HOMOTOPY SEMINAR: NOTES

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Contents

0. Category Theory

0.1. **Categories.** The theory of categories should be thought of as a reformulation/language for a birdseye view of mathematics. Perhaps too harsh of a statement, but the subject is too dry if the sake is to understand only categories and not to fervently apply it to well-known categories to see how such things blossom in their respective contexts.

Definition 0.1. A morphism **category** $\mathscr{C}$ consists of a collection of **objects**, denoted as $\mathrm{Ob}(\mathscr{C})$, a collection of **morphisms** given any $A, B \in \mathrm{Ob}(\mathscr{C})$, written $\mathrm{Hom}_{\mathscr{C}}(A, B)$, and a function for any $A, B, C \in \mathrm{Ob}\,\mathscr{C}$

$$\mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}}(B, C) \to \mathrm{Hom}_{\mathscr{C}}(A, C)$$
$$(f, g) \mapsto g \circ f,$$

satisfying the following properties:

(i) For any $A \in \mathrm{Ob}$, an identity map $1_A \in \mathrm{Hom}_{\mathscr{C}}(A, A)$ such that $1_A \circ f = f$ and $g \circ 1_A = g$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(B, A)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(A, B)$.
(ii) For all $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathscr{C}}(B, C)$, and $h \in \mathrm{Hom}_{\mathscr{C}}(C, D)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
(iii) For all $f \in \mathrm{Hom}_{\mathscr{C}}(A, B)$, we have $f \circ 1_A = f = 1_B \circ f$.

Example 0.1.

(i) The category of groups, Grps, where the objects are groups and have as morphisms between them group homomorphisms. Similarly, the category of abelian groups, Ab, and the category of rings, Rings, that has objects rings and morphisms are ring homomorphisms.
(ii) The category of topological spaces, Top, where the objects are topological spaces and the morphisms are continuous maps.
(iii) The category of sets, Sets, which has as objects sets and morphisms are usual functions.
(iv) For a ring A, the category of A-modules, Mod_A, that has as objects A-modules and morphisms are A-linear maps. If $A = k$ is a field, then this is the category of k-vector spaces, Vec_k.

(v) The category of pure Hodge structures for a given weight n, $\mathsf{pHod_n}$, where the objects are pure Hodge structures of weight n and the morphisms are homomorphisms of the underlying abelian groups.

Definition 0.2. A category $\mathscr{C}$ is a **locally small** if $\mathrm{Hom}_{\mathscr{C}}(A, B)$ forms a set for $A, B \in \mathrm{Ob}(\mathscr{C})$. Furthermore, $\mathscr{C}$ is **small** if $\mathscr{C}$ is locally small and any object A in $\mathscr{C}$ forms a set.

Example 0.2.

(i) All the categories from Example 0.1 are locally small.
(ii) The category Cat that has as objects categories and morphisms as *functors* (we will see what this is soon) is not locally small.

Definition 0.3. A morphism $f\colon A \to B$ in a category $\mathscr{C}$ is an **isomorphism** if there exists a morphism $g\colon B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. We write $A \simeq B$ if there exists an isomorphism between A and B. An **automorphism on** A is a morphism $A \to A$ that is an isomorphism.

Definition 0.4. A category $\mathscr{C}$ where every morphism is an isomorphism is a **groupoid**.

Exercise 0.1. Let $\mathscr{C}$ be a locally small category. If $A \in \mathscr{C}$, then show that the invertible morphisms of $\mathrm{Hom}_{\mathscr{C}}(A, A)$ form a group, which is denoted $\mathrm{Aut}(A)$. Furthermore, show that if $X \simeq Y$ in $\mathscr{C}$, then $\mathrm{Aut}(X) \simeq \mathrm{Aut}(Y)$. (This is a nice *invariant* which is extremely relevant in Galois theory: That is, if we know that $\mathrm{Aut}(X) \not\simeq \mathrm{Aut}(Y)$, e.g. if they don't have the same size, then we can conclude that $X \not\simeq Y$.)

Exercise 0.2. Understand the following statement: A *group* is a groupoid $\mathscr{G}$ with a single object. (That is, if $x \in \mathscr{G}$ is the single object of the groupoid, then what is $\mathrm{Hom}_{\mathscr{G}}(x, x) = \mathrm{Aut}_{\mathscr{G}}(x)$.)

Definition 0.5. A **subcategory** $\mathscr{D}$ of a cateogry $\mathscr{C}$ is a category that has some of its objects and morphisms of $\mathscr{C}$ such that the morphisms of $\mathscr{D}$ include the identity morphisms of the objects of $\mathscr{D}$, and are closed under composition. A subcategory $\mathscr{D}$ of $\mathscr{C}$ is a **full subcategory** if $\mathrm{Hom}_{\mathscr{D}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(A, B)$ for all $A, B \in \mathscr{D}$.

We've already seen quite a lot of full subcategories. Namely, the category of abelian groups is a full category of the category of groups, and for a field k the category Vec_k forms a full subcategory of the category of k-modules. Lastly, for those that know about the theory of schemes, the category of affine schemes forms a full subcategory of the category of schemes.

