MATH 425A HW4, DUE 09/23/2022, 6PM

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Chapter 2. §5.

Exercise 0.1 (5.2.)**.** Let $a_1, a_2, \dots$ be any enumeration of the negative rational numbers; let $b_1, b_2, \dots$ be any enumeration of the positive rational numbers. Show that the following two equalities hold:

$$\bigcap_{j=1}^{\infty}(a_j, b_j) = \{0\}, \bigcup_{j=1}^{\infty}(a_j, b_j) = \mathbf{R}$$

Proof. WLOG, for the first equality, it will suffice to show that $\bigcap_{j=1}^{\infty}(a_j, b_j) \subseteq \{0\}$. Take $\ell \in T = \bigcap_{j=1}^{\infty}(a_j, b_j)$. Then $a_j < \ell < b_j$ for every $j \geq 1$. Now we can find some $\epsilon > 0$ with $b_j < \epsilon$. As $\ell < b_j$ then we have that $\ell < \epsilon$, and as $\epsilon > 0$, then $\ell \leq 0$. Similarly, we can find some $\epsilon < 0$ (i.e. $0 > -\epsilon$) with $a_j > -\epsilon$. So then as $\ell > a_j$, $0 > a_j > -\epsilon$, and $\epsilon > 0$, then we must have that $\ell \geq 0$. Thus, all together, we have that $0 \leq \ell \leq 0$, and hence we have that $\ell = 0$.

For the second equality, it suffices to show WLOG that $\mathbf{R} \subseteq \bigcup_{j=1}^{\infty}(a_j, b_j)$. Suppose have that $\ell \in \mathbf{R}$ and $\ell > 0$—if $\ell = 0$ then the result is clear. Then $\ell + 1 > \ell$ and so there is some $s \in (\ell, \ell+1)$; we will write $s = b_j$ for $j \geq 1$. As $\ell > 0$, then there is some $t \in (0, \ell)$; we will write $t = b_j$ for some $j \geq 1$. So then $0 < a_j < \ell < b_j < \ell+1$, and hence $\ell \in (a_j, b_j)$. Thus $\ell \in \bigcup_{j=1}^{\infty}(a_j, b_j)$. Now, for the other case, suppose $\ell < 0$. Then $\ell - 1 < \ell$, and so we can find some $q \in (\ell - 1, \ell)$; we write $q = a_j$ for some $j \geq 1$. So as $x < 0$ then we can find some $p \in (x, 0)$; we will write $p = b_j$ for some $j \geq 1$. All together, we have that $\ell - 1 < a_j < \ell < b_j < 0$. Thus $\ell \in (a_j, b_j)$ and hence $\ell \in \bigcup_{j=1}^{\infty}(a_j, b_j)$. Therefore, as ℓ was chosen to be simply some random real number, then $\bigcup_{j=1}^{\infty}(a_j, b_j) = \mathbf{R}$. □

Chapter 2. § 6.

Exercise 0.2 (6.1.)**.** Prove that the addition and multiplication operations in $(\mathbf{C}, +, \cdot)$ satisfy the field axioms of Definition 2.1.

Proof. We essentially need to show that five axioms hold true from Definition 2.1. From now on, let $x, y, z \in \mathbf{R} \times \mathbf{R}\,(= \mathbf{C})$, which is the underlying set of $\mathbf{C}$, where $x = (a, b), y = (c, d), z = (s, t)$ where $a, b, c, d, s, t \in \mathbf{R}$.

(1) The set $\mathbf{C} := (\mathbf{C}, +, \cdot)$, as the operations are defined in Chapter 2, §6., is closed since $x+y = (a, b)+(c, d) = (a+c, b+d) \in \mathbf{R}\times\mathbf{R}$ and $xy = (a, b)\cdot(c, d) = (ac-bd, ad+bc) \in \mathbf{R}\times\mathbf{R}$ since $a + c, b + d, ac - bd, ad + bc \in \mathbf{R}$ as $\mathbf{R}$ is a field, and so $x + y \in \mathbf{C}$ and $xy \in \mathbf{C}$.

