MATH 425A HW3, DUE 09/06/2022

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Chapter 2. §1.

Exercise 0.1 (2.1.)**.** Let A be a nonempty subset of an ordered field $(F, +, \cdot, \leq)$. Assume that $\sup A$ and $\inf A$ exist in F, and let c be any element of F. Define the set $cA := \{ca : a \in A\}$.

(a) Prove that if $c \geq 0$, then $\sup(cA) = c \sup A$.
(b) What is $\sup(cA)$ if $c < 0$. Prove that your answer is correct.

Proof. (a) Suppose that $c \geq 0$. By hypothesis, we have that $x \leq \sup A$ for all $x \in A$, so $cx \leq c \sup A$ as A is a subset of an ordered field. So then $c \sup A$ is an upper bound for cA. Now suppose we have some other upper bound γ for cA. Let $c \neq 0$. Then $cq \leq \gamma$ for $q \in A$ gives us that $q \leq \gamma/c$ which is another upper bound for A; thus $\sup A \leq \gamma/c$ as $\sup A$ is the least upper bound of A, and so $c \sup A \leq \gamma$ by multiplication. Hence $c \sup A = \sup(cA)$ if $c \neq 0$. Now suppose $c = 0$. This implies that $cA = (0)A = \{0\}$. Let γ be an upper bound for cA. So $\ell \leq \gamma$ for all $\ell \in cA$, but cA consists of only 0 so $0 \leq \gamma$. Moreover, $c \sup A = (0) \sup A = 0$, so $c \sup A = 0 \leq \gamma$ is a true statement. Hence $c \sup A = \sup(cA)$ if $c = 0$. Therefore $c \sup A = \sup(cA)$ if $c \geq 0$.

(b) Let $c < 0$. By hypothesis, $x \leq \sup A$ for all $x \in A$, but then the inequality reverses after multiplication since c is negative: $xc \geq c \sup A$. So then cA is bounded below by $c \sup A$. Let λ be some other lower bound of cA. Then $ca \leq \lambda$ for all $a \in A$, and so $a \geq \frac{\lambda}{c}$. But as $\sup A \geq a \geq \frac{\lambda}{c}$, and so $c \sup A \leq \lambda$. This shows that $c \sup A$ is the least upper bound for cA, i.e. $c \sup A = \inf(cA)$. □

Exercise 0.2 (2.2.)**.** Let A and B be nonempty subsets of an ordered field $(F, +, \cdot, \leq)$. Assume $\sup A$ and $\sup B$ exist in F. Define $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup A + \sup B$ by filling in the details of the following outline:

- Denote $s = \sup A$, $t = \sup B$. Then $s + t$ is an upper bound for $A + B$.
- Let u be any upper bound for $A + B$, and let a be any element of A. Then $t \leq u - a$.
- We have $s + t \leq u$. Consequently, $\sup(A + B)$ exists in F and is equal to $s + t = \sup A + \sup B$.

Proof. Denote $s = \sup A, t = \sup B$. So $a \leq s$ and $b \leq t$ for all $a \in A$ and $b \in B$, and so $a + b \leq s + t$. Thus we have that $s + t$ is an upper bound for $A + B$. Now let u be some other upper bound for $A + B$, i.e. $a + b \leq u$ for all $a \in A$ and $b \in B$. Now this implies $b \leq u - a$ so that $u - a$ is another upper bound for B, but then as $\sup B = t$ is the least upper bound for B, then $t \leq u - a$. Moreover, we have that $a \leq u - t$ and so this is another upper bound for A; thus $s \leq u - t$ as $s = \sup A$. Therefore we have $s + t \leq u$. [Consequently, $\sup(A + B)$ exists in F and is equal to $s + t = \sup A + \sup B$.] □

Exercise 0.3 (2.3.)**.** Let f and g be functions from a set X to an ordered field $(F, +, \cdot, \leq)$. Let A be a subset of X.

