MATH 425A HW4, DUE 09/23/2022, 6PM

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2.4 in Chapter 3.

Chapter 2. §5.

Exercise 0.1 (5.2.)**.** Let $a_1, a_2, \dots$ be any enumeration of the negative rational numbers; let $b_1, b_2, \dots$ be any enumeration of the positive rational numbers. Show that the following two equalities hold:

$$\bigcap_{j=1}^{\infty}(a_j, b_j) = \{0\}, \bigcup_{j=1}^{\infty}(a_j, b_j) = \mathbf{R}$$

Proof. For the first equality, take $\ell \in T = \bigcap_{j=1}^{\infty}(a_j, b_j)$, that is, ℓ is in every $(a_j, b_j) \subseteq \mathbf{R}$. So then $a_j < \ell < b_j$ for $\ell \in \overline{\mathbf{R}}$, but as a_j is essentially a negative rational number, and b_j is a positive rational, then we have that ℓ is squished between every negative and positive rational number.

$\square$

Chapter 2. § 6.

Exercise 0.2 (6.1.)**.** Prove that the addition and multiplication operations in $(\mathbf{C}, +, \cdot)$ satisfy the field axioms of Definition 2.1.

Proof. We essentially need to show that five axioms hold true from Definition 2.1. From now on, let $x, y, z \in \mathbf{R} \times \mathbf{R} \,(= \mathbf{C})$, which is the underlying set of $\mathbf{C}$, where $x = (a, b), y = (c, d), z = (s, t)$ where $a, b, c, d, s, t \in \mathbf{R}$.

(1) The set $\mathbf{C} := (\mathbf{C}, +, \cdot)$, as the operations are defined in Chapter 2, §6., is closed since $x+y = (a, b)+(c, d) = (a+c, b+d) \in \mathbf{R}\times\mathbf{R}$ and $xy = (a, b)\cdot(c, d) = (ac-bd, ad+bc) \in \mathbf{R}\times\mathbf{R}$ since $a + c, b + d, ac - bd, ad + bc \in \mathbf{R}$ as $\mathbf{R}$ is a field, and so $x + y \in \mathbf{C}$ and $xy \in \mathbf{C}$.

(2) For commutativity: $x + y = (a, b) + (c, d) = (a + c, b + d) = (c + a, d + b) = (c, d)+(a, b) = y+x$ since $\mathbf{R}$ is a field, and, similarly, $xy = (a, b)\cdot(c, d) = (ac-bd, ad+bc) = (ca - db, cb + da) = (c, d) \cdot (a, b) = yx$ as $\mathbf{R}$ is a field. Now for associativity:

$$\begin{aligned} x + (y + z) &= (a, b) + ((c, d) + (s, t)) = (a, b) + (c + s, d + t) \\ &= (a + (c + s), b + (d + t)) = ((a + c) + s, (b + d) + t)) \qquad (\mathbf{R} \text{ is a field}) \\ &= (a + c, b + d) + (s, t) = (x + y) + z \end{aligned}$$

$$\begin{aligned} x(yz) &= (a, b) \cdot ((c, d) \cdot (s, t)) = (a, b) \cdot (cs - dt, ct + ds) \\ &= (a(cs - dt) - b(ct + ds), a(ct + ds) + b(cs - dt)) && (\mathbf{R} \text{ is a field}) \\ &= (acs - adt - bct - bds, act + ads + bcs - bdt) && (\mathbf{R} \text{ is a field}) \\ &= ((ac - bd)s - (ad + bc)t, (ad + bc)s + (ac - bd)t) && (\mathbf{R} \text{ is a field}) \\ &= (ac - bd, ad + bc) \cdot (s, t) = ((a, b) \cdot (c, d)) \cdot (s, t) \\ &= (xy)z \end{aligned}$$

Therefore we have associativity and commutativity with the defined operations on $\mathbf{C}$.

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(3) The additive identity of $\mathbf{C}$ is defined to be $0 = (0,0) \in \mathbf{R} \times \mathbf{R}$, and so $x + 0 = (a,b) + (0,0) = (a+0, b+0) = (a,b) = (0+a, 0+b) = (0,0) + (a,b) = 0 + x$. Similarly, the multiplicative identity is defined to be $1 = (1,0)$, and so $x \cdot 1 = (a,b) \cdot (1,0) = (a(1) - b(0), a(0) + b(1)) = (a,b) = x = 1 \cdot x = (1,0) \cdot (a,b) = (1(a) - 0(b), 1(b) + 0(a)) = (a,b) = x$.