(2) For commutativity: $x + y = (a, b) + (c, d) = (a + c, b + d) = (c + a, d + b) = (c, d)+(a, b) = y+x$ since $\mathbf{R}$ is a field, and, similarly, $xy = (a, b)\cdot(c, d) = (ac-bd, ad+bc) = (ca - db, cb + da) = (c, d) \cdot (a, b) = yx$ as $\mathbf{R}$ is a field. Now for associativity:

$$\begin{aligned} x + (y + z) &= (a, b) + ((c, d) + (s, t)) = (a, b) + (c + s, d + t) \\ &= (a + (c + s), b + (d + t)) = ((a + c) + s, (b + d) + t)) \qquad (\mathbf{R} \text{ is a field}) \\ &= (a + c, b + d) + (s, t) = (x + y) + z \end{aligned}$$

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$$\begin{aligned}
x(yz) &= (a,b)\cdot((c,d)\cdot(s,t)) = (a,b)\cdot(cs-dt, ct+ds)\\
&= (a(cs-dt)-b(ct+ds), a(ct+ds)+b(cs-dt)) && (\mathbf{R}\text{ is a field})\\
&= (acs-adt-bct-bds, act+ads+bcs-bdt) && (\mathbf{R}\text{ is a field})\\
&= ((ac-bd)s-(ad+bc)t, (ad+bc)s+(ac-bd)t) && (\mathbf{R}\text{ is a field})\\
&= (ac-bd, ad+bc)\cdot(s,t) = ((a,b)\cdot(c,d))\cdot(s,t)\\
&= (xy)z
\end{aligned}$$

Therefore we have associativity and commutativity with the defined operations on $\mathbf{C}$.

(3) The additive identity of $\mathbf{C}$ is defined to be $0 = (0,0) \in \mathbf{R}\times\mathbf{R}$, and so $x + 0 = (a,b)+(0,0) = (a+0,b+0) = (a,b) = (0+a,0+b) = (0,0)+(a,b) = 0+x$. Similarly, the multiplicative identity is defined to be $1 = (1,0)$, and so $x\cdot 1 = (a,b)\cdot(1,0) = (a(1)-b(0), a(0)+b(1)) = (a,b) = x = 1\cdot x = (1,0)\cdot(a,b) = (1(a)-0(b), 1(b)+0(a)) = (a,b) = x$.

(4) The multiplicative inverse of $x = (a,b)$, where $x \neq 0$, can be found to be $x^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b(\frac{a}{a^2+b^2})}{a}\right)$, and we can tediously calculate to get that

$$x\cdot x^{-1} = (a,b)\cdot\left(\frac{a}{a^2+b^2}, \frac{-b(\frac{a}{a^2+b^2})}{a}\right) = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1,0) = 1. \tag{1}$$

The additive inverse is much easier: for $y = (c,d)$, the additive inverse is $-y = (-c,-d)$, and so $y + (-y) = (c+(-c), d+(-d)) = (0,0) = 0$.

(5) Lastly, we need to check distributivity: Let $t := y + z = (c+s, d+t)$. Now

$$\begin{aligned}
x\cdot t = (a,b)\cdot(c+s,d+t) &= (a(c+s)-b(d+t), a(d+t)+b(c+s))\\
&= (ac+as-bd-bt, ad+at+bc+bs)\\
&= ((ac-bd)+(as-bt), (ad+bc)+(at+bs))\\
&= (a,b)\cdot(c,d)+(a,b)\cdot(s,t)
\end{aligned}$$

Therefore the distributive law holds.

Hence $\mathbf{C}$ is indeed a field. □

Exercise 0.3 (6.2.)**.** Prove that there exists no order $\leq$ that makes $(\mathbf{C},+,\cdot,\leq)$ into an ordered field. (Hint: If there were such an ordering, then $i = \sqrt{-1}$ would necessarily be either positive or negative.)

