# MATH 425A HOMEWORK 7 SOLUTIONS

Assignment: Chapter 4, #1.8, 1.9, 1.10, 1.14, 3.2.

Due Date: October 12, 2022

Rubric: (26 points total)

- Exercise 1.8: 4 points, Category I
- Exercise 1.9: 4 points, Category II
- Exercise 1.10: 2 points, Category II
- Exercise 1.14: 6 points (1 + 2 + 2 + 1), Category II
- Exercise 3.2: 8 points (2 + 4 + 1 + 1) , Category I
- Neatness: 2 points
- Optional: LaTeXbonus: 1 point

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1.8. Let $(X, d)$ be a metric space, and let $E$ be a subset of $X$. Prove that $\operatorname{Lim}_X(E)$ is a closed set of $X$.

Soln.: Choose $x \in \operatorname{Lim}_X(\operatorname{Lim}_X(E))$; we claim that $x \in \operatorname{Lim}_X(E)$. Let $U$ be a neighborhood of $x$ in $X$. Since $x \in \operatorname{Lim}_X(\operatorname{Lim}_X(E))$, it follows that $U \cap (\operatorname{Lim}_X(E)\backslash\{x\})$ is nonempty; choose $y$ belonging to this set. Then $U$ is a neighborhood of $y$ in $X$ and $y \in \operatorname{Lim}_X(E)$, so $U$ contains infinitely many points of $E$, by Theorem 1.11. In particular, $U \cap (E\backslash\{x\})$ is nonempty, so $x \in \operatorname{Lim}_X(E)$. Since $\operatorname{Lim}_X(\operatorname{Lim}_X(E)) \subset \operatorname{Lim}_X(E)$, we conclude that $\operatorname{Lim}_X(E)$ is closed in $X$.

1.9. Let $(X, d)$ be a metric space, and let $E$ be a subset of $X$. Prove that

$$X\backslash \operatorname{Cl}_X(E) = \operatorname{Int}_X(X\backslash E)$$

(This can be written more concisely as $(\overline{E})^c = (E^c)^\circ$, if desired.

Soln.: Suppose $x \in X\backslash \operatorname{Cl}_X(E)$. Then $x \notin E$ and $x \notin \operatorname{Lim}_X(E)$. So there exists a neighborhood $U$ of $x$ in $X$ which does not intersect $E$. Then $x \in U \subset X\backslash E$ implies that $x \in \operatorname{Int}_X(X\backslash E)$ (since $U$ is open in $X$, c.f. Proposition 2.14 in Chapter 3).

On the other hand, suppose $x \in \operatorname{Int}_X(X\backslash E)$. Then there exists a neighborhood $U$ of $x$ such that $x \in U \subset X\backslash E$; this neighborhood $U$ of $x$ does not intersect $E$, so $x \notin \operatorname{Lim}_X(E)$. Furthermore, $x \notin E$, so $x \notin \operatorname{Cl}_X(E)$, i.e., $x \in X\backslash \operatorname{Cl}_X(E)$. This proves the claim.

1.10. Let $(X, d)$ be a metric space. Let $E$ and $Y$ be subsets of $X$ such that $E \subset Y$. Prove that

$$\operatorname{Cl}_Y(E) = \operatorname{Cl}_X(E) \cap Y.$$

Soln.: By Exercise 1.2, we can write the following:

$$\operatorname{Cl}_Y(E) = E \cup \operatorname{Lim}_Y(E) = E \cup (\operatorname{Lim}_X(E) \cap Y) = (E \cup \operatorname{Lim}_X(E)) \cap Y = \operatorname{Cl}_X(E) \cap Y.$$

1.14. Let $(X, d)$ be a metric space.

(a) Prove that for any $x \in X$ and $r > 0$, we have $\overline{B_X(x,r)} \subset \{y \in X : d(x,y) \leq r\}$. (Hint: Take complements and draw a picture.) Note that the inclusion $\overline{B_X(x,r)} \subset B_X(x, r+\varepsilon)$ follows for any $\varepsilon > 0$.

(b) Give an example using the discrete metric that demonstrates that equality need not hold in the inclusion $\overline{B_X(x,r)} \subset \{y \in X : d(x,y) \le r\}$ that you proved in part (a).
(c) Prove that in $\mathbb{R}^n$ under the Euclidean metric $d(x,y) = \|x-y\|$, we have $\overline{B_{\mathbb{R}^n}(x,r)} = \{y \in \mathbb{R}^n : \|x-y\| \le r\}$. (Again, a picture may be useful.)
(d) Using part (a), prove that if $A$ is bounded in $(X,d)$, then $\overline{A}$ is also bounded in $(X,d)$.

Soln.: (a) We prove that $X \backslash \overline{B_X(x,r)} \supset X \backslash \{y \in X : d(x,y) \le r\} = \{y \in X : d(x,y) > r\}$, which is equivalent to the desired statement (by Exercise 1.9, for example).

Suppose $d(x,y) > r$ and put $\delta = d(x,y) - r$. Then $B_X(y,\delta)$ is a neighborhood of $y$ that does not intersect $B_X(x,r)$. (Indeed, if $z \in B_X(y,\delta) \cap B_X(x,r)$, then $d(x,y) \le d(x,z) + d(z,y) < r + \delta = d(x,y)$, a contradiction. A picture will clarify this step.) Therefore $y \notin \overline{B_X(x,r)}$. This proves the claim.

(b) Let $X$ be any set with more than one element, and let $d$ be the discrete metric. (That is, $d(x,y) = 1$ if $x \ne y$, $d(x,x) = 0$.) Every subset of $X$ is both open and closed with respect to $X$; in particular $\overline{A} = A$ for all subsets $A$ of $X$. Thus $\overline{B_X(x,1)} = B_X(x,1) = \{x\}$. But $\{y \in X : d(x,y) \le 1\} = X \supsetneq \{x\}$ by definition of the discrete metric.