Date: September 11, 2022
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(a) Prove that the following inequality holds, assuming the relevant suprema all exist.

$$\sup_{x\in A}(f(x)+g(x)) \le \sup_{x\in A} f(x) + \sup_{x\in A} g(x). \tag{1}$$

(b) Show by way of an example that equality might not hold in (1) , even if the suprema all exist. (Hint: This is probably easiest if you choose X to be a set with two elements, and $F = \mathbf{Q}$.)

Proof. (a) We have that $f, g\colon X \to F$ are functions and $A \subseteq X$ by hypothesis. Now let $s = \sup f(A) = \sup_{x\in A} f(x)$ and $t = \sup g(A) = \sup_{x\in A} g(x)$. Then $f(q) \le s$ and $g(q) \le t$ for all $q \in A$, and so $f(q) + g(q) = (f + q)(q) \le s + t = \sup_{x\in A} f(x) + \sup_{x\in A} g(x)$.

(b) We define a function from $f\colon \mathbf{Q} \to \mathbf{Q}$ where $x \mapsto 0$ if $x \ne -2$ or $x \mapsto 1$ if $x = -2$, and $g\colon \mathbf{Q} \to \mathbf{Q}$ where $x \mapsto 1$ if $x = 2$ or $x \mapsto 0$ if $x \ne 2$. Now, clearly, $L = \sup(\{f(x) + g(x) : x \in \mathbf{Q}\}) = f(2) + g(2) = 0 + 1 = 1$ but $R = \sup(\{f(x) : x \in A\}) + \sup(\{g(x) : x \in A\}) = 1 + 1 = 2$ where $A \subseteq \mathbf{Q}$. So the equality $L + 1 \ne 2 = R$ doesn't hold always hold. □

Chapter 2. §3.

Exercise 0.4 (3.1.)**.** Using the strategies similar to those of the proofs in this section, prove the following statements.

(a) There is no rational number whose square is 20.
(b) The set $A := \{r \in \mathbf{Q} : r^2 < 20\}$ has no least upper bound in $\mathbf{Q}$.

Proof. (a) Suppose there is a rational number whose square is 20, i.e. let $q = \sqrt{20}$ where $q \in \mathbf{Q}$. Then $q = \sqrt{20} = \sqrt{2^2 \cdot 5} = 2\sqrt{5}$ and we aim for a contradiction that this $q = 2\sqrt{5}$ cannot possibly be rational, and it suffices to show that $\sqrt{5}$ isn't rational to show that q is irrational since the product of a rational and irrational number is irrational. We suppose that $p = \sqrt{5}$ is rational, i.e. $p = \frac{a}{b}$ with some $a, b \in \mathbf{Z}$ and $b \ne 0$. Furthermore, we may assume that $\gcd(a, b) = 1$, which is to say that a and b have no common factors. Then $p^2 = 5$ and $a^2 = 5b^2$; thus $5 \mid a$ and we can write $a = 5k$ for some $k \in \mathbf{Z}$. Hence $a^2 = (5k)^2 = 25k^2 = 5b^2$ and so $5k^2 = b^2$. We have that $5 \mid b$ as well and so we have found a common factor of a and b that is greater than 1. Thus we have a contradiction and we conclude that no such p exists.

(b) The set A is nonempty since $0^2 = 0 < 20$. Let $p \in \mathbf{Q}$ and $q > 0$. We define $q = p - \frac{p^2+20}{p+20}$, which is clearly another rational number. Now, $p - q = \frac{p^2+20}{p+20}$, and so $p - q$ and $p^2 - 20$ have the same sign, i.e. if $p^2 > 20$ then $p > q$, or if $p^2 < 20$ then $p < q$. Now

$$\begin{aligned} q = p - \frac{p^2+20}{p+20} &= \frac{p(p+20) - (p^2+20)}{p+20} && (2)\\ &= \frac{p^2 + 20p - p^2 - 20}{p+20} = \frac{20(p+1)}{p+20}, && (3) \end{aligned}$$