Proof. Suppose that there does exists an ordering that makes $\mathbf{C}$ into an ordered field. Then, by definition, we have that either $i \leq 0$ or $i \leq 0$, but we do not have that $i = 0$, so we simply have that either i is negative or positive. Suppose, for the first case, that $i < 0$. Then $0 < -i$ so $0^2 < (-i)^2 = 1(-1) = -1$ and once again, $0^2 < (-1)^2 = 1$; hence a contradiction. Thus we cannot have that i is negative. Now, for the second/last case, then assume that $i > 0$. Then $i^2 = -1 > 0^2 = 0$ and so $(-1)+1 = 0 > 0+1 = 1$, and multiplying by 1, $i\cdot 0 = 0 > 1\cdot i = i$; thus a contradiction. Hence we cannot have that i is not positive either. Therefore we cannot have that there exists an order on $\mathbf{C}$ that makes it into an ordered field. □

1. Chapter 3. § 1

Exercise 1.1 (1.1.)**.** Let $\|\cdot\|$ be a norm on a real vector space V. Prove the *reverse triangle inequality:*

$$|\|x\| - \|y\|| \leq \|x-y\|$$

Proof. Firstly, as we have that $\|\cdot\|$ is a norm, then: $\|x-y\| = \|-(y-x)\| = |-1|\|y-x\| = \|y-x\|$. Now $\|x\| = \|(x-y)+y\| \leq \|x-y\|+\|y\|$, and so $\|x\|-\|y\| \leq \|x-y\|$. Similarly, $\|y\| = \|(y-x)+x\| \leq \|y-x\|+\|x\|$, so $\|y\|-\|x\| \leq \|y-x\|$, which can be rewritten as

$-\|y-x\| \leq \|x\| - \|y\|$. Thus we can write $-\|y-x\| \leq \|x\| - \|y\| \leq \|x-y\|$. And this can finally be rewritten as $-\|x-y\| \leq \|x\| - \|y\| \leq \|x-y\|$. Hence $|\|x\| - \|y\|| \leq \|x-y\|$. □

Exercise 1.2 (1.2.)**.** Prove that any complex inner product is conjugate linear in its second argument; that is,

$$\langle x, \lambda y + z\rangle = \overline{\lambda}\langle x, y\rangle + \langle x, z\rangle,$$

for any scalar λ. (Note that this implies that any real inner product is linear in its second argument.)

Proof. We are considering a complex inner product and so we have a mapping $\langle\cdot,\cdot\rangle\colon V \times V \to \mathbf{C}$ with some properties. Let $x, y, z \in V$ and $\lambda \in \mathbf{C}$. Then $\langle x, \lambda y+z\rangle = \overline{\langle \lambda y + z, x\rangle} = \overline{\lambda\langle y, x\rangle} + \overline{\langle z, x\rangle} = (\overline{\lambda})\overline{\langle y, x\rangle} + \overline{\langle z, x\rangle} = \overline{\lambda}\langle x, y\rangle + \langle x, z\rangle$. □

Exercise 1.3 (1.3.-Polarization identity)**.** If $(V, \langle\cdot,\cdot\rangle)$ is a real inner product space, then

$$\langle v, w\rangle = \frac{1}{4}\left[\|v+w\|^2 - \|v-w\|^2\right], \text{ for all } v, w \in V.$$

If $(V, \langle\cdot,\cdot\rangle)$ is a complex inner product space, then

$$\langle v, w\rangle = \frac{1}{4}\left[(\|v+w\|^2 - \|v-w\|^2) + i(\|v+iw\|^2 - \|v-iw\|^2)\right]$$