(c) In light of part (a), it suffices to show that if $\|x-y\| = r$, then every $\varepsilon$-neighborhood of $y$ intersects $B_X(x,r)$. To this end, choose $y \in \mathbb{R}^n$ such that $\|x-y\| = r$, and choose $\varepsilon > 0$. Put $z = y + \frac{\varepsilon}{2} \cdot \frac{x-y}{r}$ (a picture will motivate this choice). Then

$$\|y-z\| = \left\| y - \left( y + \frac{\varepsilon}{2} \cdot \frac{x-y}{r} \right) \right\| = \frac{\varepsilon}{2} \cdot \frac{\|x-y\|}{r} = \frac{\varepsilon}{2},$$

so $z \in B_X(y,\varepsilon)$, and

$$\|x-z\| = \left\| x - \left( y + \frac{\varepsilon}{2} \cdot \frac{x-y}{r} \right) \right\| = \left\| (x-y)\left(1 - \frac{\varepsilon}{2r}\right) \right\| = r \cdot \left(1 - \frac{\varepsilon}{2r}\right) = r - \frac{\varepsilon}{2} < r,$$

so $z \in B_X(x,r)$. Thus $B_Y(y,\varepsilon)$ intersects $B_X(x,r)$, so $y \in \overline{B_X(x,r)}$, as needed.

(d) Assume $A$ is bounded in $(X,d)$. Then there exists $M > 0$ and $x \in X$ such that $A \subset B_X(x,M)$. But then

$$\overline{A} \subset \overline{B_X(x,M)} \subset B_X(x,M+1)$$

(where we have used part (a) to get the second inclusion), so $\overline{A}$ is also bounded in $(X,d)$.

3.2. Prove Proposition 3.9 (copied below for convenience).

**Proposition**: Let $(X,d)$ be a metric space.
(a) If $X$ is totally bounded, then it is bounded.
(b) If $Y \subset X$, then $(Y,d)$ is totally bounded if and only if for any $\varepsilon > 0$, there exist $a_1, \dots, a_J \in X$ such that $Y \subset \bigcup_{j=1}^J B_X(a_j,\varepsilon)$. (The point is that the $a_j$'s do not have to lie in $Y$.)
(c) If $X$ is totally bounded and $Y \subset X$, then $Y$ is totally bounded.
(d) If $Y \subset X$ and $(Y,d)$ is totally bounded, then $(\overline{Y},d)$ is totally bounded.

Soln.: (a) Pick a finite subset $\{x_1, \dots, x_n\}$ of $X$ such that $X = \bigcup_{j=1}^n B_X(x_j,1)$. Put $M = \max\{d(x_1,x_j) : j = 1, \dots, n\}$. Given $y \in X$, we must have $y \in B_X(x_j,1)$ for some $j \in \{1, \dots, n\}$, so that

$$d(x_1,y) \le d(x_1,x_j) + d(x_j,y) \le M + 1.$$

Since $y$ was an arbitrary point of $X$, it follows that $X = B_X(x_1,M+1)$ and thus that $X$ is bounded.

(b) Assume that $(Y,d)$ is totally bounded. Then for any $\varepsilon > 0$, there exist $a_1, \dots, a_J$ in $Y \subset X$ such that $Y = \bigcup_{j=1}^J B_Y(a_j,\varepsilon) \subset \bigcup_{j=1}^J B_X(a_j,\varepsilon)$.

Assume on the other hand that for every $\varepsilon > 0$, there exist $a_1, \dots, a_J$ in $X$ such that $Y \subset \bigcup_{j=1}^J B_X(a_j,\varepsilon)$. Choose $\varepsilon > 0$, then choose $a_1, \dots, a_J \in X$ such that $Y \subset \bigcup_{j=1}^J B_X(a_j,\frac{\varepsilon}{2})$. Assume without loss of generality that $B_X(a_j, \frac{\varepsilon}{2}) \cap Y$ is nonempty for each $j$; otherwise remove the offending $a_j$'s from the original list. Choose $b_j \in B_X(a_j, \frac{\varepsilon}{2}) \cap Y$; then $B_Y(b_j, \varepsilon) = B_X(b_j, \varepsilon) \cap Y \supset B_X(a_j, \frac{\varepsilon}{2}) \cap Y$, so

$$Y \subset \bigcup_{j=1}^{J} B_X(a_j, \tfrac{\varepsilon}{2}) \cap Y \subset \bigcup_{j=1}^{J} B_Y(b_j, \varepsilon).$$

Thus $Y$ is totally bounded with respect to $d$, as claimed.

(c) Choose $\varepsilon > 0$. Since $(X, d)$ is totally bounded, there exist $a_1, \ldots, a_J$ such that

$$Y \subset X = \bigcup_{j=1}^{J} B_X(a_j, \varepsilon).$$

Applying part (b) finishes the argument.

(d) Choose $\varepsilon > 0$, then choose $a_1, \ldots, a_J$ in $X$ such that $Y \subset \bigcup_{j=1}^{J} B_X(a_j, \frac{\varepsilon}{2})$. Then

$$\overline{Y} \subset \bigcup_{j=1}^{J} \overline{B_X(a_j, \tfrac{\varepsilon}{2})} \subset \bigcup_{j=1}^{J} B_X(a_j, \varepsilon).$$

(The second inclusion follows from part (a) of Exercise 1.14.) It follows that $(\overline{Y}, d)$ is totally bounded.