# MATH 425A HW7, OCT. 12, 6PM

## 1. Chapter 4.

**Chapter 4 §1.2: Exercise 1.8.**

Let $(X, d)$ be a metric space, and let $E$ be a subset of $X$. Prove that $\mathrm{Lim}_X(E)$ is a closed set of $X$.

*Proof.* Our strategy is to show that $\mathrm{Lim}_X(\mathrm{Lim}_X(E)) \subseteq \mathrm{Lim}_X(E)$. Take $x \in \mathrm{Lim}_X(\mathrm{Lim}_X(E))$. Then we have that $\mathrm{Lim}_X(E) \cap B_X(x, \frac{\epsilon}{2}) \neq \varnothing$ and let $\ell$ be this intersection. Then $d(x, \ell) < \epsilon/2$ and $\ell \in \mathrm{Lim}_X(E)$. So then $E \cap B_X(\ell, d(x, \ell)) \setminus \{\ell\} \neq \varnothing$ and take $p$ to be in this intersection. And thus $d(x, p) \leq d(p, \ell) + d(\ell, x) < d(x, \ell) + d(x, \ell) = 2d(x, \ell) < 2(\frac{\epsilon}{2}) = \epsilon$ as $p \in B_X(\ell, d(x, \ell))$ and $\ell \in B_X(x, \frac{\epsilon}{2})$. Now $p \neq x$ as $d(p, \ell) < d(x, \ell)$. So thus $p \in E \cap B_X(x, \epsilon) \setminus \{x\}$. Hence $x \in \mathrm{Lim}_X(E)$ and we can conclude the claim of the exercise. $\square$

**Chapter 4 §1.2: Exercise 1.9.**

Let $(X, d)$ be a metric space, and let $E$ be a subset of $X$. Prove that $X \setminus \mathrm{Cl}_X(E) = \mathrm{Int}_X(X \setminus E)$

*Proof.* ($\supseteq$) Note that $\mathrm{Int}_X(X \setminus E) \subseteq X \setminus E$, and also $E = X \setminus (X \setminus E) \subseteq X \setminus \mathrm{Int}_X(X \setminus E)$. Thus $X \setminus \mathrm{Int}_X(X \setminus E)$ is a closed set that contains $E$, which means that $\mathrm{Cl}_X(E) \subseteq X \setminus \mathrm{Int}_X(X \setminus A)$ and so $X \setminus (X \setminus \mathrm{Int}_X(X \setminus A)) = \mathrm{Int}_X(X \setminus A) \subseteq X \setminus \mathrm{Cl}_X(E)$. Hence we have the backwards inclusion.
($\subseteq$) As $E \subseteq \mathrm{Cl}_X(E)$, then we have that $X \setminus \mathrm{Cl}_X(E) \subseteq E$. And as $\mathrm{Cl}_X(E)$ is a closed set, then we have an open set $X \setminus \mathrm{Cl}_X(E)$ contained in $E$. $\square$

**Chapter 4 §1.2: Exercise 1.10.**

Let $(X, d)$ be a metric space. Let $E$ and $Y$ be subsets of $X$ such that $E \subseteq Y$ . Prove that

$$\mathrm{Cl}_Y(E) = \mathrm{Cl}_X(E) \cap Y.$$

*Proof.* ($\subseteq$) Suppose $p \in \mathrm{Cl}_Y(E)$. Then $p \in \mathrm{Lim}_Y(E) \cup E$. If $p \in E$, then $p \in \mathrm{Cl}_X(E) = \mathrm{Lim}_X(E) \cup E$, as $p \in E$, and as $E \subseteq Y$ we also have $p \in Y$. Thus $p \in \mathrm{Cl}_X(E) \cap Y$. On the other hand, if $p \in \mathrm{Lim}_Y(E)$, then we have that $p \in Y$ such that for any open neighborhood $p \in V$ of $Y$ such that $E \setminus \{p\} \cap Y \neq \varnothing$. As $V$ is open in $Y$ and $Y \subseteq X$, then $V = U \cap Y$ for some open set $U$ of $X$. But as $U \cap Y \subseteq U$, then $p \in U$ and $U$ is an open neighborhood of $p$ such that $U \cap E \setminus \{p\}$. Therefore $p \in \mathrm{Lim}_X(E)$, and hence $p \in \mathrm{Cl}_X(E) \cap Y$. The forward inclusion follows from these two cases.
($\supseteq$) Suppose that $\ell \in \mathrm{Cl}_X(E) \cap Y$. Then $\ell \in \mathrm{Lim}_X(E) \cup E$ and $\ell \in Y$. If $\ell \in E$, then $\ell \in \mathrm{Cl}_Y(E) = \mathrm{Lim}_Y(E) \cup E$, as $\ell \in E$. On the other hand, let $\ell \in \mathrm{Lim}_X(E)$. Then $\ell \in X$ and for any open neighborhood $\ell \in W$ of $X$ we have $W \cap E \setminus \{\ell\} \neq \varnothing$. As $W$ is open in $X$, then $L = W \cap Y$ is open in $Y$. But as $\ell$ in both $Y$ and $W$ then $L$ is an open neighborhood of $\ell$ in $X$. Lastly, as $L = W \cap Y \subseteq W$, then $L \cap E \setminus \{\ell\} \neq \varnothing$. Therefore $\ell \in \mathrm{Cl}_Y(E)$. Hence the backwards inclusion follows. $\square$