Proof. Suppose that $(V, \langle\cdot,\cdot\rangle)$ is a real inner product space. Then $\|v+w\|^2 = \langle v + w, v+w\rangle = \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle$, and, similarly, $\|v-w\|^2 = \langle v-w, v-w\rangle = \langle v, v\rangle - \langle v, w\rangle - \langle w, v\rangle + \langle w, w\rangle = \langle v, v\rangle - 2\langle v, w\rangle + \langle w, w\rangle$. Thus:

$$\begin{aligned}
\frac{1}{4}\left[\|v+w\|^2 - \|v-w\|^2\right] &= \frac{1}{4}\left[\langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle - (\langle v, v\rangle - 2\langle v, w\rangle + \langle w, w\rangle)\right] \\
&= \frac{1}{4}\left[2\langle v, w\rangle + 2\langle v, w\rangle\right] \\
&= \frac{1}{4}\left[4\langle v, w\rangle\right] = \langle v, w\rangle.
\end{aligned}$$

Suppose that $(V, \langle\cdot,\cdot\rangle)$ is a complex inner product. Similar to the first computations we did for the real case, we can find that $\|v+w\|^2 = \langle v, v\rangle = \langle v, w\rangle + \overline{\langle v, w\rangle} + \langle w, w\rangle$, and $\|v-w\|^2 = \langle v, v\rangle - \langle v, w\rangle - \overline{\langle v, w\rangle} + \langle w, w\rangle$. Moreover, $\|v+iw\| = \langle w, w\rangle + i\langle w, v\rangle - i\langle v, w\rangle + \langle v, v\rangle$, and $\|v-iw\| = \langle w, w\rangle - i\langle w, v\rangle + i\langle v, w\rangle + \langle v, v\rangle$. Now:

$$\begin{aligned}
\|v+w\|^2 - \|v-w\|^2 &= 2\langle v, w\rangle + 2\langle w, v\rangle, \text{ and} \\
\|v+iw\|^2 - \|v-iw\|^2 &= 2i\langle w, v\rangle - 2i\langle v, w\rangle = 2i\left[\langle w, v\rangle - \langle v, w\rangle\right]
\end{aligned}$$

Thus:

$$\begin{aligned}
\frac{1}{4}\left[(2\langle v, w\rangle + 2\langle w, v\rangle) + i\left(2i(\langle w, v\rangle - \langle v, w\rangle)\right)\right] &= \frac{1}{4}\left[2\langle v, w\rangle + 2\langle w, v\rangle + (-2\langle w, v\rangle + 2\langle v, w\rangle)\right] \\
&= \frac{1}{4}\left[4\langle v, w\rangle + 2\langle w, v\rangle - 2\langle w, v\rangle\right] \\
&= \frac{1}{4}\left[4\langle v, w\rangle\right] = \langle v, w\rangle.
\end{aligned}$$

□

2. Chapter 3 §2

Exercise 2.1 (2.2)**.** For each of (a), (b), and (c), determine whether the given function d_j is a metric on $\mathbf{R}$, and prove that your answer is correct.

(a) $d_1(x, y) = \sqrt{|x-y|}$
(b) $d_2(x, y) = |x - 2y|$.
(c) $d_3(x, y) = \frac{|x-y|}{1+|x-y|}$

Proof. (a) Indeed a metric. We need to show that $d_1 \colon X \times X \to \mathbf{R}$ satisfies nonnegativity, symmetry, and the triangle inequality:

Firstly, let's fix some arbitrary $(x,y) \in X \times X$. Then $d_1(x,y) = \sqrt{|x-y|}$, which is the root of some positive number, or 0, in $\mathbf{R}$, and so $d_1(x,y) \geq 0$; if $x = y$, then $\sqrt{|x-y|} = \sqrt{|x-x|} = 0$, and if we first assumed $d_1(x,y) = 0$, then $d_1(x,y) = \sqrt{|x-y|} = 0$ and so $|x-y| = 0$ and in either case of $x - y \geq 0$ or $x - y < 0$, we get that $x = y$.

