# Course Notes for Math 521: Analysis I

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# Introduction

Defining what ‘analysis’ means as a mathematical discipline is a bit of a difficult task, largely because of its vast scope. However, it is perhaps useful to have a working definition of the subject of these notes before beginning in earnest. Therefore we’ll give a brief discussion of what analysis is, with the caveat that it just a first approximation. Analysis involves drawing qualitative (and sometimes quantitative) conclusions about functions when only limited information is initially available. What exactly does ‘limited information’ mean, then? One of the most relevant situations here is when an exact formula for a function is not available or not useful. Alternatively, deriving a formula may involve a computation that is impractical or impossible. The basic tools of analysis are designed so that it is still possible in many cases to extract useful information.

The main concepts of analysis are convergence, continuity, differentiation, and integration. The reader should be familiar with each of these concepts from courses in Calculus. However, these concepts are often treated in a less-than-rigorous manner in Calculus courses; even when rigor is not lacking, these concepts are introduced in a very limited context, which needs to be expanded before one can go deeper into analysis. These notes aim to put the main theorems of Calculus on a rigorous footing and broaden their scope somewhat. However, the reader should be aware that the context provided by these notes is only the ‘tip of the iceberg’, so to speak. The immense power that analysis can bring to mathematical and physical problems is only really accessible after several more passes.

In the interest of rigor, it would be appealing to develop the theory ‘from scratch’, so to speak. However, these notes are intended to accompany a one-semester course; therefore such a development is practically impossible. We choose a starting point that is compatible with the goal of developing the main concepts in analysis, while sacrificing as little rigor as possible. We assume to begin with that the reader is familiar with the arithmetic of the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, and the rational numbers $\mathbb{Q}$. Furthermore, we will take a ‘naive’ rather than ‘axiomatic’ approach to set theory (these are terms of art). However, the shortcuts we take here can be justified (and are subtle enough that most readers will not notice); the interested reader can fill in the gaps by taking courses in logic and in algebra.

While we assume knowledge of the basic properties of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$, we assume essentially nothing about the set $\mathbb{R}$ of real numbers. This is because $\mathbb{R}$ is a more complicated set. Indeed, starting from $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$, can you write down a definition of $\mathbb{R}$? Almost any way you try to define it will implicitly make use of limits. For example, suppose we try to define $\mathbb{R}$ in terms of infinite decimals. The question then arises: What exactly is an infinite decimal? The natural definition is as a limit or supremum of finite decimals. But then we must ask what exactly is meant by a ‘limit’, and whether the supremum is guaranteed to exist. In short, rigorously defining $\mathbb{R}$ starting from $\mathbb{Q}$ is far from straightforward. Giving the ‘right’ definition of $\mathbb{R}$ will be the subject of this second chapter.

When writing these notes, the author referred frequently to several textbooks, including Walter Rudin’s *Principles of Mathematical Analysis*, Stephen Abbott’s *Understanding Analysis*, James Munkres’ *Topology*, and Andrew Browder’s *Mathematical Analysis: An Introduction*. A few sections in these notes follow various parts of the above texts rather closely. No copyright violation is intended; however, as these notes will be posted publicly, please email me if you are a publisher or author who happens upon these notes and has any objections to the presentation.

Finally, these notes are a work in progress. Please email lesliet@usc.edu for any corrections or suggestions for improvements.

# Part 1

# Basics and Preliminaries

CHAPTER 1

# Naive Set Theory

In this chapter, we build the basic set-theoretic terminology and machinery needed for the rest of the course. For a combination of reasons (time and space considerations, pedagogical sensibility, expertise of the instructor), We take the approach of 'Naive Set Theory' rather than an axiomatic approach. Basically, all this means is that we assume that the notion of a 'set' is intuitively clear. The study of what exactly a set is belongs to the realm of logic and mathematical foundations. To illustrate that this issue is a non-triviality, consider the following.

Russell's Paradox: Let $R$ be the set of all sets that are not members of themselves. Is $R \in R$?

Well-known Layman's Reformulation: Suppose a barber cuts the hair of exactly those people who do not cut their own hair. Does the barber cut his or her own hair?

Both of these questions seem impossible to answer. The logical resolution is that such an object $R$ cannot actually be a set, and that such a barber cannot exist. That is, the extent to which one can abstractly manipulate sets to create new ones is not without limitations. Therefore, when constructing new sets, one has to use a bit of care.

The above might induce some existential worry in the minds of more dramatic readers. What *is* a set, anyway? What are the 'rules' of mathematics? Has math been lying to me all this time? Fortunately, logicians have got us covered. They have carefully and painstakingly crafted a set of axioms on which all of mathematics can be based. Unfortunately, acquiring a thorough understanding of these axioms and their consequences is a major undertaking. We will forgo a discussion of these axioms in these notes. Instead, we assure the reader that the logic we use is justified by the axioms. The reader who wishes for a more careful and systematic treatment of set theory should consider taking a course in logic or mathematical foundations.

## 1. Sets and Set Operations

### 1.1. Sets and Subsets.

DEFINITION 1.1. A *set* is any collection of objects, or *elements*. Usually a set is denoted by a capital letter, such as $A$, and its elements are denoted by lowercase letters, such as $x$. The notation $x \in A$ is equivalent to the statement "$x$ is an element of the set $A$". If $A$ and $B$ are sets and every element of $A$ belongs to $B$ (i.e., $x \in A$ implies $x \in B$), then we say that $A$ is a *subset* of $B$, or that $A$ is *contained in* $B$, and we write $A \subset B$. We say that two sets are the same, or *equal*, if $B \subset A$ and $A \subset B$; in this case we write $A = B$.

Note that all the notation in the definition above is 'reversible': $x \in A$ is the same as $A \ni x$, and $A \subset B$ is the same as $B \supset A$.

If $A \subset B$ and $A \neq B$, we sometimes write $A \subsetneq B$. This is more specific than $A \subset B$, since the latter notation allows for the possibility that $A = B$. However, the notation $A \subset B$ is usually used unless we wish to explicitly stress the non-equality.

An emphatically different notation is $A \not\subset B$. This means that $A \subset B$ *fails*, i.e. $A$ is *not* a subset of $B$, i.e. there exists $a \in A$ such that $a \notin B$. Do not confuse the two notations $A \not\subset B$ and $A \subsetneq B$.

We assume familiarity with the following sets:

- The integers $\mathbb{Z}$.
- The natural numbers (i.e. positive integers) $\mathbb{N}$.
- The rational numbers $\mathbb{Q}$.

Later, we will also assume familiarity with the usual algebraic operations on these sets (e.g., addition, multiplication, etc.)

Sometimes it is useful to write down all the elements of a set, either explicitly, by suggesting a pattern, or by specifying a rule. The usual notation here is illustrated in the following examples.

$$A = \{1, 2, \frac{7}{2}, 9, 3, 14\}$$

$$B = \{2, 4, 6, \ldots, 100\}.$$

$$C = \{1, 4, 7, 10, \ldots\}.$$

$$D = \{x \in A : x \text{ is an even integer}\} = \{2, 14\}.$$

In the last example, the colon is read "such that". The notation used here is called *set-builder notation*. It should be familiar and will not be explained further.

DEFINITION 1.2. The *empty set* is the set with zero elements, usually denoted either by $\emptyset$ or $\{\}$. Any set that is not empty is called *nonempty*.

Note that if $A$ is any set whatsoever, then $\emptyset \subset A$, since the statement "every element of $\emptyset$ is an element of $A$" is vacuously true. If $\emptyset \subsetneq A \subsetneq B$, then $A$ is said to be a *proper subset* of $B$.

**1.2. Collections of Sets.** Certain kinds of sets often carry slightly different terminology and notation. For example, it may be the case that each element of a set is itself a set! In this case the "set of sets" is often referred to as a *collection* of sets, and is usually denoted with a scripted capital letter, such as $\mathcal{A}$.

EXAMPLE 1.3. If $A = \{1, 2, 3\}$, $B = \{1, 2\}$, $C = \{1\}$, then $\mathcal{A} = \{A, B, C\} = \{\{1, 2, 3\}, \{1, 2\}, \{1\}\}$ is a collection of sets. To clarify the notation, note that we write (for example) $B \in \mathcal{A}$, $B \subset A$, $1 \in B$. This is consistent with the definition of the symbols $\in$ and $\subset$.

EXAMPLE 1.4. Given a set $A$, one can form its *power set* $\mathcal{P}(A)$, which consists of all subsets of $A$. For example, if $A = \{1, 2, 3\}$, then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

**1.3. Binary Set Operations.**

DEFINITION 1.5. Given two sets $A$ and $B$, their *union* $A \cup B$ is defined to be the set consisting of all elements belonging either to $A$ or to $B$. The *intersection* $A \cap B$ of $A$ and $B$ is defined to be the set of elements belonging to *both* $A$ and $B$.

$$A \cup B = \{x : x \in A \text{ or } x \in B\}, \qquad A \cap B = \{x : x \in A \text{ and } x \in B\},$$

If $A \cap B = \emptyset$, we say that $A$ and $B$ are *disjoint*. If $A \cap B \neq \emptyset$, then we say that $A$ and $B$ *intersect*.

DEFINITION 1.6. Let $A$ and $B$ be sets. The *relative complement* of $A$ in $B$, denoted $B \backslash A$, is defined to be the set of elements of $B$ which are not in $A$.

$$B \backslash A = \{x \in B : x \notin A\}.$$

Sometimes one works within the context of some very large set $X$, and all other sets under consideration are understood to be subsets of $X$. In this case (and if $X$ is clear from context), we define the *absolute complement* of $A$ by

$$A^c = \{x \in X : x \notin A\}.$$

Note that when $A$ and $B$ are both subsets of a large set such as the set $X$ in the definition above, one has

$$B \backslash A = B \cap A^c.$$

In practice, one often drops the qualifiers 'relative' and 'absolute' from the terminology

EXERCISE 1.7. Let $A$ and $B$ be subsets of another set $X$. Prove the following statements.

(1) $A \cap B = A \backslash (A \backslash B)$
(2) $A \subset B$ if and only if $X \backslash A \supset X \backslash B$.

Note: For any sets $A$ and $B$, we have

$$A = (A \backslash B) \cup (A \cap B).$$

In particular, $A = (A \backslash B) \cup B$ if and only if $B \subset A$ (in which case $A \cap B = B$).

The intersection and union satisfy the following basic properties.

- (Commutativity) $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
- (Associativity) $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$
- (Distributivity) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

In light of the commutativity and associativity, we may define the union of $n$ sets inductively by

$$A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_{n-1} \cup A_n := (\cdots((A_1 \cup A_2) \cup A_3) \cup \cdots \cup A_{n-1}) \cup A_n,$$

and similarly for intersections.

DEFINITION 1.8. An *ordered pair* in a set $X$ is a pair of elements of $X$ listed in an order. We use the notation $(x_1, y_1)$ to denote ordered pairs. We say that two ordered pairs $(x_1, y_1)$ and $(x_2, y_2)$ are *equal* if $x_1 = x_2$ and $y_1 = y_2$.

DEFINITION 1.9. Given two sets $A$ and $B$, their pairwise *product* $A \times B$ consists of all ordered pairs $(a, b)$ in $A \cup B$, such that $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

### 1.4. General Unions and Intersections.

DEFINITION 1.10. If $\mathcal{A}$ is a collection of subsets of a set $X$, then the *union* of all sets in $\mathcal{A}$ consists of all elements of every set in $\mathcal{A}$:

$$\bigcup_{A \in \mathcal{A}} A = \{x \in X : x \in A \text{ for at least one } A \in \mathcal{A}\}.$$

The *intersection* of all sets in $\mathcal{A}$ consists of those elements of $X$ which are in every set of $\mathcal{A}$:

$$\bigcap_{A \in \mathcal{A}} A = \{x \in X : x \in A \text{ for every } A \in \mathcal{A}\}.$$

If $\mathcal{A} = \{A_1, \ldots, A_n\}$, then clearly

$$\bigcup_{A \in \mathcal{A}} A = A_1 \cup \cdots \cup A_n,$$

and similarly for the intersection. We will introduce some more convenient notation later on.

A rather annoying example which is worth mentioning explicitly is the case where $\mathcal{A} = \emptyset$. In this case the union over $\mathcal{A}$ is $\emptyset$, while the intersection over $\mathcal{A}$ is all of $X$. The reader can check that these statements are (vacuously) true.

General unions and intersections satisfy properties analogous to those listed in the previous subsection for binary unions and intersections (commutativity, associativity, distributivity). These will be used without mention throughout the notes. However, we point out two particularly useful equalities:

THEOREM 1.11 (DeMorgan's Laws). *If $\mathcal{A}$ is a collection of sets and $X$ is another set, then the following two equalities hold.*

$$X \backslash \left( \bigcup_{A \in \mathcal{A}} A \right) = \bigcap_{A \in \mathcal{A}} (X \backslash A). \tag{1}$$

$$X \backslash \left( \bigcap_{A \in \mathcal{A}} A \right) = \bigcup_{A \in \mathcal{A}} (X \backslash A). \tag{2}$$

PROOF. Assume $x$ is an element on the left side of (1). Then $x \in X$, but $x$ does not belong to any $A \in \mathcal{A}$. Thus $x \in X \backslash A$, for every $A \in \mathcal{A}$; that is, $x$ belongs to the set on the right side of (1). This shows that LHS$\subset$RHS for (1). To prove the opposite inclusion, assume that $x$ belongs to the set on the right side of (1). Then $x \in X \backslash A$ for every $A \in \mathcal{A}$. Thus $x \in X$, but $x \notin A$ for any $A \in \mathcal{A}$, i.e. $x \notin \bigcup_{A \in \mathcal{A}} A$. Therefore $x$ belongs to the set on the left side of (1). This shows that LHS = RHS for (1).

To prove (2), we use (1). Both sets in (2) are subsets of $X$, therefore

$$X \backslash \left( \bigcup_{A \in \mathcal{A}} (X \backslash A) \right) = \bigcap_{A \in \mathcal{A}} X \backslash (X \backslash A) = \bigcap_{A \in \mathcal{A}} A.$$

Taking complements in $X$ one more time, we thus obtain

$$\bigcup_{A \in \mathcal{A}} (X \backslash A) = X \backslash \left( \bigcap_{A \in \mathcal{A}} A \right),$$

as claimed. $\square$

## 2. Relations

DEFINITION 2.1. A *relation* $R$ from a set $A$ to a set $B$ is a subset of $A \times B$. We sometimes write $aRb$ to mean that $(a, b) \in R$. If $A$ is a set, a relation 'on $A$' means a relation between $A$ and itself, i.e. a subset of $A \times A$.

DEFINITION 2.2. Let $R$ be a relation on a set $A$. Then

- $R$ is said to be *reflexive* if $aRa$ for every $a \in A$.
- $R$ is said to be *symmetric* if $aRb$ implies $bRA$.
- $R$ is said to be *antisymmetric* if $aRb$ and $bRa$ imply that $a = b$.
- $R$ is said to be *transitive* if $aRb$ and $bRc$ together imply that $aRc$.

Though you may not realize it, several familiar concepts can be described using relations. Since the notation $aRb$ may be new, we allow some redundancy in the following definitions for the sake of clarity.

DEFINITION 2.3. A *partial order* $\preceq$ on a set $A$ is a relation on $A$ that satisfies the following properties:

- (Reflexivity) $a \preceq a$, for all $a \in A$.
- (Antisymmetry) If $a \preceq b$ and $b \preceq a$, then $a = b$.
- (Transitivity) If $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

A *total order* $\leq$ on a set $A$ is a partial order which has the following additional property:

- (Comparability) If $a, b \in A$, then at least one of the statements $a \leq b$ or $b \leq a$ must hold.

EXAMPLE 2.4. The usual meaning 'less than or equal to' of the symbol $\leq$ constitutes a total ordering on $\mathbb{Q}$, or any subset thereof. For small sets, the relation can be given explicitly. If $A = \{1, 2\}$, then $\leq$ is the set $\{(1, 1), (1, 2), (2, 2)\}$.

If $\mathcal{A}$ is a collection of sets, then $\subset$ is a partial ordering on $\mathcal{A}$; we also say that $\mathcal{A}$ is *partially ordered by inclusion*. However, $\subset$ may or may not be a total order on $\mathcal{A}$. For example, if $\mathcal{A} = \{\{1, 2\}, \{1\}, \{2\}\}$, then neither of the statements $\{1\} \subset \{2\}$ or $\{2\} \subset \{1\}$ is true (the sets $\{1\}$ and $\{2\}$ are not *comparable*); therefore $\mathcal{A}$ is not ordered by inclusion. However, the collection $\mathcal{B} = \{\{1\}, \{1,2\}, \{1,2,3\}\}$ is (totally) ordered by inclusion. If a collection $\mathcal{A}$ is totally ordered by inclusion, it is said to be a collection of *nested sets*.

REMARK 2.5. Technically, any symbol $R$ can be used to denote an order relation (partial or total). However, it is often useful to have compact notation for the situation where $aRb$ but $a \neq b$. Whenever the symbols $\preceq$, $\leq$, $\subset$ are used to denote (partial or total) orders, it should be understood that the corresponding symbols $\prec$, $<$, $\subsetneq$ specify non-equality of the related elements.

The symbols usually used for order relations are also 'reversible': $b \succeq a$ means $a \preceq b$; $b \geq a$ means $a \leq b$, etc.

DEFINITION 2.6. An *equivalence relation* $\sim$ on a set $A$ is a relation on $A$ that satisfies the following properties.

- (Reflexivity) $a \sim a$ for all $a \in A$.
- (Symmetry) $a \sim b$ implies $b \sim a$.
- (Transitivity) If $a \sim b$ and $b \sim c$, then $a \sim c$.

If $A$ is the set of all people on the planet, then the relation defined by 'Person 1$\sim$ Person 2 if and only if Person 1 and Person 2 have the same birthday' is an equivalence relation.

DEFINITION 2.7 (Function). A *function* $f$ from $A$ to $B$ is a relation between two sets $A$ and $B$ that satisfies the following properties:

- For each $a \in A$, we have $afb$ for some $b \in B$.
- If $afb$ and $afc$ hold, then $b = c$.

We almost always use the notation $f(a) = b$ instead of $afb$. $A$ is called the *domain*, and $B$ is called the *codomain*; we express this by writing $f : A \to B$.

The definition of 'function' above can easily be seen to be equivalent to the following probably more familiar variant: Let $A$ and $B$ be sets, and suppose that to each $x \in A$ there is exactly one associated element $f(x) \in B$. Then the association $f$ is called a *function* from $A$ to $B$.

We now enter into a much deeper discussion of functions.

## 3. Functions

### 3.1. Basic Definitions and Notation.

DEFINITION 3.1 (Some Important Kinds of Functions).

- For any set $A$, the map $\mathrm{id}_A : A \to A$ defined by $\mathrm{id}_A(a) = a$ for all $a \in A$ is called the *identity* map.
- If $A \subset B$, the map $\iota : A \to B$ defined by $\iota(a) = a$ for all $a \in A$ is called the *inclusion* map from $A$ to $B$.
- If $f : A \to B$ is a function and $C \subset A$, then the function $f\big|_C : C \to B$ defined by $(f\big|_C)(c) = f(c)$ for all $c \in C$ is called the *restriction* of $f$ to $C$.
- If $f : A \to B$ is a function and $f(A) \subset D \subset B$, the map $g : A \to D$ defined by $g(a) = f(a)$ is referred to as the function formed by *restricting the codomain* of $f$ to $D$. (There is no standard notation for this function. $f(A)$ is defined below.)
- If $f : A \to B$ and $g : B \to C$ are functions, then the *composition* of $f$ and $g$ is the function $g \circ f : A \to C$ defined by $(g \circ f)(a) = g(f(a))$ for all $a \in A$. (If $f : A \to B$ and $g : B' \to C$ are functions with $B \subset B'$, we use the notation $g \circ f$ to denote $(g\big|_B) \circ f$.)

DEFINITION 3.2. Let $f : A \to B$ be a function.

- If $E \subset A$, then the *image* $f(E)$ of $E$ under $f$ is defined by
$$f(E) = \{f(x) : x \in E\}.$$
The image $f(A)$ of the entire domain $A$ is called the *image* of $f$, or the *range* of $f$. The elements of $f(A)$ are called the *values* of $f$. If $f(A) = B$, we say that $f$ maps $A$ *onto* $B$, or that $f : A \to B$ is *surjective*.
- If $G \subset B$, then the *inverse image* $f^{-1}(G)$ of $G$ under $f$ is defined as
$$f^{-1}(G) = \{x \in A : f(x) \in G\}.$$
The inverse image is sometimes called the *preimage*. If $y \in B$, then we define $f^{-1}(y) = f^{-1}(\{y\})$. If $f^{-1}(y)$ contains at most one element of $A$ for each $y \in B$, then $f$ is said to be *injective*, or a *one-to-one* mapping of $A$ into $B$.
- If $f$ is one-to-one and onto (injective and surjective), then we say that $f$ is *bijective*.

Notes: (i) Another formulation of injectivity is the following: $f$ is injective if and only if $f(x) = f(y)$ implies $x = y$. (ii) If $f : A \to B$ is any function, then the function formed by restricting the codomain of $f$ to $f(A)$ is a surjection. In particular, such a restriction results in a bijection if the original function $f$ is injective. (iii) If $\iota : C \to A$ is the inclusion map, then $f\big|_C = f \circ \iota$. (iv) The image of a function $f : A \to B$ is sometimes denoted by $\operatorname{Im} f$ rather than by $f(A)$.

Important: In general, one can't 'cancel' $f$ and $f^{-1}$, the way the notation might tempt one to do. For example, let $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$ and define $f : A \to B$ by $f(1) = f(2) = f(3) = 5$. Then
$$f^{-1}(f(\{1, 2\})) = f^{-1}(\{5\}) = \{1, 2, 3\} \neq \{1, 2\}$$
$$f(f^{-1}(\{4, 5\})) = f(\{1, 2, 3\}) = \{5\} \neq \{4, 5\}.$$
However, if $f : A \to B$ is a function and $C \subset A$ and $D \subset B$, we *do* have that[1]
$$f^{-1}(f(C)) \supset C \qquad \text{and} \qquad f(f^{-1}(D)) \subset D.$$
In fact, a little more is true:

EXERCISE 3.3. Let $f : A \to B$ be a function. Prove the following statements:

(1) $f$ is injective if and only if $f^{-1}(f(C)) = C$ for every subset $C$ of $A$.
(2) $f$ is surjective if and only if $f(f^{-1}(D)) = D$ for every subset $D$ of $B$.

EXERCISE 3.4. Let $f : A \to B$ and $g : B \to C$ be functions. Prove the following statements.

(1) If $f$ and $g$ are both injective, then so is $g \circ f$.
(2) If $f$ and $g$ are both surjective, then so is $g \circ f$.
(3) If $g \circ f$ is surjective, then so is $g$.
(4) Surjectivity of $g \circ f$ does not imply surjectivity of $f$.
(5) If $g \circ f$ is injective, then so is $f$.
(6) Injectivity of $g \circ f$ does not imply injectivity of $g$.

Hint: For each of parts (4) and (6), you should prove the statement by giving an example of two functions $f$ and $g$ which demonstrate the statement. These examples are easiest to construct if you choose $A$, $B$, and $C$ to be sets with very few elements.

[1]Proof: If $a \in C$, then $f(a) \in f(C)$, which means that $a \in f^{-1}(f(C))$ by definition of the inverse image. If $b \in f(f^{-1}(D))$, then there exists $a \in f^{-1}(D)$ such that $f(a) = b$. But since $a \in f^{-1}(D)$, we have $f(a) \in D$ by definition of the inverse image. Thus $b \in D$.

**3.2. Functions and Set Operations.** In this section, we summarize the relationship between the image/preimage of a function and the set operations union, intersection, and complement.

PROPOSITION 3.5. *Let $f : X \to Y$ be a function.*

- *For any collection $\mathcal{A}$ of subsets of $X$, we have*

$$f\left(\bigcup_{A\in\mathcal{A}} A\right) = \bigcup_{A\in\mathcal{A}} f(A), \qquad f\left(\bigcap_{A\in\mathcal{A}} A\right) \subset \bigcap_{A\in\mathcal{A}} f(A).$$

- *$f$ is injective if and only if for every nonempty collection $\mathcal{A}$ of subsets of $X$, we have*

$$f\left(\bigcap_{A\in\mathcal{A}} A\right) = \bigcap_{A\in\mathcal{A}} f(A).$$

- *If $\mathcal{B}$ is a collection of subsets of $Y$, then*

$$f^{-1}\left(\bigcup_{B\in\mathcal{B}} B\right) = \bigcup_{B\in\mathcal{B}} f^{-1}(B). \qquad f^{-1}\left(\bigcap_{B\in\mathcal{B}} B\right) = \bigcap_{B\in\mathcal{B}} f^{-1}(B).$$

PROOF. The proofs of the first and third statements follow basically by unraveling the definitions and are omitted. We prove only the second statement, considering the contrapositive of both implications.

( $\implies$ ) Let $\mathcal{A}$ be a nonempty collection of subsets of $X$. For convenience, we write

$$Y_1 = f\left(\bigcap_{A\in\mathcal{A}} A\right), \qquad Y_2 = \bigcap_{A\in\mathcal{A}} f(A).$$

Assume that $Y_2 \neq Y_1$. Then by the first point of the Proposition, we have $Y_2 \not\subset Y_1$. Then there exists $y \in Y_2$ such that $y \notin Y_1$. Suppose such a $y$ has been chosen. Then $y \in \mathrm{Im}(f)$; therefore choose $x \in X$ such that $y = f(x)$. This $x$ cannot be an element of $\bigcap_{A\in\mathcal{A}} A$; if it were, than we would have $y = f(x) \in Y_1$, contradicting our assumption. Therefore there exists $\widetilde{A} \in \mathcal{A}$ such that $x \notin \widetilde{A}$. But it is still true that $y \in f(\widetilde{A})$, so we can pick $\widetilde{x} \in \widetilde{A}$ such that $f(\widetilde{x}) = y = f(x)$. On the other hand, $x \neq \widetilde{x}$, since $\widetilde{x}$ is an element of $\widetilde{A}$ and $x$ is not. This proves that $f$ is not injective, completing the proof of the implication ( $\implies$ ).

( $\impliedby$ ) Assume that $f$ is not injective, i.e. there exist $x_1, x_2 \in X$ and $y \in Y$ such that $f(x_1) = f(x_2) = y$, but $x_1 \neq x_2$. Consider the collection $\mathcal{A} = \{\{x_1\}, \{x_2\}\}$. Then

$$f\left(\bigcap_{A\in\mathcal{A}} A\right) = f(\{x_1\} \cap \{x_2\}) = f(\emptyset) = \emptyset,$$

while

$$\bigcap_{A\in\mathcal{A}} f(A) = f(\{x_1\}) \cap f(\{x_2\}) = \{y\} \cap \{y\} = \{y\}.$$

That is, the equality under consideration does not hold for the collection $\mathcal{A}$. This completes the proof of the implication ( $\impliedby$ ). $\square$

EXERCISE 3.6. Let $f : X \to Y$ be a function. Prove the following statements.

(1) If $A$ and $C$ are subsets of $X$, then $f(C\backslash A) \supset f(C)\backslash f(A)$.
(2) $f$ is injective if and only $f(C\backslash A) = f(C)\backslash f(A)$ for any two subsets $A$ and $C$ of $X$.
(3) If $B$ and $D$ are subsets of $Y$, then $f^{-1}(D\backslash B) = f^{-1}(D)\backslash f^{-1}(B)$.

### 3.3. Function Inverses.

DEFINITION 3.7 (Function Inverses). Let $f : A \to B$ be a function.

- A *left inverse* for $f$ is a function $g : B \to A$ such that $g \circ f = \mathrm{id}_A$.
- A *right inverse* for $f$ is a function $h : B \to A$ such that $f \circ h = \mathrm{id}_B$.
- A *two-sided inverse* (or simply an *inverse*) for $f$ is a function $k : B \to A$ which is a right inverse and a left inverse.
- If $f$ has a two-sided inverse $k$, we sometimes use the notation $f^{-1}$ to denote the function $k$.

Note: $g$ is a left inverse for $f$ if and only if $f$ is a right inverse for $g$.

Important! The inverse of a function does not always exist. If $f : A \to B$ does not have an inverse, then the notation $f^{-1}(a)$ refers instead to the inverse image of the set $\{a\}$.

REMARK 3.8. Let $f : A \to B$ and $g : B \to A$ be such that $g \circ f = \mathrm{id}_A$. Then since $\mathrm{id}_A$ is bijective, Exercise 3.4 tells us that $g$ must be surjective and $f$ must be injective. Thus any left inverse is surjective, and any right inverse is injective. Consequently, any two-sided inverse must be bijective.

THEOREM 3.9. *Let $f : A \to B$ be a function.*

*(1) $f$ is injective if and only if it has a left inverse.*
*(2) $f$ is surjective if and only if it has a right inverse.*

PROOF. Remark 3.8 above essentially proves the direction ( $\Longleftarrow$ ) for both statements. Indeed, if $f$ has a left inverse $g$, then $f$ is a right inverse for $g$, so $f$ is injective. On the other hand, if $f$ has a right inverse $h$, then $f$ is a left inverse for $h$, so $f$ is surjective.

Now we prove the direction ( $\Longrightarrow$ ) for both statements.

(1) Assume $f$ is injective. Then for each $y \in B$, the set $f^{-1}(y)$ contains at most one element. Pick $x_0 \in A$ arbitrarily. Define a function $g : B \to A$ as follows. If $y \in f(A)$, define $g(y)$ to be the (unique) element of the set $f^{-1}(y)$; otherwise put $g(y) = x_0$. We claim that $g \circ f = \mathrm{id}_A$. Indeed, for any $x \in A$, we have $\{x\} = f^{-1}(f(x))$ since $f$ is injective. On the other hand, $g(f(x))$ is by definition the unique element of the set $f^{-1}(f(x)) = \{x\}$, thus $g(f(x)) = x$. That is, $g \circ f = \mathrm{id}_A$.

(2) Assume $f$ is surjective. Then for each $y \in B$, the set $f^{-1}(y)$ is nonempty. Define a function $h : B \to A$ as follows[2]. For each $y \in B$, let $h(y)$ be an arbitrary element of $f^{-1}(y)$. We claim that $f \circ h = \mathrm{id}_B$. Indeed, if $y \in B$, then $h(y) \in f^{-1}(y)$, so that $f(h(y)) = y$. This proves the claim. $\square$

PROPOSITION 3.10. *If $f : A \to B$ has a left inverse $g$ and a right inverse $h$, then $g = h$. That is, $g = h$ is actually a two-sided inverse for $f$.*

PROOF. For any $b \in B$, we have $g(b) = g((f \circ h)(b)) = g(f(h(b))) = (g \circ f)(h(b)) = h(b)$. $\square$

[2]For the benefit of readers who know some Logic, we note that this argument uses what's called the *Axiom of Choice*.

We summarize the relationship between inverses and injectivity/surjectivity/bijectivity as follows:

$$\begin{aligned}
f : X \to Y \text{ is injective} \iff& f \text{ has a (necessarily surjective) left inverse } g : Y \to X \\
\iff& f \text{ is a right inverse for } g \\
\iff& f^{-1}(f(C)) = C \text{ for all } C \subset X \\
\iff& f\Bigg(\bigcap_{A\in\mathcal{A}} A\Bigg) = \bigcap_{A\in\mathcal{A}} f(A) \text{ for any nonempty collection } \mathcal{A} \text{ of subsets of } X \\
\iff& f(C\backslash A) = f(C)\backslash f(A) \text{ for any two subsets } A \text{ and } C \text{ of } X.
\end{aligned}$$

$$\begin{aligned}
f : A \to B \text{ is surjective} \iff& f \text{ has a (necessarily injective) right inverse } g : B \to A \\
\iff& f \text{ is a left inverse for } g \\
\iff& f(f^{-1}(D)) = D \text{ for all } D \subset B.
\end{aligned}$$

$$\begin{aligned}
f : A \to B \text{ is bijective} \iff& f \text{ has a (necessarily bijective) two-sided inverse } g : B \to A \\
\iff& f \text{ is a two-sided inverse for } g \\
\iff& f^{-1}(f(C)) = C \text{ and } f(f^{-1}(D)) = D \text{ for all } C \subset A \text{ and } D \subset B.
\end{aligned}$$

### 3.4. $J$-tuples and Cartesian Products.

#### 3.4.1. *$J$-tuples.*

DEFINITION 3.11. Let $J$ and $X$ be sets. The set of all functions from $J$ to $X$ is denoted $X^J$. A *$J$-tuple* of elements of $X$ is a function $x : J \to X$, i.e. an element of $X^J$. However, when we call a function a $J$-tuple, we usually denote its values by $x_j$ instead of $x(j)$, and we denote the function itself by $x = (x_j)_{j\in J}$. We use the term *index set* to refer to the domain $J$.

Technically, the difference between a function and a $J$-tuple is purely notational. However, in practice, the two are used differently, and certain kinds of sets $J$ are used more frequently than others. Denote

$$J_n := \{1, 2, \ldots, n\}, \quad n \in \mathbb{N}. \tag{3}$$

(This notational convention will be used throughout the notes.) A $J_n$-tuple is often just called an $n$-tuple, and several possible notations are commonly used:

$$(x_j)_{j\in J_n} = (x_j)_{j=1}^n = (x_1, \ldots, x_n).$$

Furthermore, the set $X^{J_n}$ of all $n$-tuples is often just denoted by $X^n$. For $n \geq 3$, we define the notation $X \times \cdots \times X$ to mean $X^n$. The set $X^2$ can be thought of as the same thing as $X \times X$ (but we have already defined the latter). Note that, for example, the set $X^3 = X \times X \times X$ means something slightly different from $(X \times X) \times X$. However, for most purposes, these sets can be treated as if they were the same[3].

#### 3.4.2. *Sequences and Subsequences.*

DEFINITION 3.12. An $\mathbb{N}$-tuple of elements of $X$ is called a *sequence* in $X$, and is denoted

$$(x_j)_{j\in\mathbb{N}} = (x_j)_{j=1}^\infty = (x_1, x_2, \ldots).$$

REMARK 3.13. The image of a $J$-tuple $(x_j)_{j\in J}$ is denoted by

$$\{x_j\}_{j\in J} = \{x_j : j \in J\}.$$

---

[3]We give one caveat in the form of an example. Writing $(a, b) \in A^2 \times A$ implies that $a \in A^2$ and $b \in A$; this is *not* the same as writing $(a, b) \in A \times A^2$, which implies that $a \in A$ and $b \in A^2$. This demonstrates the difference between $(A \times A) \times A$ and $A \times (A \times A)$; the difference between these two sets and $A \times A \times A$ is similar in spirit.

For $n$-tuples and sequences, the natural additional notation is available:

$$\{x_j\}_{j\in J_n} = \{x_j : j \in J_n\} = \{x_j\}_{j=1}^n = \{x_1, x_2, \ldots, x_n\}.$$
$$\{x_j\}_{j\in\mathbb{N}} = \{x_j : j \in \mathbb{N}\} = \{x_j\}_{j=1}^\infty = \{x_1, x_2, \ldots\}$$

Do not confuse the two notations $(x_j)_{j=1}^\infty$ and $\{x_j\}_{j=1}^\infty$. The first is a sequence (which is a function); the second is the image of the sequence, which is a set. For example, if $x_i = 2$ for odd $i$ and $x_i = 3$ for even $i$, then the sequence $(x_i)_{i=1}^\infty$ is $(2, 3, 2, 3, 2, 3, \ldots)$, while the set $\{x_i\}_{i=1}^\infty$ is just $\{2, 3\}$. Similar remarks hold for ordered $n$-tuples. Unfortunately, some authors (including myself, previously, and also including Rudin) do not make this distinction, which can lead to a lot of confusion.

DEFINITION 3.14. Let $x = (x_j)_{j=1}^\infty$ be a sequence, and let $k = (k_\ell)_{\ell=1}^\infty$ be a strictly increasing sequence in $\mathbb{N}$ (i.e., $k(1) < k(2) < k(3) < \cdots$). Then the sequence $y := x \circ k$ is called a *subsequence* of $x$, and we often write $y = (y_\ell)_{\ell=1}^\infty = (x_{k_\ell})_{\ell=1}^\infty$.

REMARK 3.15. This is the commonly accepted definition of a subsequence, but it might seem rather opaque at first. In practice, given a sequence $x = (x_1, x_2, x_3, \ldots)$, we think of a subsequence as being formed by deleting some of the entries of $x$ and preserving the original order, i.e. $y = (x_2, x_3, x_5, x_7, \ldots)$. The above definition is just a way of writing this process. However, writing down the following equivalent notations may help clarify the picture:

$$y_\ell = y(\ell) = (x \circ k)(\ell) = x(k(\ell)) = x_{k(\ell)} = x_{k_\ell}.$$

EXAMPLE 3.16. If the sequence $(x_j)_{j=1}^\infty$ is given by $(x_j)_{j=1}^\infty = (1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots)$, then the subsequence $(y_\ell)_{\ell=1}^\infty = (x_{2\ell})_{\ell=1}^\infty$ is given by $(y_\ell)_{\ell=1}^\infty = (2, 1, 3, 2, 1, 3, \ldots)$.

3.4.3. *Indexed Sets.*

DEFINITION 3.17. If $X$ is a set and $X$ can be written $X = \{x_j\}_{j\in J}$ for some set $J$, then we say that $X$ is *indexed by* $J$, and we refer to the notation $\{x_j\}_{j\in J}$ as an *indexed set*.

Note that the function implicit in this definition is necessarily surjective when considered as a function from $J \to X$. This function could also be considered as a $J$-tuple of elements in some space $Y$ containing $X$; if this is the case we do *not* say that $Y$ is indexed by $J$. The function $J \to X$ (or $J \to Y$) is not in general required to be injective.

The most useful indexed sets are actually indexed collections: $\mathcal{A} = \{A_j\}_{j\in J}$. If $A_j$ is a set for each $j \in J$, we sometimes we refer to $\{A_j\}_{j\in J}$ as a *family of sets indexed by* $J$.

Indexed families of sets yield convenient notation for unions. If $\mathcal{A} = \{A_j\}_{j\in J}$, we write

$$\bigcup_{A\in\mathcal{A}} A = \bigcup_{j\in J} A_j.$$

For sets indexed by $J_n$ (for some $n \in \mathbb{N}$) or by $\mathbb{N}$, we often use the following notation:

$$\bigcup_{j\in J_n} A_j = \bigcup_{j=1}^n A_j = A_1 \cup A_2 \cup \cdots \cup A_n,$$

$$\bigcup_{j\in\mathbb{N}} A_j = \bigcup_{j=1}^\infty A_j = A_1 \cup A_2 \cup \cdots .$$

Entirely similar notation is used for intersections.

REMARK 3.18. Any set can be indexed by itself (although this is rather silly). For example, if $A = \{3, 4, 7\}$, then we can define a function $x : A \to A$ by the identity map (denoted temporarily by $x$ rather than by $\mathrm{id}_A$, so $x(3) = x_3 = 3$, etc.). In this case we have $A = \{x_a\}_{a\in A} = \{a\}_{a\in A}$. The purpose of this is to show that we don't lose anything if we consider unions, etc. only over indexed sets.

3.4.4. *Cartesian Products.*

DEFINITION 3.19. Let $X$ be a set; let $\mathcal{A} = \{A_j\}_{j\in J}$ be a collection of subsets of $X$ set indexed by $J$. The *cartesian product* of $\{A_j\}_{j\in J}$ is the set of all $J$-tuples $(x_j)_{j\in J}$ in $X$ such that $x_j \in A_j$ for each $j \in J$, and is denoted

$$\prod_{j\in J} A_j.$$

The above general definition is useful to have in some contexts; however, we will most often consider it in the special case where $J = J_n$. In that context, the definition reads as follows:

DEFINITION 3.20. Let $\{A_j\}_{j=1}^n$ be a collection of subsets of a set $X$. The *cartesian product* of the sets $\{A_j\}_{j=1}^n$ is the set of all $n$-tuples $(x_1, \ldots, x_n)$ in $X^n$ such that $x_j \in A_j$ for each $j \in J_n$.

We use the notation

$$\prod_{j\in J_n} A_j = \prod_{j=1}^n A_j = A_1 \times \cdots \times A_n.$$

Notice that this definition of $A_1 \times A_2$ agrees with the one we already have. We discussed the relationship between the sets $X^3 = X \times X \times X$ and $(X \times X) \times X$ above; similar comments hold for $A \times B \times C$ and $(A \times B) \times C$, for example.

## 4. Cardinality

### 4.1. Definitions and Notation.

DEFINITION 4.1. Let $A$ and $B$ be nonempty sets. We use the notation

$$\operatorname{card}(A) \le \operatorname{card}(B), \quad \operatorname{card}(A) \ge \operatorname{card}(B), \quad \operatorname{card}(A) = \operatorname{card}(B)$$

to mean that there exists a function $f : A \to B$ which is injective, surjective, or bijective, respectively. If there exists an injection $A \to B$ but not a bijection, then we write $\operatorname{card}(A) < \operatorname{card}(B)$; if there exists a surjection but not a bijection, we write $\operatorname{card}(A) > \operatorname{card}(B)$. The notation $\operatorname{card}(A)$ is called the *cardinality* of the set $A$.

The notation here is reminiscent of that of an ordering. This is mostly a manner of notational convenience. However, we do give some partial justification for the notation that is independently useful.

THEOREM 4.2. *Let $A$ and $B$ be sets.*

*(1) If $A \subset B$, then* $\operatorname{card}(A) \le \operatorname{card}(B)$.
*(2) If* $\operatorname{card}(A) \le \operatorname{card}(B)$ *and* $\operatorname{card}(B) \le \operatorname{card}(C)$, *then* $\operatorname{card}(A) \le \operatorname{card}(C)$.
*(3)* $\operatorname{card}(A) \le \operatorname{card}(B)$ *if and only if* $\operatorname{card}(B) \ge \operatorname{card}(A)$.
*(4) At least one of the statements* $\operatorname{card}(A) \le \operatorname{card}(B)$ *or* $\operatorname{card}(B) \le \operatorname{card}(A)$ *holds.*
*(5) (Schröder-Bernstein) If* $\operatorname{card}(A) \le \operatorname{card}(B)$ *and* $\operatorname{card}(B) \le \operatorname{card}(A)$, *then* $\operatorname{card}(A) = \operatorname{card}(B)$.

Statement (1) follows from the fact that the inclusion map $\iota : A \to B$ is an injection. Statement (2) follows from the fact that the composition of two injections is again an injection. Statement (3) follows from the fact that any injection $f : A \to B$ has a (necessarily surjective) left inverse $g : B \to A$, and any surjection $g : B \to A$ has a (necessarily injective) right inverse $f : A \to B$.

Statement (4) says that given any two sets $A$ and $B$, either there exists an injection from $A$ to $B$, or there exists an injection from $B$ to $A$ (or both). This statement may seem intuitive; however, for general sets $A$ and $B$, the proof of (4) requires the use of some set-theoretic machinery that we have not developed[4]. We will not prove statement (4) in general; however, we will treat some special cases below.

[4]Namely, it requires Zorn's Lemma. It turns out that Zorn's Lemma is equivalent to the Axiom of Choice. However, unlike the Axiom of Choice, invoking Zorn's Lemma is not something that will go unnoticed.

We do have the tools to prove the Schröder-Bernstein Theorem, but its proof is not very enlightening, so we omit it also.

This Theorem implies, roughly speaking, that $\leq$ 'acts like' an order relation.

We sometimes use the notation $A \sim B$ as shorthand for $\operatorname{card}(A) = \operatorname{card}(B)$. Note that $\sim$ satisfies the properties of an equivalence relation.

- (Reflexivity) $A \sim A$, for every set $A$.
- (Symmetry) If $A \sim B$, then $B \sim A$.
- (Transitivity) If $A \sim B$ and $B \sim C$, then $A \sim C$.

Therefore we sometimes say that $A$ and $B$ are *equivalent* if $A \sim B$.

**4.2. Generating Equivalences.** The following statement gives an extremely useful way to construct sets with the same cardinality from existing ones.

PROPOSITION 4.3. *Let $A$, $B$, $X$, and $Y$ be sets. If $A \cap B = X \cap Y = \emptyset$ and $A \sim X$, $B \sim Y$, then $A \cup B \sim X \cup Y$.*

The proof of this Proposition follows directly from Lemma 4.4 below. A particularly useful case of this Proposition is the case where $A = X$.

LEMMA 4.4. *Assume $A \cap B = X \cap Y = \emptyset$. Let $f : A \to X$ and $g : B \to Y$ be bijections, and define a function $h : (A \cup B) \to (X \cup Y)$ by the following rule:*

$$h(c) = \begin{cases} f(c) & \text{if } c \in A \\ g(c) & \text{if } c \in B. \end{cases}$$

*Then $h$ is a bijection.*

PROOF. Note that since $f$ and $g$ are bijections, we can define their inverses $f^{-1}$ and $g^{-1}$, respectively. Define a function $k : X \cup Y \to A \cup B$ by

$$k(z) = \begin{cases} f^{-1}(z) & \text{if } z \in X \\ g^{-1}(z) & \text{if } z \in Y. \end{cases}$$

One can easily check[5] that $k$ is a two-sided inverse for $h$. $\square$

A word of caution: If $X$ and $Y$ are not disjoint, then the function $h$ may not be bijective. If $A$ and $B$ are not disjoint, then the object $h$ as defined above might not even be a function!

Recall that the notation $B^A$ means the set of all functions from $A$ to $B$. The following exercise gives a useful way to generate equivalences.

EXERCISE 4.5. Assume that $\operatorname{card}(A) \leq \operatorname{card}(X)$ and $\operatorname{card}(B) \leq \operatorname{card}(Y)$. Prove that $\operatorname{card}(B^A) \leq \operatorname{card}(Y^X)$. Hint: Consider a function $\Phi : B^A \to Y^X$ of the form $\Phi(f) = h \circ f \circ k$, where $k : X \to A$ and $h : B \to Y$ are certain functions. Theorem 3.9 might be useful for your final step.

[5] To see that $k$ is a right inverse, note the following:

- If $z \in X$, then $k(z) = f^{-1}(z) \in A$, so $h(k(z)) = h(f^{-1}(z)) = f(f^{-1}(z)) = z$.
- If $z \in Y$, then $k(z) = g^{-1}(z) \in B$, so $h(k(z)) = h(g^{-1}(z)) = g(g^{-1}(z)) = z$.

The proof that $k$ is a left inverse is entirely similar.

Recall that for any set $A$, the notation $\mathcal{P}(A)$ is used to denote the *power set* of $A$, i.e., the collection of all subsets of $A$. Here is a useful *non*-equivalence:

PROPOSITION 4.6. *For any set $A$,* $\operatorname{card}(A) < \operatorname{card}(\mathcal{P}(A))$.

PROOF. Let $g : A \to \mathcal{P}(A)$ be any map; we show it is not surjective. Define $B := \{x \in A : x \notin g(x)\}$. Then $B \notin g(A)$. For if $g(x) = B$ for some $x \in A$, then $x$ cannot be in $B$ by definition of $B$; on the other hand, if $x \notin B = g(x)$, then $x \in B$ by definition of $B$. This contradiction establishes the claim. □

Thinking about cardinality as a notion of (relative) 'size', this Proposition shows that there is no limit to how 'big' sets can get; given any set $A$ we can always construct a 'bigger' one by taking the power set of $A$.

EXERCISE 4.7. Prove that for any set $A$, one has $\mathcal{P}(A) \sim \{0,1\}^A$.

**4.3. Finite and Infinite Sets.** For each $n \in \mathbb{N}$, let $J_n$ denote the set

$$J_n := \{1, \ldots, n\}.$$

It can be proved rigorously (by a rather tedious induction argument[6] that for $m, n \in \mathbb{N}$, one has

$$\operatorname{card}(J_m) \leq \operatorname{card}(J_n) \iff m \leq n.$$

Therefore it makes sense to define $\operatorname{card}(J_n) = n$ for all $n \in \mathbb{N}$. We also define $\operatorname{card}(\emptyset) = 0$.

DEFINITION 4.8. A set $A$ is called *finite* if $\operatorname{card}(A) = \operatorname{card}(J_n)$ for some $n \in \mathbb{N}$, or if $A = \emptyset$. A set is called *infinite* if it is not finite.

In light of these conventions, it makes sense to think of the cardinality of a finite set $A$ as the number of elements it contains. However, one should keep in mind that the statement $\operatorname{card}(A) = n$ really means that there exists a bijection $f : J_n \to A$. Furthermore, if $A$ is an infinite set, then the concept of 'number of elements in $A$' becomes rather vague.

For finite sets, we sometimes use the notation $|A| = \operatorname{card}(A)$.

PROPOSITION 4.9. *If $A$ and $B$ are finite, disjoint sets, then $A \cup B$ is finite, and* $|A \cup B| = |A| + |B|$.

PROOF. Since $A$ and $B$ are finite, there exist integers $m$ and $n$ such that

$$A \sim J_n,$$

$$B \sim J_m \sim \{m+1, \ldots, m+n\}.$$

Since both $A \cap B$ and $J_n \cap \{m+1, \ldots, m+n\}$ are empty, Lemma 4.4 tells us that

$$A \cup B \sim J_n \cup \{m+1, \ldots, m+n\} = J_{n+m},$$

i.e. $|A \cup B| = n + m = |A| + |B|$. □

[6] If $m \leq n$, then the inclusion map $\iota : J_m \to J_n$ is an injection, so $\operatorname{card}(J_m) \leq \operatorname{card}(J_n)$. We prove other direction, the statement "$\operatorname{card}(J_m) \leq \operatorname{card}(J_n)$ implies $m \leq n$", by induction on $n$. The only function $f : J_m \to J_1$ is defined by the rule $f(j) = 1$ for all $j \in J_m$. If $f$ is injective, then the fact that $f(1) = f(m)$ implies that $m = 1$. Next, assume the statement is true for $n = k \in \mathbb{N}$; we prove it for $n = k+1$. So, we start with an injection $f : J_m \to J_{k+1}$ and try to prove that $m \leq k+1$. If $m = 1$, we are already done, so we assume that $m \geq 2$ without loss of generality. It actually suffices to show that there exists a bijection $g : J_{m-1} \to J_k$. Indeed, assume such a $g$ exists. Then by our inductive hypothesis, we have $m - 1 \leq k$, i.e. $m \leq k+1$, which is what we are trying to prove. Therefore, let us construct an injection $g : J_{m-1} \to J_k$. We consider two cases. First, if $f(m) = k$, then we can simply define $g$ by $g(j) = f(j)$ for all $j \in J_{m-1}$ (i.e., by restricting both the domain and codomain of $f$). If $f(m) \neq k$, then define two functions $g : J_{m-1} \to J_k \backslash \{f(m)\}$ and $h : J_{k+1} \backslash \{f(m)\} \to J_k$ as follows. Define $g(j) = f(j)$ for each $j$; define $h(j) = j$ if $j \neq k$ and $h(k) = f(m)$. Then $g$ is an injection and $h$ is a bijection, so $h \circ g : J_{m-1} \to J_k$ is an injection.

COROLLARY 4.10. *If $A$ and $B$ are any finite sets (not necessarily disjoint), then $A \cup B$ is finite.*

PROOF. $A \cup B$ can also be written as the union of the two finite, disjoint sets $A \backslash B$ and $B$. □

PROPOSITION 4.11. *Let $A$ and $B$ be finite sets. Then $|A^B| = |A|^{|B|}$, and $|\mathcal{P}(A)| = 2^{|A|}$.*

PROOF. Since $A$ and $B$ are finite, there exist $n, m \in \mathbb{N}$ such that $A \sim J_n$ and $B \sim J_m$. By Exercise 4.5, it follows that $A^B \sim \{1, \ldots, n\}^m$, i.e. the set of $m$-tuples whose entries are the numbers $1, \ldots n$. Since there are $n$ choices for each of the $m$ entries, there are $n^m$ elements in this set. Thus $|A^B| = n^m = |A|^{|B|}$. This proves the first statement.

The second statement follows from the first statement, together with Exercise 4.7: Since $\mathcal{P}(A) \sim \{0,1\}^A$, we have $|\mathcal{P}(A)| = |\{0,1\}^A| = |\{0,1\}|^{|A|} = 2^{|A|}$. □

Note that the set $\mathbb{N}$ is *not* finite. Hopefully this doesn't come as a surprise, but a few words about the proof are in order: Suppose (to obtain a contradiction) that $\mathbb{N}$ is finite. Then there exists $n \in \mathbb{N}$ and a surjection $f : J_n \to \mathbb{N}$. On the other hand, it is easy to construct a surjection $g : \mathbb{N} \to J_{n+1}$. (For example, take $g(j) = j$ for $j \in J_{n+1}$, and $g(j) = 1$ for $j > n+1$.) But then $g \circ f$ is a surjection from $J_n$ to $J_{n+1}$, which is impossible. Therefore $\mathbb{N}$ cannot be finite. However, $\mathbb{N}$ is, in a sense, the 'smallest' infinite set. The following discussion makes this statement precise.

DEFINITION 4.12. Let $A$ be a set. We say that $A$ is *countable* if $\operatorname{card}(A) \leq \operatorname{card}(\mathbb{N})$. We say $A$ is *countably infinite* if $\operatorname{card}(A) = \operatorname{card}(\mathbb{N})$. We say $A$ is *uncountable* if it is not countable.

