MATH 425A HW8, OCT. 21, 6PM

1. Chapter 4.

Chapter 4: Exercise 1.12.

Let (X, d) be a metric space, and let E be a subset of X.

(a) Show that E is dense in X if and only if any nonempty open subset of X contains a point of E.

(b) Suppose $E \subseteq Y \subseteq X$. Prove that E is dense in Y if and only if $\mathrm{Cl}_X(E) \supseteq Y$.

Proof. (a) ($\Rightarrow$) Suppose that E is dense in X. Then we have that $\mathrm{Cl}_X(E) = X$. Let W be a nonempty open subset of X, so $W \subseteq \mathrm{Cl}_X(E)$. Suppose by contradiction that there are no points of E in W, i.e. $W \cap E = \varnothing$. Thus we have solely that $W \subseteq \mathrm{Lim}_X(E)$. Take $\ell \in W$. Then $\ell \in \mathrm{Lim}_X(E)$, so for any open neighborhood U of ℓ, $U \cap (E \setminus \{\ell\}) = U \cap E \neq \varnothing$. But as W is itself an open neighborhood of ℓ then $W \cap E \neq \varnothing$. This a contradiction. Therefore W must have at least one point of E. ($\Leftarrow$) Assume that any nonempty open subset of X contains a point of E. Clearly, by construction, $E \subseteq X$ and $\mathrm{Lim}_X(E) \subseteq X$ so $\mathrm{Cl}_X(E) \subseteq X$. It remains to show the opposite inclusion. Now let $p \in X$. Then consider the open neighborhood $B_X(p, \epsilon)$. Then we have that $B_X(p, \epsilon)$ contains a point of E as it's open and a subset of X, so $B_X(p, \epsilon) \cap E \neq \varnothing$. Hence, by Remark 1.18 in the Course Notes, we have that $p \in \mathrm{Cl}_X(E)$.

(b) ($\Rightarrow$) Suppose that E is dense in Y, i.e. $\mathrm{Cl}_Y(E) = Y$. But as $Y \subseteq X$, then $\mathrm{Cl}_Y(E) = \mathrm{Cl}_X(E) \cap Y$. Hence this means that $E = \mathrm{Cl}_X(E) \cap Y$. Let $y \in Y$. Then $y \in E$, as $\mathrm{Cl}_Y(E) = E$. So $y \in \mathrm{Cl}_X(E) \cap Y$, and thus $y \in \mathrm{Cl}_X(E)$. We can conclude that $Y \subseteq \mathrm{Cl}_X(E)$. ($\Leftarrow$) Suppose that $Y \subseteq \mathrm{Cl}_X(E)$. Then $\mathrm{Cl}_Y(E) = \mathrm{Cl}_X(E) \cap Y = Y$. Hence $\mathrm{Cl}_Y(E) = Y$ and E is dense in Y. $\square$

Chapter 4: Exercise 1.13.

Previously, we said that a subset E of $\mathbb{R}$ was dense in $\mathbb{R}$ if for any real numbers a and b, there exists a number $c \in E$ which lies between a and b. Show that in $\mathbb{R}$, the new, more general definition of *dense* agrees with the old one. That is, show that a subset E of $\mathbb{R}$ is dense in $\mathbb{R}$ according to the new definition if and only if it is dense according to the old one. (Hint: Use Exercise 1.12(a).)

Proof. ($\Rightarrow$) Suppose that $E \subseteq \mathbb{R}$ and E is dense in $\mathbb{R}$, i.e. $\mathrm{Cl}_{\mathbb{R}}(E) = \mathbb{R}$. Then consider the interval $(a, b) = \{x \in \mathbb{R} \colon a < x < b\} \subseteq \mathbb{R}$. We know that this set is open in $\mathbb{R}$, and so by (a) of Exercise 1.12, then (a, b) contains a point of E. That is, there is some $q \in E$ such that $a < q < b$; this precisely our old definition. ($\Leftarrow$) Suppose that for any $a, b \in \mathbb{R}$ we can find a $p \in E \subseteq \mathbb{R}$ such that $a < p < b$. But as open sets of the form (a, b), which are just open balls, form a basis of $\mathbb{R}$, and any open subset of $\mathbb{R}$ can be decomposed into a union of these open balls, then every point $p \in E$ can be found in the composition of an open set of $\mathbb{R}$. Thus by Exercise 1.12(a), we have E is dense in X according to the new definition. $\square$

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Chapter 4: Exercise 2.1.

