MATH 425A HW5, DUE 09/27/2022, 6PM

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Chapter 3. §2.

Exercise 0.1 (2.5.)**.** Finish the proof of Proposition 2.6., by proving that

$$\mathrm{Int}_Y(U) \cap \mathrm{Int}_X(Y) \subseteq \mathrm{Int}_X(U) \qquad (U \subseteq Y \subseteq X)$$

Proof. Suppose $U \subseteq Y \subseteq X$, where (X, d) is a metric space. Let $s \in \mathrm{Int}_Y(U) \cap \mathrm{Int}_X(Y)$. Then there is some ball $B_Y(s, \alpha) \subseteq U$ and $\alpha > 0$, and there is also another ball $B_X(s, \beta) \subseteq Y$ with $\beta > 0$. We want to show that there is some ball $B_X(s, \gamma) \subseteq U$ with $\gamma > 0$. We will pick our needed ball based on two cases: whether $\alpha \geq \beta$ or $\alpha < \beta$.

Suppose $\beta = \alpha$. Then we have that given $B_Y(s, \beta) = B_X(s, \beta) \cap Y = B_X(s, \beta)$ as $B_X(s, \beta)$ is contained in Y. But as $\beta = \alpha$, then $B_Y(s, \beta) = B_Y(s, \alpha) \subseteq U$. Hence we have that the ball $B_X(s, \beta) = B_Y(s, \beta)$ is contained in U.

Suppose that $\alpha > \beta$. Let us take $t \in B_X(s, \beta)$. So we have that $t \in X$ and $d(s, t) < \beta$. We already know that $B_X(s, \beta) \subseteq Y$, and so $t \in Y$ as well. But as $\beta < \alpha$, then $d(t, s) < \beta < \alpha$, and $t \in B_Y(s, \alpha)$. Hence $B_X(s, \beta) \subseteq B_Y(s, \alpha)$. As $B_Y(s, \alpha) \subseteq U$, we thus have that $B_X(s, \beta) \subseteq U$.

Suppose that $\alpha < \beta$. Now pick $\gamma \in \mathbf{R}$ such that $0 < \gamma < \alpha$, which is valid since by the density property and the fact that we assumed $\alpha > 0$. Then consider $B_X(s, \gamma) = \{t \in X \colon d(s, t) < \gamma\}$. If $t \in B_X(s, \gamma)$, then $t \in X$ such that $d(s, t) < \gamma$, but as $0 < \gamma < \alpha$, we have that $d(s, t) < \alpha$ as well, and so $t \in B_X(s, \alpha)$. But as $\gamma < \alpha < \beta$, then we clearly also have that $d(s, t) < \beta$, i.e. $t \in B_X(s, \beta)$. But as $B_X(s, \beta) \subseteq Y$, we have that $t \in Y$. Thus $t \in B_Y(s, \alpha)$, which is indeed contained in U by hypothesis. Thus $B_X(s, \gamma) \subseteq U$. □

Exercise 0.2 (2.6.)**.** As in Example 2.4, let $X = \mathbf{R}^2$, $Y = [-1, 3] \times [2, 4]$, and let d denote the Euclidean metric on $X = \mathbf{R}^2$. Let $p = (3, 4)$ and let $q = (2, 4)$.

(a) Arguing *directly from the definition of an interior point* (i.e., without using Proposition 2.6, show that q is an interior point of $B_Y(p, 2)$ with respect to Y, but q is not an interior point of $B_Y(p, 2)$ with respect to X). In addition, draw a picture on a piece of graph paper that illustrates the idea of your proof.

(b) Give a short argument that re-establishes your conclusion from (*a*) but relies instead on Proposition 2.6.