REMARK 4.13. A completely equivalent definition of 'countable' is the following: A set $A$ is countable if it can be indexed by $\mathbb{N}$, i.e. there exists a sequence $(a_j)_{j=1}^{\infty}$ whose image $\{a_j\}_{j=1}^{\infty}$ is equal to $A$. In this case the sequence $(a_j)_{j=1}^{\infty}$ is called an *enumeration* of the elements of $A$, since all the elements of $A$ appear in the 'list' $(a_1, a_2, a_3, \ldots)$.

PROPOSITION 4.14. *Let $A$ be an infinite set. Then $A$ contains a countably infinite subset.*

PROOF. Since $A$ is infinite, $A$ is not empty; therefore we can find an element of $A_1 := A$, which we denote $a_1$. The set $A_2 := A_1 \backslash \{a_1\}$ is still infinite. (If $A_2$ is finite, then so is $A_1 = A_2 \cup \{a_1\}$, since $A_2 = A_1 \backslash \{a_1\}$ and $\{a_1\}$ are finite disjoint sets.) Therefore we can find an element $a_2 \in A_2$, and $A_3 := A_2 \backslash \{a_2\} = A_1 \backslash \{a_1, a_2\}$ is still infinite. We can continue this process indefinitely, obtaining sets $A_k$ and elements $a_k$ of $A_k$ such that $A_{k+1} = A_k \backslash \{a_k\}$, for each $k \in \mathbb{N}$. Consequently, for $m < n$, we have

$$A_n = A_{n-1} \backslash \{a_{n-1}\} = A_{n-2} \backslash \{a_{n-2}, a_{n-1}\} = \cdots = A_m \backslash \{a_m, \ldots, a_{n-1}\},$$

so that $a_m \in A_m \backslash A_n$ whenever $m < n$. Since $a_n \in A_n$, it follows that $a_m \neq a_n$ for $m < n$. Therefore, the function $f : \mathbb{N} \to A$ defined by $f(n) = a_n$ is injective. Let $B$ denote the image of $f$. Then $B$ is a countably infinite subset of $A$. □

COROLLARY 4.15. *A set $A$ is infinite if and only if* $\operatorname{card}(A) \geq \operatorname{card}(\mathbb{N})$.

PROOF. If $A$ is infinite, then $A$ contains a countably infinite subset $B$. Therefore $\operatorname{card}(A) \geq \operatorname{card}(B) = \operatorname{card}(\mathbb{N})$. On the other hand, if $\operatorname{card}(A) \geq \operatorname{card}(\mathbb{N})$, then $\operatorname{card}(A) > \operatorname{card}(J_m)$ for every $m \in \mathbb{N}$, so $A$ is not finite. □

Another useful characterization of infinite sets is the following:

THEOREM 4.16. *A set $A$ is infinite if and only if there exists a proper subset $B \subsetneq A$ such that $A \sim B$.*

PROOF. Suppose $A$ is infinite; choose $a \in A$ and define the proper subset $B = A \backslash \{a\}$ of $A$. We prove that $A \sim B$. Now, $B$ is infinite, so it contains a countably infinite subset $C$, and $a \notin C$. The set $C \cup \{a\}$ is still countably infinite. (Indeed, if $f : \mathbb{N} \to C$ is a bijection, define a bijection $g : \mathbb{N} \to C \cup \{a\}$

by $g(1) = a$, $g(n) = f(n-1)$ for $n > 1$.) Since $(B\backslash C) \cap C$ and $(B\backslash C) \cap (C \cup \{a\})$ are both empty, we conclude that

$$B = (B\backslash C) \cup C \sim (B\backslash C) \cup (C \cup \{a\}) = B \cup \{a\} = A.$$

The equivalence $\sim$ follows from Proposition 4.3.

Suppose $A$ is finite. If $A$ contains no proper subsets, then we are done; otherwise let $B$ be any proper subset of $A$. Then $A\backslash B$ is nonempty, so $|A\backslash B| \geq 1$. Therefore $|A| = |A\backslash B| \cup |B| \geq 1 + |B| > |B|$. In particular, $|A| \neq |B|$, so the statement $A \sim B$ cannot hold. $\square$

We summarize the last few results:

$$\begin{aligned} A \text{ is infinite} &\iff \operatorname{card}(A) \geq \operatorname{card}(\mathbb{N}) \\ &\iff A \text{ contains a countably infinite subset} \\ &\iff A \sim B \text{ for some proper subset } B \text{ of } A. \end{aligned}$$

EXERCISE 4.17. Let $A$ and $B$ be sets, and assume $f : A \to B$ and $g : B \to A$ are injective functions.

(1) Assume additionally that $A$ is finite. Prove that $f$ and $g$ must actually be bijections.
(2) Show by way of an example that both $f$ and $g$ may fail to be bijective if we do not assume that $A$ is finite.

EXERCISE 4.18. Let $A$ and $B$ be sets. Assume $A$ is infinite, $B$ is countable, and $A$ and $B$ are disjoint. Prove that $A \sim A \cup B$. Hint: The strategy of Theorem 4.16 may be useful. You may also use the fact that the union of two countable sets is countable. (A more general statement is proved in the next subsection.)

EXERCISE 4.19. Let $X$ and $Y$ be sets. Assume $Y$ is countable and $X\backslash Y$ is infinite. Prove that $X \sim X \cup Y \sim X\backslash Y$. Hint: Each of the equivalences can be done extremely quickly if you use the previous exercise and some set manipulations.

### 4.4. Countable Sets.

PROPOSITION 4.20. $\mathbb{N} \times \mathbb{N}$ *is countably infinite.*

PROOF. The function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $f((n,m)) = 2^n \cdot 3^m$ is injective. Indeed, if $2^{n_1}3^{m_1} = 2^{n_2}3^{m_2}$ for some $n_1, n_2, m_1, m_2 \in \mathbb{N}$, then we must have $n_1 = n_2$ and $m_1 = m_2$, by uniqueness of prime decompositions. Therefore $(n_1, m_1) = (n_2, m_2)$, i.e., $f$ is injective. It follows that $\operatorname{card}(\mathbb{N} \times \mathbb{N}) \leq \operatorname{card}(\mathbb{N})$. On the other hand, the function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $g((n,m)) = n$ is a surjection, so $\operatorname{card}(\mathbb{N} \times \mathbb{N}) \geq \operatorname{card}(\mathbb{N})$. Thus $\operatorname{card}(\mathbb{N} \times \mathbb{N}) = \operatorname{card}(\mathbb{N})$, as needed. $\square$

If $\mathcal{A}$ is a countable collection of sets, we say that $\bigcup_{A \in \mathcal{A}} A$ is a *countable union*. If $\mathcal{A}$ is finite, we call it a *finite union*. (We use similar terminology for cartesian products.) The next proposition can be stated briefly as "A countable union of countable sets is countable."

PROPOSITION 4.21. *Let* $\{A_j\}_{j=1}^{\infty}$ *be a countable collection of countable sets. Then the union*

$$S = \bigcup_{j=1}^{\infty} A_j$$

*is also countable.*

PROOF. We define a function $f : \mathbb{N} \times \mathbb{N} \to S$ as follows. For each $j \in \mathbb{N}$, let $g_j : \mathbb{N} \to A_j$ be a surjection. Define $f((j,k)) = g_j(k)$ for each $(j,k) \in \mathbb{N} \times \mathbb{N}$. Then $f$ is surjective. Indeed, if $x \in S$, then $x \in A_j$ for some $j \in \mathbb{N}$. But then since $g_j$ is surjective, we have $x = g_j(k) = f(j,k)$ for some $k \in \mathbb{N}$. It follows that $\operatorname{card}(\mathbb{N}) = \operatorname{card}(\mathbb{N} \times \mathbb{N}) \geq \operatorname{card}(S)$. That is, $S$ is countable. $\square$

Note that this Proposition implies rather trivially that $\mathbb{Z}$ is countable. (Write $\mathbb{Z} = \mathbb{N} \cup (-\mathbb{N}) \cup \{0\}$, where $-\mathbb{N}$ denotes the set $\{-n\}_{n=1}^{\infty}$. Consequently, $\mathbb{Q}$ is countable, since the function $f : \mathbb{Z} \times (\mathbb{Z}\backslash\{0\}) \to \mathbb{Q}$ defined by $f((n,m)) = \frac{n}{m}$ is a surjection.

EXERCISE 4.22. Let $X$ be a countable set.

(1) Prove that $X^{n+1} \sim X^n \times X$ for any $n \in \mathbb{N}$. (This is not difficult, but be careful with the notation.)
(2) Prove inductively that $X^n$ is countable for any $n \in \mathbb{N}$.

Despite the result of this Exercise, we are able rather easily to prove that an infinite product of copies of a set $X$ is rarely countable, even if $X$ is finite. We make this precise in the next subsection.

### 4.5. Uncountable Sets.

PROPOSITION 4.23. *If $X$ is any set that contains at least two elements and $J$ is countably infinite, then $X^J$ is uncountable. In particular, $\{0,1\}^{\mathbb{N}}$ is uncountable.*

PROOF. Since $\operatorname{card}(X) \geq \operatorname{card}(\{0,1\})$ and $\operatorname{card}(J) \geq \operatorname{card}(\mathbb{N})$, we have

$$\begin{aligned} \operatorname{card}(X^J) &\geq \operatorname{card}(\{0,1\}^{\mathbb{N}}) && \text{by Exercise 4.5} \\ &= \operatorname{card}(\mathcal{P}(\mathbb{N})) && \text{by Exercise 4.7} \\ &> \operatorname{card}(\mathbb{N}) && \text{by Proposition 4.6.} \end{aligned}$$

□

This high-brow argument is nice in that it is concise, but doesn't give too much of an idea of what is going on here. We give a more down-to-earth proof of the following special case.

PROPOSITION 4.24. *$\{0,1\}^{\mathbb{N}}$ is uncountable.*

Remember that $X^J$ denotes the set of functions from $J$ to $X$. So $\{0,1\}^{\mathbb{N}}$ is the set of sequences in $\{0,1\}$. A typical element of $\{0,1\}^{\mathbb{N}}$ might look like $(0,1,0,0,0,1,0,1,1,\ldots)$, for example. The Proposition says that $\operatorname{card}(\{0,1\}^{\mathbb{N}}) > \operatorname{card}(\mathbb{N})$, i.e., there *does not* exist a surjection from $\mathbb{N}$ to $\{0,1\}^{\mathbb{N}}$. As with many nonexistence claims, this statement is proven by contradiction.

PROOF. Suppose $A := \{0,1\}^{\mathbb{N}}$ is countable, and let $(x_1, x_2, \ldots)$ be an enumeration of the elements of $A$. Remember that each $x_j$ is a sequence; let $a_{jk}$ denote the $k$th coordinate of the sequence $x_j$. Let the sequence $y = (y_\ell)$ be defined by the rule $y_\ell \neq a_{\ell\ell}$, for each $\ell \in \mathbb{N}$. (Since there are only two choices, $0$ and $1$, for $a_{\ell\ell}$, this definition is not ambiguous.) Then $y \neq x_j$ for any $j \in \mathbb{N}$ (since $y(j) \neq x_j(j) = a_{jj}$). Therefore, $y$ does not appear in the supposed enumeration $(x_1, x_2, \ldots)$, contradicting our assumption. □

REMARK 4.25. The idea of this proof is illustrated in the following diagram:

$$\begin{matrix} \mathbf{a_{11}} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{21} & \mathbf{a_{22}} & a_{23} & a_{24} & \cdots \\ a_{31} & a_{32} & \mathbf{a_{33}} & a_{34} & \cdots \\ a_{41} & a_{42} & a_{43} & \mathbf{a_{44}} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

We construct $y$ by flipping all the entries on the diagonal. For example, if $a_{11} = 1$, $a_{22} = 0$, $a_{33} = 0$, then we choose $y_1 = 0$, $y_2 = 1$, $y_3 = 1$, and so on. This argument is thus called *Cantor's diagonal argument*, after Georg Cantor, who introduced it in 1891.

CHAPTER 2

# The Real and Complex Number Systems

## 1. Ordered Sets and the Least Upper Bound Property

We begin by recalling the definition of an order, and introducing the concept of an ordered set.

DEFINITION 1.1. Let $S$ be a set. An *order* (or *total order*) on $S$ is a relation $\leq$, such that

- (Reflexivity) $a \leq a$, for all $a \in A$.
- (Antisymmetry) If $a \leq b$ and $b \leq a$, then $a = b$.
- (Transitivity) If $a \leq b$ and $b \leq c$, then $a \leq c$.
- (Comparability) If $a, b \in A$, then at least one of the statements $a \leq b$ or $b \leq a$ must hold.

An *ordered set* is a set $S$ together with an order $\leq$, denoted $(S, \leq)$.

Some comments on notation: First, for most of the ordered sets we consider, the specific order $\leq$ is obvious from context. In this case we refer to 'the ordered set $S$', even when we actually mean $(S, \leq)$. Other structures on sets are treated similarly. Second, we remind the reader that $x < y$ will mean $x \leq y$ and $x \neq y$.

Note that if $(S, \leq)$ is an ordered set and $T \subset S$, then $(T, \leq)$ is also an ordered set.

DEFINITION 1.2. Let $E$ be a subset of an ordered set $S$, and let $\alpha$ and $\beta$ be elements of $S$.

- If $x \leq \beta$ for all $x \in E$, then $\beta$ is called an *upper bound* for $E$, and we say that $E$ is *bounded above* if such a $\beta$ exists.
- If $\beta$ is an upper bound for $E$ in $S$ and $\beta \in E$, then $\beta$ is called the *maximum* of $E$. (This is written $\beta = \max E$.)
- Suppose $\alpha$ is an upper bound for $E$ in $S$ and that $\alpha \leq \gamma$ for any upper bound $\gamma$ of $E$ in $S$. Then $\alpha$ is called the *least upper bound* of $E$ in $S$, or the *supremum* of $E$ in $S$. For short, we write $\alpha = \sup E$ if this holds. (Usually the set $S$ is clear from context; if not, we can simply write the relevant statement in sentence form.)

The terms *lower bound*, *bounded below*, *minimum*, *greatest lower bound*, and *infimum* are defined analogously. If $\alpha$ is the infimum of a subset $E$ of $S$, we write $\alpha = \inf E$. If $\alpha$ is the minimum of $E$, we write $\alpha = \min E$. If $E$ is bounded above and bounded below, we sometimes simply say that $E$ is *bounded* (in $S$).

Notes: (i) Whenever a maximum $\alpha$ of $E$ exists, it is equal to the supremum of $E$. (More precisely, let $E$ be a subset of an ordered set $S$. Assume that $E$ has a maximum $\alpha$. Then $\alpha = \sup E$.) Indeed, if $\gamma < \alpha$, then $\gamma$ cannot be an upper bound for $E$, since $\alpha$ is an element of $E$ which is greater than $\gamma$. Similarly, whenever a minimum exists, it is equal to the infimum. Note that if $E$ is a finite subset of $S$, then the maximum and minimum of $E$ automatically exist.

(ii) One way we often prove that an upper bound $\alpha$ for $E$ in $S$ is in fact the *least* upper bound is by proving the contrapositive of the requirement in the definition: If $\gamma < \alpha$, then $\gamma$ is not an upper bound for $E$ in $S$.

(iii) If $X$ is a set, $A$ is a subset of $X$, $(S, \leq)$ is an ordered set, and $f : X \to S$ is a function, then the following notations are all considered equivalent:

$$\sup f(A) = \sup\{f(x) : x \in A\} = \sup_{x \in A} f(x) = \sup_A f.$$

Similar conventions are used for the infimum.

EXAMPLE 1.3. Let $E$ denote the set of all rational numbers of the form $2 - \frac{1}{n}$, where $n \in \mathbb{N}$. The supremum of $E$ in $\mathbb{Q}$ is 2. Indeed, $2 - \frac{1}{n} < 2$ for all $n$, so 2 is an upper bound for $E$. On the other hand, if $q$ is a rational number less than 2, let $n$ be any natural number greater than $(2-q)^{-1}$. Then $\frac{1}{n} < 2 - q$, so $2 - \frac{1}{n} > q$, which implies that $q$ is not an upper bound for $E$. Since $2 \notin E$, it follows that $E$ has no maximum. On the other hand, 1 is the minimum of $E$, therefore also the infimum of $E$ in $\mathbb{Q}$.

Notes: (i) This example shows that the least upper bound and greatest lower bound, when they exist, may or may not be elements of the set $E$.

(ii) In this example, the supremum of $E$ exists in $\mathbb{Q}$. However, there are bounded subsets of $\mathbb{Q}$ for which no least upper bound exists. (Consider for example $\{r \in \mathbb{Q} : r^2 < 2\}$. We will prove later that this set has no least upper bound in $\mathbb{Q}$.) That is, $\mathbb{Q}$ does not have the *least-upper-bound property*, defined below. Actually, this turns out to be the key difference between $\mathbb{Q}$ and $\mathbb{R}$. We will return to this point later.

DEFINITION 1.4. An ordered set $S$ is said to have the *least-upper-bound property* (LUBP) if the following statement holds: "Whenever $E$ is a nonempty subset of $S$ that is bounded above, it follows that $E$ has a least upper bound in $S$." The *greatest-lower-bound property* (GLBP) is defined similarly.

The greatest-lower-bound property won't make much more of an appearance in these notes, since it turns out to be equivalent to the least upper bound property:

THEOREM 1.5. *Let $S$ be an ordered set. Then $S$ has the least-upper-bound property if and only if it has the greatest-lower-bound property.*

PROOF. We will prove only one direction here, namely that the LUBP implies the GLBP. The other direction is quite similar.

Assume then that $S$ has the LUBP. We need to show that if $B \subset S$ is bounded below, then it has a greatest lower bound. Let $L$ denote the set of lower bounds for $B$. Since $B$ is bounded below, we know $L$ is nonempty. Furthermore, $L$ is bounded above, by any element of $B$[1]. Since $S$ has the least-upper-bound property, we may conclude that $L$ has a least upper bound $\alpha$ in $S$. We claim that $\alpha$ is also the greatest lower bound for $B$.

First, we recall that every $x \in B$ is an upper bound for $L$. Then since $\alpha$ is the *least* upper bound, we conclude that $\alpha \leq x$ for all $x \in B$. This says exactly that $\alpha$ is a lower bound for $B$.

Next, if $\gamma$ is any lower bound for $B$, then $\gamma \in L$, so that $\gamma \leq \alpha$, as $\alpha$ is an upper bound for $L$. Therefore $\alpha$ is the *greatest* lower bound for $B$. □

EXERCISE 1.6. Let $E$, $F$, and $G$ be nonempty subsets of an ordered set $(S, \leq)$. Prove the following statements.

(1) If $\alpha \in S$ is a lower bound for $E$ and $\beta \in S$ is an upper bound for $E$, then $\alpha \leq \beta$.
(2) $\sup E \leq \inf F$ if and only if $x \leq y$ for any $x \in E$, $y \in F$.
(3) If $E \subset G$, then $\sup E \leq \sup G$.

EXERCISE 1.7. Let $(S, \leq)$ be an ordered set, let $f$ and $g$ be functions from $X$ to $S$ and let $A$ be a subset of $X$. Assume that $f(x) \leq g(x)$ for all $x \in A$, and that furthermore $\sup_A f$ and $\sup_A g$ exist in $S$. Prove that $\sup_A f \leq \sup_A g$.

---

[1]After all, $L$ consists of lower bounds for $B$. So if $\gamma \in L$, then $\gamma \leq x$ for all $x \in B$, by definition of lower bound. On the other hand, if we fix $x \in B$, then $\gamma \leq x$ for *every* $\gamma \in L$. So $x$ is an upper bound for $L$.

## 2. Fields and Ordered Fields

The concept of a field should be familiar from linear algebra courses. Here we'll just review the definition.

DEFINITION 2.1. A *field* is a set $F$ which has two operations, called *addition* (denoted by $+$) and *multiplication* (denoted either by $\cdot$ or simply by juxtaposition), such that the following *field axioms* hold:

(1) $F$ is closed under addition and multiplication. That is, if $x$ and $y$ are elements of $F$, then so are $x + y$ and $xy$.
(2) Addition and multiplication are commutative and associative.
(3) Addition and multiplication each have identity elements (usually denoted $0$ and $1$, respectively) which are distinct.
(4) Each element of $F$ has both an additive and a multiplicative inverse (with the exception of $0$, which does not have a multiplicative inverse). That is, if $x \in F$, there exists $y \in F$ (the additive inverse) such that $x + y = 0$; if additionally $x \neq 0$, then there exists $z \in F$ (the multiplicative inverse) such that $xz = 1$. Usually the additive inverse of $x$ is denoted $-x$ and the multiplicative inverse of $x \neq 0$ is denoted $x^{-1}$ or $1/x$.
(5) The distributive law holds: $x(y + z) = xy + xz$.

Note that we use the notation for addition and multiplication that is usually associated to addition and multiplication of real numbers. Keep in mind that in principle these symbols could take other meanings. Furthermore, if more than one field is under consideration, alternate notation may be used.

DEFINITION 2.2. An *ordered field* is a field $F$ which is also an ordered set, whose order relation $\leq$ satisfies the following:

(1) If $x, y, z \in F$ and $y \leq z$, then $x + y \leq x + z$.
(2) If $x, y \in F$ and $x > 0$, $y > 0$, then $xy > 0$.

If $x > 0$ we say that $x$ is *positive*; if $x < 0$ we say that $x$ is *negative*.

Examples: Recall that $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are all ordered sets, under the usual ordering. But even though the usual addition and multiplication operations can be defined on $\mathbb{N}$ and $\mathbb{Z}$, neither of these is a field, since, for example, the requirement of multiplicative inverses fails for each. On the other hand, $\mathbb{Q}$ is a field under the usual addition and multiplication operations. In fact, it is an ordered field, as one can check.

Not all fields are ordered fields. For example, the usual addition and multiplication operations on the set $\mathbb{C}$ of complex numbers (introduced later) make $(\mathbb{C}, +, \cdot)$ into a field. However, it is provably impossible to define an order $\leq$ on $(\mathbb{C}, +, \cdot)$ such that $(\mathbb{C}, +, \cdot, \leq)$ is an ordered field.

EXERCISE 2.3. Let $A$ be a nonempty subset of an ordered field $(F, +, \cdot, \leq)$. Assume that $\sup A$ and $\inf A$ exist in $F$, and let $c$ be any element of $F$. Define the set $cA := \{ca : a \in A\}$.

(1) Prove that if $c \geq 0$, then $\sup(cA) = c \sup A$.
(2) What is $\sup(cA)$ if $c < 0$? Prove that your answer is correct.

EXERCISE 2.4. Let $A$ and $B$ be nonempty subsets of an ordered field $(F, +, \cdot, \leq)$. Assume $\sup A$ and $\sup B$ exist in $F$. Define $A+B := \{a+b : a \in A, b \in B\}$. Prove that $\sup(A+B) = \sup A + \sup B$ by filling in the details of the following outline:

- Denote $s = \sup A$, $t = \sup B$. Then $s + t$ is an upper bound for $A + B$.
- Let $u$ be any upper bound for $A + B$, and let $a$ be any element of $A$. Then $t \leq u - a$.
- We have $s + t \leq u$. Consequently, $\sup(A+B)$ exists in $F$ and is equal to $s+t = \sup A + \sup B$.

EXERCISE 2.5. Let $f$ and $g$ be functions from a set $X$ to an ordered field $(F, +, \cdot, \leq)$. Let $A$ be a subset of $X$.

(1) Prove that the following inequality holds, assuming the relevant suprema all exist.

$$\sup_{x\in A}(f(x)+g(x)) \le \sup_{x\in A} f(x) + \sup_{x\in A} g(x). \tag{$*$}$$

(2) Show by way of an example that equality might not hold in $(*)$, even if the suprema all exist. (Hint: This is probably easiest if you choose $X$ to be a set with two elements, and $F = \mathbb{Q}$.)

DEFINITION 2.6. We say that an ordered field $F$ has the *Archimedean property* if for every $x, y \in F$, with $x > 0$, there is a positive integer $n$ such that $nx > y$.

Note: In this definition, we think of $x > 0$ as being small and $y$ being (possibly) large.

PROPOSITION 2.7. *$\mathbb{Q}$ has the Archimedean property.*

PROOF. Let $p, q$ be rational numbers with $p > 0$. We need to show that $np > q$ for some $n \in \mathbb{N}$. If $q \le 0$ there is nothing to show, so we assume without loss of generality that $q > 0$. Write $p = a/b$ and $q = c/d$, where $a, b, c, d \in \mathbb{N}$. We seek $n \in \mathbb{N}$ such that $n(a/b) > c/d$, or clearing denominators, $nad > bc$. Since $nad \ge n$ (as $a, d \in \mathbb{N}$ and hence are each at least 1), it suffices to choose $n > bc$, say $n = bc + 1$. This is an integer because $b$ and $c$ are integers. Let's check that this choice works. We have

$$np = (bc+1)p = (pb)c + p = ac + p = a \cdot \frac{c}{d} \cdot d + p = adq + p \ge q + p > q.$$

Thus $\mathbb{Q}$ has the Archimedean property, as claimed. □

## 3. The Problem with $\mathbb{Q}$

Working in $\mathbb{Q}$ rather than $\mathbb{R}$ has its shortcomings. Most likely the reader is already convinced of this fact; however, it might be less clear what the most fundamental shortcomings actually are. First of all, square roots are not guaranteed to exist:

PROPOSITION 3.1. *There is no rational number $p$ such that $p^2 = 2$.*

PROOF. Assume that $p$ is a rational number such that $p^2 = 2$. Since $p$ is rational, we can write $p = \frac{m}{n}$, where $m$ and $n$ are integers with no common factors. The equation $p^2 = 2$ can thus be rewritten as $m^2 = 2n^2$. This implies that $m^2$ is even, which implies that $m$ is even, which in turn implies that $m^2$ is actually divisible by $4$. But then $m^2/2 = n^2$ is even, so $n$ is even. But evenness of both $m$ and $n$ is incompatible with our initial assumption that $m$ and $n$ have no common factors! We conclude that such a rational number $p$ cannot exist. □

The above proof can be modified to prove the irrationality of many square roots. However, the lack of square roots is not the end of the problem. It turns out that the fundamental shortcoming of $\mathbb{Q}$ is the fact that it does not have the least upper bound property. We prove this fact by giving an explicit example of a set nonempty subset $A$ of $\mathbb{Q}$ whose supremum does not exist in $\mathbb{Q}$. The set we use is the set of rational numbers between 0 and $\sqrt{2}$. However, we have to be a bit careful about how we state and prove our claim, since $\sqrt{2}$ is an object for which we don't yet have a rigorous definition.

THEOREM 3.2. *Define $A = \{r \in \mathbb{Q} : r^2 < 2\}$. Then $A$ has no least upper bound in $\mathbb{Q}$. Consequently, $\mathbb{Q}$ does not have the least-upper-bound property.*

We give the proof after a rather extended discussion into its strategy. Clearly $A$ is nonempty and bounded above; 2 is an upper bound, for example. So proving that $A$ has no least upper bound will prove that $\mathbb{Q}$ does not have the least-upper-bound property. We break up the proof into two steps:

(1) $p \in \mathbb{Q}$ is an upper bound for $A$ if and only if $p^2 > 2$ and $p > 0$.
(2) If $p \in \mathbb{Q}$, $p^2 > 2$ and $p > 0$, then there exists $q \in \mathbb{Q}$ such that $0 < q < p$ and $q^2 > 2$.

Suppose these two statements are proven, and let $p \in \mathbb{Q}$ be an upper bound for $A$. Then $p^2 > 2$ and $p > 0$ by (1); then by (2) there exists $q \in \mathbb{Q}$ such that $0 < q < p$ and $q^2 > 2$, which implies (by (1) again) that $q$ is an upper bound for $A$. But since $q < p$, it follows that $p$ is not the least upper bound for $A$. But $p$ was an arbitrary upper bound for $A$ in $\mathbb{Q}$; therefore no upper bound for $A$ in $\mathbb{Q}$ can be the least upper bound in $\mathbb{Q}$. Thus $A$ has no least upper bound in $\mathbb{Q}$.

Though statement (1) probably looks intuitive, it still requires a bit of untangling. It should be clear that $p^2 > 2$ and $p > 0$ together imply that $p$ is an upper bound for $\mathbb{Q}$. (We'll write this down explicitly in the actual proof, though.) To rigorously show that '$p$ is an upper bound' implies $p^2 > 2$, though, we need to argue the contrapositive. That is, we need to show that if $p^2 \leq 2$ and $p \in \mathbb{Q}$, then $p$ is not an upper bound for $A$ in $\mathbb{Q}$. We have just shown that $p^2 \neq 2$, so we essentially need to show that if $p^2 < 2$ (and $p > 0$, without loss of generality), then there exists $r \in A$ such that $p < r$. This should look similar to statement (2). It turns out that in order to prove statements (1) and (2), the relevant task is the following:

*Given $p \in \mathbb{Q}$ such that $p > 0$, find $q \in \mathbb{Q}$ strictly in between $p$ and $\sqrt{2}$.*

The reader might object that the use of $\sqrt{2}$ in this reasoning is 'cheating', since we don't yet have access to the full real number system. To this, my response is the following: 'Cheating' like this is completely fair game when trying to figure out how to complete a problem, provided that the logic of the actual proof doesn't rely on it. In this case, we will see that the quantity $\sqrt{2}$ does not actually appear anywhere in the proof.

Let's consider first the case where $p > \sqrt{2}$. We want to find a rational number $q$ such that $\sqrt{2} < q < p$. Our first attempt might be to subtract some small (rational) number $\varepsilon > 0$ from $p$. But this strategy will only work for some $p$'s; once our $p$'s get very close to $\sqrt{2}$, we will need to choose a different $\varepsilon$. That is, the quantity we subtract from $p$ will itself need to depend on $p$. So, what's a small, positive rational number that depends on $p$? we know that $p^2 - 2 > 0$, so we might try something like $q = p - \varepsilon_p$, where $\varepsilon_p$ involves the expression $p^2 - 2$ somehow. In order to guarantee that $\varepsilon_p$ is small enough, in the sense that we still have $(p - \varepsilon_p)^2 > 2$, we reverse engineer a little more. We need $p - \sqrt{2} > \varepsilon_p > 0$; we try setting $\varepsilon_p = \frac{p^2-2}{r}$, where $r$ is some positive rational number. (Then $\varepsilon_p$ will still be a rational number, since $\mathbb{Q}$ is a field.) How large must we take $r$ in order to guarantee that $\frac{p^2-2}{r} < p - \sqrt{2}$? Well, rearranging, we get

$$r > \frac{p^2 - 2}{p - \sqrt{2}} = \frac{(p - \sqrt{2})(p + \sqrt{2})}{p - \sqrt{2}} = p + \sqrt{2}.$$

So we put $r = p + 2$, and $\varepsilon_p = \frac{p^2-2}{p+2}$. Note that the '2' in the equation $r = p + 2$ could have been any rational number greater than $\sqrt{2}$. We suspect now that our choice gives $p > p - \varepsilon_p > \sqrt{2}$, which is essentially what we need. Actually, the case where $p < \sqrt{2}$ can be dealt with using the *same choice of* $q$! The reader should take a moment to convince themself that this is to be expected before reading the actual proof.

Having brainstormed the strategy of the proof, we can now write it out fairly succinctly.

PROOF OF THEOREM 3.2. Given $p \in \mathbb{Q}$, $p > 0$, set $q = p - \frac{p^2-2}{p+2}$. We claim that $q$ is a positive rational number, and that $q^2$ is always between 2 and $p^2$. Indeed, $q \in \mathbb{Q}$ follows from the fact that $\mathbb{Q}$ is a field. Next,

$$p - q = \frac{p^2 - 2}{p + 2};$$

therefore $p - q$ and $p^2 - 2$ have the same sign. (So if $p^2 > 2$, then $p > q$; if $p^2 < 2$, then $p < q$.) Next, we write

$$q = \frac{p(p+2)}{p+2} - \frac{p^2 - 2}{p+2} = \frac{2p + 2}{p + 2} = \frac{2(p+1)}{p+2}.$$

This shows that $q > 0$; furthermore,

$$q^2 - 2 = \frac{4(p+1)^2}{(p+2)^2} - \frac{2(p+2)^2}{(p+2)^2} = \frac{4(p^2+2p+1) - 2(p^2+4p+4)}{(p+2)^2}$$
$$= \frac{2p^2-4}{(p+2)^2} = \frac{2(p^2-2)}{(p+2)^2}.$$

Thus $q^2-2$ and $p^2-2$ have the same sign. (So if $p^2 > 2$, then $q^2 > 2$; if $p^2 < 2$, then $q^2 < 2$.) Combining this with the previous step, we conclude that either $p^2 < q^2 < 2$, or $p^2 > q^2 > 2$.

Now we prove that $A$ has no least upper bound in $\mathbb{Q}$, in two steps. First, we claim that if $p$ is an upper bound for $A$ in $\mathbb{Q}$ then $p^2 > 2$. Indeed, we have shown above that if $p \in \mathbb{Q}$, then $p^2$ cannot be equal to 2; therefore if $p^2 > 2$ fails, we must have $p^2 < 2$. But then $q$ as defined above satisfies $p^2 < q^2 < 2$, $q > 0$, $q \in \mathbb{Q}$. Therefore $q \in A$ and $p < q$. It follows that $p$ is not an upper bound for $A$. This proves that if $p$ is an upper bound for $A$ in $\mathbb{Q}$, then $p^2 > 2$.

Now let $p$ be any upper bound for $A$ in $\mathbb{Q}$. We show that $p$ is not the *least* upper bound for $A$ in $\mathbb{Q}$. Indeed, if $p^2 > 2$, then $q$ as defined above satisfies $p^2 > q^2 > 2$, $q > 0$, $q \in \mathbb{Q}$. It follows that $q$ is strictly less than $p$; we claim that additionally $q$ is an upper bound for $A$. Indeed, if there exists $r \in A$ such that $r > q$, then $r^2 > q^2 > 2$, contradicting the definition of $A$. Therefore $q$ is an upper bound for $A$ which is strictly less than $p$. So $p$ is not an upper bound. We conclude that $A$ has no least upper bound in $\mathbb{Q}$, as desired.

Since $A$ is nonempty (as $1 \in A$) and bounded above (by 2, for instance), but $A$ has no least upper bound in $\mathbb{Q}$, it follows that $\mathbb{Q}$ does not have the least upper bound property. $\square$

A remark on reading proofs: The extensive motivation that preceded the proof in this case is a luxury you as a reader won't always have. Without it, however, you might be convinced of the truth of the claimed statement, but without any idea of how the writer came up with the strategy. If reading a proof is to have any benefit to your ability to write similar proofs, you should make sure you understand both how *and why* each step follows logically from the previous ones. This means that you will have to deconstruct some of the proofs you see, in a manner similar to how we motivated this proof.

EXERCISE 3.3. Using the strategies similar to those of the proofs in this section, prove the following statements.

(1) There is no rational number whose square is 20.
(2) The set $A := \{r \in \mathbb{Q} : r^2 < 20\}$ has no least upper bound in $\mathbb{Q}$.

Hint: Most of the solution for both parts can be directly copied from the proof of the corresponding result in this section. The key differences are as follows: In (1), the 'common factor' of $m$ and $n$ from the proof of Proposition 3.1 needs to be modified in order to reach a contradiction in the present circumstances; in (2), the number $q$ and the associated calculations from the proof of Theorem 3.2 require modification.

## 4. Definition and Basic Properties of $\mathbb{R}$

THEOREM 4.1. *There exists an ordered field which has the least-upper-bound property. This field is unique up to isomorphism, and it contains $\mathbb{Q}$ as a subfield.*

We will not prove this Theorem. Everything except for the uniqueness is proven in Rudin's *Principles of Mathematical Analysis*, to which we refer the reader.

DEFINITION 4.2. $(\mathbb{R}, +, \cdot, \leq)$ is defined to be the field from the Theorem above.

In this section, we prove the following three properties of $\mathbb{R}$:

(1) $\mathbb{R}$ has the Archimedean property.
(2) $\mathbb{Q}$ is dense in $\mathbb{R}$.

(3) $n$th roots of positive real numbers exist in $\mathbb{R}$.

PROPOSITION 4.3. *$\mathbb{R}$ has the Archimedean property.*

PROOF. Choose real numbers $x$ and $y$ such that $x > 0$. We need to show that $nx > y$ for a sufficiently large integer $n$. This lends itself well to an argument by contradiction: If $nx \leq y$ for all $n \in \mathbb{N}$, then $y$ is an upper bound for the (nonempty) set $A := \{nx : n \in \mathbb{N}\}$. Since $\mathbb{R}$ has the least upper bound property, $A$ has a least upper bound $\alpha$ in $\mathbb{R}$. Since $\alpha$ is the least upper bound, $\alpha - x$ is not an upper bound for $A$, i.e. there is an integer $n$ for which $nx > \alpha - x$. But then $(n+1)x > \alpha$, contradicting the fact that $\alpha$ is an upper bound for $A$. $\square$

Recall that $\mathbb{Q}$ has the Archimedean property as well. This might cause one to wonder whether the LUBP is really necessary when proving that $\mathbb{R}$ has the Archimedean property. Technically, the answer to this question is negative; one does *not* need the LUBP. However, the proof that avoids the LUBP uses the explicit construction of $\mathbb{R}$ from $\mathbb{Q}$. The proof above is much simpler.

PROPOSITION 4.4. *If $x, y \in \mathbb{R}$ and $x < y$, then there exists a $p \in \mathbb{Q}$ such that $x < p < y$.*

This statement is sometimes rephrased as saying that $\mathbb{Q}$ is *dense* in $\mathbb{R}$. We will later meet another, more general definition of the term *dense*, in the context of metric (or topological) spaces. In the limited context of the real line, these two definitions are the same.

Before we write down the proof, we give the idea. For fixed $x$ and $y$ (not equal), it would be easier to prove the existence of a rational number between them if we knew they were far apart. In fact, if we know $y - x > 1$, then we can find an *integer* between them. This is a pretty obvious statement, but it's a helpful step to record explicitly.

LEMMA 4.5. *If $x$ and $y$ are real numbers with $y - x > 1$, then there exists an integer $m$ such that $x < m < y$.*

PROOF. let $m$ be the smallest integer greater than $x$. Then $m - 1 < x$ and $x < y - 1$, so $m < y$. Thus $x < m < y$. $\square$

Now, back to the discussion of the density of $\mathbb{Q}$ in $\mathbb{R}$. Even if $x$ and $y$ are close together, we can 'zoom in' by multiplying both by some large integer $n$, so that $nx$ and $ny$ are far apart. Then we can find an integer $m$ between $nx$ and $ny$, which guarantees that $m/n$ is between $x$ and $y$. The formal argument is below.

PROOF OF PROPOSITION 4.4. Choose $n$ large enough so that $ny - nx > 1$. This is possible by the Archimedean property of $\mathbb{R}$ because $ny - nx = n(y - x)$ and $y - x > 0$. Let $m$ be the smallest positive integer greater than $nx$. We claim that $x < m/n < y$; since $nx < m$ by definition of $m$, it remains to show that $m < ny$. Now, since $m$ is the *smallest* integer greater than $nx$, we have $m - 1 \leq nx$. On the other hand, our choice of $n$ guarantees that $nx < ny - 1$. Combining these two inequalities gives $m - 1 \leq nx < ny - 1$, or adding $1$ to both sides, $m < ny$. This finishes the proof. $\square$

Elements of $\mathbb{R} \backslash \mathbb{Q}$ are called *irrational numbers*. We can already prove that irrational numbers exist. Indeed, we have shown that $\mathbb{Q}$ does not have the least upper bound property; i.e., there exists a subset $A$ of $\mathbb{Q}$ which is nonempty and bounded above, such that $A$ has no least upper bound in $\mathbb{Q}$. But $A$ must have a least upper bound $x$ in $\mathbb{R}$; this $x$ cannot be rational (otherwise it would be least upper bound of $A$ in $\mathbb{Q}$, contrary to assumption), therefore $x \in \mathbb{R} \backslash \mathbb{Q}$.

EXERCISE 4.6. Prove the following statements about rational and irrational numbers.

(1) Assume $r$ is rational and $x$ is irrational. Show that $r + x$ is irrational. Show that $rx$ is irrational unless $r = 0$.
(2) Use the Archimedean property of $\mathbb{R}$ to prove that the set of irrational numbers is dense in $\mathbb{R}$. (Hint: Let $x$ be any positive irrational number. If $y$ and $z$ are are real numbers with $z - y > x$, then there exists an integer $m$ such that $y < mx < z$.)

We know that $\mathbb{R}$ is an ordered field with the least upper bound property; therefore the set $A = \{r \in \mathbb{Q} : r^2 < 2\}$ from the previous section has an upper bound in $\mathbb{R}$. Intuitively, it should be clear that the only reasonable candidate for $\sup A$ should be the object that we normally call $\sqrt{2}$. However, note carefully that we have not yet proven that $(\sup A)^2 = 2$. The following Proposition puts our intuition on rigorous footing, and actually establishes the existence of $n$th roots (not just square roots) for positive real numbers.

PROPOSITION 4.7. *For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real $y$ such that $y^n = x$.*

A couple of notes before the proof: Whenever a statement claims existence of an object satisfying some properties, the first step in the proof should be to come up with a candidate for that object. (It's pretty hard to prove properties of an object which has no specified identity.) In this case, the object should be the supremum of the set $E := \{t \in \mathbb{R},\, t > 0 : t^n < x\}$. Basically, the logic of (the existence part of) the proof is:

- $\alpha = \sup E$ exists.
- If $\beta^n \neq x$, then $\beta \neq \sup E$.
- Therefore $\alpha^n = x$.

The strategy for the first step should be clear: Show that $E$ is nonempty and bounded above, then use the LUBP of $\mathbb{R}$. The last step obviously follows from the first two. On the other hand, the second step deserves a closer look.

The statement $\beta \neq \sup E$ means that exactly one of the following two statements holds.

- $\beta$ is not an upper bound for $E$, or
- $\beta$ is an upper bound for $E$, but $\beta$ is not the *least* upper bound for $E$.

That is, either (1) $\beta$ is too small to be an upper bound, or (2) $\beta$ is too large to be the least upper bound. This dichotomy might seem pedantic, but it is useful when comparing to the statement we begin with, namely $\beta^n \neq x$, which splits into a parallel dichotomy: either $\beta^n < x$ or $\beta^n > x$. Now the strategy should be clearer.

To prove $\beta^n \neq x$ implies $\beta \neq \sup E$, it suffices to show

- If $\beta^n < x$, then $\beta$ is not an upper bound for $E$.
- If $\beta^n > x$, then $\beta$ is not the *least* upper bound for $E$.

We are now ready to begin the proof. This is another proof where some rather clever choices must be made; some of the motivation for these choices is described in the footnotes.

PROOF. Before we get to the existence part of the statement, we deal with the much easier uniqueness claim; that is, we show that there can be *at most* one positive $y$ satisfying $y^n = x$. Indeed, if $y_1$ and $y_2$ are distinct positive numbers, then one is bigger, say $y_2$, and then $y_1^n < y_2^n$. In particular, $y_1^n$ and $y_2^n$ cannot both be equal to $x$.

Now we prove existence of such a $y$, following the outline in the remarks above. As above, define $E := \{t \in \mathbb{R},\, t > 0 : t^n < x\}$.

Step 1: We show that $\alpha = \sup E$ exists. First, we show that $E$ is not empty. Indeed, define $t_1 := \frac{x}{x+1}$. Then $t_1 < 1$ implies $t_1^n < t_1$; since $t_1 < x$ we also have $t_1^n < x$, i.e. $t_1 \in E$. Next, $E$ is bounded above.

Indeed, define $t_2 := x + 1$; we show that $t \leq t_2$ for every $t \in E$, i.e. $t_2$ is an upper bound for $E$. Actually, we show the contrapositive: whenever $t > t_2$ it follows that $t \notin E$. Indeed, since $t_2 > 1$, we have $t^n > t_2^n > t_2 > x$ in this case. This proves that $x + 1$ is an upper bound for $E$. Now since $\mathbb{R}$ has the least upper bound property, it follows that $E$ has a least upper bound $\alpha$ in $\mathbb{R}$.

Step 2: We show that if $\beta^n < x$, then $\beta$ is not an upper bound for $E$. The statement is obvious if $\beta \leq 0$, so we consider only the nontrivial case $\beta > 0$. To show that $\beta$ is not an upper bound, we show that there exists $\varepsilon > 0$ such that $\gamma := \beta + \varepsilon$ is an element of $E$. In particular, we can choose[2]

$$\varepsilon = \min\left\{1, \frac{x - \beta^n}{n(\beta+1)^{n-1}}\right\}.$$

Then

$$\gamma^n - \beta^n < \varepsilon n \gamma^{n-1} \leq \varepsilon n (\beta+1)^{n-1} \leq \frac{x-\beta^n}{n(\beta+1)^{n-1}} \cdot n(\beta+1)^{n-1} = x - \beta^n.$$

Thus $\gamma^n < x$, so $\gamma \in E$. Thus $\beta$ is not an upper bound for $E$. We conclude that $\alpha^n \geq x$.

Step 3: We show that if $\beta^n > x$, then $\beta$ is not the *least* upper bound for $E$. As in the previous step, we assume without loss of generality that $\beta > 0$. If $\beta^n > x$, put $\gamma = \beta - \varepsilon$ and choose $\varepsilon > 0$ small enough so that $\beta^n - \gamma^n < \beta^n - x$. In particular, we can take[3]

$$\varepsilon = \frac{\beta^n - x}{n\beta^{n-1}}.$$

Then

$$\beta^n - \gamma^n < \varepsilon n \beta^{n-1} = \beta^n - x,$$

and rearranging gives $\gamma^n > x$. Therefore, if $t > \gamma$, then $t^n > \gamma^n > x$, so $t \notin E$. Thus $\gamma$ is an upper bound for $E$ that is strictly smaller than $\beta$, i.e. $\beta$ is not the least upper bound for $E$.

Steps 2 and 3 show collectively that if $\beta^n \neq x$, then $\beta \neq \sup E$. Therefore $\alpha^n = x$; that is, $\alpha$ is the desired $n$th root of $x$. □

EXERCISE 4.8. Assume $a, b \in \mathbb{R}$. Prove that $a \leq b$ if and only if $a \leq b + \varepsilon$ for every $\varepsilon > 0$.

EXERCISE 4.9. Let $E$ be a set of real numbers, and let $s$ be an upper bound for $E$. Prove that $s = \sup E$ if and only if for every $\varepsilon > 0$ there exists $x \in E$ such that $x > s - \varepsilon$.

EXERCISE 4.10. Let $A$ and $B$ be nonempty sets of real numbers. Decide whether the following statements are true or false. If true, give a proof; if false, give a counterexample.

(1) If $\sup A < \inf B$, then there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.
(2) If there exists a $c \in \mathbb{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

---

[2] We want to choose $\varepsilon > 0$ small enough so that $\gamma$ is still in $E$. How small does $\varepsilon$ need to be? Well, we want $\gamma = \beta + \varepsilon \in E$, i.e. $(\beta+\varepsilon)^n < x$. We also know that $\beta^n < x$, so it will be good enough to find $\varepsilon$ satisfying $(\beta+\varepsilon)^n - \beta^n < x - \beta^n$. Now we try to put something in between the left and right sides of this inequality. To do so, we rewrite the left side using a telescoping sum, then estimate $\beta < \gamma$.

$$\gamma^n - \beta^n = (\gamma - \beta)(\gamma^{n-1} + \gamma^{n-2}\beta + \cdots + \gamma\beta^{n-2} + \beta^{n-1}) < \varepsilon \cdot n \cdot \gamma^{n-1}.$$

Thus, in order for $\varepsilon$ to be small enough, we just need $\varepsilon n(\beta+\varepsilon)^{n-1} < x - \beta^n$. Actually, let's make the LHS of this very last inequality just a little bigger, so that we can get $\varepsilon$ by itself. As long as $\varepsilon \leq 1$, it suffices to choose $\varepsilon$ small enough so that $\varepsilon n(\beta+1)^{n-1} \leq x - \beta^n$.

[3] Reasoning as before, we have

$$\beta^n - \gamma^n < \varepsilon n \beta^{n-1},$$

so it suffices to choose $\varepsilon > 0$ small enough so that $\varepsilon n \beta^{n-1} \leq \beta^n - x$.

## 5. Subsets of $\mathbb{R}$, and the Extended Real Number System $\overline{\mathbb{R}}$

Though working in $\mathbb{R}$ is certainly much more convenient for most purposes than working in $\mathbb{Q}$, it is rather annoying that the supremum of a set of real numbers is not always defined in $\mathbb{R}$. We correct this deficiency by defining the extended real number system $\overline{\mathbb{R}}$, where the supremum and infimum always exist for nonempty sets.

DEFINITION 5.1. The *extended real number system* $(\overline{\mathbb{R}}, \leq)$ is an ordered set, defined by the following. As a set, $\overline{\mathbb{R}}$ is simply the set formed by adjoining the two symbols $+\infty$ and $-\infty$ to the set $\mathbb{R}$ of real numbers. Within $\mathbb{R}$, the order $<$ remains the same, but we define $-\infty < x < +\infty$ for every $x \in \mathbb{R}$.

Notes: (i) By this definition, $\overline{\mathbb{R}}$, and every subset thereof, is bounded above by $+\infty$. Therefore the 'nonempty and bounded above' is equivalent to 'nonempty' when working in $\overline{\mathbb{R}}$.

(ii) Suppose $E$ is a nonempty subset of $\mathbb{R}$ that is not bounded above in $\mathbb{R}$. Then $\sup E$ is not defined in $\mathbb{R}$, but the supremum of $E$ in $\overline{\mathbb{R}}$ is $+\infty$, since $+\infty$ is in fact[4] the *only* upper bound for $E$ in $\overline{\mathbb{R}}$.

DEFINITION 5.2. Let $a, b$ be elements of $\overline{\mathbb{R}}$ such that $a < b$. We use special notation for the following subsets of $\overline{\mathbb{R}}$:

$$(a, b) = \{x \in \overline{\mathbb{R}} : a < x < b\}, \qquad [a, b) = \{x \in \overline{\mathbb{R}} : a \leq x < b\},$$
$$(a, b] = \{x \in \overline{\mathbb{R}} : a < x \leq b\}, \qquad [a, b] = \{x \in \overline{\mathbb{R}} : a \leq x \leq b\}.$$

If $a \neq -\infty$ and $b \neq +\infty$, then these sets are all called *intervals*; more specifically $(a, b)$ is an *open interval*, $(a, b]$ and $[a, b)$ are *half-open intervals*, and $[a, b]$ is a *closed interval*. For any $c \in \mathbb{R}$, the sets $(c, +\infty)$ and $(-\infty, c)$ are called *open rays* of $\mathbb{R}$, and the sets $[c, \infty)$ and $(-\infty, c]$ are called *closed rays* of $\mathbb{R}$.

Notes: (i) Whenever $A$ is a subset of $\overline{\mathbb{R}}$ that does not contain $-\infty$ or $+\infty$, we always consider it as a subset of $\mathbb{R}$ unless explicitly stated otherwise.

(ii) There is a small risk of confusion between the subset $(a, b)$ of $\mathbb{R}$ and the ordered pair $(x, y) \in A \times B$, with $x \in A$ and $y \in B$. However, the distinction should be clear from context as long as the reader realizes that the same notation is used for the two different meanings.

REMARK 5.3. Though working in $\overline{\mathbb{R}}$ has its technical advantages, the introduction of the symbols $+\infty$ and $-\infty$ has the unpleasant side effect of creating lots of extra cases in the proofs of Theorems. Furthermore, the symbols $+\infty$ and $-\infty$ are rarely dealt with directly, but rather as byproducts of the unboundedness of certain sets. Therefore we make the following conventions.

- If $E$ is a subset of $\mathbb{R}$, the statements '$E$ is bounded above' and '$E$ is not bounded above' are to be interpreted as '$E$ is bounded above in $\mathbb{R}$' and '$E$ is not bounded above in $\mathbb{R}$', respectively.
- If $E$ is a nonempty subset of $\mathbb{R}$ or $\overline{\mathbb{R}}$, the notation $\sup E$ will refer to the supremum in $\overline{\mathbb{R}}$ unless otherwise specified.
- Arithmetic: If $x \in \mathbb{R}$, then $x + \infty = +\infty$, $x - \infty = -\infty$, $\frac{x}{+\infty} = \frac{x}{-\infty} = 0$. If $y \in \overline{\mathbb{R}}$ and $y > 0$, then $y \cdot (+\infty) = +\infty$, $y \cdot (-\infty) = -\infty$; if $z \in \overline{\mathbb{R}}$ and $z < 0$, then $z \cdot (+\infty) = -\infty$, $z \cdot (-\infty) = +\infty$. Despite these conventions, we stress that $\overline{\mathbb{R}}$ is *not* a field. In particular, $\infty - \infty$ is not defined.

EXERCISE 5.4. Let $a$ and $b$ be real numbers. Show that the following three equalities hold:

$$\bigcap_{x>b}(a, x) = (a, b], \qquad \bigcup_{n=1}^{\infty}[a + \frac{1}{n},\ b - \frac{1}{n}) = (a, b), \qquad \bigcap_{n=1}^{\infty}(a + n, +\infty) = \emptyset.$$

[4] Let $M$ be any upper bound in $\overline{\mathbb{R}}$. Clearly $M \neq -\infty$, since $x > -\infty$ for all $x \in E$. Furthermore, $M \notin \mathbb{R}$, since $E$ is by assumption not bounded above in $\mathbb{R}$.

