MATH 425A HW1, DUE 09/02/2022

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1. §1.1.

Exercise 1.1 (1.1.)**.** Let A and B be subsets of another set X. Prove the following statements.

(a) $A \cap B = A \setminus (A \setminus B)$
(b) $A \subseteq B$ if and only of $X \setminus A \supseteq X \setminus B$.

Proof. (a) For the backwards inclusion, suppose $x \in A \setminus (A \setminus B)$. Then $x \in A$ but $x \notin A \setminus B$ so then we have that $x \notin \{x \in A : x \notin B\}$ which means that we must have that $x \in B$ since $x \in A$. Thus $x \in A \cap B$. Now suppose we have $x \in A \cap B$. So then $x \in A$ and $x \in B$, and so $x \notin A \setminus B$ since if this we did have $x \in A \setminus B$ this contradicts our assumption that although $x \in A$, we also have $x \in B$.

(b.) Suppose $A \subseteq B$. Then let $x \in X \setminus B$. If we had that $x \notin X \setminus A$, then this would mean that $x \in A$ but $x \notin B$, which contradicts our initial assumption that $A \subseteq B$. Thus the forward direction holds. Now for the backwards direction, suppose $X \setminus B \subseteq X \setminus A$. Now take $p \in A$. Then $p \notin X \setminus A$, and so if $p \in X \setminus B$, then this again contradicts our assumption, and so $p \in B$. Hence the backwards direction holds as well. □

§3.1.

Exercise 1.2 (3.1.)**.** Let $f : A \to B$ be a function. Prove the following statements:

(a) f is injective if and only if $f^{-1}(f(C)) = C$ for every subset $C \subseteq A$.
(b) f is surjective if and only if $f(f^{-1}(D)) = D$ for every subset $D \subseteq B$.

Proof. (a) ($\Rightarrow$) Suppose that $f : A \to B$ is injective. Let $x \in f^{-1}(f(C))$, where $C \subseteq A$. Clearly we have that $f(C) \subseteq B$ so it makes sense to consider the preimage of this set. Now as x is in the preimage of $f(C)$, then $x \in A$ such that $f(x) = f(c_1)$ for some arbitrary $c_1 \in C$. As f is injective, then we have that $x = c_1$; hence, as $c_1 \in C$ we chosen arbitrarily, then $x \in C$. Thus $f^{-1}(f(C)) \subseteq C$. For the reverse inclusion, suppose that $y \in C$ where $C \subseteq A$. Note that $f(C) = \{f(x) : x \in C \subseteq A\}$, and so $f(y) \in f(C)$. Thus we have that $y \in f^{-1}(f(C))$, by definition of the set, i.e. $y \in \{x \in A : f(x) \in f(C)\} = f^{-1}(f(C))$. Therefore if f is injective, then we have that $f^{-1}(f(C)) = C$ for every subset $C \subseteq A$. For the reverse direction ($\Leftarrow$), suppose, by contrapositive, that $f : A \to B$ is not injective. Then we have some $x_1, x_2 \in C \subseteq A$ with $f(x_1) = f(x_2)$. So then $\{x_1, x_2\} \subseteq f^{-1}(f(\{x_1\})$, but $\{x_1, x_2\} \neq f^{-1}(f(\{x\})$ since $f^{-1}(f(x_1\})$ is not contained in the singleton $\{x_1\}$ (WLOG).

(b) ($\Rightarrow$) Suppose f is surjective, although not necessary for one side of the inclusion. Let $x \in f(f^{-1}(D))$ for some subset $D \subseteq B$. Then there is some $\ell \in f^{-1}(D)$ with $f(\ell) = x$. By definition, this means that $\ell \in A$ with $f(\ell) \in D$, and so $x \in D$. Thus the forward inclusion holds. Now let $m \in D$. Then we have some $a \in A$ with $f(a) = m$ as f is surjective. This means that $a \in f^{-1}(D)$ and so $m = f(a) \in f(f^{-1}(D))$. Hence the backwards inclusion holds and we have $f(f^{-1}(D)) = D$. For the reverse direction ($\Leftarrow$), suppose $f(f^{-1}(D)) = D$ for all subsets $D \subseteq B$. But then, as B is a subset of itself, then $f(f^{-1}(B)) = B$. If we have

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$x \in f^{-1}(B)$ then $x \in A$ with $f(x) \in B$, but this set condition is just by all elements in the domain A so $f^{-1}(B) = A$. Hence $f(f^{-1}(B)) = f(A) = B$. Thus f is surjective. $\square$

Exercise 1.3 (3.2.)**.** Let $f\colon A \to B$ and $g\colon B \to C$ be functions.

