URP: HOMOLOGY AND ALGEBRAIC GEOMETRY

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Let R be a commutative ring with unit (if not stated otherwise, always assume that these are the assumptions upon a ring in this paper) and let A, B, C be R-modules. Then, we know that the machinery of (exact) sequences—also equivalent to the notion of a complex—is rich in what is tell us about the relations between our R-modules; for example, if $\varphi\colon A \longrightarrow B$ is a monomorphism and $\psi\colon B \longrightarrow C$ is an epimorphism, then we may form an exact sequence that provides us with the fact that $C \simeq B/\operatorname{im}(\varphi)$, although this may be a very basic fact about exact sequences I think it's great! However, there's another rich feature of sequences (which are not necessarily exact), that is, considering the so-called homology. We do this, using our example sequence above, and call our sequence X, by defining the homology of this group to be $H(X) = \ker(\psi)/\operatorname{im}(\varphi)$. If our sequence is in fact exact, then $\ker(\psi) = \operatorname{im}(\varphi)$, so $H(X) = \ker(\psi)/\ker(\psi)$, then we get that for an element in $H(X)$ it is of the form $x + \ker(\psi)$ for some $x \in \ker(\psi)$, but then since $x \in \ker(\psi)$ then $x + \ker(\psi) = \ker(\psi)$, as, essentially, $\psi(x + \ker(\psi)) = \psi(x) + \ker(\psi) = 0$, so $x + \ker(\psi) = \ker(\psi)$ implies that $x = 0$; thus $H(X) = 0$. But this usually not the case when our sequence isn't exact. The most common phrase you'll hear from algrebraist is that the homology group measure how *far off* the original sequence is away from being exact and also how homological algebra has much applications to prove, well, theorems; however, we should look for motivation from algebraic topology as my goal is to understand/apply homology to algebraic geometry. So, now the usual phrase you'll hear from algebraic topologists is that $H_k(X)$ counts the number of k-dimensional holes in a topological space. "In general, our approach will be to add a structure on a space or object (and thus a topology) and figure out what subsets of the space are cycles, then sort through those subsets that are holes. Of course, as many properties we care about in topology, this property is invariant under homotopy equivalence. This is the slightly weaker than homeomorphism which we before said gave us the same fundamental group."

Now, as we spoke about above, we aim towards talking about algebraic geometry, so lets talk about varieties: Let V be a variety over the complex numbers, then $V(\mathbb{C})$ is given a topological structure from that on $\mathbb{C}$, so now we obviously can take the route of algebraic topology. Another notion that's important is this context is a Betti number: for a non-negative integer k, the kth Betti number $b_k(X)$ of the space X is defined as the rank (or the number of linearly independent generators) of the abelian group $H_k(X)$. Equivalently, if we can define a Betti number in the context of the dimension of the vector space of $H_k(X;\mathbb{Q})$, as the homology group in this case is a vector space over $\mathbb{Q}$, and this generalizable for a given field F by defining $b_k(X, F)$, the kth Betti number with coefficients in F, as the vector space dimension of $H_k(X, F)$. Personally, to me, this sounds much more interesting. Going back to the geometric context, we can, once again, say that the Betti numbers $b^r(V)$ of V to be the dimensions of the vector spaces $H^r(V(\mathbb{C}), \mathbb{Q})$—the indicies switched from being on the "bottom" to the "top" since in geometry we're heavily biased towards talking about cohomology. A nice sentiment given by Reid Barton on Mathoverflow: "Homology has to do with taking the free abelian group on a set, while cohomology has to do with taking the ring of functions on a set. Algebraic geometry is all about treating an arbitrary ring as the ring of functions on something, so it's not surprising that algebraic geometers care a lot about cohomology." Thus, as this may indicate, beyond the title of this paper itself, is that we hope to treat a lot of homological ideas in algebraic geometry (e.g. schemes, sheaves, etc) as well.

1. Tor

One thing that we should immediately note here is that I have a complete bias for the tensor product over Hom, and so I think we'll have a more fruitful discussion about Tor rather than Ext as my mind can't juggle, as of now, the many things that go on when taking about Ext. So, in light of this, we'll go onto the basics that will later provide us with what Tor *is*.