EXERCISE 5.5. Let $a_1, a_2, \ldots$ be any enumeration of the negative rational numbers; let $b_1, b_2, \ldots$ be any enumeration of the positive rational numbers. Show that the following two equalities hold:

$$\bigcap_{j=1}^{\infty}(a_j, b_j) = \{0\}, \qquad \bigcup_{j=1}^{\infty}(a_j, b_j) = \mathbb{R}.$$

## 6. The Complex Field

In this section, we define complex numbers. We will use these from time to time, but our emphasis will be on the real number system.

DEFINITION 6.1. The set $\mathbb{C}$ of complex numbers is defined simply as $\mathbb{R} \times \mathbb{R}$. The *complex field* $(\mathbb{C}, +, \cdot)$ is defined by the following operations: If $x = (a, b)$, $y = (c, d)$, then

$$x + y = (a, b) + (c, d) = (a + c,\ b + d),$$
$$xy = (a, b) \cdot (c, d) = (ac - bd,\ ad + bc).$$

The additive and multiplicative identities are $(0, 0)$ and $(1, 0)$, respectively.

Note that the addition and multiplication operations on the RHS take place in $\mathbb{R}$; these constitute the definition of complex addition and multiplication written on the left.

One should check that the addition and multiplication defined here satisfy the field axioms. We will not do this here; instead we leave it as a lengthy but straightforward exercise.

Note: The definition above is probably not the definition of the complex field you are most familiar with. However, it has the advantage of being stated in terms of objects for which we already have good definitions. We will see in a moment that the definition above agrees with the one you have probably seen.

Notation:

- We sometimes use the shorthand notation[5] $(a, 0) = a$.
- We define $i = (0, 1)$.

With these notational conventions, we have

$$i^2 = (0, 1) \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1,\ 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -1.$$
$$a + bi = (a, 0) + (b, 0) \cdot (0, 1) = (a, 0) + (0, b) = (a, b).$$

DEFINITION 6.2. If $a, b \in \mathbb{R}$ and $z = a + bi$, the number $a - bi$ is called its *complex conjugate* and denoted by $\bar{z}$. The numbers $a$ and $b$ are called the *real* and *imaginary parts* of $z$, and denoted $a = \mathrm{Re}(z)$, $b = \mathrm{Im}(z)$.

Note that the number $z\bar{z}$ is always real and nonnegative; therefore the following definition makes sense.

DEFINITION 6.3. Given $z \in \mathbb{C}$, its *modulus*, or *absolute value*, is defined by $|z| = (z\bar{z})^{\frac{1}{2}}$.

The basic properties of complex numbers that follow from these definitions should be familiar and will not be reviewed in these notes. The main purpose of this section is to set definitions and notation for later use.

EXERCISE 6.4. Prove that there exists no order $\leq$ that makes $(\mathbb{C}, +, \cdot, \leq)$ into an ordered field.

[5]This notation is justified: the function $f : \mathbb{R} \to \mathbb{C}$ given by $f(a) = (a, 0)$ is what's called a field homomorphism, with image $\mathbb{R} \times \{0\}$.

CHAPTER 3

# More Structures on Sets

## 1. Vector Spaces

The concept of a vector space is one that should be familiar from Linear Algebra. However, (normed) vector spaces provide useful intuition for the more general 'metric spaces' that will be our primary focus in the following chapters. While the concept of a metric space is strictly more general than that of a normed vector space, many of the most important metric spaces are actually normed vector spaces. Therefore, we give a few definitions and examples related to vector spaces. However, the reader should be aware that there is *much* more to be learned about vector spaces than the limited presentation here suggests. For example, we will not even treat the notions of a basis or the dimension of a vector space; these are fundamental to the theory of vector spaces but do not, in the author's opinion, motivate any fundamental concepts in the metric space theory (at least not to the extent that would justify a lengthy digression into their development). Furthermore, with the exception of the definition of a vector space and some preliminary examples, we limit ourselves to the treatment of real vector spaces.

### 1.1. Definition of a Vector Space.

DEFINITION 1.1. Let $F$ be a field. A *vector space* $V$ over the field $F$, or an $F$-vector space, is a set $V$, consisting of elements called *vectors*, together with two operations:

(1) Vector addition $V \times V \to V$, denoted by $+$ (e.g. $v + w$ where $v, w \in V$), and
(2) Multiplication of a vector by a scalar[1] $F \times V \to V$, denoted by juxtaposition (e.g. $\alpha v$, where $\alpha \in F$, $v \in V$).

The vector space operations are required to satisfy the following *vector space axioms*:

- Vector addition is commutative and associative: If $u, v, w \in V$, then $(u+v)+w = u+(v+w)$ and $u + v = v + u$.
- $V$ contains an additive identity, called the *zero vector* and denoted by $0$, such that $0 + v = v$ for all $v \in V$.
- Every element $v$ of $V$ has an *additive inverse*, denoted $-v$, such that $v + (-v) = 0$.
- Multiplication by scalars is compatible with multiplication in $F$, in the sense that $\alpha(\beta v) = (\alpha\beta)v$ whenever $\alpha, \beta \in F$ and $v \in V$.
- If $1$ denotes the multiplicative identity in $F$, then $1v = v$ for any $v \in V$.
- The following distributive laws hold. For any $\alpha, \beta \in F$ and $u, v \in V$, we have $\alpha(u + v) = \alpha u + \alpha v$ and $(\alpha + \beta)v = \alpha v + \beta v$.

An $\mathbb{R}$-vector space is also called a *real vector space*; a $\mathbb{C}$-vector space is also called a *complex vector space*. A *subspace* $W$ of a vector space $V$ is a subset of $V$ which is an $F$-vector space in its own right, with the same operations as $V$. (Equivalently, a subspace is required to satisfy the condition that $\lambda v + w$ belongs to $W$ whenever $v, w \in W$ and $\lambda \in F$, where the operations in $\lambda v + w$ are the operations in $V$.)

Unfortunately, the notation here hides a lot of information. For example, consider the equality $(\alpha + \beta)v = \alpha v+\beta v$. On the LHS, the symbol $+$ denotes addition in the field $F$. On the RHS, the same symbol

[1] In the context of $F$-vector spaces, elements of $F$ are also called *scalars*.

denotes addition in $V$, *which is a completely different operation!* Therefore some care is warranted when parsing statements involving vector space operations.

We will assume basic familiarity with vector spaces; however we do give some examples before proceeding.

EXAMPLE 1.2. Let $F$ be a field. The following are examples of $F$-vector spaces.

- $F$ itself is an $F$-vector space. In this case the vector space operations are the same as the operations in $F$.
- $F^n$ can be made into an $F$-vector space, for any $n$. Vector addition and multiplication by scalars are defined component-wise: For $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n, \lambda \in F$, we define
$$(\alpha_1, \ldots, \alpha_n) + (\beta_1, \ldots, \beta_n) = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n),$$
$$\lambda(\alpha_1, \ldots, \alpha_n) = (\lambda\alpha_1, \ldots, \lambda\alpha_n).$$
The zero element in $F^n$ is $(0, \ldots, 0)$, where $0$ denotes the zero element in $F$.
- If $X$ is any set, then the set $F^X$ of functions from $X$ to $F$ can be made into a vector space, by defining addition of functions and scalar multiples of functions pointwise. (The previous example is technically a special case of this one, with $X = J_n$, but the former is important enough to mention separately.) Given functions $f$ and $g$ from $X$ to $F$ and $\alpha \in F$, we define the functions $f + g$ and $\alpha f$ to be the functions from $X$ to $F$ such that
$$(f + g)(x) = f(x) + g(x), \qquad (\alpha f)(x) = \alpha(f(x)), \qquad \text{for all } x \in X.$$
We emphasize that the notation $(f + g)(x) = f(x) + g(x)$ is a *definition* of vector addition (the symbol $+$ on the LHS, between the two vectors $f$ and $g$), defined in terms of addition in $F$ (the symbol $+$ on the RHS, between the two elements $f(x)$ and $g(x)$ of $F$). Similarly, the notation $(\alpha f)(x) = \alpha(f(x))$ is the *definition* of multiplication by scalars, in terms of multiplication in $F$.

  The zero element in $F^X$ is the function $f$ such that $f(x) = 0$ for all $x \in X$. Note that under this definition, $F^X$ cannot be a field under the usual definition $(fg)(x) = f(x)g(x)$ of pointwise multiplication (unless of course $X$ has only one element). Indeed, if $x$ and $y$ are two distinct elements of $X$ and $f(x) = 0$, $f(y) \neq 0$, then $f$ is not the zero element of $F^X$, but it cannot have a multiplicative inverse under pointwise multiplication. However, the pointwise multiplication operation does make $F^X$ into what's called a *commutative algebra* over $F$. Furthermore, even though elements of $F^X$ do not in general have multiplicative inverses, we often use the notation $\frac{1}{f}$ to denote the function taking the value $1/f(x)$ at $x$, on the restricted domain $X \backslash f^{-1}(0) = \{x \in X : f(x) \neq 0\}$. Of course, $1/f$ is only an element of $F^X$ if $f^{-1}(0)$ is empty. We also write $\frac{g}{f}$ for $g\frac{1}{f}$ in this case.

  We'll always give $F^X$ the structure discussed above in what follows.

REMARK 1.3. The concept of an $F$-vector space makes sense for any field $F$, and the definitions and examples given above are completely independent of the structure of $F$. However, any further development will require specialized treatment that depends on exactly what $F$ is. Therefore we restrict ourselves to the case $F = \mathbb{R}$, which is the most important case for our purposes. The case $F = \mathbb{C}$ is also quite important in analysis; in most (but not all) of the following discussion, the replacement of $\mathbb{R}$ by $\mathbb{C}$ requires only minor changes.

### 1.2. Normed Vector Spaces over $\mathbb{R}$.

DEFINITION 1.4. Let $V$ be a real vector space. A *norm* on $V$ is a function $\|\cdot\|$ from $V$ to $\mathbb{R}$, satisfying the following properties:

(1) (Nonnegativity) $\|v\| \geq 0$, and $\|v\| = 0$ if and only if $v$ is the zero vector in $V$.

(2) (Homogeneity) $\|\alpha v\| = |\alpha|\|v\|$ for any $\alpha \in \mathbb{R}$ and $v \in V$.
(3) (The Triangle Inequality) $\|u + v\| \leq \|u\| + \|v\|$ whenever $u, v \in V$.

A *normed vector space* over $\mathbb{R}$, or a *normed linear space* over $\mathbb{R}$, is a vector space $V$ on which a norm $\|\cdot\|$ is defined. We denote this normed vector space by $(V, \|\cdot\|)$, or simply by $V$, when the norm is understood from context.

REMARK 1.5. One might wonder what the 'dot' in the notation $\|\cdot\|$ means. The dot is just a placeholder, which says we're going to put something there. We could use similar notation for other functions, e.g., $f(\cdot)$ instead of $f$. In fact, this notation is sometimes also used if a function is defined on some product space $A \times B$, and its formula depends on the individual components of its argument. For example, consider the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $g(x, y) = x^2 + 3y$. One can define a new function $h : \mathbb{R} \to \mathbb{R}$ by the formula $h(y) = 5^2 + 3y$, or one can simply use the notation $h = g(5, \cdot)$. More generally, one might wish to use the notation $k = g(x, \cdot)$ to mean that $k$ is defined by the formula $k(y) = x^2 + 3y$, where $x$ is unspecified but fixed.

EXAMPLE 1.6. In the case $F = \mathbb{R}$, we can define norms for some of the vector spaces in Example 1.2 as follows:

- $\mathbb{R}$ itself is a real normed vector space, with the absolute value serving as its norm.
- $\mathbb{R}^n$ is a real normed vector space, with the *Euclidean norm*
$$\|(x_1, \dots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$
The fact that the triangle inequality holds for this norm requires some justification; this will be provided below.
- We can't make $\mathbb{R}^X$ into a real normed vector space for general sets $X$. However, we can define a norm on the subspace of *bounded* functions $B(X)$. (A real-valued function is called *bounded* if its image is a bounded subset of $\mathbb{R}$.) The *supremum norm* (or the *sup* norm for short), also called the *uniform norm*, is defined by
$$\|f\|_u = \sup_{x \in X} |f(x)|, \qquad f \in B(X).$$
Exercise 2.5 in Chapter 2 shows that this norm satisfies the triangle inequality.
- As a special case of the previous point, we consider the uniform norm on $\mathbb{R}^n$. Since $J_n$ is a finite set, the supremum becomes a maximum:
$$\|x\|_u = \max\{|x_1|, \dots, |x_n|\}.$$

EXERCISE 1.7. Let $\|\cdot\|$ be a norm on a real vector space $V$. Prove the *reverse triangle inequality*:
$$\big|\|x\| - \|y\|\big| \leq \|x - y\|.$$

### 1.3. Real Inner Product Spaces.

DEFINITION 1.8. An inner product on a real vector space $V$ is a function $\langle \cdot, \cdot \rangle$ from $V \times V$ to $\mathbb{R}$, which satisfies the following properties:

(1) (Symmetry) $\langle x, y \rangle = \langle y, x \rangle$ for all $x, y \in V$,
(2) (Linearity in the first component) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$,
(3) (Positive-definiteness) $\langle x, x \rangle \geq 0$ for all $x \in V$, with $\langle x, x \rangle = 0$ if and only if $x$ is the zero vector.

A real vector space on which an inner product is defined is called a *real inner product space*.

Notes: (i) One can also define inner products on a complex vector space. In this case the symmetry requirement becomes *conjugate symmetry*: $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (where the bar denotes the complex conjugate). (ii) The requirement of linearity in the first component automatically gives linearity in the second component. (The situation is subtler for complex inner products.)

EXAMPLE 1.9. The *dot product* on $\mathbb{R}^n$ is an inner product: If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, then their dot product $x \cdot y$ is defined by

$$x \cdot y = (x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1 y_1 + \cdots + x_n y_n.$$

The symmetry and linearity requirements follow from basic properties of real numbers. The positive definiteness follows from the fact that $r^2 \geq 0$ for all $r \in \mathbb{R}$, with $r^2 > 0$ unless $r = 0$. Note that

$$x \cdot x = x_1^2 + \cdots + x_n^2 = \|x\|^2,$$

where $\|\cdot\|$ denotes the Euclidean norm. Thus Theorem 1.11 below will complete the verification that the Euclidean norm is in fact a norm.

EXAMPLE 1.10. We have a long way to go before we rigorously define the integral operator. However, the idea of the integral should be familiar from Calculus courses. On a certain class of functions (which we will not specify at this time), the integral is an inner product. Indeed, if $f$ and $g$ are (nice) functions, then (being intentionally ambiguous with the notation), we have $\int fg \,\mathrm{d}x = \int gf \,\mathrm{d}x$, $\int \lambda f + g \,\mathrm{d}x = \lambda \int f \,\mathrm{d}x + \int g \,\mathrm{d}x$, and $\int ff \,\mathrm{d}x = \int |f|^2 \,\mathrm{d}x \geq 0$, with equality if and only if $f$ is the zero element of this class of functions.

Even though we don't have the terminology to be more rigorous here, we give this example to show that the dot product is not the only useful inner product.

THEOREM 1.11. *Let $V$ be real vector space, and let $\langle \cdot, \cdot \rangle$ be an inner product on $V$. Then $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$ is a norm on $V$.*

We prove this Theorem after several preparatory and hopefully familiar results. We use the notation $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ introduced in the statement of the Theorem.

DEFINITION 1.12. Let $V$ be an inner product space with inner product $\langle \cdot, \cdot \rangle$, and let $u$ and $v$ be vectors in $V$. We say that $u$ and $v$ are *orthogonal* if $\langle u, v \rangle = 0$.

THEOREM 1.13 (Pythagorean Theorem for Real Inner Products). *Let $V$ be an inner product space with inner product $\langle \cdot, \cdot \rangle$, and let $u$ and $v$ be orthogonal vectors in $V$. Then $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.*

PROOF. This follows immediately by expanding $\|u + v\|^2$:

$$\|u + v\|^2 = \langle u + v,\ u + v \rangle = \langle u, u \rangle + \underbrace{\langle u, v \rangle}_{=0} + \underbrace{\langle v, u \rangle}_{=0} + \langle v, v \rangle = \|u\|^2 + \|v\|^2.$$

□

PROPOSITION 1.14 (Existence of Orthogonal Projections). *Let a real vector space $V$ have inner product $\langle \cdot, \cdot \rangle$, and let $u$ and $v$ be vectors in $V$. There exists a number $c$ and a vector $z$ such that $u = cv + z$ and $\langle v, z \rangle = 0$. The number $c$ is given by the formula*

$$c = \frac{\langle u, v \rangle}{\|v\|^2}.$$

The vector $cv$ is called the *orthogonal projection* of $u$ onto $v$. This is most easily understood in the context of $\mathbb{R}^2$; the reader is encouraged to draw a picture in $\mathbb{R}^2$ to accompany the following proof, to build their intuition.

PROOF. Define $c = \langle u, v \rangle / \|v\|^2$ as in the Theorem statement, and define $z = u - cv$. Then

$$\langle z, v \rangle = \langle u, v \rangle - c\langle v, v \rangle = \langle u, v \rangle - \frac{\langle u, v \rangle}{\|v\|^2} \|v\|^2 = 0.$$

□

THEOREM 1.15 (Cauchy-Schwarz inequality). *If $\langle \cdot, \cdot \rangle$ is an inner product on a real vector space $V$, then*

$$|\langle u, v\rangle| \le \|u\|\|v\|,$$

*for any vectors $u$, $v$ in $V$.*

PROOF. Define $c = \langle u, v\rangle/\|v\|^2$, so that $cv$ is the orthogonal projection of $u$ onto $v$ and $z := u - cv$ is orthogonal to $v$ (thus also to $cv$). Then we can apply the Pythagorean Theorem to $cv$ and $z$ in the following calculation:

$$\|u\|^2 = \|cv + z\|^2 = \|cv\|^2 + \|z\|^2 \ge |c|^2\|v\|^2 + 0 = \frac{|\langle u, v\rangle|^2}{\|v\|^4}\|v\|^2 = \frac{|\langle u, v\rangle|^2}{\|v\|^2}.$$

Multiplying both sides by $\|v\|^2$ and taking square roots gives the desired inequality. $\square$

REMARK 1.16. In $\mathbb{R}^2$, we have the formula $u \cdot v = \|u\|\|v\| \cos\theta$, where $\theta$ is the angle between $u$ and $v$. In this case the Cauchy-Schwarz inequality follows immediately from the fact that $|\cos\theta| \le 1$.

Using the Cauchy-Schwarz inequality, we finally prove that $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ is a norm.

PROOF OF THEOREM 1.11. Expand $\|u + v\|^2$ to get

$$\|u+v\|^2 = \langle u+v,\ u+v\rangle = \langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle \le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2.$$

Take square roots to obtain the triangle inequality. The other requirements for a norm are clear. $\square$

## 2. Metric Spaces

The norm in a vector space give a notion of the 'size' of an element, or its 'distance' from the zero element in that space. Furthermore, one can talk about the 'distance' between two vectors $v$ and $u$ in terms of $\|u - v\|$, the 'distance' of $u - v$ from the zero element. However, the structure of a vector space is rather rigid, and we would like a sensible notion of distance for sets which are not vector spaces. The notion of a *metric space* will serve this need.

DEFINITION 2.1. Let $X$ be a set. A *metric* on $X$ is a function $d : X \times X \to \mathbb{R}$, satisfying the following properties:

1. (Nonnegativity) $d(x, y) \ge 0$ for all if $x, y \in X$, with $d(x, y) = 0$ if and only if $x = y$.
2. (Symmetry) $d(x, y) = d(y, x)$, for all $x, y \in X$.
3. (Triangle Inequality) $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The set $X$ together with the metric $d$ is called a *metric space*, denoted $(X, d)$ (though we simply refer to 'the metric space $X$' when $d$ is understood from context). Elements of the set $X$ are called *points*. The number $d(x, y)$ is called the *distance* from $x$ to $y$, and a metric is sometimes called a *distance function*.

EXAMPLE 2.2.

- Any normed vector space is a metric space. More specifically, if $V$ is a normed vector space with norm $\|\cdot\|$, then the function $d : V \times V \to \mathbb{R}$ given by $d(x, y) = \|x - y\|$ is a metric on $V$. Only the triangle inequality for $d$ is not obvious. But, as one might expect, it follows from the triangle inequality for $\|\cdot\|$: Let $x, y, z$ be points of $X$. then

  $$d(x, y) = \|x - y\| = \|(x - z) - (z - y)\| \le \|x - z\| + \|z - y\| = d(x, z) + d(z, y).$$

- Not every metric space is a normed vector space. This is true for rather trivial reasons: If $(X, d)$ is a metric space, $X$ might not even be a vector space. However, even if $X$ is a vector space, $d$ might not define a norm on $X$. In fact, consider the function $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = 1$ if $x \ne y$, $d(x, x) = 0$. Then $d$ is a metric on *any* set, called the *discrete metric*. Clearly the discrete metric does not satisfy any kind of homogeneity requirement on $\mathbb{R}^n$, for example.

EXERCISE 2.3. Let $X$ be any set. Prove that the discrete metric $d : X \times X \to \mathbb{R}$ (defined by $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$ for $x \in X$) satisfies the triangle inequality and is therefore a metric on $X$.

EXERCISE 2.4. Determine which of the following functions are metrics on $\mathbb{R}$. Prove your answer in each case.

- $d_1(x, y) = \sqrt{|x - y|}$.
- $d_2(x, y) = |x - 2y|$.
- $d_3(x, y) = \frac{|x-y|}{1+|x-y|}$.

**2.1. Open Balls in a Metric Space.** It is often useful to talk about the points in a metric space $(X, d)$ which are 'close' to a given point $x$. We introduce the following notation for the open 'ball' of radius $r$ centered at $x$ in the metric space $(X, d)$:

$$B_{(X,d)}(x, r) := \{y \in X : d(x, y) < r\}.$$

If the metric $d$, or the set $X$ is understood from context, we simply write $B_X(x, r)$ or $B_d(x, r)$. If both are understood, we simply write $B(x, r)$.

EXAMPLE 2.5. The set $B_{(X,d)}(x, r)$ may or may not 'look' like a ball, even in Euclidean spaces. For example, consider the following metrics on $\mathbb{R}^2$.

- $d_1(x, y) = \|x - y\|$, where $\| \cdot \|$ denotes the Euclidean norm. (The metric defined this way is called the *Euclidean metric*.) Then $B_{d_1}(x, r)$ is the set of points $y \in \mathbb{R}^2$ such that $\|x - y\| < r$. This really does look like a ball of radius $r$ centered at $x$. (Note that it does not include the 'boundary' of the ball.)
- $d_2(x, y) = \|x - y\|_u = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. (The notation $\| \cdot \|_u$ was introduced in Example 1.6 and denotes the uniform norm.) Then

$$\begin{aligned} d_2(x, y) < r &\iff |x_1 - y_1| < r \text{ and } |x_2 - y_2| < r \\ &\iff x_1 - r < y_1 < x_1 + r \text{ and } x_2 - r < y_2 < x_2 + r \\ &\iff y = (y_1, y_2) \in (x_1 - r, x_1 + r) \times (x_2 - r, x_2 + r). \end{aligned}$$

(Note that on the RHS we have the product of intervals, not ordered pairs.) Thus

$$B_{d_2}(x, r) = (x_1 - r, x_1 + r) \times (x_2 - r, x_2 + r).$$

This is (the 'interior' of) a square of side length $2r$. For this reason the metric $d_2$ is sometimes called the *square metric* (as are its analogs in higher-dimensional Euclidean spaces).

EXERCISE 2.6. Consider the function $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, defined by

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|, \qquad (x = (x_1, x_2), \quad y = (y_1, y_2)).$$

(1) Prove that $d$ is a metric on $\mathbb{R}^2$.
(2) On a sheet of graph paper, draw the set $B_d((5, 1), 3)$. Use dotted lines to indicate the 'boundary', which is not included in the set you are drawing. (Hint: it may be easier to figure out what the set looks like if you first consider $B_d((0, 0), 3)$.)
(3) On the same graph as in the previous part, draw $B_{d_u}((-3, 2), 1)$, where $d_u$ denotes the square metric.

If $(X, d)$ is a metric space and $Y$ is a subset of $X$, then $d$ is still a metric on the set $Y$. (And we always consider $Y$ to be a metric space with the same metric $d$, unless explicitly noted otherwise.) If $x \in Y$ is 'near the edge' of $Y$, then $B_{(Y,d)}(x, r)$ might be a smaller set than $B_{(X,d)}(x, r)$. Indeed,

$$B_{(Y,d)}(x, r) = \{z \in Y : d(x, z) < r\} = \{z \in X : d(x, z) < r\} \cap Y = B_{(X,d)}(x, r) \cap Y.$$

EXAMPLE 2.7. For example, let $X = \mathbb{R}^2$, $Y = [-1, 3] \times [2, 4]$, and let $d$ denote the Euclidean metric on $X = \mathbb{R}^2$. Let $p$ denote the point $(3, 4)$, the upper right corner of $Y$. Then $B_{(Y,d)}(p, 2)$ looks like a quarter of the ball $B_{(X,d)}(p, 2)$. (Draw a picture to see this.)

EXERCISE 2.8. Let $(X, d)$ be a metric space, and let $E$ be a subset of $X$. The *diameter* of $E$ in $(X, d)$ is defined by the formula

$$\operatorname{diam}_d(E) = \sup\{d(x, y) : x, y \in E\}.$$

(Usually we just write $\operatorname{diam}(E)$ when $d$ is clear.)

(1) Prove that for any $r > 0$ and $x \in X$, we have $\operatorname{diam}(B(x, r)) \leq 2r$.
(2) If $X$ is any set and $d$ is the discrete metric, show that $\operatorname{diam}(B(x, r)) = 0$ for any $r \leq 1$, while $\operatorname{diam}(B(x, r)) = 1$ for any $r > 1$.
(3) If $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$ and $d$ is the Euclidean metric, prove that $\operatorname{diam}(B(x, r)) = 2r$.

**2.2. Interior Points.** We have used the term 'interior' rather loosely in the preceding discussion. We can now clarify what we mean.

DEFINITION 2.9. Let $(X, d)$ be a metric space and let $U$ be a subset of $X$. A point $x$ in $U$ is called an *interior point* of $U$ with respect to $X$ if there exists $r > 0$ such that $B_X(x, r) \subset U$. The set of all interior points of $U$ with respect to $X$ is called the *interior* of $U$ with respect to $X$, and we will denote it $\operatorname{Int}_X(U)$.

$$\operatorname{Int}_X(U) = \{x \in U : x \text{ is an interior point of } U \text{ with respect to } X\}.$$

If $(X, d)$ is clear from context and it is the only metric space under consideration, we sometimes write $U^\circ$ instead of $\operatorname{Int}_X(U)$.

Suppose $(X, d)$ is a metric space and $U \subset Y \subset X$. If $x$ is an interior point of $U$ with respect to $X$, then it is an interior point of $U$ with respect to $Y$. However, the converse is not true. These two statements are justified by the following Proposition and Exercises.

PROPOSITION 2.10. *Let $(X, d)$ be a metric space; assume $U \subset Y \subset X$. Then $\operatorname{Int}_X(U) \subset \operatorname{Int}_Y(U)$, and $\operatorname{Int}_X(U) \subset \operatorname{Int}_X(Y)$.*

PROOF. Choose $x \in \operatorname{Int}_X(U)$. Then there exists $r > 0$ such that $B_X(x, r) \subset U$. But then $B_Y(x, r) = B_X(x, r) \cap Y \subset U \cap Y \subset U$, so $x$ is an interior point of $U$ with respect to $Y$ as well, i.e. $x \in \operatorname{Int}_Y(U)$. This proves the first statement. The second statement follows from the fact that $U \subset Y$ (so $B_X(x, r) \subset Y$ whenever $B_X(x, r) \subset U$). $\square$

EXERCISE 2.11. As in Example 2.7, let $X = \mathbb{R}^2$, $Y = [-1, 3] \times [2, 4]$, and let $d$ denote the Euclidean metric on $X = \mathbb{R}^2$. Let $p = (3, 4)$ and let $q = (2, 4)$. Arguing *directly from the definition of an interior point* (i.e., without using Exercise 2.12), show that $q$ is an interior point of $B_Y(p, 2)$ with respect to $Y$, but $q$ is not an interior point of $B_Y(p, 2)$ with respect to $X$. In addition, draw a picture on a piece of graph paper that illustrates the idea of your proof.

EXERCISE 2.12. Let $(X, d)$ be a metric space, and let $Y$ be a subset of $X$. Prove that for any subset $U$ of $Y$, we have

$$\operatorname{Int}_X(U) = \operatorname{Int}_Y(U) \cap \operatorname{Int}_X(Y). \tag{$*$}$$

In the notation of Exercise 2.11, the equality $(*)$ gives an alternate explanation of why $q$ is not an interior point of $B_Y(p, 2)$ with respect to $X$: It is because $q \notin \operatorname{Int}_X(Y)$, as can be seen from the picture you drew in that Exercise.

Let $(X, d)$ be a metric space. In light of the exercises above, we must be careful about whether we are considering a set $U$ as a subset of $X$ or a smaller set $Y$. However, if $X$ is the only metric space under consideration, we often just say '$x$ is an interior point of $U$', and leave off the specification 'with respect to $X$'.

### 2.3. Open Sets in Metric Spaces.

DEFINITION 2.13. Let $(X, d)$ be a metric space. A subset $U$ of $X$ is said to be *open* in $X$ if every point of $U$ is an interior point of $U$ (with respect to $X$), that is, if $U = \mathrm{Int}_X(U)$.

In this subsection, we answer two questions:

(1) What are the open sets in a metric space $(X, d)$?
(2) If $(X, d)$ is a metric space and $Y \subset X$, what is the relationship between the open sets in $X$ and the open sets in $Y$?

REMARK 2.14. Note that in any metric space $(X, d)$, the empty set is (vacuously) an open set in $X$, and $X$ is (trivially) an open set in $X$.

PROPOSITION 2.15. *Let $(X, d)$ be a metric space. Then $B_X(x, r)$ is open in $X$.*

PROOF. Choose $y \in B_X(x, r)$. We must show that $y$ is an interior point of $B_X(x, r)$. Put[2] $\delta = r - d(x, y)$. We know that $\delta$ is positive, since $y \in B_X(x, r)$ implies $d(x, y) < r$. We claim that $B_X(y, \delta) \subset B_X(x, r)$, and that therefore $y$ is an interior point of $B_X(x, r)$. To see this, let $z$ be any point of $B_X(y, \delta)$. To show that $z \in B_X(x, r)$, we estimate its distance from $x$:

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \delta = r,$$

where we have used the fact that $z \in B_X(y, \delta)$ to estimate $d(y, z) < \delta$, and we used the definition of $\delta$ to obtain the last equality. We have proved that $d(x, z) < r$, which means $z \in B_X(x, r)$, which means $B_X(y, \delta) \subset B_X(x, r)$, which means that $y$ is an interior point of $B_X(x, r)$, which (finally) means that $B_X(x, r)$ is open in $X$, since $y$ was an arbitrary point of $B_X(x, r)$. □

EXERCISE 2.16. Let $(X, d)$ be a metric space, and let $U$ be a subset of $X$. Use Proposition 2.15 to prove that $\mathrm{Int}_X(U)$ is open in $X$.

PROPOSITION 2.17. *Let $(X, d)$ be a metric space.*

- *Let $\mathcal{U}$ be a collection of open sets of $X$. Then the set $\bigcup_{U \in \mathcal{U}} U$ is open in $X$.*
- *Let $\{U_1, \ldots U_n\}$ be a finite collection of open sets of $X$. Then $\bigcap_{i=1}^n U_i$ is an open set of $X$.*

PROOF. Denote $V = \bigcup_{U \in \mathcal{U}} U$, and let $x$ be a point of $V$. Then $x \in U$ for some $U \in \mathcal{U}$. Since $U$ is open, $x$ is an interior point of $U$, so there exists $r > 0$ such that $B_X(x, r) \subset U \subset V$. Thus $x$ is an interior point of $V$. Thus every point of $V$ is an interior point, i.e. $V$ is open in $X$.

Next, denote $W = \bigcap_{i=1}^n U_i$, and pick $y \in W$. Then $y \in U_i$ for every $i \in J_n$; since each $U_i$ is open in $X$, we can choose $r_i > 0$ such that $B_X(x, r_i) \subset U_i$. Take $r = \min\{r_1, \ldots, r_n\}$. Then

$$B_X(x, r) = \bigcap_{i=1}^n B_X(x, r_i) \subset \bigcap_{i=1}^n U_i = W.$$

Thus $x$ is an interior point of $W$, so $W$ is open in $X$. □

Note that the above Proposition does *not* claim that an arbitrary intersection of open sets is open. The latter statement is in fact false, as can be seen by the fact that $\bigcap_{n=1}^\infty (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ is not an open set in $\mathbb{R}$, even though each of the sets $(-\frac{1}{n}, \frac{1}{n})$ is.

THEOREM 2.18. *Let $(X, d)$ be a metric space, and let $U$ be a subset of $X$. Then $U$ is open in $X$ if and only if $U$ can be written as a union of open balls $B_X(x, r_x)$ of $X$.*

The notation $r_x$ indicates that the radius of each ball may depend on its center point $x$.

[2] To see why we made this choice, draw a picture.

PROOF. ( $\Longrightarrow$ ) Assume $U$ is open in $X$. Then for each $x \in U$, there exists $r_x > 0$ such that $B_X(x, r_x) \subset U$. Therefore

$$\bigcup_{x \in U} B_X(x, r_x) \subset U.$$

On the other hand, every $x$ in $U$ is contained in $B_X(x, r_x)$, so $U$ is contained in the union of these sets. It follows that

$$\bigcup_{x \in U} B_X(x, r_x) = U,$$

which establishes the direction ( $\Longrightarrow$ ).

( $\Longleftarrow$ ) On the other hand, if $U$ can be written as the union of open balls $B_X(x, r)$, then the fact that $U$ is open in $X$ follows immediately from Proposition 2.15 (balls are open) and the first part of Proposition 2.17 (unions of open sets are open). $\square$

We are now in a position to clarify the relationship between open sets in a metric space $(X, d)$ and those in a smaller metric space $(Y, d)$.

THEOREM 2.19. *Let $(X, d)$ be a metric space, and let $U$ and $Y$ be subsets of $X$ such that $U \subset Y \subset X$. Then $U$ is open in $Y$ if and only if $U = Y \cap V$ for some set $V$ which is open in $X$.*

This Theorem should not be too surprising in light of the fact that $B_Y(x, r) = B_X(x, r) \cap Y$, as we have already observed. In fact, this observation plays a key role in the proof.

PROOF. ( $\Longrightarrow$ ) Assume first that $U$ is open in $Y$. Then we can write $U$ as a union of open balls $B_Y(x, r_x)$ of $Y$; by Theorem 2.18. Consequently,

$$U = \bigcup_{x \in U} B_Y(x, r_x) = \bigcup_{x \in U} (B_X(x, r_x) \cap Y) = Y \cap \bigcup_{x \in U} B_X(x, r_x).$$

Define $V$ to be the union $\bigcup_{x \in U} B_X(x, r_x)$ on the RHS of this equality. Then $V$ is open in $X$, as it has been written as a union of open balls of $X$, and $U = Y \cap V$. This finishes the proof of the forward implication.

( $\Longleftarrow$ ) On the other hand, assume that $U = Y \cap V$, where $V$ is open in $X$. Choose $y \in U$; we show $y \in \mathrm{Int}_X(U)$. Since $y \in V$ and $V$ is open in $X$, there exists $r > 0$ such that $B_X(y, r) \subset V$; as $V \subset U$ we have $B_X(y, r) \subset U$, i.e. $y \in \mathrm{Int}_X(U)$, which shows that $U$ is open in $X$. $\square$

EXERCISE 2.20. Let $(X, d)$ be a metric space. Assume that $U \subset Y \subset X$, and additionally that $Y$ is open in $X$. Prove that $U$ is open in $Y$ if and only if $U$ is open in $X$. (Note: There at least two possible solutions; one uses Theorem 2.19, the other uses Exercise 2.12.)

We make one final remark before moving on:

PROPOSITION 2.21. *Let $(X, d)$ be a metric space, and assume $U \subset X$. Then $x \in \mathrm{Int}_X(U)$ if and only if there exists an open subset $V$ of $X$ such that $x \in V \subset U$.*

PROOF. If $x \in \mathrm{Int}_X(U)$, then there exists $r > 0$ such that $B_X(x, r) \subset U$. Put $V = B_X(x, r)$. On the other hand, if $x \in V \subset U$ and $V$ is open in $X$, then $x \in V = \mathrm{Int}_X(V) \subset \mathrm{Int}_X(U)$. $\square$

## 3. Topological Spaces

**3.1. Definition of a Topology.** In the previous subsection, we characterized the collection of open subsets of a metric space in terms of the open balls of that metric space. A topology, is, more generally, a choice of what subsets of a given set $X$ to call 'open', even if that set $X$ is not a metric space.

The concept of a topological space is an abstract one, and we will not work with it too much; instead we will mostly stay within the context of metric spaces. However, we will use the concept of a topology (rather tangentially) when we work with the space $\overline{\mathbb{R}}$.

DEFINITION 3.1. Let $X$ be a set. A *topology* on the set $X$ is a subset $\mathcal{T}$ of $\mathcal{P}(X)$, i.e., a collection of subsets of $X$. The collection $\mathcal{T}$ is required to satisfy certain properties:

(1) $\emptyset, X \in \mathcal{T}$.
(2) $\mathcal{T}$ is closed under (arbitrary) unions: If $\mathcal{U} \subset \mathcal{T}$, then $\bigcup_{U \in \mathcal{U}} U$ is an element of $\mathcal{T}$.
(3) $\mathcal{T}$ is closed under finite intersections: If $U_1, \ldots U_n \in \mathcal{T}$, then $\bigcap_{i=1}^n U_i$ is an element of $\mathcal{T}$.

A *topological space* is a set $X$ together with a topology $\mathcal{T}$ on $X$, denoted $(X, \mathcal{T})$, or simply $X$ when $\mathcal{T}$ is understood. The elements of $\mathcal{T}$ are called *open* subsets of $X$.

DEFINITION 3.2. A *basis* for a topology $\mathcal{T}$ is a subset $\mathcal{B}$ of $\mathcal{T}$ such that every element of $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$.

These definitions may seem opaque, but hopefully they will seem less so after it is pointed out that the results of the previous subsection combine to give us the following fact: Any metric space, together with its open sets, is a topological space, and a basis for the topology is given by the open balls of $X$.

THEOREM 3.3. *Let $(X, d)$ be a metric space. Let $\mathcal{T}$ denote the collection of all subsets of $X$ which are open according to Definition 2.13, and let $\mathcal{B}$ denote the collection $\mathcal{B} = \{B_X(x, r) \in \mathcal{T} : x \in X,\ r > 0\}$. Then $\mathcal{T}$ is a topology on $X$, and $\mathcal{B}$ is a basis for $\mathcal{T}$.*

The topology $\mathcal{T}$ consisting of sets which are open according to the metric space definition is called the *metric topology*. The metric topology on any Euclidean space $\mathbb{R}^n$ is called the *standard topology* on $\mathbb{R}^n$.

# Part 2

# Analysis on Metric Spaces

CHAPTER 4

# Basic Topological Concepts in a Metric Space

## 1. Fundamental Notions

Recall from the previous Chapter that a subset $U$ of a metric space $(X, d)$ is called *open* if every point of $U$ is an interior point of $U$ with respect to $X$. That is, if $x \in U$, then $B_X(x, r) \subset U$ for some $r > 0$. We now define and investigate several related concepts.

### 1.1. Limit Points and Limits of Sequences.

1.1.1. *Definition of a Limit Point.*

DEFINITION 1.1. Let $(X, d)$ be a metric space. If $U$ is an open set of $X$ containing $x$, we say that $U$ is a *neighborhood* of $x$ in $X$. The set $B_X(x, \varepsilon)$ is called an *$\varepsilon$-neighborhood* of $x$ in $X$.

If $(X, d)$ is understood and is the only metric space under consideration, we sometimes refer simply to a 'neighborhood of $x$', or an '$\varepsilon$-neighborhood of $x$', without mention of the underlying space $X$.

DEFINITION 1.2. Let $(X, d)$ be a metric space; let $E$ be a subset of $X$. A point $x$ is said to be a *limit point* of $E$ with respect to $X$ if every neighborhood $U$ of $x$ in $X$ intersects $E \backslash \{x\}$. We will denote the set of all limit points of $E$ with respect to $X$ by $\mathrm{Lim}_X(E)$:

$$\mathrm{Lim}_X(E) = \{x \in X : x \text{ is a limit point of } E \text{ with respect to } X\}.$$

A point $x$ is called an *isolated point* of $E$ with respect to $X$ if $x \in E$ and $x$ is not a limit point of $E$ with respect to $X$.

If $(X, d)$ is understood from context and it is the only metric space under consideration, we sometimes write $E'$ instead of $\mathrm{Lim}_X(E)$.

We make a trivial but useful observation: If $x \notin E$, then $E = E \backslash \{x\}$. So if $x \notin E$, then $x$ is a limit point of $E$ if and only if every neighborhood of $x$ intersects $E$.

EXAMPLE 1.3. Let $A$ be the subset of $\mathbb{R}$ given by $A = (3, 5) \cup \{9\}$. Then $\mathrm{Lim}_{\mathbb{R}}(A) = [3, 5]$. The point 9 is an isolated point of $A$. We prove these claims in several steps.

First, 9 is an isolated point of $A$ with respect to $\mathbb{R}$, since $(8, 10)$ is a neighborhood of 9 in $\mathbb{R}$ which does not intersect $A \backslash \{9\}$.

Next, we show that $\mathrm{Lim}_{\mathbb{R}}(A) \subset [3, 5]$ by taking complements. Assume that $x \notin [3, 5]$; we want to show that $x \notin \mathrm{Lim}_{\mathbb{R}}(A)$. We may assume without loss of generality that $x \neq 9$, as we have already shown that 9 is an isolated point (and isolated points cannot be limit points). Then $x \in (-\infty, 3) \cup (5, 9) \cup (9, \infty)$, which is an open set of $\mathbb{R}$ (and thus a neighborhood of $x$ in $\mathbb{R}$) that does not intersect $A$. Therefore $x$ is not a limit point of $A$ with respect to $\mathbb{R}$. Since $x$ was an arbitrary point of $\mathbb{R} \backslash [3, 5]$, we conclude that $\mathrm{Lim}_{\mathbb{R}}(A) \subset [3, 5]$.

To prove that $\mathrm{Lim}_{\mathbb{R}}(A) \supset [3, 5]$, we argue as follows. Choose $x \in [3, 5]$, and let $U$ be a neighborhood of $x$. We consider three cases.

- Case 1: $x \in (3, 5)$. Then $U \cap (3, 5)$ is a neighborhood of $x$, so there exists $r > 0$ such that $B_{\mathbb{R}}(x, r) = (x - r,\ x + r)$ is contained in $U \cap (3, 5)$. Then $U \cap (A \backslash \{x\}) \supset (x, x + r) \ni x + \frac{r}{2}$. So $x$ is a limit point of $A$ with respect to $\mathbb{R}$.
- Case 2: $x = 3$. Then there exists $r \in (0, 2)$ such that $(3 - r,\ 3 + r) \subset U$. Thus $U \cap A \supset (3, 3 + r) \ni 3 + \frac{r}{2}$. Thus 3 is a limit point of $A$ with respect to $\mathbb{R}$.

- Case 3: $x = 5$. Then there exists $r \in (0, 2)$ such that $(5 - r,\ 5 + r) \subset U$. Thus $U \cap A \supset (5 - r, 5) \ni 5 - \frac{r}{2}$. Thus 5 is a limit point of $A$ with respect to $\mathbb{R}$.

One can show in the same manner as above that whenever $a, b \in \mathbb{R}$ and $a < b$, we have

$$\mathrm{Lim}_{\mathbb{R}}((a, b)) = \mathrm{Lim}_{\mathbb{R}}((a, b]) = \mathrm{Lim}_{\mathbb{R}}([a, b)) = \mathrm{Lim}_{\mathbb{R}}([a, b]) = [a, b].$$

Furthermore, for any $c \in \mathbb{R}$, we have

$$\mathrm{Lim}_{\mathbb{R}}((c, +\infty)) = \mathrm{Lim}_{\mathbb{R}}([c, +\infty)) = [c, +\infty), \qquad \mathrm{Lim}_{\mathbb{R}}((-\infty, c)) = \mathrm{Lim}_{\mathbb{R}}((-\infty, c]) = (-\infty, c].$$

1.1.2. *Tools for Computing* $\mathrm{Lim}_X(E)$. The process by which we computed the limit points of a given set in the Example above seems unduly complicated. The goal of this section is to introduce some tools that make the process much less cumbersome in most cases. We begin with the following questions, which the reader may have already asked themself while working the example above.

(1) Can the 'neighborhoods' in the definition of a limit point be replaced with $\varepsilon$-neighborhoods?
(2) Is there a relationship between the interior of a set and the set of its limit points?

The answer to the first question is 'yes'; the answer to the second is (a tiny bit) more subtle but is also 'yes'. We make each of these statements precise below.

PROPOSITION 1.4. *Let $E$ be a subset of $X$. A point $x \in X$ is a limit point of $E$ with respect to $X$ of $E$ if and only if every $\varepsilon$-neighborhood of $x$ intersects $E \backslash \{x\}$.*

PROOF. If $x \in \mathrm{Lim}_X(E)$, then every neighborhood of $x$, in particular every $\varepsilon$-neighborhood of $x$, intersects $E\backslash\{x\}$. For the non-trivial direction, assume every $\varepsilon$-neighborhood of $x$ intersects $E\backslash\{x\}$, and let $U$ be a neighborhood of $x$ in $X$. Then there exists $r > 0$ such that $B_X(x, r) \subset U$. Therefore $U \cap (E\backslash\{x\}) \supset B_X(x, r) \cap (E\backslash\{x\})$ is nonempty, since $B_X(x, r)$ intersects $E\backslash\{x\}$ by assumption. ☐

The reader might wonder now why we defined limit points in terms of neighborhoods rather than $\varepsilon$-neighborhoods in the first place; actually, each formulation has its advantage.

Advantages of open set definition of limit point:

- Easier to prove that a given point is *not* a limit point.
- Easier to prove things about points already known to be limit points.
- This definition is valid in a general topological space.

Advantages of $\varepsilon$-neighborhood formulation of a limit point:

- Easier to prove that a point is a limit point in the first place.

The reader may feel (justifiably) that we are belaboring this point. However, this theme of open sets versus open balls will come back again and again. Oftentimes, we will define some concept in terms of general open sets, then prove that a formulation in terms of open balls is equivalent. This way, we can use the advantages of each. For example, we can use the $\varepsilon$-neighborhood formulation to prove that a given point is a limit point of a certain set, then use the open set formulation to prove that the point in question satisfies certain properties.

PROPOSITION 1.5. *Let $(X, d)$ be a metric space, and let $E$ be a subset of $X$.*

- *$\{x\}$ is open in $X$ if and only if there exists $r > 0$ such that $B_X(x, r) = \{x\}$.*
- *If $\{x\}$ is open in $X$ and $x \in E$, then $x \in \mathrm{Int}_X(E)$, and $x$ is an isolated point of $E$ with respect to $X$.*
- *If $\{x\}$ is not open in $X$ and $x \in \mathrm{Int}_X(E)$, then $x \in \mathrm{Lim}_X(E)$.*

PROOF. If $B_X(x, r) = \{x\}$ for some $r > 0$, then clearly $\{x\}$ is open in $X$. If $\{x\}$ is open in $X$, then there exists $r > 0$ such that $B_X(x, r) \subset \{x\}$, by definition of an open set. But $\{x\} \subset B_X(x, r)$ always holds. So $\{x\} = B_X(x, r)$ for this choice of $r > 0$.

If $\{x\}$ is open in $X$, then by the previous statement, there exists $r > 0$ such that $B_X(x,r) = \{x\} \subset E$, which shows that $x \in \mathrm{Int}_X(E)$. On the other hand, $\{x\}$ is a neighborhood of $x$ in $X$ which does not intersect $E\backslash\{x\}$, so $x \notin \mathrm{Lim}_X(E)$. Since $x \in E$, this means that $x$ is an isolated point of $E$ with respect to $X$.

Assume that $\{x\}$ is not open in $X$ and $x \in \mathrm{Int}_X(E)$. Choose $r > 0$ so that $B_X(x,r) \subset E$; since $\{x\}$ is not open in $X$ we can find $y \neq x$ in $B_X(x,r)$. This $y$ belongs to $B_X(x,r) \cap (E\backslash\{x\})$. This shows that every $\varepsilon$-neighborhood of $x$ intersects $E\backslash\{x\}$, so that $x \in \mathrm{Lim}_X(E)$. □

COROLLARY 1.6. *Give $\mathbb{R}^k$ the Euclidean metric, and let $E$ be a subset of $\mathbb{R}^k$. Then* $\mathrm{Int}_{\mathbb{R}^k}(E) \subset \mathrm{Lim}_{\mathbb{R}^k}(E)$.

PROOF. This statement follows immediately from the previous proposition, together with the fact that $\{x\}$ is not open in $\mathbb{R}^k$ for any $x \in \mathbb{R}^k$. □

Note that in the above Corollary, $\mathbb{R}^k$ can be replaced by any metric space in which singleton sets $\{x\}$ are never open.

REMARK 1.7. The discussion above tells us that $\mathrm{Int}_X(E)$ is 'usually' a subset of $\mathrm{Lim}_X(E)$. However, $\mathrm{Int}_X(E)$ is rarely all of $\mathrm{Lim}_X(E)$. This will be clearer when we talk about closed sets later on.

Here is another straightforward consequence of the definition:

PROPOSITION 1.8. *Let $(X,d)$ be a metric space and assume $E \subset F \subset X$. Then*

$$\mathrm{Lim}_X(E) \subset \mathrm{Lim}_X(F).$$

PROOF. Assume $x \in \mathrm{Lim}_X(E)$, and let $U$ be a neighborhood of $x$ in $X$. Then $\emptyset \neq U \cap (E\backslash\{x\}) \subset U \cap (F\backslash\{x\})$, so $x \in \mathrm{Lim}_X(F)$. □

The following Exercise (and subsequent discussion) gives a way of computing the limit points of a finite union of sets, provided one knows the limit points of each of the sets over which the union is taken.

EXERCISE 1.9. Let $E_1$ and $E_2$ be subsets of a metric space $(X,d)$. Prove that

$$\mathrm{Lim}_X(E_1 \cup E_2) = \mathrm{Lim}_X(E_1) \cup \mathrm{Lim}_X(E_2).$$

With the result of this exercise in hand, an easy induction argument shows that

$$\mathrm{Lim}_X\left(\bigcup_{i=1}^{n} E_j\right) = \bigcup_{j=1}^{n} \mathrm{Lim}_X(E_j),$$

for any finite collection $\{E_j\}_{j=1}^n$ of subsets of $X$. However, this does *not* extend to infinite collections. Indeed,

$$\mathrm{Lim}_{\mathbb{R}}\left(\bigcup_{n=1}^{\infty}[\frac{1}{n},1]\right) = \mathrm{Lim}_{\mathbb{R}}((0,1]) = [0,1];$$

$$\bigcup_{n=1}^{\infty}\mathrm{Lim}_{\mathbb{R}}\left([\frac{1}{n},1]\right) = \bigcup_{n=1}^{\infty}[\frac{1}{n},1] = (0,1].$$

Finally, whether a point $x$ is a limit point of a given set $E$ depends on the underlying metric space. See the Exercise below.

EXERCISE 1.10. Let $(X,d)$ be a metric space, and assume $E \subset Y \subset X$. Prove that

$$\mathrm{Lim}_Y(E) = \mathrm{Lim}_X(E) \cap Y.$$

Hint: Use Theorem 2.19 in Chapter 3.

1.1.3. *Sequences and Limit Points.*

DEFINITION 1.11. Let $(X, d)$ be a metric space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in $X$. We say that $(x_n)_{n=1}^{\infty}$ *converges* in $X$ to a point $x \in X$ if for every neighborhood $U$ of $x$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $x_n \in U$. A sequence that converges is called *convergent*. If a sequence does not converge, we say that it *diverges*, or is *divergent*.

REMARK 1.12. The following are all standard ways to write '$(x_n)_{n=1}^{\infty}$ converges in $X$ to $x \in X$', provided $(X, d)$ is understood from context.

- $\lim_{n\to\infty} x_n = x$.
- $x_n \to x$ as $n \to \infty$.

If $(s_n)_{n=1}^{\infty}$ is a sequence of real numbers, one often write '$s_n \to s$ as $n \to \infty$', without even mentioning $\mathbb{R}$ or the Euclidean metric.

Whether or not a given sequence converges can depend on the metric space in which the sequence lives. However, if $(X, d)$ is a metric space and $Y$ is a subset of $X$, considered (as usual) with the same metric, then the relationship between convergence in $X$ and convergence in $Y$ is rather straightforward, as demonstrated by the following Exercise.

EXERCISE 1.13. Let $(X, d)$ be a metric space, and assume $Y \subset X$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $Y$ and let $x$ be a point of $X$. Prove that the following two statements are equivalent:

(1) $x_n \to x$ in $X$, and $x \in Y$.
(2) $x_n \to x$ in $Y$.

Hint: Use Theorem 2.19 in Chapter 3.

