MATH 425A HW4, DUE 09/23/2022, 6PM

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1.3, 2.2, 2.3, 2.4 in Chapter 3.

Chapter 2. §5.

Exercise 0.1 (5.2.)**.** Let $a_1, a_2, \ldots$ be any enumeration of the negative rational numbers; let $b_1, b_2, \ldots$ be any enumeration of the positive rational numbers. Show that the following two equalities hold:

$$\bigcap_{j=1}^{\infty}(a_j, b_j) = \{0\}, \bigcup_{j=1}^{\infty}(a_j, b_j) = \mathbf{R}$$

Proof. For the first equality, take $\ell \in T = \bigcap_{j=1}^{\infty}(a_j, b_j)$, that is, ℓ is in every $(a_j, b_j) \subseteq \mathbf{R}$. So then $a_j < \ell < b_j$ for $\ell \in \overline{\mathbf{R}}$, but as a_j is essentially a negative rational number, and b_j is a positive rational, then we have that ℓ is squished between every negative and positive rational number. □

Chapter 2. §6.

Exercise 0.2 (6.1.)**.** Prove that the addition and multiplication operations in $(\mathbf{C}, +, \cdot)$ satisfy the field axioms of Definition 2.1.

Proof. We essentially need to show that five axioms hold true from Definition 2.1. From now on, let $x, y, z \in \mathbf{R} \times \mathbf{R}\,(= \mathbf{C})$, which is the underlying set of $\mathbf{C}$, where $x = (a, b), y = (c, d), z = (s, t)$ where $a, b, c, d, s, t \in \mathbf{R}$.

(1) The set $\mathbf{C} := (\mathbf{C}, +, \cdot)$, as the operations are defined in Chapter 2, §6., is closed since $x+y = (a, b)+(c, d) = (a+c, b+d) \in \mathbf{R}\times\mathbf{R}$ and $xy = (a, b)\cdot(c, d) = (ac-bd, ad+bc) \in \mathbf{R}\times\mathbf{R}$ since $a + c, b + d, ac - bd, ad + bc \in \mathbf{R}$ as $\mathbf{R}$ is a field, and so $x + y \in \mathbf{C}$ and $xy \in \mathbf{C}$.

(2) For commutativity: $x + y = (a, b) + (c, d) = (a + c, b + d) = (c + a, d + b) = (c, d)+(a, b) = y+x$ since $\mathbf{R}$ is a field, and, similarly, $xy = (a, b)\cdot(c, d) = (ac-bd, ad+bc) = (ca - db, cb + da) = (c, d) \cdot (a, b) = yx$ as $\mathbf{R}$ is a field. Now for associativity:

$$\begin{aligned} x + (y + z) &= (a, b) + ((c, d) + (s, t)) = (a, b) + (c + s, d + t) \\ &= (a + (c + s), b + (d + t)) = ((a + c) + s, (b + d) + t)) \qquad (\mathbf{R} \text{ is a field}) \\ &= (a + c, b + d) + (s, t) = (x + y) + z \end{aligned}$$

$$\begin{aligned} x(yz) &= (a, b) \cdot ((c, d) \cdot (s, t)) = (a, b) \cdot (cs - dt, ct + ds) \\ &= (a(cs - dt) - b(ct + ds), a(ct + ds) + b(cs - dt)) && (\mathbf{R} \text{ is a field}) \\ &= (acs - adt - bct - bds, act + ads + bcs - bdt) && (\mathbf{R} \text{ is a field}) \\ &= ((ac - bd)s - (ad + bc)t, (ad + bc)s + (ac - bd)t) && (\mathbf{R} \text{ is a field}) \\ &= (ac - bd, ad + bc) \cdot (s, t) = ((a, b) \cdot (c, d)) \cdot (s, t) \\ &= (xy)z \end{aligned}$$

Therefore we have associativity and commutativity with the defined operations on $\mathbf{C}$.

