MATH 425A HW12, NOV. 22, 6PM

Chapter 7

Exercise 2.1.

For each of the following sequences $(a_n)_{n=1}^{\infty}$, prove whether the series $\sum_{n=1}^{\infty} a_n$ converges or diverges. (If it converges, you do not need to find the limit.)

(1) $a_n = \sqrt{n+1} - \sqrt{n}$.
(2) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$.
(3) $a_n = (\sqrt[n]{n} - 1)^n$.
(4) $a_n = \frac{(-1)^n}{\log n}$ for $n \geq 2$ (and $a_1 = 0$).

Proof. (1) We make an observation to the series defined by a_n, that is, we note that it is a telescoping:

$$\sum_{n=1}^{M} \sqrt{n+1} - \sqrt{n} = (\sqrt{2}-1) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + \cdots + \sqrt{M+1} - \sqrt{M}$$
$$= \sqrt{M+1} - 1.$$

Then as $M \to \infty$, we have that the sum tends to infinity. Hence we have the series diverges.
(2) For $|a_n|$, we have $\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} < \frac{1}{n\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}$. Hence $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$ converges as we know that $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges and so we apply the Comparison Test.
(3) We apply the Root Test.

$$|a_n|^{\frac{1}{n}} = |(\sqrt[n]{n} - 1)^n|^{1/n} = |\sqrt[n]{n} - 1|.$$

If $\sqrt[n]{n} - 1 > 0$, then $\limsup(\sqrt[n]{n} - 1) = 1 - 1 - 0 < 1$, by Theorem 5.1; similarly, if $\sqrt[n]{n} - 1 < 0$, then $\limsup(1 - \sqrt[n]{n}) = 1 - 1 = 0 < 1$. Therefore we have that the sequence converges.
(4) Consider $a_k = \frac{1}{\log k}$ for $k \geq 2$ (as $\log 1 = 0$ and $\log 0$ isn't defined). As we have $\log k < \log k + 1$, then $\frac{1}{\log k} > \frac{1}{\log k+1}$, and so we have a monotonically decreasing sequence and clearly $\lim_{n\to\infty} \frac{1}{\log k} = 0$. Therefore after applying the Alternating Series Test we get that $\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{\log k}\right) = \sum_{n=1}^{\infty} a_n$ converges. ∎

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Exercise 2.2.

Consider the series
$$\sum_{n=1}^{\infty} \frac{1}{1+z^n}.$$
Determine the values of $z \in \mathbb{R}$ ($z \neq -1$) make the series convergent and which make it divergent. Prove your answers are correct.

Proof. To start off, for $z = 0$, we have $\sum_{n=1}^{\infty} \frac{1}{1+0^n} = 1 + 1 + 1 \cdots$ which makes it divergent. For $z = 1$, $\sum_{n=1}^{\infty} \frac{1}{2} = \frac{1}{2} + \frac{1}{2} + \cdots$ which makes it divergent. Furthermore, if $|z| < 1$, and $z \neq -1$, then $\lim_{n\to\infty} \frac{1}{1+z^n} = \frac{1}{1+0} = 1 \neq 0$ and so we cannot have convergence. Hence it remains to look at $z \in (-\infty, -1) \cup (1, \infty)$. Fix $z > 1$. Then $z^n < z^n + 1$ and so $\frac{1}{z^n} > \frac{1}{z^n+1}$. Now $\sum_{n=1}^{\infty} \frac{1}{z^n}$ converges by the Root Test as $\limsup \left| \left(\frac{1}{z^n}\right)^{\frac{1}{n}} \right| = \limsup |1/z| = 0 < 1$. Hence we have that $\sum_{n=1}^{\infty} \frac{1}{z^n+1}$ converges by the Comparison Test for fixed $z > 1$. Lastly, fix $z < 1$. Then $(\frac{1}{1+z^n})_{n=1}^{\infty}$ is monotonically decreasing sequence and $\frac{1}{1+z^n} \to 0$ as $n \to \infty$. By Alternating Series Test we have that $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+z^n}$ converges and thus makes $\sum_{n=1}^{\infty} \frac{1}{1+z^n}$ converge with $z < 1$. □

Exercise 3.1.

Assume that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Prove that $\sum_{n=1}^{\infty} \frac{\sqrt{|a_n|}}{n}$ converges. (Hint: Use the inequality $2AB \leq A^2 + B^2$, valid for any real numbers A, B.)

Proof. We use the AM-GM inequality as follows.
$$\sqrt{a_n \cdot \frac{1}{n^2}} \leq \frac{a_n + \frac{1}{n^2}}{2} = \frac{a_n}{2} + \frac{\frac{1}{n^2}}{2} = \frac{a_n}{2} + \frac{1}{2n^2}$$
$$\implies \sqrt{\frac{a_n}{n^2}} = \frac{\sqrt{a_n}}{n} \leq \frac{a_n}{2} + \frac{1}{2n^2}.$$
Now $\sum_{n=1}^{\infty} \frac{a_n}{2} = \frac{1}{2} \sum_{n=1}^{\infty}$ converges as we assumed absolute converges of $\sum_{n=1}^{\infty} a_n$, and so $\sum_{n=1}^{\infty} \frac{1}{2n^2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges as this is a p-series with $p = 2 > 1$. Hence we have that their sum converges, i.e.
$$\sum_{n=1}^{\infty} \left(\frac{a_n}{2} + \frac{1}{2n^2} \right)$$
converges, which forces $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ to converge as well by the Comparison Test and AM-GM. □

Exercise 3.2.

