MATH 425A HW12, DEC. 5, 11:59PM

Chapter 8

Exercise 1.2.

Let $f\colon \mathbb{R} \to \mathbb{R}$, and assume $\lim_{x\to\infty} x|f'(x)| = 0$. Define a sequence $(a_n)_{n=1}^{\infty}$ in $\mathbb{R}$ by $a_n = f(2n) - f(n)$ for each $n \in \mathbf{N}$. Prove that $a_n \to 0$ as $n \to \infty$.

Proof. By MVT, we have $\frac{f(2n)-f(n)}{2n-n} = \frac{f(2n)-f(n)}{n} = f'(x_n)$ for a sequence $x_n \in (n, 2n)$, which implies $f(2n) - f(n) = nf'(x_n)$, so $a_n = nf'(x_n)$. Let $x_n \to x$ in $\mathbb{R}$ as $n \to \infty$. Then $0 \leq n|f'(x_n)| \leq x_n|f'(x_n)|$, and so $0 \leq \lim_{n\to\infty} n|f'(x_n)| \leq \lim_{n\to\infty} x_n|f'(x_n)| = 0$. Hence $\lim_{n\to\infty} n|f'(x_n)| = 0 \implies \lim_{n\to\infty} nf'(x_n) = a_n = 0$. Therefore $a_n \to 0$ as $n \to \infty$. $\square$

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Exercise 1.3.

Let $f\colon (a,b) \to \mathbb{R}$ be a differentiable function with $f'(x) > 0$ for all $x \in (a,b)$.

(a) Prove that f is injective, and and argue that its image must be an open interval (c,d) (with c and/or d possibly infinite).

(b) By part (a), there exists a function $g\colon (c,d) \to (a,b)$ such that $g(f(x)) = x$ for all $x \in (a,b)$. Prove that g is continuous. (Hint: Use Theorem 2.19 in Chapter 5, and use the proof of Proposition 2.20 of Chapter 5 as a model for your answer.)

(c) Prove that g is differentiable, and that $g'(f(x)) = 1$, for all $x \in (a,b)$. (Hint: Pick $y \in (c,d)$, $f'(x)$ and let $(y_n)_{n=1}^{\infty}$ be a sequence in $(c,d) \setminus \{y\}$ that converges to y. Write the difference quotient $\frac{g(y_n)-g(y)}{y_n-y}$ in terms of f and a sequence $(x_n)_{n=1}^{\infty}$ in (a,b).)

Proof. (a) Firstly we argue that f is injective. Suppose we have $f(x) = f(y)$ such that $x \neq y$ with $x,y \in (a,b)$. Then by MVT there exists some $\gamma \in (a,b)$ such that $\frac{f(y)-f(x)}{y-x} = f'(\gamma)$, which is of course defined as $x \neq y$ so $x - y \neq 0$. But then $f'(\gamma) = 0$ which is thus a contradiction. Hence f must be injective. Next we argue that $\mathrm{im}(f)$ is open an interval of $\mathbb{R}$. As f is continuous and (a,b) is connected, then $f((a,b))$ is also connected, so $f((a,b))$ is an interval and we still need to check that it is indeed open. We have that f is a monotonically increasing function, and so for all $x,y \in (a,b)$ we have $f(x) < f(y)$, so $f((x,y)) = (f(x), f(y))$ by the fact that f is monotonically increasing and IVT (note that if we would've considered closed intervals, or half closed intervals, then these would've been absurd). Therefore we're done.

(b) As f is injective then we have an inverse function $g\colon (c,d) \to (a,b)$ such that $g(f(x)) = x$ for all $x \in (a,b)$. Pick $\delta > 0$ such that $a < x - \delta < b$. Then define $h\colon [x-\delta, x+\delta] \to (f(x-\delta), f(x+\delta))$, which is just a restriction and so this map is also continuous. Note also that as we've constructed the map then this map is also a bijection, and so we have a continuous bijection. By Theorem 2.19, we have a continuous map $\tilde{h}\colon (f(x-\delta), f(x+\delta)) \to [x-\delta, x+\delta]$. But this is new map is just g, so $g = \tilde{h}$, which gives that g is in fact continuous.

