

MATH 425A HW6, OCT. 7, 6PM

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1. CHAPTER 2.

Chapter 2 §2.4: Exercise 2.9.

Suppose X a finite, nonempty, set and suppose d be a metric on X . Let \mathcal{T} denote the topology generated by d . Show that $\mathcal{T} = \mathcal{P}(X)$. Conclude that any metric on X is equivalent to the discrete metric. (Hint: To show that $\mathcal{T} = \mathcal{P}(X)$, start by proving that $\{x\} = B_{(X,d)}(x, r_x)$ for some sufficiently small r_x , for each $x \in X$.)

Proof. (\subseteq) As X is finite, then it has finitely many subsets. Recall that the topology on a metric space is the collection of all open sets on the metric space X generated by the corresponding metric. Let $E \in \mathcal{T}$, i.e. E is an open set in X . Then obviously E must be subset of X by definition of an open set, and so $E \in \mathcal{P}(X)$.

(\supseteq) Now suppose that $L \in \mathcal{P}(X)$, i.e. $L \subseteq X$. Then, as X is finite, then there are finitely many points in L ; we can enumerate L so that $L = \{p_1, p_2, \dots, p_n\}$. As X is itself open, then $L \subseteq X = \text{Int}_X(X)$. Then for every point $p_i \in L$, $1 \leq i \leq n$, there is some ball $B_X(p_i, r_i)$ with $r_i > 0$ such that $B_X(p_i, r_i) \subseteq X$. Thus it suffices to show that for any $p_i \in L$,

$$L = \bigcup_{i=1}^n B_X(p_i, r_i).$$

Suppose that $q \in L$. Then $q = p_\beta$, where $1 \leq \beta \leq n$, and, moreover, there is some open ball $B_X(p_\beta, r_\beta)$ with $r_\beta > 0$ such that $B_X(p_\beta, r_\beta) \subseteq X$. Thus we see that $p_\beta \in \bigcup_{i=1}^n B_X(p_i, r_i)$. Now suppose that $\ell \in \bigcup_{i=1}^n B_X(p_i, r_i)$. Then $\ell \in B_X(p_\alpha, r_\alpha) \subseteq X$ for some α . We claim that $\{p_\alpha\} = B_X(p_\alpha, r_\alpha)$ for some sufficiently small r_α . In general, let $x \in X$. Then we take can take $r_x = \min_{y \in X \setminus \{x\}}(d(x, y))$ as X is assumed to be finite. Then $B_X(x, r_x) = \{x\}$ as each open ball of the form $B_X(x, r_x)$, where $r_x = \min_{y \in X \setminus \{x\}}(d(x, y))$, has the property that $t \in B_X(x, r_x)$ if and only if $d(x, t) < \min_{y \in X \setminus \{x\}}(d(x, y))$ and the only point in X that satisfies this strict property with r_x is x itself (and the opposite direction $\{x\} \subseteq B_X(x, r_x)$ is clear). Thus the claim follows, and so $\ell \in \{p_\alpha\}$, i.e. $\ell = p_\alpha \in L$. Hence the equality of sets follows and L is a union of open balls (a posteriori, just a union of singleton open balls) and so L is open in X and thus $L \in \mathcal{T}$.

Lastly we can conclude that any metric on a finite set X is equivalent to the discrete metric as our claim in showing the reverse inclusion showed that for any point $x \in X$ there is some $r_x > 0$ which gives $B_X(x, r_x) = \{x\}$, and as open balls generate the topology in a metric space then any other metric on X will be equivalent to the discrete metric (i.e., the discrete metric is given by our established open balls $B_X(x, r_x) = \{x\}$) as the open balls $\{x\}$ will be contained in all other hypothetical open balls given by “another” metric on X . \square

Chapter 2 §2.4: Exercise 2.10.

Prove that the Euclidean metric and the square metric are equivalent on \mathbb{R}^n .

Proof. Recall that the usual Euclidean norm in \mathbb{R}^n is given taken by comparing two points $p = (a_1, \dots, a_n)$ and $q = (b_1, \dots, b_n)$ in \mathbb{R}^n is given by $d(p, q) = \sqrt{(b_1 - a_1)^2 + \dots + (b_n - a_n)^2}$. And the square norm is given by $d_u(p, q) = \max\{|b_1 - a_1|, \dots, |b_n - a_n|\}$. \square

Prove Proposition 3.4.

Proposition 3.4.