(1) Assume that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely. Prove that $\sum_{n=1}^{\infty}(a_n+b_n)$ converges absolutely as well.
(2) Assume that $\sum_{n=1}^{\infty} a_n$ converges. Does it follow that $\sum_{n=1}^{\infty} a_{2n}$ converges? Give a proof or a counterexample.
(3) Assume that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Does it follow that $\sum_{n=1}^{\infty} a_{2n}$ converges absolutely? Give a proof or counterexample.

Proof. (1) As $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} |b_n|$ both converge, then for all $\epsilon > 0$, there exists $N_1, N_2 \in \mathbb{N}$ such that $n \geq m \geq N_1, N_2$ gives $|\sum_{n=k}^{m} |a_k|| < \epsilon/2$ and $|\sum_{n=k}^{m} |b_k|| < \epsilon/2$ by Proposition 1.3. And so

$$\begin{aligned}\left|\sum_{n=k}^{m} |a_k + b_k|\right| \leq \left|\sum_{n=k}^{m} |a_k| + |b_k|\right| &= \left|\sum_{n=k}^{m} |a_k| + \sum_{n=k}^{m} |b_k|\right| \\ &\leq \left|\sum_{n=k}^{m} |a_k|\right| + \left|\sum_{n=k}^{m} |b_k|\right| < \epsilon/2 + \epsilon/2 = \epsilon,\end{aligned}$$

(2) We give a counter example. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent where $a_n = \frac{(-1)^n}{n}$, but $a_{2n} = \frac{(-1)^{2n}}{2n} = \frac{1}{2n}$ and so $\sum_{n=1}^{\infty} a_{2n} = \sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ which is the harmonic series scaled by $1/2$ and hence this series is divergent.
(3) We claim that this statements holds true. Let $\sum_{n=1}^{\infty} a_n$ be absolutely convergent. Then we have $(|a_n|)$ being a sequence of nonnegative real numbers and now $\sum_{n=1}^{\infty} |a_{2n}| \leq \sum_{n=1}^{\infty} |a_n|$ which makes $\sum_{n=1}^{\infty} |a_{2n}|$ converge absolutely by the Comparison Test.

□

Exercise 4.1.

Let $B = \{0\} \cup \{\frac{-1}{n^2}\}_{n\in\mathbb{N}}$ and $E = \mathbb{R} \setminus B$. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$$

on the set E.

(1) Prove that the series converges absolutely for all $x \in E$; therefore it converges pointwise to a function $f\colon E \to \mathbb{R}$.
(2) Prove that the series converges uniformly to f on $[-\infty, -\delta] \cup [\delta, \infty) \setminus B$ for any $\delta > 0$, but that it does not converge uniformly to f on E.
(3) Prove that f is continuous on E.
(4) Prove that $f(0+) = +\infty$, and therefore f is not a bounded function.

Proof. (1) For $x > 0$ in E, we have $\sum_{n=1}^{\infty} \left|\frac{1}{1+n^2x}\right| = \sum_{n=1}^{\infty} \frac{1}{1+nx^2} \leq \sum_{n=1}^{\infty} \frac{n^2x}{1} = \frac{1}{x}\sum_{n=1}^{\infty}\frac{1}{n^2}$ and so this is a p-series with $p = 2 > 1$, and therefore we have that $\sum_{n=1}^{\infty} \frac{1}{1+nx^2}$ converges absolutely by the Comparison Test. Let $x < 0$ in E. Note that after sufficiently large $n \geq N$, we have $1 + nx^2 < -n^2$; we pick N to be the minimum natural number such that $N^2 \geq \frac{1}{|1+x|}$. For $n \geq N$, this gives us $\left|\frac{1}{1+n^2x}\right| \leq \frac{1}{n^2}$. Hence as this is once again the same p-series as before with $p = 2$, we have that $\sum_{n=N}^{\infty} \left|\frac{1}{1+n^2x}\right|$ converges and so our original series $\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ converges as well as we did so with finitely many terms up to N.
(2) We show this by the Weierstrass M-Test. Let $\delta > 0$. We're going to be assuming that x is not in B in the following argument.
Take $x \in [\delta, \infty)$ and $\ell \in (0, \delta)$. Then $1 + n^2x > n^2x > n^2\ell$, and thus $\frac{1}{1+n^2x} < \frac{1}{n^2\ell}$. As $\sum_{n=1}^{\infty} \frac{1}{n^2\ell} = \frac{1}{\ell}\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, then f converges uniformly by Weierstrass M-Test. Now for $x \in [-\infty, -\delta]$ we've established at the end of part (1) that gives some $N \in \mathbb{N}$ such that for $m \geq N$, we have $\left|\sum_{n=m}^{\infty} \frac{1}{1+n^2x}\right| \leq \sum_{n=m}^{\infty} \frac{1}{n^2}$ and this last series converges so we uniform convergence as well. The reason that f doesn't converge uniformly on all of $E = \mathbb{R} \setminus B$ is because, for example, it does not converge uniformly on $(0, \delta]$ with $\delta > 0$ as if it did by the Cauchy Criterion we have that there is some $N \in \mathbb{N}$ such that $\sum_{n=N}^{m} \frac{1}{1+n^2x} < \frac{1}{2}$ for all $x \in (0, \delta]$. But this doesn't work as we could then choose $x = \frac{1}{N^2}$, which gives a contradiction.
(3) The function f is continuous where it it is uniformly continuous on by the Uniform Limit Theorem. As shown in (1), $f(x)$ doesn't converge in B. For $b > 0$ and $t \in [b, \infty)$ or $a < 0$ and $t \in (-\infty, a]$ we have that f converges uniformly as established in (2). Hence f is continuous at t.
(4) Write $x_k = \frac{1}{k}$, then $\sum_{n=1}^{\infty} \frac{1}{1+\frac{n^2}{k}} = k\sum_{n=1}^{\infty} \frac{1}{k+n^2}$ which diverges and hence shows that the function f is not bounded. Alternatively, f cannot be bounded as $f(0) = \sum_{n=1}^{\infty} \frac{1}{1+n^2(0)} = \sum_{n=1}^{\infty} 1$ doesn't converge. □