Definition 1.1. An R-module P is called **projective** if for every epimorphism $\varphi\colon M \longrightarrow M''$ and some map $f\colon P \longrightarrow M''$, there exists a map ψ making the following diagram commute:

$$\begin{tikzcd}
 & P \arrow[dl, dashed, "\psi"'] \arrow[d, "f"] & \\
M \arrow[r, "\varphi"] & M'' \arrow[r] & 0
\end{tikzcd}$$

Remark 1.1. More generally, the map $\psi\colon P \longrightarrow M$ is called a lifting; that is, explicitly, given a map $\varphi\colon M \longrightarrow M''$ and another map $f\colon P \to M''$, a **lifting** is a map $\psi\colon P \longrightarrow M$ with $\varphi\psi = f$. Moreover, if F is a free R-module, and $\varphi\colon M \to M''$ is an epimorphism, then for every $f\colon F \longrightarrow M''$, there exists an R-map $\psi\colon P \longrightarrow M$. Thus every free R-module is projective. Note that I'm most definitely abusing notation here for sake of compactness.

If we wanted to classify some projective R-modules, then we are going to be heavily dependent on the ring R of our choosing. However, if we have that R is a PID, then every submodule of a free module is itself free, and this also admits that every projective R-module is free (we state this simply as fact). Moreover, a possibly more interesting ring, and much harder result to prove, is the case where $R = k[x_1, \ldots, x_n] := k[X]$, i.e. R is the polynomial ring in n variables over a field k—this question was raised by Jean-Pierre Serre. This also induces the fact that every projective $R = k[X]$-module is also free. Another useful property of projective modules is that every module M is the image of some projective (even free) module P, i.e. there is an epimorphism $\eta\colon P \to M$. Now, why should we care about this? Well, and you should get used to hearing this sort of reponse, it essentially comes down to wanting to understand how much a module fails to be projective—maybe homological algebra should actually be prescribed the name measure theory, huh? Seeking to measure how much a module fails to be projective relates to a notion of a module being *flat*: this was introduced by Jean-Pierre Serre in his 1956 paper *Géometrie Algébrique et Géométrie Analytique.* An R-module M is flat if whenever we take the tensor product over R with M it preserves the exact sequence. However, we can get away with defining flat in terms of only requiring a monomorphism:

Definition 1.2. An R-module M is called **flat** if for every injective linear map $\varphi\colon K \to L$ of R-modules, the map

$$\varphi \otimes_R M\colon K \otimes_R M \longrightarrow L \otimes_R M$$

is also injective, where $\varphi \otimes_R M$ is the map induced by $k \otimes m \longmapsto \varphi(k) \otimes m$.

Remark 1.2. If M is an R-module and $M' \subseteq M$ is a submodule with an injective map $\varphi\colon M' \longrightarrow M$, then for any flat module R-module N, the map $M' \otimes_R N \longrightarrow M \otimes_R N$ obtained by tensoring over R with N remains injective. Thus, in this case, $M' \otimes_R$ can be viewed as a submodule of $M \otimes_R N$.

So, as spoken about above, we may want to measure how much a module deviates from being projective, and we conduct this measurement with something called a *projective resolution.*

Definition 1.3. Given an R-module A, a **projective resolution** of A is any exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

in which every P_i is a projective R-module.

Construction 1.1. As hinted in Remark 1.1., every R-module has a projective resolution. We show this by letting P_0 be a free R module on a set of generators for an R-module B. Then B is a direct summand of P_0—we should note here that an R-module X is projective if and only if it is a direct summand of a free R-module. Now, let $\epsilon\colon P_0 \longrightarrow B$ be the projection map. Then, since ϵ is surjective, we have that that $P_0 \xrightarrow{\epsilon} B \longrightarrow 0$ is exact, i.e. $\operatorname{im}(\epsilon) = \ker(0)$. As we want to construct objects in the leftward direction of the exact sequence to form a projective resolution, then for the homomorphism d_1, define $\operatorname{im}(d_1) = d_1(P_1) = \ker(\epsilon)$, and let P_1 be a free R-module (thus projective) mapping onto $\ker(\epsilon) \subseteq P_0$. This provides us with an exactness at P_0. And we can rinse and repeat this process to get a resolution of B that is free at every P_i; we call this a *free resolution.* Moreover, we should note the following chain of consequences if a module is free:

$$\text{free} \Longrightarrow \text{projective} \Longrightarrow \text{flat} \Longrightarrow \text{torsion-free}$$

We'll talk about what it means for a module to be "torsion free" soon.

