MATH 425A HW9, OCT. 28, 6PM

JUAN SERRATOS

1. Chapter 4

Chapter 4; §6.1: Exercise 6.1.

Let $\mathcal{A}$ be a collection of convex subsets of a real vector space V. Show that $B := \bigcap_{A \in \mathcal{A}} A$ is convex.

Proof. Clearly, $\bigcap_{A \in \mathcal{A}} A$ is still a subset of V as for all $A \in \mathcal{A}$ we have A is a subset of V. Now let $a, b \in \bigcap_{A \in \mathcal{A}} A$ and $t \in \mathbb{R}$ such that $0 < t < 1$. Then we have that $(1-t)a + tb \in A$ for all $A \in \mathcal{A}$ by construction of $\mathcal{A}$. Thus we have that $(1-t)a + tb \in \bigcap_{A \in \mathcal{A}} A$. $\square$

Chapter 4; §6.2: Exercise 6.2.

Let (X, d) be a metric space and let A and B be disjoint subsets of X. Prove that if A and B are both open in X, then A and B are separated.

Proof. Suppose that A and B are both open in X. WLOG, suppose that $\mathrm{cl}_X(A) \cap B \neq \varnothing$. Then we have some $\ell \in \mathrm{cl}_X(A) \cap B$; thus we have $\ell \in B$ and for any open neighborhood of $\ell \in U$ then $U \cap A \neq \varnothing$. But as B is itself an open neighborhood of ℓ as ℓ is in the intersection, then $B \cap A \neq \varnothing$; this contradicts our assumption that A and B are disjoint. Therefore we have that $\mathrm{cl}_X(A) \cap B = \varnothing$. To show that $\mathrm{cl}_X(B) \cap A = \varnothing$ follows nearly the exact same argument. Hence we have that $\mathrm{cl}_X(A) \cap B = \varnothing$ and $\mathrm{cl}_X(B) \cap A = \varnothing$, i.e. A and B are separated. $\square$

Chapter 4; §6.2: Exercise 6.3.

Let E be a connected subset of a metric space (X, d). Show that $\overline{E}$ is connected.

Proof. Suppose we can write $\overline{E} = A \cup B$ where $A \cap \overline{B} = \varnothing = \overline{A} \cap B$. We can write $E = (A \cap E) \cup (B \cap E)$, and we still have that $(A \cap E)$ and $(B \cap E)$ are separated. Thus as E is connected, either $A \cap E = \varnothing$ or $B \cap E = \varnothing$. WLOG, suppose $A \cap E = \varnothing$. Then $E = B \cap E$, which implies that $E \subseteq B$ and so $\overline{E} \subseteq \overline{B}$. So as $A \cap \overline{B} = \varnothing$, then $A \cap \overline{E} = \varnothing$. Thus if we let $\ell \in \overline{E}$, then $\ell \in B$ (as if $\ell \in A$ then we have a contradiction), and hence $\overline{E} = B$, which means that $\overline{E}$ is connected. $\square$

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Department of Mathematics, University of Southern California.

Chapter 4; §6.2: Exercise 6.4.

Let (X, d) be a metric space, and let $\mathcal{C}$ be a collection of connected subsets of X. Assume $A = \bigcap_{C \in \mathcal{C}} C$ is nonempty. Show that $B = \bigcup_{C \in \mathcal{C}} C$ is connected.

Proof. By contradiction, suppose that B is not connected. That is, suppose we can write $B = E \cup F$ where $\overline{E} \cap F = \overline{F} \cap E = \varnothing$. Then for all C_k in $\mathcal{C}$, we have $C_k \subseteq E \cup F$. By Theorem 6.4, we have that $C_k \subseteq E$ or $C_k \subseteq F$. WLOG, let $C_k \subseteq E$. Then $C_k = (C_k \cap E) \cup (C_k \cap F)$, meaning that $C_k \cap F = \varnothing$ and $C_k = C_k \cap E$. As we assumed $B = E \cup F$ with both nonempty, then we have $C_k \subseteq E$ and $C_l \subseteq F$ where $k \neq l$. But this additionally means that $C_k \cap C_l = \varnothing$ by Theorem 6.4, which is a contradiction. Thus B is connected. □

Chapter 4; §6.2: Exercise 6.5.

