MATH 425A HW6, OCT. 7, 6PM

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1. Chapter 3.

Chapter 3 §2.4: Exercise 2.9.

Suppose X a finite, nonempty, set and suppose d be a metric on X. Let $\mathcal{T}$ denote the topology generated by d. Show that $\mathcal{T} = \mathcal{P}(X)$. Conclude that any metric on X is equivalent to the discrete metric. (Hint: To show that $\mathcal{T} = \mathcal{P}(X)$, start by proving that $\{x\} = B_{(X,d)}(x, r_x)$ for some sufficiently small r_x, for each $x \in X$.)

Proof. ($\subseteq$) As X is finite, then it has finitely many subsets. Recall that the topology on a metric space is the collection of all open sets on the metric space X generated by the corresponding metric. Let $E \in \mathcal{T}$, i.e. E is an open set in X. Then obviously E must be subset of X by definition of an open set, and so $E \in \mathcal{P}(X)$.
($\supseteq$) Now suppose that $L \in \mathcal{P}(X)$, i.e. $L \subseteq X$. Then, as X is finite, then there are finitely many points in L; we can enumerate L so that $L = \{p_1, p_2, \ldots, p_n\}$. As X is itself open, then $L \subseteq X = \mathrm{Int}_X(X)$. Then for every point $p_i \in L$, $1 \leq i \leq n$, there is some ball $B_X(p_i, r_i)$ with $r_i > 0$ such that $B_X(p_i, r_i) \subseteq X$. Thus it suffices to show that for any $p_i \in L$,

$$L = \bigcup_{i=1}^{n} B_X(p_i, r_i).$$

Suppose that $q \in L$. Then $q = p_\beta$, where $1 \leq \beta \leq n$, and, moreover, there is some open ball $B_X(p_\beta, r_\beta)$ with $r_\beta > 0$ such that $B_X(p_\beta, r_\beta) \subseteq X$. Thus we see that $p_\beta \in \bigcup_{i=1}^{n} B_X(p_i, r_i)$. Now suppose that $\ell \in \bigcup_{i=1}^{n} B_X(p_i, r_i)$. Then $\ell \in B_X(p_\alpha, r_\alpha) \subseteq X$ for some α. We claim that $\{p_\alpha\} = B_X(p_\alpha, r_\alpha)$ for some sufficiently small r_α. In general, let $x \in X$. Then we take can take $r_x = \min_{y \in X \setminus \{x\}}(d(x, y))$ as X is assumed to be finite. Then $B_X(x, r_x) = \{x\}$ as each open ball of the form $B_X(x, r_x)$, where $r_x = \min_{y \in X}(d(x, y))$, has the property that $t \in B_X(x, r_x)$ if and only if $d(x, t) < \min_{y \in X \setminus \{x\}}(d(x, y))$ and the only point in X that satisfies this strict property with r_x is x itself (and the opposite direction $\{x\} \subseteq B_X(x, r_x)$ is clear). Thus the claim follows, and so $\ell \in \{p_\alpha\}$, i.e. $\ell = p_\alpha \in L$. Hence the equality of sets follows and L is a union of open balls (a posteriori, just a union of singleton open balls) and so L is open in X and thus $L \in \mathcal{T}$.
Lastly we can conclude that any metric on a finite set X is equivalent to the discrete metric as our claim in showing the reverse inclusion showed that for any point $x \in X$ there is some $r_x > 0$ which gives $B_X(x, r_x) = \{x\}$, and as open balls generate the topology in a metric space then any other metric on X will be equivalent to the discrete metric (i.e., the discrete metric is given by our established open balls $B_X(x, r_x) = \{x\}$) as the open balls $\{x\}$ will be contained in all other hypothetical open balls given by "another" metric on X. $\square$

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Chapter 3 §2.4: Exercise 2.10.

Prove that the Euclidean metric and the square metric are equivalent on $\mathbb{R}^n$.