Proof. (a) Firstly, we claim that $q = (2, 4) \in B_Y(p, 2) = B_Y((3, 4), 2)$. This is easy to see since $d(p, q) = d((3, 4), (2, 4)) = \sqrt{(4-4)^2 + (3-2)^2} = 1 < 2$. Now, let $\epsilon = 2 - d(p, q) = 2 - 1 = 1$. We claim that $B_Y(q, \epsilon) = B_Y(q, 1) \subseteq B_Y(p, 2)$. Let $t \in B_Y(q, 1)$, where $t = (t_1, t_2) \in \mathbf{R}^2$. Then $d(q, t) = \sqrt{(2-t_1)^2 + (4-t_2)^2} < 1$. If $t \in B_Y(p, 2)$, then we need that $d = (p, t) = \sqrt{(3-t_1)^2 + (4-t_2)^2} < 2$; we claim that $t \in B_Y(p, 2)$. By the triangle inequality, we have that $d(p, t) \leq d(t, q) + d(q, p) = d(t, q) + 1$. But as $d(t, q) < 1$, then we have that $d(t, q) + 1 < 2$, and hence $d(p, t) < 2$. Therefore $t \in B_Y(p, 2)$, and we have that $B_Y(q, 1) \subseteq B_Y(p, 2)$. Thus q is an interior point of $B_Y(p, 2)$.

We claim that q is not an interior point of $B_Y(p, 2)$ with respect to X, i.e. we cannot find a ball $B_X(q, \alpha) \subseteq B_Y(p, 2)$ with $\alpha > 0$. The simple reason of this is the fact that

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the ball of such a hypothetical $B_X(q,\alpha)$ is 'too big' to be contained in $B_Y(p,2)$. To show that this sort of containment isn't possible, we shall find such an element that is in $B_X(q,\alpha)$ but not in $B_Y(p,2)$. Suppose that an open ball $B_X(q,\alpha)$, where $\alpha > 0$ exists, and let $B_X(q,\alpha) \subseteq B_Y(p,2)$. Now suppose that $\alpha \geq 2$. Then pick $s = (2,5)$, and so $\sqrt{(2-2)+(4-5)^2} = 1 < \alpha$; thus we have that $s \in B_X(q,\alpha)$ but is not in $B_Y(p,2)$ since $s \notin Y = [-1,3]\times[2,4]$. Now suppose that $\alpha < 2$. Then as $\alpha > 0$, we can find $\gamma \in \mathbf{R}$ such that $0 < \gamma < \alpha$. Consider $s = (2, 4+\gamma)$, which is clearly not in Y. But $\sqrt{(2-2)^2+(4-(4+\gamma))^2} = \sqrt{\gamma^2} = \gamma < \alpha$, and so $s \in B_X(q,\alpha)$, but yet once again $s \notin B_Y(p,2)$ since $s \notin Y$. Therefore the initial claim follows.

(b) Using Proposition 2.6., to establish our conclusion, we use the fact that $\mathrm{Int}_X(B_Y(p,2)) = \mathrm{Int}_Y(B_Y(p,2)) \cap \mathrm{Int}_X(Y)$. Thus q is not an interior point of $B_Y(p,2)$ if and only if q is not in $\mathrm{Int}_Y(B_Y(p,2)) \cap \mathrm{Int}_X(Y)$. From (a) we've already established that q is an interior point of $B_Y(p,2)$ with respect to Y, and so we must show that $q \notin \mathrm{Int}_X(Y)$. We argue in aim of contradiction. Suppose that $q \in \mathrm{Int}_X(Y)$. So then there exists some ball $B_X(q,\gamma)$ for some $\gamma > 0$ with $B_X(q,\gamma) \subseteq Y$; we want to show that there exists some $s \in B_X(q,\gamma)$ such that $s \notin Y$. As $\gamma > 0$, then we can find some $\beta \in \mathbf{R}$ such that $0 < \beta < \gamma$. Now pick $s = (2, 4+\beta)$. Then $s \notin Y = [-1,3]\times[2,4]$ as $4+\beta > 4$. But $s \in X = \mathbf{R}^2$ and $d(s,q) = \sqrt{(2-2)^2+(4-(4+\beta))^2} = \beta < \gamma$, and thus $s \in B_X(q,\gamma)$. Therefore we have a contradiction and $q \notin \mathrm{Int}_X(Y)$. Hence we have that $q \notin \mathrm{Int}_X(B_Y(p,2))$ by Proposition 2.6.

