MATH 425A HW11, NOV. 16, 10PM

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1. Chapter 6

Exercise 3.1.

A collection $\mathcal{A}$ of real-vlaued functions on a set E is said to be **uniformly bounded** on E if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in E$, for all $f \in \mathcal{A}$. (So each function is bounded, and the same bound works for all functions in $\mathcal{A}$.) Let (f_n) be a sequence of bounded functions which converges uniformly to a limit function f. Prove that $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded subset of $(B(X), d_u)$.

Proof. As $f_n \to f$, then we may pick $\epsilon = 1$ such that there exists some $N \in \mathbb{N}$ where $n \geq N$ gives $|f_n - f| < 1$. For each f_n, we have that it is bounded by some corresponding $M_n > 0$, i.e. $|f_n| \leq M_n$. Now write $M = \max\{M_1, \ldots, M_N\}$. Then, for $n \geq N$, $|f_n| \leq |f_n - f| + |f - f_N| + |f_N| < 2 + M$. Therefore we have that $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded subset of $B(X)$. □

Exercise 3.2.

Let $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on a set E, which converges uniformly on E to limit functions f and g, respectively.

(a) Prove that $(f_n + g_n)_{n=1}^{\infty}$ converges to $f + g$ uniformly on E.
(b) If each f_n and each g_n is bounded, show that $(f_n g_n)_{n=1}^{\infty}$ converges uniformly to fg on E.

Proof. (a) As $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ both converge uniformly, we have some $N \in \mathbb{N}$ and $M \in \mathbb{N}$ such that for $n \geq N$ and $n \geq M$ we get $|f_n(x) - f(x)| < \epsilon/2$ and $|g_n(x) - g(x)| < \epsilon/2$, respectively, for (all) $\epsilon > 0$. Then

$$\begin{aligned}|(f_n + g_n) - (f + g)| &= |(f_n - f) + (g_n - g)| \\ &\leq |f_n - f| + |g_n - g| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

for $n \geq \max\{M, N\}$. Hence we have $(f_n + g_n)_{n=1}^{\infty}$ uniformly converging to $f + g$ on E.

(b) Suppose that f_n and g_n are bounded; that is, for all f_n and g_n, we have $|f_n(x)| \leq M$ and $|g_n(x)| \leq T$ for some $M, P \in \mathbb{R}$ and all $x \in X$. The idea is two get an $\epsilon/2$ demonstration after applying the triangle inequality many times. As $g_n \to g$, for $\epsilon > 0$, there is some $N_1 \in \mathbb{N}$ such that for $n \geq N_1$ we can write $|g_n - g| < \frac{\epsilon}{2M}$. Additionally, for $\epsilon > 0$, there exists some $N_2 \in \mathbb{N}$ such that $|f_n - f| < \frac{\epsilon}{2(1+T)}$ for $n \geq N_2$. Note that there exists an $N_3 \in \mathbb{N}$ such that $|g_n - g| < 1$ for $n \geq N_3$. Then $|g| \leq |g_n - g| + |g_n| < 1 + T$. Then for $n \geq \max\{N_1, N_2, N_3\}$,

$$\begin{aligned}|(f_n g_n) - fg| &= |(f_n g_n) - fg + (f_n g - f_n g)| = |(f_n g_n - f_n g) + (f_n g - fg)| \\ &\leq |f_n g_n - f_n g| + |f_n g - fg| = |f_n(g_n - g)| + |g(f_n - f)| \\ &= |f_n||g_n - g| + |g||f_n - f| < M\left(\frac{\epsilon}{2M}\right) + (1 + T)\left(\frac{\epsilon}{2(T+1)}\right) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

□

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Exercise 1.1.

Prove parts (b) and (c) of the Proposition.

(a) Let A be subset of $\mathbb{R}$, let $p \in \mathbb{R}$ be a limit point of A with respect to $\mathbb{R}$, and let $f\colon A \to \overline{\mathbb{R}}$ be a function. Then
$$\lim_{x\to p} f(x) = +\infty$$
if and only if for every $L \in \mathbb{R}$, there exists $\delta > 0$ such that $0 < |x-p| < \delta$ and $x \in A$ together imply that $f(x) > L$.

(b) Let B be a subset of $\mathbb{R}$ that is not bounded above in $\mathbb{R}$, and let $g\colon B \to \overline{\mathbb{R}}$ be a function. Let q be a real number. Then
$$\lim_{x\to +\infty} g(x) = q$$
if and only if for every $\epsilon > 0$, there exists $M \in \mathbb{R}$ such that $x > M$ and $x \in B$ together imply that $|g(x) - q| < \epsilon$.

(c) Let C be a subset of $\mathbb{R}$ that is not bounded above in $\mathbb{R}$; let $h\colon C \to \overline{\mathbb{R}}$ be a function. Then
$$\lim_{x\to +\infty} h(x) = +\infty$$
if and only if for every $N \in \mathbb{R}$, there exists $P \in \mathbb{R}$ such that $x > P$ and $x \in C$ together imply $h(x) > N$.

Proof. Recall that a neighborhood of $+\infty$ is given by for any $c \in \mathbb{R}$ and $x > c$, the set $(c, +\infty)$ is a neighborhood of $+\infty$.

