FINAL REVIEW: DECEMBER 12TH, 11AM-1PM

Contents

1. Multiple Choice/True & False

Proposition 1.1. Let $A \subset \mathbb{R}$, and assume that every term in the sequence $\{x_n\}_{n\in\mathbb{N}}$ is an upper bound for A. Show that if $x_n \to x$, then x is also an upper bound for A.

Proof. (**True.**) We proceed by contradiction. Assume $x_n \to x$. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - x| < \varepsilon$. Suppose that x is not an upper bound for A, meaning that there exists $\alpha \in A$ such that $\alpha > x$. Now pick $\varepsilon = \alpha - x$. Then $|x_n - x| < \alpha - x$, which implies $x - \alpha < x_n - x < \alpha - x$, but then $x_n < \alpha$, and thus a contradiction. □

Proposition 1.2. Can there exist a continuous function $f\colon [0,1] \to \mathbb{R}$ such that f is not constant and all values are rational?

Proof. (**False.**) As $f\colon [0,1] \to \mathbb{R}$ is continuous, and $[0,1]$ is connected, then $E = f([0,1]) \subset \mathbb{R}$ is connected and so if $x, y \in E$ then $x < z < y$ implies $z \in E$. Let $\alpha, \beta \in f([0,1])$, and WLOG, let $\alpha < \beta$. As the irrational numbers (and also rational numbers) are dense in $\mathbb{R}$, then there exists an irrational $q \in \mathbb{R}$ such that $\alpha < q < \beta$. As E is connected, then $q \in f([0,1])$ and $[\alpha, \beta] \subset E$, so we have an irrational $q \in [\alpha, \beta]$ such that $f(x) = q$ for some $x \in [0,1]$. □

Proposition 1.3. If f and g are continuous on $[a,b]$, then $\int_a^b f(x)g(x)dx = \int_a^b f(x)dx \cdot \int_a^b g(x)dx$.

Proof. (**False.**) Consider $\int_1^2 x^2 dx = \frac{x^3}{3}|_1^2 = (2^3)/3 - 1/3 = 7/3 \neq (\int_1^2 x dx)^2 = (x^2/2|_1^2)^2 = (2 - 1/2)^2 = 9/4$. □

Proposition 1.4. If f is continuous on $[a,b]$, then $\int_a^b x f(x)dx = x \int_a^b f(x)dx$

Proof. (**False.**) $f(x) = x^2$ again on $[0,1] \to \mathbb{R}$. Then $\int_0^1 x f(x)dx = \int_0^1 x^3 dx = 1$, but $x \int_0^1 f(x)dx = x$. □

Proposition 1.5. If f' is continuous on $[-1,4]$, then $\int_{-1}^4 f'(x)dx = f(4) - f(-1)$

Proof. (**True.**) As $f'(x)$ is continuous on $[-1,4]$, and $[-1,4]$ is a compact interval, then f is bounded; thus $f'(x)$ is Riemann integrable. By FTC 1, $F(x) = \int_a^x f'(t)dt$ is differentiable on $[-1,4]$ and $F'(x) = f'(x)$. For $x = 4$ and $a = -1$, we have $\int_{-1}^4 f'(x)dt = f(t)|_{-1}^4 = f(4) - f(-1)$.

□

Proposition 1.6. $\int_{-2}^1 \frac{1}{x^4} dx = -\frac{3}{8}$.

Proof. (**False.**) The function $f(x) = \frac{1}{x^4}$ has a vertical asymptote at $x = 0$, and $0 \in [-2,1]$ so we cannot apply FTC. □

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Proposition 1.7. All continuous functions have derivatives.

Proof. (**False**.) Consider $f(x) = |x|$. This functions is continuous at $x = 0$, but the function is not differentiable at this point. $f'(0) = \lim_{t\to 0} \frac{f(t)-f(0)}{t-0} = \lim_{t\to 0} \frac{|t|}{t} = DNE$ as $\lim_{t\to 0^+} = 1$ and$\lim_{t\to 0^-} = -1$. □

Proposition 1.8. Even though the function

$$f(x) = \begin{cases} x^2 & x < 1 \\ 3 + x & x > 1 \end{cases}$$

is not continuous at $x = 1$, we can compute $\int_0^2 f(x)dx$

Proof. The function $f(x)$ is not continuous at $x = 1$, but we can still compute its definite integral over the interval $[0,2]$ by splitting the integral into two parts:

$$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$$

The first integral on the right-hand side is the definite integral of the function $f(x)$ over the interval $[0,1]$. Because the function $f(x)$ is defined as x^2 for all values of x in this interval, we can compute this integral directly as:

$$\int_0^1 x^2 dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}\cdot 1^3 - \frac{1}{3}\cdot 0^3 = \frac{1}{3}$$

The second integral on the right-hand side is the definite integral of the function $f(x)$ over the interval $[1,2]$. Because the function $f(x)$ is defined as $3 + x$ for all values of x in this interval, we can compute this integral directly as:

$$\int_1^2 (3+x)dx = \left[3x + \frac{1}{2}x^2\right]_1^2 = (3\cdot 2 + \frac{1}{2}\cdot 2^2) - (3\cdot 1 + \frac{1}{2}\cdot 1^2) = 6$$

Therefore, we can compute the definite integral of $f(x)$ over the interval $[0,2]$ as the sum of these two integrals:

$$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx = \frac{1}{3} + 6 = \boxed{\frac{19}{3}}$$

□

Proposition 1.9.

(a) If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.
(b) If $\sum a_n$ converges and (b_n), then $\sum a_n b_n$ converges.
(c) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

Solution. (a) This is **true**. Assume $\sum a_n$ converges absolutely. Then $\lim_{n\to\infty} |a_n| \to 0$ as $n \to \infty$, which means that for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $0 < ||a_n| - 0| = a_n < \varepsilon$. Then $0 < a_n^2 < \varepsilon a_n$, and as ε is just a constant, then $\sum a_n^2$ converges by Comparison Test.

