RISING SEA, VAKIL: SOLUTIONS

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Contents

1. Chapter 1: Preliminaries

1.1. Motivation [skipped].

1.2. Categories and functors [skipped].

1.3. Universal properties determine an object up to unique isomorphism.

Exercise 1.1 (1.3A)**.** Show that any two initial objects are uniquely isomorphic. Show that any two final objects are uniquely isomorphic.

Proof. Let $\mathscr{C}$ be some category that has two initial objects, say, A and B. Then let $f: A \to B$ and $g: B \to A$ via unique maps, as they're both initial objects. But then, $A \to B \to A$ so $1_A = g \circ f$ and $1_B = f \circ g$ by uniqueness of maps. This makes A and B isomorphic, and a similar argument shows that final objects are unique. □

Exercise 1.2 (1.3B)**.** What are the initial and final objects in Sets, Rings, and Top (if they exist)?

Proof. The initial object of Sets is the empty set, while the terminal object is the singleton set. Similarly, the empty space is the initial object in Top and final object is the one-point space. The initial object of Ring is $\mathbb{Z}$ with unique map $f: \mathbb{Z} \to A$ where $f(n) = 1_A \cdot n$ for any given ring A, and has final object the trivial ring $\{0\}$. □

Exercise 1.3 (1.3C)**.** Show that $A \to S^{-1}A$ is injective if and only if S contains no zerodivisors.

Proof. Assume that $\pi: A \to S^{-1}A$ is injective, i.e. $\ker \pi = (0)$ where $\pi(a) = \frac{a}{1}$. Then, $\ker \pi = \{x \in A: \pi(x) = 0\} = \{x \in A: \frac{x}{1} = 0\} = \{x \in A: \exists s \in S, sx = 0\} = (0)$, so S contains no zerodivisors. Assume that S contains no zerodiviors. Then $\pi(x) = \frac{x}{1} = 0$, so there exists $u \in S$, which implies $ux = 0$ and thus $x = 0$ as S contains no zerodivisors. □

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Exercise 1.4 (1.3D). Let $A \to B$ be a ring map such that every element of S gets mapped to a unit of B. Then there is a unique homomorphism $g\colon S^{-1}A \to B$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & B \\ & S^{-1}A \arrow[ur, "g"'] & \end{tikzcd}$$

Proof. Define a map $g\colon S^{-1} \to B$ by $x/s \in S^{-1}A \mapsto f(x)f(s)^{-1} \in B$. The diagram commutes as $g \circ j(x) = g(x/1) = f(x)f(1)^{-1} = f(x)$. Now we want to show that the map g is unique. So, assume that there exists another map π such that $\pi \circ j = f$ as outlined above. Then $\pi(x/1) = f(x)$ and $\pi(s/1) = f(s)$ which implies $\pi(1/s) = f(s)^{-1}$. Lastly, $\pi(x/s) = \pi(x/1)\pi(1/s) = f(x)f(s)^{-1} = g(x/s)$, and thus we conclude $g = \varphi$. ☐

Exercise 1.5 (1.3F).

(a) Show that localization commutes with finite products, or equivalently, with finite direct sums. In other words, if $M_1, \cdots, M_n$ are A-modules, describe an isomorphism (of A-modules, and of $S^{-1}A$-modules) $S^{-1}(M_1 \times \cdots \times M_n) \to S^{-1}M_1 \times \cdots \times S^{-1}M_n$.

(b) Show that localization commutes with arbitrary direct sums.

Proof. (a) Let $M_1, \cdots, M_n$ be A-modules. Define a map

$$\begin{aligned} \pi\colon S^{-1}(M_1 \times \cdots \times M_n) &\to S^{-1}M_1 \times \cdots \times S^{-1}M_n \\ (x_1, \dots, x_n)\frac{1}{s} &\mapsto (x_1/s, \dots, x_n/s) \end{aligned}$$

This map is a linear map as

$$\begin{aligned} \pi\left((x_1,\dots,x_n)\frac{1}{s_1} + (y_1,\dots,y_n)\frac{1}{s_2}\right) = \pi\left(s_2\overline{x} + \overline{y}s_1\frac{1}{s_1s_2}\right) &= \pi\left((s_2x_1 + s_1y_1, \dots, s_2x_n + s_1y_n)\frac{1}{s_1s_2}\right) \\ &= \left(\frac{s_2x_1+s_1y_1}{s_1s_2}, \dots, \frac{s_2x_n+s_1y_n}{s_1s_2}\right) = \left(\frac{x_1}{s_1} + \frac{y_1}{s_2}, \dots, \frac{x_n}{s_1} + \frac{y_n}{s_2}\right) \\ &= \pi\left((x_1,\dots,x_n)\frac{1}{s_1}\right) + \pi\left((y_1,\dots,y_n)\frac{1}{s_2}\right), \quad \text{and, for } r \in A, \end{aligned}$$

$$\pi\left(r(x_1,\dots,x_n)\frac{1}{s_1}\right) = \left(\frac{rx_1}{s}, \dots, \frac{rx_n}{s}\right) = r\left(\frac{x_1}{s}, \dots, \frac{x_n}{s}\right) = r\pi\left((x_1,\dots,x_n)\frac{1}{s_1}\right).$$