(c) Let $y \in (c,d)$ and let $(y_n)_{n=1}^{\infty}$ be a sequence in $(c,d) \setminus \{y\}$ such that $y_n \to y$ for $n \to \infty$. As $y \in (c,d)$ and (c,d) is the image of f, then we have $x \in (a,b)$ such that $f(x) = y$, and we can also write $x_n \in (a,b)$ with $f(x_n) = y_n$. Then

$$\frac{g(y_n) - g(y)}{y_n - y} = \frac{g(f(x_n)) - g(f(x))}{y_n - y} = \frac{x_n - x}{y_n - y} = \frac{x_n - x}{f(x_n) - f(x)}.$$

Note that as $f'(x) > 0$ for all $x \in (a,b)$ then f is a monotonically increasing function and so with our assumptions we can conclude $x_n \to x$ as $n \to \infty$. So

$$\lim_{n\to\infty} \frac{x_n - x}{f(x_n) - f(x)} = \lim_{n\to\infty} \left[\frac{f(x_n) - f(x)}{x_n - x}\right]^{-1} = \left[\lim_{n\to\infty} \frac{f(x_n) - f(x)}{x_n - x}\right]^{-1} = [f'(x)]^{-1} = \frac{1}{f'(x)}.$$

Hence we've shown that g is differentiable and that $g'(f(x)) = \frac{1}{f'(x)}$ for all $x \in (a,b)$! $\square$

Exercise 2.2.

Show that if $f\colon [a,b] \to \mathbb{R}$ is continuous and $F(x) := \int_a^x f(t)dt = 0$ for all $x \in [a,b]$, then $f(x) = 0$ for all $x \in [a,b]$. Prove an example to show that the statement may fail if f is not continuous.

Proof. Assume that $f\colon [a,b] \to \mathbb{R}$ is continuous, and let $F(x) = \int_a^x f(t)dt = 0$ for all $x \in [a,b]$ (note that we have $f \in \mathcal{R}([a,b])$). By FTC we have that $F'(x) = f(x) = 0$ and we're done. Alternatively, we don't need the hypothesis that $F(x) = 0$, as if we had $F'(x) = f(x)$ only, then we get that $f(a) = 0$, and reading this $F'(x) = f(x)$ means that the function is the same once we take its derivative and so $F(x) = ke^x$ but $f(a) = ke^a = 0$, so $k = 0$, and hence $f(x) = 0$. □

Exercise 2.3.

Assume that f and g are differentiable functions on $[a,b]$, and assume $f', g' \in \mathcal{R}([a,b])$. Show that the integration by parts formula is valid:

$$\int_a^b fg'dx = f(b)g(b) - f(a)g(a) - \int_a^b f'gdx$$

Proof. As f and g are differentiable, then $(fg)' = f'g + fg'$. Note here that as $f, g\colon [a,b] \to \mathbb{R}$ are differentiable, and thus continuous, with compact domain, then f, g are both each bounded by Weierstrass and hence Riemann integrable by Theorem 2.8. Now we can integrate both sides as follows:

$$\int_a^b (fg)'dx = \int_a^b (f'g + fg')dx = \int_a^b f'gdx + \int_a^b fg'dx,$$

and we have $\int_a^b (fg)'dx = f(b)g(b) - f(a)g(a)$ by FTC. Again, note that as $f', g' \in \mathcal{R}([a,b])$ and $f, g \in \mathcal{R}([a,b])$, then we have $f'g \in \mathcal{R}([a,b])$ and $fg' \in \mathcal{R}([a,b])$ by Theorem 2.10. Now, we have

$$\begin{aligned} f(b)g(b) - f(a)g(a) &= \int_a^b f'gdx + \int_a^b fg'dx \\ \implies \int_a^b fg'dx &= f(b)g(b) - f(a)g(a) - \int_a^b f'gdx. \end{aligned}$$

□

Exercise 2.4.