(b) This is **false**. Consider the sequence $a_n = \frac{(-1)^n}{\sqrt{n}}$ and $b_n = \frac{(-1)^n}{\sqrt{n}}$, and as this is a p-series with $p = 1/2$, we get $\sum \frac{(-1)^n}{\sqrt{n}} \cdot \frac{(-1)^n}{\sqrt{n}} = \sum \frac{(-1)^{2n}}{n^{1/2+1/2}} = \sum \frac{1}{n}$, which diverges.

(c) This is **true**. Let $\sum a_n$ converge conditionally (i.e. the series here converges but does not converge absolutely). By contradiction, assume that $\sum n^2 a_n$ converges, and so $\lim_{n\to\infty} n^2 a_n = 0$, which means that $|n^2 a_n| < \varepsilon$ for all $\varepsilon > 0$. So $|n^2||a_n| < \varepsilon \Rightarrow |a_n| < \varepsilon/n^2$ (pick $\varepsilon = 1$), then $|a_n| < 1/n^2$. But then this means that $\sum a_n$ converges absolutely. Thus a contradiction.

Proposition 1.10. The series $\sum_{n=1}^{\infty} \frac{n!}{3^n}$ converges.

Proof. (**True**.) We can check this by the ratio test

$$\lim_{n\to\infty} \left|\frac{(n+1)!/3^{n+1}}{n!/3^n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!}\right| = \lim_{n\to\infty} \frac{n+1}{3} = \infty$$

□

Proposition 1.11. A bounded sequence $\{a_n\}$ of real numbers always has a convergent subsequence

Proof. (**True**.) Rudin, pg.51 Theorem 3.6(b). □

Proposition 1.12. A closed and bounded subset of a complete metric space must be compact.

Proof. (**False**.) Consider the unit sphere in ℓ_2. □

Proposition 1.13. If A and B are compact subsets of a metric space, then $A \cup B$ is also compact.

Proof. (**True**.) This is a quick proof. Let A and B both be compact subsets, where $\mathcal{A} = \{U_i : i \in I\}$ and $\mathcal{B} = \{V_i : i \in I\}$ are open covers, respectively, of A and B. As A and B are compact, then we can do with finitely many, i.e. $\mathcal{A} = \{U_i\}_{i=1}^n$ and $\mathcal{B} = \{V_i\}_{i=1}^m$ still cover A and B, respectively. Then $A \cup B \subset (\bigcup_{i=1} U_i) \cup (\bigcup_{i=1} V_i)$ cover $A \cup B$, which admits a finite subcover given by $\mathcal{A}$ and $\mathcal{B}$ and thus $A \cup B$ is compact. □

Proposition 1.14. If $\mathcal{X}$ is any metric space and $f : \mathcal{X} \to \mathbb{R}$ is any continuous real-valued function, then the function $g : M \to \mathbb{R}$ defined by $g(x) = (f(x))^2$ is always continuous.

Proof. (**True**.) Consider $\varphi : \mathcal{X} \to \mathbb{R}$ where $x \in \mathcal{X} \mapsto x^2$. Then $g(x) = (\varphi \circ f)(x)$ and as φ is continuous on all of $\mathbb{R}$, then g is a composition of two continuous function and hence g is itself continuous. Alternatively, let $\varepsilon > 0$. Then we have $\delta > 0$ such that $d_{\mathcal{X}}(x,y) < \delta \Rightarrow d_{\mathbb{R}}(f(x), f(y)) = |f(x) - f(y)|$. Now

$$|g(x) - g(y)| = |(f(x))^2 - (f(y))^2| = |(f(x) - f(y))(f(x) - f(y) + 2f(x))| \leq \varepsilon(\varepsilon + 2M),$$

where $M = |f(y)|$. □

Proposition 1.15. If $f : X \to Y$ is a continuous map between metric spaces, and $f(X)$ is compact, then X is compact.

Proof. (**False**.) Recall that any finite metric space is compact (and also any subset of the finite metric space is also going to be compact). Now consider $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 0$ for all $x \in \mathbb{R}$. Then $f(\mathbb{R}) = \{0\}$, which is compact. But $\mathbb{R}$ itself is not. □

Proposition 1.16. A compact subset of a metric space is always complete.

Proof. (**True.**) Recall that if (X,d) is a metric space and (x_n) is a Cauchy sequence in X, then if (x_n) has a subsequential limit to a point x, then the sequence (x_n) also converges to x. Hence as: compact if and only if subsequentially compact for any metric space (X,d), then we're done as then any any sequence in X has a convergent sequence. □

Proposition 1.17. Let $\{x_n\}$ be a sequence of points in a metric space $\mathcal{X}$. If two subsequences of (x_n) converge, then they must converge to the same number.

Proof. (**False**.) This is saying that, essentially, the set of subsequential limits of $S = (x_n)$ is at most 1, i.e. $|S_\infty| = 1$, which is obviously false. Consider $\mathcal{X} = \mathbb{R}$ and $S = (3, 3.1, 3, 3.14, 3, 3.141, 3, \ldots)$, which has two subsequential limits $S_\infty = \{3, \pi\} \subset \mathbb{R}$. Alternatively, the sequence $((-1)^n)_{n=1}^\infty = S$ also has two subsequential limits. □

Proposition 1.18. If $f : [0,1] \to \mathbb{R}$ is a continuous function and $\int_0^1 f(x)dx = 0$, then $f(x)$ is positive somewhere and negative somewhere in this interval (unless it is identically zero).

Proof. (**True.**) □

Proposition 1.19. $f(x) = \sum_{n=1}^\infty \frac{\sin(3^n \pi x)}{2^n}$ is a continuous function on $\mathbb{R}$.

Proof. (**True**.) By M-test, $\left|\frac{\sin(3^n \pi x)}{2^n}\right| \leq \frac{1}{n^2}$ which makes $f(x)$ converge uniformly and hence is continuous. □

Proposition 1.20.

