MATH 425A HW5, DUE 09/27/2022, 6PM

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Chapter 3. §2.

Exercise 0.1 (2.5.)**.** Finish the proof of Proposition 2.6., by proving that

$$\mathrm{Int}_Y(U) \cap \mathrm{Int}_X(Y) \subseteq \mathrm{Int}_X(U) \qquad (U \subseteq Y \subseteq X)$$

Proof. Suppose $U \subseteq Y \subseteq X$, where (X, d) is a metric space. Now take $t \in \mathrm{Int}_Y(U) \cap \mathrm{Int}_X(Y)$. Then we have that there is some $t \in U \cap Y$ such that $r_\alpha > 0$ with $B_Y(t, r_\alpha) \subseteq U$, and there is some r_β such that $B_X(t, r_\beta) \subseteq Y$. Using ball notation, we know that $t \in U \cap Y \cap X$, and that $B_X(t, r_\beta) = \{y \in X \colon d(t, y) < r_\beta\}$ and $B_Y(t, r_\alpha) = \{q \in Y \colon d(t, q) < r_\alpha\}$. So we need to show that $t \in \mathrm{Int}_X(U)$, i.e. $t \in X$ and there is some $r_\gamma > 0$ such that $B_X(t, r_\gamma) \subseteq U$. $\square$

Exercise 0.2 (2.6.)**.**

Exercise 0.3 (2.7.)**.** Let (X, d) be a metric space, and let U be a subset of X. Use Proposition 2.9 to prove that $\mathrm{Int}_X(U)$ is open in X.

Proof. We want to show that $\mathrm{Int}_X(U) \subseteq \mathrm{Int}_X(\mathrm{Int}_X(U))$. So let $t \in \mathrm{Int}_X(U)$. Then $t \in U$ with $B(t, r) \subseteq U$ for some $r > 0$. Now let $y \in B(t, r)$, but as open balls are open then $y \in \mathrm{Int}(B(t, r))$, and so we can find an another open ball $y \in B(t, \epsilon) \subseteq B(t, r)$. All together, we have that $B(t, \epsilon) \subseteq B(t, r) \subseteq U$. Thus if we have some point $y \in B(t, r)$ then $y \in \mathrm{Int}(U)$, and hence $B(t, r) \subseteq \mathrm{Int}(U)$. Therefore $\mathrm{Int}(U)$ is an open set. $\square$

Exercise 0.4 (2.8.)**.** Let (X, d) be a metric space. Assume that $U \subseteq Y \subseteq X$, and additionally that Y is open in X. Prove that U is open in Y if and only if U is open in X. (Note: There at least two possible solutions; one uses Theorem 2.13, the other uses Exercise 2.5.)

Proof. Suppose that U is open in Y. Then $U = Y \cap V$ for some open set $V \subseteq X$. But as finite intersections of open sets are open, and $Y \subseteq X$ and $V \subseteq X$ are both open in X, then $U = Y \cap V$ is open in X by Theorem 2.13. Now suppose that U is open in X. Then $U = Y \cap U$ as $U \subseteq Y$, but as U and Y are both open in X, then by Theorem 2.13 U is open in Y as well. $\square$

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