and so $q > 0$ as p was assumed to be greater than zero. Moreover,

$$\begin{aligned} q^2 - 2 = \left(\frac{20(p+1)}{p+20}\right)^2 - 20 &= \frac{400(p+1)^2}{(p+20)^2} - 20 && (4)\\ &= \frac{400(p+1)^2 - 20(p+20)^2}{(p+20)^2} = \frac{400(p^2+2p+1) - 20(p^2+40p+400)}{(p+20)^2} && (5)\\ &= \frac{400p^2 + 800p + 400 - 20p^2 - 800p - 800}{(p+20)^2} = \frac{380p^2 - 400}{(p+20)^2} && (6)\\ &= \frac{20(19p^2 - 20)}{(p+20)^2}, && (7) \end{aligned}$$

and thus this shows that $p^2 - 2$ and $q^2 - 2$ have the same sign, i.e. if $p^2 > 20$ then $q^2 > 20$, or if $p^2 < 2$ then $q^2 < 20$. Putting together what we've established so far then $p^2 < q^2 < 20$ or $p^2 > q^2 > 20$.

Now we want to show that A has no least upper bound. We will do this in two steps. Firstly, we want to show that if p is an upper bound for A then $p^2 > 20$. By contrapositive, if $p^2 \leq 20$, then p is not an upper bound for A. We've established that p^2 cannot be equal to 20 since $p \in \mathbf{Q}$. Thus we assume that $p^2 < 20$. But then this implies that $p^2 < q^2 < 20$ where $q > 0$ and $q \in \mathbf{Q}$. Hence $q \in A$ and $p < q$ so p is not an upper bound for A.

Lastly, assume that p is any upper bound for A in $\mathbf{Q}$. If $p^2 > 20$, then $p^2 > q^2 > 20$. So then $q < p$. Now, by contradiction, if $r \in A$ such that $r > q$, then $r^2 > q^2 > 20$; thus we cannot have that $r \in A$. Therefore q is an upper bound for A that is less than p so o is not an upper bound. We can conclude that $\mathbf{Q}$ has no least upper bound. $\square$

1. Chapter 2. §4

Exercise 1.1 (4.1.)**.** Prove the following statements about rational and irrational numbers.

(a) Assume r is rational and x is irrational. Show that $r + x$ is irrational. Show that rx is irrational unless $r = 0$.
(b) Use the Archimedean property of $\mathbf{R}$ to prove that the set of irrational numbers is dense in $\mathbf{R}$. (Hint: Let x be any positive irrational number. If y and z are real numbers with $z - y > x$, then there exists an integer m such that $y < mx < z$.)

Proof. (a) Let $r = \frac{a}{b} \in \mathbf{Q}$ with $a, b \in \mathbf{Z}$ and $b \neq 0$, and suppose x is irrational. Now, by way of contradiction, assume that $r + x$ is rational. Then we can write $r + x = \frac{c}{d}$ for some $c, d \in \mathbf{Z}$ and $d \neq 0$. Then $x = \frac{c}{d} - r = \frac{c}{d} - \frac{a}{b} = \frac{cb-ad}{db}$ and so $x \in \mathbf{Q}$, which is a contradiction as we assumed that x is irrational. Therefore $r + x$ is irrational.

Now, using the same notation, suppose that rx is rational with $r \neq 0$. Then $rx = \frac{a}{b}(x) = \frac{s}{t}$ for some $s, t \in \mathbf{Z}$ and $t \neq 0$. So then $rx = \frac{s}{t} \implies rxt = s \implies x = \frac{s}{rt}$ and so $x \in \mathbf{Q}$, which is a contradiction. Thus rx is irrational. Now if $r = 0$, then $xr = 0 \in \mathbf{Q}$. Hence rx is rational unless $r = 0$.

