

MATH 425A HW5, DUE 09/27/2022, 6PM

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CHAPTER 3. §2.

**Exercise 0.1** (2.5.). Finish the proof of Proposition 2.6., by proving that

$$\text{Int}_Y(U) \cap \text{Int}_X(Y) \subseteq \text{Int}_X(U) \quad (U \subseteq Y \subseteq X)$$

*Proof.* Suppose  $U \subseteq Y \subseteq X$ , where  $(X, d)$  is a metric space. Now take  $t \in \text{Int}_Y(U) \cap \text{Int}_X(Y)$ . Then we have that there is some  $r_\alpha > 0$  with  $B_Y(t, r_\alpha) \subseteq U$ , and there is some  $r_\beta$  such that  $B_X(t, r_\beta) \subseteq Y$ . Using ball notation, we know that  $t \in U \cap Y \cap X$ , and that  $B_X(t, r_\beta) = \{y \in X : d(t, y) < r_\beta\}$  and  $B_Y(t, r_\alpha) = \{q \in Y : d(t, q) < r_\alpha\}$ . So we need to show that  $t \in \text{Int}_X(U)$ , i.e.  $t \in X$  and there is some  $r_\gamma > 0$  such that  $B_X(t, r_\gamma) \subseteq U$ .  $\square$

**Exercise 0.2** (2.6.). As in Example 2.4, let  $X = \mathbf{R}^2$ ,  $Y = [-1, 3] \times [2, 4]$ , and let  $d$  denote the Euclidean metric on  $X = \mathbf{R}^2$ . Let  $p = (3, 4)$  and let  $q = (2, 4)$ .

- (a) Arguing *directly from the definition of an interior point* (i.e., without using Proposition 2.6, show that  $q$  is an interior point of  $B_Y(p, 2)$  with respect to  $Y$ , but  $q$  is not an interior point of  $B_Y(p, 2)$  with respect to  $X$ . In addition, draw a picture on a piece of graph paper that illustrates the idea of your proof.
- (b) Give a short argument that re-establishes your conclusion from (a) but relies instead on Proposition 2.6.

*Proof.* (a) Firstly, we claim that  $q = (2, 4) \in B_Y(p, 2) = B_Y((3, 4), 2)$ . This is easy to see since  $d(p, q) = d((3, 4), (2, 4)) = \sqrt{(4-4)^2 + (3-2)^2} = 1 < 2$ . Now, let  $\epsilon = 2 - d(p, q) = 2 - 1 = 1$ . We claim that  $B_Y(q, \epsilon) = B_Y(q, 1) \subseteq B_Y(p, 2)$ . Let  $t \in B_Y(q, 1)$ , where  $t = (t_1, t_2) \in \mathbf{R}^2$ . Then  $d(q, t) = \sqrt{(2-t_1)^2 + (4-t_2)^2} < 1$ . If  $t \in B_Y(p, 2)$ , then we need that  $d = d(p, t) = \sqrt{(3-t_1)^2 + (4-t_2)^2} < 2$ ; we claim that  $t \in B_Y(p, 2)$ . By the triangle inequality, we have that  $d(p, t) \leq d(p, q) + d(q, t) = 1 + 1 = 2$ . But as  $d(p, t) < 2$ , then we have that  $d(p, t) < 2$ , and hence  $d(p, t) < 2$ . Therefore  $t \in B_Y(p, 2)$ , and we have that  $B_Y(q, 1) \subseteq B_Y(p, 2)$ . Thus  $q$  is an interior point of  $B_Y(p, 2)$ .

We claim that  $q$  is not an interior point of  $B_Y(p, 2)$  with respect to  $X$ , i.e. we cannot find a ball  $B_X(q, \alpha) \subseteq B_Y(p, 2)$  with  $\alpha > 0$ . The simple reason of this is the fact that the ball of such a hypothetical  $B_X(q, \alpha)$  is ‘too big’ to be contained in  $B_Y(p, 2)$ . To show that this sort of containment isn’t possible, we shall find such an element that is in  $B_X(q, \alpha)$  but not in  $B_Y(p, 2)$ . Suppose that an open ball  $B_X(q, \alpha)$ , where  $\alpha > 0$  exists, and let  $B_X(q, \alpha) \subseteq B_Y(p, 2)$ . Now suppose that  $\alpha \geq 2$ . Then pick  $s = (2, 5)$ , and so  $\sqrt{(2-2)^2 + (4-5)^2} = 1 < \alpha$ ; thus we have that  $s \in B_X(q, \alpha)$  but is not in  $B_Y(p, 2)$  since  $s \notin Y = [-1, 3] \times [2, 4]$  a priori. Now suppose that  $\alpha < 2$ . Then as  $\alpha > 0$ , we can find  $\gamma \in \mathbf{R}$  such that  $0 < \gamma < \alpha$ . Consider  $s = (2, 4 + \gamma)$ , which is clearly not in  $Y$ . But  $\sqrt{(2-2)^2 + (4-(4+\gamma))^2} = \sqrt{\gamma^2} = \gamma < \alpha$ , and so  $s \in B_X(q, \alpha)$ , but yet once again  $s \notin B_Y(p, 2)$  since  $s \notin Y$ . Therefore the initial claim follows.  $\square$