EXERCISE 1.14. Let $(X, d)$ be a metric space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in $X$. Prove that the following statements are equivalent:

(1) $x_n \to x$ in $X$.
(2) For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in B_X(x, \varepsilon)$ (i.e. $d(x, x_n) < \varepsilon$).
(3) $d(x, x_n) \to 0$ as $n \to \infty$

EXERCISE 1.15. Let $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ be sequences of real numbers, with $t_n > 0$ for each $n \in \mathbb{N}$. Assume that $t_n \to 0$ as $n \to \infty$.

- Prove that if $|s_n - s| < t_n$ for all $n \in \mathbb{N}$, then $s_n \to s$ as $n \to \infty$.
- Prove that $1/n \to 0$ as $n \to \infty$.

The next Theorem spells out an important relationship between limits of sequences and limit points.

THEOREM 1.16. *Let $(X, d)$ be a metric space and let $E$ be a subset of $X$. The following are equivalent:*

*(1) $x$ is a limit point of $E$ with respect to $X$.*
*(2) Every neighborhood of $x$ in $X$ contains infinitely many points of $E$.*
*(3) There exists a sequence $(x_n)_{n=1}^{\infty}$ in $E\backslash\{x\}$ that converges in $X$ to $x$.*

PROOF. (1) $\implies$ (2) Let $U$ be an open set of $X$ that contains $x$, and assume that $U$ contains only finitely many points of $E$; then we can write $U \cap (E\backslash\{x\}) = \{x_i\}_{i=1}^{n}$ for some $n \in \mathbb{N}$. Pick $r > 0$ such that $B(x, r) \subset U$, and put $\varepsilon = \min\{r, d(x, x_1), \ldots, d(x, x_n)\}$. Then $B_X(x, \varepsilon)$ is an open set of $X$, and

$$B_X(x, \varepsilon) \cap (E\backslash\{x\}) \subset U \cap (E\backslash\{x\}) = \{x_i\}_{i=1}^{n}.$$

On the other hand, $B_X(x, \varepsilon)$ contains none of the $x_i$, since $d(x, x_i) \geq \varepsilon$ by definition of $\varepsilon$. It follows that $B_X(x, \varepsilon) \cap (E\backslash\{x\})$ is empty, so $x$ is not a limit point of $E$. This establishes that (1) $\implies$ (2).

(2) $\implies$ (1) follows directly from the definition of a limit point. Indeed, if $U$ is an open set of $X$ that contains $x$, and (2) holds, then $U$ contains infinitely many points of $E$, therefore infinitely many

points of $E\backslash\{x\}$, and certainly at least one such point, which implies that $x$ is a limit point of $E$ with respect to $X$.

(1) $\implies$ (3) Assume $x$ is a limit point of $E$ with respect to $X$. For each $n \in \mathbb{N}$, let $x_n$ be a point of $B_X(x, \frac{1}{n}) \cap (E\backslash\{x\})$. (This is possible since $x$ is a limit point of $E$.) Then the sequence $(x_n)_{n=1}^{\infty}$ is a sequence in $E\backslash\{x\}$; this sequence converges in $X$ to $x$, since $d(x, x_n) < \frac{1}{n}$ for each $n \in \mathbb{N}$.

(3) $\implies$ (1) Assume $(x_n)_{n=1}^{\infty}$ is a sequence in $E\backslash\{x\}$ such that $x_n \to x$ as $n \to \infty$. Choose $\varepsilon > 0$, and then choose $n$ large enough so that $d(x, x_n) < \varepsilon$. Then $x_n \in B_X(x, \varepsilon) \cap (E\backslash\{x\})$. Thus $x$ is a limit point of $E$ with respect to $X$. ☐

Note that statement (2) in the Proposition above requires that $E$ has infinitely many points to begin with. Therefore we have the following Corollary:

COROLLARY 1.17. *Let $(X, d)$ be a metric space, and let $E$ be a finite subset of $X$. Then $E$ has no limit points with respect to $X$.*

### 1.2. Limit Points, Closed Sets, and Closure.

DEFINITION 1.18. Let $(X, d)$ be a metric space, and let $F$ be a subset of $X$. We say that $F$ is *closed* in $X$ if $X\backslash F$ is open in $X$.

Important! ‘Closed’ is *not* the same thing as ‘not open’! A set can be open, closed, neither, or both. For example,

- $(a, b)$ is open in $\mathbb{R}$ but not closed in $\mathbb{R}$;
- $[a, b]$ is closed in $\mathbb{R}$ but not open in $\mathbb{R}$;
- $\mathbb{R}$ is both open and closed in $\mathbb{R}$;
- $(a, b]$ is neither open nor closed in $\mathbb{R}$.

PROPOSITION 1.19. *Let $(X, d)$ be a metric space. Any intersection of closed subsets of $X$ is a closed set set of $X$. Any finite union of closed subsets of $X$ is a closed set of $X$.*

PROOF. Combine Proposition 2.17 in Chapter 3 with DeMorgan’s Laws. ☐

PROPOSITION 1.20. *Let $(X, d)$ be a metric space, and let $E$ be a subset of $X$. Then $E$ is closed in $X$ if and only if it contains all its limit points, i.e.,* $\mathrm{Lim}_X(E) \subset E$.

PROOF. ( $\implies$ ) Assume that $E$ is closed, and choose $x \in \mathrm{Lim}_X(E)$. If $x \notin E$, $X\backslash E$ is an open set containing $x$ which does not intersect $E$, contradicting the definition of a limit point. We conclude that $x$ must be an element of $E$, proving that $\mathrm{Lim}_X(E) \subset E$.

( $\impliedby$ ) On the other hand, assume that $\mathrm{Lim}_X(E) \subset E$; we show that $X\backslash E$ is open in $X$. To this end, let $x$ be a point of $X\backslash E$. Then $x \notin \mathrm{Lim}_X(E)$, so there is an open set $U$ of $X$ such that $U$ does not intersect $E\backslash\{x\} = E$, i.e. $x \in U \subset X\backslash E$. Since $U$ is open, it follows that $x \in \mathrm{Int}_X(U) \subset \mathrm{Int}_X(X\backslash E)$. Since $x$ was an arbitrary point of $X\backslash E$, we conclude that every point of $X\backslash E$ is an interior point; that is, $X\backslash E$ is open. Therefore $E$ is closed. ☐

EXERCISE 1.21. Let $(X, d)$ be a metric space, and let $E$ be a subset of $X$. Prove that $\mathrm{Lim}_X(E)$ is a closed set of $X$.

DEFINITION 1.22. Let $(X, d)$ be a metric space and let $E$ be a subset of $X$. The *closure* of $E$ in $X$ is the set $E$, together with the set of its limit points in $X$ adjoined, denoted $\mathrm{Cl}_X(E)$. That is,

$$\mathrm{Cl}_X(E) = E \cup \mathrm{Lim}_X(E).$$

If $(X, d)$ is clear from context and it is the only metric space under consideration, we may write $\overline{E}$ instead of $\mathrm{Cl}_X(E)$.

PROPOSITION 1.23. *Let $(X, d)$ be a metric space and let $E$ be a subset of $X$.*

- *$E$ is closed in $X$ if and only if $E = \mathrm{Cl}_X(E)$.*
- *$\mathrm{Cl}_X(E)$ is always a closed set.*
- *If $F$ is a closed set of $X$ and $E \subset F$, then $\mathrm{Cl}_X(E) \subset F$.*

Combining the second and third points tells us that $\mathrm{Cl}_X(E)$ is the 'smallest' closed set in $X$ that contains $E$.

PROOF. Assume $E$ is closed in $X$. Then $\mathrm{Lim}_X(E) \subset E$, so $\mathrm{Cl}_X(E) = E \cup \mathrm{Lim}_X(E) \subset E$. But $E \subset \mathrm{Cl}_X(E)$ by definition, so $E = \mathrm{Cl}_X(E)$. On the other hand, assume $E = \mathrm{Cl}_X(E)$. Then $\mathrm{Lim}_X(E) = E \cup \mathrm{Lim}_X(E) = \mathrm{Cl}_X(E) = E$, so $E$ is closed.

To show that $\mathrm{Cl}_X(E)$ is closed in $X$, we write

$$\mathrm{Lim}_X(E \cup \mathrm{Lim}_X(E)) = \mathrm{Lim}_X(E) \cup \mathrm{Lim}_X(\mathrm{Lim}_X(E)) \subset \mathrm{Lim}_X(E) \subset \mathrm{Cl}_X(E).$$

To get the first inclusion, we used the fact that $\mathrm{Lim}_X(E)$ is closed in $X$, by Exercise 1.21.

Finally, let $F$ be a closed set of $X$ such that $E \subset F$. Then $\mathrm{Lim}_X(E) \subset \mathrm{Lim}_X(F)$, so

$$\mathrm{Cl}_X(E) = E \cup \mathrm{Lim}_X(E) \subset F \cup \mathrm{Lim}_X(F) = \mathrm{Cl}_X(F) = F.$$

□

Note that in the course of proving the third point above, we have shown that if $E \subset F$, then $\overline{E} \subset \overline{F}$.

Another note: $x \in \mathrm{Cl}_X(E)$ if and only if every neighborhood $U$ of $x$ in $X$ intersects $E$. Proof: If every neighborhood of $x$ in $X$ intersects $E$, then either $x \in E$ or every neighborhood intersects $E \backslash \{x\}$, in which case $x \in \mathrm{Lim}_X(E)$. Conversely, if $x \in \mathrm{Cl}_X(E)$, then either $x \in E$, in which case every neighborhood of $x$ intersects $E$ at $x$, or $x \in \mathrm{Lim}_X(E)$, in which case every neighborhood intersects the smaller set $E \backslash \{x\}$, and therefore $E$. As is often the case, neighborhoods may be replaced by $\varepsilon$-neighborhoods in this criterion.

EXERCISE 1.24. Let $(X, d)$ be a metric space, and let $E$ be a subset of $X$. Prove that

$$X \backslash \mathrm{Cl}_X(E) = \mathrm{Int}_X(X \backslash E)$$

(This can be written more concisely as $(\overline{E})^c = (E^c)^\circ$, if desired.

EXERCISE 1.25. Let $(X, d)$ be a metric space. Let $E$ and $Y$ be subsets of $X$ such that $E \subset Y$. Prove that

$$\mathrm{Cl}_Y(E) = \mathrm{Cl}_X(E) \cap Y.$$

EXERCISE 1.26. Let $(X, d)$ be a metric space. Let $(X, d)$ be a metric space.

(1) Prove that for any collection $\mathcal{E}$ of subsets of $X$, we have
$$\bigcup_{E \in \mathcal{E}} \overline{E} \subset \overline{\bigcup_{E \in \mathcal{E}} E},$$
and equality holds if $\mathcal{E}$ is finite.

(2) Prove that for any collection $\mathcal{E}$ of subsets of $X$, we have
$$\bigcap_{E \in \mathcal{E}} \overline{E} \supset \overline{\bigcap_{E \in \mathcal{E}} E}.$$

(3) Give examples that demonstrate that equality might fail in part (1) if $\mathcal{E}$ is not finite, and equality might fail in part (2) even if $\mathcal{E}$ is finite.

### 1.3. Closure, Boundedness, and Density.

DEFINITION 1.27. Let $(X, d)$ be a metric space. A subset $E$ of $X$ is called *bounded* in $(X, d)$ if there exists $x \in X$ and $R > 0$ such that $E \subset B_{(X,d)}(x, R)$. If $Z$ is any set and $f : Z \to X$ is a function, we say that $f$ is *bounded* if its image $f(Z)$ is bounded in $(X, d)$.

EXERCISE 1.28. Let $(X, d)$ be a metric space.

(1) Prove that for any $x \in X$ and $r > 0$, we have $\overline{B_X(x, r)} \subset \{y \in X : d(x, y) \leq r\}$. (Hint: Take complements and draw a picture.) Note that the inclusion $\overline{B_X(x, r)} \subset B_X(x, r + \varepsilon)$ follows for any $\varepsilon > 0$.
(2) Give an example using the discrete metric that demonstrates that equality need not hold in the inclusion $\overline{B_X(x, r)} \subset \{y \in X : d(x, y) \leq r\}$ that you proved in part (1).
(3) Prove that in $\mathbb{R}^n$ under the Euclidean metric $d(x, y) = \|x-y\|$, we have $\overline{B_{\mathbb{R}^n}(x, r)} = \{y \in \mathbb{R}^n : \|x - y\| \leq r\}$. (Again, a picture may be useful.)
(4) Using part (1), prove that if $A$ is bounded in $(X, d)$, then $\overline{A}$ is also bounded in $(X, d)$.

DEFINITION 1.29. Let $(X, d)$ be a metric space. A subset $E$ of $X$ is said to be *dense* in $X$ if $\mathrm{Cl}_X(E) = X$.

EXERCISE 1.30. Let $(X, d)$ be a metric space, and let $E$ be a subset of $X$.

(1) Show that $E$ is dense in $X$ if and only if any nonempty open subset of $X$ contains a point of $E$.
(2) Suppose $E \subset Y \subset X$. Prove that $E$ is dense in $Y$ if and only if $\mathrm{Cl}_X(E) \supset Y$.

EXERCISE 1.31. Previously, we said that a subset $E$ of $\mathbb{R}$ was dense in $\mathbb{R}$ if for any real numbers $a$ and $b$, there exists a number $c \in E$ which lies between $a$ and $b$. Show that in $\mathbb{R}$, the new, more general definition of 'dense' agrees with the old one. That is, show that a subset $E$ of $\mathbb{R}$ is dense in $\mathbb{R}$ according to the new definition if and only if it is dense according to the old one. (Hint: Use Exercise 1.30(1).)

## 2. Sequences in Metric Spaces

Consider the following sequence in $\mathbb{R}$:

$$(x_n)_{n=1}^{\infty} = (3, 1, 3, \frac{1}{2}, 3, \frac{1}{3}, 3, \frac{1}{4}, 3, \frac{1}{5}, 3, \frac{1}{6}, \ldots)$$

The sequence $(x_n)_{n=1}^{\infty}$ diverges. However, its subsequences $(3, 3, 3, 3, 3, \ldots)$ and $(\frac{1}{n})_{n=1}^{\infty}$ do converge, to $3$ and $0$, respectively. We would like a concept that captures this information; the limit point notion is insufficient. Indeed, one can show that $\mathrm{Lim}_{\mathbb{R}}(\{x_n\}_{n=1}^{\infty}) = \{0\}$; the point $3$ is an isolated point of the image of the sequence $(x_n)_{n=1}^{\infty}$. The notion of a subsequential limit is the concept we are looking for.

DEFINITION 2.1. Let $(x_n)_{n=1}^{\infty}$ be a sequence in a metric space $(X, d)$, and let $(x_{n_k})_{k=1}^{\infty}$ be a subsequence of $(x_n)_{n=1}^{\infty}$. If $x_{n_k} \to x$ as $k \to \infty$, then $x$ is called a *subsequential limit* of the original sequence $(x_n)_{n=1}^{\infty}$.

In this section, after establishing some basic properties of sequences and convergence, we will show that a subsequential limit of a sequence $(x_n)_{n=1}^{\infty}$ is either a limit point of $\{x_n\}_{n=1}^{\infty}$ or a term of the sequence that appears infinitely many times. Next, we will discuss *Cauchy sequences*, which act like convergent sequences in many ways but may or may not have a limit.

**2.1. Convergent Sequences.** Recall that we say that a function is bounded if its image is bounded. Since a sequence is in particular a function, this terminology applies here as well.

PROPOSITION 2.2. *(Convergent sequences are bounded.) Let $(x_n)_{n=1}^{\infty}$ be a sequence in a metric space $(X, d)$. If $(x_n)_{n=1}^{\infty}$ converges in $X$, then it is bounded.*

PROOF. Assume $x_n \to x$ as $n \to \infty$. Choose $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in B_X(x, 1)$. Then put $R = 1 + \max\{1, d(x, x_1), \ldots, d(x, x_N)\}$. Then $\{x_n\}_{n=1}^{\infty} \subset B_X(x, R)$. Thus $\{x_n\}_{n=1}^{\infty}$ is bounded. $\square$

In the following discussion, we write $(x_n)$ for $(x_n)_{n=1}^{\infty}$ and $\{x_n\}$ for $\{x_n\}_{n=1}^{\infty}$.

PROPOSITION 2.3. *Let $(p_n)_{n=1}^{\infty}$ be a sequence in a metric space $(X, d)$, and assume that $p_n \to p$ as $n \to \infty$. The the following statements hold.*

- $(\{p_n\})' \subset \{p\}$.
- $(\{p_n\})' = \emptyset$ *if and only if* $p_n = p$ *for all but finitely many* $n \in \mathbb{N}$.

PROOF. Assume $q \neq p$; we show that $q$ is not a limit point of $\{p_n\}_{n=1}^{\infty}$. Put $R = \frac{1}{2} d(p, q)$, so that $B_X(p, R)$ and $B_X(q, R)$ are disjoint[1]. Choose $N$ large enough so that $n \geq N$ implies that $p_n \in B_X(p, R)$. Then $B_X(q, R)$ contains at most finitely many points of $\{p_n\}_{n=1}^{\infty}$, so $q$ cannot be a limit point of $\{p_n\}_{n=1}^{\infty}$.

If $(\{p_n\})' = \emptyset$, then (since $p$ is not a limit point of $\{p_n\}$), there exists a neighborhood $U$ of $p$ such that $U$ contains no points of $\{p_n\}$ other than $p$. On the other hand, $U$ must contain all but finitely many of the $p_n$, by definition of the limit. These $p_n$ must therefore all be equal to $p$.

Conversely, if $p_n = p$ for all but finitely many $n$, then $\{p_n\}_{n=1}^{\infty}$ is actually a finite set; therefore it has no limit points. $\square$

This Proposition says that for a convergent sequence $(p_n)$, there are only two possibilities for $(\{p_n\})'$. Either $(\{p_n\})'$ is a single point—the point $p$ to which the sequence converges, or $(\{p_n\})'$ is empty, in which case the 'tail' of the sequence looks like $(p, p, p, p, \ldots)$.

COROLLARY 2.4. *(Limits are unique) Let $(p_n)_{n=1}^{\infty}$ be a sequence in a metric space $(X, d)$. If $p_n \to p$ and $p_n \to q$ as $n \to \infty$, then $p = q$.*

PROOF. If $p \neq q$, then $(\{p_n\})' \subset \{p\} \cap \{q\} = \emptyset$. Therefore all but finitely many of the $p_n$ must be equal to $p$, and all but finitely many of the $p_n$ must be equal to $q$. This is impossible. $\square$

**2.2. Subsequential Limits of a Sequence.** We now expand our view to include sequences that may or may not converge.

PROPOSITION 2.5. *Let $(p_n)_{n=1}^{\infty}$ be a sequence in a metric space $(X, d)$. If $p \in (\{p_n\})'$, then there exists a subsequence $(p_{n_k})_{k=1}^{\infty}$ of $(p_n)$ such that $p_{n_k} \to p$ as $k \to \infty$.*

PROOF. We construct a subsequence $(p_{n_k})$ by induction. Choose $n_1 \in \mathbb{N}$ such that $p_{n_1} \in B_X(p, 1)$. Assume that $n_1 < n_2 < \cdots < n_k$ are given. We claim that there exists $n_{k+1} \in \mathbb{N}$ such that $n_{k+1} > n_k$ and $p_{n_{k+1}} \in B_X(p, \frac{1}{k+1})$. Indeed, since $p$ is a limit point of $\{p_n\}$, we know that $B_X(x, \frac{1}{k+1})$ contains infinitely many points of $\{p_n\}_{n=1}^{\infty}$, so it must contain infinitely many points of $\{p_n\}_{n=n_k+1}^{\infty}$.

The subsequence we have constructed converges to $p$, since $d(p_{n_k}, p) < \frac{1}{k}$ for each $k \in \mathbb{N}$. $\square$

The reader might wonder whether or not this proof was really necessary in light of Theorem 1.16, which guarantees under our hypotheses that there exists a sequence $(x_n)_{n=1}^{\infty}$ in $\{p_n\} \backslash \{p\}$ that converges to $p$. However, the order of the terms of the sequence $(x_n)$ that this Theorem provides might be completely different than the order of the terms of the sequence $(p_n)$. Therefore the sequence provided by the Theorem might not be a subsequence of $(p_n)$. This is why we paid special attention to the order of the terms when writing the proof of the Proposition.

THEOREM 2.6. *Let $S = (p_n)_{n=1}^{\infty}$ be a sequence in a metric space $(X, d)$. Let $S^*$ denote the set of all subsequential limits of $S$, and let $S_\infty$ denote the set of all terms of $S$ that appear infinitely many times in the sequence (i.e., $S_\infty = \{p \in X : p_n = p$ for infinitely many $n \in \mathbb{N}\}$). Then*

$$S^* = (\operatorname{Im} S)' \cup S_\infty,$$

[1]Proof: If $z \in B_X(x, R) \cap B_X(y, R)$, then $d(x, y) \leq d(x, z) + d(z, y) < R + R = 2R = d(x, y)$, a contradiction.

*and $S^*$ is closed in $X$.*

PROOF. We have essentially already proved the inclusion $(\operatorname{Im} S)' \cup S_\infty \subset S^*$. Indeed, Proposition 2.5 implies that $(\operatorname{Im} S)' \subset S^*$, and the inclusion $S_\infty \subset S^*$ is clear[2]. Thus $S^* \supset (\operatorname{Im} S) \cup S_\infty$.

To prove the opposite inclusion, choose $p \in S^*$. If $p \in (\operatorname{Im} S)'$, then we are done; therefore assume without loss of generality that $p \notin (\operatorname{Im} S)'$. It suffices to show, then, that $p \in S_\infty$. Since $p \in S^*$, we can find a subsequence $(p_{n_k})_{k=1}^\infty$ of $(p_n)$ that converges to $p$ in $X$. Then $(\{p_{n_k}\})' \subset (\operatorname{Im} S)'$ (simply because $(p_{n_k})$ is a subsequence of $(p_n)$), and $(\{p_{n_k}\})' \subset \{p\}$ (by Proposition 2.5), so $(\{p_{n_k}\})' \subset (\operatorname{Im} S)' \cap \{p\}$. But $p \notin (\operatorname{Im} S)'$ by assumption, so $(\{p_{n_k}\})' = \emptyset$. Thus, by Proposition 2.3, it follows that $p_{n_k} = p$ for all but finitely many $k$, whence $p_n = p$ for infinitely many $n$, whence $p \in S_\infty$. This completes the proof of the equality $S^* = (\operatorname{Im} S)' \cup S_\infty$.

To prove that $S^*$ is closed in $X$, we look at $(S^*)'$.

$$\begin{aligned}
(S^*)' &= ((\operatorname{Im} S)' \cup S_\infty)' && \text{Definition of } S^* \\
&= ((\operatorname{Im} S)')' \cup S_\infty' && \text{Exercise 1.9} \\
&\subset (\operatorname{Im} S)' && \text{Exercise 1.21, plus } S_\infty \subset \operatorname{Im} S \\
&\subset S^*.
\end{aligned}$$

This completes the proof. □

Note that as a trivial Corollary of the above Proposition, we have $S^* \subset \overline{\operatorname{Im} S}$. This fact is sometimes useful when determining $S^*$.

EXERCISE 2.7. Let $S = (p_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$ whose image is $(\mathbb{Q} \cap (0,1)) \cup \{5\}$. What are the two possibilities for $S^*$? Justify your answer.

PROPOSITION 2.8. *Let $S = (p_n)_{n=1}^\infty$ be a sequence in a metric space $X$, and let $S^*$ denote the set of subsequential limits of $S$. The following are equivalent.*

*(1) $p_n \to p$ in $X$ as $n \to \infty$.*
*(2) Every subsequence of $(p_n)_{n=1}^\infty$ converges to $p$ in $X$.*
*(3) $S^* = \{p\}$, and every subsequence of $(p_n)_{n=1}^\infty$ converges in $X$.*

EXERCISE 2.9. Prove Proposition 2.8.

**2.3. Sequential Compactness and Limit Point Compactness.** In this subsection, we give a name to the condition that $S^*$ is nonempty for all sequences $S$, and we compare it to a related notion for limit points.

DEFINITION 2.10. A metric space $(X, d)$ is called *sequentially compact* if every sequence has a convergent subsequence. We say that $X$ is *limit point compact* if every infinite subset of $X$ has a limit point in $X$.

We show that the two notions are in fact the same in any metric space.

PROPOSITION 2.11. *Let $(X, d)$ be a metric space, and assume that $X$ is limit point compact. Then $(X, d)$ is sequentially compact. More specifically, if $(x_n)_{n=1}^\infty$ is a sequence in $X$, then it has at least one subsequential limit in $X$.*

PROOF. If $\{x_n\}_{n=1}^\infty$ is finite, then there exists $x \in X$ such that $x_n = x$ for infinitely many $n$, which implies by Theorem 2.6 that there is a subsequence of $(x_n)$ that converges to $x$ in $X$. If $\{x_n\}$ is infinite, then it has a limit point $x$ in $X$ (as $X$ is limit point compact), so again by Theorem 2.6, $x$ is a subsequential limit of the sequence $(x_n)$. □

[2]In case the reader disagrees, here is a proof: If $p \in S_\infty$, then $p_n = p$ for infinitely many $n$. Let $n_1$ be the smallest integer such that $p_{n_1} = p$. Having chosen $n_1 < n_2 < \cdots < n_k$, let $n_{k+1}$ be the smallest integer larger than $n_k$ such that $p_{n_{k+1}} = p$. Then $(p_{n_k})_{k=1}^\infty = (p, p, p, p, \dots)$ clearly converges to $p$ in $X$.

PROPOSITION 2.12. *Let $(X, d)$ be a sequentially compact metric space. Then $(X, d)$ is limit point compact. More specifically, let $A$ be an infinite subset of $X$. Then $A$ has a limit point in $X$.*

PROOF. Let $B$ be a countably infinite subset of $A$, and let $(x_n)_{n=1}^{\infty}$ be an enumeration of $B$; assume without loss of generality that the $x_n$ are all distinct. Then $(x_n)$ has a subsequential limit $x$, which does not appear infinitely many times in the sequence $(x_n)$; therefore $x$ is a limit point of $\{x_n\}_{n=1}^{\infty} = B$ and thus of $A$. □

We have thus proved:

THEOREM 2.13. *A metric space $(X, d)$ is limit point compact if and only if it is sequentially compact.*

### 2.4. Cauchy Sequences.

DEFINITION 2.14. Let $(X, d)$ be a metric space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in $X$. We say that $(x_n)_{n=1}^{\infty}$ is a *Cauchy sequence* in $X$ (or simply *Cauchy* in $X$) if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m \geq n \geq N$ implies $d(x_n, x_m) < \varepsilon$. We say that $(X, d)$ is *complete* if every Cauchy sequence in $X$ converges in $X$.

Note that the key difference between a convergent sequence and a Cauchy sequence is that there is no mention of a limit in the definition of a Cauchy sequence. However, a Cauchy sequence behaves rather like a convergent sequence, in that it 'wants' to converge somewhere. If the point to which it 'wants' to converge lies in the metric space in question, then it does converge; however, this need not be the case. Here are two illustrative examples.

EXAMPLE 2.15. Consider the sequence $S = (\frac{1}{n})_{n=1}^{\infty}$ in $\mathbb{R}$ in the Euclidean metric. Let $Y = (0, 2)$ in the same metric. Then $S$ converges in $\mathbb{R}$, to $0$, but $S$ does not converge in $Y$. Indeed, if $S$ converges to $y$ in $Y$, then $S$ also converges to $y$ in $X$. But $0 \notin Y$, so $y \neq 0$. This contradicts the uniqueness of limits of a convergent sequence.

However, $S$ *is* Cauchy in $Y$. Indeed, choose $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \frac{2}{\varepsilon}$. Then $m \geq n \geq N$ implies that $\left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} + \frac{1}{n} \leq \frac{2}{N} < \varepsilon$. Thus $S$ is Cauchy in $Y$.

This example also shows that $Y = (0, 2)$ is not complete and not sequentially compact.

Note that if a sequence $(x_n)_{n=1}^{\infty}$ in $(X, d)$ is Cauchy in $X$, then it is also Cauchy in $Y$ whenever $\{x_n\}_{n=1}^{\infty} \subset Y \subset X$. Similarly, if $(x_n)_{n=1}^{\infty}$ is a sequence in $X$ which is Cauchy in $Y$, then it is also Cauchy in $X$. These claims follow immediately from the definition of a Cauchy sequence.

EXERCISE 2.16. Let $(X, d)$ be a metric space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in $X$. Prove the following statements.

(1) If $(x_n)_{n=1}^{\infty}$ converges in $X$, then it is Cauchy in $X$
(2) If $(x_n)_{n=1}^{\infty}$ is Cauchy in $X$, then it is bounded in $X$.

In light of the first part of this Exercise, we see another way to conclude that the sequence $S = (\frac{1}{n})_{n=1}^{\infty}$ is Cauchy in $Y = (0, 2)$ under the Euclidean metric. We know that $S$ converges in $X = \mathbb{R}$, so it is Cauchy in $\mathbb{R}$, so it is Cauchy in $Y$.

EXAMPLE 2.17. Let $X = \mathbb{R}$, and let $d$ be the Euclidean metric. Consider the sequence $S = (1, 1.4, 1.41, 1.414, \ldots)$ which converges to $\sqrt{2}$ in $\mathbb{R}$. Then, reasoning as before, we see that $S$ converges in $\mathbb{R}$, but $S$ does not converge in $\mathbb{Q}$ (though it is Cauchy in $\mathbb{Q}$).

One way to generalize the examples above is via the first part of the Exercise below:

EXERCISE 2.18. Let $(X, d)$ be a metric space, and let $Y$ be a subset of $X$. Prove the following.

(1) If $Y$ is complete, then $Y$ is closed in $X$. (Hint: Use Theorem 1.16 and Proposition 1.20.)
(2) If $X$ is complete and $Y$ is closed in $X$, then $Y$ is complete. (Hint: Exercise 1.13 is relevant.)

PROPOSITION 2.19. *Let $(X,d)$ be a metric space, and let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in $X$. If $(x_n)_{n=1}^{\infty}$ has a subsequence that converges in $X$ to a point $x \in X$, then $(x_n)_{n=1}^{\infty}$ converges in $X$ to $x$.*

PROOF. Choose $\varepsilon > 0$. Since $(x_n)$ is Cauchy, we can pick $N \in \mathbb{N}$ large enough so that $m \geq n \geq N$ implies $d(x_m, x_n) < \varepsilon/2$. Next, let $(x_{n_k})_{k=1}^{\infty}$ be a subsequence of $(x_n)_{n=1}^{\infty}$ such that $x_{n_k} \to x$ in $X$; pick $M \in \mathbb{N}$ large enough so that $k \geq M$ implies $d(x, x_{n_k}) < \varepsilon/2$. Put $P = \max\{N, M\}$. We claim that $m \geq P$ implies $d(x, x_m) < \varepsilon$. (This will prove $x_n \to x$ in $X$.) In order to take advantage of both the fact that $(x_n)$ is Cauchy and that $x_{n_k} \to x$, we pick $k \geq m$ and apply the triangle inequality:

$$d(x, x_m) \leq d(x, x_{n_k}) + d(x_{n_k}, x_m).$$

Since $n_k \geq M$, we have $d(x, x_{n_k}) < \frac{\varepsilon}{2}$; since $n_k \geq m \geq N$, we have $d(x_{n_k}, x_m) < \frac{\varepsilon}{2}$. Thus $d(x, x_m) < \varepsilon$, and we are done. ☐

COROLLARY 2.20. *A Cauchy sequence has at most one subsequential limit.*

PROOF. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in a metric space $(X,d)$. If $x$ and $y$ are subsequential limits, then $x_n \to x$ and $x_n \to y$, by the previous Proposition. By uniqueness of limits, we must have $x = y$. Therefore $(x_n)$ has at most one subsequential limit in $X$. ☐

COROLLARY 2.21. *A sequentially compact metric space is complete.*

PROOF. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in a sequentially compact metric space $(X,d)$. By sequential compactness, $(x_n)$ has a subsequence which converges in $X$; let $x$ denote the subsequential limit. By Proposition 2.19, it follows that $x_n \to x$ in $X$. Thus $X$ is complete. ☐

### 2.5. More on Sequential Compactness.

DEFINITION 2.22. Let $\mathcal{A}$ be a collection of subsets of a set $X$, and let $E$ be a subset of $X$. We say the collection $\mathcal{A}$ *covers* $E$ (or is a *cover* of $E$) if

$$E \subset \bigcup_{A \in \mathcal{A}} A.$$

DEFINITION 2.23. Let $(X,d)$ be a metric space. $X$ is said to be *totally bounded* (with respect to $d$) if for every $\varepsilon > 0$, $X$ can be covered by finitely many balls of radius $\varepsilon$, that is, if there exist finitely many points $x_1, \ldots, x_n \in X$ such that

$$X = \bigcup_{j=1}^{n} B_{(X,d)}(x_j, \varepsilon).$$

Of course, $n$ may depend on $\varepsilon$, but $\varepsilon$ is the same for each $j$.

EXERCISE 2.24. Let $(X,d)$ be a metric space. Show that if $X$ is totally bounded, then it is bounded.

THEOREM 2.25. *Let $(X,d)$ be a sequentially compact metric space. Then $X$ is complete and totally bounded.*

REMARK 2.26. In fact, the converse of this Theorem is true: A metric space is sequentially compact if and only if it is complete and totally bounded. We even have the tools to prove it; however, the proof is rather involved, so we will omit it. Furthermore, we will not use it in what follows.

PROOF. In light of Corollary 2.21, it suffices to prove that $X$ is totally bounded. We argue by contradiction. Assume that there exists $\varepsilon > 0$ such that no finite collection of $\varepsilon$-balls covers $X$. We construct a sequence $(x_n)$ as follows. Pick $x_1 \in X$ arbitrarily. Assume $x_1, \ldots, x_k$ have been chosen so that $d(x_i, x_j) \geq \varepsilon$ for all $i, j \in J_k$. Then $\bigcup_{i=1}^{k} B_X(x_i, \varepsilon)$ is not all of $X$; pick $x_{k+1}$ outside this set, so that now $d(x_i, x_j) \geq \varepsilon$ for all $i, j \in J_{k+1}$. We claim that the sequence $(x_n)$ has no subsequential limits. Indeed, no subsequence can be Cauchy, as $d(x_i, x_j) \geq \varepsilon$ for all $i, j \in \mathbb{N}$; therefore no subsequence may converge. This contradicts the sequential compactness of $X$; we conclude therefore that $X$ is totally bounded. □

The following special case of the preceding Theorem is useful enough to single out:

COROLLARY 2.27. *Let $(X,d)$ be a metric space and let $Y$ be a subset of $X$. If $(Y,d)$ is sequentially compact, then $Y$ is closed and bounded in $X$.*

PROOF. By the previous Theorem, $Y$ is complete; therefore it is closed in $X$ (Exercise 2.18. Furthermore, $Y$ is totally bounded, therefore bounded (Exercise 2.24). □

## 3. Compactness

### 3.1. Basic Definitions and Examples.

DEFINITION 3.1. Let $\mathcal{A}$ be a collection of subsets of a set $X$, and let $E$ be a subset of $X$. We say the collection $\mathcal{A}$ *covers* $E$ (or is a *cover* of $E$) if

$$E \subset \bigcup_{A\in\mathcal{A}} A.$$

A *subcover* of $E$ corresponding to the cover $\mathcal{A}$ is a subcollection $\mathcal{B} \subset \mathcal{A}$ that still covers $E$.

EXAMPLE 3.2. Let $X = \mathbb{R}$, $E = (0,2)\cup\{5\}$, and $\mathcal{A} = \{(-1,1),(0,1),\{8\},(0,6)\}\cup\{(n,n+1)\}_{n=1}^{\infty}$. Then $E \subset \bigcup_{A\in\mathcal{A}} A$, so $\mathcal{A}$ is a cover of $E$. But some (in fact, most) of the sets of $\mathcal{A}$ were unnecessary for the purpose of covering $E$; if we take $\mathcal{B} = \{(-1,1),(0,6)\}$, then $\mathcal{B} \subset \mathcal{A}$, and $E \subset \bigcup_{B\in\mathcal{B}} B$, so $\mathcal{B}$ is a *subcover* of $E$ associated to the cover $\mathcal{A}$. In fact, the set $\mathcal{B}$ is finite subcollection of $\mathcal{A}$; to emphasize this fact, we say that $\mathcal{B}$ is a *finite subcover* of $E$ associated to the cover $\mathcal{A}$.

DEFINITION 3.3. Let $(X,d)$ be a metric space. A collection $\mathcal{A}$ of subsets of $X$ is called an *open cover* of $E$ in $X$ if $\mathcal{A}$ covers $E$ and each $A \in \mathcal{A}$ is open in $X$. A subset $K$ of $X$ is called *compact* in $X$ if every open cover of $K$ in $X$ has a finite subcover.

What the definition of compactness says, less concisely, is the following. Suppose you give me an arbitrary open cover $\mathcal{A}$ of $K$. If $K$ is compact, then no matter what collection $\mathcal{A}$ you gave me, I can choose a finite subcollection $\mathcal{B}$ of $\mathcal{A}$ that still covers $K$.

EXAMPLE 3.4. Let $(X,d)$ be a metric space and let $K$ be any finite subset of $X$. Then $K$ is compact in $X$. Proof: Write $K = \{x_1,\dots,x_n\}$, and let $\mathcal{A}$ be an arbitrary open cover of $K$ in $X$. For each $j \in J_n$, let $A_j$ be an element of $\mathcal{A}$ such that $x_j \in A_j$. Then $K = \bigcup_{j=1}^{n}\{x_j\} \subset \bigcup_{j=1}^{n} A_j$, so $\mathcal{B} = \{A_1,\dots,A_n\}$ is a finite subcover of $K$ associated to the (arbitrary) open cover $\mathcal{A}$. Thus $K$ is compact in $X$, as claimed.

EXAMPLE 3.5. The set $G = \{\frac{1}{n}\}_{n=1}^{\infty}$ is not compact in $\mathbb{R}$. Proof: For each $n \in \mathbb{N}$, define $A_n = (\frac{1}{n},+\infty)$. Consider the open cover $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$. Then $\mathcal{A}$ is an open cover of $G$, since $\frac{1}{n} \in A_{n+1}$ for each $n \in \mathbb{N}$. But no finite subcollection of $\mathcal{A}$ will cover $G$. Indeed, if $\mathcal{B}$ is a finite subcollection of $\mathcal{A}$, then $\mathcal{B}$ must be of the form

$$\mathcal{B} = \{(\tfrac{1}{m_1},+\infty),(\tfrac{1}{m_2},+\infty),\dots,(\tfrac{1}{m_k},+\infty)\}$$

for some $m_1,\dots,m_k \in \mathbb{N}$. Denoting $m := \max\{m_1,\dots,m_k\}$, it follows that $\bigcup_{B\in\mathcal{B}} B = (\frac{1}{m},+\infty)$, and thus that $\mathcal{B}$ does not cover $\frac{1}{m}$; therefore does not cover $G$. Since there exists an open cover of $G$ in $\mathbb{R}$ which has no finite subcover, we conclude that $G$ is not compact in $\mathbb{R}$.

On the other hand, $K = G\cup\{0\}$ *is* compact in $\mathbb{R}$. Indeed, let $\mathcal{A}$ be an open cover of $K$ in $\mathbb{R}$. Then some $A_0 \in \mathcal{A}$ is a neighborhood of $0$ in $\mathbb{R}$. Since $\frac{1}{n} \to 0$ as $n \to \infty$, it follows that only finitely many of the numbers $\frac{1}{n}$ may lie outside of $A_0$. Let $N$ denote the largest $N$ such that $\frac{1}{N} \notin A_0$. Then for each $j \in J_N$, we can find $A_j \in \mathcal{A}$ such that $\frac{1}{j} \in A_j$. Then $\{\frac{1}{j}\}_{j=1}^{N} \subset \bigcup_{j=1}^{N} A_j$, and $\{0\}\cup\{\frac{1}{n}\}_{n=N+1}^{\infty} \subset A_0$.

Letting $\mathcal{B} = \{A_0, A_1, \ldots, A_N\}$, we see now that $\mathcal{B}$ is a finite subcollection of $\mathcal{A}$ that still covers $K$. Since $\mathcal{A}$ was an arbitrary open cover of $K$ in $\mathbb{R}$, we conclude that $K$ is compact in $\mathbb{R}$.

One nice property of compactness is that it behaves nicely when passing to subsets (especially in comparison with the notions of open, closed, etc.)

PROPOSITION 3.6. *Let $(X, d)$ be a metric space, and let $K$ and $Y$ be subsets of $X$ such that $K \subset Y \subset X$. Then $K$ is compact in $X$ if and only if it is compact in $Y$.*

PROOF. Assume $K$ is compact in $X$, and let $\mathcal{B}$ be an open cover of $K$ in $Y$. To each $B \in \mathcal{B}$ associate an open set $A_B$ of $X$ such that $B = A_B \cap Y$. Then $\{A_B\}_{B\in\mathcal{B}}$ is[3] an open cover of $K$ in $X$. Since $K$ is compact in $Y$, we can choose a finite subcover $\{A_{B_1}, \ldots, A_{B_n}\}$ associated to the open cover $\{A_B\}_{B\in\mathcal{B}}$ of $K$ in $X$. But then $K \subset Y$ implies

$$K \subset \left(\bigcup_{i=1}^{n} A_{B_i}\right) \cap Y = \bigcup_{i=1}^{n} (A_{B_i} \cap Y) = \bigcup_{i=1}^{n} B_i.$$

Thus $\{B_1, \ldots, B_n\}$ covers $K$, so it is a finite subcover of $K$ associated to the (arbitrary) open cover $\mathcal{B}$ of $K$ in $Y$. Thus $K$ is compact in $Y$.

On the other hand, assume that $K$ is compact in $Y$. Let $\mathcal{A}$ be an open cover of $K$ in $X$, and put $\mathcal{B} = \{A \cap Y : A \in \mathcal{A}\}$. Then $\mathcal{B}$ is an open cover of $K$ in $Y$; since $K$ is compact in $Y$ we can find a finite subcover $\{A_j \cap Y\}_{j=1}^n$. But then

$$K \subset \bigcup_{i=1}^{n} (A_j \cap Y) \subset \bigcup_{i=1}^{n} A_j.$$

That is, $\{A_j\}_{j=1}^n$ covers $K$. Therefore $\{A_j\}_{j=1}^n$ is a finite subcover of $K$ associated to the open cover $\mathcal{A}$ of $K$ in $X$. Thus $K$ is compact in $X$. □

In light of this Proposition, we may treat compact subsets of metric spaces in their own right, without paying attention to exactly what metric space they came from; a compact set (in a given metric) is compact, regardless of where it came from. Contrast this with the situation for open and closed sets; the notion of an 'open metric space' does not make sense, but a 'compact metric space' does.

### 3.2. Compactness and Limit Point Compactness.

THEOREM 3.7. *Let $(X, d)$ be a compact metric space. Then $X$ is limit point compact.*

REMARK 3.8. Actually, compactness and limit point compactness are equivalent in any metric space. However, we do not prove this here. (The proof that limit point compactness implies compactness is rather involved.)

PROOF. We argue by contradiction. Assume $E$ is an infinite subset of $X$ that has no limit point in $X$. Then for each $x \in X$, $x$ is not a limit point of $E$, so we can choose a neighborhood $U_x$ of $x$ such that $U_x$ contains no points of $E \backslash \{x\}$. Then $\mathcal{A} = \{U_x\}_{x\in X}$ is an open cover for $X$ (since each $x \in U_x$ for each $x \in X$). Since $X$ is compact, we can choose a finite subcover $\mathcal{B} = \{U_{x_1}, \ldots, U_{x_n}\}$ of $X$, associated to the open cover $\mathcal{A}$. Since $\mathcal{B}$ covers $X$, it certainly covers $E$; that is,

$$E \subset \bigcup_{i=1}^{n} U_{x_i}.$$

[3]Indeed, each $A_B$ is open in $X$ by construction; furthermore,

$$K \subset \bigcup_{B\in\mathcal{B}} B = \bigcup_{B\in\mathcal{B}} (A_B \cap Y) \subset \bigcup_{B\in\mathcal{B}} A_B.$$

But then

$$E = E \cap \bigcup_{i=1}^{n} U_{x_i} = \bigcup_{i=1}^{n} (E \cap U_{x_i}) \subset \bigcup_{i=1}^{n} \{x_i\} = \{x_1, \dots, x_n\}.$$

Here the second-to-last equality is justified by our choice of $U_{x_i}$, which contains no points of $E$ except possibly $x_i$. The fact that $E$ is contained in the finite set $\{x_1, \dots, x_n\}$ implies that $E$ itself is finite, which contradicts our initial assumption that $E$ was infinite. In light of this contradiction, we conclude that $E$ must in fact be limit point compact. □

COROLLARY 3.9. *Let $(X, d)$ be a metric space, and let $K$ be a compact subset of $X$. Then $K$ is closed and bounded in $X$.*

PROOF. Since $K$ is compact, it is limit point compact, therefore sequentially compact. Thus it is closed and bounded in $X$ by Corollary 2.27. □

THEOREM 3.10. *Let $(X, d)$ be a compact metric space and let $K$ be a closed subset of $X$. Then $K$ is compact.*

PROOF. Let $\mathcal{A}$ be an open cover of $K$ in $X$; denote $\mathcal{B} = \mathcal{A} \cup \{K^c\}$. Then $\mathcal{B}$ is an open cover of $X$ and $X$ is compact; therefore we can find a finite subcover $\mathcal{B}'$ of $X$ associated to the open cover $\mathcal{B}$. This new collection $\mathcal{B}'$ still covers $K$ of course (since it covers $X$), but it might or might not be a subcollection of $\mathcal{A}$ (depending on whether $K^c \in \mathcal{B}'$). But removing $K^c$ from $\mathcal{B}'$ will not affect whether $\mathcal{B}'$ covers $K$; therefore we define $\mathcal{A}' = \mathcal{B}' \backslash \{K^c\}$. Then $\mathcal{A}'$ is a finite subcover of $K$ associated to the open cover $\mathcal{A}$ of $K$ in $X$. Therefore $K$ is compact. □

COROLLARY 3.11. *Let $(X, d)$ be a metric space. Assume $F$ and $K$ are subsets of $X$, with $F$ closed and $K$ compact. Then $F \cap K$ is compact.*

EXERCISE 3.12. Prove Corollary 3.11.

**3.3. Compact Subsets of $\mathbb{R}^k$.** We have shown that in any metric space, compact sets are closed and bounded. In $\mathbb{R}^k$, the converse holds; this is the goal of the present subsection.

DEFINITION 3.13. A $k$-cell in $\mathbb{R}^k$ is a subset of $\mathbb{R}^k$ of the form $[a_1, b_1] \times \cdots \times [a_k, b_k]$.

THEOREM 3.14. *Every $k$-cell is compact in $\mathbb{R}^k$.*

Instead of proving this Theorem, we will prove the following special case:

THEOREM 3.15. *$[0, 1]$ is a compact subset of $\mathbb{R}$.*

The proof of Theorem 3.15 uses all the ideas of its more general counterpart for arbitrary $k$-cells, but the notation is less cumbersome. Before we embark on the proof, however, let us observe some of the consequences of Theorem 3.14.

THEOREM 3.16 (Heine-Borel). *Let $K$ be a subset of $\mathbb{R}^k$. Then $K$ is compact if and only if it is closed and bounded in $\mathbb{R}^k$.*

PROOF. As noted above, we have already proved that compactness of $K$ implies it is closed and bounded. Therefore it suffices to prove the opposite implication. Assume then that $K$ is closed and bounded in $\mathbb{R}^k$. Then $K$ is contained in some $k$-cell $C$; the latter is compact by Theorem 3.14. Since $K$ is closed in $X$, it follows from Corollary 3.11 that $K = K \cap C$ is compact. □

COROLLARY 3.17. *Let $K$ be a subset of $\mathbb{R}^k$. The following are equivalent:*

*(1) $K$ is compact.*
*(2) $K$ is limit point compact.*
*(3) $K$ is sequentially compact.*

*(4) $K$ is closed and bounded.*

COROLLARY 3.18. *$\mathbb{R}^k$ is a complete metric space.*

PROOF. Let $S = (x_n)_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^k$. Then $S$ is bounded, so $K := \overline{\operatorname{Im} S}$ is closed and bounded (see Exercise 1.28), therefore compact, therefore sequentially compact, therefore complete. Since $S$ is a Cauchy sequence in the complete metric space $K$, it follows that $S$ converges in $K$, therefore in $\mathbb{R}^k$. $\square$

We now work on proving Theorem 3.15. First we need a Lemma:

LEMMA 3.19. *If $(I_n)_{n\in\mathbb{N}}$ is a sequence of closed intervals in $\mathbb{R}$, such that $I_n \supset I_{n+1}$ for each $n \in \mathbb{N}$, then $A := \bigcap_{n=1}^{\infty} I_n$ is not empty.*

PROOF. Write $I_n = [a_n, b_n]$ for each $n$. Since $I_n \subset I_1$, we have $b_n \leq b_1$; therefore $a_n \leq b_1$ for each $n \in \mathbb{N}$. The set of $a_n$ is therefore nonempty and bounded above (by $b_1$), so $\sup a_n$ exists in $\mathbb{R}$; call it $x$. We claim that $x \in A$; to show this we must prove that $x \in I_m$ for each $m \in \mathbb{N}$. To this end, choose $m \in \mathbb{N}$. Then $x \geq a_m$ by definition of the supremum; to show that $x \leq b_m$ it suffices to show that $b_m$ is an upper bound for the $a_n$s. That the latter is true follows from the inequalities $a_n \leq a_{n+m} \leq b_{n+m} \leq b_m$, which hold for all $n \in \mathbb{N}$. $\square$

PROOF OF THEOREM 3.15. Suppose $\{G_\alpha\}$ is an open cover of $[0, 1]$ which has no finite subcover. Then at least one of the intervals $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$ cannot be covered by finitely many of the $G_\alpha$; call it $I_1$. (If both $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ could be covered by finitely many of the $G_\alpha$, then $[0, 1]$ could also be covered by finitely many of the $G_\alpha$'s.) Subdivide $I_1$ and continue the process, constructing a sequence of closed intervals $I_1, I_2, \ldots$ such that

- $[0, 1] \supset I_1 \supset I_2 \supset \cdots$,
- $I_n$ is not covered by any finite subcollection of the $G_\alpha$s,
- $I_n = [k \cdot 2^{-n}, (k + 1)2^{-n}]$ for some $k \in \{0, 1, \ldots, 2^n - 1\}$.

By the Lemma above, there is some point $x^*$ which belongs to every $I_n$. Furthermore, $x^*$ belongs to some $G_\alpha$. Choose $r > 0$ so that $|x^* - y| < r$ implies $y \in G_\alpha$ whenever $y \in [0, 1]$. Then choose $n$ large enough so that $2^{-n} < r$. Then for any $y \in I_n$, we have $|x^* - y| \leq 2^{-n} < r$; this means that $I_n \subset G_\alpha$, which contradicts the construction of $I_n$. (Recall that $I_n$ was chosen so that it could not be covered by finitely many of the $G_\alpha$s, whereas we have shown that it can be covered by a single $G_\alpha$.) This contradiction establishes the claim; $[0, 1]$ must be compact. $\square$

Note that the Heine-Borel Theorem is not true in general metric spaces. See for example the following.

EXAMPLE 3.20. Put $X = \mathbb{R}$, under the Euclidean metric. Put $Y = \mathbb{Q}$ and $F = [\sqrt{2}, 5] \cap \mathbb{Q}$. Then $F$ is closed and bounded in $\mathbb{Q}$, but it is not compact, since it is not closed in $\mathbb{R}$. Indeed,

$$\mathrm{Cl}_{\mathbb{R}}(F) = [\sqrt{2}, 5] \neq F.$$

Thus $F$ is not closed in $\mathbb{R}$, hence not compact.

### 3.4. Compact Sets and the Finite Intersection Property.

DEFINITION 3.21. Let $(X, d)$ be a metric space and let $\mathcal{A}$ be a collection of subsets of $X$. Then $\mathcal{A}$ is said to have the *finite intersection property* if $\bigcap_{A\in\mathcal{B}} A$ is nonempty whenever $\mathcal{B}$ is a finite subset of $\mathcal{A}$.

EXAMPLE 3.22. Let $\mathcal{A} = \{A_n\}_{n=1}^{\infty}$ be a countable collection of nonempty nested sets (i.e., $A_n \supset A_{n+1}$ for each $n \in \mathbb{N}$). Then $\mathcal{A}$ has the finite intersection property. Indeed, let $C$ be a finite subset of $\mathbb{N}$. Then $C$ has a largest element $N$, and $\bigcap_{n\in C} A_n = A_N$.