(b) Let q be a positive irrational number, and suppose $y, z \in \mathbf{Q}$ such that $x < y$. Then $x - q < y - q$, and, by the Archimedean property, there is some $n \in \mathbf{N}$ such that $n > \frac{1}{y-x}$ so $n(y - x) > 1$. Now $n(x - q) < n(y - q)$ by multiplication and as $n(y - z) > 1$ there is some $m \in \mathbf{Z}$ such that $n(x - q) < m < n(y - q)$. Thus $x - q < m/n < y - q$, and so $x < m/n + q < y$. Clearly, as $m/n \in \mathbf{Q}$, then $\frac{m}{n} + q$ is irrational by part (a). Hence the set of irrational numbers are dense in $\mathbf{R}$. $\square$

Exercise 1.2 (4.2.)**.** Assume $a, b \in \mathbf{R}$. Prove that $a \leq b$ if and only if $a \leq b + \epsilon$ for every $\epsilon > 0$.

Proof. Suppose that $a \leq b$. Consider the case of $a < b$. Then $b < b + \epsilon$ for some $\epsilon > 0$, and so $a < b < b + \epsilon$ so $a < b + \epsilon$. If $a = b$, then $b \leq b + \epsilon$, clearly, from the work before. So then both cases are done. Now for the opposite direction, we will proceed by contrapositive. Suppose that $a > b$. Then we need to show that there exists some ϵ such that $a > b + \epsilon$. Now we pick $\epsilon = \frac{a-b}{2}$. So then $a > b + \epsilon = b + \frac{a-b}{2}$. $\square$

Exercise 1.3 (4.3.)**.** Let E be a subset of real numbers, and let s be an upper bound for E. Prove that $s = \sup E$ if and only if for every $\epsilon > 0$ there exists $x \in E$ such that $x > s - \epsilon$.

Proof. ($\Rightarrow$) Suppose that $s = \sup E$, i.e. s is the least upper bound for E. By contradiction, suppose that for some $\epsilon > 0$ and for all $x \in E$, we have that $x \leq s - \epsilon$. Then this is another upper bound for E, and so $\sup E = s \leq s - \epsilon$, which is a contradiction. ($\Leftarrow$) Now for the opposite direction, suppose that $s \neq \sup E$, i.e s is not the least upper bound.

Then, by contradiction, assume that for every $\epsilon > 0$, there exists some $x \in E$, such that $x > s - \epsilon$. Since $E \subseteq \mathbf{R}$, then E has a least upper bound; we will call it ℓ. Then we have that $\ell < s$, and as $\mathbf{R}$ is dense in itself, then there is some real number $t \in \mathbf{R}$ such that $\ell < t < s$. For $\epsilon = s - t > 0$, then $x > s - \epsilon = s - (s - t) = t > \ell$. But this is a contradiction as ℓ is an upper bound of E. So then we must have that $s = \ell$. $\square$

Exercise 1.4 (4.4.)**.** Let A and B be nonempty sets of real numbers. Decide whether the following statements are true or false. If true, give a proof; if false, give a counterexample.

(a) If $\sup A < \inf B$, then there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$.

(b) If there exists a $c \in \mathbf{R}$ satisfying $a < c < b$ for all $a \in A$ and $b \in B$, then $\sup A < \inf B$.

Proof. (a) Suppose $\sup A < \inf B$. Then we define $c = (\sup A + \inf B)/2$. So then $c > \sup A$ since $\sup A + \sup A = 2 \sup A < \sup A + \inf B$, and so $c > a$ for all $a \in A$. Now, similarly, $c < \inf B$ since $\inf B + \sup A < \inf B + \inf B = 2 \inf B$, and so $c < b$ for all $b \in B$. All together, we have that $a < c < b$.

(b) This is a false statement. For example, if we take $A = (0, 1)$ and $b = (1, 2)$. Then $a < 1 < b$ for all $b \in B$ and $a \in A$ and $c = 1$. We see that $\sup A = 1$ and $\inf B = 1$. Hence $\sup A = 1 = \inf B$. $\square$

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