0.2. **Functors.** Lets start from the motivating standpoint of the category of Grps. For two groups $(G, *)$ and $(H, \cdot)$, which may have distinct binary operations on them, a group homomorphism is a set-theoretic map $\varphi\colon G \to H$ such that $\varphi(a * b) = \varphi(a) \cdot \varphi(b)$, i.e. information from the group G gets transferred over nicely to the group H by way of (loosely) preservation of binary operations. In a similar way of preservation of binary operations for groups, we define a "map" from a category to another category that preserves the operations from the original category to the category we're mapping to; this is what a *functor* is.

Definition 0.6. A **covariant functor** (resp. **contravariant functor**) $F\colon \mathscr{C} \to \mathscr{H}$ is a map that associates every object $X \in \mathscr{C}$ to an object $F(X) \in \mathscr{H}$ and every morphism $g\colon X \to Y$ in $\mathscr{C}$ is associated to a map $F(g)\colon F(X) \to F(Y)$ in $\mathscr{H}$ (resp. $F(g)\colon F(Y) \to F(X)$), which satisfies $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, and for a composition of maps $g\colon X \to Y$ and $h\colon Y \to Z$ in $\mathscr{C}$ we have $F(h \circ g) = F(h) \circ F(g)$ (resp. $F(h \circ g) = F(g) \circ F(h)$).

Example 0.3.

(i) Let A be a ring and fix and an A-module M. The assignment $\mathrm{Hom}_A(M, -)\colon \mathsf{Mod}_A \to \mathsf{Mod}_A$ via $N \mapsto \mathrm{Hom}_A(M, N)$ defines a covariant functor, while $\mathrm{Hom}_A(-, M)\colon \mathsf{Mod}_A \to \mathsf{Mod}_A$ via $N \mapsto \mathrm{Hom}_A(N, M)$ defines a contravariant functor. For A-linear maps $f\colon N \to L$, the map $\mathrm{Hom}_A(M, f) := f_*\colon \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, L)$ is given by $f_*(g) = f \circ g$, while for the contravariant version we have $\mathrm{Hom}_A(f, M) := f^*\colon \mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(L, M)$ given by $f^*(g) = g \circ f$.
(ii) We have a contravariant functor $\mathrm{Spec}(-)\colon \mathsf{Ring} \to \mathsf{Top}$ given by $A \mapsto \mathrm{Spec}\, A$, where the right hand side is the spectrum of the ring considered with the Zariski topology. For a given map of rings $\varphi\colon A \to B$, we have $\mathrm{Spec}(\varphi)\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ where $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$.
(iii) Let X be a topological space. A continuous map $f\colon [0,1] \to X$ is called a *path* from $f(0)$ to $f(1)$, we denote f^{-1} to be the map from $f(1)$ to $f(0)$ where $f^{-1}(t) = f(1-t)$, and $i_{x_0}\colon [0,1] \to X$ be the constant map to $x_0 \in X$. We define an equivalence relation on X by saying $x_0 \sim x_1$ if and only if there exists a path from x_0 to x_1, and we write $\pi_0(X) = X/\sim$ with this equivalence relation. Then $\pi_0\colon \mathsf{Top}_* \to \mathsf{Set}$ defines a covariant functor.

(iv) The *forgetful functor* is a functor that essentially drops some of the added object and map structures: The forgetful functor $F\colon \mathsf{Ring} \to \mathsf{Ab}$ is the functor that takes a ring $A \in \mathsf{Ring}$ and maps it to $F(A) = A$ but now considered only as an abelian group, i.e. viewing a ring A as a triple $(A, +, \cdot)$ our functor views A as the double $F(A) = A = (A, +)$.

(v) Let G be a group. For elements $g, h \in G$ we define the *commutator* of g and h by $[g, h] := g^{-1}h^{-1}gh \in G$, and we define the *commutator subgroup* $D(G)$, often also called the *derived subgroup* and notated by $[G, G]$, as the subgroup of G generated by all commutators, o.e.

$$D(G) = [G, G] = \langle [g, h] \colon g, h \in G \rangle.$$

A good exercise here is the following: Let G be a group and denote $D(G)$ as its derived subgroup. Let N be a subgroup of G. Then prove that the N is normal in G and G/N is an abelian group if and only if $D(G) \subset N$. Now, relevant to the example we want to make, we have a functor $(-)^{\mathrm{ab}}\colon \mathsf{Grps} \to \mathsf{Ab}$, often called the *abelianization*, where $G \mapsto G/D(G)$ and the morphisms are the natural quotient ones.

We check here that Example 0.3 (i) does indeed define functors... well we check that the contravariant one does and leave the covariant one as an exercise. Let A be a ring and fix an A-module M. Let $N \in \mathsf{Mod}_A$. Then $\mathrm{Hom}_A(N, M)$ does indeed define an object of Mod_A as $\mathrm{Hom}_A(N, M)$ is a module with structure given pointwise, i.e. $(f + g)(x) := f(x) + g(x)$, and we define a map $\cdot\colon A \times \mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(N, M)$ where $(r, f) \mapsto rf$; these satisfy the module axioms and makes $\mathrm{Hom}_A(N, M)$ an A-module. Now let $f\colon N \to L$ in Mod_A. Then $f^*\colon \mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(L, M)$ is such that $f^*(g) = g \circ f$ which defines an A-linear map of $\mathrm{Hom}_A(L, M)$ as

$$M \xrightarrow{g} N \xrightarrow{f} L,$$

and lastly $\mathrm{id}_N^*\colon \mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(N, M)$ gives $\mathrm{id}_N^*(f) = f \circ \mathrm{id}_N = f$. Hence we have $\mathrm{Hom}_A(-, M)$ defines a contravariant functor as claimed.

