MATH 425A HOMEWORK 6 SOLUTIONS

Assignment: Chapter 3, #2.9, 2.10, 3.1, 3.2, and Chapter 4, #1.1, 1.2, 1.3, 1.4, (1.5), (1.6).
Due Date: October 7, 2022
Rubric: (42 points total)

- Exercise 2.9. Category I (5 points)
- Exercise 2.10. Category II (5 points)
- Exercise 3.1. Category II (7 points)
- Exercise 3.2. Category II (6 points)
- Exercise 1.1. Category II (2 points)
- Exercise 1.2. Category I (5 points)
- Exercise 1.3. Category II (4 points)
- Exercise 1.4. Category I (5 points)
- Neatness: 3 points
- Optional: 1 LaTeXbonus point.

Please report any corrections, etc. to lesliet@usc.edu

CHAPTER 3

2.9. Suppose X is a finite, nonempty set and suppose d be a metric on X. Let $\mathcal{T}$ denote the topology generated by d. Show that $\mathcal{T} = \mathcal{P}(X)$. Conclude that any metric on X is equivalent to the discrete metric. (Hint: To show that $\mathcal{T} = \mathcal{P}(X)$, start by proving that $\{x\} = B_{(X,d)}(x, r_x)$ for some sufficiently small r_x, for each $x \in X$.)

Soln.: Given $x \in X$, define $r_x = \min_{y \in X \setminus \{x\}} d(x,y)$. This number is well-defined (since $X \setminus \{x\}$ is finite, so the minimum exists) and positive (since $d(x,y) > 0$ for $y \neq x$). Then $\{x\} = B_{(X,d)}(x, r_x)$, since $z \in B_{(X,d)}(x, r_x)$ implies $d(x,z) < r_x = \min_{y \in X \setminus \{x\}} d(x,y)$, which in turn implies that $z \notin X \setminus \{x\}$ and forces $z = x$. We conclude that $\{x\}$ is open in X, for any $x \in X$. But then any $E \in \mathcal{P}(X)$ is open in X, since $E = \bigcup_{x \in E} \{x\}$ is a union of open sets. Therefore the topology $\mathcal{T}$ generated by d is all of $\mathcal{P}(X)$, as needed. This argument applies to any metric, including the discrete metric. That is, any two metrics generate the same topology (namely $\mathcal{P}(X)$) on X and are therefore equivalent. We conclude that any metric is equivalent to the discrete metric on X.

2.10. Prove that the Euclidean metric and the square metric are equivalent on $\mathbb{R}^n$.

Soln.: Let $\mathcal{T}_E$ denote the topology generated by the Euclidean metric d_E, and let $\mathcal{T}_s$ denote the topology generated by the square metric d_s. We show that $\mathcal{T}_E \subset \mathcal{T}_s$ and $\mathcal{T}_s \subset \mathcal{T}_E$. We start by noting that given $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, we have

$$\max_{j \in J_n} |x_j - y_j|^2 \leq \sum_{i=1}^{n} (x_i - y_i)^2 \leq n \max_{j \in J_n} |x_j - y_j|^2.$$

That is,

$$d_s(x,y) \leq d_E(x,y) \leq \sqrt{n} d_s(x,y).$$

It follows that, given $r > 0$ and $x \in X$, we have

$$(*) \qquad B_{d_s}(x, \tfrac{r}{\sqrt{n}}) \subset B_{d_E}(x, r) \subset B_{d_s}(x, r).$$

(Indeed, if $d_s(x,y) < \frac{r}{\sqrt{n}}$, then $d_E(x,y) \leq \sqrt{n} d_s(x,y) < r$; similarly, if $d_E(x,y) < r$, then $d_s(x,y) \leq d_E(x,y) < r$ as well.)

With $(*)$ in hand, the inclusions $\mathcal{T}_E \subset \mathcal{T}_s$ and $\mathcal{T}_s \subset \mathcal{T}_E$ are easy to prove: Assume $U \in \mathcal{T}_E$. Choose $x \in U$, then choose $r > 0$ so that $B_{d_E}(x,r) \subset U$. Then $B_{d_s}(x, \frac{r}{\sqrt{n}}) \subset B_{d_E}(x,r) \subset U$, so x is an interior point of U with respect to (X, d_s). Since x was an arbitrary point of U, we conclude that U is open in (X, d_s), i.e., $U \in \mathcal{T}_s$. This shows that $\mathcal{T}_E \subset \mathcal{T}_s$. On the other hand, if $V \in \mathcal{T}_s$, choose $y \in V$ and $r > 0$ such that $B_{d_s}(y,r) \subset V$. Then $B_{d_E}(y,r) \subset B_{d_s}(y,r) \subset V$, whence y is an interior point of V with respect to (X, d_E); since y was arbitrary, we conclude that $V \in \mathcal{T}_E$. This shows that $\mathcal{T}_s \subset \mathcal{T}_E$.