The map π is injective as $\ker\pi = \{x/s \in S^{-1}(M_1 \times \cdots \times M_n) \colon \pi(x/s) = 0\} = (0)$, and lastly the map is surjective as if we take some $(x_1/s_1, \dots, x_n/s_n) \in S^{-1}M_1 \times \cdots \times S^{-1}M_n$, we can take $y = \left((x_1(s_2\cdots s_n), \dots, x_n(s_1\cdots s_{n-1}))\frac{1}{s_1\cdots s_n}\right)$, so that $\pi(y) = (x_1/s_1, \dots, x_n/s_n)$. Thus we have an isomorphism.

(b) We want to show that

$$S^{-1}\left(\bigoplus_{i\in I} M_i\right) \simeq \bigoplus_{i\in I}(S^{-1}M_i).$$

This can be shown using a similar morphism argument from part (a), or we can use the fact that the direct sum is the coproduct in the category Mod_A and Exercise 1.4. That is, we write $M = \oplus_{i\in I}M_i$ and we have a corresponding localization $S^{-1}M$ and injective maps $j_i\colon M_i \to M$ so that there is an induced isomorphism

$$\bigoplus_{i\in I} S^{-1}M_i \to S^{-1}M = S^{-1}\left(\bigoplus_{i\in I} M_i\right).$$

☐

Exercise 1.6 (1.3G). Show that $\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12) \simeq \mathbb{Z}/(2)$.

Proof. Recall that $\mathbb{Z}/(mn) \oplus \mathbb{Z}/(m) \times \mathbb{Z}/(n)$ if $m, n \geq 2$ and $(m,n) = 1$. We will, somewhat cheaply, use Exercise 1.3M and some other facts. There is definitely an easier way, and I should be somewhat embarrassed not to see it, but this is what I wrote down the first time and got a correct solution. We can get the decomposition that $\mathbb{Z}/(10) \simeq \mathbb{Z}/(5) \oplus \mathbb{Z}/(2)$ and $\mathbb{Z}/(12) \simeq \mathbb{Z}/(4) \oplus \mathbb{Z}/(3)$, so

$$\begin{aligned}\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12) &\simeq (\mathbb{Z}/(5) \oplus \mathbb{Z}/(2)) \otimes_{\mathbb{Z}} (\mathbb{Z}/(4) \oplus \mathbb{Z}/(3)) \\ &\simeq [(\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(4)) \oplus (\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(3))] \oplus [(\mathbb{Z}/(5) \otimes_{\mathbb{Z}} \mathbb{Z}/(4)) \oplus (\mathbb{Z}/(5) \otimes_{\mathbb{Z}} \mathbb{Z}/(3))] .\end{aligned}$$

Now its easy to determine $\mathbb{Z}/(s) \otimes_{\mathbb{Z}} \mathbb{Z}/(t)$ with $\gcd(s,t) = 1$: this is as $as + tb = 1$, so $as = tb - 1$ if and only $tb \equiv 1 \pmod{a}$, and thus for $x \otimes y$ of $\mathbb{Z}/(s) \otimes_{\mathbb{Z}} \mathbb{Z}/(t)$

$$x \otimes y = 1x \otimes y = (tb)x \otimes y = x \otimes (tb)y = x \otimes 0 = 0.$$

We conclude that $\mathbb{Z}/(s) \otimes_{\mathbb{Z}} \mathbb{Z}/(t) \simeq \{0\}$. Applying this to the mess we had above,

$$\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12) \simeq (\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(4) \oplus \{0\}) \oplus (\{0\} \oplus \{0\}) \simeq \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(4).$$

It remains to inspect $\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(4)$. We have the following exact sequence

$$0 \to (2) \to \mathbb{Z} \to \mathbb{Z}/(2) \to 0,$$

and as $- \otimes_{\mathbb{Z}} \mathbb{Z}/(4)$ is right exact, then

$$(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(4) \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/(4) \to \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(4) \to 0$$

which corresponds to

$$(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(4) \to \mathbb{Z}/(4) \to \mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(4) \to 0$$

and we lastly get an induced isomorphism $\mathbb{Z}/(2) \otimes_{\mathbb{Z}} \mathbb{Z}/(4) \simeq \mathbb{Z}/2\mathbb{Z}$. Hence $\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12) \simeq \mathbb{Z}/(2)$. □