Lemma 0.1. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a covariant (resp. contravariant) functor. If $X \simeq Y$ in $\mathscr{C}$, then $F(X) \simeq F(Y)$.

Proof. Let $f\colon X \to Y$ be an isomorphism in $\mathscr{C}$, and as such there exists $g\colon Y \to X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$. As F is a covariant functor, then $F(f) \circ F(g) = 1_{F(Y)}$ and $F(g) \circ F(f) = 1_{F(X)}$, and thus we conclude $F(X) \simeq F(Y)$. To get a diagrammatic point of view of the statement, look at the following commuting diagrams:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "1_X"] & Y \arrow[d, "g"] \arrow[dr, "1_Y"] & \\ & X \arrow[r, "f"] & Y \end{tikzcd} \quad \overset{F}{\Longrightarrow} \quad \begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[dr, "1_{F(X)}"] & F(Y) \arrow[d, "F(g)"] \arrow[dr, "1_{F(Y)}"] & \\ & F(X) \arrow[r, "F(f)"] & F(Y) \end{tikzcd}$$

□

Definition 0.7. A morphism $f\colon X \to Y$ in a category $\mathscr{C}$ is an **epimorphism** if for all objects Z in $\mathscr{C}$ and all morphisms $g_1, g_2\colon Y \to Z$ we have $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$. A morphism $f\colon X \to Y$ in $\mathscr{C}$ is a **monomorphism** if for all objects Z and $g_1, g_2\colon Z \to X$ we have that $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

Lemma 0.2. Let $f\colon X \to Y$ and $h\colon Y \to Z$ be monomorphisms (resp. epimorphisms), then their composition $h \circ f\colon X \to Z$ is a mononmorphism (resp. epimorphism).

Proof. We prove the epimorphism case and the other one follows by nearly the exact same argument. Write $q = h \circ f$, and let L be any object in $\mathscr{C}$ with $g_1, g_2\colon Z \to L$. Then $g_1 \circ q = g_2 \circ q$ is the same as $g_1 \circ (h \circ f) = g_2 \circ (h \circ f)$ and $(g_1 \circ h) \circ f = (g_2 \circ h) \circ f$ as f is epi and implies $g_1 \circ h = g_2 \circ h$ implies $g_1 = g_2$ as h is epi as well. □

The above definition attempts to generalize the situation in Sets, i.e. a map is surjective if and only if it has a right inverse, and a map is injective if and only if it has a left inverse. In some ways, we should see monomorphisms as usual injections and epimorphisms as surjections, but we caution here that in a lot of categories we cannot treat these things as being the same. Even in the category of Grps, we don't have the case that the notions coincide. But we do in the case of Mod_A for some ring A. Lastly, we should note that being both a monomorphism and an epimorphism will not be an isomorphism in a lot of categories. For example, in Ring, the embedding $\mathbb{Z} \to \mathbb{Q}$ is both epi and mono, but of course not an isomorphism.

Lemma 0.3. An A-module morphism $f\colon M \to N$ is

(i) injective if and only if its kernel is trivial.
(ii) surjective if and only if its cokernel is trivial.

Proof. (i) Standard, omitted. For (ii), note that $\operatorname{coker} f = N/\operatorname{im} f = (0)$ if and only if $N = \operatorname{im} f$ if and only if f is surjective. ☐

Proposition 0.1. In Mod_A,

(i) A morphism is injective if and only if it is a monomorphism.
(ii) A morphism is surjective if and only if it is an epimorphism.

Proof. (i) By contrapositive, assume that $f\colon M \to N$ is not an injective A-linear map. As f is not injective, then $\ker f$ is non-trivial. This implies that the zero map $0\colon \ker f \to M$, where $x \mapsto 0$ for all $x \in \ker f$, and $i\colon \ker f \to M$, with $x \mapsto x$, are distinct from each other. The compositions, $f \circ 0 = 0$ and $f \circ i = 0$, are both the zero map, but as i maps every element to $\ker f$, and hence the compositions are equal, and we conclude that f is not a monomorphism. ☐

Definition 0.8. An object Y in a category $\mathscr{C}$ is **initial** if $\operatorname{Hom}_{\mathscr{C}}(Y, Z) = \{*\}$ for all $Z \in \mathscr{C}$; an object K in $\mathscr{C}$ is **terminal** if $\operatorname{Hom}_{\mathscr{C}}(Z, K) = \{*\}$ for all $Z \in \mathscr{C}$. If Y is both initial and final, then it is called a **zero object**.

Example 0.1. The category of sets, the empty set is an initial object, and every singleton set is a terminal object. (Thereby we conclude that there is no zero object in Sets.)