(a) The set $\{x \in \mathbb{Q} : 0 < x < 1\}$ is uncountable.
(b) The collection of all possible function $f : \mathbb{N} \to \{2,3,4\}$ is finite.
(c) The collection of all possible function $f : \{2,3,4\} \to \mathbb{N}$ is uncountable.
(d) The collection of all possible function $f : C \to D$ is finite

Proof. (a) False. This is countably infinite.
(b) False. This is uncountable.
(c) False. This is countably infinite.
(d) True. □

2. Examples of Properties

Example 2.1. Assume (f_n) and (g_n) are uniformly convergent sequences of functions. Then the product $(f_n g_n)$ may not converge uniformly.

Consider $f_n(x) = g_n(x) = \frac{1}{x} + \frac{1}{n}$, where $f_n, g_n \colon (0, \infty) \to \mathbb{R}$.

Example 2.2. Give an example of a sequence of functions that converges uniformly (on $E = [0,1)$).

Consider $f_n(x) = x^n$. For $x = 0$, we get $f_n(0) = 0$, and now for $x \in (0,1)$, then $f_n(x) = x^n \to 0$ as $n \to \infty$.

Example 2.3. Give an example a of a metric space that is not compact.

Consider $X = \mathbb{R}$ endowed with the Euclidean metric. Then this space is not compact.

Example 2.4. Give an example of a metric space (X,d) with a Cauchy sequence that does not converge.

Consider the subspace $(\mathbb{Q}, d_{\text{Euc}}) \subset \mathbb{R}$. Then the sequence $S = (3, 3.1, 3.14, 3.141, \dots)$ is Cauchy in $\mathbb{Q}$ but does not converge; in $\mathbb{R}$, which is a complete metric space (i.e. all Cauchy sequences converge), we have $S \to \pi$ as $n \to \infty$.

Another example is consider $\mathcal{X} = \mathbb{R} \setminus \{0\}$ with distance $d_{\mathcal{X}}(x,y) = |x-y|$, where the sequence $x_n = \frac{1}{n}$ is Cauchy in $\mathcal{X}$ but not convergent.

Example 2.5. All continuous functions have antiderviatives.

(**True**.) This is just FTC.

Example 2.6. Give an example of two sets A and B such that $A, B, A \cap B$, and $A \setminus B$ are all infinite sets.

Consider $A = \mathbb{Z}$ and $B = \mathbb{N}$.

Example 2.7. Let $a = (a_n)_{n=1}^{\infty}$ denote the following sequence in $\mathbb{Q}$:

$$a = \left(3, 1, 3, \frac{1}{2}, 3, \frac{1}{3}, 3, \frac{1}{4}, \dots\right).$$

Write down a strictly decreasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that the image $\{a_{n_k}\}_{k=1}^{\infty}$ of the sequence $(a_{n_k})_{n=1}^{\infty}$ contains exactly two elements. You do not need to justify your answer, but do state explicitly what the image is.

Take the sequence $(n_k)_{k=1}^{\infty} = (1, 2, 3, 5, 7, 9, \dots)$, then $(a_{n_k}) = (3, 1, 3, 3, 3, 3, \dots)$, so the image is just $\{1, 3\}$.

Example 2.8. Example of a series that converges, but does not converge absolutely.

Consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

Example 2.9. Give examples of subsets of $\mathbb{R}$ that have $1, 2, 3$, and 4 limit points.

Consider $\{\frac{1}{n}\}_{n \in \mathbb{N}}$ has only one limit point (namely, 0). The set $\{\frac{1}{n}\}_{n \in \mathbb{N}} \cup \{1 - \frac{1}{n}\}_{n \in \mathbb{N}}$ has two limit points (that is, 0 and 1). And $\{\frac{1}{n}\}_{n \in \mathbb{N}} \cup \{1 - \frac{1}{n}\}_{n \in \mathbb{N}} \cup \{\frac{1}{n} - 1\}_{n \in \mathbb{N}}$ has three limit points $(0, 1, -1)$. And a set that has four would be something more unioned.

Example 2.10. Is every closed set a perfect set?

False. Consider $[0,1] \cup \{2\} \subset \mathbb{R}$, which is closed but not perfect.

3. Set, topology of metric spaces

Exercise 3.1.

Let (X,d) be a metric space, and let x_n be a convergent sequence in X. Show that x_n is also Cauchy.

Proof. Let $x_n \to x$ as $n \to \infty$ in X. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - x| < \varepsilon/2$, and similarly, for $m \geq n \geq N$, we have $|x_m - x| < \varepsilon/2$. By the triangle inequality,

$$|x_m - x_n| \leq |x_m - x| + |x_n - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus we have a Cauchy sequence as well. $\square$

Exercise 3.2.

If (x_n) and (y_n) are both Cauchy sequences in $\mathbb{R}$, then the sequence $(|x_n - y_n|)$ converges.

Proof. Assume that (x_n) and (y_n) are both Cauchy sequences. There there exists $N \in \mathbb{N}$ and $M \in \mathbb{N}$ such that $s \geq t \geq N$ and $p \geq q \geq M$ implies, respectively, that $|x_s - x_t| < \varepsilon/2$ and $|y_p - y_q| < \varepsilon/2$. Now to show $(|x_n - y_n|)$ converges, it suffices to show that it is Cauchy as $\mathbb{R}$ is complete. Now

$$|(x_m - y_m) - (x_n - y_n)| = |(x_m - x_n) - (y_m - y_n)| \leq |x_m - x_n| + |y_m - y_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore the sequence is Cauchy and hence converges in $\mathbb{R}$. $\square$

Exercise 3.3.

If X is a connected metric space and $f \colon X \to Y$ is a continuous surjection, then Y is connected.

Proof. Suppose X is connected where $f \colon X \to Y$ a continuous surjection. As f is surjective, then $f(X) = Y$. Now assume that Y is not connected, i.e. we can write $Y = A \cup B$ where A and B are separated sets (A and B are open and nonempty). Then $f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) = X$. As f is continuous then both $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty open sets of X. Hence a contradiction. $\square$

Exercise 3.3.