Proof. 2. We instead prove the more general statement $\mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n) \simeq \mathbb{Z}/(\gcd(m,n))$, from which we conclude the statement of the exercise. Define a bilinear map $P\colon \mathbb{Z}/(m) \times \mathbb{Z}/(n) \to \mathbb{Z}/(\gcd(m,n))$ is well-defined. By the universal property of tensor product, there is a linear map $f\colon \mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n) \to \mathbb{Z}/(\gcd(m,n))$ such that $f(x + (m), y + (n)) = xy + (\gcd(m,n))$. We define a linear a map $g\colon \mathbb{Z}/(\gcd(m,n)) \to \mathbb{Z}/(m) \otimes_{\mathbb{Z}} \mathbb{Z}/(n)$ defined by $g(z + (\gcd(m,n))) = (z + (m)) \otimes (1 + (n))$ is well-defined, so $g \circ f = 1$ and $f \circ g = 1$, and thus f is an isomorphism. As $\gcd(10, 12) = 2$, then $\mathbb{Z}/(10) \otimes_{\mathbb{Z}} \mathbb{Z}/(12) \simeq \mathbb{Z}/(2)$. □

Exercise 1.7 (1.3L)**.** If S is a multiplicative subset of A and M is an A-module, describe a natural isomorphism $(S^{-1}A) \otimes_A M \simeq S^{-1}M$ (as $S^{-1}A$-modules and as A-modules)

Proof. Let S be a multiplicative subset of A and M is an A-module. We can define a bilinear map $S^{-1}A \times M \to S^{-1}M$ by $(a/s, m) \mapsto am/s$ which induces, by the universal property of tensor products, a linear map $f\colon S^{-1}A \otimes_A M \to S^{-1}M$ by $f(a/s \otimes m) = am/s$. Similarly, there is a linear map $g\colon S^{-1}M \to S^{-1}A \otimes_A M$ by $g(m/s) = m/s \otimes 1$. These maps are inverse to each other as $f(g(m/s)) = f(m/s \otimes 1) = m/s$ and $g(f(a/s \otimes m)) = g(am/s) = am/s \otimes 1 = a/s \otimes m$. Hence we get the desired isomorphism. □

Exercise 1.8 (1.3M)**.** Show that tensor products commute with arbitrary direct sums: if M and $\{N_i\}_{i \in I}$ are all A-modules, describe an isomorphism

$$M \otimes_A \left(\bigoplus_{i \in I} N_i \right) \simeq \bigoplus_{i \in I} (M \otimes_A N_i)$$

Proof. Define the map $f\colon M \times (\oplus_{i \in I} N_i) \to \oplus_{i \in I}(M \otimes_A N_i)$ by $f(m, (x_1, \ldots, x_n, \ldots 0, \ldots)) = ((m \otimes x_1), \ldots, (m \otimes x_n), \ldots, (0), \ldots))$. This map can be checked to be bilinear and thus induces a linear map $\tilde{f}\colon M \otimes_A (\oplus_{i \in I} N_i) \to \oplus_{i \in I}(M \otimes_A N_i)$ where $\tilde{f}(m \otimes (x_1, \ldots, x_n, 0, \ldots)) = (m \otimes x_1, \ldots, m \otimes x_n, 0, \ldots)$. We now define an inverse map □

2. Chapter 2: Sheaves

2.1. Definition of sheaf and presheaf.

Exercise 2.1 (2.2A)**.** Given any topological space X, we have a "category of open sets" (Example 1.2.9), where the objects are the open sets and the morphisms are inclusions. Verify that the data of a presheaf is precisely the data of a contravariant functor from the category of open sets of X to the category of sets. (This interpretation is surprisingly useful.)

Proof. We claim that the presheaf $\mathscr{F}: X_{\mathsf{open}} \to \mathscr{C}$ is a contravariant functor. For a given open set $U \in X_{\mathsf{open}}$, we have an associated object $\mathscr{F}(U) = \Gamma(U, \mathscr{F}) = H^0(U, \mathscr{F})$ in $\mathscr{C}$. Note that given two open sets of X, U and V, either $\mathrm{Hom}_{X_{\mathsf{open}}}(U, V) = i$ or $\varnothing$ where $i: U \to V$ is the canonical injection. Assume $U \subseteq V \subseteq W$, then we have the following diagram in X_{top}

$$\begin{tikzcd} U \arrow[rr, "j\circ i"] \arrow[dr, "i"'] & & W \\ & V \arrow[ur, "j"'] & \end{tikzcd}$$