0.3. **Colimits and limits.** A *directed set* I is a poset such that for all $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$ (i.e. every pair of elements in I has an upper bound). Fix a directed set I and a category $\mathscr{C}$. Let $(A_i)_{i \in I}$ be a family of objects in $\mathscr{C}$. This is called an *inverse system over I in $\mathscr{C}$* if for every $i \leq j$ in I, there exists a morphism $f_{ji}\colon A_j \to A_i$ such that $f_{ii} = \mathrm{id}_{A_i}$ and such that $f_{ki} = f_{ji} \circ f_{kj}$ whenever $i \leq j \leq k$. Then the **limit** (or projective/inverse limit), denoted as $\lim A_i$, has the property that it possesses morphisms $\pi_i\colon \lim A_i \to X_i$ which make the diagram

$$\begin{tikzcd}
A_i & A_j \arrow[l, "f_{ji}"'] \\
 & \lim A_i \arrow[ul, "\pi_i"] \arrow[u, "\pi_j"']
\end{tikzcd}$$

commute for all $i, j \in I$ with $i \leq j$. Additionally, it has the *universal property* that whenever there is an object $Z \in \mathscr{C}$ and morphisms $h_i\colon Z \to X_i$ for all $i \in I$ that makes the diagrams

$$\begin{tikzcd}
A_i & A_j \arrow[l, "f_{ji}"'] \\
 & Z \arrow[ul, "h_i"] \arrow[u, "h_j"']
\end{tikzcd}$$

for all $i \leq j$, then there exists a unique morphism $\theta\colon Z \to \lim X_i$ that makes the diagrams

$$\begin{tikzcd}
A_i & \lim X_i \arrow[l, "\pi_i"'] \\
 & Z \arrow[ul, "h_i"] \arrow[u, "\theta"']
\end{tikzcd}$$

commute for all $i \in I$.

In the instance that $\mathscr{C}$ is either Top, (Ab)Grps, Rings, Mod_A for a ring A, or Alg_k over a ring k, then the inverse limit is described simply as

$$\lim_{i \in I} A_i = \left\{ (a_i) \in \prod_{i \in I} : f_{ji}(a_j) = a_i \text{ for every pair } i \leq j \right\} \tag{1}$$

Example 0.4. The *ring of p-adic integers* is defined as the projective limit $\mathbb{Z}_p := \lim_n \mathbb{Z}/p^n\mathbb{Z}$ given by the sequence of of ring maps

$$\begin{tikzcd}
\mathbb{Z}_p \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\
\cdots \arrow[r] & \mathbb{Z}/p^3\mathbb{Z} \arrow[r] & \mathbb{Z}/p^2\mathbb{Z} \arrow[r] & \mathbb{Z}/p\mathbb{Z}
\end{tikzcd}$$

where the sequence of maps are given by $a + p^n \mapsto a + p^{n-1} \colon \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$. By construction, since we're taking an inverse limit of ring $\mathbb{Z}/p^n\mathbb{Z}$ for $n \geq 1$, we have that $\mathbb{Z}_p$ will inherit a ring structure from its family of rings which constitute it. We have the following characterization:

$$\mathbb{Z}_p := \lim n\mathbb{Z}/p^n\mathbb{Z} = \{(a_n)_{n \geq 1} \colon a_n \in \mathbb{Z}/p^n\mathbb{Z},\ a_{n+1} \equiv a_n \pmod{p^n}\}$$

That is, an element $x \in \mathbb{Z}_p$ is a sequence $x = (x_1, x_2, \ldots)$ whereby $x_{n+1} \equiv x_n \pmod{p^n}$ for all $n \geq 1$. For $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ in $\mathbb{Z}_p$, we write $x + y = (x_1 + y_1, x_2 + y_2, \ldots)$ and $xy = (x_1 y_1, x_2 y_2, \ldots)$, and our multiplicative identity is $1 = (1, 1, \ldots)$.

Example 0.5. On the positive integers, $\mathbb{N}^*$, we give it a poset structure by declaring that $n \leq m$ if $n \mid m$. We give the collection $(\mathbb{Z}/n\mathbb{Z})_{n \in \mathbb{N}^*}$ transition maps by defining them to be $f_{mn} \colon \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ by $f_{mn}(a + m\mathbb{Z}) = a + n\mathbb{Z}$, the natural projection. This gives rise to the the ring,

$$\widehat{\mathbb{Z}} = \lim n \in \mathbb{N}^* \mathbb{Z}/n\mathbb{Z} \tag{2}$$

1. Introduction: ∞-categories and related prerequsities

1.1. **(Pre)Sheaves.** A recurring term you will forever hear is that of a *presheaf* and a *sheaf*, where the latter is an extension of a presheaf that satisfies some extra properties. The reason this is heard so often is that the concept isn't actually a very hard one (it is also very general), and will be heard in many concepts (especially in the algebraic contexts).

Definition 1.1. Let $\mathscr{C}$ be a category. A **presheaf over** $\mathscr{C}$ is a functor of the form $\mathscr{F} : \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$.