Let X be a set, and let \mathcal{B} be a collection of subsets of X , which has the following properties:

- (1) Every $x \in X$ is contained in at least one element B of \mathcal{B} .
- (2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Then the following collection \mathcal{T} is a topology on X :

$$\mathcal{T} = \left\{ U \in \mathcal{P}(X) : U = \bigcup_{B \in \mathcal{A}} B \text{ for some subcollection } \mathcal{A} \subseteq \mathcal{B} \right\}.$$

and \mathcal{B} is a basis for \mathcal{T} . On the other hand, if \mathcal{T} is a topology on X and \mathcal{B} is a subcollection of \mathcal{T} such that the preceding equation holds, then \mathcal{B} must satisfy both properties (1) and (2).

Proof. (\Rightarrow) For the forward direction, suppose that for all $x \in X$ are contained in some $B_\alpha \in \mathcal{B}$, and if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there exists another $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. We want to show that the space elements of \mathcal{T} , as presented in Proposition 3.4, do indeed satisfy the topology axioms, and that \mathcal{B} is a basis for \mathcal{T} . Firstly, it is easy to see that $\emptyset \in \mathcal{T}$ since $\emptyset \in \mathcal{P}(X)$ and \emptyset is in the subcollection $\mathcal{A} \subseteq \mathcal{B}$ as it's a collection of sets, so \emptyset is vacuously a union of sets in the subcollection \mathcal{A} . Now $X \in \mathcal{P}(X)$, and we claim that $X = \bigcup_{W \in \mathcal{A}} W$ for a subcollection $\mathcal{A} \subseteq \mathcal{B}$. If we take $x \in X$, then we have that $x \in B_\alpha$ for some $B_\alpha \in \mathcal{B}$. So then if we take \mathcal{A} to be the subcollection which has $B_\alpha \in \mathcal{A}$, then $x \in \bigcup_{W \in \mathcal{A}} W$ as $x \in B_\alpha \in \mathcal{A}$. For the reverse inclusion, let $\bigcup_{W \in \mathcal{A}} W$ be such that \mathcal{A} has the element $B_\beta \in \mathcal{B}$ that posses every $x \in X$. Then clearly $\bigcup_{W \in \mathcal{A}} W \subseteq X$. Hence the claim holds and $X \in \mathcal{T}$. Let $\mathfrak{U} \subseteq \mathcal{T}$. We must show that $\bigcup_{U \in \mathfrak{U}} U \in \mathcal{T}$. As each $U \in \mathfrak{U} \subseteq \mathcal{T}$, then $U \in \mathcal{P}(X)$ such that $U = \bigcup_{T \in \mathcal{A}_U} T$ for some subcollection $\mathcal{A}_U \subseteq \mathcal{B}$. So then for each $V \in \mathfrak{U}$ there is a corresponding $V = \bigcup_{T \in \mathcal{A}_V} T$, where \mathcal{A}_V denotes the associated subcollection of \mathcal{B} . Now, for every $U \in \mathfrak{U}$, then we can denote the union of all corresponding subcollections \mathcal{A}_U as \mathfrak{A} , i.e. $\mathfrak{A} = \bigcup_{U \in \mathfrak{U}} \mathcal{A}_U$ where \mathcal{A}_U is a subcollection collection of \mathcal{B} and $U = \bigcup_{T \in \mathcal{A}_U} T$ for all $U \in \mathfrak{U}$. Then

$$\bigcup_{U \in \mathfrak{U}} U = \bigcup_{U \in \mathfrak{U}} \left(\bigcup_{T \in \mathcal{A}_U} T \right) = \bigcup_{E \in \mathfrak{A} \subseteq \mathcal{B}} E.$$

Thus we have that $\bigcup_{U \in \mathfrak{U}} U \in \mathcal{T}$ as the preceding rewriting of the union shows and as all $U \in \mathfrak{U} \subseteq \mathcal{T}$ by assumption then $U \in \mathcal{P}(X)$ (i.e., $U \subseteq X$) so the union of all such U are contained in X once again, that is, in X 's power set. Lastly, to show that \mathcal{T} is indeed a topology we have to show that finite intersections satisfy the corresponding topology axiom. Let $\mathfrak{D} = \{V_1, \dots, V_n\}$ be a finite subset of \mathcal{T} . Now consider $\bigcap_{i=1}^n V_i$, where $V_i \in \mathfrak{D}$. Then, for all $1 \leq i \leq n$, we can write $V_i = \bigcup_{B \in \mathcal{A}_i} B$ where \mathcal{A}_i is a subcollection of \mathcal{B} ; write \mathcal{A}_{V_i} for each corresponding subcollection of \mathcal{B} for $V_i = \bigcup_{B \in \mathcal{A}_{V_i}} B$. Then