Proof. Recall that the usual Eucidean norm in $\mathbb{R}^n$ is given taken by comparing two points $p = (a_1, \dots, a_n)$ and $q = (b_1, \dots, b_n)$ in $\mathbb{R}^n$ where the metric is given by $d_E(p,q) = \sqrt{(b_1-a_1)^2 + \cdots + (b_n-a_n)^2}$. And the square norm is given by $d_\infty(p,q) = \max\{|b_1-a_1|, \dots, |b_n-a_n|\}$ where $p, q \in \mathbb{R}^n$ again. We claim that $\frac{1}{\sqrt{n}} d_E(p,q) \leq d_\infty(p,q)$. By definition of d_E, we have that $d_E(p,q) = \sqrt{(b_1-a_1)^2 + \cdots + (b_n-a_n)^2} \leq \sqrt{\sum_{\ell=1}^n \max\{|b_1-a_1|^2, \dots, |b_n-a_n|^2\}} = \sqrt{n \max\{|b_1-a_1|^2, \dots, |b_n-a_n|^2\}} = \sqrt{n} d_\infty(p,q)$, and so $\frac{1}{\sqrt{n}} d_E(p,q) \leq d_\infty(p,q)$ and the claim follows. Furthermore, we have that $d_\infty(p,q) = \max_{1 \leq i \leq n}\{|b_i - a_i|\} \leq d_E(p,q) = \sqrt{\sum_{i=1}^n (b_i - a_i)}$ since $d_\infty(p,q) = |b_\alpha - a_\alpha|$ for some $\alpha \in \{1, \dots, n\}$ so $d_\infty^2(p,q) = (b_\alpha - a_\alpha)^2$ and thus $d_\infty^2(p,q) \leq d_E(p,q)^2$ which implies $d_\infty(p,q) \leq d_E(p,q)$. So then if $t \in B_{\mathbb{R}^n}^E(p,\epsilon)$, where this is the open ball with the Euclidean metric, then $t \in B_{\mathbb{R}^n}^\infty(p,\epsilon)$, where this is the open ball with respect to the square metric, by the previous sentence; that is, $B_{\mathbb{R}^n}^E(p,\epsilon) \subseteq B_{\mathbb{R}^n}^\infty(p,\epsilon)$ for any $p \in \mathbb{R}^n$ and $\epsilon > 0$. Lastly we claim that $B_{\mathbb{R}^n}^\infty(p, \frac{\epsilon}{\sqrt{n}}) \subseteq B_{\mathbb{R}^n}^E(p,\epsilon)$ for any $p \in \mathbb{R}^n$ and $\epsilon > 0$. Take $s \in B_{\mathbb{R}^n}^\infty(p, \frac{\epsilon}{\sqrt{n}})$. So then $d_\infty(p,s) < \epsilon/\sqrt{n}$, and by a claim we made earlier, we have that $d_E(p,s) \leq \sqrt{n} d_\infty(p,s) < \epsilon$ and thus $s \in B_{\mathbb{R}^n}^E(p,\epsilon)$. This establishes our claim and we have that $B_{\mathbb{R}^n}^\infty(p, \frac{\epsilon}{\sqrt{n}}) \subseteq B_{\mathbb{R}^n}^E(p,\epsilon)$. The proposition of the Exercise follows. $\square$

Proposition 3.4.

Let X be a set, and let $\mathcal{B}$ be a collection of subsets of X, which has the following properties:

(1) Every $x \in X$ is contained in at least one element B of $\mathcal{B}$.

(2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Then the following collection $\mathcal{T}$ is a topology on X:

$$\mathcal{T} = \left\{ U \in \mathcal{P}(X) \colon U = \bigcup_{B \in \mathcal{A}} B \text{ for some subcollection } \mathcal{A} \subseteq \mathcal{B} \right\}.$$

and $\mathcal{B}$ is a basis for $\mathcal{T}$. On the other hand, if $\mathcal{T}$ is a topology on X and $\mathcal{B}$ is a subcollection of $\mathcal{T}$ such that the preceding equation holds, then $\mathcal{B}$ must satisfy both properties (1) and (2).