For symmetry, suppose we have $d_1(x,y)$ and $(x,y) \in X \times X$. Now, consider $d_1(x,y) - d_1(y,x)$, and so $\sqrt{|x-y|} - \sqrt{|y-x|}$—if $x - y > 0$ then $0 > y - x$, which implies $\sqrt{x-y} - \sqrt{-(y-x)} = \sqrt{x-y} - \sqrt{x-y} = 0$ and so $d_1(x,y) = d_1(y,x)$ if $x - y > 0$; if $x - y < 0$ then $0 < y - x$, so $\sqrt{-(x-y)} - \sqrt{y-x} = \sqrt{y-x} - \sqrt{y-x} = 0$ and so $d_1(x,y) = d_1(y,x)$ if $x - y < 0$; if $x - y = 0$ then $x = y$ and $d_1(x,y) = \sqrt{|x-y|} = \sqrt{|y-x|} = d_2(y,x)$. Thus d_1 is symmetric. [Could use instead the fact that $|\cdot|$ is a metric, and so $(d_1(x,y))^2 = |x-y| = |y-x| = (d_1(y,x))^2$, and so $d_1(x,y) = d_1(y,x)$.]

Lastly, we need to show that the triangle inequality holds. This is shown easiest if we show that, for any $s, t \in \mathbf{R}$ such that $s, t \geq 0$, we have that $\sqrt{s} + \sqrt{t} \geq \sqrt{s+t}$. This is true as we clearly have that $2\sqrt{st} \geq 0$ and so this leads to $s + \sqrt{2st} + t \geq s + t$ which is the same as $(\sqrt{s} + \sqrt{t})^2 \geq s + t$, and thus $\sqrt{s} + \sqrt{t} \geq \sqrt{s+t}$. Now, $d_1(x,y) = \sqrt{|x-y|} = \sqrt{|(x-z)+(z-y)|} \leq \sqrt{|x-z|} + \sqrt{|z-y|} = d_1(x,z) + d_1(z,y)$—note that if $x - y < 0$ then the inequality would still work out in the end.

Therefore we have that $d_1 \colon X \times X \to \mathbf{R}$ where $d_1 \colon (x,y) \mapsto \sqrt{|x-y|}$ does define a metric.

(b) $d_2(x,y) = |x - 2y|$ does not define a metric on $\mathbf{R}$, since it does not, at the very least, satisfy the symmetry condition: Let $X = \mathbf{R}$. Then $d_2 \colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, where $d_2 \colon (x,y) \mapsto |x - 2y|$, is not a metric since, for example, $d_2(2,3) = |2 - 2(3)| = |2 - 6| = |-4| = 4$ but $d_2(3,2) = |3 - 2(2)| = |3 - 4| = |-1| = 1$, and so a counter example against the symmetry property.

(c) Indeed a metric.

Suppose $(x,y) \in X \times X$. Then $\frac{|x-y|}{1+|x-y|}$ is always positive since $|x-y| \geq 1$ for any choice $x \neq y$ and $x = y$ gives us that $d_3(x,y) = 0$. Now if $x = y$, then $\frac{0}{1+0} = 0$. If instead assumed firstly that $\frac{|x-y|}{1+|x-y|} = 0$, then: $x - y > 0$ implies $\frac{x-y}{1+x-y} = 0$ and so $x = y$ clearly, and similarly, $x - y < 0$ gives us $\frac{y-x}{1+y-x} = 0$ and so $x = y$; if $x - y = 0$, then the result follows immediately. Thus $d_3(x,y) \geq 0$ for all $x, y \in X$ and $d_3(x,y) = 0$ if and only if $x = y$.