Let $X = \mathbb{R}^2$. Give an example of a connected subset E of X, such that $\operatorname{Int}_X(E)$ is not connected. Prove both that your set E is connected and that its interior is not. (Hint: Consider the union of two convex sets joined at a point. You may assume without proof the fact that convexity implies connectedness in $\mathbb{R}^2$.)

Proof. Consider $\mathcal{M}_1 = \{(x, y) \in \mathbb{R}^2 \colon x \geq 0, y \geq 0\}$ and $\mathcal{M}_2 = \{(x, y) \in \mathbb{R}^2 \colon x \leq 0, y \leq 0\}$. We glue these two $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \subseteq \mathbb{R}^2$. Firstly, let $(a, b), (x, y) \in \mathcal{M}_1$ where $\ell_1 = (a, b)$ and $\ell_2 = (x, y)$, and let $t \in \mathbb{R}$ such that $0 < t < 1$. Then $(1 - t)\ell_1 + t\ell_2$ will clearly be always be bigger than or equal to 0 as $t - 1 > 0$ and $\ell_1, \ell_2 \geq 0$. So $\mathcal{M}_1$ is convex. For $\mathcal{M}_2$, let $\alpha_1 = (a, b)$ and $\alpha_2 = (x, y)$ be in $\mathcal{M}_2$ with $t \in \mathbb{R}$ such that $0 < t < 1$. Then $(1 - t)\alpha_1 + t\alpha_2$ is always less than or equal to zero clearly. So, by Exercise 6.4, $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ is connected since convexity implies connected in $\mathbb{R}^2$. We claim that $\operatorname{Int}_{\mathbb{R}^2}(\mathcal{M}) = \operatorname{Int}_{\mathbb{R}^2}(\mathcal{M}_1) \cup \operatorname{Int}_{\mathbb{R}^2}(\mathcal{M}_2)$. Let $x \in \operatorname{Int}_{\mathbb{R}^2}(\mathcal{M})$. Then $x \in \mathcal{M}$ and there exists an open ball $B_{\mathbb{R}^2}(x, \epsilon) \subseteq \mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$. We know that, as $\mathcal{M}_1$ and $\mathcal{M}_2$ are tangent to one another, if $B_{\mathbb{R}^2}(x, \epsilon) \subseteq \mathcal{M}_1 \cap \mathcal{M}_2 = \{(0, 0)\}$, then $B_{\mathbb{R}^2}(x, \epsilon) = \{(0, 0)\}$. So $B_{\mathbb{R}^2}(x, \epsilon) \subseteq \mathcal{M}_1$ (and also $B_{\mathbb{R}^2}(x, \epsilon) \subseteq \mathcal{M}_2$). Thus we have the forward inclusion since (if we had the only other case, the trivial case) $B_{\mathbb{R}^2}(x, \epsilon) \subseteq \mathcal{M}_1 \cup \mathcal{M}_2$ where $B_{\mathbb{R}^2}(x, \epsilon) \subseteq \mathcal{M}_1$ (or $B_{\mathbb{R}^2}(x, \epsilon) \subseteq \mathcal{M}_2$), then $x \in \operatorname{Int}_{\mathbb{R}^2}(\mathcal{M}_1)$. Now for the backwards direction, let $\ell \in \operatorname{Int}_{\mathbb{R}^2}(\mathcal{M}_1) \cup \operatorname{Int}_{\mathbb{R}^2}(\mathcal{M}_2)$. WLOG, let $\ell \in \operatorname{Int}_{\mathbb{R}^2}(\mathcal{M}_1)$. Then we have $\ell \in \mathcal{M}_1$ such that there is an open ϵ-neighborhood $B_{\mathbb{R}^2}(\ell, \epsilon) \subseteq \mathcal{M}_1$, but as $\mathcal{M}_1 \subseteq \mathcal{M}_1 \cup \mathcal{M}_2 = \mathcal{M}$, then we have that $\ell \in \mathcal{M}$ and $B_{\mathbb{R}^2}(\ell, \epsilon) \subseteq \mathcal{M}$. Thus $\ell \in \operatorname{Int}_{\mathbb{R}^2}(\mathcal{M})$. Therefore we have the equality of sets. Lastly, $\overline{\operatorname{Int}_{\mathbb{R}^2}(\mathcal{M}_1)} \cap \operatorname{Int}_{\mathbb{R}^2}(\mathcal{M}_2) = \mathcal{M}_1 \cap \operatorname{Int}_{\mathbb{R}^2}(\mathcal{M}_2) = \varnothing$, and similarly, $\overline{\operatorname{Int}_{\mathbb{R}^2}(\mathcal{M}_2)} \cap \operatorname{Int}_{\mathbb{R}^2}(\mathcal{M}_1) = \mathcal{M}_2 \cap \operatorname{Int}_{\mathbb{R}^2}(\mathcal{M}_1) = \varnothing$. Thus we have written $\operatorname{Int}_{\mathbb{R}^2}(\mathcal{M})$ as the union of two disjoint, nonempty, separated sets, and so $\operatorname{Int}_{\mathbb{R}^2}(\mathcal{M})$ is not connected. □

