.

For (d):  $\ell(x) = \frac{(x+2)^p-2^p}{x} = \frac{(\sum_{i=0}^p \binom{p}{i} 2^{p-i} x^i) - 2^p}{x} = \sum_{i=1}^p \binom{p}{i} 2^{p-i} x^{i-1}$ , and so every coefficient of  $\ell(x)$  is divisible by  $p$ , except for the leading term. The constant term at the end of the sum is  $p2^{p-1}$  and so as  $p > 2$ , we have that  $p^2$  does not divide  $2p^{p-1}$ . Therefore it is irreducible by Eisenstein.  $\square$

**Exercise 7.11 (7).** Prove that  $\mathbb{R}[x]/(x^2+1) \simeq \mathbb{C}$ .

*Proof.* Define  $\varphi: \mathbb{R}[x] \rightarrow \mathbb{C}$  by the evaluation map  $\varphi(i) = \sqrt{-1}$ . Then let  $f, g \in \mathbb{R}[x]$  where  $f = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  and  $g = b_m x^m + b_{m-1} x^{m-1} + \dots + b_1 x + b_0$ . Then  $\varphi(f) + \varphi(g) = f(i) + g(i) = \sum_k a_k i^k + \sum_k b_k i^k = \sum_k (a_k + b_k) i^k = (f+g)(i) = \varphi(f+g)$  and  $\varphi(f)\varphi(g) = f(i)g(i) = \sum_k (\sum_{u+v=k} (a_u i^u)(b_v i^v)) = \sum_k (\sum_{u+v=k} a_u b_v) i^k = (fg)(i) = \varphi(fg)$ . So then  $\varphi$  is a homomorphism. Now  $\varphi(x^2+1) = 0 \in \ker \varphi = \{f \in \mathbb{R}[x]: \varphi(f) = 0\}$ . Now by Proposition 6.10, we have the polynomial is irreducible as it has no roots in  $\mathbb{R}[x]$ . As  $x^2+1$  is irreducible, we have that  $(x^2+1)$  is prime and  $\mathbb{R}[x]$  is a PID so  $(x^2+1)$  is maximal. Thus we have a field  $\mathbb{R}[x]/(x^2+1)$  is a field, and additionally, we have that, as  $(x^2+1) \subseteq \ker \varphi$  and  $(x^2+1)$  is maximal, then  $(x^2+1) = \ker \varphi$  as we cannot have  $\ker \varphi = \mathbb{R}[x]$  as then this would imply  $\varphi(1) = 0$  which is false. Thus as  $\ker \varphi = (x^2+1)$ ,  $\varphi$  is a homomorphism, and  $\mathbb{R}[x]/(x^2+1)$ , we have a isomorphism of fields  $\mathbb{R}[x]/(x^2+1) \simeq \mathbb{C}$ .  $\square$

**Exercise 7.12 (11).** Prove that  $x^2 + y^2 - 1$  is irreducible in  $\mathbb{Q}[x, y]$ .

*Proof.* Consider the ring  $(\mathbb{Q}[x])[y]$  where we're taking coefficients in  $\mathbb{Q}[x]$ . Then  $y^2 + (x^2 - 1)$  implies that we can apply Eisenstein where we pick our prime ideal to be  $(x+1)$ . Now, clearly  $x^2 - 1 = (x+1)(x-1)$  is divisible by  $x+1$ , however,  $x^2 - 1$  is not divisible by  $(x+1)^2$ . Thus this polynomial is irreducible.  $\square$

**Exercise 7.13 (18).** Show that  $6x^5 + 14x^3 - 21x + 35$  and  $18x^5 - 30x^2 + 120x + 360$  are irreducible in  $\mathbb{Q}[x]$ .

