MATH 425A HOMEWORK 9 SOLUTIONS

Assignment: Exercises 6.1, 6.2, 6.3, 6.4, 6.5 in Chapter 4; Exercises 1.1, 2.1, 2.2, 2.3, 2.4 in Chapter 5 (Solution for Exercise 2.5 also given below).

Due Date: October 28, 2022

Rubric: (40 points total)

- Exercise 6.1. Category II (2 points)
- Exercise 6.2. Category II (3 points)
- Exercise 6.3. Category I (5 points)
- Exercise 6.4. Category I (5 points)
- Exercise 6.5. Category II (4 points)
- Exercise 1.1. Category II (3 points)
- Exercise 2.1. Category II (3 points)
- Exercise 2.2. Category II (2 points)
- Exercise 2.3. Category II (5 points)
- Exercise 2.4. Category I (5 points)
- Neatness: 3 points
- Optional LaTeXbonus: 1 point extra credit.

Please report any corrections, etc. to lesliet@usc.edu

CHAPTER 4

6.1. Let $\mathcal{A}$ be a collection of convex subsets of a real vector space V. Show that $B := \bigcap_{A\in\mathcal{A}} A$ is convex.

Soln.: Choose $x, y \in B$ and $t \in [0,1]$, and put $z = (1-t)x + ty$. Since each $A \in \mathcal{A}$ is convex and contains both x and y, it follows that $z \in A$ for each $A \in \mathcal{A}$; therefore $z \in B$. Thus B is convex as well.

6.2. Let (X, d) be a metric space and let A and B be disjoint subsets of X. Prove that if A and B are both open in X, then A and B are separated.

Soln.: Since A is open and does not intersect B, it follows that $X\backslash A$ is closed and contains B; therefore $X\backslash A$ contains $\overline{B}$ as well, meaning that $A \cap \overline{B}$ is empty. Reversing the roles of A and B in this argument gives us that $\overline{A} \cap B$ is also empty, and thus that A and B are separated.

6.3. Let E be a connected subset of a metric space (X, d). Show that $\overline{E}$ is connected.

Let A and B be two separated sets whose union is $\overline{E}$. We show that one of A or B must be empty, which will prove that $\overline{E}$ is connected. Put $C = A \cap E$ and $D = B \cap E$. We claim that (i) C and D are separated and (ii) $C \cup D = E$. Once we have proven (i) and (ii), the fact that E is connected will guarantee that either C or D must be empty; without loss of generality, suppose that it is $C = A \cap E$. Then $E = C \cup D = \emptyset \cup D = D = B \cap E \subset B$, so $\overline{E} \subset \overline{B}$. Now $A \subset A \cup B = \overline{E}$, so $A = A \cap \overline{E} \subset A \cap \overline{B} = \emptyset$, since A and B are separated.

It remains to prove (i) and (ii). To prove (i), note that $\overline{C} \cap D \subset \overline{A} \cap B = \emptyset$, and similarly $C \cap \overline{D} = \emptyset$. To prove (ii), note that $C \cup D = (A \cap E) \cup (B \cap E) = (A \cup B) \cap E = \overline{E} \cap E = E$.

6.4. Let (X, d) be a metric space, and let $\mathcal{C}$ be a collection of connected subsets of X. Assume $A = \bigcap_{C\in\mathcal{C}} C$ is nonempty. Show that $B = \bigcup_{C\in\mathcal{C}} C$ is connected.

Soln.: Assume D and E are separated subsets whose union is B. We show that one of D or E must be empty. Now, B is not empty, as it contains the nonempty set A. Since $A \subset B = D \cup E$, one of D or E must contain a point of A; assume D contains the point x of A.

Each $C \in \mathcal{C}$ is connected and $C \subset B = D \cup E$, so (by Theorem 6.8) C is contained entirely in D or entirely in E. But $x \in C \cap D$, so it must be the case that $C \subset D$. Since $C \in \mathcal{C}$ was arbitrary, we conclude that $B = \bigcup_{C \in \mathcal{C}} C \subset D$ and $E = \emptyset$. This proves the claim.