3.1. Prove Proposition 3.4. The following may be helpful.

- Given $\mathcal{B}$ satisfying the two properties of the Proposition, you can verify that $\mathcal{T}$ as defined in (9) satisfies each of the three points in the definition of a topology.
- In the converse statement, you are given a topology $\mathcal{T}$ and a subcollection $\mathcal{B}$ for which (9) holds. You need to show that $\mathcal{B}$ satisfies (1) and (2) in the Proposition. For (1), start your (short) argument by noting that X must belong to $\mathcal{T}$. For (2), it is helpful to remember that $\mathcal{B} \subset \mathcal{T}$, so $B_1 \cap B_2$ must be an element of $\mathcal{T}$ if B_1 and B_2 are. Argue that if $B_1 \cap B_2 = \bigcup_{B \in \mathcal{A}} B$ for some $\mathcal{A} \subset \mathcal{B}$ and $x \in B_1 \cap B_2$, then x must belong to some $B \in \mathcal{A}$, and this B must be contained in $B_1 \cap B_2$.

Soln.: For convenience, we rewrite the statement of the Proposition:

Proposition: Let X be a set, and let $\mathcal{B}$ be a collection of subsets of X which has the following properties:

(1) Every $x \in X$ is contained in at least one element B of $\mathcal{B}$.
(2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Then the following collection $\mathcal{T}$ is a topology on X:

$$\mathcal{T} = \left\{ U \in \mathcal{P}(X) : U = \bigcup_{B \in \mathcal{A}} B \text{ for some subcollection } \mathcal{A} \subset \mathcal{B} \right\}, \tag{9}$$

and $\mathcal{B}$ is a basis for $\mathcal{T}$. On the other hand, if $\mathcal{T}$ is a topology on X and $\mathcal{B}$ is a subcollection of $\mathcal{T}$ such that (9) holds, then $\mathcal{B}$ must satisfy properties (1) and (2).

Proof. Suppose $\mathcal{B}$ satisfies properties (1) and (2). We show that the corresponding $\mathcal{T}$ defined in (9) satisfies the properties of a topology. First, $\emptyset$ is the empty union and thus belongs to $\mathcal{T}$. Next, for each $x \in X$, choose $B_x \in \mathcal{B}$ such that $x \in B_x$. Then $X = \bigcup_{x \in X} B_x \in \mathcal{T}$.

Next, assume $\mathcal{U} \subset \mathcal{T}$. For each $U \in \mathcal{U}$, let $\mathcal{B}_U$ be a subset of $\mathcal{B}$ such that $U = \bigcup_{B \in \mathcal{B}_U} B$. Then

$$\bigcup_{U \in \mathcal{U}} U = \bigcup_{\substack{U \in \mathcal{U}, \\ B \in \mathcal{B}_U}} B \in \mathcal{T}.$$

Now, let $\{U_1, \ldots, U_n\}$ be a *finite* subset of $\mathcal{T}$, and denote $U = U_1 \cap \cdots \cap U_n$. For each $x \in U$ and each $j \in J_n$, choose $B_j \in \mathcal{B}$ such that $x \in B_j \subset U_j$. (Such a choice of B_j is possible by the definition of $\mathcal{T}$.) Then, pick $B_x \in \mathcal{B}$ such that $x \in B_x \subset B_1 \cap \cdots \cap B_n$ (this is possible by property (2) of $\mathcal{B}$, by induction). Then $x \in B_x \subset U$ for each x, with $B_x \in \mathcal{B}$. It follows that

$$U = \bigcup_{x \in U} B_x,$$

which finishes the proof that $\mathcal{T}$ is a topology on X.

Now, let us assume that $\mathcal{T}$ is a topology on X, and $\mathcal{B}$ is a subcollection of $\mathcal{T}$ for which (9) holds. To see that $\mathcal{B}$ satisfies (1), we note that $X \in \mathcal{T}$ (since $\mathcal{T}$ is a topology), so we can find a subcollection $\mathcal{B}_X$ such that $X = \bigcup_{B \in \mathcal{B}_X} B$. But then any $x \in X$ must belong to some $B \in \mathcal{B}_X \subset \mathcal{B}$.