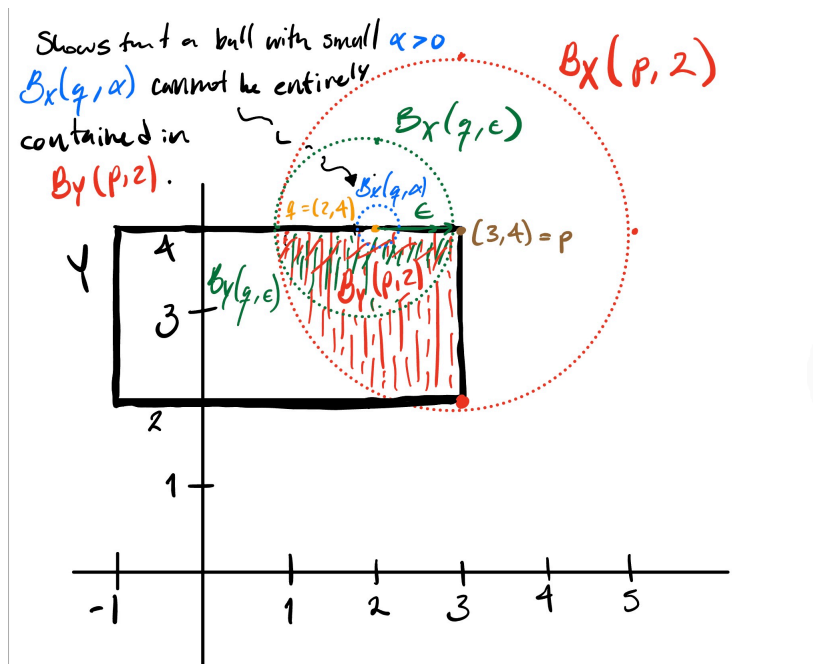


FIGURE 1. The illustration of the proof given for part (a) of Exercise 2.6.

**Exercise 0.3** (2.7.). Let  $(X, d)$  be a metric space, and let  $U$  be a subset of  $X$ . Use Proposition 2.9 to prove that  $\text{Int}_X(U)$  is open in  $X$ .

*Proof.* We want to show that  $\text{Int}_X(U) \subseteq \text{Int}_X(\text{Int}_X(U))$ . So let  $t \in \text{Int}_X(U)$ . Then  $t \in U$  with  $B(t, r) \subseteq U$  for some  $r > 0$ . Now let  $y \in B(t, r)$ , but as open balls are open then  $y \in \text{Int}(B(t, r))$ , and so we can find another open ball  $y \in B(t, \epsilon) \subseteq B(t, r)$ . All together, we have that  $B(t, \epsilon) \subseteq B(t, r) \subseteq U$ . Thus if we have some point  $y \in B(t, r)$  then  $y \in \text{Int}(U)$ , and hence  $B(t, r) \subseteq \text{Int}(U)$ . Therefore  $\text{Int}(U)$  is an open set.  $\square$

**Exercise 0.4** (2.8.). Let  $(X, d)$  be a metric space. Assume that  $U \subseteq Y \subseteq X$ , and additionally that  $Y$  is open in  $X$ . Prove that  $U$  is open in  $Y$  if and only if  $U$  is open in  $X$ . (Note: There at least two possible solutions; one uses Theorem 2.13, the other uses Exercise 2.5.)

*Proof.* Suppose that  $U$  is open in  $Y$ . Then  $U = Y \cap V$  for some open set  $V \subseteq X$ . But as finite intersections of open sets are open, and  $Y \subseteq X$  and  $V \subseteq X$  are both open in  $X$ , then  $U = Y \cap V$  is open in  $X$  by Theorem 2.13. Now suppose that  $U$  is open in  $X$ . Then  $U = Y \cap U$  as  $U \subseteq Y$ , but as  $U$  and  $Y$  are both open in  $X$ , then by Theorem 2.13  $U$  is open in  $Y$  as well.  $\square$

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