6.5. Let $X = \mathbb{R}^2$. Give an example of a connected subset E of X, such that $\mathrm{Int}_X(E)$ is *not* connected. Prove both that your set E is connected and that its interior is not. (Hint: Consider the union of two convex sets joined at a point. You may assume without proof the fact that convexity implies connectedness in $\mathbb{R}^2$.)

Soln.: Define $C = B((-1,0),1)$ and $D = B((1,0),1)$, and let $E = \overline{C} \cup \overline{D}$. Then C and D are both convex, hence connected. Thus $\overline{C}$ and $\overline{D}$ are both connected (by Exercise 6.3), and $(0,0) \in \overline{C} \cap \overline{D}$. Therefore E is connected, by Exercise 6.4. On the other hand, $\mathrm{Int}_X(E) = C \cup D$ is not connected. Indeed, C and D are both open in $\mathbb{R}^2$ and they are disjoint, hence separated, by Exercise 6.2. Thus $\mathrm{Int}_X(E) = C \cup D$ is a representation of $\mathrm{Int}_X(E)$ as a union of nonempty separated sets, so $\mathrm{Int}_X(E)$ is not connected.

CHAPTER 5

1.1. Let (X, d_X) and (Y, d_Y) be metric spaces, and let E be a subset of X. Let $f : E \to Y$ be a function, and let p be a limit point of E in X. Prove that $f(x) \to q$ as $x \to p$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in E$ and $0 < d_X(x,p) < \delta$ imply together that $d_Y(f(x), q) < \varepsilon$.

Soln.: Assume $f(x) \to q$ as $x \to p$ and choose $\varepsilon > 0$. Put $V = B_Y(q, \varepsilon)$ and choose a neighborhood U of p in X such that $f(E \cap U \backslash \{p\}) \subset U$. Choose $\delta > 0$ such that $B_X(p, \delta) \subset U$. Then $x \in E$ and $0 < d_X(x,p) < \delta$ together imply $x \in E \cap B_X(p,\delta) \backslash \{p\} \subset E \cap U \backslash \{p\}$, which implies $f(x) \in V$, i.e., $d_Y(f(x), q) < \varepsilon$.

On the other hand, assume that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in E$ and $0 < d_X(x,p) < \delta$ together imply $d_Y(f(x), q) < \varepsilon$. Let V be a neighborhood of q in Y, and choose $\varepsilon > 0$ so that $B_Y(q, \varepsilon) \subset V$. Then choose $\delta > 0$ so that $0 < d_X(x,p) < \delta$ and $x \in E$ together imply that $f(x) \in V$. Put $U = B_X(p, \delta)$. Then $f(U \backslash \{p\}) \subset V$ by construction of U. Therefore $f(x) \to q$ as $x \to p$.

2.1. Let (X, d_X) and (Y, d_Y) be metric spaces; let $f : X \to Y$ be a function. Prove that f is continuous at $p \in X$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in B_X(p, \delta)$ implies $f(x) \in B_Y(f(p), \varepsilon)$.

Soln.: Assume f is continuous at $p \in X$, and choose $\varepsilon > 0$. Put $V = B_Y(f(p), \varepsilon)$ and let U be a neighborhood of p in X such that $f(U) \subset V$. Choose $\delta > 0$ so that $B_X(p, \delta) \subset U$. Then $x \in B_X(p, \delta)$ implies $f(x) \in f(U) \subset V = B_Y(f(p), \varepsilon)$.

Conversely, assume that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in B_X(p, \varepsilon)$ implies $f(x) \in B_Y(f(p), \varepsilon)$. Let V be a neighborhood of $f(p)$ in Y and choose $\varepsilon > 0$ such that $B_Y(f(p), \varepsilon) \subset V$. Then choose $\delta > 0$ so that $x \in B_X(p, \delta)$ implies $f(x) \in B_Y(f(p), \varepsilon)$. Put $U = B_X(p, \delta)$. Then U is a neighborhood of p in X, and $f(U) \subset B_Y(f(p), \varepsilon) \subset V$. So f is continuous at p.

