MATH 425A HOMEWORK 9 SOLUTIONS

Assignment: Exercises 2.6, 2.7, 2.8, 2.9 in Chapter 5

Due Date: November 1, 2022

Rubric: (22 points total)

- Exercise 2.6. Category I (6 points)
- Exercise 2.7. Category II (5 points)
- Exercise 2.8. Category II (3 points)
- Exercise 2.9. Category I (6 points)
- Neatness: 2 points
- Optional LaTeXbonus: 1 point extra credit.

Please report any corrections, etc. to lesliet@usc.edu

2.6. Complete the following tasks.

(a) Find a closed subset E of $\mathbb{R}$ and a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(E)$ is not closed.

(b) Find a bounded subset E of $\mathbb{R}$ and a continuous function $f : E \to \mathbb{R}$ such that $f(E)$ is not bounded.

(c) Show that if E is a bounded subset of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f(E)$ is bounded.

Soln.: (a) $f(x) = \frac{1}{1+x^2}$, $E = \mathbb{R}$. Then f is continuous (being a rational function with everywhere positive denominator), and $\mathbb{R}$ is closed in itself. But $f(E) = (0, 1]$, which is not closed in $\mathbb{R}$.

(b) Let $E = (0, 1)$ and $f : E \to \mathbb{R}$ be given by $f(x) = \frac{1}{x}$ for all $x \in E$. Then E is bounded in $\mathbb{R}$, but $f(E) = (1, \infty)$ is not bounded in $\mathbb{R}$.

(c) If E is bounded in $\mathbb{R}$, then $\overline{E}$ is closed and bounded in $\mathbb{R}$, hence compact. Therefore $f(\overline{E})$ is compact, therefore bounded, so the subset $f(E)$ of $\overline{f(E)}$ is also bounded.

2.7. Prove that the set $\mathbb{R}^2\backslash\{(0, 0)\}$ is connected. Then, use the function $x/|x|$ to show that $S = \{x \in \mathbb{R}^2 : |x| = 1\}$ is connected.

Soln.: consider the following four sets:

$$A = \{(x, y) \in \mathbb{R}^2 : x > 0\} \qquad B = \{(x, y) \in \mathbb{R}^2 : y > 0\}$$
$$C = \{(x, y) \in \mathbb{R}^2 : x < 0\} \qquad D = \{(x, y) \in \mathbb{R}^2 : y < 0\}.$$

Then all four of A, B, C, D are convex. (Whenever z_1 and z_2 are numbers with the same sign, then $(1-t)z_1+tz_2$ is another number with the same sign as z_1 and z_2. Applying this fact to each of A, B, C, D, we conclude that each is convex and hence connected.) Since A and B have a point in common (for example, $(1, 1)$), $A\cup B$ is connected (see HW 5). Similarly, $C\cup D$ is connected. Finally, $(A\cup B)\cup(C\cup D)$ is connected, because $A \cup B$ and $C \cup D$ are connected sets which have a point in common (for example, $(1, -1)$). But $A \cup B \cup C \cup D = \mathbb{R}^2\backslash\{(0, 0)\}$, so we are done.

(Alternatively, one can show that $\mathbb{R}^2\backslash\{(0, 0)\}$ is path connected by explicitly constructing a path from any point to some base point (say $(1, 0)$, for example). This requires breaking things into cases, though, since one has to make sure none of the paths pass through the origin.)

Now, S is the image of the connected set $\mathbb{R}^2\backslash\{(0, 0)\}$ under the function $f : \mathbb{R}^2\backslash\{(0, 0)\} \to \mathbb{R}^2$, $f(x) = x/|x|$. This function is continuous on $\mathbb{R}^2\backslash\{(0, 0)\}$; indeed, we can write $f(x) = (\frac{x_1}{|x|}, \frac{x_2}{|x|})$; the function $x \mapsto x_1$ and $x \mapsto |x|$ are both continuous, and $|x|$ is never zero on $\mathbb{R}^2\backslash\{(0, 0)\}$; therefore the first component is a continuous function. Reasoning similarly, we conclude that the second component is also continuous. Putting this together, we conclude that f is continuous, as claimed, and that therefore the image S is connected.

(Note: Technically one should also prove that S is in fact the image of f. This is pretty much obvious, but let's do it anyway. If $x \in S$, then $|x| = 1$, so $f(x) = \frac{x/|x|}{|x/|x||} = x$. Thus $x \in \operatorname{Im} f$. On the other hand, if $y = \operatorname{Im} f$, then there exists $x \in \mathbb{R}^2 \backslash \{(0,0)\}$ such that $y = x/|x|$, so $|y| = |x/|x|| = 1$, so $y \in S$. Thus $S = \operatorname{Im} f$.)

2.8. Assume $f : X \to Y$ and $g : Y \to Z$ are uniformly continuous functions, where (X, d_X), (Y, d_Y), and (Z, d_Z) are metric spaces. Prove that $g \circ f$ is uniformly continuous.

Soln.: Choose $\varepsilon > 0$; let $\eta > 0$ be such that $d_Y(c, d) < \eta$ implies $d_Z(g(c), g(d)) < \varepsilon$. Then choose $\delta > 0$ so that $d_X(a, b) < \delta$ implies $d_Y(f(a), f(b)) < \delta$. Then $d_X(a, b) < \delta$ implies $d_Y(f(a), f(b)) < \eta$, which in turn implies that $d_Z(g(f(a)), g(f(b)) < \varepsilon$. Thus $g \circ f$ is uniformly continuous, as claimed.

2.9. Let E be a bounded subset of $\mathbb{R}^k$ and let $f : E \to \mathbb{R}$ be a uniformly continuous function. Show that f is bounded. (Hint: You will need to use compactness of $\overline{E}$ at some point.)

Soln.: Choose $\delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$ for any $x, y \in X$. Then $\mathcal{B} = \{B_\delta(x)\}_{x \in E}$ is an open cover for $\overline{E}$. Indeed, if $x \in E$, then clearly $x \in B_\delta(x)$; if $x \in \overline{E}$ but $x \notin E$, then $x \in E'$, in which case there exists $y \in E$ such that $|x - y| < \delta$, so $x \in B_\delta(y)$. Thus $\mathcal{B}$ covers $\overline{E}$. Since E is bounded, it follows that $\overline{E}$ is compact, and we can therefore choose a finite subcover; let $\{x_i\}_{i=1}^N$ be such that $\mathcal{B}' = \{B_\delta(x_i)\}_{i=1}^N$ is a finite subcover of $\overline{E}$ associated to $\mathcal{B}$. Put $M = \max\{|f(x_i)|\}_{i=1}^N$. We claim that $|f(x)| \leq M + 1$ for all $x \in E$, and therefore that f is bounded. Indeed, if $x \in E$, then $|x - x_i| < \delta$ for some $i \in \{1, \ldots, N\}$. Thus $|f(x)| = |f(x) - f(x_i) + f(x_i)| \leq |f(x) - f(x_i)| + |f(x_i)| \leq 1 + M$, i.e. $|f(x)| \leq 1 + M$ for all $x \in X$, so f is bounded.