*Proof.* For the first polynomial, we have that all the coefficients other than the leading term are divisible by 7 and applying Eisenstein we see that it is irreducible as we have that  $7^2 \nmid 35$ . For the second polynomial, we can reduce the coefficients by 6 to get  $3x^5 - 5x^2 + 20x + 60$ , and so now we can apply Eisenstein with prime  $p = 5$ : all but the leading term are divisible by 5 but  $5^2 = 25 \nmid 60$ , and so this polynomial is thus irreducible as well. We should note here that Eisenstein isn't only useful when a polynomial is monic; we simply need for the prime term to divide all coefficients besides the leading term and it works the same.  $\square$

**Exercise 7.14** (Found this in a qualifying exam). Prove that  $p(x) = x^4 + 5x^2 + 3x + 2$  is irreducible over  $\mathbb{Q}$ .

*Proof.* Will will be working on this over  $\mathbb{Z}$  instead. Observe that as the polynomial is monic and with integer coefficients, then we can use the rational root theorem to deduce that we cannot produce a root in  $\mathbb{Z}$  as the divisors of the constant term 2 do not provide a root, i.e.  $p(\pm 2) \neq 0$  and  $p(\pm 1) \neq 0$ . This implies that we cannot have a linear root. Thus assume that we can factor into two quadratics:  $p(x) = x^4 + 5x^2 + 3x + 2 = (x^2 + ax + b)(x^2 + cx + d) \in \mathbb{Z}[x]$ . Then after expanding the RHS, we have  $x^4 + x^3(a+c) + x^2(ac+b+d) + x(ad+bc) + bd$ . Matching coefficients gives us  $bd = 2$ ;  $ad+bc = 3$ ;  $ac+b+d = 5$ ;  $a+c = 0$ . Now this system takes a bit of work, but these were my concluding results:  $c = -a$ ;  $b = 3/a$ ;  $d = 6/a$ , and so using the third item in our list of matching coefficients, we have:  $-a^2 + 6/a + 3/a = 5$ , which means that  $a^3 + 5a = 9$  and this has no integer solutions! Therefore we have that  $p(x)$  is irreducible over  $\mathbb{Q}$ .  $\square$

## 8. FIRST MIDTERM:

**Exercise 8.1.** Let  $\varphi: G \rightarrow G'$  be a group homomorphism, and let  $H' \leq G'$ . Show that  $\varphi^{-1}(H') \leq G$ .

*Proof.* The set is nonempty as  $\varphi(1) = 1 \in \varphi^{-1}(H')$ . Now take  $x, y \in \varphi^{-1}(H')$ . Then as  $\varphi(y) = h$  for some  $h \in H'$ , then  $\varphi(1) = \varphi(y)\varphi(y^{-1}) = h\varphi(y^{-1}) = 1$ , and so we have that  $\varphi(y^{-1}) = h^{-1} \in H'$ . Thus we trivially get that  $\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1}) \in H'$ . So we have that  $\varphi^{-1}(H') \leq G$ . For normality, take  $g \in G$  and  $\varphi(a \in G) = k \in H'$ . As  $H'$  is normal for  $G'$ , then we have  $\varphi(gkg^{-1}) = \varphi(g)k\varphi(g^{-1}) \in H'$ . Thus  $\varphi^{-1}(H') \leq G$ .  $\square$

**Exercise 8.2.** Let  $G$  be a finite group of order  $p$ , a prime. Show that  $G \simeq \mathbb{Z}/p\mathbb{Z}$ .

*Proof.* As  $G$  has order  $p$ , then there exists an element, say,  $q \in G$  such that  $q^p = 1$ . Thus we have a cyclic subgroup  $\langle q \rangle$  with order  $p$ , and so we have that  $|\langle q \rangle| = p \mid p$ . (The only other subgroup of  $G$  would be  $\{1\}$ , the trivial subgroup.) This forces  $G$  to be cyclic and so let  $\langle x \rangle = G$ . Now we define  $\varphi: G \rightarrow \mathbf{Z}/p\mathbf{Z}$  by  $\varphi(x^i) = \bar{i}$  for some  $x^i \in G$ , and  $0 \leq i \leq p$ . This map is seen to be surjective. Now let  $x^i, x^j \in G$ . Then  $\varphi(x^i \cdot x^j) = \varphi(x^{ij}) = \bar{ij} = \bar{i} \cdot \bar{j} = \varphi(x^i)\varphi(x^j)$ , and so we have a homomorphism. Now,  $\ker \varphi = \{x^i \in G: \varphi(x^i) = \bar{0} = \bar{p}\}$ . Thus  $\ker \varphi$  has only one element in  $G$  that maps to  $\bar{0}$  is  $x^p = 1$  (by the nature of how the map is defined). So  $\varphi$  is injective and we have an isomorphism  $G \simeq \mathbf{Z}/p\mathbf{Z}$ .  $\square$