Let $S = (p_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ whose image is $(\mathbb{Q} \cap (0,1)) \cup \{5\}$. What are the two possibilities for S^*? Justify your answer.

Proof. By Theorem 2.6., we have that $S^* = (\{p_n\}_{n=1}^{\infty})' \cup S_\infty$. Moreover it's clear that $(\{p_n\}_{n=1}^{\infty})' = [0,1]$. Then two possibilities for S^* are $S^* = [0,1] \cup \{5\}$ or $S^* = [0,1]$, depending on S_∞; that is, if 5 appears in S infinitely many times, then we have $S^* = [0,1] \cup \{5\}$, but if 5 only appears finitely many times then $S^* = [0,1]$. $\square$

Chapter 4: Exercise 2.2.

Let $S = (p_n)_{n=1}^{\infty}$ be a sequence in a metric space X, and let S^* denote the set of subsequential limits of S. The following are equivalent.

(1) $p_n \to p$ as $n \to \infty$.

(2) Every subsequence of $(p_n)_{n=1}^{\infty}$ converges to p in X.

(3) $S^* = \{p\}$, and every subsequence of $(p_n)_{n=1}^{\infty}$ converges in X.

Proof. (1) $\Rightarrow$ (2) Suppose that $p_n \to p$ as $n \to \infty$, i.e. S converges to p in X. Let $A = (p_{n_k})_{k=1}^{\infty}$ be a subsequence of S. As S converges to p in X then for some $N \in \mathbb{N}$ we have that $n \geq N$ implies $p_n \in B_X(p, \epsilon)$ for all $\epsilon > 0$. We claim that $n_k \geq k$, in terms of indices. Firstly $n_1 \geq 1$ by construction of being a strictly increasing sequence of indices. Now suppose that $n_l \geq l$ for some $l \in \mathbb{N}$. Then $n_{l+1} > n_l \geq l$, by definition. So then $n_{l+1} \geq n_l + 1 \geq l + 1$ as if $s, t \in \mathbb{N}$ and $s > t$ then $s \geq t + 1$. Thus we can pick k sufficiently large so that $k \geq N$. This implies that $n_k \geq k \geq N$ by our claim and hence $d(p, p_{n_k}) < \epsilon$; that is, A converges to p as well.
(2) $\Leftrightarrow$ (3) For the forward direction, suppose that every subsequence $A = (p_{n_k})_{k=1}^{\infty}$ of S converges to p in X. Then as S^* denotes the set of all subsequential limits, and all of the subsequences of S converge to p by hypothesis, then $S^* = \{p\}$. For the backwards direction, suppose that $S^* = \{p\}$, and every subsequence of S converges in X. To rephrase this assumption, every subsequence of S converges in X so the set of subsequences are the same as the set of all subsequential limits as they all converge, and given that $S^* = \{p\}$, then every subsequence converges to p in X.
(2) $\Rightarrow$ (1) Suppose $A = (p_{n_k})_{k=1}^{\infty}$ is a subsequence of S that converges to p. Then we have some $N \in \mathbb{N}$ such that $n_k \geq N$ guarantees that $p_{n_k} \in B_X(p, \epsilon)$ for any $\epsilon > 0$. But, trivially, we have that S is a subsequence as itself given by the rule for $A = (p_{n_k})_{k=1}^{\infty}$ we let $n_k = k$, i.e. $n_1 = 1$, $n_2 = 2$, and so on. Thus S converges to p as well. $\square$

Chapter 4: Exercise 2.3.

Let (X, d) be a metric space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in X. Prove the following statements.

(a) If $(x_n)_{n=1}^{\infty}$ converges in X, then it is Cauchy in X.

(b) If $(x_n)_{n=1}^{\infty}$ is Cauchy in X, then it is bounded in X.

Proof. (a) Suppose that $(x_n)_{n=1}^{\infty}$ converges to x in X. Then there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $x_n \in U$ for any open neighborhood U of x. Let $\epsilon > 0$ and $m \geq n$. Then $x_n, x_m \in B_X(x, \frac{\epsilon}{2})$. Thus we have that $d(x, x_n) < \epsilon/2$ and $d(x, x_m) < \epsilon/2$. So then $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \epsilon/2 + \epsilon/2 = \epsilon$. Therefore $d(x_n, x_m) < \epsilon$ and $(x_n)_{n=1}^{\infty}$ is Cauchy in X.
(b) Suppose that $(x_n)_{n=1}^{\infty}$ is Cauchy in X. Then for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $m \geq n \geq N$ implies $d(x_n, x_m) < \epsilon$. Let $\epsilon > 0$ and pick some point $p \in X$. Write $\ell = \max_{N \geq i} d(x_i, p)$ where $N \geq i \geq 1$. Then $d(x_\alpha, p) < \ell + \epsilon$ if $\alpha \leq N$. Now supppose $\alpha > N$. Then $d(x_\alpha, p) \leq d(x_\alpha, x_N) + d(x_N, p) < \epsilon + \ell$. Thus $\{x_n\}_{n=1}^{\infty}$ is bounded. $\square$

Chapter 4: Exercise 2.4.