(b) ($\Rightarrow$) Let $\lim_{x\to+\infty} g(x) = q$ and $\epsilon > 0$. As $q \in B(q,\epsilon)$ is a neighborhood of q so we have a neighborhood U of $+\infty$, and we can write $U = (M, +\infty)$ by Proposition 1.1(a), such that $x \in U \cap B$ (i.e. $x > M$ and $x \in B$) implies $g(x) \in V$, that is, $|g(x) - q| < \epsilon$. ($\Leftarrow$) Suppose that for every $\epsilon > 0$ there is some $M \in \mathbb{R}$ such that $x > M$ and $x \in B$ implies $|g(x) - q| < \epsilon$. Let $V = B(q, \epsilon)$ be a neighborhood of q. Then as we have $x > M$ and $x \in B$ (and B is not bounded), then $U = (M, +\infty)$ is a neighborhood of $+\infty$ (by Proposition 1.1(a)) for which $x \in U \cap B$ implies $g(x) \in V$ as we assumed $|g(x) - q| < \epsilon$ holds true for the preceding hypotheses. Side note: we're even able to even let $x \to +\infty$ since we chose B to be unbounded (by Proposition 1.1(b)) in either case.

(c) ($\Rightarrow$) Let $\lim_{x\to+\infty} h(x) = +\infty$, and let $N \in \mathbb{R}$. Then $(N, +\infty]$ is an open neighborhood of $+\infty$ in $\overline{\mathbb{R}}$ by Proposition 1.1(a). So we have an open neighborhood $U = (P, +\infty)$ of $+\infty$ such that $x \in U \cap C$ gives $h(x) \in (N, +\infty]$, which means that $x > P$ for all $x \in C$ and also $h(x) > N$. ($\Leftarrow$) Assume that for every $N \in \mathbb{R}$, there exists some $P \in \mathbb{R}$ such that $x > P$ and $x \in C$ implies $h(x) > N$. Then for every $N \in \mathbb{R}$, we have a neighborhood $U = (P, +\infty)$ of $+\infty$ by Proposition 1.1(a), and we also have that for $x > P$ and $x \in C$ (i.e. $\in U \cap C$) we get $h(x) > N$, i.e. $h(x) \in (N, +\infty)$, a neighborhood of $+\infty$. Therefore we're done as for any neighborhood $V = (N, +\infty)$ of $+\infty$ (by Proposition 1.1(a)) there exists an open neighborhood $U = (P, +\infty)$ of $+\infty$ such that $x \in U \cap C$ implies $h(x) \in V$. Side note: we're even able to even let $x \to +\infty$ since we chose C to be unbounded (by Proposition 1.1(b)) in either case. $\square$

Exercise 4.1.

Let $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ be sequences in $\overline{\mathbb{R}}$. Prove the following statements.

(a) If $s_n \leq t_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = +\infty$, then $\lim_{n\to\infty} t_n = +\infty$ as well.

(b) If $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ converge in $\overline{\mathbb{R}}$ to s and t, respectively, and if $s_n \leq t_n$ for each $n \in \mathbb{N}$, then $s \leq t$.

Proof. (a) Let $s_n \leq t_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = +\infty$. Then we have that (s_n) is an unbounded sequence, meaning that for every $M \in \mathbb{R}$ such that $M > 0$ we have $s_n > M$. Then this forces $M \leq s_n \leq t_n$ for each $n \in \mathbb{N}$, and therefore t_n is also unbounded forcing $\lim_{n\to\infty} t_n = +\infty$ in $\overline{\mathbb{R}}$.

(b) Suppose $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ with $s_n \to s$ and $t_n \to t$ as $n \to \infty$. Further, assume $s_n \leq t_n$ for all $n \in \mathbb{N}$. By way of contradiction, suppose that $s > t$. Then $s - t > 0$, and now we write $\epsilon = \frac{s-t}{2}$. As $s_n \to s$, then there is some $N_1 \in \mathbb{N}$ such that $n \geq N$ gives $s_n \in (s-\epsilon, s+\epsilon)$, i.e. $s - \epsilon < s_n < s + \epsilon$; similarly, we have some $N_2 \in \mathbb{N}$ where $n \geq N_2$ produces $t - \epsilon < t_n < t + \epsilon$. Now we have $t_n < t + \epsilon$, and so

$$t_n < t + \epsilon = t + \frac{s-t}{2} = \frac{t+s}{2} = s - \epsilon < s_n,$$

which is a contradiction. Therefore we have $s \leq t$. □

Exercise 4.2.

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Provided that the RHS isn't of the form $\infty - \infty$.

Proof. We proceed by contradiction; suppose

$$\limsup(a_n + b_n)_{n\to\infty} > \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

As a short hand, we write $A = \limsup_{n\to\infty} a_n$, $B = \limsup_{n\to\infty}$, and
$\ell = \limsup_{n\to\infty}(a_n + b_n)$. Then we have $\ell > A + B$ so $\ell - B > A$, and so there exists some $\gamma \in (A, \ell - B)$, so $\gamma > A = \limsup_{n\to\infty} a_n$. By Theorem 4.2.(b), there exists some $N \in \mathbb{N}$ such that $n \geq N$ we get $a_n < \gamma$ (i.e. for all but finitely many, as stated in the Theorem). Hence $a_n + b_n < \gamma + b_n$ implies $\ell = \limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty}(\gamma + b_n) = \gamma + \limsup_{n\to\infty} b_n = \gamma + B$ (by the previous exercise and Theorem 4.2.(b)) when $n \geq N$. So $\ell \leq \gamma + B < \ell - B + B = \ell$, as we assumed $\gamma \in (A, \ell - B)$, and so we have a contradiction. Hence we have the desired inequality

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

□

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