MATH 425A HW6, OCT. 7, 6PM

JUAN SERRATOS

1. Chapter 2.

Chapter 2 §2.4: Exercise 2.9.

Suppose X a finite, nonempty, set and suppose d be a metric on X. Let $\mathcal{T}$ denote the topology generated by d. Show that $\mathcal{T} = \mathcal{P}(X)$. Conclude that any metric on X is equivalent to the discrete metric. (Hint: To show that $\mathcal{T} = \mathcal{P}(X)$, start by proving that $\{x\} = B_{(X,d)}(x, r_x)$ for some sufficiently small r_x, for each $x \in X$.)

Proof. ($\subseteq$) As X is finite, then it has finitely many subsets. Recall that the topology on a metric space is the collection of all open sets on the metric space X generated by the corresponding metric. Let $E \in \mathcal{T}$, i.e. E is an open set in X. Then obviously E must be subset of X by definition of an open set, and so $E \in \mathcal{P}(X)$.
($\supseteq$) Now suppose that $L \in \mathcal{P}(X)$, i.e. $L \subseteq X$. Then, as X is finite, then there are finitely many points in L; we can enumerate L so that $L = \{p_1, p_2, \ldots, p_n\}$. As X is itself open, then $L \subseteq X = \mathrm{Int}_X(X)$. Then for every point $p_i \in L$, $1 \leq i \leq n$, there is some ball $B_X(p_i, r_i)$ with $r_i > 0$ such that $B_X(p_i, r_i) \subseteq X$. Thus it suffices to show that for any $p_i \in L$,

$$L = \bigcup_{i=1}^{n} B_X(p_i, r_i).$$

Suppose that $q \in L$. Then there is some open ball $B_X(q, r_q)$ with $r_q > 0$ such that $B_X(q, r_q) \subseteq X$. Thus we see that $q \in \bigcup_{i=1}^{n} B_X(p_i, r_i)$. Now suppose that $\ell \in \bigcup_{i=1}^{n} B_X(p_i, r_i)$. Then $\ell \in B_X(p_\alpha, r_\alpha) \subseteq X$ for some α. But $B_X(p_\alpha, r_\alpha)$ is constructed by some $p_\alpha \in L$, and $\{p_\alpha\} = B_X(p_\alpha, r_\alpha)$ for some sufficiently small r_α. Then $\ell \in \{p_\alpha\}$, i.e. $\ell = p_\alpha \in L$. Thus the equality of sets holds true, and as the right side is a union of open balls in X, then L is open in X. Hence $L \in \mathcal{T}$. □

Chapter 2 §2.4: Exercise 2.10.

Prove that the Euclidean metric and the square metric are equivalent on $\mathbb{R}^n$.

Proof. Recall that the usual Eucidean norm in $\mathbb{R}^n$ is given taken by comparing two points $p = (a_1, \ldots, a_n)$ and $q = (b_1, \ldots, b_n)$ in $\mathbb{R}^n$ is given by $d(p, q) = \sqrt{(b_1 - a_1)^2 + \cdots + (b_n - a_n)^2}$. And the square norm is given by $d_u(p, q) = \max\{|b_1 - a_1, \ldots, |b_n - a_n|\}$. □

Date: October 3, 2022
Department of Mathematics, University of Southern California.

Chapter 2 §3.2: Exercise 3.1.

Prove Proposition 3.4.

Proposition 3.4.

Let X be a set, and let $\mathcal{B}$ be a collection of subsets of X, which has the following properties:

(1) Every $x \in X$ is contained in at least one element B of $\mathcal{B}$.

(2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Then the following collection $\mathcal{T}$ is a topology on X:

$$\mathcal{T} = \left\{ U \in \mathcal{P}(X) \colon U = \bigcup_{B \in \mathcal{A}} B \text{ for some subcollection } \mathcal{A} \subseteq \mathcal{B} \right\}.$$

and $\mathcal{B}$ is a basis for $\mathcal{T}$. On the other hand, if $\mathcal{T}$ is a topology on X and $\mathcal{B}$ is a subcollection of $\mathcal{T}$ such that the preceding equation holds, then $\mathcal{B}$ must satisfy both properties (1) and (2).