Next, we prove that $\mathcal{B}$ satisfies (2). To this end, choose $B_1, B_2 \in \mathcal{B}$ and assume $x \in B_1 \cap B_2$. Since $\mathcal{B} \subset \mathcal{T}$, we have in particular that $B_1 \cap B_2$ can be written as a union of the form

$$B_1 \cap B_2 = \bigcup_{B \in \mathcal{A}} B,$$

for some subcollection $\mathcal{A}$ of $\mathcal{B}$. Then x must belong to some $B \in \mathcal{A}$, and on the other hand $B \subset B_1 \cap B_2$. Therefore $x \in B \subset B_1 \cap B_2$, and we have established (2). $\square$

3.2. Prove that the collection $\mathcal{R}$ of all *open rectangles* of the form

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n), \qquad a_j, b_j \in \mathbb{R},\ a_j < b_j, \text{ for all } j.$$

is a basis for the standard topology on $\mathbb{R}^n$. You are encouraged to use the following (partial) outline:

- Prove that $\mathcal{R}$ satisfies the hypotheses of Proposition 3.4 and is therefore a basis for *some* topology on $\mathbb{R}^n$.
- Show that if $R \in \mathcal{R}$, then R is open with respect to the Euclidean metric.
- Show that if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $r > 0$, then $B(x, r)$ can be written as a union of open rectangles in $\mathcal{R}$. (This is easier than it seems—mimic part of the proof of Theorem 2.12.) Alternatively, use Exercise 2.10.
- Finish by invoking Proposition 3.6.

Soln.: Given $x = (x_1, \ldots, x_n) \in \mathcal{R}^n$, we have

$$x \in (x_1 - 1, x_1 + 1) \times \cdots \times (x_n - 1, x_n + 1).$$

Thus $\mathcal{R}$ satisfies property (1) from Proposition 3.4. As for property (2), we note the general identity

$$(A_1 \times \cdots \times A_n) \cap (B_1 \times \cdots \times B_n) = (A_1 \cap B_1) \times \cdots \times (A_n \cap B_n),$$

which holds for any subsets A_j, B_j of any set X.

Since the intersection of two open intervals is always either an open interval or the empty set, it follows from the above identity that the intersection of two open rectangles is another open rectangle, or the empty set.

Next, given $x \in R^n$ and $r > 0$, we show that the Euclidean ball $B(x, r)$ can be written as the union of elements of $\mathcal{R}$. To this end, choose $y = (y_1, \ldots, y_n) \in B(x, r)$. Choose $\delta > 0$ such that $B(y, \delta) \subset B(x, r)$. Put $\eta = \frac{\delta}{\sqrt{n}}$. Then (by a short calculation similar to the one in the solution to Exercise 2.10)

$$R_y = (y_1 - \eta, y_1 + \eta) \times \cdots \times (y_n - \eta, y_n + \eta)$$

is an element of $\mathcal{R}$ which contains y and is entirely contained in $B(x, r)$. (Note that R_y is the ball of radius η, centered at y, with respect to the square metric.) Choosing R_y similarly for each $y \in B(x, r)$, we conclude that $B(x, r) = \bigcup_{y \in B(x,r)} R_y$. We have thus shown that every element of the basis $\mathcal{B}$ of Euclidean balls for the Euclidean topology $\mathcal{T}$ is in fact open with respect to the topology $\mathcal{T}_R$ on $\mathbb{R}^n$ which we generated from $\mathcal{R}$. That is, $\mathcal{B} \subset \mathcal{T}_R$. By Proposition 3.6, it follows that $\mathcal{T} \subset \mathcal{T}_R$. For the opposite inclusion, we note that every $R \in \mathcal{R}$ is open with respect to the usual topology on $\mathbb{R}^n$; that is, $\mathcal{R} \subset \mathcal{T}$. (Proof: If $R = (a_1, b_1) \times \cdots \times (a_n, b_n)$ and $x = (x_1, \ldots, x_n) \in R$, then $B(x, r) \subset R$ for $r = \min\{x_1 - a_1, b_1 - x_1, \ldots, x_n - a_n, b_n - x_n\}$.) Using Proposition 3.6 again, we conclude that $\mathcal{T}_R \subset \mathcal{T}$. Thus $\mathcal{R}$ is a basis for the standard topology on $\mathbb{R}^n$.