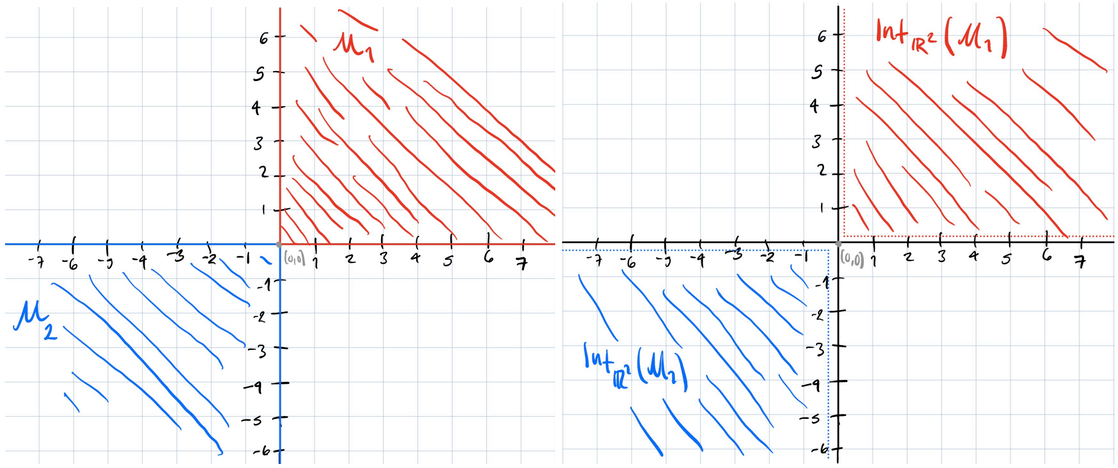

FIGURE 1. The image to the left depicts $\mathcal{M}_1$ and $\mathcal{M}_2$ for Exercise 6.5, and image to the left depicts the interiors of the sets.

2. CHAPTER 5

Chapter 5; §1.1: Exercise 1.1.

Let (X, d_X) and (Y, d_Y) be metric spaces, and let E be a subset of X. Let $f\colon E \to Y$ be a function, and let p be a limit point of E in X. Prove that $f(x) \to q$ as $x \to p$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $x \in E$ and $0 < d_X(x,p) < \delta$ imply together that $d_Y(f(x), q) < \epsilon$.

Proof. ($\Rightarrow$) Suppose that $\lim_{x\to p} f(x) = q$. Consider the open neighborhood $V = B_Y(q, \epsilon)$, where $\epsilon > 0$. Then we have some open neighborhood $p \in B_X(p, \delta) = U$ of X. So take $x \in B_X(p,\delta) \cap E \setminus \{p\}$. Then $x \in E$ and $x \neq p$ with $d_X(x,p) < \delta$ (and as $x \neq p$, then we have $0 < d_X(x,p) < \delta$). Moreover we have that $f(x) \in V = B_Y(q,\epsilon)$, and so $d_Y(f(x), q) < \epsilon$. Thus the forward direction is established. ($\Leftarrow$) Suppose that for any $\epsilon > 0$ we have some $\delta > 0$ such that $x \in E$ and $0 < d_X(x,p) < \delta$ implies that $d_Y(f(x), q) < \epsilon$. Note that as $0 < d_X(x,p)$ for all such $x \in E$ then $x \neq p$. WLOG, consider the open neighborhood $B_Y(q,\epsilon)$ with $\epsilon > 0$ as assumed. Then by hypothesis we have an induced open neighborhood $B_X(p,\delta)$ where $0 < d_X(x,p) < \delta$ for all $x \in E$. As $x \neq p$, and $p \in \text{Lim}_X(E)$, then $B_X(p,\delta) \cap E \setminus \{p\} \neq \varnothing$. Lastly, by assumption, we have that $d_Y(f(x), q) < \epsilon$, which means that $f(x) \in B_Y(q, \epsilon)$. Hence the backwards assumption is established and we are done. □

Chapter 5; §2.1: Exercise 2.1.