If one prefers, the equivalent definition is that $\mathscr{F} : \mathscr{C} \to \mathsf{Sets}$ is a *presheaf* if $\mathscr{F}$ is a contravariant functor (in Definition 1.1, it is instead defined to be covariant but we preemptively flip the arrows when we consider the opposite category). Additionally, in our definition we explicitly use the category of sets, but it is often that we change this category to something like the category of abelian groups, or rings, and so on. We say that $\mathscr{F} : \mathscr{C} \to \mathsf{Ab}$ is an *abelian-valued presheaf* for an extra level of clarity, although it is often clear from context. Given a presheaf $\mathscr{F}$ over $\mathscr{C}$ and an object $U \in \mathscr{C}$, we call the elements of $\mathscr{F}(U)$ *sections*. A morphism of presheaves $\eta \colon \mathscr{F} \to \mathscr{L}$, where $\mathscr{F}$ and $\mathscr{L}$ are both presheaves over $\mathscr{C}$, is simply a natural transformation. Lastly, we can build a category out of presheaves: it's a functor category $\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets}) := \mathsf{PSh}(\mathscr{C})$, i.e it's the category that has objects as presheaves $\mathscr{F} \colon \mathscr{C} \to \mathsf{Sets}$ and has as morphisms natural transformations between presheaves.

In algebraic geometry, one often fixes a topological space X and considers the category X_{open}, where the objects are open subsets of X and, for any given two open subsets of X, say, U and V, the morphism is either the inclusion map $i \colon U \to V$ if $U \subset V$ or no morphism at all if neither is contained in the other. The theory of schemes is built out of considering sheaves $\mathscr{O}_X : X_{\text{open}} \to \mathsf{Rings}$ with a lot more structure.

Definition 1.2. The **Yoneda embedding** is the functor $h \colon \mathscr{C} \to \mathsf{PSh}(\mathscr{C})$ where $h(A) := h_A = \mathrm{Hom}_{\mathscr{C}}(-, A)$ for an object $A \in \mathscr{C}$.

Exercise 1.1. Verify that $\mathrm{Hom}_{\mathscr{C}}(-, A)$ is a presheaf.

Theorem 1.1 (Yoneda Lemma)**.** For any presheaf $\mathscr{F}$ over $\mathscr{C}$, there is a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{PSh}(\mathscr{C})}(h_A, \mathscr{F}) \xrightarrow{\sim} \mathscr{F}(A)$$

...[To be completed]

1.2. **The category of simplicial sets.** Consider the category whose objects are are finite subsets $[n] = \{0, 1, 2, \ldots, n\} = \{i \in \mathbb{Z} \colon 1 \leq i \leq n\}$, for $n \geq 0$, which inherit the natural ordering of integers, and has morphisms $f \colon [n] \to [m]$ that are (non strict) order-perserving maps, i.e. functions $f \colon [n] \to [m]$ with the property that $a < b$ in $[n]$ implies $f(a) \leq f(b)$ in $[m]$. We denote this category by Δ and is called the *simplicial category*, which is a small category by construction.s

Definition 1.3. A **simplicial set** is a presheaf over Δ. We denote $\mathsf{Set}_\Delta = \mathsf{PSh}(\Delta)$ to be the *category of simplicial sets*.

Given $n \geq 0$, we write $\Delta^n = \mathrm{Hom}_\Delta(-, [n])$ and call it the *standard n simplex*, that is, Δ^n is the simplicial set represented by $[n] \in \Delta$. So, we write $(\Delta^n)_m = \mathrm{Hom}_\Delta([m], [n])$. For a simplicial set $X \colon \Delta \to \mathsf{Sets}$ and an integer $n \geq 0$, we write $X_n = X([n]) \simeq \mathrm{Hom}_{\mathsf{Sets}_\Delta}(\Delta^n, X)$, by Theorem 1.1, for the set of *n-simplices* of X. Lastly, a *simplex* of X is an element of X_n for some nonnegative integer n.

We define, for $0 \leq i \leq n$, the pair of following maps:

$$\partial_i \colon [n-1] \to [n], \qquad \partial_i(j) = \begin{cases} j & j < i, \\ j+1 & i \leq j \end{cases}$$

$$\sigma_i : [n+1] \to [n], \qquad \sigma_i(j) = \begin{cases} j & j \leq i, \\ j-1 & i \leq j \end{cases}$$

The map $\partial_i \colon [n-1] \to [n]$ is called the i-th *face map*, which is injective, and $\sigma_i \colon [n+1] \to [n]$ is called the i-th *degeneracy map*, which is surjective.