(4) The multiplicative inverse of $x = (a,b)$, where $x \neq 0$, can be found to be $x^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b(\frac{a}{a^2+b^2})}{a}\right)$, and we can tediously calculate to get that

$$x \cdot x^{-1} = (a,b) \cdot \left(\frac{a}{a^2+b^2}, \frac{-b(\frac{a}{a^2+b^2})}{a}\right) = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (1,0) = 1. \tag{1}$$

The additive inverse is much easier: for $y = (c,d)$, the additive inverse is $-y = (-c,-d)$, and so $y + (-y) = (c + (-c), d + (-d)) = (0,0) = 0$.

(5) Lastly, we need to check distributivity: Let $t := y + z = (c+s, d+t)$. Now

$$\begin{aligned} x \cdot t = (a,b) \cdot (c+s, d+t) &= (a(c+s) - b(d+t), a(d+t) + b(c+s)) \\ &= (ac + as - bd - bt, ad + at + bc + bs) \\ &= ((ac - bd) + (as - bt), (ad + bc) + (at + bs)) \\ &= (a,b) \cdot (c,d) + (a,b) \cdot (s,t) \end{aligned}$$

Therefore the distributive law holds.

Hence $\mathbf{C}$ is indeed a field. □

Exercise 0.3 (6.2.)**.** Prove that there exists no order $\leq$ that makes $(\mathbf{C}, +, \cdot, \leq)$ into an ordered field. (Hint: If there were such an ordering, then $i = \sqrt{-1}$ would necessarily be either positive or negative.)

Proof. Suppose that there does exists an ordering that makes $\mathbf{C}$ into an ordered field. Then, by definition, we have that either $i \leq 0$ or $i \leq 0$, but we do not have that $i = 0$, so we simply have that either i is negative or positive. Suppose, for the first case, that $i < 0$. Then $0 < -i$ so $0^2 < (-i)^2 = 1(-1) = -1$ and once again, $0^2 < (-1)^2 = 1$; hence a contradiction. Thus we cannot have that i is negative. Now, for the second/last case, then assume that $i > 0$. Then $i^2 = -1 > 0^2 = 0$ and so $(-1) + 1 = 0 > 0 + 1 = 1$, and multiplying by 1, $i \cdot 0 = 0 > 1 \cdot i = i$; thus a contradiction. Hence we cannot have that i is not positive either. Therefore we cannot have that there exists an order on $\mathbf{C}$ that makes it into an ordered field. □

1. Chapter 3. § 1

Exercise 1.1 (1.1.)**.** Let $\|\cdot\|$ be a norm on a real vector space V. Prove the *reverse triangle inequality:*

$$|\|x\| - \|y\|| \leq |\|x - y\||$$

Exercise 1.2 (1.2.)**.** Prove that any complex inner product is conjugate linear in its second argument; that is,

$$\langle x, \lambda y + z \rangle = \overline{\lambda} \langle x, y \rangle + \langle x, z \rangle,$$

for any scalar λ. (Note that this implies that any real inner product is linear in its second argument.)

Proof. We are considering a complex inner product and so we have a mapping $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbf{C}$ with some properties. Let $x, y, z \in V$ and $\lambda \in \mathbf{C}$. Then $\langle x, \lambda y + z \rangle = \overline{\langle \lambda y + z, x \rangle} = \overline{\lambda \langle y, x \rangle} + \overline{\langle z, x \rangle} = (\overline{\lambda}) \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} = \overline{\lambda} \langle x, y \rangle + \langle x, z \rangle$. □

Exercise 1.3 (1.3.-Polarization identity)**.** If $(V, \langle \cdot, \cdot \rangle)$ is a real inner product space, then

$$\langle v, w \rangle = \frac{1}{4} \left[\|v + w\|^2 - \|v - w\|^2 \right], \text{ for all } v, w \in V.$$

If $(V, \langle \cdot, \cdot \rangle)$ is a complex inner product space, then

$$\langle v, w\rangle = \frac{1}{4}\left[(\|v+w\|^2 - \|v-w\|^2) + i(\|v+iw\|^2 - \|v-iw\|^2)\right]$$

Proof. Suppose that $(V, \langle \cdot, \cdot \rangle)$ is a real inner product space. Then $\|v+w\|^2 = \langle v + w, v+w\rangle = \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle$, and, similarly, $\|v-w\|^2 = \langle v-w, v-w\rangle = \langle v, v\rangle - \langle v, w\rangle - \langle w, v\rangle + \langle w, w\rangle = \langle v, v\rangle - 2\langle v, w\rangle + \langle w, w\rangle$. Thus:

$$\begin{aligned}\frac{1}{4}\left[\|v+w\|^2 - \|v-w\|^2\right] &= \frac{1}{4}\left[\langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle - (\langle v, v\rangle - 2\langle v, w\rangle + \langle w, w\rangle\right] \\ &= \frac{1}{4}\left[2\langle v, w\rangle + 2\langle v, w\rangle\right] \\ &= \frac{1}{4}\left[4\langle v, w\rangle\right] = \langle v, w\rangle.\end{aligned}$$