Let (X, d_X) and (Y, d_Y) be metric spaces; let $f\colon X \to Y$ be a function. Prove that f is continuous at $p \in X$ if and only if $\epsilon > 0$, there exists $\delta > 0$ such that $x \in B_X(p,\delta)$ implies $f(x) \in B_Y(f(p), \epsilon)$.

Proof. ($\Rightarrow$) Suppose that f is continuous at $p \in X$. Then for the neighborhood $B_Y(f(p), \epsilon)$, where $\epsilon > 0$, we have that there exists a neighborhood $B_X(p,\delta)$ such that $x \in B_X(p,\delta)$, i.e. $x \in X$ such that $d_X(p,x) < \delta$, gives us that $f(x) \in B_Y(f(p),\epsilon)$. Thus the forward direction follows. ($\Leftarrow$) Let $\epsilon > 0$. Consider the neighborhood $f(p) \in V = B_Y(f(p), \epsilon) \subseteq Y$. Then we have some $\delta > 0$ such that for x being in the neighborhood $U = B_X(p,\delta)$, we have $f(x) \in V$. Thus, by definition, we have that f is continuous at the point $p \in X$. □

Chapter 5; §2.1: Exercise 2.2.

Assume $f\colon \mathbb{R} \to \mathbb{R}$ is a function satisfying $\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$, for all $x \in \mathbb{R}$. Does it follow that f must be continuous? If so, give a proof; if not, give a counterexample.

Proof. We construct a function that satisfies this property: consider the function $\varphi\colon \mathbb{R} \to \mathbb{R}$, where

$$\varphi(x) = \begin{cases} 0 & \text{if } x > 0 \text{ or } x < 0 \\ 1 & \text{if } x = 0x \end{cases}$$

Firstly, we can check that φ satisfies a simple property: We claim that $\varphi(x) = \varphi(-x)$ for $x \in \mathbb{R}$. WLOG, if $x > 0$, then $\varphi(x) = 0$, but multiplying the inequality by -1 gives us $-x < 0$ so $\varphi(-x) = 0$; thus $\varphi(x) = \varphi(-x)$ for $x \in \mathbb{R} \setminus \{0\}$. If $x = 0 \in \mathbb{R}$, then $\varphi(x) = \varphi(0) = \varphi(-0) = \varphi(-x) = 0$. Now fix $x = 0$, then $\lim_{h\to 0}(\varphi(x+h) - \varphi(x-h)) = \lim_{h\to 0}(\varphi(h) - \varphi(-h)) = \lim_{h\to 0}(\varphi(h) - \varphi(h)) = \lim_{h\to 0}(0-0) = 0$. However, if we suppose that φ is continuous at $x = 0$, then $\lim_{h\to 0} \varphi(h) = \varphi(0)$, but $\lim_{h\to 0} \varphi(h) = 0$ and $\varphi(0) = 1$, which is contradictory. Thus φ cannot be continuous. □

Chapter 5; §2.1: Exercise 2.3.

Let (X, d_X) and (Y, d_Y) be metric spaces and $f\colon X \to Y$ a function.

(a) Show that f is contiunous if and only if $f^{-1}(C)$ is closed in X whenever C is closed in Y.
(b) Show that $f\colon X \to Y$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X.
(c) Consider the (continuous) function $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = \frac{1}{1+x^2}$. Give an example of a subset A of $\mathbb{R}$ such that $g(\overline{A}) \neq \overline{g(A)}$,