(a) Prove that if f and g are both injective, then so is $g \circ f$.
(b) Prove that if f and g are both surjective, then so is $g \circ f$.
(c) Prove that if $g \circ f$ is surjective, then so is g.
(d) Argue that surjectivity of $g \circ f$ does not imply surjectivity of f, by providing explicit examples of functions f and g for which $g \circ f$ is surjective but f is not. You should explicitly demonstrate that your functions have the desired properties.
(e) Prove that if $g \circ f$ is injective, then so is f.
(f) Argue that injectivity of $g \circ f$ does not imply injectivity of g. Format your answer similarly to part (d).

Proof. (a) Suppose that f and g are both injective. Now consider the composed map $g \circ f\colon A \to C$. Assume that $g \circ f(x) = g \circ f(y)$ for some $x, y \in A$. Then, $g(f(x)) = g(f(y))$ implies $f(x) = f(y)$ as g is injective, and, lastly, as f is injective then $x = y$; thus $g \circ f$ is itself injective.

(b) Suppose f and g are both surjective. As f is surjective then $f(A) = B$, and as g is surjective, then $g(B) = g(f(A)) = C$. Thus the last equality tells us that given some $c \in C$, we have some $a \in A$ such that $g(f(a)) = g \circ f(a) = c$. Thus $g \circ f$ is surjective.

(c) Suppose $g \circ f$ is surjective. The for all $c_1 \in C$, we have $g(f(a)) = c$ for all $a \in A$, and so define $f(a) = b \in B$, which then means that $g(b) = c$. Thus we see that we see that for every $c \in C$ there is some $b \in B$ with $g(b) = c$. Hence g is surjective.

(d) A simple example which shows that given that $g \circ f$ surjective doesn't necessarily imply that f is surjective is one where we define $f\colon \mathbf{R} \to \mathbf{R}$ by $x \mapsto x^2$ and $g\colon \mathbf{R} \to \{0\}$ where this is zero mapping which takes ever element $r \in \mathbf{R} \mapsto 0 \in \mathbf{R}$. We see that $g \circ f(\mathbf{R}) = \{0\}$, easily, but f itself isn't surjective since, for example, $2 \in \mathbf{R}$ isn't hit by f since 2 isn't the square of a real number.

(e) Suppose that f is not injective. Then we must show that $g \circ f\colon A \to C$ is not injective by contrapositive. Now since f is not injective, then there is some x_1, x_2 such that $x_1 \neq x_2$ but $f(x_1) = f(x_2)$. Composing with g, then $g \circ f(x_1) = g(f(x_1)) = g(f(x_2)) = g \circ f(x_2)$. Hence $g \circ f$ is not injective.

(d) An example where we have that $g \circ f$ is injective but g itself isn't injective is the case where we take $A = \{x\}$, $B = \{s, t\}$ and $C = \{v\}$, where $f\colon a \mapsto s$, $g\colon s, t \mapsto v$. Obviously given that $g \circ f\colon A \to C$, the preimage of v in the codomain can only have one element in the domain by construction so the composition is an injection. But by construction again g is not an injection since it maps two distinct elements in the domain $B = \{s, t\}$ to the same image in $C = \{v\}$. $\square$

Exercise 1.4 (3.3.)**.** Let $f\colon X \to Y$ be a function.

(a) If A and C are subsets of X, then $f(C \setminus A) \supseteq f(C) \setminus f(A)$.
(b) f is injective if and only if $f(C \setminus A) = f(C) \setminus f(A)$ for any two subsets A and C of X.
(c) If B and D are subsets of Y, then $f^{-1}(D \setminus B) = f^{-1}(D) \setminus f^{-1}(B)$.

Proof. (a) Suppose $A, C \subseteq X$. Let $x \in f(C) \setminus f(A)$. So $x \in f(C)$ and $x \notin f(A)$. This means that we have some $c_1 \in C$ with $x = f(c_1)$. Then $c_1 \notin A$ since if $c_1 \in A$, then $x = f(c_1) \in f(A)$, which is a contradiction. Thus $c_1 \in C \setminus A$ and $x \in f(C \setminus A)$.

(b) Suppose that A and C are subsets of X. ($\Rightarrow$) We've already shown the reverse inclusion for this, in general, in part (a), and so it remains to show the opposite inclusion. Assume that $f\colon X \to Y$ is injective. Let $x \in f(C \setminus A)$. Then there is some $c \in C \setminus A$ with $x = f(c)$, and so $f(x) \in f(C)$. If we have that $f(x) \in A$, then this implies that there is some $y \in A$ with $f(c) = x = f(y)$, and since f is injective, then $c = y$. This implies $y \notin A$ as $c = y$ cannot be in A by assumption, and so we have a contradiction. Hence $x \notin f(A)$

and $x \in f(C) \setminus f(A)$. For the opposite direction ($\Leftarrow$), suppose we have that f is not injective. Then we must show the equivalence $f(C \setminus A) = f(C) \setminus f(A)$ is not satisfied. If f is not injective, then there are some $x, y \in X$ with $x \neq y$ such that $f(x) = f(y)$. .