For symmetry, we proceed as follows. If $x - y = 0$ then the result is clear. Now if $x - y > 0$, then

$$\begin{aligned} \frac{|x-y|}{1+|x-y|} - \frac{|y-x|}{1+|y-x|} &= \frac{x-y}{1+x-y} - \frac{-(y-x)}{1+(-1)(y-x)} \\ &= \frac{x-y}{1+x-y} + \frac{y-x}{1+(x-y)} = 0. \end{aligned}$$

Thus $d_1(x,y) = d_1(y,x)$ if $x - y > 0$. If $x - y < 0$, then

$$\begin{aligned} \frac{|x-y|}{1+|x-y|} - \frac{|y-x|}{1+|y-x|} &= \frac{-(x-y)}{1+(-1)(x-y)} - \frac{y-x}{1+y-x} \\ &= \frac{y-x}{1+y-x} + \frac{-y+x}{1+y-x} = 0. \end{aligned}$$

Thus $d_3(x,y) = d_3(y,x)$ if $x - y < 0$. Lastly, we have that if $x - y = 0$, then $x = y$, and so, trivially, $d_3(x,y) = d_3(y,x)$. Therefore d_3 is symmetric. [It would have been easier to see that, as $|x-y|$ itself defines a metric, then $\frac{|x-y|}{1+|x-y|} = \frac{|y-x|}{1+|y-x|}$, and so $d(x,y) = d(y,x)$.]

For the triangle inequality

$$\begin{aligned} d(x,y)+d(y,z) = \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} &\geq \frac{|x-y|}{1+|x-y|+|y-z|} + \frac{|y-z|}{1+|x-y|+|y-z|} \\ &= \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|} \\ &= 1 - \frac{1}{1+|x-y|+|y-z|} \\ &\geq 1 - \frac{1}{1+|x-z|} = \frac{1+|x-z|-1}{1+|x-z|} = \frac{|x-z|}{1+|x-z|} = d_3(x,z). \end{aligned}$$

Therefore the triangle inequality holds and $d_3\colon X \times X \to \mathbf{R}$ defines a metric. $\square$

Exercise 2.2 (2.3)**.** Consider the function $d\colon \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ defined by

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|, \quad [\, x = (x_1, x_2),\ y = (y_1, y_2)\,]$$

(a) Prove that d is a metric on $\mathbf{R}^2$.
(b) On a sheet of graph paper, draw the set $B_d((5,1),3)$. Use dotted lines to indicate the boundary, which is not included in the set you are drawing. (Hint: it may be easier to figure out what the set looks like if you first consider $B_d((0,0),3)$.)
(c) On the same graph as in the previous part, draw $B_{d_u}((-3,2),1)$, where d_u denotes the square metric.

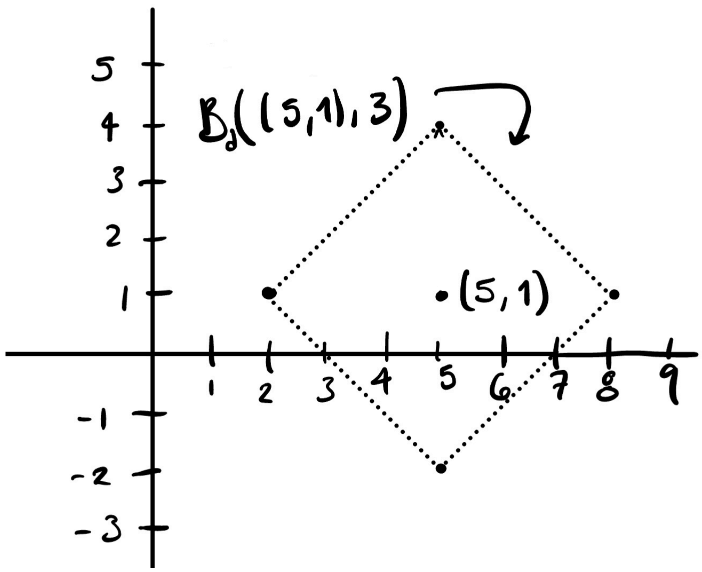
Proof. (a) We always have that $d(x,y) \geq 0$ since $|\cdot|$ is itself a metric, and so $|x_1 - y_1| \geq 0$ and $|x_2 - y_2| \geq 0$. Now if $d(x,y) = 0$, then we must have that $|x_1 - y_1| = 0$ and $|x_2 - y_2| = 0$, and so we have that $x_1 = y_1$ and $x_2 = y_2$. Hence $x = y$.