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Chapter 4 §1.2: Exercise 1.14.

Let $(X,d)$ be a metric space.

(a) Prove that for any $x \in X$ and $r > 0$, we have $\overline{B_X(x,r)} \subseteq \{y \in X : d(x,y) \leq r\}$. Note that the inclusion $\overline{B_X(x,r)} \subseteq B_X(x, r+\epsilon)$ follows for any $\epsilon > 0$.

(b) Give an example using the discrete metric that demonstrates that equality need not hold in the inclusion $\overline{B_X(x,r)} \subseteq \{y \in X : d(x,y) \leq r\}$ that you proved in part $(a)$

(c) Prove that in $\mathbb{R}^n$ under the Euclidean metric $d(x,y) = \|x-y\|$, we have $\overline{B_{\mathbb{R}^n}(x,r)} = \{y \in \mathbb{R}^n : \|x-y\| \leq r\}$.

(d) Using part $(a)$, prove that if $A$ is bounded in $(X,d)$, then $\overline{A}$ is also bounded in $(X,d)$.

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*Proof.* (a) Let $x \in X$ and $r > 0$. Then by Remark 1.18 in the course notes, if $p \in \overline{B_X(x,r)}$ if and only if for any $\epsilon > 0$ we have that $B_X(x,r) \cap B_X(p,\epsilon) \neq \varnothing$. By contrapositive suppose $d(x,y) > r$. Now consider $\epsilon = d(x,y) - r > 0$. So then $B_X(x,r) \cap B_X(p, d(x,y) - r) = \varnothing$. Thus we have that $p \notin \overline{B_X(x,r)}$.

(b) Consider $\mathbb{R}_{\text{disc}}$ with the discrete metric $d_{\text{disc}}(x,y) = 0$ if $x = y$ in $\mathbb{R}$ or 1 if $x \neq y$. Now consider $S = \{y \in \mathbb{R} : d_{\text{disc}}(3,y) \leq 1\}$. WLOG, consider $\ell \in S$ such that $\ell \neq 3$. Then $d_{\text{disc}}(3,\ell) = 0 \leq 1$. So then clearly $\ell \notin B_{\mathbb{R}}(3,1) = \{3\}$. Now we claim that $\ell \notin \text{Lim}_{\mathbb{R}}(B_{\mathbb{R}}(3,1))$. This is simple as $\text{Lim}_{\mathbb{R}}(B_{\mathbb{R}}(3,1)) = \text{Lim}_{\mathbb{R}}(\{3\})$. Now suppose that $\ell \in \text{Lim}_{\mathbb{R}}(\{3\})$. Then this would mean that $B_{\mathbb{R}}(\ell,1) \cap (\{3\} \setminus \{\ell\})$ intersect as $B_{\mathbb{R}}(\ell,1)$ is an open neighborhood of $\ell$. So then if $p \in B_{\mathbb{R}}(\ell,1) = \{\ell\}$ (i.e. $p = \ell$) and $p \in \{3\} \setminus \{\ell\}$ (i.e. $p = 3$ and $p \neq \ell$), then we have a contradiction. Therefore the equality in part (a) need not hold.

(c) The forward inclusion follows from part (a). Thus is remains to show the backwards inclusion: It suffices to show that that if we take $p \in \{y \in X : d(x,y) = r\}$. But $B_X(x,r) \subseteq \overline{B_X(x,r)} \subseteq \{y \in X : d(x,y) \leq r\}$.

(d) Suppose $A$ is bounded in $X$, i.e. we have $A \subseteq B_X(x,\epsilon)$, which is to say that $d(x,q) < \epsilon$ for all $q \in A$. So then for any $a \in \overline{A}$ we have that we have that $B_1(a,1)$ intersects with $A$; let $t$ be a point in this intersection. Then $d(a,x) \leq d(a,t) + d(t,x) < 1 + \epsilon$. Therefore we are done.