**Exercise 8.3.** Let  $H \trianglelefteq G$  and  $K \leq G$ , where  $G$  is a group. Show that  $HK = \{hk: h \in H, k \in K\}$  is a subgroup of  $G$ .

*Proof.* Note that  $HK$  is trivially nonempty as  $0 \cdot 0 \in HK$ . Let  $x = h_1 k_1 \in HK$  and  $y = h_2 k_2 \in HK$ . Then, consider  $y^{-1} = (h_2 k_2)^{-1} = k_2^{-1} h_2^{-1}$ . Then  $xy^{-1} = (h_1 k_1)(k_2^{-1} h_2^{-1}) = h_1(k_1 k_2^{-1})h_2^{-1}$ , and let  $k_1 k_2^{-1} = k_3$ . As  $H \trianglelefteq G$  and  $K \leq G$ , we have that  $h_1 k_2 h_2^{-1} = h_1 h_2 k_2 \in HK$ . Thus  $HK$  is a subgroup of  $G$ .  $\square$

**Exercise 8.4.** Consider the action of a group  $G$  on a set  $X$ , and let  $x_0 \in X$ . Show that there is a bijection between  $G/G_{x_0}$  and  $G \cdot x_0$ .

*Proof.* Define  $\varphi: G \cdot x_0 \rightarrow G/G_{x_0}$  given by, for all  $g \in G$ ,  $\varphi(g \cdot x) = gG_{x_0}$ , and  $\cdot$  in the LHS is the group action on  $X$ . Suppose that  $g \cdot x = h \cdot x$  for some  $g, h \in G$ . Then we have that  $h^{-1}g \cdot x = 1 \cdot x = x$ , which means that  $h^{-1}g \in G_{x_0}$ , so then  $gG_{x_0} = hG_{x_0}$ ; thus we have that  $\varphi$  is well defined. Let  $\varphi(g_1 \cdot x) = \varphi(g_2 \cdot x)$ , then  $g_1 G_{x_0} = g_2 G_{x_0}$ , and so we have that  $g_2^{-1}g_1 \in G_{x_0}$ . This means that  $g_2^{-1}g_1 \cdot x = x$ , so then we may write  $g_2 \cdot x = g_2 \cdot (g_2^{-1}g_1 \cdot x) = (g_2 g_2^{-1}g_1) \cdot x = g_1 \cdot x$ , and so we conclude that  $\varphi$  is injective. Moreover, we can see, by definition of our map, that  $\varphi$  is clearly surjective. Therefore we have a bijection.  $\square$

**Exercise 8.5.** Let  $G$  be a group and let  $x$  and  $y$  be two elements of  $G$  such that  $|xy| = n$ . Show that  $|yx| = n$ .

*Proof.* So,  $(xy)(xy) \cdots (xy) = 1$  for  $n$  many  $xy$  terms, which implies that  $x(yx)(yx) \cdots (yx)y = 1$ , where there are  $n-1$  many  $yx$  terms. So then  $(yx)(yx) \cdots (yx) = (yx)^{n-1} = x^{-1}y^{-1} = (yx)^{-1}$ , and hence  $(yx)^{n-1}(yx) = (yx)^n = 1$ . Thus  $|yx| = n$ .  $\square$

**Exercise 8.6.** Let  $G$  be a  $p$ -group of order  $p^2$ . We wish to classify the possible group structures on  $G$  up to isomorphism.

