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1. Establishing (Some) Topology

Definition 1.1. Let X be a set. A **topology** on X is a collection τ of subsets of X that satisfy the following three requirements:

(i) X and $\varnothing$ belong to τ;
(ii) the union of any (finite or infinite) number of sets in τ belong to τ again; and
(iii) the intersection of any finite number of sets in τ belongs to τ.

The pair (X, τ) is called a **topological space**, and the members of the topology are called **open sets**.

Example 1.1. Let X be any non-empty set and let τ be the collection of all subsets of X, i.e. $\tau = \mathcal{P}(X)$, the power set of X. Then τ is called the *discrete topology* on the set X, and the topological space (X, τ) is called the *discrete space*. Another trivial example is when $\tau = \{\varnothing, X\}$, and τ is called the *indiscrete topology.*

Proposition 1.1. If (X, τ) is a topological space such that, for every $x \in X$, the singleton set $\{x\}$ is in τ, then τ is a discrete topology.

Proof. Suppose (X, τ) is a topological space such that for every $x \in X$, the singleton set $\{x\} \in \tau$. Note that every set is the union of all its singleton subsets, and so if S is any subset of X, then $S = \bigcup_{x \in S}\{x\}$. Since we are given that each $\{x\} \in \tau$ then the above equation implies that $S \in \tau$. As S is an arbitrary subset of X, we have that τ is the discrete topology. $\square$

Definition 1.2. Let X be a set and let τ_1 and τ_2 be topologies on X. If $\tau_1 \subset \tau_2$, then τ_1 is **coarser** than τ_2. If $\tau_2 \subset \tau_1$,is **finer** than τ_2.

Example 1.2. We endow the set $\mathbf{R}^n$ with a topology, denoted by $||\cdot||_n$, which depics the natural topology induced by the Euclidean metric. Our open sets are given by the Euclidean metric: a set $U \subset \mathbf{R}$ is open if for any $x \in U$ there is an $\epsilon \in \mathbf{R}_{\geq 0}$ such that the set $(x - \epsilon, x + \epsilon) \subset U$.

Definition 1.3. Let (X, τ) be a topological space and $x \in X$. A set $N \subset X$ is a **neighbourhood** of x if there exists some $U \in \tau$ with $x \in U \subset N$. The set of all neighbourhoods of an element x will be denoted by $\mathcal{N}_x$.

Definition 1.4. Let (X, τ) be a topological space. We say that a subset F of X is **closed** when $X \setminus F$ is in τ.

Example 1.3. In $(\mathbf{R}, ||\cdot||_n)$, the set $[0, 1]$is closed. This is because $\mathbf{R} \setminus [0, 1] = (-\infty, 0) \cup (1, \infty)$, which is the union of two open intervals. Moreover, we should begin to notice the motivation for the names of open and closed (as open and closed intervals); although we begun with the abstract notion of a topology, its motivation comes from the abstraction of the topology on $(\mathbf{R}, ||\cdot||_n)$.

Example 1.4. In $(X, \mathcal{P}(X))$ every subset of X is closed. The reason for this being that for any $F \subset X$, we have $X \setminus F \in \mathcal{P}(X)$.

Remark 1.1. Given a topological space (X, τ), subsets of X can be: open, closed, both open and closed, or neither open nor closed. If we have that a subset of X is both open and closed, we will say that is it *clopen.* Thus we say that, in the discrete space, every subset of X is a clopen set.

Definition 1.5. Let (X, τ) be a topological space and S a setset of X. The **closure** of S, denoted by $\bar{S}$, is defined to be $\bar{S} = \bigcap\{F \colon F \subset X \text{ is closed and contains } S\}$.

Symbolically, the closure can be rewritten as $\bar{S} = \{x \in X \colon N \cap S \neq \varnothing \text{ for all } N \in \mathcal{N}_x\}$. However, the fact that these two sets presentations of $\bar{S}$, as defined in Definition 1.5, are the same isn't obvious, and the proof will be omitted here. But we should remark that we should think of $\bar{S}$ as being the smallest closed set that contains S.

Definition 1.6. Let (X, τ) be a topological space. We say that a set $D \subset X$ is **dense** in X when $\bar{D} = X$.

Example 1.5. $\mathbf{Q}$ is dense in $\mathbf{R}$ with the Euclidean topology. This is usually presented early on in a first real analysis course when one proves that between any two real numbers, there is a rational number between them. Additionally, we have that the set of irrational numbers are dense in $\mathbf{R}$ as well.

Definition 1.7. Let (X, τ) be a topological space. A **base** $\mathcal{B}$ for τ is a subset of τ where each open set in τ can be written as a union of elements in $\mathcal{B}$

Definition 1.8. Let (X, τ) be a topological space. A **subbase** for τ is a set $\mathcal{J} \subset \tau$ with the property that the set of all finite intersections of sets in $\mathcal{J}$ is a base for τ.

Definition 1.9. A topological space (X, τ) is called **Hausdorff** space when distinct points can be seperated by open sets. That is, for all $x, y \in X$ and $x \neq y$, there exists $U, v \in \tau$ such that $x \in U$ and $y \in V$ with $U \cap V = \varnothing$.

Remark 1.2. The real numbers $\mathbf{R}$ are Hausdorff with the Euclidean topology as if we let $x, y \in \mathbf{R}$ and $x \neq y$, then letting $\epsilon = (|x - y|)/3$ implies that $(x - \epsilon, x + \epsilon)$ and $(y - \epsilon, y + \epsilon)$ are disjoint open sets that contain x and y, respectively.

Definition 1.10. Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$ be a map. We call f **continuous** at $x_0 \in X$ if the following condition is true: $N \in \mathcal{N}_{f(x_0)}$ implies $f^{-1}(N) \in \mathcal{N}_{x_0}$.

Proposition 1.2. Let X and Y be topological spaces and let $f : X \to Y$ be a function. THe following three statements are equivalent.

(i) $f : X \to Y$ is continuous on X,
(ii) $f^{-1}(U)$ is open in X for all open sets U in Y,
(iii) $f^{-1}(F)$ is closed in X for all closed sets F of Y.

Proposition 1.3. Let X, Y, and Z be topological spaces and let $f : X \to Y$ be continuous on all X and $g : Y \to Z$ be continuous on all of Y. Then $g \circ f : X \to Z$ is continuous of all of X.

Proof. Since f is continuous on all of X we have that for every open set U in Y, $f^{-1}(U)$ is open in X. Similarily, we have that for all open sets of V in Z, we have that $g^{-1}(V)$ is open in Y. Now consider the composition of g and f, that is, $g \circ f : X \to Z$. Let W be an open set in Z. Then $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$, and since $g^{-1}(W)$ is open in Y, so let $V = g^{-1}(W)$. Then $(gf)^{-1}(W) = f^{-1}(V)$, but as V is open in Y, then $f^{-1}(V)$ is open in X. Thus $(gf)^{-1}(W)$ is open in X. Hence gf is continuous on all of X. □