THEOREM 3.23. *Let $\mathcal{A}$ be a nonempty collection of compact subsets of a metric space $(X,d)$. Assume that $\mathcal{A}$ has the finite intersection property. Then $\bigcap_{K\in\mathcal{A}} K$ is nonempty.*

PROOF. We argue by contradiction. Assume that $\bigcap_{K\in\mathcal{A}} K$ is empty. Then

$$\bigcup_{K\in\mathcal{A}} K^c = \left(\bigcap_{K\in\mathcal{A}} K\right)^c = X.$$

That is, the collection $\{K^c\}_{K\in\mathcal{A}}$ is an open cover for $X$, and thus for any fixed $K^* \in \mathcal{A}$. Fix such a $K^* \in \mathcal{A}$, and choose a finite subcover $\{K_1^c, \ldots, K_n^c\}$ of $K^*$, associated to the cover $\{K^c\}_{K\in\mathcal{A}}$ of $K^*$. Then

$$K^* \subset K_1^c \cup \cdots \cup K_n^c,$$

and consequently $(K^*)^c \supset K_1 \cap \cdots \cap K_n$. But then $K^* \cap K_1 \cap \cdots \cap K_n = \emptyset$; this contradicts our hypothesis since $\{K^*\} \cup \{K_j\}_{j=1}^n$ is a finite subcollection of elements of $\mathcal{A}$, and $\mathcal{A}$ has the finite intersection property. $\square$

COROLLARY 3.24. *Let $(K_n)_{n=1}^\infty$ be a sequence of nonempty compact subsets of a metric space $(X,d)$ and assume that $K_n \supset K_{n+1}$ for all $n \in \mathbb{N}$. Then*

$$\bigcap_{n=1}^\infty K_n \neq \emptyset.$$

PROOF. Since the $K_n$s are nested, the set $\{K_n\}$ has the finite intersection property (see the example above). Then by Theorem 3.23, the intersection of all of the $K_n$s is nonempty. $\square$

EXAMPLE 3.25. Let $\mathcal{A}$ be a collection of subsets of $\mathbb{R}$ defined by $\mathcal{A} = \{[n,\infty)\}_{n=1}^\infty$. Then $\mathcal{A}$ has the finite intersection property (see the example above), but

$$\bigcap_{A\in\mathcal{A}} A = \bigcap_{n=1}^\infty [n,\infty) = \emptyset.$$

In light of Theorem 3.23, we observe that the sets $[n,\infty)$ are not bounded in $\mathbb{R}$, hence not compact in $\mathbb{R}$.

EXERCISE 3.26. Give an example of a collection $\mathcal{A}$ of *bounded* subsets of $\mathbb{R}$ such that $\mathcal{A}$ has the finite intersection property, but $\bigcap_{A\in\mathcal{A}} A = \emptyset$. Hint: If $A \subset \mathbb{R}$ is bounded in $\mathbb{R}$, what else can prevent it from being compact?

We will use Theorem 3.23 and its consequences in our discussion of perfect sets below. This will allow us to give a proof of the fact that $\mathbb{R}$ is uncountable.

## 4. Perfect Sets in $\mathbb{R}^k$

DEFINITION 4.1. Let $(X,d)$ be a metric space. A subset $E$ of $X$ is called *perfect* if $E = E'$.

REMARK 4.2. Closed sets are those which contain all their limit points (i.e. $E' \subset E$). Thus every perfect set is closed. But a perfect set cannot have isolated points. (If $x \in E$ is an isolated point, then $x \notin E'$, so $E \neq E'$.)

EXAMPLE 4.3. $E = [0,1]$ is a perfect set of $\mathbb{R}$, but $F = [0,1] \cup \{2\}$ is not, even though it is closed. The problem is that 2 is an isolated point of $F$.

### 4.1. Perfect Sets and the Uncountability of $\mathbb{R}^k$.

THEOREM 4.4. *Let $P$ be a nonempty perfect subset of $\mathbb{R}^k$. Then $P$ is uncountable.*

PROOF. Since $P$ is nonempty, it contains some element $p$, which must be a limit point of $P$ since $P$ is perfect. Since finite sets have no limit points and $P$ does have limit points, $P$ cannot be finite. We claim that $P$ is also not countable. To see this, we argue by contradiction. Assume that $P$ is countable, and let $(x_1, x_2, \ldots)$ be an enumeration of its elements. Our goal is to construct a sequence $(K_n)$ of nonempty compact sets such that

- $K_{n+1} \subset K_n \subset P$ for each $n \in \mathbb{N}$ and
- For each $n \in \mathbb{N}$, we have $x_n \notin K_{n+1}$.

Assume for a moment we have done this. Then by the first two points, $K := \bigcap_{n=1}^{\infty} K_n$ is a nonempty subset of $P$. However, by the second point, $K$ cannot contain any of the $x_n$, which means that it *is* empty. Clearly this is a contradiction. Therefore, to finish the proof, it suffices to construct compact sets $K_n$ with the properties above.

Choose $p_1 \in P$, $p_1 \neq x_1$. Put $\varepsilon_1 = \frac{1}{2}|p_1 - x_1|$; then $\overline{B(p_1, \varepsilon_1)}$ does not contain $x_1$. However, since $p_1 \in P$ and $P$ is perfect, $p_1$ must be a limit point of $P$. Therefore $B(p_1, \varepsilon_1)$ contains another point of $P$; call it $p_2$. Put $\varepsilon_2 = \min\{\frac{1}{2}|p_2 - x_2|, \varepsilon - |p_1 - p_2|\}$. Then $B(p_2, \varepsilon_2)$ is contained in $B(p_1, \varepsilon_1)$ and $x_2 \notin \overline{B(p_2, \varepsilon_2)}$. We continue in this way, obtaining $p_j$, $\varepsilon_j$ so that $\overline{B(p_j, \varepsilon_j)}$ does not contain $x_j$, and $B(p_{j+1}, \varepsilon_{j+1}) \subset B(p_j, \varepsilon_j)$.

Put $K_n = \overline{B(p_n, \varepsilon_n)} \cap P$. Then $\{K_n\}_{n \in \mathbb{N}}$ consists of nonempty compact sets that satisfy the two bullet points above. □

COROLLARY 4.5. *Any open or closed interval in $\mathbb{R}$ is uncountable. In particular, $\mathbb{R}$ is uncountable.*

**4.2. The Cantor Set.** Next, we will construct a perfect set *which contains no intervals*. This set is called the *Cantor set* after Georg Cantor. The Cantor set is an example of a fractal; its "dimension" is non-integer valued in most senses of the word dimension, and it can be put in bijective correspondence with $\mathbb{R}$.

Take $E_0 = [0, 1]$. Construct $E_1$ by removing the open middle third of $E_0$:

$$E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1].$$

Construct $E_2$ by removing the open middle third of each of the closed intervals that make up $E_1$:

$$E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1].$$

We continue this process, obtaining a sequence of compact sets $(E_n)$, such that $E_n \supset E_{n+1}$ for each $n$, and $E_n$ is the disjoint union of $2^n$ closed intervals of length $3^{-n}$. In particular, each $E_n$ is nonempty. The set

$$P = \bigcap_{n=1}^{\infty} E_n$$

is called the *Cantor set*. Note that since the $E_n$s are nested nonempty compact sets, $\{E_n\}$ has the FIP, so $P$ is nonempty and compact.

We claim that $P$ contains no segment. Indeed, each interval of the form

$$I_{k,m} = \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$$

is disjoint from $E_m$, therefore from $P$. On the other hand, any open interval contains some $I_{k,m}$. Indeed, choose any $a, b \in \mathbb{R}$ with $a < b$. We want to ensure that $m$ is large enough so that it is possible to pick $k$

satisfying

$$a < \frac{3k+1}{3^m} \quad \text{and} \quad \frac{3k+2}{3^m} < b.$$

These two inequalities yield

$$\frac{3^m a - 1}{3} < k < \frac{3^m b - 2}{3}.$$

In order to ensure that it is possible to pick an *integer* $k$ satisfying both of these inequalities, we pick $m$ large enough so that the LHS and RHS are distance greater than 1 apart. That is,

$$\frac{3^m b - 2}{3} - \frac{3^m a - 1}{3} > 1.$$

Or simplifying, $3^m(b-a) > 4$, i.e. $3^{-m} < \frac{b-a}{4}$. It follows from this construction that $I_{k,m} \in (a,b)$.

Since any interval $(a,b)$ contains an interval of the form $I_{k,m}$, yet $I_{k,m}$ is disjoint from $P$, it follows that $P$ contains no segment.

Next, we prove that $P$ contains no isolated points, which implies that $P$ is perfect (since we know $P$ is closed). Choose $x \in P$, then choose $\varepsilon > 0$. Since $x \in P$, we know by definition of $P$ that $x \in E_n$ for all $n \in \mathbb{N}$; let $I_n$ denote the closed interval of $E_n$ that contains $x$, and choose $n$ large enough so that $I_n \subset (x-\varepsilon, x+\varepsilon) = B(x,\varepsilon)$. Let $x_n$ be an endpoint of $I_n$, with $x_n \neq x$. Then $x_n \in P$ (the endpoints of the closed intervals in $E_n$ are never removed from $P$), and $x_n \in I_n \subset B(x,\varepsilon)$. Thus $x$ is a limit point of $P$. Since $x$ was chosen arbitrarily, we conclude that $P$ is perfect.

## 5. Connectedness

Connectedness is a crucial notion in the analysis of metric spaces, and more generally, topological spaces. However, the general definition is a bit abstract, so we preface it with a discussion of the notion of convexity, which is sort of a concrete prototype for connectedness. Unfortunately, convexity does not make sense in general metric spaces without some extra structure. Therefore we make this discussion in concrete setting of $\mathbb{R}^k$.

### 5.1. Convexity in $\mathbb{R}^k$.

DEFINITION 5.1. Let $E$ be a subset of $\mathbb{R}^k$. We say that $E$ is *convex* if whenever $a, b \in E$ and $t \in (0,1)$, one has $(1-t)a + tb \in E$ as well.

Informally, this definition says that given any two points $a$ and $b$ in $E$, one can connect $a$ and $b$ with a line segment which remains in $E$. (Explanation: $p(t) = (1-t)a + tb$, $t \in [0,1]$, is a parametrization of the line segment joining $a$ and $b$.)

EXAMPLE 5.2. $B(x,\varepsilon)$ is convex in the usual metric $d(x,y) = |x-y|$ on $\mathbb{R}^k$. Indeed, pick $a, b \in B(x,\varepsilon)$ and choose $t \in (0,1)$. We need to show that $z = (1-t)a + tb$ is distance less than $\varepsilon$ from $x$. To see this, we write $x = (1-t)x + tx$, so that

$$\begin{aligned} |x - z| &= |[(1-t)x + tx] - [(1-t)a + tb]| \\ &= |(1-t)(x-a) + t(x-b)| \\ &\leq (1-t)|x-a| + t|x-b| < (1-t)\varepsilon + t\varepsilon = \varepsilon. \end{aligned}$$

EXERCISE 5.3. Let $\mathcal{A}$ be a collection of convex subsets of $\mathbb{R}^k$. Show that $B := \bigcap_{A \in \mathcal{A}} A$ is convex.

### 5.2. Connectedness in Metric Spaces; Connected Subsets of $\mathbb{R}$.

DEFINITION 5.4. Let $(X, d)$ be a metric space, and let $A$ and $B$ be subsets of $X$. $A$ and $B$ are said to be *separated* in $X$ if $\mathrm{Cl}_X(A) \cap B$ and $A \cap \mathrm{Cl}_X(B)$ are both empty. That is, $A$ and $B$ are separated in $X$ if they are disjoint, and neither contains a limit point of the other. A set $E \subset X$ is called *connected* if it cannot be written as a union of two nonempty sets which are separated in $X$. Equivalently, $E$ is connected if and only if the following statement holds: Whenever $A$ is nonempty and $A$ and $B$ are separated sets whose union is $E$, it follows that $A = E$ and $B = \emptyset$.

REMARK 5.5. In order for two sets $A$ and $B$ to be separated, they must be disjoint. But *disjointness is not enough to guarantee separatedness!* See the following Example.

EXAMPLE 5.6. If $A = (0, 1)$, $B = (1, 2)$, and $C = [1, 2)$, then $A$ and $B$ are separated, while $A$ and $C$ are not separated (even though $A$ and $C$ are disjoint).

EXERCISE 5.7. Let $(X, d)$ be a metric space and let $A$ and $B$ be disjoint subsets of $X$. Prove that if $A$ and $B$ are both open in $X$, then $A$ and $B$ are separated.

EXERCISE 5.8. Let $E$ be a connected subset of a metric space $(X, d)$. Show that $\overline{E}$ is connected.

It turns out that connectedness and convexity are equivalent in $\mathbb{R}$. Furthermore, the only connected subsets of $\mathbb{R}$ are the intervals, the rays, the single point sets, and the empty set. We show the equivalence of convexity and connectedness in $\mathbb{R}$ after a Lemma about real numbers.

LEMMA 5.9. *Let $A \subset \mathbb{R}$ be nonempty and bounded above. Then $\alpha := \sup A \in \overline{A}$.*

PROOF. (By contradiction.) If $\alpha \notin \overline{A}$, then there exists $\varepsilon > 0$ so that $B(\alpha, \varepsilon)$ contains no points of $A$; this implies that $\alpha - \varepsilon$ is an upper bound for $A$, contradicting the fact that $\alpha$ is the *least* upper bound. $\square$

THEOREM 5.10. *Let $E$ be a subset of $\mathbb{R}$. Then $E$ is connected if and only if whenever $x, y \in E$ and $x < z < y$, one has $z \in E$ as well.*

Note that the second condition in the Theorem is equivalent to saying that $E$ is convex. (Indeed, $z = (1-t)x + ty$ if $t = \frac{z-a}{b-a}$).

PROOF. We prove the contrapositive statement in each direction. Suppose the second condition fails, i.e. there exists $x, y \in E$ and $z \in \mathbb{R}$ such that $x < z < y$ but $z \notin E$. Put $A = (-\infty, z) \cap E$ and $B = (z, \infty) \cap E$. Then $A$ and $B$ are nonempty, since $x \in A$ and $y \in B$, and $A \cup B = E$. Furthermore, $\overline{A} \cap B \subset (-\infty, z] \cap (z, \infty) = \emptyset$ and similarly $A \cap \overline{B}$ is empty. Therefore $A$ and $B$ are separated, so $E$ is not connected.

Next, suppose that $E$ is not connected; pick nonempty sets $A$ and $B$ such that $A \cup B = E$ but $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Choose $x \in A$ and $y \in B$; assume without loss of generality that $x < y$. We find an element of $(x, y)$ that does not belong to $E$, proving that the second condition in the statement of the Theorem must fail.

Put $z = \sup(A \cap [x, y])$. Then by the Lemma, $z \in \overline{A \cap [x, y]} \subset \overline{A}$, so that $z \notin B$. Further, $z \neq y$ because $y \in B$. Thus $x \leq z < y$. If additionally $z \notin A$, then we are done, since this will imply that $z \neq x$ (so $x < z < y$) and $z \notin A \cup B = E$. Therefore we assume without loss of generality that $z \in A$.

Since $z \in A$, we know $z \notin \overline{B}$, since $A$ and $B$ are separated. Since $\overline{B}$ is closed, we can find an $\varepsilon$-neighborhood $B(z, \varepsilon) = (z - \varepsilon, z + \varepsilon)$ of $z$ which is disjoint from $\overline{B}$. Choose $z_1 \in (z, z + \varepsilon)$. Then $z + \varepsilon \leq y$ (otherwise $y \in B(z, \varepsilon) \cap B$, a contradiction), so $z_1 < y$. We claim that also $z_1 \notin A$; to see this, we argue by contradiction. Suppose $z_1 \in A$. Then since we already know $z_1 \in [x, y)$, it follows that $z_1 \in A \cap [x, y]$, so $z_1 \leq \sup(A \cap [x, y]) = z$. On the other hand, $z_1 > z$ by construction, giving us a contradiction. We conclude that $z_1 \notin A$, proving the auxiliary claim. It follows that $z_1 \neq x$ (as $x \in A$), so $x < z_1 < y$ (by combining $z_1 \neq x$ and $z_1 \in [x, y)$). Since also $z_1 \notin A \cup B = E$, this finishes the proof. $\square$

As noted above, the previous Theorem says that convexity and connectedness are equivalent in $\mathbb{R}$ (under the Euclidean metric). It is in fact true that convexity implies connectedness in $\mathbb{R}^k$ for any $k \in \mathbb{N}$. We will prove this in the next Chapter using the notion of *path connectedness*, which lies 'in between' convexity and connectedness in the context of $\mathbb{R}^k$. However, connectedness does *not* imply convexity in $\mathbb{R}^k$ for $k \geq 2$. For example, the circle $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected, but it is not convex. To see that it is not convex, note that $a := (1, 0) \in S$ and $b := (-1, 0) \in S$, but $O = (0, 0) \notin S$, even though $O = (1 - \frac{1}{2})a + \frac{1}{2}b$. We will soon give a way to show that this set is in fact connected.

THEOREM 5.11. *Let $(X, d)$ be a metric space, and let $E$ be a connected subset of $X$. If $A$ and $B$ are separated sets in $X$ and $A \cup B \supset E$, then either $E \subset A$ or $E \subset B$.*

PROOF. Put $C = A \cap E$ and $D = B \cap E$. Then $E = C \cup D$, and $\overline{C} \cap D \subset \overline{A} \cap B = \emptyset$ and similarly $C \cap \overline{D} = \emptyset$. Thus $C$ and $D$ are separated. Since $E$ is connected, it follows that either $C$ or $D$ must be empty; that is, one of $C$ or $D$ must contain all of $E$. □

EXERCISE 5.12. Let $(X, d)$ be a metric space, and let $\mathcal{C}$ be a collection of connected subsets of $X$. Assume $A = \bigcap_{C \in \mathcal{C}} C$ is nonempty. Show that $B = \bigcup_{C \in \mathcal{C}} C$ is connected.

EXERCISE 5.13. Let $X = \mathbb{R}^2$. Give an example of a connected subset $E$ of $X$, such that $\mathrm{Int}_X(E)$ is *not* connected. Prove both that your set $E$ is connected and that its interior is not. (Hint: Consider the union of two convex sets joined at a point. You may assume without proof the fact that convexity implies connectedness in $\mathbb{R}^2$.)

To end this section, we give one final characterization of connectedness which is sometimes useful.

PROPOSITION 5.14. *Let $(X, d)$ be a metric space. Then $X$ is connected if and only if the only subsets of $X$ which are both open and closed in $X$ are $\emptyset$ and $X$.*

PROOF. Assume that $X$ is connected and let $E$ be a subset of $X$ which is both open and closed in $X$. Then $E$ and $X \backslash E$ are both open in $X$, therefore they are separated (Exercise 5.7). Since $X$ is connected, it follows that $E$ is either $X$ or $\emptyset$.

On the other hand, assume that $X$ is not connected. Then $X$ can be written as a union of two nonempty separated sets $A$ and $B$. Separatedness of $A$ and $B$ implies $A \cap \overline{B} = \emptyset$, so $A \subset X \backslash \overline{B}$. On the other hand, $A \cup B = X$, so $X \backslash B \subset A$. Therefore $A = X \backslash \overline{B} = X \backslash B$. It follows that $A$ is open in $X$, since $\overline{B}$ is closed in $X$. Reasoning entirely similarly for $B$, we conclude that $B$ is open in $X$ as well. So $A$ and $B$ are subsets of $X$, neither of which is $\emptyset$ or $X$, and both of which are open and closed in $X$. □

CHAPTER 5

# Functions Between Metric Spaces

## 1. Limits of Functions

### 1.1. Definition of the Limit.

DEFINITION 1.1. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and let $E$ be a subset of $X$. Let $f : E \to Y$ be a function, and let $p$ be a limit point of $E$ in $X$. We write

$$\lim_{x \to p} f(x) = q$$

to mean the following: For every neighborhood $V$ of $q$ in $Y$, there exists a neighborhood $U$ of $p$ in $X$ such that $x \in U \cap E \backslash \{p\}$ implies $f(x) \in V$. In this case we say that $q$ is the *limit* of the function $f(x)$ as $x$ approaches $p$ in $E$.

Note that another common way to write the statement $\lim_{x \to p} f(x) = q$ is '$f(x) \to q$ in $Y$ as $x \to p$ in $X$', or simply '$f(x) \to q$ as $x \to p$', if $X$ and $Y$ are understood.

Of course, a point $q$ as in the definition of the limit may or may not exist, depending on the function $f$ and the point $p$. Therefore we sometimes say that 'the limit exists' or 'the limit does not exist', or '$f$ has/does not have a limit at $p$'. There are many other ways to express this notion; most are self-explanatory.

You are probably accustomed to seeing an $\varepsilon$-$\delta$ version of this definition. In metric spaces, the two notions are equivalent. Both say that by considering $x$ close enough to $p$ (but not equal to $p$, and also in the domain of $f$), you can make $f(x)$ as close as you like to $q$. The only difference is what 'close enough' means a priori.

EXERCISE 1.2. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and let $E$ be a subset of $X$. Let $f : E \to Y$ be a function, and let $p$ be a limit point of $E$ in $X$. Prove that $f(x) \to q$ as $x \to p$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in E$ and $0 < d_X(x, p) < \delta$ imply together that $d_Y(f(x), q) < \varepsilon$.

The $\varepsilon$-$\delta$ formulation is often more useful in practice for proving that a given limit holds. Note that in either formulation, the function $f$ may or may not be defined at $p$ (i.e., $p$ may or may not be in $E$). Furthermore, even if $p \in E$, it may be the case that $f(p) \neq q$.

Another useful formulation of the limit concept is in terms of sequences:

PROPOSITION 1.3. *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $E$ be a subset of $X$, and let $p$ be a limit point of $E$ in $X$. The following are equivalent:*

*(1)* $\lim_{x \to p} f(x) = q$.
*(2)* $\lim_{n \to \infty} f(x_n) = q$ *for any sequence* $(x_n)_{n=1}^{\infty}$ *in* $E \backslash \{p\}$ *that converges to* $p$ *in* $X$.

Note that the assumption that $p$ is a limit point of $E$ guarantees that at least one sequence of the type considered in the second condition must exist.

PROOF. $(1) \implies (2)$. Assume $\lim_{x \to p} f(x) = q$ and let $(x_n)$ be a sequence in $E \backslash \{p\}$ that converges to $p$ in $X$. Let $V$ be a neighborhood of $q$ in $Y$, and let $U$ be the corresponding neighborhood of $p$ in $X$ such that $x \in U \cap E \backslash \{p\}$ implies $f(x) \in V$. Choose $N$ large enough so that $n \geq N$ implies $x_n \in U$; then $n \geq N$ also implies that $f(x_n) \in V$. Thus $f(x_n) \to q$ as $n \to \infty$.

(2) $\implies$ (1). Assume (1) fails; we show (2) must as well. The failure of (1) implies that there exists a neighborhood $V$ of $q$ in $Y$ such that for any neighborhood $U$ of $p$ in $E$, there exists $x \in U \backslash \{p\}$ with $f(x) \notin V$. Assume a neighborhood $V$ is so chosen, and for each $n \in \mathbb{N}$, let $x_n$ be a point of $B_E(p, \frac{1}{n}) \backslash \{p\}$ such that $f(x_n) \notin V$. Then $d(x_n, p) < \frac{1}{n}$ for each $n$, so $x_n \to p$ as $n \to \infty$, but $(f(x_n))$ does not converge to $q$, since its image lies entirely outside the neighborhood $V$ of $q$. $\square$

COROLLARY 1.4 (Uniqueness of Limits of Functions). *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $E$ be a subset of $X$, and let $p$ be a limit point of $E$ in $X$. Let $f : E \to Y$ be a function. If $\lim_{x\to p} f(x) = q$ and $\lim_{x\to p} f(x) = r$, then $q = r$.*

The proof of this statement follows immediately from the sequence formulation of limits, together with the fact that limits of sequences are unique in metric spaces.

**1.2. Some Limit Rules for Real-Valued Functions.** The limit concept is compatible with the commutative algebra structure on $\mathbb{R}^X$ (the set of real-valued functions on the set $X$) when $X$ is given a metric space structure.

THEOREM 1.5. *Let $(X, d)$ be a metric space. Let $E$ be a subset of $X$, and let $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be functions. Assume that $p$ is a limit point of $E$, and $\lim_{x\to p} f(x)$, $\lim_{x\to p} g(x)$ both exist. Then*

$$\lim_{x\to p}(f+g)(x) = \lim_{x\to p} f(x) + \lim_{x\to p} g(x),$$
$$\lim_{x\to p}(fg)(x) = \lim_{x\to p} f(x) \lim_{x\to p} g(x).$$

*If additionally $\lim_{x\to p} f(x) \neq 0$, then*

$$\lim_{x\to p} \left(\frac{1}{f}\right)(x) = \frac{1}{\lim_{x\to p} f(x)}.$$

PROOF. We prove only the statement on multiplication of limits; the others are similar. We write

$$\lim_{x\to p} f(x) = A; \qquad \lim_{x\to p} g(x) = B.$$

Choose $\varepsilon > 0$. We write

$$\begin{aligned} |(fg)(x) - AB| &= |(f(x) - A)g(x) + A(g(x) - B)| \\ &\leq |f(x) - A||g(x)| + |A||g(x) - B| \\ &\leq |f(x) - A|\big(|g(x) - B| + |B|\big) + |A||g(x) - B|. \end{aligned}$$

We want to make each of the terms on the RHS less than $\frac{\varepsilon}{2}$. To this end, we choose $\delta > 0$ small enough so that $|g(x) - B| < \min\{\frac{\varepsilon}{2|A|}, 1\}$ and $|f(x) - A| < \frac{\varepsilon}{2(1+|B|)}$, for all $x \in E$ such that $0 < d(x, p) < \delta$. Then for such $x$, we have

$$\begin{aligned} |(fg)(x) - AB| &\leq |f(x) - A|\big(|g(x) - B| + |B|\big) + |A||g(x) - B| \\ &< \frac{\varepsilon}{2(1 + |B|)} \cdot (1 + |B|) + |A| \cdot \frac{\varepsilon}{2|A|} = \varepsilon. \end{aligned}$$

$\square$

## 2. Continuous Functions

**2.1. Definition of Continuity.**

DEFINITION 2.1. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces; let $f : X \to Y$ be a function. We say that $f$ is *continuous* at $p \in X$ if for every neighborhood $V$ of $f(p)$, there exists a neighborhood $U$ of $p$ such that $x \in U$ implies $f(x) \in V$.

Once again, this is equivalent to the $\varepsilon$-$\delta$ formulation in metric spaces.

EXERCISE 2.2. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces; let $f : X \to Y$ be a function. Prove that $f$ is continuous at $p \in X$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in B_X(p, \delta)$ implies $f(x) \in B_Y(f(p), \varepsilon)$.

REMARK 2.3. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces; let $f : X \to Y$ be a function.

- $f$ must be defined at $p$ in order to be continuous at $p$.
- If $p$ is not a limit point of $X$, then $f$ is automatically continuous at $p$. Indeed, let $U$ be a neighborhood of $p$ that does not intersect $X \backslash \{p\}$. Then $p$ is the only point in $U$, so trivially, we have $f(x) = f(p) \subset V$ for all $x \in U$ and for any neighborhood $V$ of $f(p)$.
- If $p$ is a limit point of $X$, then $f$ is continuous at $p$ if and only if $p \in X$ and $\lim_{x \to p} f(x) = f(p)$. This should be clear after comparing the definitions of the two concepts. The only discrepancy is the lack of mention of an auxiliary set $E$ in the definition of continuity at a point; the set $E$ is unnecessary in the present situation because we are assuming explicitly that $p$ is both an element of $X$ and a limit point of $X$.

REMARK 2.4. We give one more way of looking at continuity at a point before proceeding. $f : X \to Y$ being continuous at $p \in X$ means that for every neighborhood $V$ of $f(p)$, there exists a neighborhood $U$ of $p$ such that $f(U) \subset V$, i.e. $p \in U \subset f^{-1}(V)$. Thus, $p$ is an interior point of $f^{-1}(V)$ for every neighborhood $V$ of $f(p)$. This viewpoint will be useful in what follows.

DEFINITION 2.5. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A function $f : X \to Y$ is said to be *continuous* if it is continuous at every $x \in X$.

PROPOSITION 2.6. *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and let $f : X \to Y$ be a function. Then $f$ is continuous if and only if $f^{-1}(V)$ is open in $X$ whenever $V$ is open in $Y$.*

PROOF. Assume $f$ is continuous, and let $V$ be an open set of $Y$. Choose $x \in f^{-1}(V)$; then $V$ is a neighborhood of $f(x)$ in $Y$. Since $f$ is continuous at $x$, it follows so $x$ is an interior point of $f^{-1}(V)$ (see Remark 2.4). Since $x$ was an arbitrary point of $f^{-1}(V)$, we conclude that $f^{-1}(V)$ is open in $X$.

On the other hand, assume that $f^{-1}(V)$ is open in $X$ whenever $V$ is open in $Y$. Pick $p \in X$, and let $W$ be a neighborhood of $f(p)$ in $Y$. Then $U := f^{-1}(W)$ is open in $X$. Since $f(p) \in W$, we have $p \in U$, and $f(U) \subset W$ by definition of the inverse image. That is, $U$ is a neighborhood of $p$ in $X$ such that $x \in U$ implies $f(x) \in W$. Thus $f$ is continuous at $x$. □

As you might guess by this point, the open set formulation of continuity can be reduced (in metric spaces) to a statement about open balls in $Y$:

PROPOSITION 2.7. *Let $f : X \to Y$ be a function between the metric spaces $(X, d_X)$ and $(Y, d_Y)$. Then $f$ is continuous if and only if for every $r > 0$ and every $y \in Y$, the set $f^{-1}(B_Y(y, r))$ is open in $X$.*

PROOF. If $f$ is continuous, then clearly $f^{-1}(B_Y(y, r))$ is open in $X$ for all $y \in Y$ and $r > 0$, as $B_Y(y, r)$ is an open set in $Y$. On the other hand, assume that $f^{-1}(B_Y(y, r))$ is open in $X$ for every $y \in Y$, $r > 0$. Let $V$ be an open set in $Y$ and choose $x \in f^{-1}(V)$. Then $f(x)$ is an element of the open set $V$, so there exists $r > 0$ such that $B_Y(f(x), r) \subset V$. Thus $f^{-1}(B_Y(f(x), r)) \subset f^{-1}(V)$. Since $f^{-1}(B_Y(f(x), r))$ is open and contains $x$, it follows that $x$ is an interior point of $f^{-1}(B_Y(f(x), r))$ and thus of the larger set $f^{-1}(V)$. As $x$ was an arbitrary point of $f^{-1}(V)$, we conclude that $f^{-1}(V)$ is open in $X$. □

This Proposition implies in particular that if the codomain of a function is $\mathbb{R}$, then proving continuity amounts to checking the inverse images of all open intervals of $\mathbb{R}$.

REMARK 2.8. The reader might wonder why we trouble ourselves with the open set formulations of these concepts when the $\varepsilon$-$\delta$ version is equivalent. We give two (related) responses to this question, though the reasons below are not the only possible justifications.

- When proving that a given function between metric spaces is continuous, it is often easier to check the $\varepsilon$-$\delta$ criterion than the open set criterion, because there are strictly fewer things to verify. However, once we know a given function is continuous, the open set formulation gives us strictly more information about our function. Therefore, Theorems involving functions that we know are continuous to begin with are easier to prove when we have access to the open set formulation of continuity. We will see two major examples below, involving the images of compact and connected sets under a continuous function.
- The $\varepsilon$-$\delta$ formulation of continuity does not generalize to topological spaces, whereas the open set formulation requires essentially no modification.

EXERCISE 2.9. Assume $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$, for all $x \in \mathbb{R}$. Does it follow that $f$ must be continuous? If so, give a proof; if not, give a counterexample.

EXERCISE 2.10. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and $f : X \to Y$ a function.

(1) Show that $f$ is continuous if and only if $f^{-1}(C)$ is closed in $X$ whenever $C$ is closed in $Y$.
(2) Show that $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for every subset $A$ of $X$.
(3) Consider the (continuous) function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \frac{1}{1+x^2}$. Give an example of a subset $A$ of $\mathbb{R}$ such that $g(\overline{A}) \neq \overline{g(A)}$.

EXERCISE 2.11. Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and let $f$ and $g$ be continuous functions from $X$ to $Y$. Assume $E$ is a dense subset of $X$.

(1) Prove that $f(E)$ is dense in $f(X)$. (Hint: Use Exercise 1.30 in Chapter 4 and Exercise 2.10 above.)
(2) Prove that if $f(x) = g(x)$ for all $x \in E$, then $f(x) = g(x)$ for all $x \in X$.

This Exercise shows, for example, that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and if we know what $f(x)$ is for all $x \in \mathbb{Q}$, then we can determine what $f(x)$ is for any $x \in \mathbb{R}$.

**2.2. Generating Continuous Functions.** We now identify a few ways to obtain continuous functions. Some of these are specific to real-valued functions; others are valid for any metric space. Most of these statements are obvious or nearly so and are presented without proof.

PROPOSITION 2.12 (Constant Functions). *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and let $f : X \to Y$ be defined by $f(x) = c$ for some $c \in Y$. Then $f$ is continuous.*

PROPOSITION 2.13 (Restricting the Domain or Codomain). *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces, and let $f : X \to Y$ be a function.*

- *If $f$ is continuous and $E$ is a subset of $X$, then $f\big|_E : E \to Y$ is continuous.*
- *If $E$ is a subset of $X$ and $f\big|_E : E \to Y$ is continuous, then $f$ is continuous at each point of $\mathrm{Int}_X(E)$.*
- *If $f(X) \subset Z \subset Y$, then the map $g : X \to Z$ obtained by restricting the codomain of $f$ is continuous if and only if $f$ is continuous.*

PROOF. The first and third statements are trivial to prove using the $\varepsilon$-$\delta$ criterion. (Purely open-set proofs are also easy but not completely trivial.) The second statement is a tiny bit subtler; therefore we prove this statement only. Choose $x \in \mathrm{Int}_X(E)$ and $\varepsilon > 0$; let $\delta_1 > 0$ be such that $d_E(x, y) < \delta_1$ implies that $d_Y(f(x), f(y)) < \varepsilon$. Choose $r > 0$ such that $B_X(x, r) \subset E$, and put $\delta_2 = \min\{\delta_1, r\}$. Then $d_X(x, y) < \delta_2$ implies that $y \in B_X(x, r) \subset E$, so $d_E(x, y) = d_X(x, y) < \delta_1$, so $d_Y(f(x), f(y)) < \varepsilon$. Thus $f$ is continuous at $x$. $\square$

The reader might wonder why it is necessary to write $\mathrm{Int}_X(E)$ in the second statement instead of just $E$. The problem is illustrated by the following example.

EXAMPLE 2.14. Define $f : \mathbb{R} \to \mathbb{R}$ by the rule $f(x) = 0$ if $x < 0$ and $f(x) = 1$ if $x \geq 0$. Put $E = [0, \infty)$. Then $f|_E$ is a constant function, therefore continuous. But $f$ is not continuous at $0$.

PROPOSITION 2.15 (Identity and Inclusion Maps). *Let $(X, d)$ be a metric space. The identity map $\mathrm{id}_X : X \to X$ is continuous. If $Y \subset X$, then the inclusion map $\iota : Y \to X$ is continuous.*

Note that $\iota : Y \to X$ is the same as $\mathrm{id}_X \big|_Y : Y \to X$. Also note that we assume the same metric is used in the domain and codomain. (This is in line with our usual convention, so we don't include this in the hypotheses; however, even a function as nice as the identity map can be discontinuous if we use different metrics in the domain and codomain.)

PROPOSITION 2.16 (Composition of Continuous Functions). *Let $(X, d_X)$, $(Y, d_Y)$, and $(Z, d_Z)$ be metric spaces; let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then $g \circ f : X \to Z$ is continuous.*

This proof is easier using the open-set formulation of continuity.

PROOF. Let $V$ be an open set of $Z$. Then $g^{-1}(V)$ is open in $Y$ by continuity of $g$; thus $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in $X$, by continuity of $f$. $\square$

Note: If $f : X \to Y$ and $g : Y' \to Z$ are continuous functions with $Y \subset Y'$, then $g\big|_Y$ is continuous, so the Proposition above still guarantees continuity of $g \circ f$ (which, remember, technically means $(g\big|_Y) \circ f$ according to the definition of composition of functions in Chapter 1.

PROPOSITION 2.17 (Coordinate Functions). *Let $\phi_k$ denote the function $\phi_k : \mathbb{R}^n \to \mathbb{R}$ given by $\phi_k((x_1, \ldots, x_n)) = x_k$. ($\phi_k$ is called the $k$th coordinate function on $\mathbb{R}^n$). Then $\phi_k$ is continuous.*

PROOF. Choose $\varepsilon > 0$, then put $\delta = \varepsilon$. Assume $x, y \in \mathbb{R}^n$ and $|x - y| < \delta$. Write $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then

$$|\phi_k(x) - \phi_k(y)| = |x_k - y_k| \leq \sqrt{|(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2|} = |x - y| < \delta = \varepsilon.$$

Thus $\phi_k$ is continuous. $\square$

PROPOSITION 2.18 (Sums, Products, and Quotients of Continuous Functions). *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $f : X \to Y$ and $g : X \to Y$ be continuous functions. Then $f + g$ and $fg$ are continuous, and $\frac{1}{f}$ is continuous on the smaller set $X \backslash f^{-1}(0)$ where it is defined.*

PROOF. Choose $p \in X$. If $p$ is an isolated point of $X$, then $f + g$ is automatically continuous at $p$. Otherwise $p$ is a limit point of $X$ and we have

$$\lim_{x \to p} (f + g)(x) = \lim_{x \to p} f(x) + \lim_{x \to p} g(x) = f(p) + g(p) = (f + g)(p).$$

Note that the first equality above requires the existence of the two limits $\lim_{x \to p} f(x)$ and $\lim_{x \to p} g(x)$ (guaranteed by continuity), and the second equality relies on the continuity of $f$ and $g$ at $p$. Comparing the LHS and RHS, we see that $f + g$ is continuous at $p$. This shows that $f + g$ is continuous at every point of $X$ and is therefore continuous. The proof for $fg$ is entirely similar. The proof for $\frac{1}{f}$ is also entirely similar, once we note that $f\big|_{X \backslash f^{-1}(0)}$ is continuous. $\square$

PROPOSITION 2.19 (Polynomials and Rational Functions). *Any polynomial in $n$ variables is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$. If $r = \frac{f}{g}$ is any rational function (a quotient of the polynomials $f$ and $g$) of $n$ variables, then $r$ is continuous on the set $\mathbb{R}^n \backslash g^{-1}(0)$.*

This Proposition follows from the fact that any polynomial or rational function in $n$ variables can be written entirely in terms of sums, products, and quotients of coordinate functions.

PROPOSITION 2.20 ($\mathbb{R}^n$-valued functions). *Let $f_1, \dots, f_n$ be real-valued functions on a metric space $(X, d)$. The function $f : X \to \mathbb{R}^n$ defined by $f(x) = (f_1(x), \dots, f_n(x))$ for all $x \in X$ is continuous if and only if each of the functions $f_j$ is continuous.*

PROOF. Assume $f$ is continuous. Then $f_j = \phi_j \circ f$ is also continuous. Assume conversely that each of the $f_j$'s is continuous. Choose $\varepsilon > 0$ and $x \in X$, and let $\delta > 0$ be such that $|f_j(x) - f_j(y)| < \frac{\varepsilon}{\sqrt{n}}$ whenever $d(x, y) < \delta$, for each $j \in J_n$. Then for such $y$, we have

$$|f(x) - f(y)| = \sqrt{(f_1(x) - f_1(y))^2 + \cdots + (f_n(x) - f_n(y))^2} < \sqrt{\frac{\varepsilon^2}{n} + \cdots + \frac{\varepsilon^2}{n}} = \varepsilon.$$

$\square$

### 2.3. Continuity and Compactness.

THEOREM 2.21 (Continuous Image of a Compact Set is Compact). *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $f : X \to Y$ be a continuous function, and let $E$ be a compact subset of $X$. Then $f(E)$ is compact.*

PROOF. Let $\mathcal{B}$ be an open cover of $f(E)$ in $Y$. We claim that $\mathcal{A} = \{f^{-1}(B)\}_{B \in \mathcal{B}}$ is an open cover of $E$ in $X$. Indeed, each $f^{-1}(B)$ is open in $X$ by continuity; furthermore, if $x \in E$, then $f(x) \in f(E)$, so $f(x) \in B_0$ for some $B_0 \in \mathcal{B}$, so $x \in f^{-1}(B_0)$. Let $\mathcal{A}' = \{f^{-1}(B_1), \dots, f^{-1}(B_n)\}$ be a finite subcover for $E$ associated to the open covering $\mathcal{A}$. Then

$$E \subset \bigcup_{j=1}^{n} f^{-1}(B_j)$$

implies

$$f(E) \subset f\left(\bigcup_{j=1}^{n} f^{-1}(B_j)\right) = \bigcup_{j=1}^{n} f(f^{-1}(B_j)) \subset \bigcup_{j=1}^{n} B_j.$$

Thus $\mathcal{B}' = \{B_1, \dots, B_n\}$ is then a finite subcovering of $f(E)$ associated to the covering $\mathcal{B}$. Since $\mathcal{B}$ was an arbitrary open cover of $f(E)$, we conclude that $f(E)$ is compact. $\square$

COROLLARY 2.22 (Extreme Value Theorem). *Let $f : X \to \mathbb{R}$ be a continuous function and let $E$ be a compact subset of $X$. Then there exist $x, y \in X$ such that*

$$f(x) = \sup f(E), \qquad f(y) = \inf f(E).$$

PROOF. Since $f(E)$ is compact, it is closed and bounded in $\mathbb{R}$. Therefore $\sup f(E) \in \overline{f(E)} = f(E)$. This shows that there exists $x \in E$ such that $f(x) = \sup f(E)$. The other claim follows in the same way. $\square$

EXERCISE 2.23.

(1) Find a closed subset $E$ of $\mathbb{R}$ and a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(E)$ is not closed.
(2) Find a bounded subset $E$ of $\mathbb{R}$ and a continuous function $f : E \to \mathbb{R}$ such that $f(E)$ is not bounded.
(3) Show that if $E$ is a bounded subset of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous, then $f(E)$ is bounded.

THEOREM 2.24 (Continuity of Inverse). *Let $X$ and $Y$ be metric spaces, with $X$ compact, and let $f : X \to Y$ be a continuous bijection. Then $f^{-1} : Y \to X$ is a continuous function.*

PROOF. Let $C$ be a closed subset of $X$; we show that $(f^{-1})^{-1}(C) = f(C)$ is closed in $Y$. (This suffices by Exercise 2.10.) As $C$ is a closed subset of the compact set $X$, it follows that $C$ is compact. Therefore $f(C)$ is compact, hence closed in $Y$. $\square$

Note that the utility of the Theorem above is not limited to the context of compact sets. Here is an example:

PROPOSITION 2.25. *The function $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = x^{\frac{1}{n}}$ is continuous for any $n \in \mathbb{N}$.*

PROOF. Choose $x \in [0, \infty)$. Then $f$ is continuous at $x$ if and only if the function $\widetilde{f} : [0, x+1] \to [0, (x+1)^{\frac{1}{n}}]$ is continuous at $x$. (See Proposition 2.13.) On the other hand, $g : [0, (x+1)^{\frac{1}{n}}] \to [0, x+1]$ given by $g(x) = x^n$ is a continuous function on the compact set $[0, x+1]$, whose inverse is $\widetilde{f}$. So by Theorem 2.24, it follows that $\widetilde{f}$ is continuous (in particular, at $x$), so that $f$ is continuous at $x$. Since $x$ was arbitrary, we conclude that $f$ is continuous. □

### 2.4. Continuity and Connectedness.

THEOREM 2.26 (Continuous Image of a Connected Set is Connected). *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Let $f : X \to Y$ be a continuous function, and let $E$ be a connected subset of $X$. Then $f(E)$ is connected.*

PROOF. Assume without loss of generality that $E$ is nonempty. Let $C$ and $D$ be separated sets whose union is $f(E)$; we show that either $C$ or $D$ must be empty. Put $A = f^{-1}(C) \cap E$ and $B = f^{-1}(D) \cap E$. Then[1] $f(A) = C$ and $f(B) = D$. So it suffices to show that $A$ or $B$ must be empty. Since $f^{-1}(\overline{C})$ is a closed set containing $A$, it must contain $\overline{A}$ as well. Thus

$$\overline{A} \cap B \subset f^{-1}(\overline{C}) \cap f^{-1}(D) = f^{-1}(\overline{C} \cap D) = f^{-1}(\emptyset) = \emptyset.$$

Similarly, $A \cap \overline{B} = \emptyset$. It follows that one of $A$ or $B$ must be empty, as needed. □

COROLLARY 2.27 (Intermediate Value Theorem). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. If $f(a) < c < f(b)$, then there exists $x \in (a, b)$ such that $f(x) = c$.*

PROOF. $f([a, b])$ is a connected set containing $f(a)$ and $f(b)$; therefore it must contain $c$ as well. □

DEFINITION 2.28. Let $(X, d)$ be a metric space and let $a$ and $b$ be points in $X$. A *path* in $X$ from $a$ to $b$ is a continuous function $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$.

DEFINITION 2.29. Let $(X, d)$ be a metric space. $X$ is said to be *path connected* if for any two points $x$ and $y$ in $X$, there exists a path in $X$ from $x$ to $y$.

THEOREM 2.30. *Let $(X, d)$ be a path connected metric space. Then $X$ is connected.*

PROOF. If $X$ is empty, there is nothing to prove. Otherwise, choose $x \in X$. For each $y \in X$, let $p_{x,y} : [0, 1] \to X$ be a path in $X$ from $x$ to $y$. Then $\operatorname{Im} p_{x,y}$ is connected and contains the point $x = p_{x,y}(0)$ for every $y \in X$. Thus $\bigcap_{y \in X} \operatorname{Im} p_{x,y}$ is nonempty. Thus, by Exercise 5.12 in Chapter 4, it follows that $X = \bigcup_{y \in X} \operatorname{Im} p_{x,y}$ is connected, as needed. □

COROLLARY 2.31. *Let $E$ be a convex subset of $\mathbb{R}^k$, for some $k \in \mathbb{N}$. Then $E$ is path-connected, therefore connected.*

This Corollary follows from the observation that the function $p(t) = (1-t)x + ty$ is a (straight-line) path from $x$ to $y$. The definition of convexity guarantees that this path exists in $E$ for any $x, y \in E$.

EXERCISE 2.32. Prove that the set $\mathbb{R}^2 \backslash \{(0, 0)\}$ is connected. Then, use the function $x/|x|$ to show that $S = \{x \in \mathbb{R}^2 : |x| = 1\}$ is connected. (You may use results from the Supplementary section below if you want, but it is also possible to do this Exercise without it.)

It is emphatically *not* the case that connectedness implies path connectedness. However, the two notions are equivalent for open subsets of Euclidean space. The supplementary material below elaborates on these points.

[1]Since $A \subset f^{-1}(C)$, it follows that $f(A) \subset C$. On the other hand, $C \subset f(E)$; therefore, if $y \in C$, then $y = f(x)$ for some $x \in E \cap f^{-1}(C) = A$, so $y = f(x) \in f(A)$.

2.4.1. *Supplement on Path Connectedness.* Note: This section will not be covered in class or tested in any way. However, it provides some useful context.

EXAMPLE 2.33 (Topologist's Sine Curve). Let's take as a given that the function $\sin x$ is continuous on all of $\mathbb{R}$. Consider the continuous functions $g : (0, +\infty) \to \mathbb{R}$ and $h : (0, +\infty) \to \mathbb{R}^2$ defined by $g(x) = \sin(x^{-1})$ and $h(x) = (x, g(x))$; denote $S = h((0, +\infty))$. Then $S$ is connected, by Theorem 2.26, therefore $\overline{S}$ is connected, by Exercise 5.8.

Denote $S_0 = \{0\} \times [-1, 1] \subset \mathbb{R}^2$. We claim that $\overline{S} = S \cup S_0$. We first show that $S_0 \subset S'$. Indeed, pick $y \in [-1, 1]$, then pick $\theta_0 \in \mathbb{R}$ such that $\sin \theta_0 = y$. Let $x_k = (\theta_0 + 2\pi k)^{-1}$, for each $k \in \mathbb{N}$. Then $g(x_k) = \sin(\theta_0+2\pi k) = y$ for each $k \in \mathbb{N}$ and $x_k \to 0$ in $\mathbb{R}$ as $k \to \infty$, so $(h(x_k))_{k=1}^{\infty} = ((x_k, g(x_k)))_{k=1}^{\infty}$ is a sequence in $S$ that converges to $(0, y)$. Since any point in $S_0$ can be expressed in the form $(0, y)$ for some $y \in [-1, 1]$, this proves that $S_0 \subset S'$ (and therefore that $S \cup S_0 \subset \overline{S}$).

To see that $S\cup S_0$ is *all* of $\overline{S}$, we show that $S\cup S_0$ is closed in $\mathbb{R}^2$, or equivalently, that its complement is open. For each $n \in \mathbb{N}$, let $C_n = h([\frac{1}{n}, n])$ and $U_n = ((\frac{1}{n}, n) \times \mathbb{R})\backslash C_n$. Since $[\frac{1}{n}, n]$ is compact, its image $C_n$ is compact as well (Theorem 2.21); therefore $C_n$ is closed, so $U_n$ is open. Define $V = \mathbb{R} \times ((-\infty, -1) \cup (1, +\infty)$ and $W = (-\infty, 0) \times \mathbb{R}$. Then

$$\mathbb{R}^2\backslash(S \cup S_0) = V \cup W \cup \bigcup_{n=1}^{\infty} U_n,$$

so $S \cup S_0$ is closed, as needed. This proves that $\overline{S} = S \cup S_0$.

Next, we claim that $\overline{S}$ is *not* path connected. To see this, we argue by contradiction. Assume $f$ is a path in $\overline{S}$ with $f(0) = (\frac{1}{\pi}, 0)$ and $f(1) = (0, 0)$. Let $\phi_1 : \mathbb{R}^2 \to \mathbb{R}$ denote projection onto the first coordinate (which is continuous by Proposition 2.17): $\phi_1((x_1, x_2)) = x_1$. Construct a sequence $t_k$ inductively as follows. Choose $t_1 \in [0, 1]$ such that $\phi_1(f(t_1)) = \frac{2}{3\pi}$. This is possible, since $(\phi_1 \circ f)([0, 1])$ is a connected subset of $\mathbb{R}$ that contains both $\phi_1(f(0)) = \phi_1((\frac{1}{\pi}, 0)) = \frac{1}{\pi}$ and $\phi_1(f(1)) = \phi_1((0, 0)) = 0$, which implies that $\frac{2}{3\pi} \in [0, \frac{1}{\pi}] \subset (\phi_1 \circ f)([0, 1])$. Having chosen $t_1 < \ldots < t_k$ in $(0, 1)$ such that $\phi_1(f(t_k)) = (\frac{\pi}{2} + k\pi)^{-1}$, choose $t_{k+1}$ such that $\phi_1(f(t_{k+1})) = (\frac{\pi}{2} + (k + 1)\pi)^{-1}$. This is possible because $(\phi_1 \circ f)([t_k, 1])$ is a connected subset of $\mathbb{R}$ that contains both $\phi_1(f(t_k)) = (\frac{\pi}{2} + k\pi)^{-1}$ and $\phi_1(f(1)) = \phi_1((0, 0)) = 0$, which implies that $(\frac{\pi}{2}+(k+1)\pi)^{-1} \in [0, (\frac{\pi}{2}+k\pi)^{-1}] \subset (\phi_1\circ f)([0, t_k])$. The sequence $(t_k)_{k=1}^{\infty}$ is increasing and bounded above in $\mathbb{R}$ (by 1), so it converges to $t_* = \sup\{t_k\} \in [0, 1]$. (See Theorem 3.6 in Chapter 6 below.) But $|f(t_k) - f(t_{k+1})| > 2$ for each $k \in \mathbb{N}$, so $(f(t_k))_{k=1}^{\infty}$ is *not* a Cauchy sequence, so it cannot converge. This contradicts the continuity of $f$, allowing us to conclude that $\overline{S}$ is not path connected.

The Example above also shows that the closure of a path-connected set might not be path connected.

PROPOSITION 2.34. *Let $U$ be a nonempty open subset of $\mathbb{R}^n$ under the Euclidean metric. Then $U$ is connected if and only if it is path connected.*

To prove this Proposition, we make use of the following Lemma:

LEMMA 2.35. *Let $(X, d)$ be a metric space. Write $x \sim y$ if there exists a path in $X$ from $x$ to $y$. Then $\sim$ is an equivalence relation*

PROOF. If $x \in X$, then the function $f : [0, 1] \to X$ defined by $f(t) = x$ for all $t$ is a path from $x$ to $x$. (This proves reflexivity.) If $x \sim y$, then there exists a path $g : [0, 1] \to X$ from $x$ to $y$; in this case the 'reverse' path $\widetilde{g} : [0, 1] \to X$ defined by $\widetilde{g}(t) = g(1-t)$ is a path from $y$ to $x$, so that $y \sim x$. (This proves symmetry.) Finally, if $x \sim y$ and $y \sim z$, let $h$ and $k$ be paths from $x$ to $y$ and $y$ to $z$, respectively. Define $p : [0, 1] \to X$ by $p(t) = h(2t)$ if $t \in [0, \frac{1}{2}]$ and $p(t) = k(2t - 1)$ if $t \in [\frac{1}{2}, 1]$. Then $p(\frac{1}{2})$ is well-defined, since $h(1) = y = k(0)$, and $p$ is continuous. Indeed, $p$ is continuous on $[0, \frac{1}{2})\cup(\frac{1}{2}, 1]$ by the second point of Proposition 2.13; it remains to show that $p$ is continuous at $t = \frac{1}{2}$. Let $U$ be a neighborhood of $y$ in $X$, then choose $\delta > 0$ so that $h((1-2\delta, 1]) \subset k([0, 2\delta)) \subset U$. (This is possible because $h$ is continuous at 1, $k$ is continuous at 0, and $h(1) = k(0) = y$.) Then $p((\frac{1}{2} - \delta,\ \frac{1}{2} + \delta)) = h((1-2\delta, 1] \cup k([0, 2\delta)) \subset U$. This proves that $p$ is continuous at $\frac{1}{2}$, thus continuous everywhere. Finally, noting that $p(0) = x$ and $p(1) = z$, we conclude that $p$ is a path from $x$ to $z$ in $X$, so that $x \sim z$. (This proves transitivity.) $\square$

PROOF OF PROPOSITION 2.34. Write $x \sim y$ if there exists a path in $U$ from $x$ to $y$. Pick $a \in U$ and let $E = \{x \in U : a \sim x\}$, $F = U \backslash E$. We claim that $E$ and $F$ are both open in $\mathbb{R}^n$. Since $U$ is open in $\mathbb{R}^n$, this will prove that $E$ and $F$ are both open in $U$, which will prove (by Proposition 5.14 in Chapter 4) that one of $E$ or $F$ is empty; since $x \in E$ it will follow that $E = U$, which says that $U$ is path connected.