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(3) The additive identity of $\mathbf{C}$ is defined to be $0 = (0,0) \in \mathbf{R} \times \mathbf{R}$, and so $x + 0 = (a,b) + (0,0) = (a+0, b+0) = (a,b) = (0+a, 0+b) = (0,0) + (a,b) = 0 + x$. Similarly, the multiplicative identity is defined to be $1 = (1,0)$, and so $x \cdot 1 = (a,b) \cdot (1,0) = (a(1) - b(0), a(0) + b(1)) = (a,b) = x = 1 \cdot x = (1,0) \cdot (a,b) = (1(a) - 0(b), 1(b) + 0(a)) = (a,b) = x$.

(4) The multiplicative inverse of $x = (a,b)$, where $x \neq 0$, can be found to be $x^{-1} = \left(\frac{a}{a^2+b^2}, \frac{-b(\frac{a}{a^2+b^2})}{a} \right)$, and we can tediously calculate to get that

$$x \cdot x^{-1} = (a,b) \cdot \left(\frac{a}{a^2+b^2}, \frac{-b(\frac{a}{a^2+b^2})}{a} \right) = \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2} \right) = (1,0) = 1. \tag{1}$$

The additive inverse is much easier: for $y = (c,d)$, the additive inverse is $-y = (-c,-d)$, and so $y + (-y) = (c + (-c), d + (-d)) = (0,0) = 0$.

(5) Lastly, we need to check distributivity: Let $t := y + z = (c+s, d+t)$. Now

$$\begin{aligned} x \cdot t = (a,b) \cdot (c+s, d+t) &= (a(c+s) - b(d+t), a(d+t) + b(c+s)) \\ &= (ac + as - bd - bt, ad + at + bc + bs) \\ &= ((ac - bd) + (as - bt), (ad + bc) + (at + bs)) \\ &= (a,b) \cdot (c,d) + (a,b) \cdot (s,t) \end{aligned}$$

Therefore the distributive law holds.

Hence $\mathbf{C}$ is indeed a field. □

Exercise 0.3 (6.2.)**.** Prove that there exists no order $\leq$ that makes $(\mathbf{C}, +, \cdot, \leq)$ into an ordered field. (Hint: If there were such an ordering, then $i = \sqrt{-1}$ would necessarily be either positive or negative.)

Proof. Suppose that there does exists an ordering that makes $\mathbf{C}$ into an ordered field. Then, by definition, we have that either $i \leq 0$ or $i \leq 0$, but we do not have that $i = 0$, so we simply have that either i is negative or positive. Suppose, for the first case, that $i < 0$. Then $0 < -i$ so $0^2 < (-i)^2 = 1(-1) = -1$ and once again, $0^2 < (-1)^2 = 1$; hence a contradiction. Thus we cannot have that i is negative. Now, for the second/last case, then assume that $i > 0$. Then $i^2 = -1 > 0^2 = 0$ and so $(-1) + 1 = 0 > 0 + 1 = 1$, and multiplying by 1, $i \cdot 0 = 0 > 1 \cdot i = i$; thus a contradiction. Hence we cannot have that i is not positive either. Therefore we cannot have that there exists an order on $\mathbf{C}$ that makes it into an ordered field. □

1. Chapter 3. §1

Exercise 1.1 (1.1.)**.** Let $\|\cdot\|$ be a norm on a real vector space V. Prove the *reverse triangle inequality:*

$$|\|x\| - \|y\|| \leq |\|x - y\||$$

Exercise 1.2 (1.2.)**.** Prove that any complex inner product is conjugate linear in its second argument; that is,

$$\langle x, \lambda y + z \rangle = \overline{\lambda} \langle x, y \rangle + \langle x, z \rangle,$$

for any scalar λ. (Note that this implies that any real inner product is linear in its second argument.)