2.2. Assume $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$, for all $x \in \mathbb{R}$. Does it follow that f must be continuous? If so, give a proof; if not, give a counterexample.

Soln.: No. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 1$ if $x \neq 0$, $f(0) = 0$. If $x \neq 0$, then $f(x+h) - f(x-h) = 1 - 1 = 0$ whenever $0 < |h| < |x|$. On the other hand, $f(0+h) - f(0-h) = 1 - 1 = 0$ for all $h \neq 0$. Therefore $\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$ in all cases, but f is not continuous at 0, since $\lim_{x \to 0} f(x) = 1 \neq 0 = f(0)$.

2.3. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ a function.

(a) Show that f is continuous if and only if $f^{-1}(C)$ is closed in X whenever C is closed in Y.
(b) Show that $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for every subset A of X.
(c) Consider the (continuous) function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \frac{1}{1+x^2}$. Give an example of a subset A of $\mathbb{R}$ such that $g(\overline{A}) \neq \overline{g(A)}$.

Soln.: (a) Assume f is continuous, and let C be a closed subset of Y. Then $f^{-1}(C) = f^{-1}(Y \backslash (Y \backslash C)) = X \backslash f^{-1}(Y \backslash C)$. Since $Y \backslash C$ is open in Y, it follows that $f^{-1}(Y \backslash C)$ is open in X, so the complement $f^{-1}(C) = X \backslash f^{-1}(Y \backslash C)$ is closed in X.

On the other hand, assume $f^{-1}(C)$ is closed in X whenever C is closed in Y. Let V be an open subset of Y. Then $f^{-1}(V) = f^{-1}(Y \backslash (Y \backslash V)) = X \backslash f^{-1}(Y \backslash V)$. Since $Y \backslash V$ is closed in Y, it follows that $f^{-1}(Y \backslash V)$ is closed in X, so the complement $f^{-1}(V) = X \backslash f^{-1}(Y \backslash V)$ is open in X, proving continuity of f.

(b) Assume f is continuous and let A be a subset of X. Then $f^{-1}(\overline{f(A)})$ is a closed subset of X (by part (a)) that contains A and therefore $\overline{A}$. Thus $f(\overline{A}) \subset \overline{f(A)}$.

On the other hand, assume $f(\overline{A}) \subset \overline{f(A)}$ for every subset A of X. Let C be any closed subset of Y. Then $f(\overline{f^{-1}(C)}) \subset \overline{f(f^{-1}(C)} \subset \overline{C} = C$. Thus $\overline{f^{-1}(C)} \subset f^{-1}(C)$, which implies that $f^{-1}(C)$ is closed in X.

(c) Take $A = \mathbb{R}$. Then $g(\overline{A}) = (0, 1] \neq [0, 1] = \overline{g(A)}$. (The point is that A cannot be bounded. If it is, then $\overline{A}$ is a closed, bounded subset of $\mathbb{R}$, therefore compact, whence $g(\overline{A})$ is compact, therefore closed, therefore equal to $\overline{g(\overline{A})}$, which contains $\overline{g(A)}$. Putting this together, we have $g(\overline{A}) = \overline{g(A)}$ whenever A is bounded.)

2.4. Let (X, d_X) and (Y, d_Y) be metric spaces, and let f and g be continuous functions from X to Y. Assume E is a dense subset of X.

(a) Prove that $f(E)$ is dense in $f(X)$. (Hint: Use Exercise 1.12 in Chapter 4 and Exercise 2.3 above.)
(b) Prove that if $f(x) = g(x)$ for all $x \in E$, then $f(x) = g(x)$ for all $x \in X$.

This Exercise shows, for example, that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and if we know what $f(x)$ is for all $x \in \mathbb{Q}$, then we can determine what $f(x)$ is for any $x \in \mathbb{R}$.

Soln.: (a) By continuity of f and Exercise 2.3, we have $\overline{f(E)} \supset f(\overline{E})$. Since E is dense in X, we have $\overline{E} = X$ and therefore $f(\overline{E}) = f(X)$. Putting this together gives $\overline{f(E)} \supset f(X)$, so that (by Exercise 1.12 in Chapter 4), we have that $f(E)$ is dense in $f(X)$.

