MATH 425A NOTES

JUAN SERRATOS

Contents

1. Sequences of Functions

1.1. **Pointwise and Uniform Convergence for Real-Valued Functions.** Consider the following graph of $f_n(x) = x^n$, this denotes for any given $n \in \mathbb{Z}_{\geqslant 0}$ we have a corresponding map $f_n : x \mapsto x^n$ from $\mathbb{R} \to \mathbb{R}$. In essence, there's a *convergence* to another function for larger and larger n:

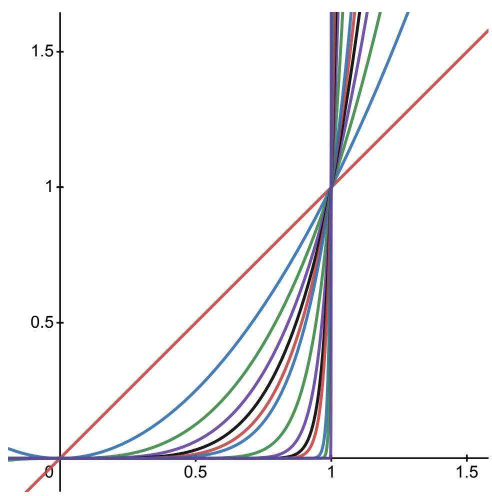

FIGURE 1. We are particularly focused on the behavior of the first quadrant as this function is not symmetric and the same cannot be said for the the behavior of the second quadrant gets messy.

Definition 1.1. Let X be any set, and let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions defined on X. If for each $x \in E$ the limit $\lim_{n\to\infty} f_n(x)$ exists , then we can define the (**pointwise**) **limit function** by $f(x) = \lim_{n\to\infty} f_n(x)$, for $x \in E$. In this case, we say that (f_n) **converges pointwise** to f on E.

Date: September 17, 2022
Department of Mathematics, University of Southern California.

Example 1.1. For $n \in \mathbb{Z}_{\geqslant 0}$, define $f_n\colon [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$. Then for all $x \in [0,1) \subset [0,1]$, we have $f_n(x) = 0$ while $f_n(1) = 1$. Hence we form a pointwise function

$$f_n(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \in [0,1) \end{cases}$$

Definition 1.2. Let (X, d_X) and (Y, d_Y) be metric spaces; let $f_n\colon X \to Y$ be functions. We say that $(f_n)_{n=1}^{\infty}$ **converges uniformly to a function** $f\colon X \to Y$ on $E \subset X$ if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for $n \geqslant N$ implies $d_Y(f_n(x), f(x)) < \epsilon$ for all $x \in E$.

Remark 1.1. This definition is easier to parse when we choose $Y = \mathbb{R}^k$; that is, we have a family of functions $f_n\colon X \to \mathbb{R}^k$ such that we say $(f_n)_{n=1}^{\infty}$ converges uniformly to a function $f\colon X \to \mathbb{R}^k$ on $E \subset X$ where for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ with $n \geqslant N$ producing $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$.

Lemma 1.1. Assume $f_n \to f$ is pointwise on E, and set

$$M_n = \sup_{x \in E} |f_n(x) - f(x)|.$$

Assume additionally that $M_n < +\infty$ for all $n \in \mathbb{N}$. Then $f_n \to f$ uniformly on E if and only if $M_n \to 0$ as $n \to \infty$.

Proof. ($\Rightarrow$) Let $f_n \to f$ be uniform. We essentially we want to show that $\sup_{x \in E} |f_n(x) - f(x)| < \epsilon$ for any given $\epsilon > 0$ (this is what we mean when we converge to zero, that is, we can pick arbitrarily small ϵ). By uniformity we get some sufficient $n \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon/2$ which gives $M_n \leqslant \epsilon/2 < \epsilon$; hence $M_n \to 0$. ($\Leftarrow$) Let $M_n \to 0$ as $n \to \infty$. Pick $\epsilon > 0$ such that for $N \in \mathbb{N}$ we have $M_n = |\sup_{x \in E} f_n(x) - f(x)| < \epsilon$ and so $|f_n(x) - f(x)| \leqslant M_n < \epsilon$. Hence $f_n \to f$ uniformly on E as $n \to \infty$. $\square$

Example 1.2. If $f_n(x) = x^n$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, then $f_n(x) \to 0$, as shown in Example 1.1 (meaning that $f_n(x) \to 0$ for every $x \in [0,1)$ as $n \to \infty$, i.e. f_n converges to the function $f\colon [0,1) \to \mathbb{R}$ defined by $f(x) = 0$ for all $x \in [0,1)$, which is denoted by 0). Now, it turns out that $f_n \to 0$, as defined, converges uniformly on any interval $[0,c]$ for an $0 < c < 1$, but (f_n) does not converge uniformly on $[0,1)$.