Prove that if E is a nonempty, bounded subset of $\mathbb{R}$, then $D = \mathbb{R} - E$ is not connected.

Proof. Recall that subset $A \subset \mathbb{R}$ is connected if and only if whenever $x, y \in A$ and $x < z < y$ then $z \in A$ as well. Now, as E is bounded, then we get $E \subset (-\alpha, \alpha)$ for some $\alpha \in \mathbb{R}$ such that $\alpha > 0$. Then, noticeably, we have $-\alpha, \alpha \in D$. For $x \in E$ we have $-\alpha < x < \alpha$ but $x \notin D$ by construction, and therefore we have that E is not connected. $\square$

Exercise 3.4.

Let (X,d) be metric space; let $\mathbb{R}^2$ have the usual metric. Let $f \colon \mathbb{R}^2 \to X$ be a function, and let A be a bounded subset of $\mathbb{R}^2$. Prove that $f(A)$ is bounded. (Hint: Consider $\overline{A}$.)

Proof. As A is bounded, then so is its closure $\overline{A}$. By construction $\overline{A}$ is closed, and so as $\overline{A}$ is closed and bounded then it is thus compact. Hence $f(\overline{A})$ is compact, which implies that it is (totally) bounded. As $A \subset \overline{A}$ then $f(A) \subset f(\overline{A})$, which makes $f(A)$ bounded. $\square$

4. Convergence, Absolute Convergence, Power series, radius of convergence (including $\limsup / \liminf$), Ratio/Root Test.

Exercise 4.1.

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers. Assume that the series

$$\sum_{n\geq 1} |a_n - a_{n-1}|$$

converges. Show that the sequence $\{a_n\}$ converges to a limit in $\mathbb{R}$

Proof. As we're assuming the series converges, then there exists $N \in \mathbb{N}$ such that $m \geq n \geq N$ implies $\sum_{k=n}^{m} |a_k - a_{k-1}| < \varepsilon$. Lastly,

$$|a_m - a_n| = \left|\sum_{k=n+1}^{m} (a_n - a_{n-1})\right| \leq \sum_{k=n+1}^{m} |a_n - a_{n-1}| < \varepsilon.$$

□

Exercise 4.2.

Assume $a_n > 0$ and $\lim_{n\to} n^2 a_n$ exists. Show that $\sum_{n\geq 1} a_n$ converges.

Proof. As $a_n > 0$ then we have a positive sequence of real numbers. As $\lim_{n\to} n^2 a_n$ and $a_n > 0$ is always positive, then $\lim_{n\to} n^2 a_n = \ell \geq 0$. If $\ell = 0$, then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|n^2 a_n| < \varepsilon$, but note that $n^2 a_n$ is always positive so $n^2 a_n < \ell$ and so $a_n < \varepsilon / n^2$. In particular, pick $\varepsilon = 1$. As $(a_n)_{n\in\mathbb{N}}$ and $(1/n^2)_{n\in\mathbb{N}}$ are both sequences of nonnegative real numbers, then we can apply the Comparison Test: The fact that $\sum_{n\geq 1} a_n$ converges follows quickly as $\sum_{n\geq 1} \frac{1}{n^2}$ is a p-series with $p = 2$. Now assume $\ell \neq 0$, so $\ell > 0$. Then we can apply a similar argument: we get that for $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|n^2 a_n - \ell| < 1$. If $n^2 a_n - \ell > 0$, then $n^2 a_n - \ell < 1$ and so $a_n < \frac{1}{n^2} + \frac{\ell}{n^2}$, which implies a_n converges as this is a sum of two p-series. Similarly, if $n^2 a_n - \ell < 0$, then $\ell - n^2 a_n < 1$ so $a_n < \frac{1-\ell}{-n^2} = \frac{(-1)}{n^2} + \frac{\ell}{n^2}$, which is another sum of p-series and thus converge. □

Exercise 4.2.

Assume $a_n > 0$ and $\lim_{n\to\infty} n^2 a_n$ exists. Show that $\sum_{n\geq 1} a_n$ converges.

Proof. As $a_n > 0$ then we have a positive sequence of real numbers. As $\lim_{n\to} n^2 a_n$ and $a_n > 0$ is always positive, then $\lim_{n\to} n^2 a_n = \ell \geq 0$. If $\ell = 0$, then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|n^2 a_n| < \varepsilon$, but note that $n^2 a_n$ is always positive so $n^2 a_n < \ell$ and so $a_n < \varepsilon / n^2$. In particular, pick $\varepsilon = 1$. As $(a_n)_{n\in\mathbb{N}}$ and $(1/n^2)_{n\in\mathbb{N}}$ are both sequences of nonnegative real numbers, then we can apply the Comparison Test: The fact that $\sum_{n\geq 1} a_n$ converges follows quickly as $\sum_{n\geq 1} \frac{1}{n^2}$ is a p-series with $p = 2$. Now assume $\ell \neq 0$, so $\ell > 0$. Then we can apply a similar argument: we get that for $\varepsilon = 1$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|n^2 a_n - \ell| < 1$. If $n^2 a_n - \ell > 0$, then $n^2 a_n - \ell < 1$ and so $a_n < \frac{1}{n^2} + \frac{\ell}{n^2}$, which implies a_n converges as this is a sum of two p-series. Similarly, if $n^2 a_n - \ell < 0$, then $\ell - n^2 a_n < 1$ so $a_n < \frac{1-\ell}{-n^2} = \frac{(-1)}{n^2} + \frac{\ell}{n^2}$, which is another sum of p-series and thus converge.
Alternatively, let $\lim_{n\to\infty} n^2 a_n = \ell$. Then, for all $\varepsilon > 0$, we have $n^2 a_n \in (\ell - \varepsilon, \ell + \varepsilon)$. As $n^2 a_n > 0$, then $\ell \geq 0$. Then picking $\varepsilon = \ell$, we get $n^2 a_n \in (0, 2\ell)$, which means $n^2 a_n < 2\ell \Rightarrow a_n < 2\ell / n^2$, so $\sum_{n\geq 1} a_n$ converges. □

Exercise 4.3.