Lemma 1.1. The following relations hold:

$$\begin{aligned} \partial_j \circ \partial_i &= \partial_i \circ \partial_{j-1}, \qquad i < j, \\ \sigma_j \circ \sigma_i &= \sigma_i \circ \sigma_{j+1}, \qquad i \leq j, \text{ and} \\ \sigma_j \circ \partial_i &= \begin{cases} \partial_i \circ \sigma_{j-1}, & i < j, \\ 1_{[n]}, & i \in \{j, j+1\}, \\ \partial_{i-1} \circ \sigma_j, & i > j+1. \end{cases} \end{aligned}$$

Proof. These are quite annoying to explicitly compute, so we do the first and leave it up to you to do the rest (or at least convince yourself of the rest). Let $i < j$. Then $\partial_j(\partial_i(k)) = k$ if $k < i$ or $k+1$ if $i \leq k$. If $k < i$, then $\partial_j(\partial_j(k)) = \partial_j(k) = k$, and $\partial_i \circ \partial_{j-1}(k) = \partial_i(k)$, as $k < i < j$ implies $k < j-1$, and $\partial_i(k) = k$. If $i \leq k$, then $\partial_j(\partial_i(k)) = \partial_j(k+1)$: (1.) If $i = k$ and $k+1 < j$, then $\partial_j(k+1) = k+1$ and this aligned with $\partial_i(\partial_{j-1}(k)) = \partial_i(k) = k+1$, and if $i = k$ with $j \leq k+1$, then $\partial_j(k+1) = k+2$ and this is aligned with $\partial_i(\partial_{j-1}(k)) = \partial_i(k+1) = k+2$; (2.) If $i < k$ and $k+1 < j$, then $\partial_j(k+1) = k+1$ and we have $\partial_i(\partial_{j-1}(k)) = \partial_i(k) = k+1$, and if $i < k$ and $j \leq k+2$, then $\partial_j(k+1) = k+2$ and $\partial_i(\partial_{j-1}(k)) = \partial_i(k+1) = k+2$. ☐

Lemma 1.2. Every morphism $f \neq 1_{[n]} \in \mathrm{Hom}_\Delta([n], [m])$ can be written as a composition

$$f = \partial_{i_1} \circ \cdots \partial_{i_r} \circ \sigma_{j_1} \circ \cdots \circ \sigma_{j_s}$$

with $0 \leq i_r < \cdots < i_1 \leq m$ and $0 \leq j_1 < \cdots < j_s < n$, where $m = n - s + r$. This decomposition is unique.

Section ω: Homotopy Theory and Homological Algebra

Throughout this section, we will develop the notion of a *derived category*, but to remain real to the subject, we develop the definition of *derived categories*. Namely, given a "nice" category $\mathcal{A}$, we can associate its derived category, denoted $\mathbf{D}(\mathcal{A})$.

We hear this story quite frequently: We want to be able to do homological algebra in better and better settings. The abstraction usually begins with developing homological algebra in Ab or Mod_A for some given ring A, but then we abstract the main source of homological properties that are specific to these categories to develop an abstracted theory that instead of working in the categories just described, we work in an *abelian category*. Although I may be mistaken, the next jump of a more ideal setting to do homological algebra is in derived settings instead now. To develop the framework and definition is the goal of this section.

1.3. **Abelian Categories.**

1.3.1. *Additive categories.* As remarked in the introduction to this section, we take Mod_A for a ring A to be our motivation towards a notion of an abelian categories, and we define an *additive category* as a preliminary step, but these categories are in themselves interesting.

So, in the category Mod_A, given some A-modules M and N, the set of maps $\mathrm{Hom}(M,N)$ forms an abelian group: we define a binary operation of addition on this set as we take some $f,g\in\mathrm{Hom}(M,N)$ and define $(f+g)(x):=f(x)+g(x)$, which does indeed define abelian group. Furthermore, we have $\mathbb{Z}$-bilinear property with $\mathrm{Hom}(M,N)$, that is, we have the following:

$$(f_1+f_2)\circ g=(f_1\circ g)+(f_2\circ g)$$
$$f\circ(g_1+g_2)=(f\circ g_1)+(f\circ g_2).$$

We're essentially stating that we have that composition of maps in Mod_A distributes over addition. Additionally, the category Mod_A has a zero object, namely, the trivial module (0), where given any A-module B, we have $\mathrm{Hom}((0),B)$ consists of only a zero map where $0\mapsto 0$ and it is terminal as $\mathrm{Hom}(B,(0))$ consists of a zero map with $x\mapsto 0$ for all $x\in B$. Lastly, given any two A-modules M and N we have a product between the them, i.e. we have another A-module $M\times N$. We make the following definition to generalize these properties.

Definition 1.4. A category $\mathcal{A}$ is **additive** if it satisfies the following properties:

(i) For $A,B\in\mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(A,B)$ is an abelian group, and composition of morphisms are $\mathbb{Z}$-bilinear, that is, the $\mathrm{Hom}_{\mathcal{A}}(Y,Z)\times\mathrm{Hom}_{\mathcal{A}}(X,Y)\to\mathrm{Hom}_{\mathcal{A}}(X,Z)$ with $(g,f)\mapsto g\circ f$ are $\mathbb{Z}$-bilinear.
(ii) There exists a zero object in $\mathcal{A}$, denoted 0.
(iii) For any $X,Y\in\mathcal{A}$, the categorical project $X\times Y$ exists in $\mathcal{A}$.

A functor $F\colon\mathcal{A}\to\mathcal{B}$ of additive categories is **additive** if the induced map $\mathrm{Hom}_{\mathcal{A}}(A,B)\to\mathrm{Hom}_{\mathcal{B}}(F(A),F(B))$ is a homomorphism of abelian groups for any $A,B\in\mathcal{A}$.