CHAPTER 4

1.1. Let E_1 and E_2 be subsets of a metric space (X, d). Prove that

$$\mathrm{Lim}_X(E_1 \cup E_2) = \mathrm{Lim}_X(E_1) \cup \mathrm{Lim}_X(E_2).$$

Soln.: Since E_1 and E_2 are both subsets of $E_1 \cup E_2$, Proposition 1.8 immediately gives that $\mathrm{Lim}_X(E_1) \cup \mathrm{Lim}_X(E_2) \subset \mathrm{Lim}_X(E_1 \cup E_2)$. For the opposite inclusion, assume that $x \in \mathrm{Lim}_X(E_1 \cup E_2)$. Let U be a neighborhood of x in X, and choose $y \in U \cap ((E_1 \cup E_2)\backslash\{x\}) = (U \cap (E_1\backslash\{x\})) \cup (U \cap (E_2\backslash\{x\}))$. Then y must belong to either $U \cap (E_1\backslash\{x\})$ (in which case $x \in \mathrm{Lim}_X(E_1)$) or $U \cap (E_2\backslash\{x\})$ (in which case $x \in \mathrm{Lim}_X(E_2)$). In either case, we have $x \in \mathrm{Lim}_X(E_1) \cup \mathrm{Lim}_X(E_2)$, proving the claim.

1.2. Let (X, d) be a metric space, and assume $E \subset Y \subset X$. Prove that

$$\mathrm{Lim}_Y(E) = \mathrm{Lim}_X(E) \cap Y.$$

Hint: Use Theorem 2.13 in Chapter 3.

Soln.: Assume $x \in \mathrm{Lim}_Y(E)$. Then $x \in Y$ by definition of a limit point. Let U be a neighborhood of x in X. Then $U \cap Y$ is a neighborhood of x in Y, so there exists $y \in E \cap ((U \cap Y)\backslash\{x\}) = E \cap (U\backslash\{x\})$. So $x \in \mathrm{Lim}_X(E)$ as well.

Assume now that $x \in \mathrm{Lim}_X(E) \cap Y$, and let V be a neighborhood of x in Y. Then (by Theorem 2.13 in Chapter 3) $V = U \cap Y$ for some open set U of X, and U is a neighborhood of x in X. Since $x \in \mathrm{Lim}_X(E)$, it follows that $U \cap (E\backslash\{x\})$ is nonempty; suppose y belongs to this set. Note that since $E \subset Y$, we have $E = E \cap Y$, so $U \cap (E\backslash\{x\}) = V \cap (E\backslash\{x\})$. Thus $y \in V \cap (E\backslash\{x\})$, which implies that $x \in \mathrm{Lim}_Y(E)$, as needed.

1.3. If $(X, \mathcal{T})$ is a topological space and E is a subset of X, we say that x is a limit point of E with respect to X if every neighborhood of x in X (that is, every $U \in \mathcal{T}$ such that $U \ni x$) intersects $E\backslash\{x\}$.

(a) Suppose $\mathcal{B}$ is a basis for the topology $\mathcal{T}$ on X. Show that x is a limit point of E with respect to X if and only if every $B \in \mathcal{B}$ containing x intersects $E\backslash\{x\}$.
(b) Show that if E is any subset of $\mathbb{R}$ which is not bounded above (with respect to the usual order relation on $\mathbb{R}$), then $+\infty$ is a limit point of E with respect to $\overline{\mathbb{R}}$ (in its standard topology).

Soln.: (a) Suppose $x \in E'$. If $B \in \mathcal{B}$ contains x, then (since in particular $B \in \mathcal{T}$) B must intersect $E\backslash\{x\}$. On the other hand, assume that every $B \in \mathcal{B}$ containing x intersects $E\backslash\{x\}$, and let U be a neighborhood of x in X. Then there exists $B \in \mathcal{B}$ such that $x \in B \subset U$, and this B intersects $E\backslash\{x\}$; therefore U intersects $E\backslash\{x\}$ as well (since $B \cap (E\backslash\{x\}) \subset U \cap (E\backslash\{x\})$). Thus $x \in E'$.