Construction 1.2. Let B be an R-module. Take a projective resolution of B

$$\mathbf{P} = \cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} B \longrightarrow 0.$$

If we let $\mathbf{P}^\bullet$ denote the truncated projective resolution, i.e. the original resolution where we delete the module B, then we let $\mathrm{Tor}_i^R(B, X) = H_i(\mathbf{P}^\bullet \otimes_R X)$ denote the ith homology group of this complex after applying the tensor $- \otimes_R X$ to the truncated resolution. This later sequence looks like the following:

$$\mathbf{P}^\bullet \otimes_R X = \cdots \longrightarrow P_n \otimes_R X \xrightarrow{d_n \otimes 1} P_{n-1} \otimes_R X \xrightarrow{d_{n-1}\otimes 1} \cdots \xrightarrow{d_2 \otimes 1} P_1 \otimes_R X \xrightarrow{d_1 \otimes 1} P_0 \otimes X \xrightarrow{d_0 \otimes 1} 0.$$

Proposition 1.1. For $\mathrm{Tor}_i^R(B, X) = H_i(\mathbf{P}^\bullet \otimes_R X)$, as above:

(i) $\mathrm{Tor}_0^R(B, X) = B \otimes_R X$
(ii) $\mathrm{Tor}_n^R(B, X) = 0$ for $n < 0$.
(iii) If B is projective, then $\mathrm{Tor}_n^R(B, X) = 0$ for $n > 0$.
(iv) If X is flat, then $\mathrm{Tor}_n^R(B, X) = 0$ for $n > 0$.

Proof. (i) Since $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} B \longrightarrow 0$ is an exact sequence, then by right exactness, $P_1 \otimes_R X \xrightarrow{d_1 \otimes 1} P_0 \otimes_R X \xrightarrow{\epsilon \otimes 1} B \otimes_R X \longrightarrow 0$ is an exact sequence as well. Now, by an elementary fact about exact sequences, in fact brought up in the introduction, we have that $B \otimes_R X \simeq P_0 \otimes_R X/(d_1 \otimes 1)(P_1 \otimes_R X)$, which is exactly what is meant by taking $\mathrm{Tor}_0^R(B, X) = H_0(\mathbf{P}^\bullet \otimes_R X) = \ker(d_0 \otimes 1)/\operatorname{im}(d_1 \otimes 1) = P_0 \otimes X/(d_1 \otimes 1)(P_1 \otimes A)$—note here that $\ker(d_0 \otimes 1) = P_0 \otimes X$ as $d_0 \otimes 1 \colon P_0 \otimes X \longrightarrow 0$ and so the kernel of the map is all of $P_0 \otimes_R X$.

(ii) The convention is that for all $n < 0$ in the projective resolution, we have trivial modules, and so we have that $P_n \otimes_R X = 0$; thus $H_n(P_n \otimes_R X) = 0$.

(iv) Suppose that X is flat. For the sequence $E = P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1$, we know that $E \otimes_R X$ is exact everywhere, as X is flat, so the homology is trivial at every module, i.e. $H_n(E \otimes_R X) = 0$ for $n > 0$. $\square$

2. Schemes

The history of schemes is a particularly fruitful one, I think. It gives us a possible first introduction to the colorful and brilliant person that is Alexander Grothendieck. Grothendieck's invention of *scheme* laid a new foundation for algebraic geometry by "making intrinsic spaces ('spectra') and associated rings the primary object of study. " In essence, I say this naively, a scheme is a topological space in which we associate a ring to its every open subset—as hinted by the preceding sentence, this arises from gluing together spectra (spaces of prime ideals, $\mathrm{Spec}(R)$) of commutative rings together with their open subsets. This is all to say that a scheme is a ringed space that is locally a spectrum of a commutative ring. The theory of schemes compliments a study of homological algebra as methods from it are very important and applicable to scheme theory. From what I can recall to the best of my ability is that a scheme *generalizes* the notion of an algebraic variety, and another complimentary fact is that a possible motivation for scheme, albeit maybe a small one, is that a scheme provides us with a category, the category of schemes, denoted by **Sch**, whereas varieties do not—this sentiment I heard from Deligne.

Construction 2.1 (Zariski topology)**.** Let R be a ring. Recall that the spectrum of a ring, $\mathrm{Spec}(R)$, is the set of all prime ideals of the ring R. By convention, the unit ideal is not a prime ideal, and so the trivial ring where $1 = 0$ has an empty spectrum, i.e. $\mathrm{Spec}(\{0\}) = \emptyset$. We hope to give this spectrum a topological structure: For any ideal I of R, let $V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \colon I \subseteq \mathfrak{p}\}$. Now, if $f \in A$, let $D(f) := \mathrm{Spec}(R) - V(fA)$