Proof. ($\Rightarrow$) Firstly, it is easy to see that $\varnothing \in \mathcal{T}$ since $\varnothing \in \mathcal{P}(X)$ and $\varnothing$ is in the subcollection $\mathcal{A} \subseteq \mathcal{B}$ as it's a collection of sets, so $\varnothing$ is vacuously a union of sets in the subcollection $\mathcal{A}$. Now $X \in \mathcal{P}(X)$, and we claim that $X = \bigcup_{W \in \mathcal{A}} W$ for a subcollection $\mathcal{A} \subseteq \mathcal{B}$. If we take $x \in X$, then we have that $x \in B_\alpha$ for some $B_\alpha \in \mathcal{B}$. So then if we take $\mathcal{A}$ to be the whole collection $\mathcal{B}$, then $x \in \bigcup_{W \in \mathcal{A}} W$ as $x \in B_\alpha \in \mathcal{A}$ for all $x \in X$. For the reverse inclusion, let $\bigcup_{W \in \mathcal{A}} W$ be such that $\mathcal{A}$ has the element $B_\beta \in \mathcal{B}$ that posses every $x \in X$. Then clearly $\bigcup_{W \in \mathcal{A}} \subseteq X$. Hence the claim holds and $X \in \mathcal{T}$. Let $\mathfrak{U} \subseteq \mathcal{T}$. We must show that $\bigcup_{U \in \mathfrak{U}} U \in \mathcal{T}$. As each $U \in \mathfrak{U} \subseteq \mathcal{T}$, then $U \in \mathcal{P}(X)$ such that $U = \bigcup_{T \in \mathcal{A}} T$ for some subcollection $\mathcal{A} \subseteq \mathcal{B}$. So then for each $V \in \mathfrak{U}$ there is a corresponding $V = \bigcup_{T \in \mathcal{A}_V} T$, where $\mathcal{A}_V$ denotes the associated subcollection of $\mathcal{B}$. Now, for every $U \in \mathfrak{U}$, then we can denote the union of all corresponding subcollections $\mathcal{A}_U$ as $\mathfrak{A}$, i.e. $\mathfrak{A} = \bigcup_{U \in \mathfrak{U}} \mathcal{A}_U$ where $\mathcal{A}_U$ is a subcollection collection of $\mathcal{B}$ and $U = \bigcup_{T \in \mathcal{A}_U} T$ for all $U \in \mathfrak{U}$. Then $\bigcup_{U \in \mathfrak{U}} U = \bigcup \left(\bigcup_{T \in \mathcal{A}_U} T \right) = \bigcup_{E \in \mathfrak{A} \subseteq \mathcal{B}}$ Thus we have that $\bigcup_{U \in \mathfrak{U}} U \in \mathcal{T}$ as the preceding rewriting of the union shows and as all $U \in \mathfrak{U} \subseteq \mathcal{T}$ by assumption then $U \in \mathcal{P}(X)$ (i.e., $U \subseteq X$) so the union of all such U are contained in X once again, that is, in X's power set. Lastly, to show that $\mathcal{T}$ is indeed a topology we have to show that finite intersections satisfy the corresponding topology axiom. Let $\mathfrak{D} = \{V_1, \dots, V_n\}$ be a finite subset of $\mathcal{T}$. Now consider $\bigcap_{i=1}^n V_i$, where $V_i \in \mathfrak{D}$. Then, for all $1 \leq i \leq n$, we can write $V_i = \bigcup_{B \in \mathcal{A}} B$ where $\mathcal{A}$ is a subcollection of $\mathcal{B}$; write $\mathcal{A}_i$ for each corresponding subcollection of $\mathcal{B}$ for $V_i = \bigcup_{B \in \mathcal{A}_i} B$. Then

$$\bigcap_{i=1}^n V_i = \bigcap_{i=1}^n \left(\bigcup_{B \in \mathcal{A}_i} B \right),$$

and we claim that $\mathcal{A} = \{B \in \mathcal{B} \colon \exists B_i \in \mathcal{A}_i, B \subseteq \bigcap_{i=1}^n B_i\}$ gives a satisfactory subcollection for $\bigcap_{i=1}^n V_i$ to be in $\mathcal{T}$. That is, we show that $\bigcap_{i=1}^n V_i = \bigcup_{B \in \mathcal{A}} B$, where $\mathcal{A}$ is the subcollection we just defined. Let $x \in \bigcap_{i=1}^n V_i$. Then $x \in V_j$ for all $1 \leq j \leq n$. Then $x \in B_1, B_2, \dots, B_n$ for all $B_j \in \mathcal{A}_j$ by assumption of $\mathcal{A}$. So then $x \in B_1 \cap B_2 \cap \cdots \cap B_n$, and so by (2) property, we have that there is some $B_\alpha \in \mathcal{B}$ such that $x \in B_\alpha \subseteq B_1 \cap B_2 \cap \cdots \cap B_n$, and so $B_\alpha \in \mathcal{A}$ and $x \in \bigcup_{B \in \mathcal{A}} B$. For the reverse inclusion, suppose that $\ell \in \bigcup_{B \in \mathcal{A}} B$. WLOG, let $x \in B$ for some $B \in \mathcal{A}$. Then $B \subseteq B_1 \cap B_2 \cap \cdots \cap B_n$ for $B_1, B_2, \dots, B_n \in \mathcal{A}_i$. Thus $x \in B_i$, for all $1 \leq i \leq n$, so we have that $x \in V_j$ as $\mathcal{D} = \{V_1, \dots, V_n\} \subseteq \mathcal{T}$, and so $x \in \bigcap_{i=1}^n V_i$. Lastly, $\mathcal{B}$ is a basis for $\mathcal{T}$ as we have shown that $\mathcal{T}$ is a topological space, and every element of $\mathcal{T}$ can be written as a union $\bigcup_{B \in \mathcal{A}} B$ for some subcollection $\mathcal{A} \subseteq \mathcal{B}$, i.e. every element of the topology $\mathcal{T}$ is written as union of elements from $\mathcal{B}$