As we have that $\mathscr{F}$ is a presheaf, then the following diagram commutes in $\mathscr{C}$,

$$\begin{tikzcd} U & & W \arrow[ll, "\mathscr{F}(j\circ i) = \mathrm{res}_{W,U}"'] \arrow[dl, "\mathrm{res}_{W,V} = \mathscr{F}(j)"] \\ & V \arrow[ul, "\mathscr{F}(i) = \mathrm{res}_{V,U}"] & \end{tikzcd}$$

where $\mathscr{F}(j \circ i) = \mathrm{res}_{W,U} = \mathrm{res}_{V,U} \circ \mathrm{res}_{W,V} = \mathscr{F}(i) \circ \mathscr{F}(j)$. Lastly, note that $\mathrm{res}_{U,U} = \mathrm{id}_{\mathscr{F}(U)}$ guarantees us the last axiom we needed to check that $\mathscr{F}$ is a contravariant functor. □

Exercise 2.2. [2.2.E] Now let $\mathscr{F}(U)$ be the maps to S that are locally constant, i.e. for any point $p \in U$, there is an open open neighborhood of p where the function is constant. Show that this is a sheaf

Proof. To summarize, for every $p \in U$, there is some neighborhood V of p with $V \subset U$ and $f|_V$ is constant. We verify the sheaf axioms. Let $\{U_i\}_{i\in I} = \mathfrak{U}$ be an open cover of U, and assume that for all pairs $f_1, f_2 \in \mathscr{F}(U)$ we have $f_1|_{U_i} = f_2|_{U_i}$ for all i. We have that f_1 and f_2 are the same on all of U since if not then there would exist some $t \in U$ such that $f_1(t) \neq f_2(t)$, but as $t \in U$ we actually have $t \in U_\alpha$ with $\alpha \in I$ and $f_1|_{U_\alpha}(t) = f_2|_{U_\alpha}(t)$, and we get a contradiction. Hence we conclude that $f_1 = f_2$ in $\mathscr{F}(U)$. Now, suppose were given $f_i \in \mathscr{F}(U_i)$ for all i such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all i, j. Define the map $f: U \to S$ defined by

$$f(x) = \begin{cases} f_i(x) & x \in U_i \\ f_j(x) & x \in U_j \end{cases}$$

(note that we continue the cases of f arbitrarily so that we use the whole covering and this is thus a map on $U \to S$) and this is indeed an element of $\mathscr{F}(U)$ as given some point $p \in U$, we have, as $\mathfrak{U}$ is a covering of U, some $p \in U_\gamma$ with $\gamma \in I$ and thereby some neighborhood V of p with $V \subset U_\gamma$ such that $f|_V$ is constant. □

Exercise 2.3 (2.2.F)**.** Suppose Y is a topological space. Show that "continuous maps to Y" form a sheaf of sets on X. More precisely, to each open set U of X, we associate the set of continuous maps of U to Y.

Proof. This is essentially the same proof as the last. Let $\{U_i\}_{i\in I}$ be a covering of U and $f_1, f_2 \in \mathscr{F}(U)$ be such that $f_1|_{U_i} = f_2|_{U_j}$ for all i. Using the same argument Exercise 2.2, we have $f_1 = f_2$ in $\mathscr{F}(U)$. For the gluing axiom, we apply the Pasting Lemma, and we're done. □

Exercise 2.4 (2.2H)**.** Suppose $\pi: X \to Y$ is a continuous map, and $\mathscr{F}$ is a presheaf on X. Then define $\pi_*\mathscr{F}$ by $\pi_*\mathscr{F}(V) = \mathscr{F}(\pi^{-1}(V))$, where V is an open subset of Y. Show that $\pi_*\mathscr{F}$ is a presheaf on Y, and is a sheaf if $\mathscr{F}$ is. This is called the **pushfoward** or **direct image** of $\mathscr{F}$. More precisely, $\pi_*\mathscr{F}$ is called the **pushfoward of $\mathscr{F}$ by** π.

Proof. We assume that $\mathscr{F}: X_{\mathsf{open}} \to \mathsf{Ab}$ where X_{open} denotes the category of open sets of X is our sheaf. This exercise is rather simple: Let $\{V_i\}_i$ cover V. Then $\pi^{-1}(V_i) \subseteq X$ is open as π is continuous and $\pi^{-1}(V_i \cap V_j) = \pi^{-1}(V_i) \cap \pi^{-1}(V_j)$ and $\pi^{-1}(V) = \bigcup_i \pi^{-1}(V_i)$. Hence we inherit that the following is an exact sequence from $\mathscr{F}$ and get that $\pi_*\mathscr{F}$ is indeed a sheaf:

$$0 \to \mathscr{F}(\pi^{-1}(V)) \to \prod_i \mathscr{F}(\pi^{-1}(V_i)) \rightrightarrows \prod_{i,j} \mathscr{F}(\pi^{-1}(V_i) \cap \pi^{-1}(V_j)).$$

□

Exercise 2.5. [2.2.I.] Suppose $\pi: X \to Y$ is a continuous map, and $\mathscr{F}$ is a sheaf of sets (or rings of A-modules) on X. If $\pi(p) = q$, describe the natural morphism of stalks $(\pi_*\mathscr{F})_q \to \mathscr{F}_p$.