Suppose that $(V, \langle \cdot, \cdot \rangle)$ is a complex inner product. Similar to the first computations we did for the real case, we can find that $\|v+w\|^2 = \langle v, v\rangle = \langle v, w\rangle + \overline{\langle v, w\rangle} + \langle w, w\rangle$, and $\|v-w\|^2 = \langle v, v\rangle - \langle v, w\rangle - \overline{\langle v, w\rangle} + \langle w, w\rangle$. Moreover, $\|v+iw\| = \langle w, w\rangle + i\langle w, v\rangle - i\langle v, w\rangle + \langle v, v\rangle$, and $\|v-iw\| = \langle w, w\rangle - i\langle w, v\rangle + i\langle v, w\rangle + \langle v, v\rangle$. Now:

$$\begin{aligned}\|v+w\|^2 - \|v-w\|^2 &= 2\langle v, w\rangle + 2\langle w, v\rangle, \text{ and} \\ \|v+iw\|^2 - \|v-iw\|^2 &= 2i\langle w, v\rangle - 2i\langle v, w\rangle = 2i\left[\langle w, v\rangle - \langle v, w\rangle\right]\end{aligned}$$

Thus:

$$\begin{aligned}\frac{1}{4}\left[(2\langle v, w\rangle + 2\langle w, v\rangle) + i\left(2i(\langle w, v\rangle - \langle v, w\rangle)\right)\right] &= \frac{1}{4}\left[2\langle v, w\rangle + 2\langle w, v\rangle + (-2\langle w, v\rangle + 2\langle v, w\rangle)\right] \\ &= \frac{1}{4}\left[4\langle v, w\rangle + 2\langle w, v\rangle - 2\langle w, v\rangle\right] \\ &= \frac{1}{4}\left[4\langle v, w\rangle\right] = \langle v, w\rangle.\end{aligned}$$

□

2. Chapter 3 §2

Exercise 2.1 (2.2)**.** For each of (a), (b), and (c), determine whether the given function d_j is a metric on $\mathbf{R}$, and prove that your answer is correct.

(a) $d_1(x, y) = \sqrt{|x-y|}$
(b) $d_2(x, y) = |x - 2y|$.
(c) $d_3(x, y) = \frac{|x-y|}{1+|x-y|}$

Proof. (a) Indeed a metric. We need to show that $d_1 \colon X \times X \to \mathbf{R}$ satisfies nonnegativity, symmetry, and the triangle inequality:

Firstly, let's fix some arbitrary $(x, y) \in X \times X$. Then $d_1(x, y) = \sqrt{|x-y|}$, which is the root of some positive number, or 0, in $\mathbf{R}$, and so $d_1(x, y) \geq 0$; if $x = y$, then $\sqrt{|x-y|} = \sqrt{|x-x|} = 0$, and if we first assumed $d_1(x, y) = 0$, then $d_1(x, y) = \sqrt{|x-y|} = 0$ and so $|x - y| = 0$ and in either case of $x - y \geq 0$ or $x - y < 0$, we get that $x = y$.

For symmetry, suppose we have $d_1(x, y)$ and $(x, y) \in X \times X$. Now, consider $d_1(x, y) - d_1(y, x)$, and so $\sqrt{|x-y|} - \sqrt{|y-x|}$—if $x - y > 0$ then $0 > y - x$, which implies $\sqrt{x-y} - \sqrt{-(y-x)} = \sqrt{x-y} - \sqrt{x-y} = 0$ and so $d_1(x, y) = d_1(y, x)$ if $x - y > 0$; if $x - y < 0$ then $0 < y - x$, so $\sqrt{-(x-y)} - \sqrt{y-x} = \sqrt{y-x} - \sqrt{y-x} = 0$ and so $d_1(x, y) = d_1(y, x)$ if $x - y < 0$; if $x - y = 0$ then $x = y$ and $d_1(x, y) = \sqrt{|x-y|} = \sqrt{|y-x|} = d_2(y, x)$. Thus d_1 is symmetric. [Could use instead the fact that $|\cdot|$ is a metric, and so $(d_1(x, y))^2 = |x - y| = |y - x| = (d_1(y, x))^2$, and so $d_1(x, y) = d_1(y, x)$.]

Lastly, we need to show that the triangle inequality holds. This is shown easiest if we show that, for any $s, t \in \mathbf{R}$ such that $s, t \geq 0$, we have that $\sqrt{s} + \sqrt{t} \geq \sqrt{s+t}$. This is true as we clearly have that $2\sqrt{st} \geq 0$ and so this leads to $s + \sqrt{2st} + t \geq s + t$ which is the same as $(\sqrt{s} + \sqrt{t})^2 \geq s + t$, and thus $\sqrt{s} + \sqrt{t} \geq \sqrt{s+t}$. Now, $d_1(x, y) = \sqrt{|x-y|} = \sqrt{|(x-z)+(z-y)|} \leq \sqrt{|x-z|} + \sqrt{|z-y|} = d_1(x, z) + d_1(z, y)$—note that if $x - y < 0$ then the inequality would still work out in the end.