We have that the symmetry property holds as a consequence of the fact that $|\cdot|$ is a metric, and so $d(x,y) = d((x_1,x_2),(y_1,y_2)) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d((y_1,y_2),(x_1,x_2)) = d(y,x)$.

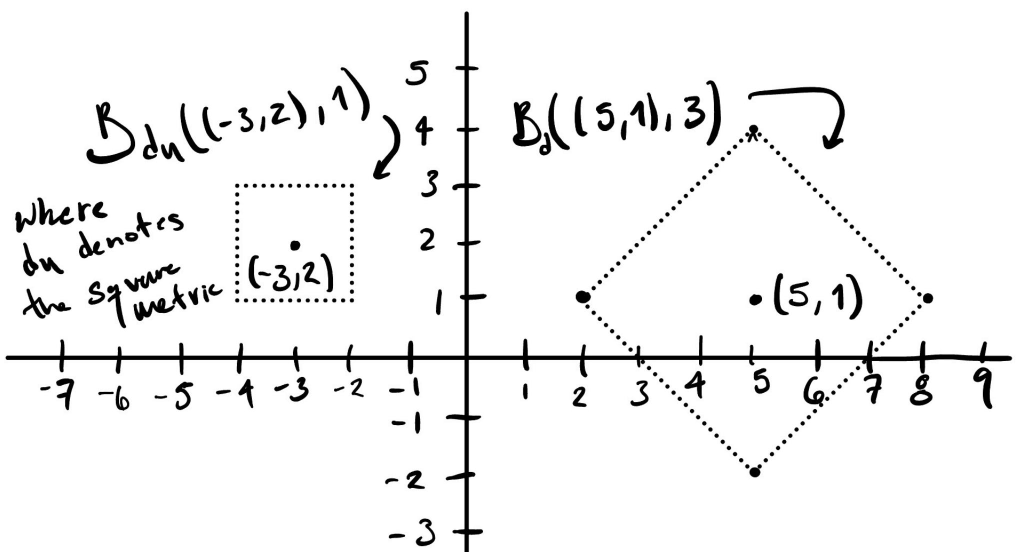
Lastly, let $z = (z_1, z_2) \in \mathbf{R}^2$. Then using the triangle inequality for the absolute value on $\mathbf{R}$, then we can show that it works for this metric: $d(x,y) = |x_1 - y_1| + |x_2 - y_2| = |x_1 - z_1 + z_1 - y_1| + |x_2 - z_2 + z_2 - y_2| \leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| = d(x,z) + d(z,y)$. $\square$

Hence we have that $d\colon \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ defines a metric.

(b)


$B_d((5,1),3)$
(5, 1)
5
4
3
2
1
-1
-2
-3
1
2
3
4
5
6
7
8
9

(c)


$B_{du}((-3,2),1)$
where du denotes the square metric
(-3,2)
$B_d((5,1),3)$
(5, 1)
5
4
3
2
1
-1
-2
-3
-7
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
7
8
9

Exercise 2.3 (2.4.)**.** Let (X, d) be a metric space, and let E be a subset of X. The *diameter* of E in (X, d) is defined by the formula

$$\operatorname{diam}_d(E) = \sup\{d(x, y) \colon x, y \in E\}.$$

(a) Prove that for any $r > 0$ and $x \in X$, we have $\operatorname{diam}(B(x, r)) \leq 2r$.
(b) If X is any set and d is the discrete metric, show that $\operatorname{diam}(B(x, r)) = 0$ for any $r \leq 1$, while $\operatorname{diam}(B(x, r)) = 1$ for any $r > 1$.
(c) If $X = \mathbf{R}^n$ for some $n \in \mathbf{N}$ and d is the Euclidean metric, prove that $\operatorname{diam}(B(x, r)) = 2r$.