Proof. Firstly, we know that $(\pi_*\mathscr{F})_q = \varinjlim_{q \in V} \mathscr{F}(\pi^{-1}(V))$, or, alternatively, this is the set $\{(g, U): g \in \mathscr{F}(\pi^{-1}(U)), q = \pi(p) \in U\}/\sim$ where $(f, L) \sim (g, U)$ if and only if there exists an open set $W \subset U \cap L$ with $p \in W$ and $\mathrm{res}_{U,W}\, g = \mathrm{res}_{L,W}\, f$. Define the map $\varphi: (\pi_*\mathscr{F})_q \to \mathscr{F}_p$ via $\varphi((g, U)) = (g, \pi^{-1}(U))$. Alternatively, we get an induced map via direct limits; this is because $(\pi_*\mathscr{F})_q = \varinjlim_{q \in V} \mathscr{F}(\pi^{-1}(V)) = \varinjlim_{\pi(p) \in V} \mathscr{F}(\pi^{-1}(V))$, so if $q \in U$ then $p \in \pi^{-1}(U))$, which means we have subsystem of $\mathscr{F}_p$ by $(\pi_*\mathscr{F})_q$ and thereby get an induced map by the universal property. □

Exercise 2.6 (2.2.J)**.** If $(X, \mathscr{O}_X)$ is a ringed space, and $\mathscr{F}$ is an $\mathscr{O}_X$-module, describe how for each $p \in X$, $\mathscr{F}_p$ is an $\mathscr{O}_{X,p}$-module.

Proof. As $\mathscr{F}$ is an $\mathscr{O}_X$-module, then for each open $U \subset X$, $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module, so that we already have an action $\cdot: \mathscr{O}_X(U) \times \mathscr{F}(U) \to \mathscr{F}(U)$ with $(r, f) \mapsto r \cdot f$ satisfying the module axioms. The stalk $\mathscr{F}_p$ is essentially made into an $\mathscr{O}_{X,p}$-module through this. That is, as $\mathscr{F}_p = \{(f, U): f \in \mathscr{F}(U), p \in U\}/\sim$ and $\mathscr{O}_{X,p} = \{(g, V): g \in \mathscr{O}_X(U), p \in U\}/\sim$ where the equivalence relation is described in Exercise 2.5. Thereby we make $\mathscr{F}_p$ into an $\mathscr{O}_{X,p}$ module via the following (but note that $\mathscr{F}_p$ is an abelian group via $(s, U) + (t, V) = (s|_{U\cap V} + t|_{U \cap V}, (U \cap V))$

$$\begin{aligned} \mathscr{O}_{X,p} \times \mathscr{F}_p &\to \mathscr{F}_p \\ ((r, V), (f, U)) &\mapsto (r|_{U\cap V} \cdot f|_{U\cap V}, U \cap V) \end{aligned}$$

□

Exercise 2.7 (2.3.A)**.** If $\phi: \mathscr{F} \to \mathscr{G}$ is a morphism of presheaves on X, and $p \in X$, describe an induced morphism of stalks $\phi_p: \mathscr{F}_p \to \mathscr{G}_p$.

Proof. There is a clear map $\phi_p: \mathscr{F}_p \to \mathscr{G}_p$ via $\phi_p((f, U)) = (\phi(U)(f), U)$. Alternatively, we could define the map $f_U: \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{G}_p$ as being the composition $\pi_p^{\mathscr{G}} \circ \phi(U) = f_U$ and this commutes with restriction $\mathrm{res}_{U,V}: \mathscr{F}(U) \to \mathscr{F}(V)$ (i.e. $f_V \circ \mathrm{res}_{U,V} = f_U$) as $\phi: \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, so by the universal property of the colimit/direct limit, we have a unique map $\phi_p: \mathscr{F}_p \to \mathscr{G}_p$ that f_U factors through. □

Exercise 2.8 (2.3.B)**.** Suppose $\pi: X \to Y$ is a continuous map of topological spaces (i.e., a morphism in the category of topological spaces). Show that pushforward gives a functor $\pi_*: \mathsf{Sets}_X \to \mathsf{Sets}_Y$.

Proof. Recall that $\mathsf{Sets_X}$ denotes the (functor) category of sheaves of sets on X. Let $\mathscr{F} \in \mathsf{Sets}_X$, then $\pi_*\mathscr{F}$ defines a sheaf of sets on Y as for every open set $V \subset Y$, we have $\pi_*\mathscr{F}(V) = \mathscr{F}(\pi^{-1}(V))$ being a set, since $\mathscr{F}$ is sheaf of sets on X and $\pi^{-1}(V) \subset X$ is open. Given a restriction $D \to V$ in Y we have $\pi_*\mathscr{F}(V) \to \pi_*\mathscr{F}(D)$ with $\mathscr{F}(\pi^{-1}(V)) \to \mathscr{G}(\pi^{-1}(D))$ given by the induced restriction $\pi^{-1}(V) \to \pi^{-1}(D)$ on X. Given a map $\Phi: \mathscr{F} \to \mathscr{G}$ where $\mathscr{F}, \mathscr{G} \in \mathsf{Sets}_X$, we have an induced map (a natural transformation) $\pi_*\Phi: \pi_*\mathscr{F} \to \pi_*\mathscr{G}$ where