Compute $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ for the following:

(a) $a_n = (-1)^n$
(b) $a_n = (-1)^n + \frac{2}{n}$
(c) $a_n = (-1)^n \cdot \frac{(n+2)}{n}$
(d) $a_n = n$
(e) $a_n = (-1)^n \cdot n$
(f) $a_n = (1+(-1)^n)n = n+(-1)^n n$

Proof. (a) We start with a way to do the rest with a clear methodology: $\{a_k : k \geq n\} = \{(-1)^n : k \geq n\}$, and so $M_n = \sup\{a_k : k \geq n\} = \{(-1)^k : k \geq n\} = 1$. Hence $\limsup_{n\to\infty} = 1$, and similarly, $\liminf_{n\to\infty} -1$.
(b) $\{(-1)^k + \frac{2}{k} : k \geq n\} \rightsquigarrow M_n = (1+\frac{2}{n})_{n=1}^{\infty} \to 1$ as $n \to \infty$, and also $m_n = (-1+\frac{2}{k})_{n=1}^{\infty} = (\frac{2}{k} - 1)_{n=1}^{\infty} \to -1$ as $n \to \infty$. Hence $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$.
(c) $\limsup_{n\to\infty} a_n = 1$ and $\liminf_{n\to\infty} a_n = -1$
(d) $\limsup_{n\to\infty} a_n = +\infty$, and $m_n = \inf\{k : k \geq n\} = (n)_{n=1}^{\infty}$ which gives $\liminf_{n\to\infty} = +\infty$ as well.
(e) $M_n = (n)_{n=1}^{\infty}$ so $\limsup_{n\to\infty} a_n = +\infty$, and $m_n = (-n)_{n=1}^{\infty}$ so $\liminf_{n\to\infty} a_n = -\infty$.
(f) $M_n = \sup\{n+(-1)^n n\} = (2n)_{n=1}^{\infty}$ so $\limsup_{n\to\infty} a_n = +\infty$, while $m_n = \inf\{n+(-1)^n n.\} \rightsquigarrow (n-n)_{n=1}^{\infty} = (0)_{n=1}^{\infty}$ so $\liminf_{n\to\infty} a_n = 0$. □

Exercise 4.4.

Consider the series

$$f_n(x) = \sum_{n\geq 1} \frac{x^n}{n^2}$$

(a) Show that the series converges uniformly for $|x| \leq a$ for any $a < 1$.
(b) Does the series converge uniformly for $|x| < 1$.

Proof. (a) As $|x| \leq a$ and $a < 1$, $\left|\frac{x^2}{n^2}\right| \leq \frac{1}{n^2}$, and by the M-test we have that the series $f_n(x)$ converges uniformly as $\sum_{n\geq 1} \frac{1}{n^2}$ converges.
(b) The series uniformly converges for $|x| < 1$.

$$\lim_{n\to\infty} \sup_{x\in(-1,1)} |f_n(x)| = \lim_{n\to\infty} \sup_{x\in(-1,1)} \left|\frac{x^n}{n^2}\right| = \lim_{n\to\infty} \frac{1}{n^2} \to 0$$

□

Exercise 4.5.

Consider the series

$$f_n(x) = \sum_{n\geq 0} x^n$$

(a) Use the Weierestrass M-test to show that the series converges uniformly for $|x| \leq a$ for all $a < 1$.
(b) Does the series $f(x) = \sum_{n\geq 0} x^n$ converge uniformly for $|x| < 1$.

Proof. As $|x| \leq a < 1$, then $|x^n| \leq a^n$. Now as $a < 1$, then $\sum_{n\geq 0} a^n$ is geometric and thus converges. Hence, by M-test, $f_n(x)$ converges uniformly.
(b) We can use the following test.

$$\lim_{n\to\infty} \sup_{x\in(-1,1)} |f_n(x)| = \lim_{n\to\infty} \sup_{x\in(-1,1)} x^n = 1 \neq 0$$

and thus the series doesn't converge uniformly. □

Exercise 4.6.

Show that if $a_n > 0$ and $\lim_{n\to\infty} na_n = \ell$ with $\ell \neq 0$, then the series $\sum_{n\geq 1} a_n$ diverges.

Proof. Let $\lim_{n\to\infty} na_n = \ell \neq 0$. Then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $na_n \in (\ell - \varepsilon, \ell + \varepsilon)$. Pick $\varepsilon = \ell/2$, and so $\ell - \ell/2 = \ell = \ell/2$ and $\ell + \ell/2 = 3\ell/2$. As $a_n > 0$ (i.e. nonegative sequence) and $na_n > \ell/2 \Leftrightarrow a_n > (\ell/2) \cdot 1/n$, and as $(\ell/2)$ is a constant and the sum $1/n$ is a divergent series, then the sum a_n diverges by the Comparison Test. □

Exercise 4.7.

Find the radius of convergence of each of the following power series:
(a) $\sum n^3 z^n$
(b) $\sum \frac{2^n}{n!} z^n$
(c) $\sum \frac{2^n}{n^2} z^n$
(d) $\sum \frac{n^3}{3^n} z^n$

Proof. (a) $a_n = n^3$, and so $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} |n^3|^{1/n} = \limsup_{n\to\infty} n^{3/n} = \lim_{n\to\infty} n^{3/n} = \lim_{n\to\infty} (n^{1/n})^3 = 1^3 = 1$. Thus $R = 1/\alpha = 1$.
(b) Using the root test, $\limsup_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right| = \lim_{n\to\infty} \frac{2^n}{n+1} = \infty$, and so $R = 1/\alpha = 0$.
(c) Identify $a_n = \frac{2^n}{n^2} \rightsquigarrow \limsup_{n\to\infty} \left(\frac{2^n}{n^2}\right)^{1/n} = \limsup_{n\to\infty} \left(\frac{2}{n^{2/n}}\right) = 2\lim_{n\to\infty} \left(\frac{1}{n^{2/n}}\right) = 2\lim_{n\to\infty} 1/(n^{1/n})^2 = 2 \cdot 1 = 2$.
(d) $a_n = \frac{n^3}{3^n}$, and so $\limsup_{n\to\infty} \left(\left|\frac{n^3}{3^n}\right|\right)^{1/n} = \limsup_{n\to\infty} \frac{n^{3/n}}{3} = 1/3 \lim_{n\to\infty} (n^{1/n})^3 = \frac{1}{3} \cdot 1^3 = \frac{1}{3}$. Hence $R = 1/\alpha = \frac{1}{1/3} = 3$. □

5. Uniform convergence, uniform continuity, etc.

Exercise 5.1.