($\Leftarrow$) For the opposite direction, suppose that $\mathcal{T}$ is a topology on X and $\mathcal{B} \subseteq \mathcal{T}$ such that $\mathcal{T}$ is what it is presented as in Proposition 3.4. So as $X \in \mathcal{T}$, then $X = \bigcup_{Y \in \mathcal{A}} Y$ for some subcollection $\mathcal{A} \subseteq \mathcal{B}$, so if $x \in X$ then $x \in Y_\alpha$ for some $Y_\alpha \in \mathcal{A}$, but as $\mathcal{A} \subseteq \mathcal{B}$, then x is in at least one element of $\mathcal{B}$. For the second property, let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. Then, as $\mathcal{B} \subseteq \mathcal{T}$, then $B_1 = \bigcup_{Y \in \mathcal{Q}} Y$ and $B_2 = \bigcup_{T \in \mathcal{W}} T$, where $\mathcal{Q}$ and $\mathcal{W}$ are subcollections of $\mathcal{B}$. So then $x \in \left(\bigcup_{Y \in \mathcal{Q}} Y\right) \cap \left(\bigcup_{T \in \mathcal{W}} T\right)$, and so $x \in Y_\alpha$ for some $Y_\alpha \in \mathcal{Q}$ and $x \in T_\beta$ for some $T_\beta \in \mathcal{W}$. And $\left(\bigcup_{Y \in \mathcal{Q}} Y\right) \cap \left(\bigcup_{T \in \mathcal{W}} T\right) = \bigcup_{B \in \mathcal{A}} B := E$ where $\mathcal{A} = \{B \in \mathcal{B} \colon B \subseteq Y_\alpha \cap T_\beta, \text{ for } T_\beta \in \mathcal{W}, Y_\alpha \in \mathcal{Q}\}$ by the previous work done showing that $\mathcal{T}$ is a topology. Thus $x \in E$ such that $x \in B$ for some $B \in \mathcal{A}$ and $B \subseteq Y_\alpha \cap T_\beta$. Hence $x \in B \in \mathcal{B}$ and $B \subseteq B_1 \cap B_2$. $\square$

Chapter 3 §3.2: Exercise 3.2.

Prove that the collection $\mathcal{R}$ of all *open rectangles* of the form

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \quad a_j, b_j \in \mathbb{R}, a_j < b_j, \text{ for all } j$$

is a basis for the standard topology on $\mathbb{R}^n$.

Proof. Let $X = \mathbb{R}^n$. First we show that X is a union of so-called open rectangles. For all $x \in X$, where $x = (x_1, \ldots, x_n)$, consider $\mathfrak{R}_X(x, \epsilon) := (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$, where $\epsilon > 0$. Then we claim that $\bigcup_{x \in X} \mathfrak{R}_X(x, 1) = X$. Firstly, it is clear that $\mathfrak{R}_X(x, 1) \subseteq X$, and so we have that $\bigcup_{x \in X} \mathfrak{R}_X(x, 1) \subseteq X$. Now suppose we have some $x = (x_1, \ldots, x_n) \in X$. Then $x \in \mathfrak{R}(x, 1)$ as for all x_i where $1 \leq i \leq n$, we have that $x_i - 1 < x_i < x_i + 1$. But then $x \in \mathfrak{R}_X(x, 1) \subseteq \bigcup_{x \in X} \mathfrak{R}(x, \epsilon)$. Moreover, we've only used open rectangles with $\epsilon = 1$, and so we have that