Let (X, d) be a metric space; let Y be subset of X. The following statements hold.

(a) If Y is complete, then Y is closed in X.

(b) If X is complete and Y is closed in X, then Y is complete.

Hint for (a): Theorem 1.11 and Proposition 1.15

Proof. (a) Suppose that Y is complete, i.e. every Cauchy sequence in Y converges in Y. Our strategy will be to show that $\mathrm{Lim}_X(Y) \subseteq Y$. Let $p \in \mathrm{Lim}_X(Y)$. Then there exists a sequence $Z = (p_n)_{n=1}^{\infty}$ in $Y \setminus \{p\} \subseteq Y$ that converges in X to p. This sequence is itself Cauchy in X and as $\operatorname{im} Z \subseteq Y \subseteq X$ then it is also Cauchy in Y. As Y is complete, then Z converges to, say, a priori ℓ in Y. So $p_n \to p$ in X and $p_n \to \ell$ in Y as $n \to \infty$. By Exercise 1.4, $p_n \to \ell$ in Y implies that $p_n \to \ell$ in X and $\ell \in Y$. But by Corollary 2.4, we must have that $p = \ell$. Hence $p \in Y$ and Y is thus closed.
(b) Suppose that X is complete and Y is closed in X. Let $S = (z_n)_{n=1}^{\infty}$ be a Cauchy sequence of Y. As S is Cauchy in Y then it is also Cauchy in X, and X is complete which means that S converges to some point $x \in X$. Thus $z_n \to x$ in X as $n \to \infty$. WLOG, $z_n \neq x$ for all n. We claim that as $z_n \to x$ in X, then $x \in \mathrm{Lim}_X(Y)$. Let $\epsilon > 0$. Then as $z_n \to x$ in X there is some $N \in \mathbb{N}$ such that $n \geq N$ implies that $z_n \in B_X(x, \epsilon)$. As we've chose $z_n \neq x$, then $z_n \in B_X(x, \epsilon) \setminus \{x\} \cap Y$. Hence $x \in \mathrm{Lim}_X(Y)$. Thus as Y is closed we have that $\mathrm{Lim}_X(Y) \subseteq Y$. Hence $z_n \to x$ in X and $x \in Y$. By Exercise 1.4., we have that $z_n \to x$ in Y. Therefore Y is complete. $\square$

Chapter 4: Exercise 4.1.

Let (X, d) be a metric space. Assume F and K are subsets of X, with F closed and K compact. Then $F \cap K$ is compact.

Proof. Consider the metric subspace (K, d). Then as F is closed in (X, d), then F is closed in (K, d) (this fact is given by the analogue of being open in a topological subspace is the same as being open in the topological space intersected with the subspace and some open set of the original topological space). Hence $F \cap K$ is closed as K is itself closed, and so $F \cap K \subseteq K$ and thus have that $F \cap K$ is compact by Theorem 4.9. $\square$

Chapter 4: Exercise 4.2.

Give an example of a collection $\mathcal{A}$ of bounded subsets of $\mathbb{R}$ such that $\mathcal{A}$ has the finite intersection property, but $\bigcap_{A \in \mathcal{A}} A = \varnothing$. Hint: If $A \subseteq \mathbb{R}$ is bounded in $\mathbb{R}$, what else can prevent it from being compact?

Proof. Consider the collection $\mathcal{M} = \left\{(0, \frac{1}{n}]_{n=2}^{\infty}\right\}$. Then $\mathcal{M}$ has the finite intersection property as we can always pick a real number $\ell \in \mathbb{R}$ such that is it is squished between $0 < \ell < 1/N$ where $\bigcap_{k=1}^{N}(0, \frac{1}{k}]$. But this cannot be applied to $\bigcap_{A \in \mathcal{M}} A$, clearly, i.e. $\bigcap_{A \in \mathcal{M}} A = \varnothing$. $\square$

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