MATH 425A HOMEWORK 8 SOLUTIONS

Assignment: Chapter 4, #1.12, 1.13, 2.1, 2.2, 2.3, 2.4, 4.1, 4.2

Due Date: October 21, 2022

Rubric: (35 points total)

- Exercise 1.12. Category I, 6 points (4+2)
- Exercise 1.13. Category II, 3 points
- Exercise 2.1. Category II, 3 points
- Exercise 2.2. Category II, 4 points
- Exercise 2.3. Category II, 3 points
- Exercise 2.4. Category I, 8 points (4 + 4)
- Exercise 4.1. Category II, 3 points
- Exercise 4.2. Category II, 2 points
- Neatness: 3 points
- Optional LaTeXbonus: 1 point extra credit.

Please report any corrections, etc. to lesliet@usc.edu

1.12. Let (X, d) be a metric space, and let E be a subset of X.

(a) Show that E is dense in X if and only if any nonempty open subset of X contains a point of E.
(b) Suppose $E \subset Y \subset X$. Prove that E is dense in Y if and only if $\mathrm{Cl}_X(E) \supset Y$.

Soln.: (a) ($\implies$) Assume there exists a nonempty open subset U of X that contains no points of E. Then E is a subset of the closed set $X \backslash U$ (which is not all of X, since U is nonempty by assumption). That is,

$$\overline{E} \subset X \backslash U \subsetneq X.$$

So E cannot be dense in X.

($\impliedby$) For the other direction, assume that every nonempty open subset of X contains a point of E. Choose $x \in X$; we claim that $x \in \overline{E}$. If $x \in E$, we are done; therefore assume without loss of generality that $x \notin E$. Let U be a neighborhood of x in X; then U contains a point y of E (by assumption) which is not equal to x (since $x \notin E$). Therefore $x \in \mathrm{Lim}_X(E) \subset \overline{E}$. This shows that $\overline{E} = X$, i.e., E is dense in X.

(b) Assume E is dense in Y. Then $Y = \mathrm{Cl}_Y(E) = \mathrm{Cl}_X(E) \cap Y \subset \mathrm{Cl}_X(E)$. On the other hand, if $\mathrm{Cl}_X(E) \supset Y$, then $\mathrm{Cl}_Y(E) = \mathrm{Cl}_X(E) \cap Y \supset Y$. So E is dense in Y.

1.13. Previously, we said that a subset E of $\mathbb{R}$ was dense in $\mathbb{R}$ if for any real numbers a and b, there exists a number $c \in E$ which lies between a and b. Show that in $\mathbb{R}$, the new, more general definition of 'dense' agrees with the old one. That is, show that a subset E of $\mathbb{R}$ is dense in $\mathbb{R}$ according to the new definition if and only if it is dense according to the old one. (Hint: Use Exercise 1.12(a).)

Soln.: Assume that E satisfies the 'old' definition; that is, assume that for any real numbers a and b, there exists a number $c \in E$ that lies between them. Let U be a nonempty open subset of $\mathbb{R}$. Then U contains some interval of the form (a, b), where $a < b$, so that there exists $c \in E$ such that $a < c < b$. By Exercise 1.12(a), it follows that E is dense in $\mathbb{R}$ in the 'new' sense.

Conversely, suppose that $\overline{E} = \mathbb{R}$, and choose $a, b \in \mathbb{R}$ with $a < b$. Then by Exercise 1.12(a), (a, b) contains a point c of E, which satisfies $a < c < b$. Thus the 'old' characterization of density holds for E.

2.1. Let $S = (p_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$ whose image is $(\mathbb{Q} \cap (0,1)) \cup \{5\}$. What are the two possibilities for S^*? Justify your answer.

Soln.: First of all, we have $(\operatorname{Im} S)' = ((\mathbb{Q} \cap (0,1) \cup \{5\}))' = [0,1]$. So S^* is a closed subset of $\mathbb{R}$ that contains $[0,1]$. On the other hand, $\overline{\operatorname{Im} S} = [0,1] \cup \{5\}$. Thus

$$[0,1] \subset S^* \subset [0,1] \cup \{5\}.$$

Thus the two candidates for S^* are $[0,1]$ and $[0,1] \cup \{5\}$. Both of these can be achieved; note that in order to have $5 \in S^*$ we must have $5 \in S_\infty$, since $5 \notin (\operatorname{Im} S)'$.

2.2. Prove Proposition 2.7 (restated here for convenience).

Proposition: Let $S = (p_n)_{n=1}^\infty$ be a sequence in a metric space X, and let S^* denote the set of subsequential limits of S. The following are equivalent.

(1) $p_n \to p$ in X as $n \to \infty$.
(2) Every subsequence of $(p_n)_{n=1}^\infty$ converges to p in X.
(3) $S^* = \{p\}$, and every subsequence of $(p_n)_{n=1}^\infty$ converges in X.

Soln.: (1) $\implies$ (2) Assume $p_n \to p$ in X as $n \to \infty$. Let U be any neighborhood of p in X, and choose $N \in \mathbb{N}$ large enough so that $n \geq N$ implies $p_n \in U$. Let $(p_{n_k})_{k=1}^\infty$ be a subsequence of $(p_n)_{n=1}^\infty$. Then $k \geq N$ also implies $n_k \geq k \geq N$, which implies $p_{n_k} \in U$. Thus $p_{n_k} \to p$ in X as $k \to \infty$.

(2) $\implies$ (1) $(p_n)_{n=1}^\infty$ is a subsequence of itself. (Put $n_k = k$ for each $k \in \mathbb{N}$.) So statement (1) is a special case of statement (2).