$$\mathbb{R}^n = \bigcup_{x \in X} \mathfrak{R}_X(x, 1) \subseteq \bigcup_{x \in X, \epsilon > 0} \mathfrak{R}_X(x, \epsilon),$$

and so we have the equality holds in equation above as $\mathfrak{R}_X(x, \epsilon) \subseteq \mathbb{R}^n$ for any $\epsilon > 0$ and $x \in \mathbb{R}$. Therefore the claim holds and we have satisfied the (1) property needed to use Proposition 3.4. Moving on, suppose we have $R_1, R_2 \in \mathcal{R}$ where $R_1 = (a_1, b_1) \times \cdots \times (a_n, b_n)$ and $R_2 = (c_1, d_1) \times \cdots \times (c_n, d_n)$ such that for each $1 \leq i \leq n$ we have $a_i < b_i$ and $c_i < d_i$. Let $x \in R_1 \cap R_2$. Then $x = (x_1, \ldots, x_n)$ implies that $x_i \in (a_i, b_i)$ and $x_i \in (c_i, d_i)$ for all i so we thus construct $R_3 = ((a_1, b_1) \cap (c_1, d_1)) \times \cdots \times ((a_n, b_n) \cap (c_n, d_n))$ for which $x \in R_3 \subseteq R_1 \cap R_2$. Therefore we have that (2) is satisfied and can conclude that $\mathcal{R}$ is basis for some topology on X. Now let $F \in \mathcal{R}$; say, $F = (a_1, b_1) \times \cdots \times (a_n, b_n)$. Then we want to show that $F = \operatorname{Int}_X(F)$ with respect to the Euclidean metric. We know that all open intervals $x \in (a_i, b_i)$ are open with respect to the Euclidean metric, and so there is some $\epsilon_i > 0$ for all i such that $x_i \in (x - \epsilon_i, x + \epsilon_i) \subseteq (a_i, b_i)$, and so following this argumentation we get that $(x_1, \ldots, x_n) \in (x_1 - \epsilon, x_1 + \epsilon_1) \times (x_2 - \epsilon, x_2 + \epsilon_2) \times \cdots \times (x_n - \epsilon, x_n + \epsilon) \subseteq F$. Hence all points of F are interior points of F with respect to X (the opposite inclusion is immediate by definition). Furthermore, consider $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and suppose $r > 0$. We claim that $B_X(x, r)$ can be written as a union of elements of $\mathcal{R}$. As explained Exercise 2.10, we have that the square metric d^∞ (which has "open balls" as open rectangles as described in Example 2.3. of pages $45-46$ in the Course Notes) and d_E are equivalent on $\mathbb{R}^n$, and so in particular $d_E(p, q) \leq \sqrt{n} d_\infty(p, q)$ for any points $p, q \in X$, and so $\mathfrak{R}_X(x, \frac{r}{\sqrt{n}}) \subseteq B_X(x, r)$ which implies that $\bigcup_{x \in X} \mathfrak{R}_X(x, \frac{r}{\sqrt{n}}) \subseteq B_X(x, r)$. Using Exercise 2.10 again, we get an equivalence in the last line so that we can write $\bigcup_{x \in X} \mathfrak{R}_X(x, \frac{r}{\sqrt{n}}) = B_X(x, r)$; that is, the we can write an open ball $B_X(x, r)$ as a union of elements of $\mathcal{R}$, so the claim follows. Therefore, by Proposition 3.6, we must have that $\mathcal{R}$ is in fact the basis for the standard topology on $\mathbb{R}^n$ as $\mathcal{T} = \mathcal{R}$, where $\mathcal{T}$ denotes the standard topology on $\mathbb{R}^n$. $\square$

Chapter 4: Exercise 1.1.