Exercise 4.2.

Find the radius of convergence for each of the following power series:

$$\sum_{n=0}^{\infty} n^n z^n \quad \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \sum_{n=0}^{\infty} z^n \quad \sum_{n=1}^{\infty} \frac{z^n}{n} \quad \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

Proof. For the first power series, $c_n = n^n$ and so $\alpha = \limsup(|n^n|)^{\frac{1}{n}} = \limsup |n|$, and thus we have that $R = \frac{1}{\alpha} = 0$. Hence the radius of convergence is $R = 0$.

For the second, $c_n = \frac{1}{n!}$, so $\alpha = \limsup \left(\left|\frac{1}{n!}\right|\right)^{\frac{1}{n}} = \lim \left(\left|\frac{1}{n!}\right|\right)^{\frac{1}{n}} = 0$. Hence the radius of convergence is $R = +\infty$.

For the third, $c_n = 1$ for all n, and so $\alpha = \limsup |1|^{\frac{1}{n}} = 1$. Thus the radius of convergence is $R = 1$.

For the fourth, $c_n = \frac{1}{n}$, and so $\alpha = \limsup |\frac{1}{n}|^{\frac{1}{n}} = \lim |\frac{1}{n}|^{\frac{1}{n}} = 1$. Therefore the radius of convergence is $R = 1$.

Lastly, for the fifth, $c_n = \frac{1}{n^2}$ which makes $\alpha = \limsup |\frac{1}{n^2}|^{\frac{1}{n}} = \lim |\frac{1}{n^2}|^{\frac{1}{n}} = \lim \left|\frac{1}{\sqrt[n]{n^2}}\right| = 1$. Hence the radius of convergence is $R = 1$. □

Exercise 4.3.

Consider the power series $\sum_{n=0}^{\infty} c_n z^n$. Let R be the radius of convergence of the power series, and assume $R > 0$. Let $f\colon (-R, R) \to \mathbb{R}$ be the function defined by $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Prove the following statements, which refine Theorem 4.4.

(1) For any $r \in (0, R)$, the series $\sum_{n=0}^{\infty} c_n z^n$ converges uniformly on $(-r, r)$ to f.
(2) f is continous on all of $(-R, R)$.

Proof. (1) We proceed by applying the Weierstrass M-Test. By Theorem 4.4, since $f\colon (-R, R) \to \mathbb{R}$, i.e. the domain of the function is the radius of convergence of $\sum_{n=0}^{\infty} c_n z^n$ and $f\colon z \mapsto \sum_{n=0}^{\infty} c_n z^n$ then the series converges for all z in the domain of f. As $r \in (0, R)$, then it is indeed in the radius of convergence, as $(0, R) \subset (-R, R)$, and furthermore we clearly have that $(-r, r) \subset (-R, R)$. Hence we have convergence of the power series for all $(-r, r)$ to f. Now we choose $M_n = (zr)^n = z^n r^n$ for each f_n in the sequence (f_n) as then this gives us $|f_n(z)| \leq M_n$ for each n and therefore by the Weierstrass M-Test we have that the series $\sum_{n=0}^{\infty} c_n z^n$ converges uniformly on $(-r, r)$ as for all $z \in (-r, r)$ we have z being less than R or greater than $-R$ and in either case $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} |c_n r^n|$ converges as $(-R, R)$ is our radius of convergence.

(2) By (1), we have uniform convergence of $\sum_{n=0}^{\infty} c_n z^n$ on $(-r, r)$ for an arbitrary r where $0 < r < R$, which means that we have continuity on $(-r, r)$ as we have uniform convergence, and as we have $r \in (0, R)$ for any such r then for $|z| < R$ we get that f is continuous. □

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