(b) Choose $x \in X$. If $x \in E$, then $f(x) = g(x)$ by assumption. Otherwise $x \in E'$, since E is dense in X. Therefore we can find a sequence $(x_k)_{k=1}^{\infty}$ in E which converges in X to x. By continuity of f and g, the limits $\lim_{k\to\infty} f(x_k)$ and $\lim_{k\to\infty} g(x_k)$ must both exist and be equal to $f(x)$ and $g(x)$, respectively. On the other hand, since $f(x_k) = g(x_k)$ for each $k \in \mathbb{N}$, the two limits must be equal. Therefore $f(x) = g(x)$, as claimed.

2.5. Consider the functions f and g defined in (10) and (11), respectively. (For reference, $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 1$ for $x \geq 0$ and $f(x) = 0$ otherwise; $g = f\big|_{\mathbb{R}\backslash\{0\}}$.)

(a) Prove that g is continuous, using the ε-δ formulation of continuity. (Break it into cases.)
(b) Prove that g is continuous, using the open set formulation of continuity. (Using the formulation of Proposition 2.6 is okay.)
(c) Prove that f is not continuous at 0, using the ε-δ formulation of continuity at a point.
(d) Prove that f is not continuous, using the open set formulation of continuity.
(e) Prove (using whichever method you prefer) that there does not exist *any* continuous extension of g to all of $\mathbb{R}$.

Soln.: (a) Choose $x \in \mathbb{R}\backslash\{0\}$ and choose $\varepsilon > 0$. Put $\delta = |x|$. If $x > 0$, then $|x - y| < \delta$ implies that $y > 0$, whence $|f(x) - f(y)| = |1 - 1| = 0 < \varepsilon$. If $x < 0$, then $|x - y| < \delta$ implies that $y < 0$, whence $|f(x) - f(y)| = |(-1) - (-1)| = 0 < \varepsilon$. Therefore g is continuous at every point in its domain, therefore continuous.

(b) Let V be an open subset of $\mathbb{R}$. We consider four cases: (i) V does not contain either 0 or 1. (ii) V contains 0 but not 1. (iii) V contains 1 but not 0. (iv) V contains both 0 and 1. The inverse images of V in each case are $\emptyset$, $(-\infty, 0)$, $(0, \infty)$, and $\mathbb{R}\backslash\{0\}$, all of which are open in $\mathbb{R}\backslash\{0\}$. Therefore g is continuous.

(c) Choose $\varepsilon = \frac{1}{2}$. Given $\delta > 0$, put $x = -\frac{\delta}{2}$. Then $|x - 0| < \delta$, but $|f(x) - f(0)| = |0 - 1| = 1 > \frac{1}{2} = \varepsilon$. Therefore f is not continuous at zero.

(d) $f^{-1}(B(1, \frac{1}{2})) = [0, \infty)$, which is not open in $\mathbb{R}$. Therefore f is not continuous.

(e) Let $k : \mathbb{R} \to \mathbb{R}$ be any extension of g to all of $\mathbb{R}$. We show that k cannot be continuous at 0. Indeed, put $\varepsilon = \frac{1}{2}$. Given $\delta > 0$, put $x_1 = -\frac{\delta}{2}$ and $x_2 = \frac{\delta}{2}$. Then $|x_1 - 0| = |x_2 - 0| < \delta$, but we claim that at least one of $|k(0) - k(x_1)|$, $|k(0) - k(x_2)|$ must be no smaller than $\varepsilon = \frac{1}{2}$. Indeed, if $|k(0) - k(x_1)| = |k(0)| < \frac{1}{2}$, then $|k(0) - k(x_2)| \geq |k(x_2)| - |k(0)| > 1 - \frac{1}{2} = \frac{1}{2}$. On the other hand, if $|k(0) - k(x_2)| = |k(0) - 1| < \frac{1}{2}$, then $|k(0) - k(x_1)| = |k(0)| \geq 1 - |1 - k(0)| > 1 - \frac{1}{2} = \frac{1}{2}$. This proves the claim.