Example 1.1.

- The category of rings is not additive as the initial object of Ring is $\mathbb{Z}$ which is not terminal as 0 is the terminal object in this category. (There are some other issues with this example being additive, but we just give one failure of it.)
- The category of modules over a ring A is additive, and thereby it follows that $\mathsf{Ab}=\mathsf{Mod}_{\mathbb{Z}}$ and $\mathsf{Vec}_k=\mathsf{Mod}_k$ for a field k are also additive categories.
- Consider the functor $F_N\colon\mathsf{Ab}\to\mathsf{Ab}$ where we have that N is a (normal) subgroup of G and map it to G/N and given $f\colon G\to H$ we have $F(f)=\overline{f}\colon G/N\to H/N$ and $\overline{f}\colon x+N\mapsto f(x)+N$. Then this describes an additive functor as $\mathrm{Hom}(G,H)\to\mathrm{Hom}(G/N,H/N)$ is described by the map $\pi(f+g)=\overline{f+g}=\overline{f}+\overline{g}$.

1.3.2. *Abelian Categories and (co)Kernels.* Starting in a similar fashion as to how the last section went, we're motivated by abstracting essentially one more concept from the category of modules. To define the co(homology) of a given complex, we need the notion of a kernel and image in Mod_A, i.e. if we have a (short) complex of A-modules

$$X^\bullet\colon 0\to A^{i-1}\xrightarrow{f^{i-1}}A^i\xrightarrow{f^i}A^{i+1}\to 0$$

we define the *i-th cohomology* as $H^i(X^\bullet)=\ker f^i/\operatorname{im} f^{i-1}$. (The *homology* group is defined in an analogous way where the maps are in descending order instead of the ascending as we've defined for the cohomology). As such, we abstract the notion of kernel and cokernel to abstract the notion of cohomology to other categories.

Definition 1.5. Let $\mathcal{A}$ be a category with a 0-object. A **kernel** of a morphism $f\colon B \to C$ is a map $i\colon A \to B$ such that $f \circ i = 0$, and that is universal with respect to this property. The kernel is written $\ker f \to B$. As a diagram:

$$\begin{tikzcd}
Z \arrow[d, dashed, "\exists!"'] \arrow[dr] \arrow[drr, "0"] & & \\
A \arrow[r, "i"] \arrow[rr, bend right, "0"'] & B \arrow[r, "f"] & C
\end{tikzcd}$$

A **cokernel** of a morphism $f\colon B \to C$ is a map $j\colon B \to A$ such that $j \circ f = 0$, and is universal with repsect to this property. The cokernel is written $B \to \operatorname{coker} f$.

Lets look at the category of modules. We know that for the kernel of an A-linear map $f\colon A \to B$ we write $\ker f = \{x \in A\colon \varphi(x) = 0\}$ and $\operatorname{coker} f = B/\operatorname{im} f$, but we should actually check that these satisfy the properties of Definition 1.5. Clearly given some A-linear map $f\colon A \to B$ we have a map $i\colon \ker f \to A$ as this is just a subset of A, and $f \circ i(x) = f(x) = 0$ as $x \in \ker f$. Now suppose we have some other pair (j, Z) of Mod_A such that there is a map $j\colon Z \to A$ such that $f \circ j = 0$. Lastly, there must exist a unique map $Z \to A$ as $f \circ j = 0$ which means that it factors through the usual kernel of f, i.e. as $f(j(x)) = 0$ for all $x \in Z$ and $j(x) \in A$ then $j(x) \in \ker f$.

Exercise 1.2. Verify that $\operatorname{coker} \varphi = B/\operatorname{im} \varphi$ for an A-linear map $\varphi\colon A \to B$ satisfies the properties of Definition 1.5.

Proposition 1.1. Kernels are monomorphisms

Proof. Let $k\colon K \to X$ be a kernel of a morphism $f\colon X \to Y$. Let $g, h\colon L \to K$ be such that $k \circ g = k \circ h$. □

Definition 1.6. An **abelian category** is an additive category where

(i) every morphism admits a kernel and a cokernel, and
(ii) every monomorphism is a kernel and every epimorphism is a cokernel.

A slogan to apply this definition is that we essentially want just enough axioms so that the snake lemma will hold in such a category. This is explained by the following exercise:

Exercise 1.3. An additive category $\mathcal{A}$ is abelian if every map $f\colon A \to B$ has a kernel and a cokernel, and if the canonical factorization

$$\begin{tikzcd}
\ker f \arrow[r, "f'"] & A \arrow[r, "f"] \arrow[d] & B \arrow[r, "f''"] \arrow[d] & \operatorname{coker} f \\
& \operatorname{coker} f' \arrow[r, "\overline{f}"] & \ker f'' &
\end{tikzcd}$$

of f induces an isomorphism $\overline{f}$. Show that this definition and Definition 1.6 coincide.

Definition 1.7 ([Sta18, Tag 010B])**.** Let $f\colon X \to Y$ be a morphism is an abelian category.

(i) We say f is **injective** if $\ker f = 0$.
(ii) We say f is **surjective** if $\operatorname{coker} f = 0$.