Let $\epsilon > 0$ and choose $c \in (0,1)$; additionally, write $E = [0,c]$. Then $M_n = \sup_{x \in E} |f_n(x) - f(x)| = \sup_{x \in E} |x^n - 0| = \sup_{x \in E} |x^n| = c^n$, and then $M_n \to 0$ as $n \to \infty$ since $0 < c < 1$. Thus by Lemma 1.1, we have uniform convergence on E.

1.2. **Uniform Convergence and the space** $B(X)$.

Remark 1.2. For any set X, we define $B(X)$ to be the set of (real-valued) functions on X that are bounded. We make this set into a metric space be endowing it with the norm $\|f\|_u = \sup_{x \in X} |f(x)|$ where $f \in X$, which gives a notion of distance defined by for any $f, g \in B(X)$, $d_u(f,g) = \|f - g\| = \sup_{x \in X} |f(x) - g(x)|$—and $B(X)$ also has a vector space structure that *allows* us to do this. In total, $(B(X), \|\cdot\|_u)$ denotes the metric space of (real-valued) functions. We say that two functions $f, g\colon X \to \mathbb{R}$ are distance γ apart in the uniform metric if $\sup_{x \in X} |f(x) - g(x)| \leqslant \gamma$.

Proposition 1.1. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $B(X)$. Then $f_n \to f$ uniformly on X if and only if $\|f_n - f\|_u \to f$ as $n \to \infty$.

Proof. By Lemma 1.1, $f_n \to f$ uniformly on X if and only if $\sup_{x \in X} |f_n(x) - f(x)| = \|f_n(x) - f(x)\|_u \to 0$ as $n \to \infty$ if and only if $f_n \to f$ in $B(X)$. (Note that this last if and only if statement is what it means for $d_u(f_n, f)$ to tend to zero in $B(X)$.) $\square$

Although the framework of $B(X)$ is nice, it doesn't capture everything that can be said about for a sequence of functions converge uniformly. For example, consider $f_n\colon (0,\infty) \to \mathbb{R}$ where $f_n(x) = \frac{1}{x} + \frac{1}{n}$ and write $f\colon (0,\infty) \to \mathbb{R}$ where $f(x) = \frac{1}{x}$. Then $\sup_{x \in (0,\infty)} |f_n(x) - f(x)| = \sup_{x \in (0,\infty)} |\frac{1}{x} + \frac{1}{n} - \frac{1}{x}| = \sup_{x \in (0,\infty)} |\frac{1}{n}| = \frac{1}{n}$ and we know that $1/n \to \infty$ as $n \to \infty$. Hence this sequence is uniform on $(0,\infty)$ and $f_n \to f$. Yet $f_n \notin B((0,\infty))$ for any n. But, coming back to Figure 1.1, we've just made 'sense' of what we wanted to establish; a notion of convergence to a function, particularly for example we've just established:

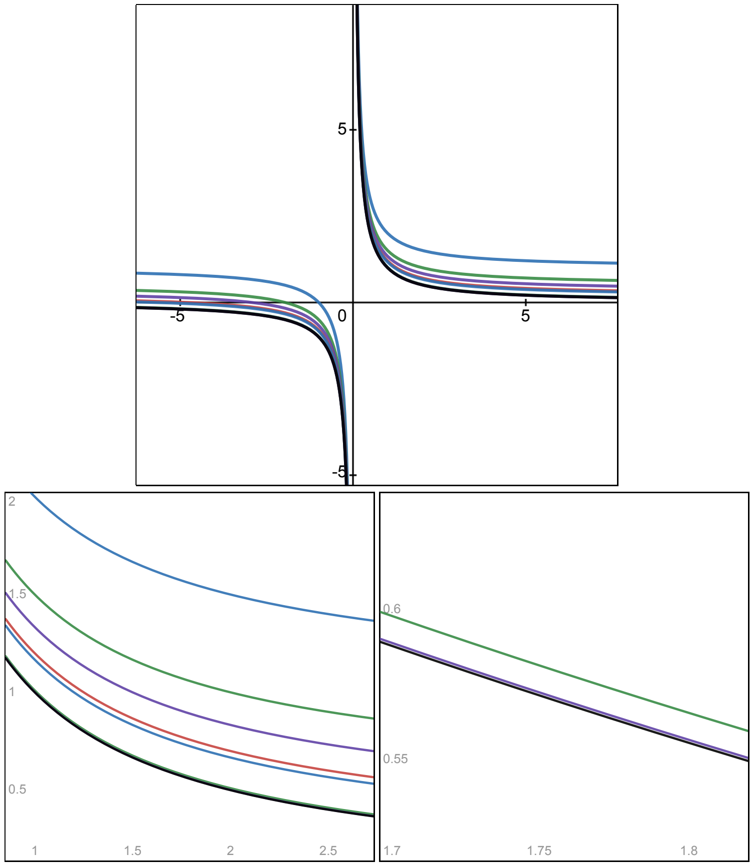

FIGURE 2. $f_n\colon (0,\infty) \to \mathbb{R}$ defined by $f_n(x) = \frac{1}{x} + \frac{1}{n}$ converges to $f(x) = \frac{1}{x}$ (which is denoted by the pure black line the images).

Exercise 3.1.

A collection $\mathcal{A}$ of real-vlaued functions on a set E is said to be **uniformly bounded** on E if there exists $M > 0$ such that $|f(x)| \leqslant M$ for all $x \in E$, for all $f \in \mathcal{A}$. (So each function is bounded, and the same bound works for all functions in $\mathcal{A}$.) Let (f_n) be a sequence of bounded functions which converges uniformly to a limit function f. Prove that $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded subset of $(B(X), d_u)$.

Proof. $\square$

Exercise 3.2.

Let $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on a set E, which converges uniformly on E to limit functions f and g, respectively.

(a) Prove that $(f_n + g_n)_{n=1}^{\infty}$ converges to $f+g$ uniformly on E.
(b) If each f_n and each g_n is bounded, show that $(f_n g_n)_{n=1}^{\infty}$ converges uniformly to fg on E.

Proof. (a) As $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ both converge uniformly, we have some $N \in \mathbb{N}$ and $M \in \mathbb{N}$ such that for $n \geqslant N$ and $n \geqslant M$ we get $|f_n(x) - f(x)| < \epsilon/2$ and $|g_n(x) - g(x)| < \epsilon/2$, respectively, for (all) $\epsilon > 0$. Then

$$\begin{aligned}|(f_n + g_n) - (f+g)| &= |(f_n - f) + (g_n - g)| \\ &\leqslant |f_n - f| + |g_n - g| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

for $n \geqslant \max\{M, N\}$. Hence we have $(f_n + g_n)_{n=1}^{\infty}$ uniformly converging to $f+g$ on E.
(b) Suppose that f_n and g_n are bounded; that is, for all f_n and g_n, we have $|f_n(x)| \leqslant M$ and $|g_n(x)| \leqslant T$ for some $M, P \in \mathbb{R}$ and all $x \in X$. The idea is two get an $\epsilon/2$ demonstration after applying the triangle inequality many times. As $g_n \to g$, for $\epsilon > 0$, there is some $N_1 \in \mathbb{N}$ such that for $n \geqslant N_1$ we can write $|g_n - g| < \frac{\epsilon}{2M}$. Additionally, for $\epsilon > 0$, there exists some $N_2 \in \mathbb{N}$ such that $|f_n - f| < (\frac{\epsilon}{2T} - T)$. Then $|f| \leqslant |f - f_n| + |f_n| = |f_n - f| + |f_n| < (\frac{\epsilon}{2T}) - T + T = \frac{\epsilon}{2T}$. Thus, for $n \geqslant \max\{N_1, N_2\}$,

$$\begin{aligned}|(f_n g_n) - fg| &= |(f_n g_n) - fg + (f_n g - f_n g)| = |(f_n g_n - f_n g) + (f_n g - fg)| \\ &\leqslant |f_n g_n - f_n g| + |f_n g - fg| = |f_n(g_n - g)| + |g(f_n - f)| \\ &= |f_n||g_n - g| + |g||f_n - f| < M\left(\frac{\epsilon}{2M}\right) + (T)\left(\frac{\epsilon}{2T}\right) = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

□

1.3. Uniformly Cauchy Sequences and Completeness of $B(X)$.

Definition 1.3. A sequence $(f_n)_{n=1}^{\infty}$ of real-valued functions on a set X is said to be **uniformly Cauchy** on $E \subset X$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $m \geqslant n \geqslant N$ implies $|f_m(x) - f_n(x)| < \epsilon$ for all $x \in E$.