$$\bigcap_{i=1}^n V_i = \bigcap_{i=1}^n \left(\bigcup_{B \in \mathcal{A}_{V_i}} B \right),$$

and we claim that [...]. Lastly, \mathcal{B} is a basis for \mathcal{T} as we have shown that \mathcal{T} is a topological space, and every element of \mathcal{T} can be written as a union $\bigcup_{B \in \mathcal{A}} B$ for some subcollection $\mathcal{A} \subseteq \mathcal{B}$, i.e. every element of the topology \mathcal{T} is written as union of elements from \mathcal{B}

(\Leftarrow) For the opposite direction, suppose that \mathcal{T} is a topology on X and $\mathcal{B} \subseteq \mathcal{T}$ such that \mathcal{T} is what it is presented as in Proposition 3.4. As \mathcal{B} is basis, and $X \in \mathcal{T}$, then $X = \bigcup_{Y \in \mathcal{A}} Y$ for some subcollection $\mathcal{A} \subseteq \mathcal{B}$, so if $x \in X$ then $x \in Y_\alpha$ for some $Y_\alpha \in \mathcal{A}$, but as $\mathcal{A} \subseteq \mathcal{B}$, then x is in at least one element of \mathcal{B} . For the second property, let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. Then, as $\mathcal{B} \subseteq \mathcal{T}$, then $B_1 = \bigcup_{Y \in \mathcal{Q}} Y$ and $B_2 = \bigcup_{T \in \mathcal{W}} T$, where \mathcal{Q} and \mathcal{W} are subcollections of \mathcal{B} . So then $x \in (\bigcup_{Y \in \mathcal{Q}} Y) \cap (\bigcup_{T \in \mathcal{W}} T)$, and so $x \in Y_\alpha$ for some $Y_\alpha \in \mathcal{Q}$ and $x \in T_\beta$ for some $T_\beta \in \mathcal{W}$; hence $x \in Y_\alpha \cap T_\beta \in \mathcal{B}$ and $Y_\alpha \cap T_\beta \subseteq B_1 \cap B_2$. □

Chapter 3: Exercise 1.1.

Let E_1, E_2 be subsets of a metric space (X, d) . Prove that

$$\text{Lim}_X(E_1 \cup E_2) = \text{Lim}_X(E_1) \cup \text{Lim}_X(E_2).$$

Proof. (\subseteq) Let $x \in \text{Lim}_X(E_1 \cup E_2)$. Then $x \in X$ and for any open neighborhood $x \in W \subseteq X$, we have that $W \cap ((E_1 \cup E_2) \setminus \{x\})$ is nonempty. Now take $\ell \in W \cap ((E_1 \cup E_2) \setminus \{x\})$. So then $\ell \in W$ and $\ell \in (E_1 \cup E_2) \setminus \{x\}$, i.e. $\ell \in E_1 \cup E_2$ and $\ell \neq x$. WLOG, suppose that $\ell \in E_1$. Then $\ell \in E_1 \setminus \{x\}$, and so as $\ell \in W$ as well we get that $\ell \in W \cap E_1 \setminus \{x\}$; that is, W intersects with $E_1 \setminus \{x\}$. Hence $x \in \text{Lim}_X(E_1)$.

(\supseteq) Let $x \in \text{Lim}_X(E_1) \cup \text{Lim}_X(E_2)$. WLOG, let $x \in \text{Lim}_X(E_1)$. Then $x \in X$ and $E_1 \setminus \{x\}$ intersects with any open neighborhood of x , say, W . So we can say that there is some $\ell \in W \cap (E_1 \setminus \{x\})$. Thus $\ell \in W$ and $\ell \in E_1 \setminus \{x\}$ (i.e. $\ell \in E_1$ and $\ell \neq x$). Trivially, $\ell \in E_1 \cup E_2$. So then $\ell \in (E_1 \cup E_2) \setminus \{x\}$ as $\ell \neq x$. As $\ell \in W$ as well, then $\ell \in W \cap ((E_1 \cup E_2) \setminus \{x\})$. Therefore $x \in \text{Lim}_X(E_1 \cup E_2)$. \square

Chapter 3: Exercise 1.2.

