Math 432 - Real Analysis II
Solutions to Test 1

Instructions: On a separate sheet of paper, answer the following questions as *completely and neatly* as possible, writing complete proofs when possible.

Question 1. Consider the following power series $L(x)$, which is also known as Euler's *dilogarithm function*:

$$L(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}.$$

(a) Compute the domain of convergence for $L(x)$. Be sure to give a full analysis of the endpoints.

(b) Show that $L(x)$ uniformly convergence on its entire domain of convergence.

(c) Explain why the $L(x)$ is continuous on its domain of convergence.

Solution 1.

(a) We can use the ratio test to compute the interior of the domain of convergence:

$$\lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)^2} \cdot \frac{k^2}{x^k} \right| = |x| < 1.$$

Thus, the radius of convergence is 1. When $x = 1$, the series converges by the p-series test. When $x = -1$, it converges by the alternating series test. So, the domain of convergence is $[-1, 1]$.

(b) We use the Weierstrass M-test to prove this. Notice that for all $x \in [-1, 1]$, we have that $\left|\frac{x^k}{k^2}\right| \leq \frac{1}{k^2}$. Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, we have that our power series converges uniformly on $[-1, 1]$, its domain of convergence.

(c) Each of the finite partial sums is a continuous function. Since these partial sums converge uniformly, the limit will be continuous as well.

Question 2. Consider the sequence of functions

$$f_n(x) = \frac{nx}{1 + nx}$$

for $x \in [0, \infty)$.

(a) Compute the pointwise limit of of $f_n(x)$. Call this limit function $f(x)$.

(b) Decide if $f_n \to f$ uniformly on $[0, 1]$. Prove your answer.

(c) Decide if $f_n \to f$ uniformly on $[1, \infty)$. Prove your answer.

Solution 2.

(a) When $x = 0$, $f_n(0) = 0$. When $x \neq 0$, then we can multiply the top and bottom by $1/n$ to get

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1/n + x} = \frac{x}{x} = 1.$$

Thus, the limit function is given by $f(0) = 0$ and $f(x) = 1$ for all $x \neq 0$.

(b) f_n cannot converge uniformly to f on $[0,1]$ because $f_n(x)$ is continuous on $[0,1]$ for all n and, if $f_n \to f$ uniformly, then its limit would also be continuous on $[0,1]$, which it is not.

(c) Taking a derivative of $f_n(x)$, we have

$$f_n'(x) = \frac{n(1+nx) - nx(n)}{(1+nx)^2} = \frac{nx}{(1+nx)^2}.$$

Thus, for all $x \in [0,1]$, $f_n(x)$ is increasing. Furthermore, $f_n(x) \leq 1$ for all $xin[1,\infty)$. Thus, $\sup\{|f_n(x)| \,|\, x \in [1,\infty)\}$ occurs at the 1. Thus,

$$\sup\{|f(x) - f_n(x)| \,|\, x \in [1,\infty)\} = 1 - f(1) = 1 - \frac{n}{1+n}.$$

Since this goes to 0, we have uniform convergence.

Question 3. Consider the power series

$$f(x) = \sum_{k=1}^{\infty} kx^k.$$

(a) Compute the domain of convergence for $f(x)$. Be sure to give a full analysis of the endpoints.

(b) Use the fact that

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

for all $|x| < 1$ to show that

$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

for all $|x| < 1$.

(c) Use (b) to compute the series

$$\sum_{k=1}^{\infty} \frac{k}{2^k}.$$

(d) Use (b) to compute the series

$$\sum_{k=1}^{\infty} \frac{k}{3^k}.$$

(e) Use (b) to compute the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k k}{3^k}.$$

Solution 3.

(a) Using the Ratio Test, we have

$$\lim_{k\to\infty} \left| \frac{(k+1)x^{k+1}}{kx^k} \right| = |x| < 1.$$

Thus, the radius of convergence for this power series is 1. For the endpoints, notice that when $x = 1$ or $x = -1$, our series will diverge by the divergence test. Thus, the complete domain of convergence is $(-1, 1)$.