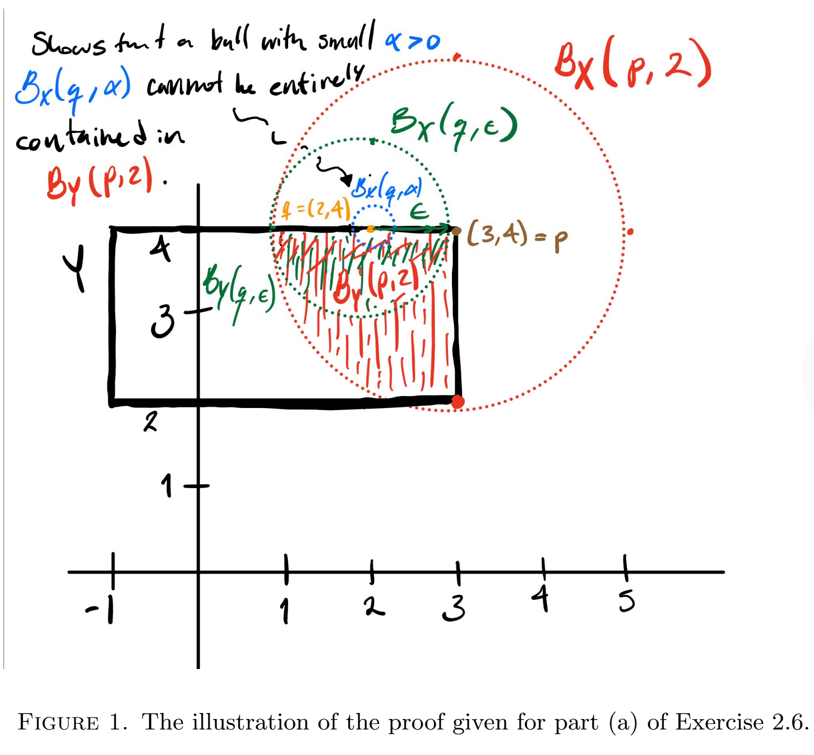

FIGURE 1. The illustration of the proof given for part (a) of Exercise 2.6.

$\square$

Exercise 0.3 (2.7.)**.** Let (X,d) be a metric space, and let U be a subset of X. Use Proposition 2.9 to prove that $\mathrm{Int}_X(U)$ is open in X.

Proof. We want to show that $\mathrm{Int}_X(U) \subseteq \mathrm{Int}_X(\mathrm{Int}_X(U))$. So let $t \in \mathrm{Int}_X(U)$. Then $t \in U$ with $B(t,r) \subseteq U$ for some $r > 0$. Now let $y \in B(t,r)$, but as open balls are open then

$y \in \text{Int}(B(t,r))$, and so we can find an another open ball $y \in B(t,\epsilon) \subseteq B(t,r)$. All together, we have that $B(t,\epsilon) \subseteq B(t,r) \subseteq U$. Thus if we have some point $y \in B(t,r)$ then $y \in \text{Int}(U)$, and hence $B(t,r) \subseteq \text{Int}(U)$. Therefore $\text{Int}(U)$ is an open set. $\square$

Exercise 0.4 (2.8.)**.** Let (X,d) be a metric space. Assume that $U \subseteq Y \subseteq X$, and additionally that Y is open in X. Prove that U is open in Y if and only if U is open in X. (Note: There at least two possible solutions; one uses Theorem 2.13, the other uses Exercise 2.5.)

Proof. Suppose that U is open in Y. Then $U = Y \cap V$ for some open set $V \subseteq X$. But as finite intersections of open sets are open, and $Y \subseteq X$ and $V \subseteq X$ are both open in X, then $U = Y \cap V$ is open in X by Theorem 2.13. Now suppose that U is open in X. Then $U = Y \cap U$ as $U \subseteq Y$, but as U and Y are both open in X, then by Theorem 2.13 U is open in Y as well. $\square$

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