(2) $\implies$ (3) Assume that every subsequence of $(p_n)_{n=1}^\infty$ converges to p. Then p a subsequential limit, and is moreover the only possible subsequential limit; this says exactly that $S^* = \{p\}$.

(3) $\implies$ (2) Assume that $S^* = \{p\}$ and that every subsequence of $(p_n)_{n=1}^\infty$ converges in X. Let $(p_{n_k})_{k=1}^\infty$ be a subsequence of $(p_n)_{n=1}^\infty$. Then (p_{n_k}) converges in X by assumption, and its limit is an element of S^* by definition of S^*. Since the only element of S^* is $\{p\}$, it follows that $p_{n_k} \to p$ in X as $k \to \infty$.

2.3. Let (X, d) be a metric space, and let $(x_n)_{n=1}^\infty$ be a sequence in X. Prove the following statements.

(a) If $(x_n)_{n=1}^\infty$ converges in X, then it is Cauchy in X.
(b) If $(x_n)_{n=1}^\infty$ is Cauchy in X, then it is bounded in X.

Soln.: (a) Assume $(x_n)_{n=1}^\infty$ converges in X, to some point $x \in X$. Choose $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that $n \geq N$ implies $d(x_n, x) < \frac{\varepsilon}{2}$. Then $m \geq n \geq N$ implies that

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus (x_n) is Cauchy in X.

(b) Assume $(x_n)_{n=1}^\infty$ is Cauchy in X, and choose $N \in \mathbb{N}$ such that $m \geq n \geq N$ implies $d(x_m, x_n) < 1$. Put $M = \max\{1, d(x_1, x_N), \ldots, d(x_{N-1}, x_N)\}$. We claim that $\{x_n\}_{n=1}^\infty \subset B_X(x_N, M)$. Indeed, if $j < N$, then $d(x_j, x_N) \leq M$ by definition of M; if $j \geq N$ then $d(x_j, x_N) < 1 \leq M$ by choice of N. This proves that $(x_n)_{n=1}^\infty$ is bounded in X.

2.4. Prove Theorem 2.12 (copied below for convenience). (Hints: For (a), use Theorem 1.11 and Proposition 1.15. For part (b), Exercise 1.4 is relevant.)

Theorem: Let (X, d) be a metric space; let Y be a subset of X. The following statements hold.

(a) If Y is complete, then Y is closed in X.
(b) If X is complete and Y is closed in X, then Y is complete.

Soln.: (a) Assume Y is complete; we show that $\mathrm{Lim}_X(Y) \subset Y$. (This is enough, by Proposition 1.15.) Choose $y \in \mathrm{Lim}_X(Y)$. By Theorem 1.11, we can find a sequence $(y_n)_{n=1}^{\infty}$ in $Y \backslash \{y\}$ such that $y_n \to y$ in X as $n \to \infty$. Thus $(y_n)_{n=1}^{\infty}$ is Cauchy in X and is a sequence in Y, so $(y_n)_{n=1}^{\infty}$ is Cauchy in Y. Since Y is complete, it follows that $(y_n)_{n=1}^{\infty}$ converges in Y to a point $z \in Y$. Consequently $y_n \to z$ in X. By uniqueness of limits, we have $z = y$, which tells us that $y \in Y$. Thus $\mathrm{Lim}_X(Y) \subset Y$, as needed.

(b) Assume X is complete and Y is closed in X. Let $(y_n)_{n=1}^{\infty}$ be a Cauchy sequence in Y; we need to show that it converges in Y. Since $(y_n)_{n=1}^{\infty}$ is Cauchy in Y, it is also Cauchy in X; since X is complete, $(y_n)_{n=1}^{\infty}$ must converge in X to a point $y \in X$. We claim first of all that $y \in Y$. If $y = y_n$ for some $n \in \mathbb{N}$, then we are done, as $(y_n)_{n=1}^{\infty}$ is a sequence in Y by assumption. Therefore we assume without loss of generality that $y \notin \{y_n\}_{n=1}^{\infty}$, which implies that $(y_n)_{n=1}^{\infty}$ is a sequence in $Y \backslash \{y\}$ that converges in X to y. By Theorem 1.16, it follows that $y \in \mathrm{Lim}_X(Y)$, which implies that $y \in Y$ since Y is closed in X.

We have thus established that $y_n \to y$ in X and $y \in Y$. By Exercise 1.4, it follows that $y_n \to y$ in Y. Thus Y is complete, as needed.

4.1. Prove Corollary 4.10. (Statement: Let (X, d) be a metric space. Assume F and K are subsets of X, with F closed and K compact. Then $F \cap K$ is compact.)

Soln.: $F \cap K$ is closed in K, since F is closed in X. Therefore it is a closed subset of a compact metric space, therefore (by Theorem 4.9) compact.

4.2. Give an example of a collection $\mathcal{A}$ of *bounded* subsets of $\mathbb{R}$ such that $\mathcal{A}$ has the finite intersection property, but $\bigcap_{A \in \mathcal{A}} A = \emptyset$. Hint: If $A \subset \mathbb{R}$ is bounded in $\mathbb{R}$, what else can prevent it from being compact?

Soln.: Put $\mathcal{A} = \{(0, \frac{1}{n})\}_{n=1}^{\infty}$. Then $\mathcal{A}$ is a countable collection of nonempty nested sets, so it has the finite intersection property. But $\bigcap_{n=1}^{\infty}(0, \frac{1}{n}) = \emptyset$.