MATH 425A HW10, NOV. 1, 6PM

JUAN SERRATOS

1. Chapter 4

Chapter 5; §2.3: Exercise 2.6.

Complete the following tasks.

(a) Find a closed subset of E and a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous such that $f(E)$ is not closed.

(b) Find a bounded subset E of $\mathbb{R}$ and a continuous function $f\colon E \to \mathbb{R}$ such that $f(E)$ is not bounded.

(c) Show that if E is a bounded subset of $\mathbb{R}$ and $f\colon \mathbb{R} \to \mathbb{R}$ is continuous, then $f(E)$ is bounded.

Proof. (a) We know from the course notes that $f\colon \mathbb{R} \to \mathbb{R}$, where $x \mapsto \frac{1}{1+x^2}$ is continuous. Consider $E = \mathbb{R}$, which is closed in $\mathbb{R}$, but $f(\mathbb{R}) = (0, 1]$ and this is not closed in $\mathbb{R}$ (nor is it open as well).

(b) The set $E = (0, 1]$ is clearly a bounded subset of $\mathbb{R}$. From the course notes, we know that $f\colon (0, 1] \to \mathbb{R}$ where $x \mapsto 1/x$ is a continuous function since f is continuous on $f\colon (0, +\infty) \to \mathbb{R}$ so the restriction $f|_E\colon (0, 1] \to \mathbb{R}$ is continuous (Proposition 2.9). But $f(E) = f((0, 1]) = [1, +\infty)$ which is of course not bounded in $\mathbb{R}$.

(c) Suppose E is a bounded subset of $\mathbb{R}$. Then $\overline{E}$ is a closed and bounded subset of $\mathbb{R}$, and so $\overline{E}$ is compact. Now $f(\overline{E})$ is compact, which gives that $f(\overline{E})$ is totally bounded by Proposition 4.8 in Course Notes, and further $f(E) \subset F(\overline{E})$, so $f(E)$ is bounded as well. □

Chapter 5; §2.3: Exercise 2.7.

Prove that the set $\mathbb{R}^2 \setminus \{(0, 0)\}$ is connected. Then, use the function $\frac{x}{|x|}$ to show that $S = \{x \in \mathbb{R}^2 : |x| = 1\}$ is connected. (You may use results from section 2.4.1 below if you want, but it is also possible to do this Exercise without it.)

Proof. Define the sets $A = \{(x, y)\colon y > 0\}$, $B = \{(x, y)\colon x > 0\}$, $C = \{(x, y)\colon x < 0\}$, $D = \{(x, y)\colon y < 0\}$ (we fix x in the set A and allow y to vary, and similarly for the rest). All of the sets A, B, C, D are clearly connected open sets, and so their union which is $\mathbb{R}^2 \setminus \{(0, 0)\}$ is thus connected by Exercise 6.4. in Chapter 4. Alternatively, we could've proved this using path connectedness of $\mathbb{R}^2 \setminus \{0, 0\}$ where given $x, y \neq 0$, then we map $f\colon [0, 1] \to \mathbb{R} \setminus \{0, 0\}$ by $f(t) = (1-t)x + ty$ where $0 \leq t \leq 1$ works if the path between the points doesn't go through $(0, 0)$, and otherwise x and y ca be connected through a path by another point, say, z in $\mathbb{R}^2 \setminus \{0, 0\}$. Now the set S is path connected: Define the map $\varphi\colon \mathbb{R}^2 \setminus \{0, 0\} \to S$ by $\varphi(x) = \frac{x}{|x|}$. This map is clearly surjective and continuous, and so $\varphi(\mathbb{R} \setminus \{(0, 0\}) = S$ Thus we have that S is connected as the image of a connected set is itself connected. □

Date: October 15, 2022

Department of Mathematics, University of Southern California.

Chapter 5; §2.3: Exercise 2.8.

Assume $f\colon X \to Y$ and $g\colon Y \to Z$ are uniformly continuous functions, where (X, d_X), (Y, d_Y), and (Z, d_Z) are metric spaces. Prove that $g \circ f$ is uniformly continuous.

Proof. We are going to show that $h := g \circ f\colon X \to Z$ is uniformly continuous. Let $\epsilon > 0$. We want to find a $\delta > 0$ such that $d_X(s,t) < \delta$ implies $d_Z(h(s), h(t)) < \epsilon$. As $g\colon Y \to Z$ is uniformly continuous, then we have that there is some $\gamma > 0$ such that for $x, y \in X$ and $f(x), f(y) \in Y$ we have $d_Y(f(x), f(y)) < \gamma$ gives us that $d_Z(g(f(x), g(f(y)) = d_Z(h(x), h(z)) < \epsilon$. Lastly, as f is uniformly continuous, then we have some $\delta > 0$ such that $d_X(x,y) < \omega$, then $d_Y(f(x), f(y)) < \epsilon$. Thus for $h\colon X \to Z$, we pick $\omega = \delta$, and therefore we have that h is uniformly continuous. □

Chapter 5; §2.3: Exercise 2.9.

Let E be a bounded subset of $\mathbb{R}^k$ and let $f\colon E \to \mathbb{R}$ be a uniformly continuous function. Show that f is bounded. (Hint: You will need to use compactness of $\overline{E}$ at some point.)

Proof. For sake of contradiction, suppose that f is not bounded. As $\mathbb{R}^k$ is totally bounded, then so is E. Now as f is not bounded, then there is a sequence $(x_n)_{n=1}^{\infty}$ such that $|f(x_n) - 0| = |f(x_n)| \to \infty$ as $n \to \infty$. In particular, we choose $(x_n)_{n-1}^{\infty}$ to be such that $|f(x_n)| > n$ for all $n \in \mathbb{N}$; we can do this as f is not bounded then for any we have $|f(x)| > \ell$ for all $x \in E$ and any $\ell > 0$ in $\mathbb{R}$. However, as $(x_n)_{n=1}^{\infty}$ is a sequence in E which is (totally) bounded then we have some convergent subsequence $(x_{n_k})_{k=1}^{\infty}$. Now we have $|x_{n_k} - x_{n_l}| \to 0$ as $k, l \to \infty$. As $(x_{n_k})_{n=1}^{\infty}$ converges, then it is a standard fact that this sequence is Cauchy as well. As f is uniformly continuous, the for any $\epsilon > 0$ there is $\delta > 0$ such that $|x - c| < \delta$ in E implies $|f(x) - f(c)| < \epsilon$ (we abuse notation here for induced metric on E as $E \subseteq \mathbb{R}^k$). As $(x_{n_k})_{k=1}^{\infty}$ is Cauchy, then there is some $N \in \mathbb{N}$ such that $s, t \geq N$ implies $|x_{n_s} - x_{n_t}| < \delta$ (as we've chosen $\delta > 0$). Thus, as f is uniformly continuous, we have $|f(x_{n_s}) - f(x_{n_t})| < \epsilon$, and so $f(x_{n_t}) \leq |f(x_{n_s})| + |f(x_{n_s}) - f(x_{n_t})| < |f(x_{n_s})| + \epsilon$, and so $|f(x_{n_t})| < f(x_{n_s}) + \epsilon$. Now if we let s vary and approach infinity and fix t, then $\lim_{t\to\infty} |f(x_{n_t})| < |f(x_{n_s})| + \epsilon$. This contradicts how we chose $(x_n)_{n=1}^{\infty}$ in sentence three. Therefore we must have that f is indeed bounded. □

Department of Mathematics, University of Southern California, Los Angeles, CA 90007