$$\begin{tikzcd} \pi_*\mathscr{F}(V) \arrow[r, "\pi_*\Phi(V)=\Phi(\pi^{-1}(V))"] \arrow[d, "\mathrm{res}^{\mathscr{F}}_{\pi^{-1}(V),\pi^{-1}(D)}"'] & \pi_*\mathscr{G}(V) \arrow[d, "\mathrm{res}^{\mathscr{G}}_{\pi^{-1}(V),\pi^{-1}(D)}"] \\ \pi_*\mathscr{F}(D) \arrow[r, "\pi_*\Phi(D)=\Phi(\pi^{-1}(D))"'] & \pi_*\mathscr{G}(D) \end{tikzcd}$$

□

Exercise 2.9 (2.3.C)**.** Suppose $\mathscr{F}$ and $\mathscr{G}$ are two sheaves of sets on X. Let $\mathcal{H}om(\mathscr{F},\mathscr{G})$ be the collection of data

$$\mathcal{H}om(\mathscr{F},\mathscr{G})(U) := \operatorname{Mor}(\mathscr{F}|_U, \mathscr{G}|_U).$$

Show that this is a sheaf of sets on X.

Proof. Let $\{U_i\}_{i\in I} = \mathfrak{U}$ be a cover of $U \subset X$ open. We want to somehow use the fact that $\mathscr{F}$ and $\mathscr{G}$ to conclude the sheaf axioms for $\mathcal{H}om(\mathscr{F},\mathscr{G})$. Firstly, we define a restriction of $\Phi \in \mathcal{H}om(\mathscr{F},\mathscr{G})(U)$ of some $E \subset U$ open as $\Phi|_E$ considered as a sheaf morphism on $\mathscr{F}|_E \to \mathscr{G}|_E$. We use an alternative definition of the identity axiom (the one found in Hartshorne).[1] Let $\Phi \in \mathcal{H}om(\mathscr{F},\mathscr{G})(U)$ be such that $\Phi_i := \Phi|_{U_i} = 0$ for all i, i.e. for all open $S \subset U_i$, we have $\Phi_i(S)\colon \mathscr{F}|_{U_i}(S) \to \mathscr{G}|_{U_i}(S)$ being the map that gives $\Phi_i(S)(g) = 0$ for all $g \in S$. Now we want to show that $\Phi(V) = 0$ for all $V \subset U$ open. Let $g \in \mathscr{F}(U)$ be a fixed section. Then, by assumption, $\Phi_i(U_i)\colon \mathscr{F}(U_i) \to \mathscr{G}(U_i)$ gives $g_i \mapsto 0$. As $\mathscr{G}$ is a sheaf and g under the image of $\Phi(U)$ restricts to zero on the open cover $\{U_i\}_{i\in I}$ then $g = 0$. Now we have to show the gluing axiom.

Let $U = \bigcup_{i\in I} U_i$ be covering of $U \subset X$ open. Let $\Phi_i \in \mathcal{H}om(\mathscr{F},\mathscr{G})(U_i)$ be such that $\Phi_i|_{U_i\cap U_j} = \Phi_j|_{U_i\cap U_j}$ for all i,j. We want to find a global section $\Phi \in \mathcal{H}om(\mathscr{F},\mathscr{G})(U)$ such that $\Phi|_{U_i} = \Phi_i$ for all i, i.e. for all open $V \subset U$, $\Phi|_{U_i}(V) = \Phi_i(V)\colon \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$. As $\{U_i\}_{i\in I}$ is a covering, then $U_i \cap V = S_i$ is a covering for V. So, if we fix $s \in \mathscr{F}(V)$ and let $s_i = s|_{S_i}$, then define $\Phi_i(S_i)(s_i) := t_i$, we get a global section $t \in \mathscr{G}(V)$, as $\mathscr{G}$ is a sheaf, such that $t|_{S_i} = t_i$ for all i. Thus we define Φ as $\Phi(V)\colon \mathscr{F}(V) \to \mathscr{G}(V)$ by $s \mapsto t$ for all open $V \subset U$. Hence $\Phi \in \mathcal{H}om(\mathscr{F},\mathscr{G})(U)$ such that $\Phi|_{U_i} = \Phi_i$ for all i. □

3. Chapter 3: Schemes

3.1. **Toward Schemes.**

3.2. **The underlying set of affine schemes.**

Exercise 3.1 (3.2.A)**.**

(a) Describe the set $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$.
(b) Describe the set $\operatorname{Spec} k[x]_{(x)}$.