To prove that $E$ is open in $\mathbb{R}^n$, choose $y \in E$. Since $y \in E \subset U$, and $U$ is open in $\mathbb{R}^n$, there exists $r > 0$ such that $B_{\mathbb{R}^n}(y, r) \subset U$. Since $B_{\mathbb{R}^n}(y, r)$ is convex, it is path connected; therefore $y \sim z$ for every $z \in B_{\mathbb{R}^n}(y, r)$. But by transitivity of $\sim$, this implies that $x \sim z$ for every such $z$, which implies (by definition of $E$) that $B_{\mathbb{R}^n}(y, r) \subset E$. This proves that the arbitrary point $y$ of $E$ is an interior point of $E$ with respect to $\mathbb{R}^n$, so $E$ is open in $\mathbb{R}^n$.

To prove that $F$ is open in $\mathbb{R}^n$, we argue similarly. If $F$ is empty then we are done; therefore assume without loss of generality that $y' \in F$. Since $y' \in F \subset U$ and $U$ is open in $\mathbb{R}^n$, it follows that there exists $r' > 0$ such that $B_{\mathbb{R}^n}(y', r') \subset U$. Then $y' \sim z'$ every $z' \in B_{\mathbb{R}^n}(y', r')$. If $x \sim z'$ for some $z' \in B_{\mathbb{R}^n}(y', r')$, then $x \sim y'$ as well by transitivity, contradicting the assumption that $y' \in F$. Therefore $B_{\mathbb{R}^n}(y', r') \subset F$. Therefore $F$ is open in $\mathbb{R}^n$. This completes the proof. $\square$

PROPOSITION 2.36 (Continuous Image of a Path Connected Set is Path Connected). *Let* $(X, d_X)$ *and* $(Y, d_Y)$ *be metric spaces. Let* $f : X \to Y$ *be a continuous function, and let* $E$ *be a path connected subset of* $X$*. Then* $f(E)$ *is path connected.*

PROOF. Choose $x, y \in E$, then choose $a \in f^{-1}(x)$, $b \in f^{-1}(b)$, and let $g : [0,1] \to E$ be a path in $E$ from $a$ to $b$. Then $f \circ g : [0,1] \to Y$ is a path in $f(E)$ from $f(a) = x$ to $f(b) = y$. Thus $f(E)$ is path connected. $\square$

### 2.5. Uniform Continuity.

DEFINITION 2.37. Let $X, Y$ be metric spaces and $f : X \to Y$ a function. We say that $f$ is *uniformly continuous* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, y) < \delta$.

Note: This is *not* the same as the $\varepsilon$-$\delta$ characterization of continuity. The key difference here is that $\delta > 0$ does not depend on $x$: The same $\delta > 0$ works *for any* $x \in X$. (That is, the $\delta$ can be taken *uniformly* for any choice of $x \in X$.)

Another note: There is no open set characterization of uniform continuity (at least, not without a vector space structure); the definition depends explicitly on the metrics in $X$ and $Y$. Further, continuity is a *local* property: one can make sense of what it means to be "continuous at a point". On the other hand, uniform continuity at a point doesn't make sense; uniform continuity is a *global* property of a function. (Remember that the same $\delta > 0$ has to work *for any* $x \in X$!)

EXAMPLE 2.38. Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = 1/x$. Then $f$ is continuous (since $g(x) = x$ is continuous and nonzero on $(0, \infty)$), but $f$ is not uniformly continuous.

PROOF. We argue by contradiction. Assume $f$ is uniformly continuous. Then putting $\varepsilon = 1$, there must exist $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$ for all $x, y \in (0, \infty)$. Assume without loss of generality[2] that $\delta < 1$. Put[3] $x = \delta$ and $y = \frac{\delta}{2}$. Then $|x - y| = \frac{\delta}{2} < \delta$, but $|f(x) - f(y)| = \delta^{-1} > 1$, contradicting our choice of $\delta$. We conclude that $f$ cannot be uniformly continuous. $\square$

[2] If some $\widetilde{\delta} \geq 1$ works, then so will any $\delta < \widetilde{\delta}$.

[3] Let's motivate this choice of $x$ and $y$: To show that $f$ is *not* uniformly continuous, it suffices to find $x, y \in (0, \infty)$ such that $|x - y| < \delta$, but $|f(x) - f(y)| > 1$. Let's look for $x$ and $y$ that make this inequality work; assume without loss of generality that $x > y$. Then $|f(x) - f(y)| = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$; to make this bigger than 1, we want $x - y > xy$, so $x > (x+1)y$,

On the other hand, $f\big|_{(1,\infty)}$ *is* uniformly continuous. Indeed, if $x, y > 1$, then $|f(x) - f(y)| = \frac{|x-y|}{|x||y|} < |x - y|$. Putting $\delta = \varepsilon$, we conclude that $|x - y| < \delta$ implies $|g(x) - g(y)| < \varepsilon$.

Note: The concept of uniform continuity is the first of many concepts in this course which involve uniformity with respect to some parameter. Uniform continuity is not the most important of these concepts, but hopefully it serves as a gentle introduction to this way of thinking.

THEOREM 2.39. *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Assume $X$ is compact and $f : X \to Y$ is continuous. Then $f$ is uniformly continuous.*

PROOF. Choose $\varepsilon > 0$. Then, for each $x \in X$, choose $\delta_x$ such that $d_X(x, y) < \delta_x$ implies $d_Y(f(x), f(y)) < \frac{\varepsilon}{2}$. It follows that $\{B_X(x, \frac{\delta_x}{2})\}_{x\in X}$ is an open cover for $X$; let $\{B_X(x_j, \frac{\delta_{x_j}}{2})\}_{j=1}^n$ be a finite subcover. Put $\delta = \frac{1}{2}\min\{\delta_{x_1}, \ldots, \delta_{x_n}\}$. We claim that $d_X(p, q) < \delta$ implies $d_Y(f(p), f(q)) < \varepsilon$. Indeed, for some $j \in \{1, \ldots, n\}$, we have $p \in B_X(x_j, \frac{\delta_{x_j}}{2})$; by the triangle inequality, $q$ belongs to $B_X(x_j, \delta_{x_j})$ (for the same $j$), since

$$d_X(x_j, q) \leq d_X(x_j, p) + d_X(p, q) < \frac{\delta_{x_j}}{2} + \delta \leq \delta_{x_j}.$$

Consequently,

$$d_Y(f(p), f(q)) \leq d_Y(f(p), f(x_j)) + d_Y(f(x_j), f(q)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $f$ is uniformly continuous. □

EXERCISE 2.40. Assume $f : X \to Y$ and $g : Y \to Z$ are uniformly continuous functions, where $(X, d_X)$, $(Y, d_Y)$, and $(Z, d_Z)$ are metric spaces. Prove that $g \circ f$ is uniformly continuous.

EXERCISE 2.41. Let $E$ be a bounded subset of $\mathbb{R}^k$ and let $f : E \to \mathbb{R}$ be a uniformly continuous function. Show that $f$ is bounded. (Hint: You will need to use compactness of $\overline{E}$ at some point.)

## 3. Sequences of Functions

In this section, we define the notions of pointwise and uniform convergence, and explain the key differences between the two concepts. For simplicity, we restrict our attention to real-valued functions; however, nearly all of our statements and proofs continue to hold (with minute changes) if $\mathbb{R}$ is replaced by any complete metric space $(Y, d_Y)$.

Large parts of this discussion make sense even for sequences of functions $(f_n)$ on a set $X$ which has no additional structure. However, the applications of uniform convergence we are most concerned with appear in the context of functions defined on metric spaces, hence the inclusion of the topic in this chapter.

### 3.1. Pointwise and Uniform Convergence for Real-Valued Functions.

DEFINITION 3.1. Let $X$ be any set, and let $(f_n)_{n=1}^\infty$ be a sequence of real-valued functions defined on $X$. If for each $x \in E$, the limit $\lim_{n\to\infty} f_n(x)$ exists, then we can define the *(pointwise) limit function* by

$$f(x) = \lim_{n\to\infty} f_n(x),$$

for each $x \in E$. In this case we say that $(f_n)$ *converges pointwise* to $f$ on $E$.

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so $y < \frac{x}{x+1}$. Let's additionally take $x < 1$, so that $y < x/2$ is good enough. To ensure that $x - y < \delta$, we require $y > x - \delta$. So we need $x < 1$ and $x/2 > x - \delta$, i.e. $x < 2\delta$.

The question that arises is this: Given information about the functions $f_n$, what can be said about the pointwise limit $f$? Obviously, this question needs to be refined a little before we can answer it. The version of this question that we are best equipped to ask is the following. Assume $(X, d)$ is a metric space and $(f_n)$ is a sequence in $\mathbb{R}^X$. If all of the $f_n$'s are continuous and $f_n \to f$ pointwise, does $f$ have to be continuous? Recall that a function $f$ is continuous at a limit point $x \in X$ if and only if $\lim_{t\to x} f(t) = f(x)$. So the limit function is continuous at $x$ if and only if

$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t).$$

The following examples show that interchanging limit processes can be perilous.

EXAMPLE 3.2.

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{m}{m+n}\right) = \lim_{n\to\infty} 1 = 1.$$

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{m}{m+n}\right) = \lim_{m\to\infty} 0 = 0.$$

Changing the order in which we take $m, n$ to infinity changes the limit!

EXAMPLE 3.3. For each $n \in \mathbb{N}$, define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = x^n$. Then for $x \in [0, 1)$, we have $\lim_{n\to\infty} f_n(x) = 0$, while $\lim_{n\to\infty} f_n(1) = 1$. So the pointwise limit function is given by

$$f(x) = \begin{cases} 1 & x = 1, \\ 0 & x \in [0, 1). \end{cases}$$

Clearly the limit function is not continuous at $x = 1$, even though all the $f_n$'s are continuous everywhere.

DEFINITION 3.4. Let $f_n : X \to \mathbb{R}$ be functions. We say that $(f_n)$ *converges uniformly* to a function $f$ on $E \subset X$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon$ for all $x \in E$.

It should be clear that uniform convergence implies pointwise convergence. However, the converse is not true. The key here is that the same $N \in \mathbb{N}$ has to work for all $x$ at the same time. For pointwise convergence, the $N$ is allowed to depend on $x$. (This should remind you of the difference between continuity and uniform continuity.) Also note that a sequence of functions might converge uniformly on some set $E$, but not on some larger set $F$.

It's usually fairly messy to show that a given sequence of functions converges uniformly directly from the definition. Therefore, before giving examples of sequences of functions which converge uniformly, we give the following very useful criterion.

PROPOSITION 3.5. *Assume $f_n \to f$ pointwise on $E$, and put*

$$M_n = \sup_{x\in E} |f_n(x) - f(x)|.$$

*Assume additionally that $M_n < +\infty$ for all $n \in \mathbb{N}$. Then $f_n \to f$ uniformly on $E$ if and only if $M_n \to 0$ as $n \to \infty$.*

PROOF. ( $\Longrightarrow$ ) Assume $f_n \to f$ uniformly on $E$. Choose $\varepsilon > 0$ and pick $N \in \mathbb{N}$ large enough so that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon/2$ for all $x \in E$. Then $n \geq N$ implies $M_n = \sup_{x\in E} |f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon$, so $M_n \to 0$ as $n \to \infty$.

( $\Longleftarrow$ ) Assume $M_n \to 0$. Choose $\varepsilon > 0$, then pick $N \in \mathbb{N}$ large enough so that $n \geq N$ implies $M_n < \varepsilon$. Then $n \geq N$ also implies that $|f_n(x) - f(x)| \leq \sup_{x\in E} |f_n(x) - f(x)| = M_n < \varepsilon$. Thus $f_n \to f$ uniformly on $E$ as $n \to \infty$. $\square$

EXAMPLE 3.6. If $f_n(x) = x^n$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, then $f_n \to 0$ pointwise[4] on $[0, 1)$, as we have seen above. The situation with regard to uniform convergence is more subtle. It turns out that

- $f_n \to 0$ uniformly on $[0, c]$ for any $c \in (0, 1)$, but
- $(f_n)$ does *not* converge uniformly on $[0, 1)$.

PROOF. Choose $c \in (0, 1)$ and $\varepsilon > 0$. Then $M_n = \sup_{x\in[0,c]} |x^n - 0| = c^n$ tends to zero as $n \to \infty$, so $f_n \to 0$ uniformly on $[0, c]$.

If $(f_n)$ were to converge uniformly on $[0, 1)$ to some function $f$, then it would also converge pointwise to that function. It follows that if $f_n \to f$ uniformly on $[0, 1)$, then $f(x) = 0$ for all $x \in [0, 1)$, as we already know $f_n \to 0$ pointwise on $[0, 1)$. However, $\widetilde{M_n} = \sup_{x\in[0,1)} |x^n - 0| = 1$, which does not tend to zero as $n \to \infty$. Therefore $(f_n)$ does *not* converge uniformly to $0$ (or to any function, for that matter) on $[0, 1)$. □

The preceding Example and Proposition suggest that the framework introduced in the following discussion is a useful way to think about uniform convergence in many cases.

**3.2. Uniform Convergence and the space $B(X)$.** Recall that if $X$ is any set, then $(B(X), \|\cdot\|_u)$ denotes the vector space of all bounded, real-valued functions on $X$, together with the supremum norm:

$$\|f\|_u = \sup_{x\in X} |f(x)|.$$

Recall also that we can make any vector space norm into a metric in a canonical way. Therefore we can consider $B(X)$ as a metric space, with metric $d_u(f, g) = \|f - g\|_u = \sup_{x\in X} |f(x) - g(x)|$. Unless otherwise stated, we always give $B(X)$ this metric.

REMARK 3.7. Two functions $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are distance at most $R$ apart in the uniform metric if $\sup_{x\in X} |f(x) - g(x)| \le R$, that is, if $|f(x) - g(x)| \le R$ for all $x \in X$. So in order for two functions to be close together in $(B(X), d_u)$, they must be close together on *all* of $X$, in a *uniform* way.

PROPOSITION 3.8. *Let $(f_n)$ be a sequence of bounded functions on a set $X$; let $f$ be another function in $B(X)$. Then $f_n \to f$ uniformly on $X$ if and only if $f_n \to f$ in $(B(X), d_u)$.*

PROOF. $f_n \to f$ uniformly on $X$ if and only if $d_u(f_n, f) = \sup_{x\in X} |f_n(x) - f(x)|$ tends to zero as $n \to \infty$, by Proposition 3.5. This occurs if and only if $f_n \to f$ in $B(X)$. □

While the space $B(X)$ is a useful way of thinking about uniform convergence, it doesn't capture everything, since a sequence of functions may converge uniformly even if none of the functions are bounded. Take for example the sequence $(f_n)_{n=1}^{\infty}$ of real-valued functions defined on $(0, \infty)$ by the rule $f_n(x) = \frac{1}{x} + \frac{1}{n}$; let $f : (0, \infty)$ be given by $f(x) = \frac{1}{x}$. Then clearly $f_n \to f$ uniformly on $(0, \infty)$ as $n \to \infty$, but $f_n \notin B((0, \infty))$ for any $n$.

EXERCISE 3.9. A collection $\mathcal{A}$ of real-valued functions on a set $E$ is said to be *uniformly bounded* on $E$ if there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in E$, for all $f \in \mathcal{A}$. (So each function is bounded, and the same bound works for all functions in $\mathcal{A}$.) Let $(f_n)$ be a sequence of bounded functions which converges uniformly to a limit function $f$. Prove that $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded subset of $(B(X), d_u)$.

EXERCISE 3.10. Let $(f_n)$ and $(g_n)$ be sequences of real-valued functions on a set $E$, which converge uniformly on $E$ to limit functions $f$ and $g$, respectively.

(1) Prove that $(f_n + g_n)$ converges to $f + g$, uniformly on $E$.
(2) If each $f_n$ and each $g_n$ is bounded, show that $(f_n g_n)$ converges uniformly to $fg$ on $E$.

[4] A notational clarification here: '$f_n \to 0$ pointwise' means that $f_n(x) \to 0$ for every $x \in [0, 1]$, as $n \to \infty$; equivalently, $f_n$ converges pointwise to the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 0$ for all $x$, i.e., the zero function on $[0, 1]$, simply denoted $0$.

### 3.3. Uniformly Cauchy Sequences and Completeness of $B(X)$.

DEFINITION 3.11. The sequence $(f_n)$ of real-valued functions on a set $X$ is said to be *uniformly Cauchy* on $E \subset X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m \geq n \geq N$ implies $|f_m(x) - f_n(x)| < \varepsilon$ for all $x \in E$.

REMARK 3.12. Clearly the definition is equivalent to the requirement that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_u(f_n, f_m) = \sup_{x \in E} |f_n(x) - f_m(x)| < \varepsilon$ whenever $m \geq n \geq N$. If $(f_n)$ is a sequence in $B(X)$, this requirement just says that $(f_n)$ is Cauchy in $B(E)$. We record this observation as the following Proposition.

PROPOSITION 3.13. *If $(f_n)$ is a Cauchy sequence in $B(X)$, then $(f_n)$ is uniformly Cauchy on $X$.*

THEOREM 3.14. *Let $(f_n)_{n=1}^\infty$ be a sequence of real-valued functions on a set $X$. Then $(f_n)$ converges uniformly on $E \subset X$ if and only if it is uniformly Cauchy on $E$.*

The key idea in the proof is the use of completeness of $\mathbb{R}$, the codomain of the functions $f_n$.

PROOF. ( $\Longrightarrow$ ) Assume that $(f_n)$ converges uniformly on $E$, and let $f$ denote the limit function. Choose $\varepsilon > 0$, then choose $N$ large enough so that $n \geq N$ implies $|f_n(x) - f(x)| < \varepsilon/2$ for all $x \in E$. Then $m \geq n \geq N$ implies that

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $(f_n)$ is uniformly Cauchy on $E$.

( $\Longleftarrow$ ) Assume that $(f_n)$ is uniformly Cauchy on $E$. Then $(f_n(x))$ is a Cauchy sequence of real numbers for each $x \in E$. Since $\mathbb{R}$ is complete, each of the sequences $(f_n(x))$ converges to some number; therefore we can define a pointwise limit function $f(x)$. We need to show that $(f_n)$ converges *uniformly* to $f$ on $E$. To this end, pick $\varepsilon > 0$. Choose $N$ large enough so that $m \geq n \geq N$ implies $|f_m(x) - f_n(x)| < \varepsilon/2$. Assume $n \geq N$, then take the limit as $m \to \infty$ on both sides of the inequality; this gives $|f(x) - f_n(x)| \leq \varepsilon/2 < \varepsilon$, for all $x \in E$. Thus $f_n \to f$ uniformly on $E$. ☐

THEOREM 3.15. *For any set $X$, the metric space $(B(X), d_u)$ is complete.*

PROOF. Let $(f_n)$ be a Cauchy sequence in $(B(X), d_u)$. Then $(f_n)$ is uniformly Cauchy, so it converges uniformly (and pointwise) to some function $f$. We need to show that $f \in B(X)$, that is, $f$ is bounded. Choose $N \in \mathbb{N}$ such that $|f_N(x) - f(x)| < 1$ for all $x \in X$. Then choose $M > 0$ so that $|f_N(x)| \leq M$ for all $x \in X$. Then for all $x \in X$, we have

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M.$$

So $f$ is bounded, as claimed. ☐

### 3.4. The Uniform Limit Theorem and Completeness of $BC(X)$.

THEOREM 3.16 (Uniform Limit Theorem). *Let $(f_n)$ be a sequence of continuous real-valued functions on a metric space $(X, d)$. Assume $f : E \to \mathbb{R}$ is a function such that $f_n \to f$ uniformly on $E \subset X$. Then $f$ is continuous.*

PROOF. Choose $\varepsilon > 0$ and $x \in E$. We need to show that there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $d(x, y) < \delta$ and $y \in E$. In light of the inequality

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|$$

(which is valid for any $N \in \mathbb{N}$), we break up the task into three parts, making each of the three terms above less than $\varepsilon/3$.

Choose $N$ large enough so that $|f(z) - f_N(z)| < \frac{\varepsilon}{3}$ for all $z \in E$. Then, for this same $N$, choose $\delta > 0$ small enough so that $d(x, y) < \delta$ and $y \in E$ together imply that $|f_N(x) - f_N(y)| < \varepsilon/3$. Then for $y \in E$ such that $d(x, y) < \delta$, we have

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

□

Where did we use uniform convergence in the proof above? Why doesn't it work for pointwise convergence? Remember that we chose $N$ so that $|f(z) - f_N(z)| < \varepsilon/3$ *for all* $z \in E$. If we only had pointwise convergence, we could guarantee that $|f(x) - f_N(x)|$ is small just as before; however, we have to choose $\delta > 0$ before we choose $y$, and then we have no way to guarantee that $|f_N(y) - f(y)|$ is small without going back and changing $N$, after which point we'll have to adjust $\delta$... it should be clear that such an argument can't work. Furthermore, we have already seen that the conclusion fails if the convergence is not uniform: a sequence of continuous functions may tend pointwise to a discontinuous limit. Uniform convergence really makes a big difference here!

DEFINITION 3.17. Let $(X, d)$ be a metric space. The set of all bounded, continuous functions on $X$ is denoted $BC(X)$.

Clearly $BC(X)$ is a vector subspace of $B(X)$. We will accordingly always consider $BC(X)$ as a metric space, together with the uniform metric.

COROLLARY 3.18. *Let $(X, d)$ be a metric space. The space $(BC(X), d_u)$ is complete.*

PROOF. To prove the statement, we only need to show that $BC(X)$ is a closed subset of the complete metric space $B(X)$. (See Exercise 2.18 in Chapter 4.) Let $f$ be a limit point of $BC(X)$ with respect to $B(X)$. Then there exists a sequence $(f_n)$ in $BC(X)$ that converges in $B(X)$ to $f$. Since $f_n \to f$ uniformly and each $f_n$ is continuous, we have by the uniform limit theorem that $f$ is continuous as well. This implies that $f \in BC(X)$, as needed. □

The statement of the Uniform Limit Theorem above can be generalized to answer an earlier-stated question about the validity of interchanging certain limit operations.

THEOREM 3.19 (Uniform Limit Theorem, version 2). *Let $(X, d)$ be a metric space. Let $E$ be a subset of $X$; let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on $E$ which converge uniformly to another function $f : E \to \mathbb{R}$. Let $x$ be a limit point of $E$, and assume that for each $n \in \mathbb{N}$, the limit $\lim_{t \to x} f_n(t)$ exists and is equal to some number $A_n$. Then the sequence $(A_n)$ of numbers converges, and*

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n.$$

*That is,*

$$\lim_{t \to x} \left( \lim_{n \to \infty} f_n(t) \right) = \lim_{n \to \infty} \left( \lim_{t \to x} f_n(t) \right).$$

REMARK 3.20. Note that version 1 of the Uniform Limit Theorem is an immediate consequence of version 2; therefore, in principle, we only needed to prove version 2. However, the reader will probably agree that it is easier to grasp the statement and proof of version 2 after seeing version 1 in detail.

PROOF. Step 1: We show that $(A_n)$ is a Cauchy sequence of numbers (and therefore converges in $\mathbb{R}$ as $n \to \infty$, since $\mathbb{R}$ is complete). To this end, choose $\varepsilon > 0$. Since the sequence $(f_n)$ converges uniformly, it is uniformly Cauchy. Choose $N \in \mathbb{N}$ large enough so that $m \geq n \geq N$ implies that $|f_m(t) - f_n(t)| < \frac{\varepsilon}{3}$ for all $t \in X$. Given such $m, n \in \mathbb{N}$, choose $t_0 \in X$ such that $|f_n(t_0) - A_n|$ and $f_m(t_0) - A_m|$ are both less than $\frac{\varepsilon}{3}$. (This is possible by choosing $t_0$ close enough to $x$.) Then

$$|A_n - A_m| \leq |A_n - f_n(t_0)| + |f_n(t_0) - f_m(t_0)| + |f_m(t_0) - A_m| < \varepsilon.$$

Let $A$ denote the limit in $\mathbb{R}$ of the $A_n$'s.

Step 2: We prove the desired equality of limits. To this end, choose $\varepsilon > 0$, then choose $n \in \mathbb{N}$ such that $|f_n(t) - f(t)| < \frac{\varepsilon}{3}$ for all $t \in X$, and such that $|A_n - A| < \frac{\varepsilon}{3}$. For this $n$, let $\delta > 0$ be such that $d_X(t, x) < \delta$ implies $|f_n(t) - A_n| < \frac{\varepsilon}{3}$. Then $d_X(t, x)$ implies

$$|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|.$$

This completes the proof. $\square$

# Part 3

# A Closer Look at the Theorems of Single-Variable Calculus

CHAPTER 6

# Working in $\mathbb{R}$ and $\overline{\mathbb{R}}$

In this short Chapter, we deal with some issues specific to the real line. The special structure of $\mathbb{R}$ (and $\overline{\mathbb{R}}$) allows us to dig a little deeper than we already have for general metric spaces. We begin with a discussion of infinite limits and limits at infinity.

## 1. Infinite Limits and Limits at Infinity

**1.1. Some Motivation and an Outline.** It is likely intuitively clear to the reader what it means for a sequence or a function to 'go off to infinity' or to 'have a limit at infinity'. It's even possible that they have met definitions which formalize this concept. For example, one often says that a sequence $(a_n)$ of real numbers 'tends to $+\infty$' if for every $M \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n \geq N$ implies $a_n \geq M$. One can make similar definitions for a sequence which 'tends to $-\infty$, or a function $f : \mathbb{R} \to (X, d)$ such that $f(t) \to x$ in $X$ as $t \to +\infty$, whatever that means. These ad hoc definitions are extremely useful, but a bit unsatisfying, precisely because they are ad hoc. However, all the relevant definitions look very similar to how we've defined limits in metric spaces, so one might wonder if we can understand the concepts of an 'infinite limit' or a 'limit at infinity' using the machinery we've already developed. The first natural question, then, is:

QUESTION 1.1. *Can we find a metric $\overline{d}$ on $\overline{\mathbb{R}}$ such that $\overline{d}(a, b) = |a - b|$ for $a, b \in \mathbb{R}$, and such that infinite limits and limits at infinity take the 'right' meaning?*

The answer, obviously, is 'no.' After all, any attempt to define $\overline{d}(0, +\infty)$ would very quickly result in a contradiction. However, keep in mind that what we're really trying to understand is a notion of convergence, and this makes sense in a general topological space. In fact, we've gone through some effort to treat many of the concepts in the previous two chapters in a way that generalizes to topological spaces. See for example the definitions below, which are essentially unchanged from their metric space versions[1].

Recall that in a topological space $(X, \mathcal{T})$, a subset $U$ of $X$ is called *open* in $X$ if $U \in \mathcal{T}$.

DEFINITION 1.2. Let $(X, \mathcal{T})$ be a topological space.

- If $U$ is an open set of $X$ containing $x$, we say that $U$ is a *neighborhood* of $x$ in $X$.
- Let $E$ be a subset of $X$. A point $x$ is said to be a *limit point* of $E$ with respect to $X$ if every neighborhood $U$ of $x$ in $X$ intersects $E\backslash\{x\}$.

[1]In light of how easily these definitions generalize to topological spaces, the reader might wonder why we didn't choose to work in the general context of topological spaces in the first place. The answer is two-fold. The first is a pedagogical reason: Metric spaces are considerably more concrete and easier to understand than general topological spaces (though this is not to say that analysis on metric spaces is easy!) The second is a practical one: Many of the statements and proofs we gave in Chapter 4 are simply not true in general topological spaces; other statements don't even make sense without a metric. Here are two examples. First, the notion of a Cauchy sequence doesn't make sense in a general topological space—this really requires the notion of a metric. Second, in general topological spaces, it is not true that if $x$ is a limit point of a set $E$ with respect to $X$, then there exists a sequence $(x_n)$ in $E\backslash\{x\}$ that converges to $x$ in $X$. Therefore our reliance on sequences throughout Chapter 4 would have to be modified extensively to discuss for exactly which kinds of topological spaces certain statements were valid. Just as we have introduced a slew of types of metric spaces, so we would also have to develop a vocabulary for different types of topological spaces.

- Let $(x_n)_{n=1}^{\infty}$ be a sequence in $X$. We say that $(x_n)$ *converges* in $X$ to a point $x \in X$ if for every neighborhood $U$ of $X$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $x_n \in U$.
- Let $(Y, \widetilde{\mathcal{T}})$ be another topological space and let $E$ be a subset of $X$. Let $f : E \to Y$ be a function, and let $p$ be a limit point of $E$ in $X$. We write
$$\lim_{x \to p} f(x) = q$$
to mean the following: For every neighborhood $V$ of $q$ in $Y$, there exists a neighborhood $U$ of $p$ in $E$ such that $x \in U \backslash \{p\}$ implies $f(x) \in V$. in this case we say that $q$ is the *limit* of the function $f(x)$ as $x$ approaches $p$ in $E$.

As noted above, these definitions are the same as the definitions we gave in metric spaces. However, there is of course no way to recast these in terms of $\varepsilon$'s and $\delta$'s without a metric space structure.

Let us return to the main question at hand, namely understanding infinite limits and limits at infinity. In light of the fact that we have the tools to discuss convergence in general topological spaces, the next question one might ask is:

QUESTION 1.3. *Does there exist a* topology *on $\overline{\mathbb{R}}$ which contains all elements of the standard topology of $\mathbb{R}$, and which gives the 'right' notion of convergence with infinite limits and limits at infinity?*

The answer to this question is yes! We will discuss this below, and we will make more precise what we mean by the 'right' notion of convergence.

Even given a positive answer to the previous question, one might not be completely satisfied with the state of affairs. In particular, for any topological space $(X, \mathcal{T})$, it makes sense to ask whether there is any metric $d$ which gives rise to the topology $\mathcal{T}$ (as discussed at the end of Chapter 3). It may seem that we have already treated this question; however, a closer look reveals that our original Question 1.1 asked for too much. If we relax the requirement that $\overline{d}(a, b) = |a - b|$ for $a, b \in \mathbb{R}$, we can ask something more realistic:

QUESTION 1.4. *Let $\overline{\mathcal{T}}$ denote the topology on $\overline{\mathbb{R}}$ that answers Question 1.3 affirmatively. Does there exist a metric $\overline{d}$ on $\overline{\mathbb{R}}$ which gives rise to the topology $\overline{\mathcal{T}}$?*

It turns out that this modified question can also be answered affirmatively, and that therefore we *can* understand infinite limits and limits at infinity in essentially the same way as we have discussed previously. This is rather satisfying, and it also turns out to be useful, for reasons that will become clear later. Before starting in earnest, we review once more our plan.

(1) Motivated by the question of making sense of infinite limits and limits at infinity, we define a topology $\overline{\mathcal{T}}$ on $\overline{\mathbb{R}}$ which is compatible with the standard topology on $\mathbb{R}$.
(2) We show that the topology $\overline{\mathcal{T}}$ is compatible with a sensible notion of infinite limits and limits at infinity.
(3) We show that there exists a metric $\overline{d}$ on $\overline{\mathbb{R}}$ which gives rise to the topology $\overline{\mathcal{T}}$[2].

Note that one technically doesn't need to pass through the topology $\overline{\mathcal{T}}$ in order to reach the metric $\overline{d}$. However, it is the author's opinion that the approach outlined above is better motivated than the approach of jumping straight to $\overline{d}$, even though it necessitates the use of general topological spaces.

**1.2. The Topology of $\overline{\mathbb{R}}$.** In this subsection, we outline the construction of a topology on $\overline{\mathbb{R}}$ which is compatible with the standard topology on $\mathbb{R}$. We mimic the approach of Section 2.3 in Chapter 3, but we do not give details.

Note that in $\mathbb{R}$, we have $(a, b) = B_{\mathbb{R}}\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$ and $B_{\mathbb{R}}(x, r) = (x - r, x + r)$. That is, the collection of open intervals of $\mathbb{R}$ is the same as the collection of open balls of $\mathbb{R}$.

[2]In topological terminology, this says that the space $(\overline{\mathbb{R}}, \overline{\mathcal{T}})$ is *metrizable*.

For the remainder of this subsection, we put $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$, where

$$\mathcal{B}_1 = \{[-\infty, a) : a \in \mathbb{R}\}, \qquad \mathcal{B}_2 = \{(b, +\infty] : b \in \mathbb{R}\}, \qquad \mathcal{B}_3 = \{(a,b) : a, b \in \mathbb{R},\ a < b\}.$$

DEFINITION 1.5. Let $E$ be a subset of $\overline{\mathbb{R}}$. We say that $x$ is an interior point of $E$ with respect to $\overline{\mathbb{R}}$ if there exists $B \in \mathcal{B}$ such that $x \in B \subset E$. The set of all interior points of $E$ with respect to $\overline{\mathbb{R}}$ is denoted $\mathrm{Int}_{\overline{\mathbb{R}}}(E)$. A subset $E$ of $\overline{\mathbb{R}}$ is called *open* in $\overline{\mathbb{R}}$ if $E = \mathrm{Int}_{\overline{\mathbb{R}}}(E)$.

THEOREM 1.6. *Let $\overline{\mathcal{T}}$ denote the collection of all subsets of $\overline{\mathbb{R}}$ which are open in the sense of Definition 1.5; let $\mathcal{T}$ denote the metric topology on $\mathbb{R}$. The following statements hold.*

- *$\overline{\mathcal{T}}$ is a topology on $\overline{\mathbb{R}}$. (That is, $\emptyset$ and $\overline{\mathbb{R}}$ are open, unions of open sets are open, finite intersections of open sets are open.)*
- *$\mathcal{B}$ is a basis for the topology $\overline{\mathcal{T}}$.*
- *The topology $\overline{\mathcal{T}}$ is compatible with the standard topology on $\mathbb{R}$ in the following sense. If $E \subset \mathbb{R}$, then $E$ is open in $\overline{\mathbb{R}}$ (i.e., $E \in \overline{\mathcal{T}}$) if and only if it is open in $\mathbb{R}$ (i.e. $E \in \mathcal{T}$).*

We will not prove this Theorem. One can prove it by following the outline of Section 2.3 in Chapter 3, with the elements of $\mathcal{B}$ replacing the open balls $B_X(x, r)$ in the relevant statements and proofs. This, of course, makes these statements and proofs much more cumbersome to write down (since arguments must be broken into cases corresponding to $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$). However, we are confident that a reader who understands the arguments in Chapter 3, Section 2.3 could in principle adapt those arguments to prove the Theorem above[3].

1.2.1. *Limits in* $(\overline{\mathbb{R}}, \overline{\mathcal{T}})$. In this subsection, we show that the topology $\overline{\mathcal{T}}$ (which we will call the 'standard' topology on $\overline{\mathbb{R}}$) gives a sensible notion of infinite limits and limits at infinity. We discuss only a few statements that we use directly. Note that we formulate everything in terms of $+\infty$, but easy modifications could give statements and proofs applicable to $-\infty$. These modifications should be obvious and we omit them.

PROPOSITION 1.7. *Let $\overline{\mathbb{R}}$ have the usual topology.*

- *If $U$ is a neighborhood of $+\infty$ in $\overline{\mathbb{R}}$, then there exists $M \in \mathbb{R}$ such that $(M, +\infty] \subset U$.*
- *If $A \subset \mathbb{R}$ and $A$ is not bounded above in $\mathbb{R}$, then $+\infty$ is a limit point of $A$ with respect to $\overline{\mathbb{R}}$.*

PROOF. (1) Let $U$ be a neighborhood of $+\infty$ in $\overline{\mathbb{R}}$. Then $U$ contains $+\infty$, and $U$ can be written as the union of sets of the form $[-\infty, a)$, $(a, b)$, and $(b, +\infty]$, where $a, b \in \mathbb{R}$. Of these, only the third contains $+\infty$, so $U$ must have a subset of this type: $(b, +\infty] \subset U$. Put $M = b$ to finish.

(2) Assume $A$ is not bounded above in $\mathbb{R}$, and let $U$ be a neighborhood of $+\infty$ in $\overline{\mathbb{R}}$, and choose $M \in \mathbb{R}$ such that $(M, +\infty] \subset U$. Since $A$ is not bounded above in $\mathbb{R}$, it follows that $M$ is not an upper bound for $A$ in $\mathbb{R}$; therefore there exists $x \in A$ such that $x > M$. This $x$ is therefore contained in $U \cap (A \backslash \{+\infty\})$. Thus $+\infty$ is a limit point of $A$ with respect to $\overline{\mathbb{R}}$. □

The 'meat' of this subsection is the following Proposition, which formalizes what we mean by the 'right' notion of convergence.

PROPOSITION 1.8 (Infinite Limits and Limits at Infinity).

- *Let $A$ be a subset of $\mathbb{R}$, let $p \in \mathbb{R}$ be a limit point of $A$ with respect to $\mathbb{R}$, and let $f : A \to \overline{\mathbb{R}}$ be a function. Then*
$$\lim_{x \to p} f(x) = +\infty$$
*if and only if for every $L \in \mathbb{R}$, there exists $\delta > 0$ such that $0 < |x - p| < \delta$ and $x \in A$ together imply that $f(x) > L$.*

[3] One can also prove this Theorem in a shorter way after developing a few more topological tools; however, it is not our purpose to develop those tools here.

- *Let $B$ be a subset of $\mathbb{R}$ that is not bounded above in $\mathbb{R}$, and let $g : B \to \overline{\mathbb{R}}$ be a function. Let $q$ be a real number. Then*

$$\lim_{x \to +\infty} g(x) = q$$

*if and only if for every $\varepsilon > 0$, there exists $M \in \mathbb{R}$ such that $x > M$ and $x \in B$ together imply that $|g(x) - q| < \varepsilon$.*
- *Let $C$ be a subset of $\mathbb{R}$ that is not bounded above in $\mathbb{R}$; let $h : C \to \overline{\mathbb{R}}$ be a function. Then*

$$\lim_{x \to +\infty} h(x) = +\infty$$

*if and only if for every $N \in \mathbb{R}$, there exists $P \in \mathbb{R}$ such that $x > P$ and $x \in C$ together imply that $h(x) > N$.*

PROOF. We prove the first statement only, leaving the proofs of the second and third statements as exercises. Assume $\lim_{x \to p} f(x) = +\infty$, and pick $L \in \mathbb{R}$. Then $(L, +\infty]$ is a neighborhood of $+\infty$ in $\overline{\mathbb{R}}$. Therefore there exists a neighborhood $U$ of $p$ in $\mathbb{R}$ such that $x \in U \cap (A \backslash \{p\})$ implies $f(x) \in (L, +\infty]$. Choose $\delta > 0$ such that $B_{\mathbb{R}}(p, \delta) \subset U$. Then $0 < |x - p| < \delta$ and $x \in A$ imply together that $f(x) \in (L, +\infty]$, i.e. $f(x) > L$.

On the other hand, assume that for every $L \in \mathbb{R}$, there exists $\delta > 0$ such that $0 < |x - p| < \delta$ and $x \in A$ together imply that $f(x) > L$. Let $V$ be a neighborhood of $+\infty$ in $\overline{\mathbb{R}}$. Since $+\infty \in V$ and $V$ is open, $V$ contains a subset of the form $(L, +\infty]$. Choose $\delta > 0$ such that $0 < |x - p| < \delta$ and $x \in A$ together imply that $f(x) > L$. Then $B_{\mathbb{R}}(p, \delta)$ is a neighborhood of $p$ in $\mathbb{R}$ such that $x \in B_{\mathbb{R}}(p, \delta) \cap (A \backslash \{p\})$ implies $f(x) \in V$. Thus $\lim_{x \to p} f(x) = +\infty$. ☐

EXERCISE 1.9. Prove the second and third points in Proposition 1.8.

REMARK 1.10.

- The second and third parts of Proposition 1.8 are applicable to sequences, since $\mathbb{N}$ is in particular a subset of $\mathbb{R}$ which is not bounded above. In fact, if we wanted to, we could use this to *define* limits of sequences in topological spaces or in metric spaces from the very beginning. This gives a satisfying unity to the limit notation; though we have defined limits in many different ways, we can now see that they are all mean the same thing. However, we didn't take this approach from the beginning because its pedagogical value is a dubious, and this viewpoint is easier to appreciate a posteriori.
- It is common to identify a function (or sequence) with values in $\mathbb{R}$ with the function (or sequence) obtained by expanding the codomain to all of $\overline{\mathbb{R}}$. Proposition 1.8 can be applied to real-valued functions and sequences in this way.
- If $(a_n)$ is a sequence of real numbers and $a_n \to +\infty$ as $n \to \infty$, then $(a_n)$ converges in $\overline{\mathbb{R}}$, but it diverges in $\mathbb{R}$. The usual convention is that sentence '$(a_n)$ diverges' means that $(a_n)$ diverges in $\mathbb{R}$; if you wish to claim that $(a_n)$ converges, you *must* specify 'in $\overline{\mathbb{R}}$'. Similarly, if $(b_n)$ is a sequence of real numbers, then '$(b_n)$ converges' means that $(b_n)$ converges in $\mathbb{R}$.
- Without getting into specifics, it's true that convergence and limits in $\mathbb{R}$ are 'the same as' convergence and limits in $\overline{\mathbb{R}}$, if $+\infty$ and $-\infty$ are not involved. This is because a subset $E$ of $\mathbb{R}$ is open in $\mathbb{R}$ if and only if it is open in $\mathbb{R}$, so the requirements for convergence in $\mathbb{R}$ and $\overline{\mathbb{R}}$ are essentially the same if $+\infty$ and $-\infty$ are not involved.
- In general, there are a lot of weird unspoken conventions in mathematics regarding $+\infty$ and $-\infty$. Writing down an exhaustive list of these conventions would be difficult and probably not very beneficial. Just make sure to use care when reading and writing arguments involving these symbols. You may sometimes have to figure out from context what is actually meant.

1.2.2. *Metrizability of* $(\overline{\mathbb{R}}, \overline{\mathcal{T}})$. Define the function $f : [-\infty, \infty] \to [-1, 1]$ by

$$f(x) = \begin{cases} 1 & x = +\infty \\ \frac{x}{1+|x|} & x \in \mathbb{R} \\ -1 & x = -\infty. \end{cases}$$

Define the function $\overline{d} : \overline{\mathbb{R}} \times \overline{\mathbb{R}} \to \mathbb{R}$ by

$$\overline{d}(x, y) = |f(x) - f(y)|.$$

Then $\overline{d}$ is a metric on $\overline{\mathbb{R}}$. (We leave this as an easy exercise for the reader.) Let $\mathcal{B}_0$ denote the collection of all open balls in $\overline{\mathbb{R}}$ under this metric (i.e., $\mathcal{B}_0 = \{B_{(\overline{\mathbb{R}},\overline{d})}(x, r) : x \in \overline{\mathbb{R}}, r > 0\}$). Let $\mathcal{T}_0$ denote the metric topology on $\overline{\mathbb{R}}$ associated to the metric $\overline{d}$, that is, the collection of all sets which are open under the metric $\overline{d}$, that is, the collection of all possible unions of elements of $\mathcal{B}_0$. We claim that $\mathcal{T}_0 = \overline{\mathcal{T}}$. To see this, it suffices to show that $\mathcal{B}_0 = \mathcal{B} \cup \{\overline{\mathbb{R}}\}$, where $\mathcal{B}$ is as in Definition 1.5, since $\mathcal{T}_0$ is the collection of all unions of elements of $\mathcal{B}_0$, and $\overline{\mathcal{T}}$ is the collection of all unions of elements of $\mathcal{B}$, or equivalently $\mathcal{B} \cup \{\overline{\mathbb{R}}\}$. In what follows, we use $d$ to denote the Euclidean metric on $\mathbb{R}$.

$$\begin{aligned}
y \in (a, b) &\iff f(y) \in (f(a), f(b)) = B_{(\mathbb{R},d)}\left(\frac{f(a)+f(b)}{2}, \frac{f(b)-f(a)}{2}\right) \\
&\iff \left|f(y) - \frac{f(a)+f(b)}{2}\right| < \frac{f(b)-f(a)}{2} \\
&\iff y \in B_{(\mathbb{R},\overline{d})}\left(f^{-1}\left(\frac{f(a)+f(b)}{2}\right), \frac{f(b)-f(a)}{2}\right). \\
y \in (b, +\infty] &\iff f(y) \in (f(b), 1] \\
&\iff |f(+\infty) - f(y)| = 1 - f(y) < 1 - f(b) \\
&\iff y \in B_{(\mathbb{R},\overline{d})}(+\infty, 1 - f(b)) \\
y \in [-\infty, a) &\iff \cdots \iff y \in B_{(\overline{\mathbb{R}},\overline{d})}(-\infty, f(a) + 1).
\end{aligned}$$

Thus

$$(a, b) = B_{(\overline{\mathbb{R}},\overline{d})}\left(f^{-1}\left(\frac{f(a)+f(b)}{2}\right), \frac{f(b)-f(a)}{2}\right),$$

$$(b, +\infty] = B_{(\overline{\mathbb{R}},\overline{d})}(+\infty, 1 - f(b)), \qquad [-\infty, a) = B_{(\overline{\mathbb{R}},\overline{d})}(-\infty, f(a) + 1).$$

This shows that $\mathcal{B} \subset \mathcal{B}_0$; it remains to show that $\mathcal{B}_0 \subset \mathcal{B} \cup \{\overline{\mathbb{R}}\}$. To see this, pick $x \in \overline{\mathbb{R}}$ and $r > 0$ and split into cases.

- If $f(x) + r < 1$ and $f(x) - r > -1$, then
$$\begin{aligned}
y \in B_{(\overline{\mathbb{R}},\overline{d})}(x, r) &\iff |f(y) - f(x)| < r \iff f(x) - r < f(y) < f(x) + r \\
&\iff y \in (f^{-1}(f(x) - r), f^{-1}(f(x) + r)),
\end{aligned}$$
so $B_{(\overline{\mathbb{R}},\overline{d})}(x, r) = (f^{-1}(f(x) - r), f^{-1}(f(x) + r))$ in this case.
- If $f(x) + r \geq 1$ and $f(x) - r > -1$, then
$$\begin{aligned}
y \in B_{(\overline{\mathbb{R}},\overline{d})}(x, r) &\iff |f(y) - f(x)| < r \iff f(x) - r < f(y) \leq 1 \\
&\iff y \in (f^{-1}(f(x) - r), f^{-1}(1)] = (f^{-1}(f(x) - r), +\infty]
\end{aligned}$$
so $B_{(\overline{\mathbb{R}},\overline{d})}(x, r) = (f^{-1}(f(x) - r), +\infty]$ in this case.
- If $f(x) + r < 1$ and $f(x) - r \leq -1$, then $B_{(\overline{\mathbb{R}},\overline{d})}(x, r) = [-\infty, f^{-1}(f(x) + r))$.
- If $f(x) + r \geq 1$ and $f(x) - r \leq -1$, then $B_{(\overline{\mathbb{R}},\overline{d})}(x, r) = [-\infty, +\infty] = \overline{\mathbb{R}}$.

This completes the proof that $\mathcal{T}_0 = \overline{\mathcal{T}}$.

COROLLARY 1.11. *$(\overline{\mathbb{R}}, \bar{d})$ is a compact metric space.*

PROOF. The function $f : [-\infty, +\infty] \to [-1, 1]$ is a bijection; both $f$ and its inverse $g : [-1, 1] \to [-\infty, +\infty]$ are continuous. (We leave this verification as an exercise for the reader.) Since $[-1, 1]$ is compact and $\overline{\mathbb{R}} = g([-1, 1])$, it follows that $(\overline{\mathbb{R}}, \bar{d})$ is compact. □

This property of $\overline{\mathbb{R}}$ will come in handy when we discuss upper and lower limits.

## 2. One-Sided Limits and Types of Discontinuities

In this subsection, we consider real-valued functions defined on some open interval $(a, b)$ of $\mathbb{R}$. (The discussion can easily be modified to the case where $a$ or $b$ is infinite, but to keep things simple, we state definitions and theorems for $a$ and $b$ finite.) For such functions, there are essentially only two kinds of ways a function can be discontinuous at a point, depending on the existence and agreement of the limits from the left and right.

DEFINITION 2.1. Let $f : (a, b) \to \mathbb{R}$ be a function and assume $x \in [a, b)$. We write $f(x+) = q$ if whenever $(t_n)$ is a sequence in $(x, b)$ which converges to $x$, we have $f(t_n) \to q$. This is called the *limit from the right*. The limit from the left is defined similarly.

Of course, $f(x+)$ and $f(x-)$ may or may not exist. Furthermore, if $x \in (a, b)$, the statement $\lim_{t \to x} f(x) = q$ is equivalent to $f(x+) = f(x-) = q$. Therefore $f$ is continuous at $x \in (a, b)$ if and only if $f(x+) = f(x-) = f(x)$.

DEFINITION 2.2. Let $f : (a, b) \to \mathbb{R}$ be a function and assume $x \in (a, b)$. Assume that $f(x+)$ and $f(x-)$ both exist.

- If $f(x+) = f(x-)$ but $f$ is not continuous at $x$, then we say that $f$ has a *removable discontinuity* at $x$.
- If $f(x+) \neq f(x-)$, then we say that $f$ has a *jump* discontinuity at $x$.

In either case, we say that $f$ has a *discontinuity of the first kind* at $x$.

DEFINITION 2.3. Let $f : (a, b) \to \mathbb{R}$ be a function and assume $x \in (a, b)$. If one of $f(x+)$, $f(x-)$ does not exist, then $f$ is said to have an *essential discontinuity*, or a *discontinuity of the second kind*, at $x$.

Simple discontinuities are fairly easy to understand; essential discontinuities are less straightforward. Therefore we give two examples of functions with essential discontinuities.

EXAMPLE 2.4. Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

Then $f$ has a discontinuity of the second kind at every point (as neither $f(x+)$ nor $f(x-)$ exists for any $x$); $g$ is continuous at $x = 0$ but has a discontinuity of the second kind at every other point. We don't prove all the claims necessary to justify these statements, but here are two such statements that give the idea.

- $f(x+)$ does not exist: Let $(x_n)$ be a sequence of numbers which converges to $x$, such that $x_n > x$ for each $n$, and $x_n \in \mathbb{Q}$ when $n$ is even, $x_n \notin \mathbb{Q}$ if $n$ is odd. Then $f(x_n) = 1$ for $n$ even and $f(x_n) = 0$ for $x$ odd, so the sequence $(f(x_n))$ cannot converge to any value as $n \to \infty$.
- $g$ is continuous at 0: Choose $\varepsilon > 0$; put $\delta = \varepsilon$. Assume $y \in B(0, \delta)$. If $y$ is rational then $f(y) = 0 \in B(0, \delta)$; if $y$ is irrational then $|f(y)| = |y| < \delta = \varepsilon$, so $f(y) \in B(g(0), \varepsilon)$.

## 3. Monotonic Functions and Sequences

DEFINITION 3.1. Let $E$ be a subset of $\mathbb{R}$ and $f : E \to \overline{\mathbb{R}}$ a function. We say that $f$ is *monotonically increasing* on $E$ if $f(x) \leq f(y)$ whenever $x, y \in E$ and $x < y$. We say that $f$ is *strictly increasing* on $E$ if $f(x) < f(y)$ whenever $x, y \in E$ and $x < y$. Monotonically decreasing and strictly decreasing functions are defined similarly. A function is called *monotonic* if it is either monotonically increasing or monotonically decreasing.

In most cases, the set $E$ above is either an interval $(a, b)$ or $\mathbb{N}$; in the latter case, $f$ is a called a *monotonic sequence*.

The results below are stated for monotonically increasing functions and sequences; there are of course analogs for decreasing functions and sequences.

THEOREM 3.2. *Let $f : (a, b) \to \mathbb{R}$ be monotonically increasing on $(a, b)$ and assume $x \in (a, b)$. Then*

$$f(x-) = \sup\{f(t) : t \in (a, x)\}; \qquad f(x+) = \inf\{f(t) : t \in (x, b)\}.$$

*In particular, $f(x-)$ and $f(x+)$ both exist in $\mathbb{R}$, and $f(x-) \leq f(x) \leq f(x+)$.*

PROOF. Denote $A = \{f(t) : t \in (a, x)\}$ and $B = \{f(t) : t \in (x, b)\}$. Then $A$ and $B$ are nonempty, and $f(x)$ is an upper bound for $A$ and a lower bound for $B$. Define $\alpha = \sup A$ and $\beta = \inf B$; then clearly $\alpha \leq f(x) \leq \beta$. We show that $f(x-) = \alpha$; the proof that $f(x+) = \beta$ is similar.