□

**Exercise 1.15.**

Let $(X, d)$ be a metric space.

(a) If $X$ is totally bounded, then it is bounded.

(b) If $Y \subseteq X$, then $(Y, d)$ is totally bounded if and only if for any $\epsilon > 0$, there exists $a_1, \ldots, a_J \in X$ such that $Y \subseteq \bigcup_{j=1}^{J} B_X(a_j, \epsilon)$.

(c) If $X$ is totally bounded and $Y \subseteq X$, then $Y$ is totally bounded.

(d) If $Y \subseteq X$ and $(Y, d)$ is totally bounded, then $(\overline{Y}, d)$ is totally bounded.

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*Proof.* (a) Suppose $X$ is totally bounded. Then $X = \bigcup_{j=1}^{n} B_X(x_j, \epsilon)$, and let $\epsilon = 1$ and $x, y \in X$. So $x \in B_X(x_j, 1)$ and $y \in B_X(x_i, 1)$ for some $1 \leq j, i \leq n$. Then $d(x, x_j) < 1$ and $d(x_i, y) < 1$. Write $\ell = \max_{1 \leq i,j \leq n}\{d(x_i, x_j)\}$. So then $d(x, y) \leq d(x, x_j) + d(x_j, x_i) + d(x_i, y) \leq 2 + \ell$. Hence $\operatorname{diam}(\bigcup_{j=1}^{n} B_X(x_j, \epsilon)) = \operatorname{diam}(X) \leq 2 + \ell$, and thus we can conclude that $X$ is bounded.
(b) Suppose $Y \subseteq X$. ($\Rightarrow$) Assume that $(Y, d)$ is totally bounded, i.e. $Y = \bigcup_{j=1}^{n} B_Y(y_j, \epsilon)$ for all $\epsilon > 0$. Each open ball $B_Y(y_j, \epsilon)$ is open in $Y$ so it can be written as $B_X(a_i, \epsilon) \cap Y$, so then $Y = \bigcup_{i=1}^{J}(B_X(a_j, \epsilon) \cap Y) = \bigcup_{i=1}^{J} B_X(a_j, \epsilon) \cap Y$. So then clearly $Y \subseteq \bigcup_{i=1}^{J} B_X(a_i, \epsilon)$. ($\Leftarrow$) Suppose that for any $\epsilon > 0$ there exists $a_1, \ldots, a_j \in X$ such that $Y \subseteq \bigcup_{j=1}^{J} B_X(a_j, \epsilon)$. Then for every open ball $B_X(a_j, \epsilon)$, we consider $B_X(a_j, \epsilon) \cap Y$ but then this just $B_Y(a_j, \epsilon)$. So then $Y = \bigcup_{j=1}^{n} B_Y(a_j, \epsilon)$ is clear.
(c) Let $X$ be totally bounded and let $Y \subseteq X$. Then $X = \bigcup_{j=1}^{n} B_X(x_i, 1)$. Then, for each 1-ball open in $X$, we have open 1-balls of $Y$ where $B_Y(x_\alpha, 1) = B_X(x_\alpha, 1) \cap Y \subseteq X$ with $1 \leq \alpha \leq n$. So then $X \cap Y = \bigcup_{j=1}^{n} B_X(x_i, 1) \cap Y = \bigcup_{i=1}^{k} B_Y(x_k, 1)$, but as $Y \subseteq X$, then $X \cap Y = Y$, so then $Y = \bigcup_{k=1}^{n} B_Y(x_j, 1)$.
(d) Suppose $Y \subseteq X$ and let $Y$ be totally bounded. Take $\epsilon > 0$. So then we have that $Y = \bigcup_{i=1}^{n} B_X(x_i, \epsilon/2)$. Now take $y \in \overline{Y}$. Then, by Remark 1.18, we have that $B_X(y, \epsilon/2)$ and $Y$ intersect. Now let $p \in B_X(y, \epsilon/2) \cap Y$. Then we have that $p \in B_X(x_\alpha, \epsilon/2)$ for some $1 \leq \alpha \leq n$ as $Y = \bigcup_{i=1}^{n} B_X(x_i, \epsilon/2)$. So then $d(x_\alpha, y) \leq d(x_\alpha, p) + d(p, y) < \epsilon/2 + \epsilon/2 = \epsilon$. Hence we have that $d(x_\alpha, y) < \epsilon$ so we have that $y \in B_X(x_\alpha, \epsilon/2) \subseteq \bigcup_{i=1}^{n} B_X(x_i, \epsilon/2)$. Hence $\overline{Y}$ is totally bounded. $\square$

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