Let E_1, E_2 be subsets of a metric space (X, d). Prove that

$$\mathrm{Lim}_X(E_1 \cup E_2) = \mathrm{Lim}_X(E_1) \cup \mathrm{Lim}_X(E_2).$$

Proof. ($\subseteq$) Let $x \in \mathrm{Lim}_X(E_1 \cup E_2)$. Then $x \in X$ and for any open neighborhood $x \in W \subseteq X$, we have that $W \cap ((E_1 \cup E_2) \setminus \{x\})$ is nonempty. Now take $\ell \in W \cap ((E_1 \cup E_2) \setminus \{x\})$. So then $\ell \in W$ and $\ell \in (E_1 \cup E_2) \setminus \{x\}$, i.e. $\ell \in E_1 \cup E_2$ and $\ell \neq x$. WLOG, suppose that $\ell \in E_1$. Then $\ell \in E_1 \setminus \{x\}$, and so as $\ell \in W$ as well we get that $\ell \in W \cap E_1 \setminus \{x\}$; that is, W intersects with $E_1 \setminus \{x\}$. Hence $x \in \mathrm{Lim}_X(E_1)$.
($\supseteq$) Let $x \in \mathrm{Lim}_X(E_1) \cup \mathrm{Lim}_X(E_2)$. WLOG, let $x \in \mathrm{Lim}_X(E_1)$. Then $x \in X$ and $E_1 \setminus \{x\}$ intersects with any open neighborhood of x, say, W. So we can say that there is some $\ell \in W \cap (E_1 \setminus \{x\})$. Thus $\ell \in W$ and $\ell \in E_1 \setminus \{x\}$ (i.e. $\ell \in E_1$ and $\ell \neq x$). Trivially, $\ell \in E_1 \cup E_2$. So then $\ell \in (E_1 \cup E_2) \setminus \{x\}$ as $\ell \neq x$. As $\ell \in W$ as well, then $\ell \in W \cap ((E_1 \cup E_2) \setminus \{x\})$. Therefore $x \in \mathrm{Lim}_X(E_1 \cup E_2)$. $\square$

Chapter 4: Exercise 1.2.

Let X be a metric space, and assume $E \subseteq Y \subseteq X$. Prove that

$$\mathrm{Lim}_Y(E) = \mathrm{Lim}_X(E) \cap Y$$

Proof. ($\subseteq$) Let $p \in \mathrm{Lim}_Y(E)$. Then $p \in Y$ and for any open neighborhood $p \in V \subseteq Y$, we have that $V \cap (E \setminus \{p\})$ intersect. As $Y \subseteq X$, then p has an open neighborhood $V \subseteq X$, and so $p \in \mathrm{Lim}_X(E)$ since $V \cap (E \setminus \{p\})$ intersect. Thus $p \in Y$ and $p \in \mathrm{Lim}_X(E)$, and so $p \in \mathrm{Lim}_X(E) \cap Y$.
($\supseteq$) Suppose that $p \in \mathrm{Lim}_X(E)$ and $p \in Y$. Then $p \in X$ and for any open neighborhood $p \in V \subseteq X$, we have that $V \cap (E \setminus \{p\})$ intersect. Now consider $W = V \cap Y$. Clearly $p \in W$, and by Theorem 2.13, W is open in Y since $W \subseteq Y \subseteq X$ and V is an open set of X. Thus $p \in W$ is an open neighborhood with respect to Y. Suppose $\ell \in V \cap (E \setminus \{p\})$. Then $\ell \in V$ and $\ell \in E \setminus \{p\}$. But as $V \subseteq W$, then $\ell \in W$, so $\ell \in W \cap (E \setminus \{p\})$. That is, W and $E \setminus \{p\}$ interesect. Thus $p \in \mathrm{Lim}_Y(E)$. $\square$

Chapter 4: Exercise 1.3.

If $(X, \mathcal{T})$ is a topological space and E is a subset of X, we say that x is a limit point of E with respect to X if every neighborhood of x in X (that is, every $U \in \mathcal{T}$ such that $x \in U$) intersects $E \setminus \{x\}$.

(a) Suppose $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X. Show that x is a limit point of E with respect to X if and only if every $B \in \mathcal{B}$ containing x intersects $E \setminus \{x\}$.

(b) Show that if E is any subset of $\mathbb{R}$ which is not bounded above (with respect to the usual order relation on $\mathbb{R}$), then $+\infty$ is a limit point of E with respect to $\overline{\mathbb{R}}$ (in its standard topology).