Choose $\varepsilon > 0$; by definition of $\alpha$ there exists $y \in (a, x)$ such that $f(y) > \alpha - \varepsilon$. Put $\delta = x - y$; then $z \in (x - \delta, x)$ implies $z > x - \delta = y$, so $f(z) \geq f(y) > \alpha - \varepsilon$. Thus $f(x-) = \alpha$, as claimed. □

COROLLARY 3.3. *Let $f : (a, b) \to \mathbb{R}$ be monotonically increasing and assume $a < x < y < b$. Then $f(x+) \leq f(y-)$.*

PROOF. $f(x+) = \inf\{f(t) : t \in (x, b)\} = \inf\{f(t) : t \in (x, y)\} \leq \sup\{f(t) : t \in (x, y)\} = \sup\{f(t) : t \in (a, y)\} = f(y-)$. Here the outer two equalities follow from the previous theorem, while the inner two equalities hold by monotonicity. □

COROLLARY 3.4. *Any monotonic function $f : (a, b) \to \mathbb{R}$ has no discontinuities of the second kind.*

THEOREM 3.5. *Let $f : (a, b) \to \mathbb{R}$ is monotonically increasing and denote by $E$ the subset of $(a, b)$ consisting of those points at which $f$ is discontinuous. Then $E$ is at most countable.*

PROOF. If $x \in E$, then $f(x-) < f(x+)$. (Otherwise, $f(x-) = f(x) = f(x+)$, contradicting the definition of $E$.) Thus for each $x \in E$ we can choose $q_x \in \mathbb{Q}$ such that $f(x-) < q_x < f(x+)$ (by density of $\mathbb{Q}$ in $\mathbb{R}$). This defines a function $g : \mathbb{Q} \to E$, $g(x) = q_x$; to show that $E$ is countable it suffices to prove that $g$ is injective. Assume that $x < y$; then $q_x < f(x+) \leq f(y-) < q_y$. Thus $g$ is injective, as needed. □

THEOREM 3.6. *Let $(s_n)$ be a monotonically increasing sequence in $\overline{\mathbb{R}}$. Then $\lim_{n\to\infty} s_n$ exists in $\overline{\mathbb{R}}$ and is equal to $\sup_{n\in\mathbb{N}} s_n$. In particular, a monotonically increasing sequence of real numbers converges if and only if it is bounded.*

PROOF. There are three cases to consider.

- Case 1: $\sup\{s_n\} = -\infty$. Then $s_n = -\infty$ for all $n \in \mathbb{N}$, in which case $\lim_{n\to\infty} s_n = -\infty$.
- Case 2: $\sup\{s_n\} = s \in \mathbb{R}$. Then choose $\varepsilon > 0$, and let $N \in \mathbb{N}$ be such that $s_N > s - \varepsilon$. (This is possible; otherwise $s - \varepsilon$ would be an upper bound for $E$, contradicting the definition of $s$.) Then $s_n \geq s_N > s - \varepsilon$ for all $n \geq N$, so $s_n \in (s - \varepsilon, s]$ for such $n$. Thus $s_n \to s$ as $n \to \infty$.
- Case 3: $\sup\{s_n\} = +\infty$. Pick $M \in \mathbb{R}$ and choose $N \in \mathbb{N}$ such that $s_N > M$. Then $n \geq N$ implies $s_n > M$. Thus $s_n \to +\infty$ as $n \to +\infty$.

□

## 4. Upper and Lower Limits

Let $(s_n)_{n=1}^{\infty}$ be a sequence of real numbers. For each $n \in \mathbb{N}$, define $d_n := \sup\{s_k : k \in \mathbb{N}, k \geq n\}$. Then $(d_n)$ is a monotonically decreasing sequence in $\overline{\mathbb{R}}$, since the set $\{s_k : k \in \mathbb{N}, k \geq n\}$ 'shrinks' as $n$ increases. (We are considering the terms of the sequence $a_n$ and *after*.) In light of the discussion in the previous section, the following definition thus makes sense.

DEFINITION 4.1. Let $(s_n)_{n=1}^{\infty}$ be a sequence in $\overline{\mathbb{R}}$. Then the *upper* and *lower* limits of $(s_n)$ are defined by

$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty}\left(\sup_{k\geq n} s_k\right) = \inf_{n\in\mathbb{N}}\left(\sup_{k\geq n} s_k\right)$$

and

$$\liminf_{n\to\infty} s_n = \lim_{n\to\infty}\left(\inf_{k\geq n} s_k\right) = \sup_{n\in\mathbb{N}}\left(\inf_{k\geq n} s_k\right),$$

respectively.

In our discussion of upper and lower limits, we will make use of the statements in the following Exercise. These statements are hopefully very believable, but they do require a bit of justification.

EXERCISE 4.2. Let $(s_n)_{n=1}^{\infty}$ and $(t_n)_{n=1}^{\infty}$ be sequences in $\overline{\mathbb{R}}$. Prove the following statements.

(1) If $s_n \leq t_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = +\infty$, then $\lim_{n\to\infty} t_n = +\infty$ as well.
(2) If $(s_n)$ and $(t_n)$ converge in $\overline{\mathbb{R}}$ to $s$ and $t$, respectively, and if $s_n \leq t_n$ for each $n \in \mathbb{N}$, then $s \leq t$.

THEOREM 4.3. *Let $(s_n)_{n=1}^{\infty}$ be a sequence in $\overline{\mathbb{R}}$ Let $S^*$ denote the set of its subsequential limits in $\overline{\mathbb{R}}$, and denote $\alpha = \limsup_{n\to\infty} s_n$. Let $x$ be an element of $\overline{\mathbb{R}}$. Then*

- *$S^*$ is nonempty, and $\alpha = \sup S^* \in S^*$.*
- *$\alpha < x$ if and only if there exists $y < x$ such that $s_n \leq y$ for all but finitely many $n \in \mathbb{N}$. In particular, if $\alpha < x$, then $s_n < x$ for all but finitely many $n \in \mathbb{N}$. And if $s_n < x$ for all but finitely many $n \in \mathbb{N}$, then $\alpha \leq x$.*
- *$\alpha > x$ if and only if $s_n > x$ for infinitely many $n \in \mathbb{N}$.*

PROOF. To see that $S^*$ is nonempty, simply note that $(\overline{\mathbb{R}}, \overline{d})$ is a compact metric space; therefore it is sequentially compact, which by definition means that $S^*$ cannot be empty. Furthermore, $S^*$ is closed in $\overline{\mathbb{R}}$, by Theorem 2.6. To see that $\sup S^* \in S^*$, we consider three cases. Case 1: $\sup S^* = -\infty$. Then $S^*$ cannot contain any elements of $\overline{\mathbb{R}}$ except possibly $-\infty$; on the other hand, $S^*$ is nonempty. The only possibility is that $S^* = \{-\infty\}$, in which case $\sup S^* \in S^*$ obviously holds. Case 2: $\sup S^* \in \mathbb{R}$. Then $S^* \cap \mathbb{R}$ is a closed subset of $\mathbb{R}$ which is bounded above in $\mathbb{R}$, and consequently $\sup S^* = \sup(S^* \cap \mathbb{R}) \in \mathrm{Cl}_{\mathbb{R}}(S^* \cap \mathbb{R}) = S^* \cap \mathbb{R} \subset S^*$. Case 3: $\sup S^* = +\infty$. If $+\infty \notin S^*$, then there exists a neighborhood of $+\infty$ of the form $(M, +\infty]$ which does not intersect $S^*$; this implies that $S^*$ is bounded above, contradicting the assumption that $\sup S^* = +\infty$.

Next, we prove that $\alpha \geq \sup S^*$. Let $s$ be any element of $S^*$, and let $(s_{n_k})_{k=1}^{\infty}$ be a subsequence of $(s_n)$ that converges to $s$ in $\overline{\mathbb{R}}$. Denote $d_n = \sup_{k\geq n} s_n$. Then for each $k \in \mathbb{N}$, we have

$$s_{n_k} \leq d_{n_k} \leq d_k,$$

where the second inequality holds because $(d_n)_{n=1}^{\infty}$ is a monotonically decreasing sequence, and $n_k \geq k$. Taking limits and using Exercise 4.2, we conclude that

$$s = \lim_{k\to\infty} s_{n_k} \leq \lim_{k\to\infty} d_k = \alpha.$$

Thus $\alpha$ is an upper bound for $S^*$, so $\sup S^* \leq \alpha$.

To prove the opposite inequality, we argue by contradiction. Assume that $\sup S^* < \alpha$, and let $\beta \in \mathbb{R}$ be such that $\sup S^* < \beta < \alpha$. We claim that there exists $N \in \mathbb{N}$ such that $k \geq N$ implies $s_n \leq \beta$. Indeed, if no such $N$ exists, then there exists a subsequence $(s_{n_k})$ whose terms all exceed $\beta$. This subsequence will in turn have a further subsequence $(s_{n_{k_\ell}})$ which converges in $\overline{\mathbb{R}}$; its subsequential limit $s$ will be at least $\beta$. But then $s \in S^*$ and $s \geq \beta > \sup S^*$, which is impossible.

We prove the second statement next. The statement '$\limsup_{n\to\infty} s_n < x$' means that $x$ is not a lower bound for the set $\{\sup_{k\geq n} s_k\}_{n=1}^{\infty}$. This in turn means that there exists $N \in \mathbb{N}$ such that $\sup_{k\geq N} s_k < x$, i.e., $s_k \leq y := \sup_{j\geq N} s_j < x$ for all $k \geq N$. So, only finitely many of the $s_k$ (the first $N-1$ terms) may possibly exceed $y$. Since we have used if and only if statements at each step, this proves the second claim completely. Let us just note that for the last part of the second claim, the assumption that $s_n < x$ for all but finitely many $n \in \mathbb{N}$ now implies that $\alpha < z$ for all $z > x$, which is equivalent to saying that $\alpha \leq x$.

The third statement follows from the first one, since we can find a subsequence $(s_{n_k})$ that converges to $\alpha$. We finish by choosing $N \in \mathbb{N}$ such that $k \geq N$ implies $s_{n_k} > x$ (i.e., $s_{n_k}$ belongs to the neighborhood $(x, +\infty]$ of $\alpha$). Then $\{n_k\}_{k=N}^{\infty}$ is the infinite set claimed to exist in the Theorem statement. $\square$

THEOREM 4.4. *Let $(s_n)$ be a sequence in $\overline{\mathbb{R}}$. Then $\lim\limits_{n\to\infty} s_n = s$ if and only if $\limsup\limits_{n\to\infty} s_n = \liminf\limits_{n\to\infty} s_n = s$.*

PROOF. Let $S^*$ denote the set of subsequential limits of $(s_n)$ in $\overline{\mathbb{R}}$. let $(t_k)$ and $(u_k)$ be subsequences of $(s_n)$ that converge to $\limsup_{n\to\infty} s_n = \sup S^*$ and $\liminf_{n\to\infty} = \inf S^*$, respectively.

If $\lim_{n\to\infty} s_n = s$, then every subsequence of $s_n$ tends to $s$; in particular this is true of the subsequences $(t_k)$ and $(u_k)$. By uniqueness of limits, this implies that $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$.

For the opposite direction, we argue by contradiction. Assume that $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$ but that $\lim_{n\to\infty} s_n = s$ fails. Then $\sup S^* = \inf S^* = s$, so $S^* = \{s\}$. But there exists a neighborhood $U$ of $s$ and a subsequence $(s_{n_k})$ such that $\{s_{n_k}\}_{k=1}^{\infty}$ lies entirely outside $U$. Since $\overline{\mathbb{R}}$ is sequentially compact, $(s_{n_k})$ has a further subsequence $(s_{n_{k_\ell}})$ which converges to some point $p$ of $\overline{\mathbb{R}}$, which must be outside $U$ and therefore cannot be equal to $s$. But then $p \in S^* = \{s\}$, which is impossible. $\square$

EXERCISE 4.5. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided that the RHS isn't of the form $\infty - \infty$. (Here we use the usual conventions involving addition and subtraction of $\pm\infty$; for example, $+\infty + r = +\infty$ for any $r \in \mathbb{R}$. But of course $+\infty - \infty$ is not defined.)

## 5. Limits of Some Special Sequences

THEOREM 5.1. *The following limits hold:*

*(1) If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.*
*(2) If $q > 0$, then $\lim_{n\to\infty} \sqrt[n]{q} = 1$.*
*(3) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.*
*(4) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.*

In the proof, we will frequently make use of the binomial theorem from elementary algebra: If $a, b \in \mathbb{R}$, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. In particular, if $a = 1$ and $b \geq 0$, we have $(1+b)^n \geq \binom{n}{k} b^k$, for $k \in J_n$.

PROOF. (1) Choose $\varepsilon > 0$, and let $N$ be any integer larger than $\varepsilon^{-1/p}$. Then $n \geq N$ implies $n^{-p} \leq N^{-p} < (\varepsilon^{-\frac{1}{p}})^p = \varepsilon$.

(2) If $p = 1$, then $\sqrt[n]{p} = 1$ for all $n$. If $p > 1$, put $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$, and $1 + nx_n \leq (1 + x_n)^n = (\sqrt[n]{p})^n = p$. It follows that $x_n \in (0, \frac{p-1}{n}]$ for all $n \in \mathbb{N}$, which implies that $x_n \to 0$, and hence $\sqrt[n]{p} \to 1$.

If $p \in (0, 1)$, then $1/p > 1$, so

$$\sqrt[n]{p} = \frac{1}{\sqrt[n]{1/p}} \to \frac{1}{1} = 1.$$

(3) Put $x_n = \sqrt[n]{n} - 1$; then

$$n = (1 + x_n)^n \geq \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2.$$

Thus $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$ for $n \geq 2$. The sequence on the right tends to zero as $n \to \infty$, so $x_n \to 0$ as well. Thus $\sqrt[n]{n} = x_n + 1 \to 1$.

(4) Assume $|x| < 1$; since $|x^n| = |x|^n$, we may assume $x \in (0, 1)$ without loss of generality. In this case, $x = (1 + p)^{-1}$ for some $p > 0$, and $(1 + p)^n \geq np$, so

$$0 < x^n = (1 + p)^{-n} \leq \frac{1}{p} \cdot \frac{1}{n}.$$

By (1), the sequence on the right tends to zero as $n \to \infty$, so we are done. $\square$

CHAPTER 7

# Series

## 1. Basic Definitions and Examples

DEFINITION 1.1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. Denote

$$s_n = \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n.$$

The notation

$$\sum_{k=1}^{\infty} a_k \tag{4}$$

is used to denote the limit of the sequence $(s_n)$ as $n \to \infty$, if this limit exists. The symbol in (4) is called a *series*. The numbers $a_k$ are called the *terms* of the series, and the numbers $s_n$ are called the *partial sums* of the series. If the partial sums converge to a limit $s \in \mathbb{R}$, we say that the series *converges* and we write $s = \sum_{k=1}^{\infty} a_k$. If the partial sums diverge, we say that the series diverges.

REMARK 1.2. We'll stick to series of real numbers here. However, basically everything we'll discuss can be applied just as well to complex numbers. The situation there, however, is sometimes more complicated.

Note: Sometimes, for convenience, we start counting the terms of the series from $k = 0$ (or $k =$ some other number) rather than $k = 1$. It should be clear that this has no bearing on the convergence or divergence of the series, etc.

Since $\mathbb{R}$ is a complete metric space, a series converges if and only if its partial sums are Cauchy. Note that if $m > n$, we have (using the notation from above)

$$s_m - s_n = \sum_{k=n+1}^{m} a_k. \tag{5}$$

PROPOSITION 1.3. *Let $(a_n)$ be a sequence of real numbers, and let $(s_n)$ denote the sequence of partial sums for the series $\sum_{k=1}^{\infty} a_k$. The following are equivalent:*

- *$\sum_{k=1}^{\infty} a_k$ converges. (I.e., $(s_n)$ converges.)*
- *$(s_n)$ is a Cauchy sequence.*
- *For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m \geq n \geq N$ implies*

$$\left| \sum_{k=n}^{m} a_k \right| < \varepsilon.$$

There's really nothing to prove here. Equivalence of the first two points follows from completeness of $\mathbb{R}$; equivalence of the second and third points follows from the definition of a Cauchy sequence and Equation (4).

COROLLARY 1.4. *If* $\sum_{k=1}^{\infty} a_k$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

PROOF. Choose $\varepsilon > 0$; then pick $N$ large enough so that $|\sum_{k=n}^{m} a_k| < \varepsilon$ for $m \geq n \geq N$. Then $n \geq N$ implies $|a_n| = |\sum_{k=n}^{n} a_k| < \varepsilon$. This says that $\lim_{n\to\infty} a_n = 0$. $\square$

The converse of this Corollary is false!!! The series $\sum_{k=1}^{\infty} a_k$ can diverge even if $a_n \to 0$ as $n \to \infty$. This is the case for the *harmonic series* with terms $a_n = 1/n$, for example. (We'll prove this soon.)

THEOREM 1.5. *A series of nonnegative terms converges if and only if the partial sums form a bounded sequence.*

PROOF. If the terms $a_k$ are nonnegative, then $(s_n)$ is monotonically increasing sequence of real numbers. Consequently, $(s_n)$ converges if and only if it is bounded; this is the desired conclusion. $\square$

EXAMPLE 1.6 (Geometric Series). Let $a_n = x^n$ for $n \in \mathbb{N} \cup \{0\}$, where $x \geq 0$. Let $s_n$ denote the $n$th partial sums of the series $\sum_{k=0}^{\infty} x^n$, i.e.

$$s_n = \sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n.$$

If $|x| \geq 1$, then $(x^n)$ does not converge to $0$ as $n \to \infty$, so the series diverges in this case. However, we claim that if $-1 < x < 1$, then the series converges. To see this, we write down a more explicit formula for $s_n$. To this end, note that

$$xs_n = x + x^2 + \cdots + x^n + x^{n+1}.$$

So

$$(1 - x)s_n = 1 - x^{n+1};$$

Therefore a more explicit formula for $s_n$ is

$$s_n = \frac{1 - x^{n+1}}{1 - x},$$

provided that $x \neq 1$. (If $x = 1$, then $s_n = n + 1$ for all $n \in \mathbb{N} \cup \{0\}$.) Therefore $s_n \to \frac{1}{1-x}$ as $n \to \infty$ if $|x| < 1$.

We end this subsection with an intuitive and basic Theorem on addition of series and multiplication of series by real numbers.

THEOREM 1.7. *Suppose* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are two convergent series; pick* $c \in \mathbb{R}$. *Then* $\sum_{n=1}^{\infty}(a_n + b_n)$ *and* $\sum_{n=1}^{\infty} ca_n$ *converge, and*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n; \qquad \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n.$$

PROOF. We prove only the statement about sums. Consider the partial sums of each of the sequences under consideration. Clearly

$$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

for each $n \in \mathbb{N}$. We take limits on both sides; since the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, we have

$$\lim_{n\to\infty} \sum_{k=1}^{n}(a_k + b_k) = \lim_{n\to\infty} \left[\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k\right] = \lim_{n\to\infty} \sum_{k=1}^{n} a_k + \lim_{n\to\infty} \sum_{k=1}^{n} b_k = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

Thus $\sum_{k=1}^{\infty}(a_k + b_k)$ converges to the claimed limit. $\square$

## 2. Tests for Convergence and Divergence of Series

### 2.1. The Comparison Test and the Series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

THEOREM 2.1 (Comparison Test). *Let $(a_n)$ be a sequence of real numbers; let $(c_n)$ and $(d_n)$ be sequences of nonnegative real numbers. Assume that $\sum_{n=1}^{\infty} c_n$ converges and $\sum_{n=1}^{\infty} d_n$ diverges.*

*(1) If there exists $N \in \mathbb{N}$ such that $|a_n| \leq c_n$ for all $n \geq N$, then $\sum_{n=1}^{\infty} a_n$ converges.*
*(2) If there exists $N \in \mathbb{N}$ such that $a_n \geq d_n$ for all $n \geq N$, then $\sum_{n=1}^{\infty} a_n$ diverges.*

PROOF. (1) Choose $\varepsilon > 0$, and pick $N \in \mathbb{N}$ large enough so that $m \geq n \geq N$ implies $\sum_{k=n}^{m} c_k < \varepsilon$. Then $m \geq n \geq N$ also implies

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} c_k < \varepsilon.$$

Thus $\sum_{k=1}^{\infty} a_k$ converges.
(2) Assume $\sum_{n=1}^{\infty} a_n$ converges. Then $\sum_{n=1}^{\infty} d_n$ converges as well, by (1), a contradiction. □

EXAMPLE 2.2. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. To see this, let $s_n$ denote the partial sums for $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Then

$$\begin{aligned} s_n &= 1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \cdots + \frac{1}{n \cdot n} \\ &\leq 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1) \cdot n} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 1 + 1 - \frac{1}{n} < 2. \end{aligned}$$

So the sequence of partial sums is bounded, which shows that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

EXAMPLE 2.3 (The Harmonic Series). On the other hand, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges*, even though $\frac{1}{n} \to 0$ as $n \to \infty$. let $(s_n)$ denote the sequence of partial sums for the series $\sum_{n=1}^{\infty} \frac{1}{n}$. Then

$$\begin{aligned} s_{2^m} &= \sum_{n=1}^{2^m} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^m - 1} + \frac{1}{2^m} \\ &= 1 + \frac{1}{2} + \Bigg(\underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}}\Bigg) + \Bigg(\underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}_{\geq \frac{1}{2}}\Bigg) + \cdots + \Bigg(\underbrace{\frac{1}{2^{m-1}+1} + \cdots + \frac{1}{2^m}}_{\geq \frac{1}{2}}\Bigg) \\ &\geq 1 + \frac{m}{2}. \end{aligned}$$

So $s_{2^m} \geq 1 + \frac{m}{2}$, which tells us that $(s_n)$ is *not* a bounded sequence, and therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges*.

The two examples above can be generalized.

THEOREM 2.4. *The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.*

We can't prove this Theorem right away; we need another convergence test.

THEOREM 2.5. *(Cauchy Condensation Test) Let $(a_n)$ be a monotonically decreasing sequence of nonnegative terms. Then the two series*

$$\sum_{k=1}^{\infty} a_k \quad \textit{and} \quad \sum_{k=0}^{\infty} 2^k a_{2^k}$$

*either both converge or both diverge.*

The proof below may remind you of the proof we just saw that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. This isn't an accident.

PROOF. We let $s_n$ and $t_k$ denote the partial sums of the two series:

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n; \\ t_k &= a_1 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k}. \end{aligned}$$

We prove the Theorem by proving two claims:

(1) For $n < 2^k$, we have $s_n \leq t_k$. This shows that if $(t_k)$ is bounded, then so is $(s_n)$. Thus if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, then so does $\sum_{k=1}^{\infty} a_k$.
(2) For $n \geq 2^k$, we have $t_k \leq 2s_n$. This shows that if $(s_n)$ is bounded, then so is $(t_k)$. Thus if $\sum_{k=1}^{\infty} a_k$ converges, the so does $\sum_{k=0}^{\infty} 2^k a_{2^k}$.

Once we've proven these two claims, the Theorem follows immediately. So, let's assume $n < 2^k$. Then

$$\begin{aligned} s_n &\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) && \text{(Adding terms can only increase the sum)} \\ &\leq a_1 + 2a_2 + \cdots + 2^k \cdot a_{2^k} && (a_n \text{ decreases with } n) \\ &= t_k. \end{aligned}$$

On the other hand, if $n \geq 2^k$, then

$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) && \text{(Left out some terms)} \\ &\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} && (a_n \text{ decreases with } n) \\ &= \frac{1}{2} t_k. \end{aligned}$$

This completes the proof. □

Finally, we can prove Theorem 2.4.

THEOREM 2.4. We are considering the series $\sum_{k=1}^{\infty} \frac{1}{n^p}$. If $p \leq 0$, then the sequence $(1/n^p)_{n=1}^{\infty}$ does not converge to zero, so the series diverges. If $p > 0$, then we apply the Theorem we just proved as follows. If $a_n = \frac{1}{n^p}$, then $2^k a_{2^k} = 2^k \cdot \frac{1}{(2^k)^p} = [2^{(1-p)}]^k$. Therefore the series $\sum_{k=1}^{\infty} \frac{1}{n^p}$ converges if and only if the following series does:

$$\sum_{k=0}^{\infty} [2^{(1-p)}]^k.$$

This is a geometric series (with $x = 2^{1-p}$), which converges if and only if $2^{1-p} < 1$; this occurs if and only if $1 - p < 0$, i.e. $p > 1$. □

**2.2. The Root, Ratio, and Alternating Series Tests: Statements and Examples.** In this subsection we discuss the Ratio and Root Test and the Alternating Series Test. Proofs are postponed until the end of the section.

THEOREM 2.6 (The Root Test). *Let $(a_n)$ be a sequence of real numbers.*

*(1) If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum_{k=1}^{\infty} a_k$ converges.*
*(2) If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.*

Important! The Root Test makes no claims about the situation where $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1$. In this case the series can converge or diverge. Indeed,

$$\limsup_{n\to\infty} \sqrt[n]{\frac{1}{n}} = 1; \quad \limsup_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = 1.$$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, while $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

THEOREM 2.7 (The Ratio Test). *Let $(a_n)$ be a sequence of (nonzero) real numbers.*

*(1) If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.*
*(2) If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.*

Important! The Ratio Test makes no claim about the situation where $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$. Indeed, as $n \to \infty$ we have

$$\frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \to 1, \qquad \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \frac{n^2}{(n+1)^2} \to 1,$$

but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, while $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

REMARK 2.8. It should be clear that the stipulation that the $a_n$'s are nonzero should not be restrictive at all. Indeed, if all but finitely many of the terms are zero, then the sum is actually finite and convergence is a non-issue (the sequence of partial sums has a tail of the form $s, s, s, s, s, \ldots$). Otherwise, one can form a subsequence of the $a_n$'s by removing all the zeros; the series formed from this subsequence will have the same behavior as the series formed by the original sequence.

EXAMPLE 2.9. Consider the following series:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

Then the sequence $\left(\left|\frac{a_{n+1}}{a_n}\right|\right)_{n=1}^{\infty}$ is

$$\left(\frac{2}{3}, \frac{3}{2^2}, \frac{2^2}{3^2}, \frac{3^2}{2^3}, \frac{2^3}{3^3}, \frac{3^3}{2^4}, \frac{2^4}{3^4}, \ldots\right).$$

The subsequence $\left(\frac{2^k}{3^k}\right)_{k=1}^{\infty}$ converges to zero, while the subsequence $\left(\frac{3^k}{2^{k+1}}\right)_{k=1}^{\infty}$ tends to $+\infty$. There can obviously be no subsequential limit of $\left(\left|\frac{a_{n+1}}{a_n}\right|\right)_{n=1}^{\infty}$ which is smaller than $0$ or larger than $+\infty$, so we conclude that

$$\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0; \qquad \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = +\infty.$$

Thus the Ratio Test gives no information here. To see if the Root Test applies, let's write down the sequence $(\sqrt[n]{|a_n|})_{n=1}^{\infty}$:

$$\left(\frac{1}{2}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt[3]{2^2}}, \frac{1}{\sqrt[4]{3^2}} = \frac{1}{\sqrt{3}}, \frac{1}{\sqrt[5]{2^3}}, \frac{1}{\sqrt[6]{3^3}} = \frac{1}{\sqrt{3}}, \cdots\right)$$

Clearly $\frac{1}{\sqrt{3}}$ is a subsequential limit, since it appears as every other term. The other terms are of the form $1/\sqrt[2k-1]{2^k}$. We claim that the subsequence consisting of these terms converges to $1/\sqrt{2}$, and therefore that $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \max\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\} = \frac{1}{\sqrt{2}} < 1$, so that the series converges. It suffices to prove that $\lim_{n\to\infty} \sqrt[2n-1]{2^n} = \sqrt{2}$. To see this, note that

$$\sqrt[2n-1]{2^n} = 2^{\frac{n}{2n-1}} = \sqrt{2} \cdot \sqrt[4n-2]{2},$$

and $(\sqrt[4n-2]{2})_{n=1}^\infty$ is a subsequence of $(\sqrt[n]{2})_{n=1}^\infty$, which converges to $1$. Therefore we have

$$\lim_{n\to\infty} \sqrt[2n-1]{2^n} = \sqrt{2} \cdot \lim_{n\to\infty} \sqrt[4n-2]{2} = \sqrt{2}.$$

This proves the claim we wanted.

REMARK 2.10. In the above example, the Root Test indicates convergence of the series, while the Ratio Test is inconclusive. One might wonder whether there are instances where the Ratio Test indicates convergence, but the Root Test does not. The following Theorem says that this is actually never the case. However, the Ratio Test is often easier to apply than the root test, so it is still useful, even though it is relevant less often.

REMARK 2.11. When using the Root Test, one might worry whether they are applying the 'right' root to the 'right' term. The convergence or divergence of a series is independent of the first $k$ terms of the series, for any $k \in \mathbb{N}$; one might like to know that the conclusion of the Root Test is also independent of the first $k$ terms. The following Proposition implies that this is the case.

PROPOSITION 2.12. *Let $(a_n)_{n=1}^\infty$ be a sequence of numbers. Then for any $j \in \mathbb{N}$, we have*

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n+j]{|a_n|}, \qquad \liminf_{n\to\infty} \sqrt[n]{|a_n|} = \liminf_{n\to\infty} \sqrt[n+j]{|a_n|}.$$

Besides showing that the Root Test is independent of the choice of indexing, the Proposition above also gives us a convenient way to evaluate certain limits. For instance, in Example 2.9, we needed to show that the limit of the sequence $(\sqrt[2n-1]{2^n})$ is $\sqrt{2}$. The above Proposition says that we can replace the sequence $(\sqrt[2n-1]{2^n})$ with the sequence whose terms are $\sqrt[2n]{2^n} = \sqrt{2}$ for all $n$. The latter sequence is simply $(\sqrt{2}, \sqrt{2}, \sqrt{2}, \dots)$ and obviously converges to $\sqrt{2}$; the computation we did in Example 2.9 was actually unnecessary.

THEOREM 2.13. *If $(c_n)$ is a sequence of (nonzero) real numbers, then*

$$\liminf_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| \le \liminf_{n\to\infty} \sqrt[n]{|c_n|} \le \limsup_{n\to\infty} \sqrt[n]{|c_n|} \le \limsup_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|.$$

*Therefore, if the Ratio Test indicates convergence or divergence of $\sum_{n=1}^\infty c_n$, so does the Root Test; if the Root test is inconclusive, then so is the Ratio Test.*

THEOREM 2.14 (Alternating Series Test). *Let $(a_n)$ be a monotonically decreasing sequence of real numbers such that $a_n \to 0$ as $n \to \infty$. Then $\sum_{n=1}^\infty (-1)^n a_n$ converges.*

EXAMPLE 2.15. The series $\sum_{n=1}^\infty \frac{(-1)^n}{n}$ converges, by the Alternating Series Test. This might be surprising in light of the fact that $\sum_{n=1}^\infty \frac{1}{n}$ diverges. We'll come back to this example later and see that the whole picture is actually much weirder even than this.

EXERCISE 2.16. For each of the following sequences $(a_n)_{n=1}^\infty$, prove whether the series $\sum_{n=1}^\infty a_n$ converges or diverges. (If it converges, you do not need to find the limit.)

(1) $a_n = \sqrt{n+1} - \sqrt{n}$.
(2) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$.
(3) $a_n = (\sqrt[n]{n} - 1)^n$.

(4) $a_n = \frac{(-1)^n}{\log n}$ for $n \geq 2$ (and $a_1 = 0$).

EXERCISE 2.17. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{1+z^n}.$$

Determine which values of $z \in \mathbb{R}$ ($z \neq -1$) make the series convergent and which make it divergent. Prove your answers are correct.

### 2.3. The Root, Ratio, and Alternating Series Tests: Proofs of the Theorems.

PROOF OF ROOT TEST. Assume $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \alpha < 1$. Choose $\beta \in (\alpha, 1)$, then pick $N$ such that $n \geq N$ implies $\sqrt[n]{|a_n|} < \beta$. Then $n \geq N$ also implies that $|a_n| \leq \beta^n$. But $\sum_{n=1}^{\infty} \beta^n$ converges, since $\beta \in (0,1)$. Thus $\sum_{n=1}^{\infty} a_n$ also converges, by the Comparison Test.

Next, assume $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \alpha > 1$. Then $|a_n| > 1$ for infinitely many $n$, so $(a_n)$ cannot converge to zero, which shows that $\sum_{n=1}^{\infty} a_n$ diverges. □

We do not prove the Ratio Test, since it follows from the Root Test and Theorem 2.13.

PROOF OF THEOREM 2.13. We prove that $\limsup_{n\to\infty} \sqrt[n]{|c_n|} \leq \limsup_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$. The proof of the other half is similar.

Put $\alpha = \limsup_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$; assume without loss of generality that $\alpha \neq +\infty$. We show that for any $\beta > \alpha$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $\sqrt[n]{|c_n|} < \beta$.

Choose $\beta > \alpha$, then choose $N \in \mathbb{N}$ such that $\left|\frac{c_{n+1}}{c_n}\right| < \beta$ for all $n \geq N$. Then for all $k \in \mathbb{N}$, we have

$$|c_{N+k}| < \beta|c_{N+k-1}| < \beta^2|c_{N+k-2}| < \cdots < \beta^k|c_N| = (\beta^{-N}|c_N|)\beta^{N+k}.$$

Thus for $n \geq N$, we have $|c_n| < |c_N|\beta^{-N} \cdot \beta^n$, i.e,

$$\sqrt[n]{|c_n|} < \beta \sqrt[n]{|c_N|\beta^{-N}}.$$

Taking the $\limsup$ of both sides, we get

$$\limsup_{n\to\infty} \sqrt[n]{|c_n|} \leq \limsup_{n\to\infty} \beta \sqrt[n]{|c_N|\beta^{-N}} = \beta \lim_{n\to\infty} \sqrt[n]{|c_N|\beta^{-N}} = \beta.$$

Since $\beta > \alpha$ was arbitrary, we conclude that $\limsup_{n\to\infty} \sqrt[n]{|c_n|} \leq \alpha = \limsup_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$, as needed. □

PROOF OF PROPOSITION 2.12. It suffices to prove the Proposition for the special case $j = 1$; the full statement then follows by induction. We prove the statement for the $\limsup$ only.

Denote $s_n = \sqrt[n]{|a_n|}$ and $t_n = \sqrt[n+1]{|a_n|}$ (so $t_n = s_n^{\frac{n}{n+1}}$, for all $n \in \mathbb{N}$). Denote $\alpha = \limsup_{n\to\infty} s_n$ and $\beta = \limsup_{n\to\infty} t_n$. We show that $\alpha \leq \beta$; the proof that $\beta \leq \alpha$ is entirely similar. Furthermore, the fact that $\beta \geq 0$ is obvious, so we may assume without loss of generality that $\alpha > 0$.

Let $(s_{n_k})_{k=1}^{\infty}$ be a subsequence of $(s_n)$, such that $s_{n_k} \to \alpha$ in $\overline{\mathbb{R}}$ as $k \to \infty$. We will prove that $t_{n_k} \to \alpha$ as well; this will show that $\alpha$ is a subsequential limit of the $t_n$'s, and therefore that $\alpha \leq \beta$, since $\beta$ is the supremum of all such subsequential limits. We consider two cases.

Case 1: $\alpha = +\infty$. Choose $M > 1$, and choose $N \in \mathbb{N}$ $k$ large enough so that $s_{n_k} > M^2$ for all $k \geq N$. Then $k \geq N$ also implies that

$$t_{n_k} = s_{n_k}^{\frac{n_k}{n_k+1}} > M^{\frac{2n_k}{n_k+1}} \geq M.$$

Thus $t_{n_k} \to +\infty$ as $k \to \infty$.

Case 2: $0 < \alpha < +\infty$. Note that for large enough $k$, we have $\frac{\alpha}{2} < s_{n_k} < \frac{3\alpha}{2}$. Taking $n_k + 1$ roots and then $\limsup$s and $\liminf$s, we get

$$1 = \lim_{k\to\infty} \sqrt[n_k+1]{\frac{\alpha}{2}} \le \liminf_{k\to\infty} \sqrt[n_k+1]{s_{n_k}}, \qquad \limsup_{k\to\infty} \sqrt[n_k+1]{s_{n_k}} \le \lim_{k\to\infty} \sqrt[n_k+1]{\frac{3\alpha}{2}} = 1.$$

Thus

$$\lim_{k\to\infty} \sqrt[n_k+1]{s_{n_k}} = 1.$$

So

$$\lim_{n\to\infty} t_{n_k} = \lim_{k\to\infty} \frac{s_{n_k}}{\sqrt[n_k+1]{s_{n_k}}} = \frac{\alpha}{1} = \alpha.$$

□

PROOF OF ALTERNATING SERIES TEST. Let $(s_n)$ denote the sequence of partial sums of the series $\sum_{n=1}^{\infty}(-1)^n a_n$; that is, let $s_n = \sum_{k=1}^{n}(-1)^k a_k$. Then the odd partial sums form a monotonically increasing sequence $s_1, s_3, s_5, \ldots$, while the even partial sums form a monotonically decreasing sequence $s_2, s_4, s_6, \ldots$. (To see this, note that $s_{2n+2} = s_{2n} - a_{2n+1} + a_{2n+2} \le a_{2n}$, since $a_{2n+2} \le a_{2n+1}$ by assumption. This shows that the even partial sums are monotonically decreasing; the proof that the odd partial sums are monotonically increasing is similar.)

Let $n$ and $m$ be positive integers; we claim that $s_{2n-1} \le s_{2m}$. To prove, this, we argue by contradiction; assume $s_{2n-1} > s_{2m}$. Then

$$\begin{aligned} s_{2m+1} &= s_{2m} - a_{2m+1} \le s_{2m} < s_{2n-1} \\ s_{2n} &= s_{2n-1} + a_{2n} \ge s_{2n-1} > s_{2m}. \end{aligned}$$

But these statements cannot both be true. For if $n \le m$, then $2n - 1 < 2m + 1$, so $s_{2n-1} \le s_{2m+1}$, contradicting the first inequality. (Remember that the odd partial sums form an increasing sequence.) Similarly, if $m < n$, then $2m < 2n$, so $s_{2m} \ge s_{2n}$, contradicting the second inequality. This proves the claim.

It follows that $(s_{2n})$ is monotonically decreasing and bounded below (by $s_{2m-1}$, for any $m \in \mathbb{N}$). Therefore it converges to its greatest lower bound $\beta$, which is greater than $s_{2m-1}$, for any $m \in \mathbb{N}$. On the other hand, the sequence $(s_{2n-1})$ is monotonically increasing and bounded above by $\beta$, so it converges to its least upper bound $\alpha$, which is no larger than $\beta$. We claim that $\alpha = \beta$. We know that $\alpha \le \beta$; to show the opposite inequality, we show that $\beta - \alpha < \varepsilon$ for any $\varepsilon > 0$. This will prove that $\beta = \alpha$ since $\varepsilon > 0$ is arbitrary.

Choose $\varepsilon > 0$; then choose $N \in \mathbb{N}$ large enough so that $a_n < \varepsilon$ for $n \ge N$. Then $s_{2N-1} \le \alpha \le \beta \le s_{2N}$, so $\beta - \alpha \le s_{2N} - s_{2N-1} = a_{2N} < \varepsilon$. This completes the proof and shows that $\sum_{n=1}^{\infty}(-1)^n a_n = \alpha = \beta$. □

## 3. Absolute Convergence and Rearrangements

DEFINITION 3.1. Let $(a_n)$ be a sequence of real numbers, and let $\phi : \mathbb{N} \to \mathbb{N}$ be a bijection. Then $(a_{\phi(n)})$ is another sequence with the same terms as $(a_n)$, but in a different order. The series $\sum_{n=1}^{\infty} a_{\phi(n)}$ is called a *rearrangement* of the series $\sum_{n=1}^{\infty} a_n$.

The partial sums of the two series in the definition above can consist of completely different numbers, and we can't in general expect the sequences of partial sums to converge to the same limit, if they converge at all. It turns out that the concept of absolute convergence is exactly what is needed to guarantee that all rearrangements converge to the same number.

DEFINITION 3.2. The series $\sum_{n=1}^{\infty} a_n$ is said to *converge absolutely* if $\sum_{n=1}^{\infty} |a_n|$ converges.

THEOREM 3.3. *If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it converges.*

PROOF. Note that $m \geq n$ implies

$$\left|\sum_{k=n}^{m} a_n\right| \leq \sum_{k=n}^{m} |a_k|.$$

The statement then follows from the Cauchy criterion. $\square$

However, there are certainly convergent series that do not converge absolutely. One example is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots .$$

This sequence converges, by the Alternating Series Test, but it does not converge absolutely, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, as we have already seen.

THEOREM 3.4. *Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers which converges, but not absolutely. Given $\alpha, \beta \in \overline{\mathbb{R}}$ with $\alpha \leq \beta$, there exists a rearrangement of $\sum_{n=1}^{\infty} a_n$, with partial sums $t_n$, such that $\liminf_{n\to\infty} t_n = \alpha$, $\limsup_{n\to\infty} t_n = \beta$.*

PROOF. Given the sequence $(a_n)$, let $(P_k)$ denote the subsequence of $(a_n)$ consisting of all the nonnegative terms of $(a_n)$, in the original order; let $(Q_k)$ denote the absolute values of the analogous sequence of negative terms. (So, if $(a_n)$ is the sequence $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots)$, then $(P_n) = (1, \frac{1}{3}, \frac{1}{5}, \ldots)$, while $(Q_n) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots)$.)

We construct a series of the following form, whose partial sums $t_n$ satisfy the desired properties:

$$P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} + \cdots .$$

Clearly this series is a rearrangement of $\sum_{n=1}^{\infty} a_n$.

Step 1: We claim that $\sum_{k=1}^{\infty} P_k$ and $\sum_{k=1}^{\infty} Q_k$ are both divergent series. To see this, we introduce two auxiliary sequences $(p_k)$ and $(q_k)$, which differ from $(P_k)$ and $(Q_k)$ only by zero terms. Therefore proving divergence of $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ also proves divergence of $\sum_{n=1}^{\infty} P_n$ and $\sum_{n=1}^{\infty} Q_n$.

Define

$$p_n = \begin{cases} |a_n| = a_n & \text{if } a_n \geq 0 \\ 0 & \text{otherwise;} \end{cases} \qquad q_n = \begin{cases} |a_n| = -a_n & \text{if } a_n \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(For example, if the sequence $(a_n)$ is $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{6}, \ldots)$, then $(p_n) = (1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \ldots)$, and $(q_n) = (0, \frac{1}{2}, 0, \frac{1}{4}, 0, \frac{1}{6}, \ldots)$.)

We show that $\sum_{n=1}^{\infty} q_n$ diverges, arguing by contradiction. Assume then that $\sum_{n=1}^{\infty} q_n$ converges. Then

$$\sum_{n=1}^{\infty} (a_n + q_n) = \sum_{n=1}^{\infty} p_n$$

also converges. But then

$$\sum_{n=1}^{\infty} (p_n + q_n) = \sum_{n=1}^{\infty} |a_n|$$

also converges, contradicting our assumption of non-absolute convergence of $\sum_{n=1}^{\infty} a_n$. We conclude that $\sum_{n=1}^{\infty} q_n$ cannot converge. The proof that $\sum_{n=1}^{\infty} p_n$ diverges is similar. This finishes the proof of Step 1.

Step 2: Choose real-valued sequences $(\alpha_n)$ and $(\beta_n)$ such that $\alpha_n \to \alpha$ and $\beta_n \to \beta$ in $\overline{\mathbb{R}}$, and such that $\alpha_n \leq \beta_n$ for all $n$. (If $\alpha$ and $\beta$ are real numbers (i.e. not $\pm\infty$), then choosing sequences is unnecessary—just put $\alpha_n = \alpha$, $\beta_n = \beta$ for all $n \in \mathbb{N}$ in this case.)

Let $m_1$ be the smallest positive integer such that $y_1 := P_1 + \cdots + P_{m_1} > \beta_1$. Such an integer exists because the series $\sum_{n=1}^{\infty} P_n$ is divergent. Next, let $k_1$ be the smallest positive integer such that

$x_1 := P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} < \alpha_1$. This is possible because $\sum_{n=1}^{\infty} Q_n$ diverges. We can continue this process to obtain integers $m_2, k_2, m_3, k_3, \ldots$, such that

$$y_2 := P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} > \beta_2,$$

$$x_2 := P_1 + \cdots + P_{m_1} - Q_1 - \cdots - Q_{k_1} + P_{m_1+1} + \cdots + P_{m_2} - Q_{k_1+1} - \cdots - Q_{k_2} < \alpha_2,$$

etc. Note that $|y_1 - \beta_1| = y_1 - \beta_1 \leq P_{m_1}$; otherwise we would have $\beta_1 > y_1 - P_{m_1} = P_1 + \cdots + P_{m_1-1}$, contradicting the definition of $m_1$ as the *smallest* positive integer such that $P_1 + \cdots + P_{m_1} > \beta_1$. Similarly, we have $|y_n - \beta_n| < P_{m_n}$ and $|x_n - \alpha_n| < Q_{k_n}$ for all $n \in \mathbb{N}$. Since $\{P_n\}$ and $\{Q_n\}$ must both converge to zero, it follows that $x_n \to \alpha$ and $y_n \to \beta$. Thus $\alpha$ and $\beta$ are subsequential limits of the sequence of partial sums $\{t_n\}$ associated to the constructed rearrangement. (Note that for each $n$, we have $y_n = t_{m_n+k_{n-1}}$, $x_n = t_{m_n+k_n}$, with the convention that $k_0 = 0$.) This shows that $\liminf_{n\to\infty} t_n \leq \alpha$, $\limsup_{n\to\infty} t_n \geq \beta$.

Step 3: We show that $\limsup_{n\to\infty} t_n = \beta$. (The fact that $\liminf_{n\to\infty} t_n = \alpha$ follows similarly.) We write

$$\begin{aligned}
\limsup_{j\to\infty} t_j &= \lim_{j\to\infty}\left(\sup_{\ell\geq j} t_\ell\right) = \lim_{j\to\infty}\left(\sup_{\ell\geq m_j+k_j} t_\ell\right)\\
&\leq \lim_{j\to\infty}\left(\sup_{\ell\geq j} t_{m_{\ell+1}+k_\ell}\right) = \limsup_{j\to\infty} t_{m_{j+1}+k_j}\\
&\lim_{j\to\infty} y_j = \beta.
\end{aligned}$$

The second equality follows by passing to a subsequence. (Denoting $d_n = \sup_{k\geq n} t_k$, the subsequence is $(d_{m_j+k_j})_{j=1}^{\infty}$. Since $(d_n)$ tends to some limit (which may be finite or infinite), every subsequence tends to the same limit.) To justify the inequality, we note that $m_j + k_j \leq \ell \leq m_{j+1} + k_{j+1}$ implies $t_\ell \leq t_{m_{j+1}+k_j}$. Indeed, if $m_j + k_j \leq \ell \leq m_{j+1} + k_j$, then

$$t_\ell = t_{m_j+k_j} + P_{m_j+1} + \cdots + P_{\ell-k_j} \leq t_{m_j+k_j} + P_{m_j+1} + \cdots + P_{m_{j+1}} = t_{m_{j+1}+k_j}.$$

On the other hand, if $m_{j+1} + k_j \leq \ell \leq m_{j+1} + k_{j+1}$, then

$$t_\ell = t_{m_{j+1}+k_j} - Q_{k_j+1} - \cdots - Q_{\ell-m_{j+1}} \leq t_{m_{j+1}+k_j}.$$

□

THEOREM 3.5. *If $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series of real numbers, then all rearrangements of $\sum_{n=1}^{\infty} a_n$ converge, and converge to the same real number.*

PROOF. Let $\phi : \mathbb{N} \to \mathbb{N}$ be a bijection; let $s_n$ and $t_n$ denote the $n$th partial sums of the series $\sum_{k=1}^{\infty} a_n$ and $\sum_{k=1}^{\infty} a_{\phi(k)}$, respectively. Since $\sum_{n=1}^{\infty} a_n$ converges absolutely, it converges; let $s$ denote the limit of the partial sums $s_n$. We need to show that $t_n \to s$ as $n \to \infty$. Since $|t_n - s| \leq |t_n - s_n| + |s_n - s|$, it suffices to show that $\lim_{n\to\infty} |t_n - s_n| = 0$.

Choose $\varepsilon > 0$, then choose $N \in \mathbb{N}$ large enough so that $m \geq n \geq N$ implies $\sum_{k=n}^{m} |a_k| < \varepsilon/2$. Then choose $M \in \mathbb{N}$ such that $\phi(\{1, \ldots, M\}) \supset \{1, \ldots, N\}$. (Note that $M \geq N$.) Pick $n \geq M$ and denote

$P = \max\{\phi(1), \dots, \phi(n)\}$. Then

$$\begin{aligned} |t_n - s_n| &= \left| \sum_{\substack{k\in\{1,\dots,n\} \\ \phi(k)>N}} a_{\phi(k)} + \sum_{\substack{k\in\{1,\dots,n\} \\ \phi(k)\le N}} a_{\phi(k)} - \sum_{k=1}^{N} a_k - \sum_{k=N+1}^{n} a_k \right| \\ &= \left| \sum_{\substack{k\in\{1,\dots,n\} \\ \phi(k)>N}} a_{\phi(k)} - \sum_{k=N+1}^{n} a_k \right| \le \sum_{\substack{k\in\{1,\dots,n\} \\ \phi(k)>N}} |a_{\phi(k)}| + \sum_{k=N+1}^{n} |a_k| \\ &\le 2 \sum_{k=N+1}^{P} |a_k| \le 2 \cdot \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that $\lim_{n\to\infty} |t_n - s_n| = 0$ and thus finishes the proof. ☐

EXERCISE 3.6. Assume that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that $\sum_{n=1}^{\infty} \frac{\sqrt{|a_n|}}{n}$ converges. (Hint: Use the inequality $2AB \le A^2 + B^2$, valid for any real numbers $A$, $B$.)

EXERCISE 3.7.

(1) Assume that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely. Prove that $\sum_{n=1}^{\infty}(a_n + b_n)$ converges absolutely as well.
(2) Assume that $\sum_{n=1}^{\infty} a_n$ converges. Does it follow that $\sum_{n=1}^{\infty} a_{2n}$ converges? Give a proof or counterexample.
(3) Assume that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Does it follow that $\sum_{n=1}^{\infty} a_{2n}$ converges absolutely? Give a proof or counterexample.

## 4. Series of Functions

Sometimes it is useful to consider series of functions. Much of the terminology and some of the Theorems from our discussion of sequences in metric spaces and series of numbers carry over to the present context.

DEFINITION 4.1. Let $A$ be a subset of $\mathbb{R}$, and for each $n \in \mathbb{N}$, let $f_n : A \to \mathbb{R}$ be a function. Then we can define the series $\sum_{n=0}^{\infty} f_n(x)$ for each $x \in A$. If the series converges for each $x \in B$, where $B$ is some subset of $A$, then we can define the limit function $f : B \to \mathbb{R}$ by $f(x) = \sum_{n=0}^{\infty} f_n(x)$, and we say that the series $\sum_{n=0}^{\infty} f_n$ *converges pointwise* to $f$ on $B$. The series $\sum_{n=0}^{\infty} f_n$ *converges uniformly* on a subset $C$ of $B$ if the sequence of functions $(s_n)_{n=0}^{\infty} = (\sum_{k=0}^{n} f_n)_{n=0}^{\infty}$ converges uniformly to $f$ on $C$.

### 4.1. The Weierstrass $M$-Test.

THEOREM 4.2 (Weierstrass $M$-Test). *Suppose $(f_n)$ is a sequence of functions defined on $E \subset \mathbb{R}$, and $|f_n(x)| \le M_n$ for all $x \in E$, for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on $E$.*

PROOF. We show that the sequence $(s_n)$ of partial sums is uniformly Cauchy. Choose $\varepsilon > 0$, then choose $N \in \mathbb{N}$ such that $m \ge n \ge N$ implies that $\sum_{k=n}^{m} M_n < \varepsilon$. Then $m \ge n \ge N$ also implies

$$|s_m(x) - s_n(x)| = \left| \sum_{k=n}^{m} f_k(x) \right| \le \sum_{k=n}^{m} |f_k(x)| \le \sum_{k=n}^{m} M_k < \varepsilon.$$

Thus $(s_n)$ is uniformly Cauchy, hence converges uniformly. That is, the series $\sum_{n=1}^{\infty} f_n$ converges uniformly. ☐

EXERCISE 4.3. Let $B = \{0\} \cup \{\frac{-1}{n^2}\}_{n\in\mathbb{N}}$ and $E = \mathbb{R}\backslash B$. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$$

on the set $E$.

(1) Prove that the series converges absolutely for all $x \in E$; therefore it converges pointwise to a function $f : E \to \mathbb{R}$.
(2) Prove that the series converges uniformly to $f$ on $(-\infty, -\delta] \cup [\delta, \infty)\backslash B$ for any $\delta > 0$, but that it does not converge uniformly to $f$ on $E$.
(3) Prove that $f$ is continuous on $E$.
(4) Prove that $f(0+) = +\infty$, and that therefore $f$ is not a bounded function.

**4.2. Power Series.** Some of the most commonly used series of functions are the *power series*. We will discuss only a few properties of these series.

DEFINITION 4.4. Given a sequence $(c_n)_{n=0}^{\infty}$ of real numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a *power series* in $z$. The numbers $c_n$ are called the *coefficients* of the series, and $z$ is some real number.