Proof. (I believe we're meant to assume that k is a field throughout the exercise.) For (a), as we have a canonical projection $\pi\colon k[\epsilon] \to k[\epsilon]/(\epsilon^2)$, then the prime ideals ideals of $k[\epsilon]/(\epsilon^2)$ corresponds to the prime ideals of $k[\epsilon]$ that contain $\ker \pi = (\epsilon^2)$. So, let $\mathfrak{q}$ be a prime ideal that contains (ϵ^2). But as k is a field then $\mathfrak{q} = (p(\epsilon))$ for some irreducible polynomial $p(\epsilon) \in k[\epsilon]$. Now $\epsilon^2 = p(\epsilon)g(\epsilon)$ for some $g(\epsilon) \in k[\epsilon]$, and as $p(\epsilon)$ is irreducible, then $p(\epsilon) = \epsilon$, and thus $\mathfrak{q} = (\epsilon)$. Hence $\operatorname{Spec} k[\epsilon]/(\epsilon^2) = \{(\epsilon)\}$. In a similar fashion, for (b), we use the fact that the prime ideals of $k[x]_{(x)}$ is determined by the prime ideals of $k[x]$ contained in (x). Thereby, if I is a prime ideal of $k[x]$ (which implies it is principal so $I = (g(x))$ for some $g(x) \in k[x]$) contained in (x), then as $(g(x))$ is prime and we're working in a PID, then $g(x)$ must be irreducible, so maximality gives that $(g(x)) \subset (x)$ implies $(g(x)) = (x)$. Now, $\operatorname{Spec} k[x]_{(x)} = \{(x), (0)\}$ □

Exercise 3.2 (3.2.B)**.** Show that for the last type of prime, of the form, $(x^2 + ax + b)$, the quotient is always isomorphic to $\mathbb{C}$.

Proof. Let $f(x) = x^2 + ax + b$ be prime in $\mathbb{R}[x]$, and so it is thus irreducible. As $f(x)$ is quadratic and irreducible, then it has no roots in $\mathbb{R}[x]$, and so let $\alpha = \beta_1 + \beta_2 i$ denote one of its roots in $\mathbb{C}$ (the other one doesn't matter per se). Then define the map $\theta\colon \mathbb{R}[x] \to \mathbb{C}$ by $g(x) \mapsto g(\alpha)$. We claim that $\ker\theta = (f(x))$. The inclusion that $(f(x)) \subset \ker\theta$ is easy as if $h(x) \in (f(x))$ then $h(x) = p(x)f(x)$ so that $h(\alpha) = p(\alpha)f(\alpha) = 0$, and thus $h(x) \in \ker\theta$. Now we claim that $f(x)$ is the minimal polynomial for α: If it weren't then there would exist $t(x) \in \mathbb{R}[x]$ such that $\deg t < \deg f$ with $t(\alpha) = 0$. But then $t(x)$ would divide $f(x)$, which contradicts the prime assumption for $f(x)$. Thus $f(x)$ is minimal for α. This implies that if $q(x) \in \ker\theta$, then $q(\alpha) = 0$ and as $f(x)$ is minimal, then $f(x) \mid q(x)$, so

[1] If U is an open set, if $\{U_i\}_{i\in I}$ is an open covering of U, and if $s \in \mathscr{F}(U)$ is an element such that $s|_{U_i} = 0$ for all i, then $s = 0$.

$\ker\theta \subset (f(x))$. Hence $\ker\alpha = (f(x))$. Now we prove surjectivity. Let $x + yi \in \mathbb{C}$. Then define the polynomial $h(x) \in \mathbb{R}[x]$ by

$$h(x) = a_1 x + a_0, \quad a_1 = \frac{y}{\beta_2}, a_0 = x - \frac{y\beta_1}{\beta_2}.$$

Then,

$$\begin{aligned} h(\alpha) = a_1\alpha + a_0 = a_1(\beta_1 + \beta_2 i) + a_0 &= \frac{y}{\beta_2}(\beta_1 + \beta_2 i) + x - \frac{y\beta_1}{\beta_2} \\ &= \frac{y\beta_1}{\beta_2} + yi + x - \frac{y\beta_1}{\beta_2} = x + yi. \end{aligned}$$

Hence we have that θ is surjective. Therefore we conclude $\mathbb{R}[x]/(x^2 + ax + b) \simeq \mathbb{C}$. □

Exercise 3.3 (3.2.C)**.** Describe the set $\mathbb{A}^1_{\mathbb{Q}}$.