Let X be a metric space, and assume $E \subseteq Y \subseteq X$. Prove that

$$\text{Lim}_Y(E) = \text{Lim}_X(E) \cap Y$$

Proof. (\subseteq) Let $p \in \text{Lim}_Y(E)$. Then $p \in Y$ and for any open neighborhood $p \in V \subseteq Y$, we have that $V \cap (E \setminus \{p\})$ intersect. As $Y \subseteq X$, then p has an open neighborhood $V \subseteq X$, and so $p \in \text{Lim}_X(E)$ since $V \cap (E \setminus \{p\})$ intersect. Thus $p \in Y$ and $p \in \text{Lim}_X(E)$, and so $p \in \text{Lim}_X(E) \cap Y$.

(\supseteq) Suppose that $p \in \text{Lim}_X(E)$ and $p \in Y$. Then $p \in X$ and for any open neighborhood $p \in V \subseteq X$, we have that $V \cap (E \setminus \{p\})$ intersect. Now V is open in Y , by Theorem 2.13, if and only if $V = W \cap Y$ for some open $W \subseteq X$. We know that $p \in V \cap Y$, and $p \in \text{Int}_X(V)$, so \square

Chapter 3: Exercise 1.3.

If (X, \mathcal{T}) is a topological space and E is a subset of X , we say that x is a limit point of E with respect to X if every neighborhood of x in X (that is, every $U \in \mathcal{T}$ such that $x \in U$) intersects $E \setminus \{x\}$.

- (a) Suppose \mathcal{B} is a basis for a topology \mathcal{T} on X . Show that x is a limit point of E with respect to X if and only if every $B \in \mathcal{B}$ containing x intersects $E \setminus \{x\}$.
- (b) Show that if E is any subset of \mathbb{R} which is not bounded above (with respect to the usual order relation on \mathbb{R}), then $+\infty$ is a limit point of E with respect to $\overline{\mathbb{R}}$ (in its standard topology).

Proof. Suppose (X, \mathcal{T}) is a topological space, and E is a subset of X .

(a) Suppose \mathcal{B} is basis for a topology \mathcal{T} , i.e. every element in \mathcal{T} can be written as union of elements of $\mathcal{B} \subseteq \mathcal{T}$.

(\Rightarrow) Assume that x is a limit point of E with respect to X . Then $x \in X$ and for any open neighborhood $x \in W \subseteq X$, we have that $W \cap (E \setminus \{x\}) \neq \emptyset$. As W is open, then it can be written as a union of basis open sets of \mathcal{B} , say, $W = \bigcup_i B_i$, where $B_i \in \mathcal{B}$. Thus $(\bigcup_i B_i) \cap (E \setminus \{x\}) \neq \emptyset$, so take ℓ in the intersection. Then $\ell \in \bigcup_i B_i$ and $\ell \in E \setminus \{x\}$. So $\ell \in \bigcup_i B_i$ implies that $\ell \in B_\alpha$ for some $B_\alpha \in \mathcal{B}$. Hence $\ell \in B_\alpha \cap (E \setminus \{x\})$. Therefore the forward direction claim follows.

(\Leftarrow) Assume that for every $x \in B \in \mathcal{B}$ intersects $E \setminus \{x\}$. Then, in any case, we can construct an open set $L = \bigcup_j B_j$ where each $B_j \in \mathcal{B}$, as \mathcal{B} is a basis. Now, as every $B_\beta \cap (E \setminus \{x\}) \neq \emptyset$ by hypothesis, where $B_\beta \in \mathcal{B}$, we have some element, say, ℓ in the intersection. So then $\ell \in B_\beta$ and $\ell \in E \setminus \{x\}$. Hence $\ell \in L = \bigcup_j B_j$ as $\ell \in B_\beta \in \mathcal{B}$. Therefore $\ell \in L \cap (E \setminus \{x\})$ since L is indeed an open neighborhood of x as $x \in B_\beta$ which intersects with $E \setminus \{x\}$; that is, x is a limit point of E with respect to X .

(b) Suppose that $E \subseteq \mathbb{R}$, and E is not bounded above. Then we must show that $+\infty \in \text{Lim}_{\overline{\mathbb{R}}}(E)$. Firstly, $+\infty \in \overline{\mathbb{R}}$, by definition, so it remains to show that for any open set $U \subseteq \overline{\mathbb{R}}$, we have that $U \cap E \setminus \{+\infty\}$ intersect. □

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