Let
$$f_n(x) = \frac{nx}{1+nx^2}.$$

(a) Find the pointwise limit of (f_n) for all $x \in (0,\infty)$.
(b) Is the convergence uniform on $(0,\infty)$?
(c) Is the convergence uniform on $(0,1)$?
(d) Is the convergence uniform on $(1,\infty)$?

Proof. (a) When $x = 0 \rightsquigarrow f_n(0) = 0$. For $x > 0$,
$$f_n(x) = \frac{nx}{1+nx^2} \cdot \frac{1/n}{1/n} = \frac{x}{\frac{1}{n}+x^2} = \frac{1}{x}, \text{ as } n \to \infty.$$
(b)
(c) A similar situation happens as with (b).
(d) As $f_n \to f$ pointwise on $(1,\infty)$ then we can use Proposition 3.6:
$$\begin{aligned} M_n = \|f_n - f\|_u = \sup_{x\in(1,\infty)} |f_n - f| = \sup_{x\in(1,\infty)} \left|\frac{-1}{x(1+nx^2)}\right| &= \sup_{x\in(1,\infty)} \left|\frac{1}{x+nx^3}\right| \\ &= \sup_{x\in(1,\infty)} \frac{1}{x(1+nx^2)} \le \frac{1}{1+n}. \end{aligned}$$
This shows that $\|f_n - f\| \to 0$ as $n \to \infty$, and therefore $f_n \to f$ does converge uniformly. □

Exercise 5.2.

Let f be uniformly continuous on all of $\mathbb{R}$, and define a sequence of functions by $f_n(x) = f(x+\frac{1}{n})$. Show that $f_n \to f$ uniformly. Give an example to show that this proposition fails if f is only assumed to be continuous and not uniformly continuous on $\mathbb{R}$.

Proof. As f is uniformly continuous on all of $\mathbb{R}$, then for all $\varepsilon > 0$, $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$. Let $\varepsilon > 0$. Now $x + \frac{1}{n} \to x$ as $n \to \infty$, and so pick $N \in \mathbb{N}$ such that $n \ge N$ implies $|(x+1/n) - x| = |1/n| = 1/n < \delta$, i.e. $1/\delta < N$. Pick $N \in \mathbb{N}$ such that $N > 1/\delta$. Then for $n \ge N$, we get $n > 1/\delta$, which implies that $1/n < \delta$, and so $|(x+1/n) - x| < \delta \Rightarrow |f(x+1/n) - f(x)| = |f_n(x) - f(x)| < \varepsilon$. Hence $f_n \to f$ uniformly.
Lets choose a solely continuous function that fails. Consider $g(x) = x^2$ where $g\colon \mathbb{R} \to \mathbb{R}$. Then $g_n(x) = g(x+\frac{1}{n}) = \frac{(xn+1)^2}{n^2} = (\frac{xn+1}{n})^2$. Now, we have
$$|f_n(x) - f(x)| = \left|\left(x+\frac{1}{n}\right)^2 - x^2\right| = \left|\frac{2x}{n} + \frac{1}{n^2}\right|.$$
□

Exercise 5.3.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function, and define $f_n(x) = f(x/n)$ for each $n \in \mathbb{N}$.

(a) Prove that $f_n(x) \to f(0)$ pointwise on $\mathbb{R}$.
(b) Prove that $f_n \to f(0)$ uniformly on any bounded subset of $\mathbb{R}$.
(c) Does $f_n \to f(0)$ uniformly on all of $\mathbb{R}$? If so, prove it; if not, give a counterexample.

Proof. (a) As $x/n \to 0$ for $n \to \infty$, then for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|x/n| < \varepsilon$. As f is continuous, then for $y \in \mathbb{R}$, there exists $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \varepsilon$. Pick $N \in \mathbb{N}$ such that $N > x/\delta$, and so $n \geq N > x/\delta$ which gives that $\delta > x/n$. Thus $|x/n - 0| = |x/n| < \delta \Rightarrow |f(x/n) - f(0)| = |f_n(x) - f(0)| < \varepsilon$. Hence pointwise convergence.
(b) Now if we want to show uniform convergence we must get rid of the dependency on delta in (a). Let $E \subset \mathbb{R}$ be bounded, say, for all $x \in E$, we have $|x| \leq M$. In particular, as E is bounded then $\sup_{x\in E} |x|$ exists, and write $\omega = \sup_{x\in E} |x|$. Pick $N \in \mathbb{N}$ such that $N > \omega/\delta$. Then let $n \geq N$, which implies that $\delta > \omega/n > x/n$. Thus $|x/n - 0| = |x/n| < \delta \Rightarrow |f_n(x/n) - f(0)| < \varepsilon$. Hence uniform convergence. $\square$

Exercise 5.4.

Consider the sequence of functions defined by

$$g_n(x) = \frac{x^n}{n}$$

(a) Show (g_n) converges uniformly on $[0,1]$ and find $g = \lim g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0,1]$
(b) Now show that (g_n') converges on $[0,1]$. Is the convergence uniform? Set $h = \lim g_n'$ and compare h and g'. Are they the same?