Proof. ($\Rightarrow$) For the forward direction, suppose that for all $x \in X$ is contained in some $B_\alpha \in \mathcal{B}$, and if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there exists another $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. We want to show that the space elements of $\mathcal{T}$, as presented in Proposition 3.4, do indeed satisfy the topology axioms, and that $\mathcal{B}$ is a basis for $\mathcal{T}$. Firstly, it is easy to see that $\varnothing \in \mathcal{T}$ since $\varnothing \in \mathcal{P}(X)$ and $\varnothing$ is in the subcollection $\mathcal{A} \subseteq \mathcal{B}$ as it's a collection of sets, so $\varnothing$ is vacuously a union of sets in the subcollection $\mathcal{A}$. Now $X \in \mathcal{P}(X)$, and we claim that $X = \bigcup_{W \in \mathcal{A}} W$ for a subcollection $\mathcal{A} \subseteq \mathcal{B}$. If we take $x \in X$, then we have that $x \in B_\alpha$ for some $B_\alpha \in \mathcal{B}$. So then if we take $\mathcal{A}$ to be the subcollection which has $B_\alpha \in \mathcal{A}$, then $x \in \bigcup_{W \in \mathcal{A}} W$ as $x \in B_\alpha \in \mathcal{A}$. For the reverse inclusion, let $\bigcup_{W \in \mathcal{A}} W$ be such that $\mathcal{A}$ has the element $B_\beta \in \mathcal{B}$ which has every $x \in X$. Then clearly $\bigcup_{W \in \mathcal{A}} \subseteq X$. Hence the claim holds and $X \in \mathcal{T}$. Let $\mathfrak{U} \subseteq \mathcal{T}$. We must show that $\bigcup_{U \in \mathfrak{U}} U \in \mathcal{T}$. As each $U \in \mathfrak{U} \subseteq \mathcal{T}$, then $U \in \mathcal{P}(X)$ such that $U = \bigcup_{T \in \mathcal{A}} T$ for some subcollection $\mathcal{A} \subseteq \mathcal{B}$. So then for each $V \in \mathfrak{U}$ there is a corresponding $V = \bigcup_{T \in \mathcal{A}_V} T$, where $\mathcal{A}_V$ denotes the associated subcollection of $\mathcal{B}$. Now, for every $U \in \mathfrak{U}$, then we can denote the union of all corresponding subcollections $\mathcal{A}_U$ as $\mathfrak{A}$, i.e. $\mathfrak{A} = \bigcup_{U \in \mathfrak{U}} \mathcal{A}_U$ where $\mathcal{A}_U$ is a subcollection collection of $\mathcal{B}$ and $U = \bigcup_{T \in \mathcal{A}_U} T$ for all $U \in \mathfrak{U}$. Then

$$\bigcup_{U \in \mathfrak{U}} U = \bigcup \left(\bigcup_{T \in \mathcal{A}_U} T \right) = \bigcup_{E \in \mathfrak{A} \subseteq \mathcal{B}} E.$$

Thus we have that $\bigcup_{U \in \mathfrak{U}} U \in \mathcal{T}$ as the preceding rewriting of the union shows and as all $U \in \mathfrak{U} \subseteq \mathcal{T}$ by assumption then $U \in \mathcal{P}(X)$ (i.e., $U \subseteq X$) so the union of all such U are contained in X once again, that is, in X's power set.

($\Leftarrow$) For the opposite direction, $\square$

Chapter 3: Exercise 1.1.

Let E_1, E_2 be subsets of a metric space (X, d). Prove that

$$\mathrm{Lim}_X(E_1 \cup E_2) = \mathrm{Lim}_X(E_1) \cup \mathrm{Lim}_X(E_2).$$

Proof. ($\subseteq$) Let $x \in \mathrm{Lim}_X(E_1 \cup E_2)$. Then $x \in X$ and for any open neighborhood $x \in W \subseteq X$, we have that $W \cap ((E_1 \cup E_2) \setminus \{x\})$ is nonempty. Now take $\ell \in W \cap ((E_1 \cup E_2) \setminus \{x\})$. So then $\ell \in W$ and $\ell \in (E_1 \cup E_2) \setminus \{x\}$, i.e. $\ell \in E_1 \cup E_2$ and $\ell \neq x$. WLOG, suppose that $\ell \in E_1$. Then $\ell \in E_1 \setminus \{x\}$, and so as $\ell \in W$ as well we get that $\ell \in W \cap E_1 \setminus \{x\}$; that is, W intersects with $E_1 \setminus \{x\}$. Hence $x \in \mathrm{Lim}_X(E_1)$.
($\supseteq$) Let $x \in \mathrm{Lim}_X(E_1) \cup \mathrm{Lim}_X(E_2)$. WLOG, let $x \in \mathrm{Lim}_X(E_1)$. Then $x \in X$ and $E_1 \setminus \{x\}$ intersects with any open neighborhood of x, say, W. So we can say that there is some $\ell \in W \cap (E_1 \setminus \{x\})$. Thus $\ell \in W$ and $\ell \in E_1 \setminus \{x\}$ (i.e. $\ell \in E_1$ and $\ell \neq x$). Trivially, $\ell \in E_1 \cup E_2$. So then $\ell \in (E_1 \cup E_2) \setminus \{x\}$ as $\ell \neq x$. As $\ell \in W$ as well, then $\ell \in W \cap ((E_1 \cup E_2) \setminus \{x\})$. Therefore $x \in \mathrm{Lim}_X(E_1 \cup E_2)$. $\square$

Chapter 3: Exercise 1.2.