(b) Suppose $E \subset \mathbb{R}$ is not bounded above in $\mathbb{R}$. In the basis $\overline{\mathcal{B}}$ for the standard topology on $\overline{\mathbb{R}}$ that we introduced in Chapter 3, the elements of $\overline{\mathcal{B}}$ that contain $+\infty$ are the ones of the form $(b, +\infty]$. Each one of these intersects $E = E\backslash\{+\infty\}$; otherwise E would be bounded above. It follows by part (a) that $+\infty$ is a limit point of E.

1.4. Let (X, d) be a metric space, and assume $Y \subset X$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in Y and let x be a point of X. Prove that the following two statements are equivalent:

(1) $x_n \to x$ in X, and $x \in Y$.
(2) $x_n \to x$ in Y.

Hint: Use Theorem 2.13 in Chapter 3.

Soln.: Assume (1) holds, and let V be a neighborhood of x in Y. Then (by Theorem 2.13 in Chapter 3) $V = U \cap Y$ for some neighborhood U of x in X. Choose $N \in \mathbb{N}$ large enough so that $n \geq N$ implies that $x_n \in U$. Since $(x_n)_{n=1}^{\infty}$ is a sequence in Y, it follows that $n \geq N$ actually implies that $x_n \in U \cap Y = V$. Thus $x_n \to x$ in Y.

Assume $x_n \to x$ in Y, and let U be a neighborhood of x in X. Then $V = U \cap Y$ is a neighborhood of x in Y, so there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in V \subset U$. Thus $x_n \to x$ in X. The fact that $x \in Y$ is part of the statement $x_n \to x$ in Y, so we are done.

1.5. Let (X, d) be a metric space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in X. Prove that the following statements are equivalent:

(1) $x_n \to x$ in X.
(2) For every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in B_X(x, \varepsilon)$ (i.e. $d(x, x_n) < \varepsilon$).
(3) $d(x, x_n) \to 0$ as $n \to \infty$

Soln.: (1) $\Longrightarrow$ (2). Assume that $x_n \to x$ in X. Then (2) follows directly from the fact that $B_X(x, \varepsilon)$ is a neighborhood of x in X.

(2) $\Longrightarrow$ (3). Assume (2) holds, and let U be a neighborhood of 0 in $\mathbb{R}$. Choose $\varepsilon > 0$ so that $(-\varepsilon, \varepsilon) = B_{\mathbb{R}}(0, \varepsilon) \subset U$, and choose $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in B_X(x, \varepsilon)$. Then $n \geq N$ implies $d(x, x_n) < \varepsilon$, so $d(x, x_n) \in (-\varepsilon, \varepsilon) \subset U$. Thus $d(x, x_n) \to 0$ as $n \to \infty$.

(3) $\Longrightarrow$ (1) Assume (3) holds, and let U be a neighborhood of x in X. Choose $\varepsilon > 0$ such that $B_X(x, \varepsilon) \subset U$, then choose $N \in \mathbb{N}$ large enough so that $n \geq N$ implies $d(x, x_n) \in (-\varepsilon, \varepsilon)$. Then $n \geq N$ implies $d(x, x_n) < \varepsilon$, which implies $x_n \in B_X(x, \varepsilon) \subset U$. So $x_n \to x$ in X.

Note: Here we have proved (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1). A slightly longer but possibly conceptually easier route would be to prove (1) $\Longleftrightarrow$ (2) and then (2) $\Longleftrightarrow$ (3).

1.6. Let (X, d) be a metric space; let $(x_n)_{n=1}^{\infty}$ be a sequence in X, and let $(t_n)_{n=1}^{\infty}$ be a sequence of positive real numbers. Assume that $t_n \to 0$ as $n \to \infty$.

(a) Prove that if there exists $x \in X$ and $N \in \mathbb{N}$ such that $d(x_n, x) \leq t_n$ for all $n \geq N$, then $x_n \to x$ in X as $n \to \infty$.
(b) (For later use) Prove that $1/n \to 0$ as $n \to \infty$.

Soln.: (a) For the first statement, choose $\varepsilon > 0$, then choose $N \in \mathbb{N}$ such that $n \geq N$ implies $t_n < \varepsilon$. Then $n \geq N$ implies that $d(x, x_n) < \varepsilon$, which proves $d(x, x_n) \to 0$, so that $x_n \to x$ as $n \to \infty$ (by the previous problem, for example).

(b) We prove that $\frac{1}{n} \to 0$ as $n \to \infty$. Indeed, choose $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that $N > \varepsilon^{-1}$. Then $n \geq N$ implies $|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. So $\frac{1}{n} \to 0$ as $n \to \infty$, as claimed.