If $X \to Y$ is injective, then we say that A is a **subobject of** Y and we use the notation $X \subset Y$. If $X \to Y$ is surjective, then we say that Y is a **quotient of** X.

Theorem 1.2. Let $f\colon A \to B$ be a morphism in an abelian category $\mathcal{A}$. Then

(i) f is injective if and only if f is a monomorphism, and
(ii) f is surjective if and only if f is an epimorphism.

1.3.3. *Homological algebra in an abelian category.* Throughout the rest of this section, we will be working an abelian category $\mathcal{A}$, and if you prefer, take the objects and morphisms to be in your favorite example of $\mathcal{A}$. We will loosely define a lot of homological terminology throughout this section.

We say a sequence

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \cdots$$

is at **complex at** B if $g \circ f = 0$, and is said to be **exact at** B if $\ker g = \operatorname{im} f$. Note that exactness at B implies is a complex at B since $g \circ f(x) = g(f(x))$ and $f(x) \in \operatorname{im} f$ for all $x \in A$ and so $g(f(x)) = 0$. A sequence is a *complex* (resp. *exact*) if it is a complex (resp. exact) at each spot. Perhaps the most common "type" of complex you'll see are so-called *short* ones. What we mean by this is that our sequence $X^\bullet$ looks like five terms where the last and initial ones is zero (sometimes they're also one):

$$X^\bullet \colon 0 \to A \to B \to C \to 0$$

For exa

1.4. **Derived Categories.** The study of derived categories came, like a lot of things, initially from Grothendieck and one of his students, Jean-Louis Verdier.

We will focused on abelian categories, and the examples best to keep in is Mod_A, Ab, or if you know some algebraic geometry, (quasi-coherent) sheaves of abelian groups on a topological space. We may also want to consider abelian subcategories of our original categories; for example, if $\mathcal{A} = \mathsf{Mod}_A$ and A is noetherian, then a good subcategory of this is the the one with objects finitely generated modules. Now, the main tool that organizes a lot of homological algebra as presented in the last section, is the category of complexes on an abelian category, that is, $\mathsf{Com}(\mathcal{A})$. Roughly speaking, we will take the category $\mathsf{Com}(\mathcal{A})$ and "flip" some arrows: the derived category $D(\mathcal{A})$ is obtained from $\mathsf{Com}(\mathcal{A})$ by a process of *localization*. The name here comes from the analogous process from the category of rings where we invert some elements, so in the category description we want to invert some morphisms. In particular, we're interested in inverting *quasi-isomorphisms*:

Definition 1.8. A chain map $A^\bullet \to B^\bullet$ is a **quasi-isomorphism** if the induced maps $H^n(A^\bullet) \to H^n(B^\bullet)$ are isomorphisms.

Example 1.2. In Mod_A, given a projective resolution of a module M, say,

$$\cdots \to P^{-1} \to P^0 \xrightarrow{\epsilon} M \to 0$$

where all the P^n are projective objects, i.e. the functor $B \mapsto \operatorname{Hom}(P^n, B)$ is exact, and $P^n = 0$ for $n > 0$. If we take the complex consisting of the projective modules, then there is a chain map:

$$\begin{tikzcd} \cdots \arrow[r] & P^{-1} \arrow[r] \arrow[d] & P^0 \arrow[r] \arrow[d, "\epsilon"] & 0 \arrow[r] & \cdots \\ & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

The cohomology of the projective truncation, denoted $P^\bullet$ has $H^n(P^\bullet) = 0$ for $n < 0$, and when $n = 0$ the cohomology is precisely M, so we get a quasi-isomorphism. Dually, there is analogous situation for injective resolutions.

Example 1.3. Let $A^\bullet$ be any complex over an abelian category. Then we define a truncation functor which preserves cohomology up to some degree and kills the rest. Define $\tau_{\leq 0}(A^\bullet)$, which is a subset complex of $A^\bullet$

$$\begin{tikzcd} \cdots \arrow[r] & A^{-1} \arrow[r] \arrow[d, dash] & A^0 \arrow[r, "d"] & A^1 \arrow[r] & \cdots \\ \cdots \arrow[r] & A^{-1} \arrow[r] & \ker d \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots \end{tikzcd}$$

Looking at the cohomology of the complexes, we have $H^n(\tau_{\leq 0}(A^\bullet)) \xrightarrow{\sim} H^n(A^\bullet)$ for $i \leq 0$ and $H^n(\tau_{\leq 0}(A^\bullet)) = 0$ for $n > 0$. So, clearly, if $H^n(A^\bullet) = 0$ for $n > 0$, then we have a quasi-isomorphism from $\tau_{\leq 0}(A^\bullet)$ to $A^\bullet$.

Now, we want to define localization but it is in fact best to pass through to the homotopy category $\mathsf{K}(\mathcal{A})$ where the objects are complexes over $\mathcal{A}$ and the morphisms $\mathrm{Hom}(A^\bullet, B^\bullet)$ are chain maps $A^\bullet \to B^\bullet$ modulo the equivalence relation of homotopy, i.e. $f \sim g$ if $f - g = dh + hd$.

References

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