Proposition 1.2. If $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $B(X)$, then $(f_n)_{n=1}^{\infty}$ is uniformly Cauchy on X.

Proof. Let $(f_n)_{n=1}^{\infty}$ be Cauchy in $B(X)$, i.e. for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $\|f_n - f_m\|_u < \epsilon$ whenever $m \geqslant n \geqslant N$. Then the claim follows when we consider $B(E)$, as the above definition requires it. □

Theorem 1.1. Let $(f_n)_{n=1}^{\infty}$ be sequence of real-valued functions on a set X. Then $(f_n)_{n=1}^{\infty}$ converges uniformly $E \subset X$ if and only it is uniformly Cauchy on E.

Proof. ($\Rightarrow$) Let $(f_n)_{n=1}^{\infty}$ converge uniformly on E to $f\colon X \to \mathbb{R}$. Let $\epsilon > 0$ and pick $N \in \mathbb{N}$ such that $n \geqslant N$ gives us $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in E$. Now pick $m \geqslant n \geqslant N$, then $|f_m(x) - f_n(x)| \leqslant |f_m(x) - f(x)| + |f_n(x) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon$. Hence uniformly Cauchy on E. ($\Leftarrow$) Let $(f_n)_{n=1}^{\infty}$ be uniformly Cauchy on E. Then $(f_n(x))_{n=1}^{\infty}$ is a Cauchy sequence of real numbers for each $x \in E$. Then as $\mathbb{R}$ is complete, each of the sequences $(f_n(x))_{n=1}^{\infty}$ converges to some numbers; thus we can have a pointwise limit function $f(x)$. Now pick $\epsilon > 0$ and $N \in \mathbb{N}$ such that $m \geqslant n \geqslant N$ implies $|f_m(x) - f_n(x)| < \epsilon/2$. Suppose $n \geqslant N$. Then letting $m \to \infty$, then $|f(x) - f_n(x)| = |f_n(x) - f(x)| \leqslant \epsilon/2 < \epsilon$. Hence $f_n \to f$ converges uniformly. □

Theorem 1.2. For any set X, the metric space $(B(X), d_u)$ is complete.

Proof. We need to show that every Cauchy sequence in $B(X)$ converges. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $B(X)$. By Proposition 1.2 $(f_n)_{n=1}^{\infty}$ is uniformly Cauchy and converges uniformly (by Theorem 1.1) to some f on $E = X \subset X$. It suffices to show that $f \in B(X)$, i.e. we have to show that $|f(x)| \leqslant M$ for some $M \in \mathbb{R}_{\geqslant 0}$. Choose $N \in \mathbb{N}$ such that $|f_N(x) - f(x)| < 1$ for all $x \in X$ (we can do this by being uniformly Cauchy). As we're working in $B(X)$, every f_n is itself bounded; choose $M > 0$ such that $|f_N(x)| \leqslant M$. Then $|f(x)| \leqslant |f(x) - f_N(X)| + |f_N(x)| < 1 + M$. Hence f is bounded and therefore $B(X)$ is a complete metric space. □

1.4. **The Uniform Limit Theorem and completeness of BC(X).**

Theorem 1.3 (Uniform Limit Theorem)**.** Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous real-valued functions on a metric space (X, d). Assume $f\colon E \to \mathbb{R}$ is a function such that $f_n \to f$ uniformly on $E \subset X$. Then f is continuous.

Proof. We proceed via a δ-ϵ proof; that is, for any $\epsilon > 0$ and $x \in E$, there is a $\delta > 0$ such that $d_X(x, y) < \delta$ implies $|f(x) - f(y)| < \epsilon$. Note that for any $N \in \mathbb{N}$

$$|f(x) - f(y)| \leqslant |f(x) - f_N(x) + |f_N(x) - f_N(y)| + |f_N(y) - f(y)|.$$

As $f_n \to f$ is uniform, then take $|f_N(s) - f(s)| < \epsilon/3$ for any $s \in E$ and this $N \in \mathbb{N}$. Moreover, pick $\delta > 0$ small enough so that $d(x, y) < \delta$ for $y \in E$ together imply $|f_N(x) - f_N(y)| < \epsilon/3$. Then

$$|f(x) - f(y)| \leqslant |f(x) - f_N(x) + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \leqslant \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

□

Definition 1.4. Let (X, d) be a metric space. The set of all bounded, continuous functions on X is denoted $\mathrm{BC}(X)$.

Corollary 1.1. Let (X, d) be a metric space. The space $(\mathrm{BC}(X), d_u)$ is complete.