The question we concern ourselves with now is the following: For which $z$ (other than $z = 0$) does the series $\sum_{n=0}^{\infty} c_n z^n$ converge? We can already give a partial answer to this question using the root Test. Let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}$. Then for any $z \in \mathbb{R}\backslash\{0\}$, we have

$$\limsup_{n\to\infty} \sqrt[n]{|c_n z^n|} = |z| \cdot \limsup_{n\to\infty} \sqrt[n]{|c_n|} = \alpha|z|.$$

So the Root Test tells us that the series $\sum_{n=1}^{\infty} c_n z^n$ converges if $\alpha|z| < 1$ and diverges if $\alpha z > 1$. If $\alpha \neq +\infty$, this guarantees convergence in an interval of the form $(-R, R)$, where $R = \frac{1}{\alpha}$ if $\alpha \in (0, +\infty)$ and $R = +\infty$ if $\alpha = 0$. (If $\alpha = +\infty$, then $\sum_{n=1}^{\infty} c_n z^n$ diverges for all $z \neq 0$.) The interval $(-R, R)$ is called the *interval of convergence* of the power series, and $R$ is called the *radius of convergence*. If $\limsup_{n\to\infty} \sqrt[n]{|c_n|} = +\infty$, we say that the radius of convergence is $0$.

We have proved the following Theorem.

THEOREM 4.5. *Let $R$ be the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n z^n$. Then the series converges whenever $|z| < R$ and diverges whenever $|z| > R$.*

A couple of notes:

- If $|z| = R$, the Theorem makes no claim about convergence. (After all, this is the inconclusive setting for the Root Test.) The series may diverge or converge in this case.
- If the $c_n$ and $z$ are allowed to take arbitrary complex values instead of being restricted to real values, the definition of radius of convergence, and the Theorem above, both hold unchanged. So, the interval of convergence becomes a disc of convergence.
- In light of Theorem 2.13, one may sometimes use the ratio test instead of the root test in order to compute the radius of convergence of a power series. That is, if $\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$ exists and is equal to $\gamma$, then $\limsup_{n\to\infty} \sqrt[n]{|c_n|}$ is also equal to $\gamma$, so the radius of convergence of the series $\sum_{n=0}^{\infty} c_n z^n$ is $\gamma^{-1}$ (with the usual conventions regarding $0$ and $\infty$).

EXERCISE 4.6. Find the radius of convergence for each of the following power series:

$$\sum_{n=0}^{\infty} n^n z^n \qquad \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \sum_{n=0}^{\infty} z^n \qquad \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

EXERCISE 4.7. Consider the power series $\sum_{n=0}^{\infty} c_n z^n$. Let $R$ be the radius of convergence of the power series, and assume $R > 0$. Let $f : (-R, R) \to \mathbb{R}$ be the function defined by $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Prove the following statements, which refine Theorem 4.5.

(1) For any $r \in (0, R)$, the series $\sum_{n=0}^{\infty} c_n z^n$ converges uniformly on $(-r, r)$ to $f$.
(2) $f$ is continuous on all of $(-R, R)$.

CHAPTER 8

# Differentiation and Riemann Integration

## 1. Differentiation

### 1.1. Definitions and Basic Properties.

DEFINITION 1.1. Let $f$ be a real-valued function defined on $[a, b]$. Pick $x \in [a, b]$; if the limit

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

exists, we say that $f$ is *differentiable* at $x$, and we denote the limit (called the *derivative* of $f$ at $x$) by $f'(x)$. If $f$ is differentiable at every point of $E \subset [a, b]$, then we say that $f$ is differentiable on $E$. If $f$ is differentiable at every point of its domain, we say simply that $f$ is differentiable.

THEOREM 1.2 (Differentiability Implies Continuity). *If $f : [a, b] \to \mathbb{R}$ is differentiable at $x \in [a, b]$, then it is continuous at $x$.*

PROOF. It is enough to show that $f(t) \to f(x)$ as $t \to x$, or equivalently $f(t) - f(x) \to 0$ as $t \to x$. To see that this is true, write

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x) \to f'(x) \cdot 0, \qquad \text{as } t \to x.$$

$\square$

THEOREM 1.3. *Assume $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are differentiable at $x \in [a, b]$. Then $f + g$ and $fg$ are differentiable at $x$, as is $f/g$ if $g(x) \neq 0$; in each case the derivatives are given by*

$$(f + g)'(x) = f'(x) + g'(x), \qquad (fg)'(x) = f'(x)g(x) + f(x)g'(x),$$
$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

PROOF. We sketch the proof of the product rule only. Write

$$(fg)(t) - (fg)(x) = (f(t) - f(x))g(t) + f(x)(g(t) - g(x)).$$

Dividing by $t - x$ and taking limits, we get

$$\frac{(fg)(t) - (fg)(x)}{t - x} = \frac{f(t) - f(x)}{t - x} g(t) + f(x) \frac{g(t) - g(x)}{t - x} \to f'(x)g(x) + f(x)g'(x), \qquad \text{as } t \to x.$$

$\square$

THEOREM 1.4 (Chain Rule). *Assume $f : [a, b] \to \mathbb{R}$ is differentiable at $x \in [a, b]$; assume $f([a, b]) \subset [c, d]$, and assume $g : [c, d] \to \mathbb{R}$ is differentiable at $f(x)$. Then $g \circ f$ is differentiable at $x$, and $(g \circ f)'(x) = g'(f(x))f'(x)$.*

A bit of motivation before we give the proof: When $f(y) \neq f(x)$, we can write

$$\frac{g(f(y)) - g(f(x))}{y - x} = \frac{g(f(y)) - g(f(x))}{f(y) - f(x)} \cdot \frac{f(y) - f(x)}{y - x};$$

thus when $f(y) \neq f(x)$, we have

$$\begin{aligned}(6)\quad &\frac{g(f(y)) - g(f(x))}{y - x} - g'(f(x))f'(x) \\ &\qquad = \left[\frac{g(f(y)) - g(f(x))}{f(y) - f(x)} - g'(f(x))\right] \cdot \frac{f(y) - f(x)}{y - x} + g'(f(x))\left[\frac{f(y) - f(x)}{y - x} - f'(x)\right]\end{aligned}$$

On the other hand, if $f(y) = f(x)$ and $y \neq x$, the LHS is clearly zero.

The functions $v$ and $u$ in the proof below are defined with the bracketed terms in mind, with some extra information so that the case $f(y) = f(x)$ can be treated.

PROOF. Define $u : [a, b] \to \mathbb{R}$ and $v : [c, d] \to \mathbb{R}$ by

$$u(y) = \begin{cases} \dfrac{f(y) - f(x)}{y - x} - f'(x) & y \neq x; \\ 0 & y = x. \end{cases} \qquad v(z) = \begin{cases} \dfrac{g(z) - g(f(x))}{z - f(x)} - g'(f(x)) & z \neq f(x); \\ 0 & z = f(x). \end{cases}$$

Then $u$ is continuous at $x$ and $v$ is continuous at $f(x)$. Further, if $y \neq x$, then

$$\frac{g(f(y)) - g(f(x))}{y - x} - g'(f(x))f'(x) = v(f(y))\frac{f(y) - f(x)}{y - x} + g'(f(x))u(y).$$

If $f(y) \neq f(x)$, this follows from the computation immediately before the proof; if $f(y) = f(x)$, then both sides are equal to $-g'(f(x))f'(x)$. (Indeed, note that $u(y) = -f'(x)$ if $f(y) = f(x)$.) Taking limits, we get

$$\begin{aligned}\lim_{y \to x} \frac{g(f(y)) - g(f(x))}{y - x} - g'(f(x))f'(x) &= \lim_{y \to x}\left[v(f(y))\frac{f(y) - f(x)}{y - x} + g'(f(x))u(y)\right] \\ &= v(f(x))f'(x) + g'(f(x))u(x) \\ &= 0 \cdot f'(x) + g'(f(x)) \cdot 0 = 0.\end{aligned}$$

□

EXERCISE 1.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a functions such that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that $f$ is constant.

### 1.2. The Mean Value Theorem.

DEFINITION 1.6. Let $(X, d)$ be a metric space. We say that $f : X \to \mathbb{R}$ has a *local maximum* in $X$ at $x \in X$ if there exists $\delta > 0$ such that $f(y) \leq f(x)$ for all $y \in B_X(x, \delta)$. *Local minimum* is defined analogously, and a *local extremum* is either a local maximum or a local minimum.

Note: A global extremum is also a local extremum, according to this definition. The words 'maximum', 'minimum', 'extremum' should be understood as 'global maximum', etc. if 'local' is not specified.

THEOREM 1.7. *If $f : [a, b] \to \mathbb{R}$ is differentiable at $x \in (a, b)$ and $f$ has a local extremum at $x$, then $f'(x) = 0$.*

PROOF. We consider only the case where $x$ is a local maximum; the other case is similar. Choose $\delta > 0$ such that $a < x - \delta < x + \delta < b$, and $f(y) \leq f(x)$ for $y \in (x - \delta, x + \delta)$. Let $(s_n)$ and $(t_n)$ be sequences in $(x - \delta, x)$ and $(x, x + \delta)$, respectively, and assume both sequences converge to $x$. Then

$$f'(x) = \lim_{n \to \infty} \frac{f(s_n) - f(x)}{s_n - x} \geq 0; \qquad f'(x) = \lim_{n \to \infty} \frac{f(t_n) - f(x)}{t_n - x} \leq 0.$$

Thus $f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 0$. □

LEMMA 1.8. *If $f : [a,b] \to \mathbb{R}$ is continuous and $f(a) = f(b)$, then $f$ has a global extremum at some point $p$ of $(a,b)$.*

PROOF. By the Extreme Value Theorem, there exist $x, y \in [a,b]$ such that $f(x) = M$, $f(y) = m$, where $M$ and $m$ denote the maximum and minimum values of $f$ on $[a,b]$. If $x$ and $y$ are not both endpoints, choose $p$ to be the one that is not an endpoint. If both are endpoints, then $M = m$, which implies that $f$ is constant. In this case any point $p \in (a,b)$ is a maximum for $f$. □

THEOREM 1.9 (Mean Value Theorem). *Assume $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on $(a,b)$. Then there exists $x^* \in (a,b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(x^*).$$

Graphically, we interpret this as follows: $\frac{f(b)-f(a)}{b-a}$ is the slope of the secant line through $(a, f(a))$ and $(b, f(b))$ on the graph of $f$. The conclusion is that there is some point $x^*$ between $a$ and $b$ such that the line tangent to the graph of $f$ at $x^*$ is parallel to the secant line through $(a, f(a))$ and $(b, f(b))$.

PROOF. Define $h : [a,b] \to \mathbb{R}$ by $h(x) = [f(b) - f(a)]x - [b-a]f(x)$. Then $h$ is continuous on $[a,b]$ and differentiable on $(a,b)$; furthermore, $h(a) = f(b)a - f(a)b = h(b)$. By the Lemma, $h$ achieves either a local max or local min at some point $x^*$ of $(a,b)$, and by the preceding Theorem, we have $h'(x^*) = 0$, which is equivalent to the statement we wanted to prove. □

COROLLARY 1.10. *Let $f : (a,b) \to \mathbb{R}$ be differentiable.*

- *If $f'(x) \geq 0$ on $(a,b)$, then $f$ is monotonically increasing.*
- *If $f'(x) \leq 0$ on $(a,b)$, then $f$ is monotonically decreasing.*
- *If $f'(x) = 0$ on $(a,b)$, then $f$ is constant.*

PROOF. We prove the first statement; the other two are similar. Assume $a < x_1 < x_2 < b$. By the MVT, there exists $x^* \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = (x_2 - x_1)f'(x^*)$. Thus $f(x_2) \geq f(x_1)$; since $x_1$ and $x_2$ were arbitrary points satisfying $x_1 < x_2$, we conclude that $f$ is monotonically increasing. □

EXERCISE 1.11. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and assume $\lim_{x\to+\infty} x|f'(x)| = 0$. Define a sequence $(a_n)$ in $\mathbb{R}$ by $a_n = f(2n) - f(n)$ for each $n \in \mathbb{N}$. Prove that $a_n \to 0$ as $n \to \infty$.

EXERCISE 1.12. Let $f : (a,b) \to \mathbb{R}$ be a differentiable function with $f'(x) > 0$ for all $x \in (a,b)$.

(1) Prove that $f$ is injective.
(2) By part (1), there exists a function $g : f((a,b)) \to (a,b)$ such that $g(f(x)) = x$ for all $x \in (a,b)$. Prove that $g$ is continuous. (Hint: Use Theorem 2.24 in Chapter 5, and use the proof of Proposition 2.25 of Chapter 5 as a model for your answer.)
(3) Prove that $g$ is differentiable, and that $g'(f(x)) = \frac{1}{f'(x)}$, for all $x \in (a,b)$. (Hint: Pick $z \in f(a,b)$, and let $(z_n)_{n=1}^{\infty}$ be a sequence in $f(a,b)$ that converges to $z$. Write the difference quotient $\frac{g(z_n)-g(z)}{z_n-z}$ in terms of $f$ and a sequence $(x_n)$ in $(a,b)$.)

### 1.3. Derivatives of Higher Order; Taylor's Theorem with Remainder.

DEFINITION 1.13. Let $f : [a,b] \to \mathbb{R}$ be a function which is differentiable at $x \in [a,b]$. If $f'$ is also differentiable at $x$, we denote its derivative at $x$ by $f''(x)$. We can continue this process, obtaining functions $f, f', f'', f^{(3)}, f^{(4)}, \ldots$, provided the relevant limits exist. $f^{(n)}$ is called the *nth derivative* of $f$, or the *derivative of order* $n$. If $f^{(n)}(x)$ exists, we say that $f$ is $n$ *times differentiable* at $x$.

Note that in order for $f^{(n+2)}(x)$ to exist, $f^{(n+1)}$ must be differentiable at $x$, so $f^{(n)}$ must be defined on some neighborhood of $x$. Note also that $f^{(0)} = f$ by convention.

LEMMA 1.14. *Let* $p(t) = c_n(t-a)^n + \cdots c_1(t-a) + c_0$ *be a polynomial of degree* $n \geq 0$ *in* $t - a$. *Then* $p^{(k)}(a) = k!c_k$ *for each* $k \in \{0, \ldots, n\}$*, and* $p^{(n+1)}(t) = 0$ *for all* $t \in \mathbb{R}$.

The proof is by induction and is omitted here.

PROPOSITION 1.15. *Let* $f : [c, d] \to \mathbb{R}$ *be* $n$ *times differentiable at* $a \in [c, d]$. *Define* $P_n : \mathbb{R} \to \mathbb{R}$ *by*

$$P_n(t) = f(a) + f'(a)(t-a) + \frac{f''(a)}{2}(t-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(t-a)^n = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(t-a)^k.$$

*Then* $P_n^{(k)}(a) = f^{(k)}(a)$ *for all* $k \in \{0, 1, \ldots, n\}$.

PROOF. In the notation of the previous Lemma, we have $c_k = \frac{f^{(k)}(a)}{k!}$, so $P_n^{(k)}(t) = k!c_k = f^{(k)}(a)$, as claimed. $\square$

THEOREM 1.16 (Taylor's Theorem with Remainder). *Let* $f : [a, b] \to \mathbb{R}$ *be* $n$ *times differentiable on* $(a, b)$*, and assume that* $f^{(n-1)}$ *is continuous on* $[a, b]$. *Choose* $\alpha, \beta \in [a, b]$ *(with* $\alpha < \beta$*); then there exists* $x^* \in (\alpha, \beta)$ *such that*

$$f(\beta) = P_{n-1}(\beta) + \frac{f^{(n)}(x^*)}{n!}(\beta - \alpha)^n,$$

*where*

$$P_{n-1}(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

This Theorem says that near $\alpha$, $f$ can be approximated by $P_{n-1}$; it also gives a sense of how big the error should be. If $\beta$ is close to $\alpha$ and $n$ is large, then $P_{n-1}$ can be seen to be a pretty good approximation if we have a bound on $f^{(n)}$ near $\alpha$. Note that if $n = 1$, this is just the MVT.

PROOF. Step 1: Define $g : [a, b] \to \mathbb{R}$ by

$$g(t) = f(t) - P_{n-1}(t) - M(t-\alpha)^n.$$

We choose $M$ so that $g(\beta) = 0$; then it suffices to show that $f^{(n)}(x^*) = n!M$ for some $x^* \in (\alpha, \beta)$. In fact, taking $n$th derivatives, we see (by the Lemma above) that

$$g^{(n)}(t) = f^{(n)}(t) - n!M.$$

So it actually suffices to show that there exists $x^* \in (\alpha, \beta)$ such that $g^{(n)}(x^*) = 0$.

Step 2: We have $g^{(k)}(\alpha) = 0$ for $k = 0, \ldots, n-1$. To see this, we first note that $f$ and $P_{n-1}$ have the same $k$th derivatives at $\alpha$ for $k = 0, \ldots, n-1$; next, the $k$th derivative of the function $M(t-\alpha)^n$, evaluated at $t = \alpha$, is zero except for when $k = n$.

Step 3: Now, since $g(\alpha) = g(\beta) = 0$, there exists $x_1 \in (\alpha, \beta)$ such that $g'(x_1) = 0$. Similarly, since $g'(\alpha) = g'(x_1) = 0$, there exists $x_2 \in (\alpha, x_1)$ such that $g''(x_2) = 0$. Continue this process to get numbers $x_1, \ldots, x_n$ such that $x_1 > x_2 > \cdots > x_n > \alpha$ and $g^{(k)}(x_k) = 0$. In particular, $g^{(n)}(x^*) = 0$ for $x^* = x_n$. This finishes the proof. $\square$

Example: Let's use this to approximate $\sin(\frac{1}{3})$ without a calculator. Writing $f(t) = \sin t$ and $P_n(t) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} t^k$, we see that

$$f\left(\frac{1}{3}\right) - P_n\left(\frac{1}{3}\right) = \frac{f^{(n+1)}(x^*)}{(n+1)!}\left(\frac{1}{3}\right)^{n+1},$$

for some $x^* \in (0, \frac{1}{3})$.

How many terms do we need in order for $P_n(\frac{1}{3})$ to approximate $f(\frac{1}{3})$ with an accuracy of within $0.001$? This question is equivalent to asking how large we must choose $n$ be in order to be certain that

$$\frac{|f^{(n+1)}(x^*)|}{3^{n+1}(n+1)!} < 0.001.$$

Well, $|f^{(n+1)}(x^*)|$ is either $|\sin(x^*)|$ or $|\cos(x^*)|$ for some $x^* \in (0, \frac{1}{3})$; this is at most $1$ in either case. So our question becomes, how big must $n$ be in order to guarantee that $3^{n+1}(n+1)! > 1000$? We get the answer simply by making a list: $3^1 \cdot 1! = 3 \cdot 1 = 3$, $3^2 \cdot 2! = 9 \cdot 2 = 18$, $3^3 \cdot 3! = 27 \cdot 6 = 162$, $3^4 \cdot 4! = 81 \cdot 24 > 1000$. So we should pick $n+1 = 4$, i.e. $n = 3$. Thus to approximate $\sin \frac{1}{3}$ to within $0.001$, we compute $P_3(\frac{1}{3})$. We make a chart to help ourselves:

| $k$ | $f^{(k)}(x)$ | $\frac{f^{(k)}(0)}{k!} \cdot \frac{1}{3^k}$ |
|---|---|---|
| 0 | $\sin x$ | $0$ |
| 1 | $\cos x$ | $1 \cdot \frac{1}{3} = \frac{1}{3}$ |
| 2 | $-\sin x$ | $0$ |
| 3 | $-\cos x$ | $-\frac{1}{6} \cdot \frac{1}{27} = -\frac{1}{162}$ |

So

$$P_3\left(\frac{1}{3}\right) = 0 + \frac{1}{3} + 0 - \frac{1}{162} = \frac{53}{162}$$

Checking our work with a calculator, we see that

$$\frac{53}{162} \approx 0.32716... \qquad \sin(1/3) \approx 0.32719...$$

Now, approximating functions by their Taylor series is something that you should in principle have done back in Calculus class. The new thing here is the statement about the remainder, which gives us a way to guarantee that we have gone far enough in the series to get an approximation of a certain accuracy, and this is all possible without using a calculator.

EXERCISE 1.17. Use Taylor's Theorem with remainder to estimate $e^{1/2}$ to an accuracy of within $10^{-3}$. Prove that your answer has the desired accuracy. You can of course use a calculator to assist you with the problem, but your solution should not contain any steps that are impossible without a calculator.

## 2. Riemann Integration

### 2.1. Basic Definitions and Properties.

DEFINITION 2.1. A *partition* of an interval $[a, b]$ is a finite set of points $x_0, x_1, \ldots, x_n$, such that $a = x_0 \le x_1 \le \cdots \le x_{n-1} \le x_n = b$. Given a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$, the usual notation for the distance between successive points in the partition is

$$\Delta x_i = x_i - x_{i-1} \quad (i = 1, \ldots, n).$$

DEFINITION 2.2. Assume $f : [a, b] \to \mathbb{R}$ is bounded, and let $P$ be a partition of $[a, b]$. Define the *upper* and *lower Riemann sums* by the formulas

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i, \qquad L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i,$$

where

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x), \qquad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x).$$

Let $\mathcal{P}$ denote the collection of *all* partitions of $[a, b]$. The *upper* and *lower Riemann integrals* of $f$ over $[a, b]$ are defined by

$$\overline{\int_a^b} f \, \mathrm{d}x = \inf_{P \in \mathcal{P}} U(P, f), \qquad \underline{\int_a^b} f \, \mathrm{d}x = \sup_{P \in \mathcal{P}} L(P, f).$$

If the upper and lower integrals are equal, we say $f$ is *Riemann integrable*, and we denote the common value of the upper and lower limits by

$$\int_a^b f \, \mathrm{d}x,$$

the *integral* of $f$ over $[a, b]$.

Notation: We write $f \in \mathcal{R}([a, b])$ for the statement "$f$ is Riemann integrable on $[a, b]$". If the interval $[a, b]$ is clear from context, we sometimes just write $f \in \mathcal{R}$.

REMARK 2.3. The upper and lower integrals always exist. Indeed, we have assumed that $f$ is bounded on $[a, b]$, so there exists $m, M \in \mathbb{R}$ such that $m \leq f(x) \leq M$ for all $x \in [a, b]$. Therefore, regardless of the partition $P$ in question, we have

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i \leq M \sum_{i=1}^{n} \Delta x_i = M(b - a),$$

and similarly $L(P, f) \geq m(b - a)$. So the set $\{U(P, f) : P \in \mathcal{P}\}$ is bounded above (by $M(b - a)$), so its supremum $\overline{\int_a^b} f \, \mathrm{d}x$ is a real number; similarly, $\{L(P, f) : P \in \mathcal{P}\}$ is bounded below, so its infimum is a real number.

DEFINITION 2.4. Given two partitions $P$ and $P^*$, we say that $P^*$ is a *refinement* of $P$ if $P \subset P^*$. If $P_1$ and $P_2$ are partitions, then $P^* = P_1 \cup P_2$ is called their *common refinement*.

THEOREM 2.5. *If $P^*$ is a refinement of $P$, then*

$$L(P, f) \leq L(P^*, f) \leq U(P^*, f) \leq U(P, f).$$

PROOF. We show that $U(P^*, f) \leq U(P, f)$; the proof that $L(P, f) \leq L(P^*, f)$ is similar, and the middle inequality follows from the definition of $U$ and $L$. It suffices to consider the case when $P^*$ contains only one more point than $P$. (If $P^*$ contains $k$ more points than $P$, then this process can be repeated $k$ times.)

Suppose then that $P = \{x_0, \ldots, x_n\}$, with $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$, and $P^* = P \cup \{y\}$ and $y \notin P$. Pick $i$ such that $x_{i-1} < y < x_i$. Put

$$M_{i,1} = \sup_{x \in [x_{i-1}, y]} f(x),$$
$$M_{i,2} = \sup_{x \in [y, x_i]} f(x).$$

Clearly $M_{i,1}$ and $M_{i,2}$ are each at most $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$. (Draw a picture if this is not clear. Note that one or both might actually be equal to $M_i$.)

Thus

$$\begin{aligned} U(P^*, f) &= \sum_{k=1}^{i-1} M_k \Delta x_k + M_{i,1}(y - x_{i-1}) + M_{i,2}(x_i - y) + \sum_{k=i+1}^{n} M_k \Delta x_k \\ &\leq \sum_{k=1}^{i-1} M_k \Delta x_k + M_i(x_i - x_{i-1}) + \sum_{k=i+1}^{n} M_k \Delta x_k \\ &= U(P, f). \end{aligned}$$

$\square$

COROLLARY 2.6. *Let $f$ be a bounded function on $[a, b]$.*

*(1) If $P_1$ and $P_2$ are any two partitions, then*

$$L(P_1, f) \le U(P_2, f). \tag{7}$$

*(2) We have*

$$\underline{\int_a^b} f \, \mathrm{d}x \le \overline{\int_a^b} f \, \mathrm{d}x.$$

PROOF. (1) Let $P^*$ denote the common refinement of $P_1$ and $P_2$. Then

$$L(P_1, f) \le L(P^*, f) \le U(P^*, f) \le U(P_2, f),$$

by the previous Theorem. This proves (1).

(2) Since $P_1$ was arbitrary, we can take the supremum over all $P_1 \in \mathcal{P}$ in (7), to get

$$\underline{\int_a^b} f(x) \, \mathrm{d}x \le U(P_2, f).$$

Then take the infimum in $P_2$ to finish the proof. □

THEOREM 2.7. *$f \in \mathcal{R}([a, b])$ if and only if for every $\varepsilon > 0$ there exists $P \in \mathcal{P}$ such that*

$$U(P, f) - L(P, f) < \varepsilon. \tag{8}$$

PROOF. ($\implies$) Assume $f \in \mathcal{R}([a, b])$ and choose $\varepsilon > 0$. Pick $P_1 \in \mathcal{P}$ such that $\int_a^b f \, \mathrm{d}x - L(P_1, f) < \varepsilon/2$. (The fact that we can pick $L(P_1, f)$ close to $\underline{\int_a^b} f \, \mathrm{d}x$ follows from the definition of $\underline{\int_a^b} f \, \mathrm{d}x$; since $f \in \mathcal{R}([a, b])$ we have $\underline{\int_a^b} f \, \mathrm{d}x = \int_a^b f \, \mathrm{d}x$.) Similarly, pick $P_2 \in \mathcal{P}$ such that $U(P_2, f) - \int_a^b f \, \mathrm{d}x < \varepsilon/2$. Let $P^*$ be the common refinement of $P_1$ and $P_2$. Then

$$\begin{aligned} U(P^*, f) - L(P^*, f) &\le U(P_2, f) - L(P_1, f) \\ &\le U(P_2, f) - \int_a^b f \, \mathrm{d}x + \int_a^b f \, \mathrm{d}x - L(P_1, f) \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

($\impliedby$) Assume that for every $\varepsilon > 0$ there exists a partition $P$ such that (8) holds. Choose $\varepsilon > 0$ and let $P$ be a partition satisfying $U(P, f) - L(P, f) < \varepsilon$. Then

$$0 \le \overline{\int_a^b} f \, \mathrm{d}x - \underline{\int_a^b} f \, \mathrm{d}x \le U(P, f) - L(P, f) < \varepsilon.$$

Since $0 \le \overline{\int_a^b} f \, \mathrm{d}x - \underline{\int_a^b} f \, \mathrm{d}x < \varepsilon$ for any $\varepsilon > 0$, we conclude that $\overline{\int_a^b} f \, \mathrm{d}x = \underline{\int_a^b} f \, \mathrm{d}x$, that is, $f \in \mathcal{R}([a, b])$. □

THEOREM 2.8. *Let $f$ be a bounded function on $[a, b]$.*

*(1) If $f$ is continuous on $[a, b]$, then $f \in \mathcal{R}([a, b])$.*

*(2) If $f$ is monotonic on $[a, b]$, then $f \in \mathcal{R}([a, b])$.*

PROOF. Choose $\varepsilon > 0$. For each of these statements, our goal will be to find a partition $P$ such that $U(P, f) - L(P, f) < \varepsilon$.

(1) Assume $f$ is continuous on $[a, b]$. Since $[a, b]$ is compact, $f$ is actually uniformly continuous. Choose $\delta > 0$ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < \eta$. (We'll choose $\eta$ in a minute.) Then choose a partition $P$ such that $\Delta x_i < \delta$ for each $i$. Then $M_i - m_i < \eta$ for each $i$. (Indeed, by

the Extreme Value Theorem, there exist $x, y \in [x_{i-1}, x_i]$ such that $f(x) = M_i$, $f(y) = m_i$. Then $|x - y| \le x_i - x_{i-1} = \Delta x_i < \delta$, so $M_i - m_i = |f(x) - f(y)| < \eta$, by choice of $\delta$.) Thus

$$U(P, f) - L(P, f) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i < \eta \sum_{i=1}^{n} \Delta x_i = \eta[b - a].$$

Choose $\eta = \frac{\varepsilon}{b-a}$ to finish the argument.

(2) Assume $f$ is monotonically increasing on $[a, b]$. Let $P_n$ denote the partition of $[a, b]$ with $n + 1$ evenly spaced points, so $\Delta x_i = (b - a)/n$ for each $i$. Then $M_i = f(x_i)$ and $m_i = f(x_{i-1})$ for all $i$. This causes most of the terms in $U(P, f) - L(P, f)$ to cancel, since we have chosen an evenly spaced partition. (Draw a picture.) Algebraically, we have

$$\begin{aligned} U(P_n, f) - L(P_n, f) &= \sum_{i=1}^{n} f(x_i)\Delta x_i - \sum_{i=1}^{n} f(x_{i-1})\Delta x_i \\ &= \frac{b-a}{n}\left[\sum_{i=1}^{n} f(x_i) - \sum_{i=0}^{n-1} f(x_i)\right] = \frac{b-a}{n}(f(b) - f(a)). \end{aligned}$$

To finish, take $n$ large enough so that the RHS is less than $\varepsilon$. □

THEOREM 2.9. *Let $M$ and $m$ be upper and lower bounds, respectively, for the Riemann integrable function $f : [a, b] \to \mathbb{R}$, and let $\phi : [m, M] \to \mathbb{R}$ be continuous. Then $h = \phi \circ f : [a, b] \to \mathbb{R}$ is in $\mathcal{R}([a, b])$. ("The composition of a continuous function with a Riemann integrable function is Riemann integrable.")*

PROOF. Choose $\varepsilon > 0$. Since $\phi$ must be uniformly continuous, we can choose $\delta > 0$ such that $|w - z| < \delta$ implies $|\phi(w) - \phi(z)| < \eta$. (We'll figure out what $\eta$ to use in a bit.) Choose a partition $P$ of $[a, b]$ such that

$$U(P, f) - L(P, f) < \xi.$$

(We'll also decide what $\xi$ is later.) Since there is more than one function under consideration here, we should set some notation that is more specific than usual. Define

$$M_i^f = \sup_{x \in [x_{i-1}, x_i]} f(x), \quad M_i^h = \sup_{x \in [x_{i-1}, x_i]} h(x),$$

$$m_i^f = \inf_{x \in [x_{i-1}, x_i]} f(x), \quad m_i^h = \inf_{x \in [x_{i-1}, x_i]} h(x).$$

Next, define

$$G = \{i : M_i^f - m_i^f < \delta\} \quad \text{'Good' indices,}$$

$$B = \{i : M_i^f - m_i^f \ge \delta\} \quad \text{'Bad' indices.}$$

Remember that we need to show that

$$U(P, h) - L(P, h) = \sum_{i \in A}(M_i^h - m_i^h)\Delta x_i + \sum_{i \in B}(M_i^h - m_i^h)\Delta x_i$$

is less than $\varepsilon$. Let's deal with $G$ and $B$ separately.

Suppose $i \in G$. We claim that $M_i^h - m_i^h \le \eta$. Indeed, if $x, y \in [x_i, x_{i-1}]$, then $|f(x) - f(y)| \le M_i - m_i < \delta$, so $|h(x) - h(y)| = |\phi(f(x)) - \phi(f(y))| < \eta$. Taking the supremum over all $x \in [x_{i-1}, x_i]$ and the infimum over all $y \in [x_{i-1}, x_i]$, we get $M_i^h - m_i^h \le \eta$, as claimed. Thus

$$\sum_{i \in G}(M_i^h - m_i^h)\Delta x_i \le \eta \sum_{i=1}^{n} \Delta x_i = \eta(b - a).$$

We don't have a good estimate for $M_i^h - m_i^h$ when $i \in B$, but our choice of partition $P$ tells us that the $\Delta x_i$s corresponding to the bad $i$s have small total length. Indeed,

$$\delta \sum_{i\in B} \Delta x_i \leq \sum_{i \in B} (M_i^f - m_i^f)\Delta x_i < \xi.$$

Thus

$$\sum_{i\in B} \Delta x_i < \frac{\xi}{\delta}.$$

Denote $K = \max_{z\in[m,M]} |\phi(z)|$. Then $M_i^h - m_i^h < 2K$ for all $i$ (whether in $A$ or $B$). So

$$\sum_{i\in B}(M_i^h - m_i^h)\Delta x_i \leq 2K \sum_{i\in B} \Delta x_i < \frac{2K}{\delta}\xi.$$

Putting everything together, we get

$$U(P,h) - L(P,h) < \eta(b-a) + \frac{2K}{\delta}\xi,$$

where $\eta$ and $\xi$ are arbitrary. Choose $\eta = \frac{\varepsilon}{2(b-a)}$ and $\xi = \frac{\varepsilon\delta}{4K}$ to finish.

□

THEOREM 2.10. *Assume $f, g \in \mathcal{R}([a,b])$, and $\alpha \in \mathbb{R}$. Then*

*(1) $f+g \in \mathcal{R}$ and $\alpha f \in \mathcal{R}$, with*

$$\int_a^b (f+g)\,\mathrm{d}x = \int_a^b f\,\mathrm{d}x + \int_a^b g\,\mathrm{d}x, \qquad \int_a^b \alpha f\,\mathrm{d}x = \alpha \int_a^b f\,\mathrm{d}x.$$

*(2) $fg \in \mathcal{R}([a,b])$.*

*(3) If $f(x) \leq g(x)$ for all $x \in [a,b]$, then*

$$\int_a^b f\,\mathrm{d}x \leq \int_a^b g\,\mathrm{d}x.$$

*(4) $|f| \in \mathcal{R}([a,b])$ and*

$$\left|\int_a^b f\,\mathrm{d}x\right| \leq \int_a^b |f|\,\mathrm{d}x.$$

*(5) If $c \in (a,b)$, then $f \in \mathcal{R}([a,c])$, $f \in \mathcal{R}([c,b])$, and*

$$\int_a^b f\,\mathrm{d}x = \int_a^c f\,\mathrm{d}x + \int_c^b f\,\mathrm{d}x.$$

The proof of each part of this Theorem is tedious but straightforward. Therefore we omit the proof. However, the reader interested in some extra practice is encouraged to fill in the details.

We have seen that a lot of functions are Riemann integrable. But there are certainly functions that are not, as illustrated by the following example:

EXAMPLE 2.11. Define $f : [a,b] \to \mathbb{R}$ by $f(x) = 0$ if $x$ is rational and $f(x) = 1$ if $x$ is irrational. Then $f$ is *not* Riemann integrable. To see this, let $P = \{x_0, \ldots, x_n\}$ be any partition of $[a,b]$. Then $M_k = 1$ and $m_k = 0$ for every $k$, so $U(P,f) = b-a$, while $L(P,f) = 0$. But $P$ was an arbitrary partition, so we conclude that

$$\overline{\int_a^b} f\,\mathrm{d}x = b - a \neq 0 = \underline{\int_a^b} f\,\mathrm{d}x.$$

Thus $f$ cannot be Riemann integrable.

EXERCISE 2.12. Which $n \in \mathbb{N}$ have the property that $f^n \in \mathcal{R}([a,b])$ implies $f \in \mathcal{R}([a,b])$? Give proof(s) and counterexample(s) to show your answer is correct and complete.

### 2.2. The Fundamental Theorem of Calculus.

THEOREM 2.13 (FTC, version 1). *Let $f \in \mathcal{R}([a,b])$ and define $F : [a,b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t.$$

*Then*

*(1) $F$ is continuous on $[a,b]$, and*
*(2) If $f$ is continuous at $x_0 \in [a,b]$, then $F$ is differentiable at $x_0$, and $F'(x_0) = f(x_0)$.*

PROOF. (1) Since $f \in \mathcal{R}$, $f$ is bounded, i.e. there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a,b]$. Note that whenever $a \leq x < y \leq b$, we have

$$\begin{aligned}|F(y) - F(x)| &= \left|\int_a^y f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t\right| = \left|\int_x^y f(t)\,\mathrm{d}t\right| \\ &\leq \int_x^y |f(t)|\,\mathrm{d}t \leq \int_x^y M\,\mathrm{d}t = M(y-x).\end{aligned}$$

Thus if $|x - y| < \frac{\varepsilon}{M}$, we have $|F(y) - F(x)| < \varepsilon$. This proves that $F$ is uniformly continuous on $[a,b]$.

(2) Assume $f$ is continuous at $x_0 \in [a,b]$. We need to show that

$$\lim_{y \to x_0} \left|\frac{F(y) - F(x_0)}{y - x_0} - f(x_0)\right| = 0.$$

This will show both parts of the desired statement.

Actually, we will show only that the limit as $y$ approaches $x_0$ from the right is zero whenever $x_0 \in [a,b)$. The left hand limit for $x_0 \in (a,b]$ is similar. (Note that if $x_0 = a$, one only needs to consider the left limit, and if $x_0 = b$, only the right limit.)

Assume then that $x_0 \neq b$. Choose $\delta > 0$ so that $[x_0, x_0 + \delta) \subset [a,b]$ and so that $|x_0 - y| < \delta$ implies $|f(y) - f(x_0)| < \varepsilon$. Choose $y \in (x_0, x_0 + \delta)$. Then

$$\frac{F(y) - F(x_0)}{y - x_0} - f(x_0) = \frac{1}{y - x_0}\int_{x_0}^y f(t)\,\mathrm{d}t - \frac{1}{y - x_0}\int_{x_0}^y f(x_0)\,\mathrm{d}t = \frac{1}{y - x_0}\int_{x_0}^y (f(t) - f(x_0))\,\mathrm{d}t.$$

(Here we just used $y > x_0$, not $y < x_0 + \delta$.) Thus

$$\left|\frac{F(y) - F(x_0)}{y - x_0} - f(x_0)\right| \leq \frac{1}{y - x_0}\int_{x_0}^y |f(t) - f(x_0)|\,\mathrm{d}t \leq \frac{1}{y - x_0}\int_{x_0}^y \varepsilon\,\mathrm{d}t = \varepsilon.$$

A similar calculation shows that the same inequality holds when $y \in (x_0 - \delta, x_0)$. Thus the desired limit has been proven. □

THEOREM 2.14 (FTC, version 2). *Let $f \in \mathcal{R}([a,b])$. If $F : [a,b] \to \mathbb{R}$ is a differentiable function on $[a,b]$ such that $F' = f$, then*

$$F(b) - F(a) = \int_a^b f\,\mathrm{d}x.$$

PROOF. Choose $\varepsilon > 0$ and let $P = \{x_0, \dots x_n\}$ be a partition of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$. For each $k = 1, \dots, n$, pick $t_k \in [x_{k-1}, x_k]$ such that $F(x_k) - F(x_{k-1}) = F'(t_k)(x_k - x_{k-1})$; this is possible by the MVT. Then

$$F(b) - F(a) = \sum_{k=1}^n (F(x_k) - F(x_{k-1})) = \sum_{k=1}^n F'(t_k)(x_k - x_{k-1}) = \sum_{k=1}^n f(t_k)\Delta x_k.$$

For each $k$, we have

$$\inf_{x\in[x_{k-1},x_k]} f(x) =: m_k \le f(t_k) \le M_k := \sup_{x\in[x_{k-1},x_k]} f(x).$$

So

$$L(P,f) = \sum_{k=1}^n m_k \Delta x_k \le \sum_{k=1}^n f(t_k)\Delta x_k \le \sum_{k=1}^n M_k \Delta x_k = U(P,f).$$

Therefore $\sum_{k=1}^n f(t_k)\Delta x_k$ and $\int_a^b f\,\mathrm{d}x$ are both numbers which are in between $L(P,f)$ and $U(P,f)$. So

$$\left|F(b)-F(a)-\int_a^b f\,\mathrm{d}x\right| = \left|\sum_{k=1}^n f(t_k)\Delta x_k - \int_a^b f\,\mathrm{d}x\right| \le U(P,f)-L(P,f) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the Theorem follows.

□

EXERCISE 2.15. Show that if $f : [a,b] \to \mathbb{R}$ is continuous and $F(x) := \int_a^x f(t)\,\mathrm{d}t = 0$ for all $x \in [a,b]$, then $f(x) = 0$ for all $x \in [a,b]$. Provide an example to show that the statement is false if $f$ is not continuous.

EXERCISE 2.16. Assume $f$ and $g$ are differentiable functions on $[a,b]$, and assume $f', g' \in \mathcal{R}([a,b])$. Show that the integration by parts formula is valid:

$$\int_a^b f g'\,\mathrm{d}x = f(b)g(b) - f(a)g(a) - \int_a^b f' g\,\mathrm{d}x.$$

Make sure you show that the relevant functions are Riemann integrable when you do the proof!

EXERCISE 2.17. Assume $g : [a,b] \to \mathbb{R}$ is differentiable, that $g'$ is continuous, and $M$ and $m$ are upper and lower bounds, respectively, for the function $g$. Assume $f : [m,M] \to \mathbb{R}$ is continuous. Show that the change of variables formula is valid:

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(t)\,\mathrm{d}t.$$

Again, part of the exercise is to check that the relevant functions are actually Riemann integrable. (Hint: By the FTC, $f$ is the derivative of some function $F$. Try to write the formula in terms of $F$.)

EXERCISE 2.18. Assume $f \in \mathcal{R}([a,b])$, but that $f$ has a jump discontinuity at $c \in (a,b)$, i.e. $f(c-) \neq f(c+)$. Show that $F(x) := \int_a^x f(t)\,\mathrm{d}t$ is not differentiable at $x = c$.

EXERCISE 2.19. Given a function $f : [a,b] \to \mathbb{R}$, define its total variation $Tf$ by

$$Tf = \sup\left\{\sum_{k=1}^n |f(x_k) - f(x_{k-1})|\right\},$$

where the supremum is taken over all partitions $P$ of $[a,b]$. Show that if $f'$ is continuous, then

$$Tf = \int_a^b |f'(x)|\,\mathrm{d}x.$$

(Hint: Use the FTC for one inequality, and use the MVT for the other direction.)

EXERCISE 2.20. Assume $g$ is bounded, $g \in \mathcal{R}([0,1])$ and $g$ is continuous at $0$. Show that

$$\lim_{n\to\infty}\int_0^1 g(x^n)\,\mathrm{d}x = g(0).$$

Hint: Consider the difference $\int_0^1 g(x^n)\,\mathrm{d}x - g(0)$; add and subtract $\int_0^c g(x^n)\,\mathrm{d}x$ for a carefully chosen $c$, and then show that $\int_0^c g(x^n)\,\mathrm{d}x$ is close to $cg(0)$ for large enough $n$.

## 3. Convergence of Integrals and Derivatives

If $(f_n)$ is a sequence of Riemann integrable functions and $f_n \to f$ pointwise, then it may or may not be the case that

$$\int_a^b f_n\,\mathrm{d}x \to \int_a^b f\,\mathrm{d}x,$$

as $n \to \infty$. As a counterexample, take $[a,b] = [0,1]$, and let

$$f_n(x) = \begin{cases} n & \text{if } x \in (0, n^{-1}) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\int_0^1 f_n\,\mathrm{d}x = 1$ for all $n \in \mathbb{N}$. But $f_n(x) \to 0$ pointwise. Indeed, $f_n(0) = 0$ for all $n$, and if $x > 0$, then $x > N^{-1}$ for some $N \in \mathbb{N}$, so that $f_n(x) = 0$ for $n \geq N$. The convergence of integrals above then reads $1 \to 0$ as $n \to \infty$, which is clearly false.

In fact, a pointwise limit of a sequence of Riemann integrable functions need not even be Riemann integrable! To see this, let $(s_n)_{n=1}^{\infty}$ be an enumeration of the set $\mathbb{Q} \cap [0,1]$, and define each $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = s_k \text{ for some } k \leq n; \\ 0 & \text{otherwise.} \end{cases}$$

Then each $f_n$ is Riemann integrable, but the limit function is not (c.f. Example 2.11).

However, if the (Riemann integrable) $f_n$ converge *uniformly* to $f$, then $f$ must be Riemann integrable, and the expected convergence of integrals does indeed hold. (Note that it can easily be seen that the convergence in the example above is not uniform, since $\sup_{x\in[0,1]} |f_n(x)| = n$ there, for each $n \in \mathbb{N}$, violating convergence to zero in the uniform norm.)

THEOREM 3.1 (Integrable Limit Theorem). *Let $f_n \in \mathcal{R}([a,b])$ for each $n \in \mathbb{N}$. If $f_n \to f$ uniformly on $[a,b]$, then $f \in \mathcal{R}([a,b])$, and*

$$\lim_{n\to\infty} \int_a^b f_n\,\mathrm{d}x = \int_a^b f\,\mathrm{d}x.$$

PROOF. Choose $\varepsilon > 0$. Choose $N \in \mathbb{N}$ large enough so that $n \geq N$ implies that $|f_n(x) - f(x)| < \eta$ (with $\eta$ to be determined later). Then for $n \geq N$, we have

$$f_n(x) - \eta < f(x) < f_n(x) + \eta, \qquad \text{for all } x \in [a,b].$$

Thus

$$0 \leq \overline{\int_a^b} f\,\mathrm{d}x - \underline{\int_a^b} f\,\mathrm{d}x < \int_a^b (f_n(x) + \eta)\,\mathrm{d}x - \int_a^b (f_n(x) - \eta)\,\mathrm{d}x = 2\int_a^b \eta\,\mathrm{d}x = 2(b-a)\eta.$$

Put $\eta = \frac{\varepsilon}{2(b-a)}$. Then we have

$$0 \leq \overline{\int_a^b} f\,\mathrm{d}x - \underline{\int_a^b} f\,\mathrm{d}x < \varepsilon,$$

for all $\varepsilon > 0$. Thus $\overline{\int_a^b} f\,\mathrm{d}x = \underline{\int_a^b} f\,\mathrm{d}x$, i.e. $f \in \mathcal{R}([a,b])$.

Next, for $n \geq N$, we have

$$\left| \int_a^b f\,\mathrm{d}x - \int_a^b f_n\,\mathrm{d}x \right| \leq \int_a^b |f - f_n|\,\mathrm{d}x \leq \eta(b-a) = \frac{\varepsilon}{2} < \varepsilon.$$

So $\int_a^b f_n\,\mathrm{d}x \to \int_a^b f\,\mathrm{d}x$, as $n \to \infty$. ☐

COROLLARY 3.2. *Assume $f_n \in \mathcal{R}([a,b])$ for each $n \in \mathbb{N}$, and assume the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a,b]$ to a function $f$. Then*

$$\int_a^b f \,\mathrm{d}x = \sum_{n=1}^{\infty} \int_a^b f_n \,\mathrm{d}x.$$

PROOF. Put $g_n = \sum_{k=1}^{n} f_k$, for each $n$. Then $g_n \to f$ uniformly on $[a,b]$ by assumption. So $\int_a^b g_n \,\mathrm{d}x \to \int_a^b f \,\mathrm{d}x$ as $n \to \infty$. On the other hand,

$$\int_a^b g_n \,\mathrm{d}x = \int_a^b \sum_{k=1}^{n} f_k \,\mathrm{d}x = \sum_{k=1}^{n} \int_a^b f_k \,\mathrm{d}x \to \sum_{k=1}^{\infty} \int_a^b f_k \,\mathrm{d}x, \qquad \text{as } n \to \infty.$$

Thus $\int_a^b f \,\mathrm{d}x = \sum_{k=1}^{\infty} \int_a^b f_k \,\mathrm{d}x$, by uniqueness of limits. $\square$

EXERCISE 3.3. Let $(f_n)$ be a sequence of real-valued, Riemann integrable functions on the interval $[a,b]$. Assume that $f_n(x) \to 0$ as $n \to \infty$ for each $x \in [a,b]$, and additionally,

$$\sum_{n=1}^{\infty} |f_{n+1}(x) - f_n(x)|$$

converges uniformly on $[a,b]$.

(1) Show that $\lim_{n\to\infty} \int_a^b f_n \,\mathrm{d}x \to 0$.
(2) Is it necessarily the case that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly? Give a proof or counterexample to support your answer.

THEOREM 3.4. *Assume $(f_n)$ is a sequence of differentiable functions on $[a,b]$, and $f_n'$ is continuous for each $n \in \mathbb{N}$. If $f_n' \to g$ uniformly on $[a,b]$ and $f_n \to f$ pointwise on $[a,b]$, then $f$ is differentiable on $[a,b]$, and $f'(x) = g(x)$ for all $x \in [a,b]$.*

PROOF. Choose $\varepsilon > 0$. Choose $x, y \in [a,b]$, with $x < y$. Since $f_n' \to g$ uniformly on $[a,b]$, we have

$$\frac{f(y) - f(x)}{y - x} = \lim_{n\to\infty} \frac{f_n(y) - f_n(x)}{y - x} = \lim_{n\to\infty} \frac{1}{y - x} \int_x^y f_n' \,\mathrm{d}t = \frac{1}{y - x} \int_x^y g \,\mathrm{d}t.$$

We claim that the RHS here tends to $g(x)$ as $y \to x+$. Indeed, since $f_n'$ are all continuous and converge to $g$ uniformly, it follows that $g$ is continuous. So we can choose $\delta > 0$ so that $|t - x| < \delta$ implies $|g(t) - g(x)| < \varepsilon$. Then $y - x < \delta$ implies

$$\begin{aligned}\left|\frac{f(y) - f(x)}{y - x} - g(x)\right| &= \left|\frac{1}{y - x}\int_x^y g \,\mathrm{d}t - g(x)\right| = \left|\frac{1}{y - x}\int_x^y (g(t) - g(x)) \,\mathrm{d}t\right| \\ &\leq \frac{1}{y - x}\int_x^y |g(t) - g(x)| \,\mathrm{d}t < \frac{1}{y - x}\int_x^y \varepsilon \,\mathrm{d}t = \varepsilon.\end{aligned}$$

The proof for $y < x$ is similar. $\square$

REMARK 3.5. The previous Theorem can actually be strengthened in at least two ways. First, uniform convergence of the $f_n'$ 'almost' implies uniform convergence of the $f_n$. We just have to assume that the $f_n$ converge somewhere at some point in $[a,b]$ (otherwise we could add a different constant to each $f_n$ without affecting convergence of the derivatives.)

THEOREM 3.6. *Assume $(f_n)$ is a sequence of differentiable functions on $[a,b]$, and assume $f_n' \to g$ uniformly on $[a,b]$. Assume there exists $x_0 \in [a,b]$ such that $(f_n(x_0))_{n=1}^{\infty}$ converges to some number $c$, as $n \to \infty$. Then there exists a function $f : [a,b] \to \mathbb{R}$ such that $f(x_0) = c$ and $f_n \to f$ uniformly on $[a,b]$.*

PROOF. We show that $(f_n)$ is uniformly Cauchy. We want to use the information about the derivatives of $f_n - f_m$ and the fact that $(f_n(x_0))$ converges. Therefore we add and subtract $f_n(x_0) - f_m(x_0)$, and apply the MVT to the function $f_n - f_m$.

$$\begin{aligned} f_n(x) - f_m(x) &= (f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0)) + (f_n(x_0) - f_m(x_0)) \\ &= (f_n'(x^*) - f_m'(x^*))(x - x_0) + (f_n(x_0) - f_m(x_0)). \end{aligned}$$

Now it's clear how to proceed. Choose $\varepsilon > 0$, then choose $N$ large enough so that $|f_n(x_0) - f_m(x_0)| < \varepsilon/2$ and $|f_n'(y) - f_m'(y)| < \frac{\varepsilon}{2(b-a)}$ whenever $m \geq n \geq N$ and $y \in [a,b]$. Then $m \geq n \geq N$ implies

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n'(x^*) - f_m'(x^*)||x - x_0| + |f_n(x_0) - f_m(x_0)| \\ &< \frac{\varepsilon}{2(b-a)} \cdot |x - x_0| + \frac{\varepsilon}{2} \leq \varepsilon. \end{aligned}$$

That is, $(f_n)$ is uniformly Cauchy, hence converges uniformly. Defining $f$ to be the limit function, we get $f(x_0) = c$ automatically. $\square$

REMARK 3.7. Actually, we don't even need to assume continuity of the $f_n'$. We state the final, strongest result as a Theorem, but we do not prove it.

THEOREM 3.8 (Differentiable Limit Theorem). *Assume $(f_n)$ is a sequence of differentiable functions on $[a,b]$ such that $f_n(x_0) \to c$ as $n \to \infty$ for some $x_0 \in [a,b]$. If $f_n' \to g$ uniformly on $[a,b]$, then there exists a function $f : [a,b] \to \mathbb{R}$ such that $f(x_0) = c$, $f_n \to f$ uniformly on $[a,b]$, $f$ is differentiable on $[a,b]$, and $f'(x) = g(x)$ for all $x \in [a,b]$.*