Proof. (a) For $x = 0$, we get $g_n(x) = 0$, and for $x = 1$, $g_n(1) = 1^n/n = 1/n \to 0$ as $n \to \infty$. Lastly, take $x \in (0,1)$. Then, $0 < x^n < 1$ so $0 < x^n/n < 1/n$, that is, $0 < g_n(x) < 1/n$, and as $n \to$, we get $0 < \lim_{n\to\infty} x^n/n < \lim_{n\to\infty} 1/n$; thus $x^n/n \to 0$ as $n \to \infty$. Hence $\lim_{n\to\infty} g_n = g = 0$, and this is obviously differentiable for which $x \in [0,1]$ gives $g'(x) = 0$ again.
(b) Here we get that $g_n'(x) = x^{n-1}$. Now $x = 0$, we get $g_n'(x) = 0$, and for $x = 1$, we get $g_n(x) = 1^{n-1} = 1$. Lastly, take $x \in (0,1)$. Then $x^{n-1} \to 0$ as $n \to$. Hence we get a piecewise function:

$$h(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$$

$\square$

Exercise 5.5.

Let
$$f_n(x) = \frac{nx}{1+n^2x^2}, \text{ for } x \in \mathbb{R}.$$

(a) Show that $f_n \to 0$ pointwise on $\mathbb{R}$.
(b) Does (f_n) converge uniformly on $[0,1]$.
(c) Does f_n converge uniformly on $[1,\infty)$. Justify.

Proof. This is almost exactly Exercise 5.1.
(a) For $x = 0$, $f_n(0) = 0$. For $x \neq 0$,

$$f_n(x) = \frac{nx}{1+n^2x^2} \cdot \frac{1/n^2}{1/n^2} = \frac{x/n}{\frac{1}{n^2}+x^2} \to 0, \text{ as } n \to \infty.$$

(b) NO (?)
(c) Let $x \geq 1$. Then

$$M_n = \sup_{x\in[1,\infty)} |f_n(x) - f(x)| = \sup_{x\in[1,\infty)} \left|\frac{nx}{1+n^2x^2}\right| \leq \frac{n}{1+n^2} \to 0.$$

So we have uniform continuity. $\square$

6. Riemann Integration, Fundamental Theorem of Calculus, etc.

Exercise 6.1.

If f is a differentiable function so that $\int_0^x f(t)dt = (f(x))^2$ for all x, find f.

Proof. Assume that f is differentiable so that $F(x) = \int_0^x f(t)dt = (f(x))^2$. As f is differentiable, then we can differentiate both sides

$$F'(x) = \frac{d}{dx}((f(x)^2) = 2f'(x)f(x).$$

By FTC, as f is differentiable, then it is continuous and so $F'(x) = f(x)$, and thus we have $f(x) = 2f'(x)f(x)$, so $f(x)(1-2f'(x)) = 0$. If $f(x) = 0$ then we're done. Now consider $1-2f'(x) = 0$. Then $1 = 2f'(x)$, which gives $f'(x) = \frac{1}{2}$. Find its antiderivative (which all continuous function do have) $f(x) = \frac{1}{2}x + C$. Now, □

Exercise 6.2.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is constant.

Proof. This inequality gives that $\left|\frac{f(x)-f(y)}{x-y}\right| \leq |x-y|$. Pick $\varepsilon > 0$. Then $0 < |x-y| < \varepsilon$ implies that $\left|\frac{f(x)-f(y)}{x-y}\right| < \varepsilon$, so $\lim_{x\to y}\frac{f(x)-f(y)}{x-y} = f'(y) = 0$. Hence f is constant as y was arbitrary. □

Exercise 6.3.

Which $n \in \mathbb{N}$ have the property that $f^n \in \mathcal{R}([a,b])$ implies $f \in \mathcal{R}([a,b])$? Give proof(s) and counterexample(s) to show your answer is correct and complete.

Proof. We claim that this holds for $n \in \mathbb{N}$ odd, but fails for n even. Define the function $\varphi\colon \mathbb{R} \to \mathbb{R}$ such that $x \mapsto x^{1/n}$. Then $\varphi \circ f^n(x) = f(x)$. Hence, since f is Riemann integrable and φ is continuous then so is f. This argument doesn't work for n even as $x \mapsto x^{1/n}$ may not be a real number unless $x \geq 0$. A function for the counter example is: $f(x) = 1$ when $x \in \mathbb{Q}$ and $f(x) = -1$ when $x \notin \mathbb{Q}$. Then f is not Riemann integrable, but f^2 is! (Another note is that $f^2 \in \mathcal{R}([a,b])$ does imply that $|f| \in \mathcal{R}([a,b])$ as $\sqrt{f^2} = |f|$... this concept generalizes to all n even.) □

Exercise 6.4.

Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to \infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Proof. As $f'(x) \to 0$ on (x, ∞) as $x \to \infty$, then for all $M > 0$, we have $|x| \geq M$. Additionally, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|f'(x) - 0| = |f'(x)| < \varepsilon$ where $x > x_0$. Now for any $x \geq x_0$ we have some $\alpha \in (x, x+1)$ such that $f(x+1) - f(x) = f'(\alpha)$ by MVT. But since $|f'(\alpha)| < \varepsilon$, then so $|f(x+1) - f(x)| < \varepsilon$. □

Exercise 6.5.

Suppose

(a) f is continuous for $x \geq 0$,
(b) $f'(x)$ exists for $x > 0$,
(c) $f(0) = 0$,
(d) f' is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x}, \ (x > 0)$$

and prove that g is monotonically increasing.

Proof. We show this by showing $g'(x) > 0$. By MVT, we have $f(x) - f(0) = f(x) = xf'(\alpha)$ where $\alpha \in (0, x)$. Also as f' is monotonically increasing, then for $x > y$ we have $f'(x) > f'(y)$, and so $f(x) < xf'(x)$. Then $g'(x) = \frac{xf'(x) - f(x)}{x^2} > 0$ and thus g is monotonically increasing. □

Exercise 6.5.

Prove that if $f\colon [0,15] \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & x \in [0,3) \cup [5,11) \cup (11,15] \\ 4 & x \in [3,5) \\ 7 & x = 11 \end{cases}$$

then $f \in \mathcal{R}([0,15])$.