Therefore we have that $d_1 \colon X \times X \to \mathbf{R}$ where $d_1 \colon (x, y) \mapsto \sqrt{|x-y|}$ does define a metric.

(b) $d_2(x, y) = |x - 2y|$ does not define a metric on $\mathbf{R}$, since it does not, at the very least, satisfy the symmetry condition: Let $X = \mathbf{R}$. Then $d_2 \colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, where $d_2 \colon (x, y) \mapsto |x - 2y|$, is not a metric since, for example, $d_2(2, 3) = |2 - 2(3)| = |2 - 6| = |-4| = 4$ but $d_2(3, 2) = |3 - 2(2)| = |3 - 4| = |-1| = 1$, and so a counter example against the symmetry property.

(c) Indeed a metric.

Suppose $(x, y) \in X \times X$. Then $\frac{|x-y|}{1+|x-y|}$ is always positive since $|x - y| \geq 1$ for any choice $x \neq y$ and $x = y$ gives us that $d_3(x, y) = 0$. Now if $x = y$, then $\frac{0}{1+0} = 0$. If instead assumed firstly that $\frac{|x-y|}{1+|x-y|} = 0$, then: $x - y > 0$ implies $\frac{x-y}{1+x-y} = 0$ and so $x = y$ clearly, and similarly, $x - y < 0$ gives us $\frac{y-x}{1+y-x} = 0$ and so $x = y$; if $x - y = 0$, then the result follows immediately. Thus $d_3(x, y) \geq 0$ for all $x, y \in X$ and $d_3(x, y) = 0$ if and only if $x = y$.

For symmetry, we proceed as follows. If $x - y = 0$ then the result is clear. Now if $x - y > 0$, then

$$\begin{aligned}\frac{|x-y|}{1+|x-y|} - \frac{|y-x|}{1+|y-x|} &= \frac{x-y}{1+x-y} - \frac{-(y-x)}{1+(-1)(y-x)} \\ &= \frac{x-y}{1+x-y} + \frac{y-x}{1+(x-y)} = 0.\end{aligned}$$

Thus $d_1(x, y) = d_1(y, x)$ if $x - y > 0$. If $x - y < 0$, then

$$\begin{aligned}\frac{|x-y|}{1+|x-y|} - \frac{|y-x|}{1+|y-x|} &= \frac{-(x-y)}{1+(-1)(x-y)} - \frac{y-x}{1+y-x} \\ &= \frac{y-x}{1+y-x} + \frac{-y+x}{1+y-x} = 0.\end{aligned}$$

Thus $d_3(x, y) = d_3(y, x)$ if $x - y < 0$. Lastly, we have that if $x - y = 0$, then $x = y$, and so, trivially, $d_3(x, y) = d_3(y, x)$. Therefore d_3 is symmetric.

For the triangle inequality

$$\begin{aligned}\frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} &\geq \frac{|x-y|}{1+|x-y|+|y-z|} + \frac{|y-z|}{1+|x-y|+|y-z|} \\ &= \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|} \\ &= 1 - \frac{1}{1+|x-y|+|y-z|} \\ &\geq 1 - \frac{1}{1+|x-z|} = \frac{1+|x-z|-1}{1+|x-z|} = \frac{|x-z|}{1+|x-z|} = d_3(x, z).\end{aligned}$$

Therefore the triangle inequality holds and $d_3 \colon X \times X \to \mathbf{R}$ defines a metric. □

Exercise 2.2 (2.3)**.** Consider the function $d \colon \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}$ defined by

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2|, \quad [\, x = (x_1, x_2),\ y = (y_1, y_2)\,]$$

(a) Prove that d is a metric on $\mathbf{R}^2$.

(b) On a sheet of graph paper, draw the set $B_d((5,1),3)$. Use dotted lines to indicate the boundary, which is not included in the set you are drawing. (Hint: it may be easier to figure out what the set looks like if you first consider $B_d((0,0),3)$.)
(c) On the same graph as in the previous part, draw $B_{d_u}((3,2),1)$, where d_u denotes the square metric.

Proof. (a) We always have that $d(x,y) \geq 0$ since $|\cdot|$ is itself a metric, and so $|x_1 - y_1| \geq 0$ and $|x_2 - y_2| \geq 0$. Now if $d(x,y) = 0$. Then

We have that the symmetry property holds as a consequence of the fact that $|\cdot|$ is a metric, and so $d(x,y) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d(y,x)$. □

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