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Proof. (a) ($\Rightarrow$) Suppose f is continuous, and let C be a closed subset in Y. Then we have that $Y \setminus C$ is open in Y, so $f^{-1}(Y \setminus C)$ is open in X. Now, by Exercise 3.3.in Chapter 1 §3.2 we have that $f^{-1}(Y \setminus C) = f^{-1}(Y) \setminus f^{-1}(C)$, which is thus open in X. Moreover, clearly, we have that $f^{-1}(Y) = \{x \in X\colon f(x) \in Y\} = X$, and so $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$ is open in X. And $f^{-1}(C)$ is closed in X if and only if $X \setminus f^{-1}(C)$ is open in X. Thus we're done.
($\Leftarrow$) Suppose that whenever C is closed in Y, then $f^{-1}(C)$ is closed in X. But U is an open set of Y if and only if $Y \setminus U$ is closed in Y. Furthermore, $f^{-1}(U)$ is open in X if and only if $X \setminus f^{-1}(U)$ is closed in X. Clearly we have the statement the proposition as $Y \setminus U$ is closed in Y and $X \setminus f^{-1}(U)$ is closed in X by assumption.
(b) ($\Rightarrow$) Suppose f is continuous, and let $A \subseteq X$. Let $f(a) \in f(\overline{A})$. Then we have that $f(a) \in f(A)$, and so as f is continuous then for any, wlog, open neighborhood V of $f(a)$ we have some open neighborhood of $a \in U$ such that $x \in U$ implies $f(x) \in V$. As $a \in \overline{A}$, then for any open neighborhood $a \in U$, we have $U \cap A \neq \varnothing$; let ℓ be in this intersection. So $\ell \in U$ and $\ell \in A$, which means that $f(\ell) \in f(A)$ and $f(\ell) \in V$, i.e. $f(A) \cap V \neq \varnothing$. This proves that $f(a) \in \overline{f(A)}$. ($\Leftarrow$) Suppose that for any subset $A \subseteq X$, we have $f(\overline{A}) \subseteq \overline{f(A)}$. Let S be a closed set of Y. Then $f^{-1}(S) \subseteq X$, and so by assumption we have $f(\overline{f^{-1}(S)}) \subseteq \overline{f(f(^{-1}(S))} \subseteq \overline{S}$. Moreover, as S is closed, then $S = \overline{S}$ and $f(\overline{f^{-1}(S)}) \subseteq S$. Now as a preimage preserves inclusions implies $\overline{f^{-1}(S)} \subseteq f^{-1}(S)$. So as $\overline{f^{-1}(S)} = f^{-1}(S) \cup (f^{-1}(S))'$, we have $f^{-1}(S) = \overline{f^{-1}(S)}$. Thus $f^{-1}(S)$ is closed in X. Hence f is continuous by part (a) above.
(c) Consider the set $A = [1, \infty)$. Then $\overline{A} = [1, \infty) = A$. So $g(\overline{A}) = \left(0, \frac{1}{2}\right] = g(A)$, while $\overline{g(A)} = \overline{\left(0, \frac{1}{2}\right]} = \left[0, \frac{1}{2}\right]$. Hence $g(\overline{A}) \neq \overline{g(A)}$. □

Chapter 5; §2.1: Exercise 2.4.

Let (X, d_X) and (Y, d_Y) be metric spaces, and let f and g be continuous functions from X to Y. Assume E is a dense subset of X.

(a) Prove that $f(E)$ is dense in $f(X)$. (Hint: Use Exercise 1.12 in Chapter 4 and Exercise 2.3 above.)

(b) Prove that if $f(x) = g(x)$ for all $x \in E$, then $f(x) = g(x)$ for all $x \in X$.

Proof. (a) As $E \subseteq X$ then $f(E) \subseteq f(X) \subseteq Y$, and so $f(E)$ is dense in $f(X)$ if and only if $f(X) \subseteq \overline{f(E)}$ (Exercise 1.12.). As f is continuous then we have that $f(\overline{E}) \subseteq \overline{f(E)}$ (Exercise 2.3 (b)), but as $\overline{E}$ is dense in X, then $\overline{E} = X$, so $f(X) \subseteq \overline{f(E)}$. Therefore we have that $f(E)$ is dense in $f(X)$.

(b) Let $\ell \in \overline{E}$. Then there is a sequence $\{l_n\}_{n=1}^{\infty}$ of E such that $l_n \to \ell$ as $n \to \infty$. As f is continuous, then $\lim_{n\to\infty} f(l_n) = f(\ell)$ and $\lim_{n\to\infty} g(l_n) = g(\ell)$. And as $l_n \in E$ for all $n \in \mathbb{N}$, then we have $f(\ell) = g(\ell)$ by assumption. But as E is dense in X, then we in fact have that for all $q \in X = \overline{E}$ there is a sequence (q_n) in E where $q_n \to q$ and so $f(q) = g(q)$, as the work showed before. $\square$

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90007