Proof. The way that we will proceed is by choosing an explicit partition of $[0,15]$ such that $U(P,f) - L(P,f) < \varepsilon$ for any $\varepsilon > 0$. Pick the partition $\mathcal{P} = \{0, 3-\eta, 3+\eta, 5-\eta, 5+\eta, 11-\eta, 11+\eta, 15\}$. Now we compute $U(\mathcal{P}, f)$ and $L(\mathcal{P}, f)$

$$\begin{aligned}
M_1 &= \sup_{x \in [0,3-\eta]} f(x) = 0 & m_1 &= \inf_{x \in [0,3-\eta]} f(x) = 0 \\
M_2 &= \sup_{x \in [3-\eta,3+\eta]} f(x) = 4 & m_1 &= \inf_{x \in [3-\eta,3+\eta]} f(x) = 0 \\
M_3 &= \sup_{x \in [3+\eta,5-\eta]} f(x) = 4 & m_3 &= \inf_{x \in [3+\eta,5-\eta]} f(x) = 4 \\
M_4 &= \sup_{x \in [5-\eta,5+\eta]} f(x) = 4 & m_4 & \inf_{x \in [5-\eta,5+\eta]} f(x) = 0 \\
M_5 &= \sup_{x \in [5+\eta,11-\eta]} f(x) = 0 & m_5 &= \inf_{x \in [5+\eta,11-\eta]} f(x) = 0 \\
M_6 &= \sup_{x \in [11-\eta,11+\eta]} f(x) = 7 & m_6 &= \inf_{x \in [11-\eta,11+\eta]} f(x) = 0 \\
M_7 &= \sup_{x \in [11+\eta,15]} f(x) = 0 & m_7 &= \sup_{x \in [11+\eta,15]} f(x) = 0
\end{aligned}$$

Then $U(\mathcal{P}, f) = \sum_{i=1}^n M_i \Delta x_i$ and $L(\mathcal{P}, f) = \sum_{i=1}^n m_i \Delta x_i$, so

$$\begin{aligned}
U(\mathcal{P}, f) - L(\mathcal{P}, f) &= \sum_{i=1}^7 M_i \Delta x_i - \sum_{i=1}^7 m_i \Delta x_i = \sum_{i=1}^7 (M_i - m_i) \Delta x_i \\
&= (M_1 - m_1)(x_1 - x_0) + \cdots \\
&= 0 + 4(2\eta) + 0 + 4(2\eta) + 0 + 7(2\eta) + 0 \\
&= 8\eta + 8\eta + 14\eta = 30\eta.
\end{aligned}$$

Now pick $\eta = \frac{\varepsilon}{31}$. Then $(\mathcal{P}, f) - L(\mathcal{P}, f) = 30\eta = 30 \cdot \frac{\varepsilon}{31} < \varepsilon$. □

7. PROVE A THEOREM FROM CLASS

Theorem 7.1 (Weierstrass M-test)**.** Suppose $(f_n)_{n=1}^{\infty}$ is a sequence of functions defined on $E \subset \mathbb{R}$, and $|f_n(x)| \leq M_n$ for all $x \in E$, for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ converges uniformy on E.

Proof. We show that the sequence $(s_n)_{n=1}^{\infty}$ of partial sums is uniformly Cauchy. Let $\varepsilon > 0$ such that $N \in \mathbb{N}$ and $m \geq n \geq N$ implies $\sum_{k=n}^{m} M_n < \varepsilon$. Then

$$|s_m - s_n| = |\sum_{k=n}^{m} f_n(x)| \leq \sum_{k=n}^{m} |f_n(x)| \leq \sum_{k=n}^{m} M_n < \varepsilon.$$

Hence (s_n) is uniformly Cauchy, and thus also uniformly convergent. □

Theorem 7.2 (Integrable Limit Theorem)**.** Let $f_n \in \mathcal{R}([a,b])$ for each $n \in \mathbb{N}$. If $f_n \to f$ uniformly on $[a,b]$, then $f \in \mathcal{R}([a,b])$, and

$$\lim_{n\to} \int_a^b f_n dx = \int_a^b f dx.$$

Proof. Assume that $f_n \to f$ uniformly. Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \eta$, where we pick η later on. Then for $n \geq N$ we have $f_n(x) - \eta < f(x) < f_n(x) + \eta$ for all $x \in [a,b]$, which implies

$$0 \leq \overline{\int_a^b} f dx - \underline{\int_a^b} f dx < \int_a^b (f_n(x) + \eta) dx - \int_a^b (f_n(x) - \eta) dx = 2\int_a^b \eta dx = 2(b-a)\eta.$$

Now pick $\eta = \frac{\varepsilon}{2(b-a)}$. Then we have $0 \leq \overline{\int_a^b} f dx - \underline{\int_a^b} f dx < \varepsilon$ for all $\varepsilon > 0$. Hence we have $\overline{\int_a^b} f dx = \underline{\int_a^b} f dx$, so $f \in \mathcal{R}([a,b])$. Lastly, for $n \geq N$, we have

$$|\int_a^b f dx - \int_a^b f_n dx| \leq \int_a^b |f - f_n| dx \leq \eta(b-a) = \varepsilon/2 < \varepsilon.$$

Thus $\int_a^b f_n dx \to \int_a^b f dx$ as $n \to \infty$. □

Theorem 7.3 (Uniform Limit Theorem)**.** Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous real-valued function on a metric space (X,d). Assume $f \colon E \to \mathbb{R}$ is a function that $f_n \to f$ uniformly on $E \subset X$. Then f is continuous.

Proof. Let $\varepsilon > 0$ and $x \in E$. Pick $N \in \mathbb{N}$ sufficiently large such that $|f(z) - f_N(z)| < \varepsilon/3$ for all $z \in E$. Then also for the same N, pick $\delta > 0$ such that $d(x,y) < \delta$ and $y \in E$ implies that $|f_N(x) - f_N(y)| < \varepsilon/3$. Then for $y \in E$ and $d(x,y) < \delta$, we get

$$|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

□

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