- (a) Case 1: Suppose  $G$  contains an element of order  $p^2$ . Conclude.
- (b) Case 2: Suppose that  $G$  does not contain an element of order  $p^2$ :
  - (b)(i) Show that  $Z(G)$  contains a subgroup  $H$  of order  $p$
  - (b)(ii) Show that  $G$  contains another subgroup  $K$  of order  $p$ ,  $K \neq H$ . Then show that  $G = HK$ .
  - (b)(iii) Show that  $H \times K \simeq G = HK$ . Conclude.

*Proof.* For part (a): If  $G$  contains an element of order  $p^2$ , then  $G$  is cyclic, as  $G$  would be generated by that element as order of cyclic group generated by that element would be  $p^2 = |G|$ , and so  $G \simeq \mathbf{Z}/p^2\mathbf{Z}$ .

(b): Suppose that  $G$  does not contain an element of order  $p^2$ .

(i): The easiest approach would be to show that the center of  $Z(G)$  has an element of order  $p$ : The center of a  $p$ -group is known to be  $G$  non trivial so let us pick an element  $x \in Z(G) - \{1\}$ . Then consider the subgroup cyclic subgroup  $\langle x \rangle$  of  $G$ . By Lagrange, we must have that  $|\langle x \rangle| \mid p^2$ , and so we have a couple of choices:  $|\langle x \rangle| = 1, p$ , or  $p^2$ . We cannot have the case of  $|\langle x \rangle| = 1$  as then would make  $x = 1$ . Now if  $|\langle x \rangle| = p^2$ , then this would generate the whole group and so we would have an element of  $p^2$  in  $G$ . Thus we must have that  $|\langle x \rangle| = p$ , which is exactly what we want. Thus we have an element of the center, and so a subgroup of  $Z(G)$ , with order  $p$ . Lastly, as we picked  $x \in Z(G)$ , then we have that  $\langle x \rangle \trianglelefteq G$  Let  $H = \langle x \rangle$

(ii) We want to show that another subgroup of order  $p$  exists, excluding  $H$ . So, take  $g \in G - H \neq \emptyset$ . In a similar fashion to (ii), by Lagrange, we have that  $|g| = p$ , and let  $\langle g \rangle = K$ , and  $K \neq H$ . Then, as  $H$  is normal in  $G$ , then  $HK \leq G$ . Additionally,  $|HK| = |H||K| = p^2 = |G|$ , and so  $HK = G$ .

(iii): Define the map  $\varphi: H \times K \rightarrow HK = G$  defined by  $a \times b \mapsto ab$ . As  $x \in Z(G)$ , we have that every element of  $H$  commutes with every element of  $K$ , and so we easily get a homomorphism: take  $(a, b), (c, d) \in H \times K$ , then  $\varphi((a, b), (c, d)) = \varphi(ac, bd) = (ac)(bd) = a(bc)d = (ac)(cd) = \varphi(a, b)\varphi(c, d)$ . If  $\varphi(h, k) = 1$  for some  $h \in H$  and  $k \in K$ , then we have that  $k$  is the inverse of  $h$  and so  $k \in H$  and as we have a trivial intersection, then  $k = 1$ , which forces  $h = 1$  as well. So  $\varphi$  is injective and thus a bijection as the order of the domain and codomain of  $\varphi$  are the same. Thus  $H \times K \simeq HK = G$ .  $\square$

**Exercise 8.7.** Let  $H$  and  $K$  be two normal subgroups of a finite group  $G$ . Recall that such conditions are more than sufficient to make  $HK \leq G$ .

- (a) Assume that the orders of  $H$  and  $K$  are coprime. Show that  $H \times K \simeq HK$ .
- (b) Let  $G$  be a group of order 33. Determine the possible group structures on  $G$  up to isomorphism.