Let X be a metric space, and assume $E \subseteq Y \subseteq X$. Prove that

$$\mathrm{Lim}_Y(E) = \mathrm{Lim}_X(E) \cap Y$$

Proof. ($\subseteq$) Let $p \in \mathrm{Lim}_Y(E)$. Then $p \in Y$ and for any open neighborhood $p \in V \subseteq Y$, we have that $V \cap (E \setminus \{p\})$ intersect. As $Y \subseteq X$, then p has an open neighborhood $V \subseteq X$, and so $p \in \mathrm{Lim}_X(E)$ since $V \cap (E \setminus \{p\})$ intersect. Thus $p \in Y$ and $p \in \mathrm{Lim}_X(E)$, and so $p \in \mathrm{Lim}_X(E) \cap Y$.
($\supseteq$) Suppose that $p \in \mathrm{Lim}_X(E)$ and $p \in Y$. Then $p \in X$ and for any open neighborhood $p \in V \subseteq X$, we have that $V \cap (E \setminus \{p\})$ intersect. Now V is open in Y, by Theorem 2.13, if and only $V = W \cap Y$ for some open $W \subseteq X$. We know that $p \in V \cap Y$, and $p \in \mathrm{Int}_X(V)$, so $\square$

Chapter 3: Exercise 1.3.

If $(X, \mathcal{T})$ is a topological space and E is a subset of X, we say that x is a limit point of E with respect to X if every neighborhood of x in X (that is, every $U \in \mathcal{T}$ such that $x \in U$) intersects $E \setminus \{x\}$.

(a) Suppose $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X. Show that x is a limit point of E with respect to X if and only if every $B \in \mathcal{B}$ containing x intersects $E \setminus \{x\}$.

(b) Show that if E is any subset of $\mathbb{R}$ which is not bounded above (with respect to the usual order relation on $\mathbb{R}$), then $+\infty$ is a limit point of E with respect to $\overline{\mathbb{R}}$ (in its standard topology).

Proof. Suppose $(X, \mathcal{T})$ is a topological space, and E is a subset of X.
(a) Suppose $\mathcal{B}$ is basis for a topology $\mathcal{T}$, i.e. every element in $\mathcal{T}$ can be written as union of elements of $\mathcal{B} \subseteq \mathcal{T}$.
($\Rightarrow$) Assume that x is a limit point of E with respect to X. Then $x \in X$ and for any open neighborhood $x \in W \subseteq X$, we have that $W \cap (E \setminus \{x\}) \neq \varnothing$. As W is open, then it can we written as a union of basis open sets of $\mathcal{B}$, say, $W = \bigcup_i B_i$, where $B_i \in \mathcal{B}$. Thus $(\bigcup_i B_i) \cap (E \setminus \{x\}) \neq \varnothing$, so take ℓ in the intersection. Then $\ell \in \bigcup_i B_i$ and $\ell \in E \setminus \{x\}$. So $\ell \in \bigcup_i B_i$ implies that $\ell \in B_\alpha$ for some $B_\alpha \in \mathcal{B}$. Hence $\ell \in B_\alpha \cap (E \setminus \{x\})$. Therefore the forward direction claim follows.
($\Leftarrow$) Assume that for every $x \in B \in \mathcal{B}$ intersects $E \setminus \{x\}$. Then, in any case, we can construct an open set $L = \bigcup_j B_j$ where each $B_j \in \mathcal{B}$, as $\mathcal{B}$ is a basis. Now, as every $B_\beta \cap (E \setminus \{x\}) \neq \varnothing$ by hypothesis, where $B_\beta \in \mathcal{B}$, we have some element, say, ℓ in the intersection. So then $\ell \in B_\beta$ and $\ell \in E \setminus \{x\}$. Hence $\ell \in L = \bigcup_j B_j$ as $\ell \in B_\beta \in \mathcal{B}$. Therefore $\ell \in L \cap (E \setminus \{x\})$ since L is indeed an open neighborhood of x as $x \in B_\beta$ which intersects with $E \setminus \{x\}$; that is, x is a limit point of E with respect to X.
(b) Suppose that $E \subseteq \mathbb{R}$, and E is not bounded above. Then we must show that $+\infty \in \mathrm{Lim}_{\overline{\mathbb{R}}}(E)$. Firstly, $+\infty \in \overline{\mathbb{R}}$, by definition, so it remains to show that for any open set $U \subseteq \overline{\mathbb{R}}$, we have that $U \cap E \setminus \{+\infty\}$ intersect. $\square$

Email address: `jserrato@usc.edu`

Department of Mathematics, University of Southern California, Los Angeles, CA 90007