Proof. We are considering a complex inner product and so we have a mapping $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbf{C}$ with some properties. Let $x, y, z \in V$ and $\lambda \in \mathbf{C}$. Then $\langle x, \lambda y + z \rangle = \overline{\langle \lambda y + z, x \rangle} = \overline{\lambda \langle y, x \rangle} + \overline{\langle z, x \rangle} = (\overline{\lambda}) \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} = \overline{\lambda} \langle x, y \rangle + \langle x, z \rangle$. □

Exercise 1.3 (1.3.-Polarization identity)**.** If $(V, \langle \cdot, \cdot \rangle)$ is a real inner product space, then

$$\langle v, w \rangle = \frac{1}{4} \left[\|v + w\|^2 - \|v - w\|^2 \right], \text{ for all } v, w \in V.$$

If $(V, \langle \cdot, \cdot \rangle)$ is a complex inner product space, then

$$\langle v, w \rangle = \frac{1}{4}\left[(\|v+w\|^2 - \|v-w\|^2) + i(\|v+iw\|^2 - \|v-iw\|^2) \right]$$

Proof. Suppose that $(V, \langle \cdot, \cdot \rangle)$ is a real inner product space. Then $\|v+w\|^2 = \langle v+w, v+w \rangle = \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle = \langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle$, and, similarly, $\|v-w\|^2 = \langle v-w, v-w \rangle = \langle v, v \rangle - \langle v, w \rangle - \langle w, v \rangle + \langle w, w \rangle = \langle v, v \rangle - 2\langle v, w \rangle + \langle w, w \rangle$. Thus:

$$\begin{aligned}
\frac{1}{4}\left[\|v+w\|^2 - \|v-w\|^2\right] &= \frac{1}{4}\left[\langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle - (\langle v, v \rangle - 2\langle v, w \rangle + \langle w, w \rangle\right] \\
&= \frac{1}{4}\left[2\langle v, w \rangle + 2\langle v, w \rangle\right] \\
&= \frac{1}{4}\left[4\langle v, w \rangle\right] = \langle v, w \rangle.
\end{aligned}$$

Suppose that $(V, \langle \cdot, \cdot \rangle)$ is a complex inner product. Similar to the first computations we did for the real case, we can find that $\|v+w\|^2 = \langle v, v \rangle = \langle v, w \rangle + \overline{\langle v, w \rangle} + \langle w, w \rangle$, and $\|v-w\|^2 = \langle v, v \rangle - \langle v, w \rangle - \overline{\langle v, w \rangle} + \langle w, w \rangle$. Moreover, $\|v+iw\| = \langle w, w \rangle + i\langle w, v \rangle - i\langle v, w \rangle + \langle v, v \rangle$, and $\|v-iw\| = \langle w, w \rangle - i\langle w, v \rangle + i\langle v, w \rangle + \langle v, v \rangle$. Now:

$$\begin{gathered}
\|v+w\|^2 - \|v-w\|^2 = 2\langle v, w \rangle + 2\langle w, v \rangle, \text{ and} \\
\|v+iw\|^2 - \|v-iw\|^2 = 2i\langle w, v \rangle - 2i\langle v, w \rangle = 2i\left[\langle w, v \rangle - \langle v, w \rangle\right]
\end{gathered}$$

Thus:

$$\begin{aligned}
\frac{1}{4}\left[(2\langle v, w \rangle + 2\langle w, v \rangle) + i\left(2i(\langle w, v \rangle - \langle v, w \rangle)\right]\right. &= \frac{1}{4}\left[2\langle v, w \rangle + 2\langle w, v \rangle + (-2\langle w, v \rangle + 2\langle v, w \rangle)\right] \\
&= \frac{1}{4}\left[4\langle v, w \rangle + 2\langle w, v \rangle - 2\langle w, v \rangle\right] \\
&= \frac{1}{4}\left[4\langle v, w \rangle\right] = \langle v, w \rangle.
\end{aligned}$$

$\square$

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