Proof. It suffices to show that $\mathrm{BC}(X)$ is a closed subset of $B(X)$. Clearly $\mathrm{BC}(X)$ is indeed a subset of $B(X)$. Now let f be a limit point of $\mathrm{BC}(X)$. Then there exists a sequence $(f_n)_{n=1}^{\infty}$ in $B(X)$ such that $f_n \to f$. By Proposition 1.1, $f_n \to f$ is uniform. Then by Uniform Limit Theorem, we have that f is continuous, and hence $f \in \mathrm{BC}(X)$ and $\mathrm{BC}(X)$ is indeed closed. □

Theorem 1.4 (Uniform Limit Theorem, version 2)**.**

2. Working in $\overline{\mathbb{R}}$ and $\mathbb{R}$

Discussion 2.1. Recall that $\overline{\mathbb{R}}$ is defined to be $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, where the *standard topology* of $\overline{\mathbb{R}}$ is given by the collection of basis set: $\mathfrak{B}_1 = \{(a, b)\colon a, b \in \mathbb{R}, a < b\}$, $\mathfrak{B}_2 = \{[-\infty, a)\colon a \in \mathbb{R}\}$, and $\mathfrak{B}_3 = \{(b, +\infty]\colon b \in \mathbb{R}\}$; write $\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_3$. We have the following proposition:

Proposition 2.1. Let $\mathcal{T}$ and $\overline{\mathcal{T}}$ denote the collection of open sets of $\mathbb{R}$ and and $\overline{\mathbb{R}}$, respectively. Then $A \in \overline{\mathcal{T}}$ if and only if $A \cap \mathbb{R} \in \mathcal{T}$, i.e. $\mathcal{T} = \{A \cap \mathbb{R}\colon A \in \overline{\mathcal{T}}\}$.

2.1. **Infinite Limits and Limits at Infinity.**

Proposition 2.2.

(a) If U is a neighborhood of $+\infty$ in $\overline{\mathbb{R}}$, then there exists an $M \in \mathbb{R}$ such that $(M, +\infty] \subset U$.
(b) If $A \subset \mathbb{R}$ and A is not bounded above in $\mathbb{R}$, then $+\infty$ is a limit point of A with respect to $\overline{\mathbb{R}}$.

Proof. (a) Let U be a neighborhood of $+\infty$ in $\overline{\mathbb{R}}$. As we have a basis described in Discussion 2.1, the open set U then contains some $(b, +\infty]$ for some $b \in \mathbb{R}$ as $U = \cup_{A \in \mathfrak{B}} A$. Put $M = b$ and we're done.
(b) Let $A \subset \mathbb{R}$ and let A not be bounded above in $\mathbb{R}$. □

Proposition 2.3 (Infinite Limits and Limits at Infinity)**.**

(a) Let A be subset of $\mathbb{R}$, let $p \in \mathbb{R}$ be a limit point of A with respect to $\mathbb{R}$, and let $f\colon A \to \overline{\mathbb{R}}$ be a function. Then
$$\lim_{x \to p} f(x) = +\infty$$
if and only if for every $L \in \mathbb{R}$, there exists $\delta > 0$ such that $0 < |x - p| < \delta$ and $x \in A$ together imply that $f(x) > L$.

(b) Let B be a subset of $\mathbb{R}$ that is not bounded above in $\mathbb{R}$, and let $g\colon B \to \overline{\mathbb{R}}$ be a function. Let q be a real number. Then

$$\lim_{x\to+\infty} g(x) = q$$

if and only if for every $\epsilon > 0$, there exists $M \in \mathbb{R}$ such that $x > M$ and $x \in B$ together imply that $|g(x) - q| < \epsilon$.

(c) Let C be a subset of $\mathbb{R}$ that is not bounded above in $\mathbb{R}$; let $h\colon C \to \overline{\mathbb{R}}$ be a function. Then

$$\lim_{x\to+\infty} h(x) = +\infty$$

if and only if for every $N \in \mathbb{R}$, there exists $P \in \mathbb{R}$ such that $X > P$ and $x \in C$ together imply $h(x) > N$.

Proof. (a) Omitted.

(b) Let $\lim_{x\to+\infty} g(x) = q$. The fact that B is not bounded above gives us that $+\infty$ is a limit point of B with respect to $\overline{\mathbb{R}}$. Then for every neighborhood V of q, there is a neighborhood U of $+\infty$ such that $x \in B \cap U \setminus +\infty$ implies $f(x) \in V$.

□

Email address: `jserrato@usc.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90007