(c) Suppose B and D are subsets of Y. Take $x \in f^{-1}(D \setminus B)$. Then we have $x \in A$ such that $f(x) \in D \setminus B$. Thus $f(x) \in D$ and $f(x) \notin B$. Which means that $x \in A$ such that $f(x) \in D$, i.e. $x \in f^{-1}(D)$; $x \in A$ such that $f(x) \notin B$, i.e. $x \notin f^{-1}(B)$. Thus $x \in f^{-1}(D) \setminus f^{-1}(B)$. For the other direction, let $x \in f^{-1}(D) \setminus f^{-1}(B)$. Then $x \in A$ with $f(x) \in D$, and $x \notin f^{-1}(B)$ with $f(x) \notin B$; that is, $f(x) \in D \setminus B$. Hence $x \in f^{-1}(D \setminus B)$. Therefore the equivalence of sets holds. $\square$

2. §4.1.

Exercise 2.1 (4.1.)**.** Assume that $\text{card}(A) \leq \text{card}(X)$ and $\text{card}(B) \leq \text{card}(Y)$. Prove that $\text{card}(B^A) \leq \text{card}(Y^X)$.

Proof. By assumption, we have maps $\varphi \colon A \to X$ and $\psi \colon B \to Y$ that are both injective. This is to say that $\text{card}(\text{Hom}(A, B)) \leq \text{card}(\text{Hom}(X, Y))$. That is, we want to find some injective function from $B^A := \text{Hom}(A, B) \to \text{Hom}(X, Y)$. For this, we define $\Phi \colon \text{Hom}(A, B) \to \text{Hom}(X, Y)$, where $\Phi \colon f \mapsto h \circ f \circ k$, where $k \colon X \to A$ is the left inverse of $\varphi \colon A \to X$ and $h \colon Y \to B$ is the left inverse of $\psi \colon B \to Y$. It remains to show that Φ is injective: Let $\Phi(f_1) = \Phi(f_2)$, then $h \circ f_1 \circ k = h \circ f_2 \circ k$. Now we invert:

$$
\begin{aligned}
h \circ f_1 \circ k &= h \circ f_2 \circ k \\
\implies h \circ \psi \circ f_1 \circ k &= h \circ \psi \circ f_1 \circ k \\
1_B \circ f_1 \circ k &= 1_B \circ f_2 \circ k \\
1_B \circ f_1 \circ k \circ \varphi &= 1_B \circ f_2 \circ k \circ \varphi \\
1_B \circ f_1 \circ 1_A &= 1_B \circ f_2 \circ 1_A \\
f_1 &= f_2
\end{aligned}
$$

Therefore we have that $f_1 = f_2$ and Φ is injective. Hence $\text{card}(B^A) \leq \text{card}(X^Y)$. $\square$

Exercise 2.2 (4.2.)**.** Prove that for any set A, one has $\mathcal{P}(A) \sim \{0, 1\}^A$.

Proof. Let A be some set. Now consider its power set, $\mathcal{P}(A)$. We want to find a bijective function from $\mathcal{P}(A) \to \{0, 1\}^A$. We do this by defining $F \colon \mathcal{P}(A) \to \{0, 1\}^A$ where $F \colon X \mapsto f_X$ and $f_X(s) = 0$ if $s \notin X$ and $f_X(s) = 1$ if $s \in X$. This map is indeed well defined since if we have some sets $M, N \in \mathcal{P}(A)$ and $M = N$, then we have that the corresponding induced maps f_M and f_N both send the set to 1 or 0 if the condition is satisfied. To show that this map is surjective, suppose that we have some function f on A taking inputs from $\{0, 1\}$. Then we can note that $f \in \{0, 1\}^A$ is F_A since $X = \{x \in X \colon f(x) = 1\}$, and so there is a surjection. Now for an injection, suppose that $F(X) = F(Y)$ for some $X, Y \in \mathcal{P}(A)$. Then $f_X = f_Y$, which means that f_X and f_Y agree on all inputs of A, which implies that $X = Y$ so the mapping is injective. Hence $\mathcal{P}(A) \sim \{0, 1\}^A$ as there is a bijection between the two. $\square$

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