Proof. (a) Suppose that $r > 0$ and $x \in X$. Then $\operatorname{diam}(B(x, r)) = \sup\{d(s, t) \colon s, t \in B(x, r)\}$. By definition of the set, without talking about the least upper bound, we're considering real numbers $d(q, p)$ where $q, p \in B(x, r)$, and so $d(x, q) < r$ and $d(x, p) < r$. Thus $d(q, p) \leq d(q, x) + d(x, p) < 2r$. And so the distance between all points is less than $2r$, i.e. $2r$ is an upper bound for the set. Thus, by the definition of the least upper bound, we have that $\sup\{d(s, t) \colon s, t \in B(x, r)\} \leq 2r$; that is, $\operatorname{diam}(B(x, r)) \leq 2r$.

(b) Suppose X is a set and d is the discrete metric and let $r \leq 1$. Now $B(x, r) = \{y \in X \colon d(x, y) < r\}$, but as we assumed $r \leq 1$, then $B(x, r) = \{x\}$ itself since we're using the discrete metric. Thus $\sup\{d(s, t) \colon s, t \in B(x, r)\} = \sup\{d(x, x) = 0\} = \sup 0 = 0$. Thus $\operatorname{diam}(B(x, r)) = 0$ for any $r \leq 1$. For the other case, suppose that $r > 1$. Then $B(x, y) = \{y \in X \colon d(x, y) < r\}$ isn't simple like the other case. However, $\{d(s, t) \colon d(x, y) < r\} = \{0, 1\}$ since we're using the discrete metric. So then clearly 1 is an upper bound for the set, but then $\sup\{0, 1\} \leq 1$, which forces us to have that $\sup\{0, 1\} = 1$. Hence $\operatorname{diam}(B(x, r)) = 1$ for $r > 1$.

(c) Suppose that $X = \mathbf{R}^n$ for some $n \in \mathbf{N}$ and d is the Euclidean metric. Now by part (a), it suffices to show that $\operatorname{diam}(B(x, r)) \geq 2r$, where $x \in \mathbf{R}^n$ and $r > 0$, since this would then imply that $\operatorname{diam}(B(x, r)) = 2r$. Assume that $\operatorname{diam}(B(x, r)) < 2r$. Now as $\operatorname{diam}(B(x, r))$ is strictly smaller than $2r$, we can find some α such that $\operatorname{diam}(B(x, r)) < \alpha < 2r$. Write $s = x + \frac{\alpha \ell}{2\|\ell\|}, t = x - \frac{\alpha \ell}{2\|\ell\|}$, where $\ell \in \mathbf{R}^n$ and $\ell \neq 0$. Then $\|s - x\| = \|\frac{\alpha \ell}{2\|\ell\|}\| = \frac{\alpha}{2} < r$, and similarly we have that $\|x - t\| = \|\frac{\alpha \ell}{2\|\ell\|} = \frac{\alpha}{2} < r$, and $s - t = (x + \frac{\alpha \ell}{2\|\ell\|}) - (x - \frac{\alpha \ell}{2\|\ell\|}) = 2\frac{\alpha \ell}{2\|\ell\|} = \frac{\alpha \ell}{\|\ell\|}$, which implies that $\|s - t\| = \|\alpha\|$. This is a contradiction as $\|s - t\| \leq \operatorname{diam}(B(x, r)) < 2r$, but we found that $\|s - t\| = \alpha \leq \operatorname{diam}(B(x, r)) < \alpha$, and we had α such that $(B(x, r)) < \alpha < 2r$. Therefore we must have that $\operatorname{diam}(B(x, r)) = 2r$. $\square$

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