HOMOTOPY SEMINAR: NOTES

JUAN SERRATOS

Contents

1. Introduction: ∞-categories and related prerequsities

1.1. **(Pre)Sheaves.** A recurring term you will forever hear is that of a *presheaf* and a *sheaf*, where the latter is an extension of a presheaf that satisfies some extra properties. The reason this is heard so often is that the concept isn't actually a very hard one (it is also very general), and will be heard in many concepts (especially in the algebraic contexts).

Definition 1.1. Let $\mathscr{C}$ be a category. A **presheaf over** $\mathscr{C}$ is a functor of the form $\mathscr{F}: \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$.

If one prefers, the equivalent definition is that $\mathscr{F}: \mathscr{C} \to \mathsf{Sets}$ is a *presheaf* if $\mathscr{F}$ is a contravariant functor (in Definition 1.1, it is instead defined to be covariant but we preemptively flip the arrows when we consider the opposite category). Additionally, in our definition we explicitly use the category of sets, but it is often that we change this category to something like the category of abelian groups, or rings, and so on. We say that $\mathscr{F}: \mathscr{C} \to \mathsf{Ab}$ is an *abelian-valued presheaf* for an extra level of clarity, although it is often clear from context. Given a presheaf $\mathscr{F}$ over $\mathscr{C}$ and an object $U \in \mathscr{C}$, we call the elements of $\mathscr{F}(U)$ *sections*. A morphism of presheaves $\eta: \mathscr{F} \to \mathscr{L}$, where $\mathscr{F}$ and $\mathscr{L}$ are both presheaves over $\mathscr{C}$, is simply a natural transformation. Lastly, we can build a category out of presheaves: it's a functor category $\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets}) := \mathsf{PSh}(\mathscr{C})$, i.e it's the category that has objects as presheaves $\mathscr{F}: \mathscr{C} \to \mathsf{Sets}$ and has as morphisms natural transformations between presheaves.

In algebraic geometry, one often fixes a topological space X and considers the category X_{open}, where the objects are open subsets of X and, for any given two open subsets of X, say, U and V, the morphism is either the inclusion map $i: U \to V$ if $U \subset V$ or no morphism at all if neither is contained in the other. The theory of schemes is built out of considering sheaves $\mathscr{O}_X : X_{\text{open}} \to \mathsf{Rings}$ with a lot more structure.

Definition 1.2. The **Yoneda embedding** is the functor $F: \mathscr{C} \to \mathsf{PSh}(\mathscr{C})$ where $F(A) = \mathrm{Hom}_{\mathscr{C}}(-, A)$ for an object $A \in \mathscr{C}$.

Exercise 1.1. Verify that $\mathrm{Hom}_{\mathscr{C}}(-, A)$ is a presheaf.

Theorem 1.1 (Yoneda Lemma). For any presheaf $\mathscr{F}$ over $\mathscr{C}$, there is a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{PSh}(\mathscr{C})}(\mathrm{Hom}_{\mathscr{C}}(-, A), \mathscr{F}) \xrightarrow{\sim} \mathscr{F}(A)$$

...[To be completed]

1.2. **The category of simplicial sets.** Consider the category whose objects are are finite subsets $[n] = \{1, 2, \ldots, n\} = \{i \in \mathbf{Z}: 1 \leq i \leq n\}$, for $n \geq 0$, which inherit the natural ordering of integers, and has morphisms $f: [n] \to [m]$ that are (non strict) order-perserving maps, i.e. functions $f: [n] \to [m]$ with the property that $a < b$ in $[n]$ implies $f(a) \leq f(b)$ in $[m]$. We denote this category by Δ.

Definition 1.3. A **simplicial set** is a presheaf over Δ. We denote $\mathsf{Set}_{\Delta} = \mathsf{PSh}(\Delta)$ to be the *category of simplicial sets*.

Given $n \geq 0$, we write $\Delta^n = \mathrm{Hom}_\Delta(-, [n])$ and call it the ***standard*** n ***simplex***, that is, Δ^n is the simplicial set represented by $[n] \in \Delta$. So, we write $(\Delta^n)_m = \mathrm{Hom}_\Delta([m], [n])$. For a simplicial set $X \colon \Delta \to \mathsf{Sets}$ and an integer $n \geq 0$, we write $X_n = X([n]) \simeq \mathrm{Hom}_{\mathsf{Sets}_\Delta}(\Delta^n, X)$, by Theorem 1.1, for the set of n-***simplices*** of X. Lastly, a ***simplex*** of X is an element of X_n for some nonnegative integer n.

We define, for $0 \leq i \leq n$, the pair of following maps:

$$d^i \colon [n-1] \to [n], \qquad d^i(j) = \begin{cases} j & j < i, \\ j+1 & i \leq j \end{cases}$$

$$s^i : [n+1] \to [n], \qquad s^i(j) = \begin{cases} j & j \leq i, \\ j-1 & i \leq j \end{cases}$$

References

[Lur12] Jacob Lurie. Higher topos theory. *Annals of Mathematics Studies*, **170**, 2012.

Email address: jserrato@usc.edu