(b) Since $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, we can differentiate term-by-term to get that

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}.$$

Multiplying both sides by x, we get

$$\sum_{k=1}^{\infty} kx^{k} = \frac{x}{(1-x)^2}.$$

(c) We can compute this series by letting $x = 1/2$ to get that

$$\sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{1/4} = 2.$$

(d) We can compute this series by letting $x = 1/3$ to get that

$$\sum_{k=1}^{\infty} \frac{k}{3^k} = \frac{1/3}{(1-1/3)^2} = \frac{1/3}{4/9} = \frac{3}{4}.$$

(e) We can compute this series by letting $x = -1/3$ to get that

$$\sum_{k=1}^{\infty} \frac{(-1)^k k}{3^k} = \frac{-1/3}{(1+1/3)^2} = \frac{-1/3}{16/9} = \frac{-3}{16}.$$

Question 4. Consider the limit

$$\lim_{x\to 0} \frac{\sin x^3 - x^3}{x^9}.$$

(a) Compute the above limit using L'Hôpital's Rule. Be sure to justify each step.

(b) Compute the Taylor series for $\sin x^3$ using the Taylor series for $\sin x$. Use this Taylor series to compute the above limit.

Solution 4.

(a) Notice that the top and bottom both limit to 0. Thus, we can use L'Hôpital's Rule to get that

$$\lim_{x\to 0} \frac{\sin x^3 - x^3}{x^9} = \lim_{x\to 0} \frac{3x^2\cos(x^3) - 3x^2}{9x^8} = \lim_{x\to 0} \frac{\cos(x^3) - 1}{3x^6}.$$

We see that we can apply the rule once again to get that the above limit is equal to

$$\lim_{x\to 0} \frac{-3x^2\sin(x^3)}{18x^5} = \lim_{x\to 0} \frac{-\sin(x^3)}{6x^3}.$$

Again, applying L'Hôpitals Rule, we get

$$\lim_{x\to 0} \frac{-3x^2\cos(x^3)}{18x^2} = \lim_{x\to 0} \frac{-\cos(x^3)}{6} = -\frac{1}{6}.$$

(b) Since the Taylor series for $\sin x$ is given by

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!},$$

we can substitute x^3 in for x to get

$$\sin(x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{6k+3}}{(2k+1)!} = x^3 - \frac{x^9}{6} + \frac{x^{15}}{5!} - \cdots.$$

If we plug this into our limit, we get in the numerator that our x^3 terms cancels out. The remaining terms all have at least an x^9 factor in them. After factoring this out and canceling with the x^9 in the denominator, we have that the limit is equal to

$$\lim_{x\to 0} -\frac{1}{6} + \frac{x^6}{5!} - \cdots = -\frac{1}{6}.$$

Question 5. Let $0 < a$. In the questions below, there is no need to justify your work; only give the requested series.

(a) Give an example of a power series that has domain of convergence equal to $(-\infty, \infty)$.

(b) Give an example of a power series that has domain of convergence equal to $(-a, a)$.

(c) Give an example of a power series that has domain of convergence equal to $[-a, a)$.

(d) Give an example of a power series that has domain of convergence equal to $[-a, a]$.

Solution 5.

(a)

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

(b)

$$\sum_{k=1}^{\infty} \frac{x^k}{a^k}$$

(c)

$$\sum_{k=1}^{\infty} \frac{x^k}{a^k k}$$

(d)

$$\sum_{k=1}^{\infty} \frac{x^k}{a^k k^2}$$

Question 6. Assume that the power series $\sum_{k=0}^{\infty} a_k x^k$ has radius of convergence 2. Let p be a fixed positive integer.

(a) Compute the radius of convergence for $\sum_{k=0}^{\infty} a_k^p x^k$.

(b) Compute the radius of convergence for $\sum_{k=0}^{\infty} p \cdot a_k x^k$.

Solution 6.

(a) Since the original power series has a radius of converge of 2, we know that

$$\limsup |a_k|^{1/k} = 1/2.$$

But,

$$\limsup |a_k^p|^{1/k} = \limsup \left(|a_k|^{1/k}\right)^p = \frac{1}{2^p}.$$

Thus, the radius of convergence is 2^p.

(b) Computing we get

$$\limsup |pa_k|^{1/k} = \limsup p^{1/k}|a_k|^{1/k} = \frac{1}{2}.$$

Extra Credit. For the dilogarithm function in Question 1, give the exact value for $L(a)$ for at least one $a \neq 0$ in the Domain of Convergence.

Solution. We know that $L(x)$ converges at $x = 1$. In fact, this value is known to us as $\pi^2/6$.