Assume $g\colon [a,b] \to \mathbb{R}$ is differentiable, that g' is continuous, and M and m are upper and lower bounds, respectively, for the function g. Assume $f\colon [m,M] \to \mathbb{R}$ is continuous. Show that the change of variables formula is valid

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt$$

Proof. As we have appropriate assumptions made for g and f, by Theorem 2.9 gives that $f \circ g\colon [a,b] \to \mathbb{R}$ is Riemann integrable. As $g \circ f \in \mathcal{R}([a,b])$ and g and f are both continuous (as g is differentiable then g is continuous) then by FTC 1 and 2 we have we have

$$\int_a^b f(g(x))g'dx = \int_a^b F'(g(x))g'(x)dx = \int_a^b (F \circ g)'(x)dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(t)dt$$

where we have $F' = f$ by FTC. □

Exercise 2.5.

Assume $f \in \mathcal{R}([a, b])$, but that f has a jump discontinuity at $c \in (a, b)$, i.e. $f(c-) \neq f(c+)$. Show that $F(x) := \int_a^x f(t)dt$ is not differentiable at $x = c$.

Proof. We prove something slightly stronger. Let $f(x) \to \alpha$ as $x \to c^+$ (something similar can be said for when $x \to c^-$). Take $\epsilon > 0$. Then we have $\delta > 0$ such that $|f(s) - \alpha| < \epsilon$ with $0 < s - c < \delta$. Now rewritting what we've done gives us $f(s) - \alpha < f(s) < \alpha + \epsilon$. Pick $r \in \mathbb{R}$ such that $0 < r < \delta$. We integrate on $[c, c + r]$, which gives

$$r(\alpha - \epsilon) < \int_c^{c+r} f(s)ds < r(\alpha + \epsilon),$$

and we can simplify this to the following:

$$\alpha - \epsilon < \frac{F(c + r) - F(c)}{r} < \alpha + \epsilon.$$

Now $\lim_{r\to 0^+} \frac{F(c+h)-F(c)}{r} = \alpha = \lim_{x\to c^+} f(x)$. Hence if f has the jump discontinuity at $c \in (a, b)$, that is $f(c-) \neq f(c+)$, then F is not differentiable at $x = c$. □

Exercise 2.7.

Assume that g is bounded, $g \in \mathcal{R}([0, 1])$ and continuous at 0. Show that

$$\lim_{n\to\infty} \int_0^1 g(x^n)dx = g(0).$$

Proof. Let $\epsilon > 0$. Since we assume g is continuous at $g(0)$, then there exists $\delta > 0$ such that $|g(x) - g(0)| < \epsilon$, where $x \in [0, \delta]$. Now pick $\omega \in (0, 1)$ such that $\omega > 1 - \epsilon$. Moreover, we have $\lim_{n\to\infty} \omega^n = 0$ as $\omega \in (0, 1)$, so there exists some $N \in \mathbf{N}$ with $n \geq N$ we have $\omega^n \in [0, \delta]$. Moving on we make the clear observation that $x \in [0, \omega]$ implies $0 \leq x^n \leq \omega^n$, so that taking limits to infinity gives us that $x^n \in [0, \delta]$. So

$$\left|\int_0^1 g(x^n) - g(0)dx\right| = \left|\int_0^1 g(x^n)dx - \int_0^1 g(0)dx\right| \leq \int_0^1 |g(x^n) - g(0)|\, dx$$
$$= \int_\omega^1 |g(x^n) - g(0)|dx + \int_0^\omega |g(x^n) - g(0)|dx$$

Now for $n \geq N$, as we picked in the previous paragraph, we have $|g(x^n) - g(0)| < \epsilon$. Then, for $n \geq N$, $\int_\omega^1 |g(x^n) - g(0)|dx \leq \int_\omega^1 2Mdx = 2M(1 - \omega) \leq (2M)\epsilon$, and $\int_0^\omega |g(x^n) - g(0)| < \int_0^\omega \epsilon dx \leq \epsilon$. Therefore $\lim_{n\to\infty} \int_0^1 g(x^n)dx = g(0)$, and we're done. □

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