*Proof.* Consider the map  $\varphi: H \times K \rightarrow HK$  defined by  $\varphi: (a, b) \mapsto ab$ . Let  $(a, b) = (c, d) \in H \times K$ . Then  $(ac^{-1}, bd^{-1}) = (1, 1)$ , which implies that  $\varphi(ac^{-1}, bd^{-1}) = (ac^{-1})(bd^{-1}) = 1$ , so then  $ab = cd$ , so the map is well defined. Moreover, the map is clearly surjective. Now we consider  $\ker \varphi = \{(a, b) \in H \times K: \varphi(a, b) = 1\}$ , so  $ab = 1$ , which means that  $b^{-1}ab = b^{-1} \in K$ , and as  $H \trianglelefteq G$ , then  $b^{-1}ab = b^{-1} \in H$ . As  $H \cap K = \{1\}$ , then we must have that  $b^{-1} = 1$ , so then  $bb^{-1} = b1 = b$ , and  $ab = 1 \implies ab(b^{-1}) = a = 1 \cdot 1 = 1$ ; thus  $a = 1$  and  $b = 1$ , so the kernel must only have  $(1, 1)$  in it, and  $\varphi$  is hence injective. Lastly we need to show that  $\varphi$  is a homomorphism: Suppose  $(a, b), (c, d) \in H \times K$ , then  $\varphi((a, b)(c, d)) = \varphi(ac, bd) = (ac)(bd) = a(cb)d = (ab)(cd) = \varphi(a, b)\varphi(c, d)$ , which is what we essentially need to show, i.e. that  $H$  commutes with any element in  $K$ . Now take  $h \in H$  and  $k \in K$ , then we have that  $hk = kh$  if and only if  $hkh^{-1}k^{-1} = 1$ . So then

$$hkh^{-1}k^{-1} = (hkh^{-1})k^{-1} \in HK \quad (4)$$

$$= h(kh^{-1}k^{-1}) \in HK, \quad (5)$$

as  $H$  and  $K$  are normal, and so  $hkh^{-1}k^{-1} \in H \cap K$ , but the intersection is trivial and so we have that  $hkh^{-1}k^{-1} = 1$ , so  $hk = kh$  and  $\varphi$  is a homomorphism. Therefore  $H \times K \simeq HK$ .

(b): As  $G$  has order  $33 = 11 \cdot 3$ , then we have that  $n_{11} \mid 3$  and  $n_{11} \equiv 1 \pmod{11}$  and so this forces  $n_{11} = 1$ , and we have that  $n_3 \mid 11$  and  $n_3 \equiv 1 \pmod{3}$  which forces  $n_3 = 1$ . This means that we have a unique, normal 11-Sylow subgroups, say,  $P$ , and a unique, normal 3-Sylow subgroup  $Q$ . As both are normal, and we have an obvious trivial intersection, we have that  $G \simeq P \times Q \simeq \mathbb{Z}/11\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \simeq \mathbb{Z}/33\mathbb{Z}$ .  $\square$

## 9. RING THEORY EXERCISES AND NOTES DONE LONG AGO

**Exercise 9.1.** All maximal ideals are prime ideals.

*Proof.* Suppose  $\mathfrak{m}$  is a maximal ideal of a ring  $R$ . Then there are no other ideals contained between  $\mathfrak{m}$  and  $R$ , i.e., if  $\mathfrak{n}$  is an ideal of  $R$  such that  $\mathfrak{m} \subseteq \mathfrak{n} \subseteq R$ , then  $\mathfrak{n} = \mathfrak{m}$  or  $\mathfrak{n} = R$ . Take  $a, b \in R$ , and suppose  $ab \in \mathfrak{m}$ . Suppose  $\mathfrak{m}$  isn't a prime ideal, so  $a \notin \mathfrak{m}$  and  $b \notin \mathfrak{m}$ . Consider the ideal  $(a) + \mathfrak{m}$  generated by  $a$  and  $\mathfrak{m}$ . Since the ideal  $(a) + \mathfrak{m}$  is strictly larger than  $\mathfrak{m}$ , we must have  $R = (a) + \mathfrak{m}$  by the maximality of  $\mathfrak{m}$ . Since  $1 \in R$ , then  $1 = ar + m$  for  $r \in R$  and  $m \in \mathfrak{m}$ , and similarly for  $(b) + \mathfrak{m}$ ,  $1 = bs + n$  for  $s \in R$  and  $n \in \mathfrak{m}$ . But then

$$1 \cdot 1 = (ar + m)(bs + n) = arbs + arn + bsm + nm = (ab)rs + (ar)n + (bs)m + (nm),$$

and so since  $ab, n, m \in \mathfrak{m}$ , the sum of this expression must be in  $\mathfrak{m}$ , i.e.,  $1 \in \mathfrak{m}$ , and hence  $\mathfrak{m} = R$ . Since a maximal ideal is proper by definition, this is a contradiction. Thus,  $\mathfrak{m}$  must be prime.  $\square$