Proof. Suppose $(X, \mathcal{T})$ is a topological space, and E is a subset of X.
(a) Suppose $\mathcal{B}$ is basis for a topology $\mathcal{T}$, i.e. every element in $\mathcal{T}$ can be written as union of elements of $\mathcal{B} \subseteq \mathcal{T}$.
($\Rightarrow$) Assume that x is a limit point of E with respect to X. Then $x \in X$ and for any open neighborhood $x \in W \subseteq X$, we have that $W \cap (E \setminus \{x\}) \neq \varnothing$. As W is open, then it can we written as a union of basis open sets of $\mathcal{B}$, say, $W = \bigcup_i B_i$, where $B_i \in \mathcal{B}$. Thus $(\bigcup_i B_i) \cap (E \setminus \{x\}) \neq \varnothing$, so take ℓ in the intersection. Then $\ell \in \bigcup_i B_i$ and $\ell \in E \setminus \{x\}$. So $\ell \in \bigcup_i B_i$ implies that $\ell \in B_\alpha$ for some $B_\alpha \in \mathcal{B}$. Hence $\ell \in B_\alpha \cap (E \setminus \{x\})$. Therefore the forward direction claim follows.
($\Leftarrow$) Assume that for every $x \in B \in \mathcal{B}$ intersects $E \setminus \{x\}$. Then, in any case, we can construct an open set $L = \bigcup_j B_j$ where each $B_j \in \mathcal{B}$, as $\mathcal{B}$ is a basis. Now, as every $B_\beta \cap (E \setminus \{x\}) \neq \varnothing$ by hypothesis, where $B_\beta \in \mathcal{B}$, we have some element, say, ℓ in the intersection. So then $\ell \in B_\beta$ and $\ell \in E \setminus \{x\}$. Hence $\ell \in L = \bigcup_j B_j$ as $\ell \in B_\beta \in \mathcal{B}$. Therefore $\ell \in L \cap (E \setminus \{x\})$ since L is indeed an open neighborhood of x as $x \in B_\beta$ which intersects with $E \setminus \{x\}$; that is, x is a limit point of E with respect to X.
(b) Suppose that $E \subseteq \mathbb{R}$, and E is not bounded above. Then we must show that $+\infty \in \operatorname{Lim}_{\overline{\mathbb{R}}}(E)$. Firstly, $+\infty \in \overline{\mathbb{R}}$, by definition, so it remains to show that for any open neighborhood $+\infty \in U \subseteq \overline{\mathbb{R}}$, we have that $U \cap E \setminus \{+\infty\}$ intersect. As U is an open neighborhood of $+\infty$, then $U = (a, +\infty)$ for some $a \in \mathbb{R}$. And as E is not bounded above, then $E = (q, +\infty)$ or $E = [q, +\infty)$ for some $q \in \mathbb{R}$. Consider $E = (q, +\infty)$, and $a > q$, then $U = (a, +\infty)$ intersects with $(q, +\infty) \setminus \{+\infty\} = (q, \ell)$ clearly; and if $a < q$, then the conclusion is the same. Similarly, if $E = [q, +\infty)$, and WLOG $a < q$, then $E = [q, +\infty) \setminus \{+\infty\}$ intersects with $(a, +\infty)$. $\square$

Chapter 4: Exercise 1.4.

Let (X, d) be a metric space, and assume $Y \subseteq X$. Let $(x_n)_{n=1}^\infty$ be a sequence in Y and let x be a point of X. Prove that the following two statements are equivalent:

(1) $x_n \to x$ in X, and $x \in Y$.

(2) $x_n \to x$ in Y.

Proof. (1) $\Rightarrow$ (2) Suppose that $x_n \to x$ in X and $x \in Y$. Then, as $(x_n)_{n=1}^\infty$ is a sequence in Y and x is a point of X, and we assume $x_n \to x$ in X, then for every open neighborhood $x \in V \subseteq X$ (so $x \in V \cap Y \subseteq X$) there is some $N \in \mathbb{N}$ such that $n \geq N$ implies that $x_n \in V$ (so $x_n \in V \cap Y$). But as $x \in Y$, then $x \in Y \cap V \subseteq Y$ so $V \cap Y$ is open in Y by Theorem 2.13 and so $x_n \in V \cap Y \subseteq Y$. Thus $x_n \to x$ in Y.
(2) $\Rightarrow$ (1) Suppose $x_n \to x$ in Y. Then every open neighborhood $x \in W \subseteq Y$ there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in W$. So $x \in Y$. But as $W \subseteq Y \subseteq X$, then W is open in Y if and only $W = E \cap Y$ for some open set E of X. So then we have some E open in X such that $x \in W = E \cap Y \subseteq X$ is an open neighborhood of x; thus $x \in E$ and $x_n \in E$ as $W \subseteq E$, an open neighborhood of x, and $x_n \to x$ in X. $\square$

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