**Exercise 9.2.** Let  $\mathfrak{p}$  be a prime ideal of a ring  $R$  that is,  $\mathfrak{p}$  is a proper ideal such that if  $ab \in \mathfrak{p}$  for  $a, b \in R$ , then  $a \in \mathfrak{p}$  or  $b \in \mathfrak{p}$ . Prove that if  $abc \in \mathfrak{p}$  for  $a, b, c \in R$ , then at least one of the three elements must be in  $\mathfrak{p}$ . More generally, use induction to prove that if  $a_1 a_2 a_3 \cdots a_n \in \mathfrak{p}$  for  $a_i \in R$ , then at least one of the factors  $a_i$  must be in  $\mathfrak{p}$ .

*Proof.* Suppose  $\mathfrak{p}$  is a prime ideal of a ring  $R$ .

For the case of three elements: Let  $a, b, c \in R$  and  $abc \in \mathfrak{p}$ . Then  $(ab)c \in \mathfrak{p}$  implies that  $ab \in \mathfrak{p}$  or  $c \in \mathfrak{p}$ , and so  $a \in \mathfrak{p}$ ,  $b \in \mathfrak{p}$ , or  $c \in \mathfrak{p}$ .

As hinted, we proceed with a view of induction. Suppose  $a_1 a_2 a_3 \cdots a_n \in \mathfrak{p}$  for  $a_i \in R$ . Then the case of  $n = 2$  holding is obviously true, and we also just did the case of  $n = 3$ . Suppose that the statement is true for  $n = k - 1$ . Then

$$a_1 a_2 a_3 \cdots a_k = (a_1 a_2 a_3 \cdots a_{k-1})a_k \in \mathfrak{p},$$

so, since  $\mathfrak{p}$  is prime,  $a_1 a_2 \cdots a_{k-1} \in \mathfrak{p}$  or  $a_k \in \mathfrak{p}$ . If  $a_k \in \mathfrak{p}$ , then we're done. Otherwise, if  $a_1 a_2 a_3 \cdots a_{k-1} \in \mathfrak{p}$ , then, by the induction hypothesis, one of the elements  $a_1, a_2, \dots, a_{k-1}$  is in  $\mathfrak{p}$ .  $\square$

**Exercise 9.3.** Any finite integral domain must be a field

*Proof.* Suppose  $D$  is a finite integral domain with  $\alpha$  elements, i.e.,  $|D| = \alpha$ . Let  $a_1, a_2, \dots, a_\alpha$  be elements of  $D$ . Fix an arbitrary element  $\ell \in D$  such that  $\ell \neq 0$ . Then, we claim that  $\ell a_1, \ell a_2, \dots, \ell a_\alpha$  are all distinct. Suppose that they aren't, for aim of contradiction. So,  $\ell a_i = \ell a_j$  for  $i \neq j$ , but then  $\ell(a_i - a_j) = 0$ , so we have that  $a_i = a_j$  — a contradiction. Thus they must all be distinct. Moreover, since  $D$  has only  $\alpha$  elements, one of which is the identity 1, one of the  $\alpha$  products has to be 1. Therefore, we can say that some  $\ell a_i$ , must be the element 1, and hence  $a_i$  is the inverse of  $\ell$ .  $\square$

**Exercise 9.4.** Let  $I$  be an ideal of a ring  $R$  that is contained in a prime ideal  $Q$ . Use Zorn's lemma to prove that there is a prime ideal  $\mathfrak{p}$  of  $R$  such that

$$I \subseteq \mathfrak{p} \subseteq Q,$$

and there are *no* prime ideals between  $I$  and  $\mathfrak{p}$ . We will call such a prime  $\mathfrak{p}$  a **minimal prime over  $I$** .