Proof. This doesn't have the neatest answer, since we can have irreducible polynomials $f(x) \in \mathbb{Q}[x]$ of any given degree, so $\mathbb{A}^1_{\mathbb{Q}} = \{(f) \colon f \text{ irreducible in } \mathbb{Q}[x]\}$. But for possibly the sought off answer by Vakil is that one can go down the line and speak about the simple orbits of $\overline{\mathbb{Q}}$ under the action of $\mathrm{Gal}_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}, \mathbb{Q})$. □

Exercise 3.4 (3.2.D)**.** If k is a field, show that $\operatorname{Spec} k[x]$ has infinitely many points

Proof. Let k be a field. This implies that $k[x]$ is a PID (and as a consequence, we have that this is also a UFD). Now suppose that $\operatorname{Spec} k[x]$ has finitely many points, say, $(p_1(x)), (p_2(x)), \ldots, (p_m(x))$ is what consists of $\operatorname{Spec} k[x]$. Then consider the polynomial $f = p_1 \cdots p_m + 1$ as a polynomial in $k[x]$. If f is a prime element in $k[x]$, then we're done. Now assume that f is not. Then f has a prime factor p in $k[x]$, as this is a UFD. We cannot have that $p = p_\alpha$ for some $1 \leq \alpha \leq m$, as if so then $f \equiv 0 \equiv 1 \pmod{p_\alpha}$ and we get a contradiction. Hence f is a new prime element that wasn't in our list. Therefore $\operatorname{Spec} k[x]$ has infinitely many points. □

Exercise 3.5 (3.2.F)**.** Show that the Nullstellensatz 3.2.5 implies the Weak Nullstellensatz 3.2.4.

Proof. Let $k = \overline{k}$, and let $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ be a maximal ideal. Then the residue field $\kappa(\mathfrak{m}) = k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite extension of k, but as k is algebraically closed, then $k = \kappa(\mathfrak{m})$ to begin with. So, a priori, we have a canonical projection $\pi \colon k[x_1, \ldots, x_n] \to \kappa(\mathfrak{m})$ by $f(x) \mapsto f(x) + \mathfrak{m}$. But now, in particular, $x_i \mapsto x_i + \mathfrak{m} = a_i$ for some $a_i \in k$. This implies that $x_i - a_i \in \mathfrak{m}$, so $I = (x_1 - a_1, \ldots, x_n - a_n) \subset \mathfrak{m}$. Now we claim that I is maximal. The map

$$\Phi \colon k[x_1, \ldots, x_n] \to k, \ f \mapsto f(a_1, \ldots, a_n)$$

gives an isomorphism, as follows: We have that $\ker\Phi = (x_1 - a_1, \ldots, x_n a_n)$ as if $f \in (x_1 - a_1, \ldots, a_n - x_n)$, then $f = (x_1 - a_1)g_1 + \cdots + (x_n - a_n)g_n$ so that $f(a_1, \ldots, a_n) = 0$ is clear, and now given $g \in k[x_1, \ldots, x_n]$, we can factor it as $g = (x_1 - a_1)q + r_1 = (x_1 - a_1)q_1 + (x_2 - a_2)q_2 + \cdots + (x_n - a_n)q_n + r_n$ for some $r_n \in k[x_1, \ldots, x_n]$ (note that we're applying the Euclidean algorithm multiple times on the remainder term), so that if $g \in \ker\Phi$, then $g(a_1, \ldots, a_n) = 0$ so $r_n = 0$ and thus $g \in (x_1 - a_1, \ldots, x_n - a_n)$. Lastly, surjectivity is obvious. Hence we have an isomorphism $k[x_1, \ldots, x_n]/I \simeq k$, so I is maximal and thus $I = \mathfrak{m}$. □

Exercise 3.6 (3.2.G)**.** Any integral domain A which is a finite k-algebra must be field.

Proof. As A is a k-algebra then it suffices to establish that it has multiplicative inverses for all $a \in A^*$. Let x be nonzero in A and assume that x is not a unit in A. Consider the map $\varphi_x \colon A \to A$ defined by $a \mapsto ax$. This map is obviously injective as A is an integral domain. As k is a field, then A can be treated as a k-vector space, and thereby we have that $\dim_k A = \dim(\ker\varphi_x) + \dim(\operatorname{im}\varphi_x) = \dim(\operatorname{im}\varphi_x)$ so our map is surjective as well. Hence $\varphi_x \colon A \xrightarrow{\sim} A$ is an isomorphism, and so given $1 \in A$, we can find $y \in A$ such that $yx = 1$. Hence after applying this to each element, we see that A is a field. □

Exercise 3.7 (3.2.H)**.** Describe the maximal ideal of $\mathbb{Q}[x, y]$ corresponding to $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. Describe the maximal ideal of $\mathbb{Q}[x, y]$ corresponding to $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$. What are the residue fields in each case?

Proof. The maximal ideals correspond to $\mathfrak{m}_1 = (x^2 - \sqrt{2}, x - y)$ and $\mathfrak{m}_2 = (x^2 - \sqrt{2}, x + y)$. They are both maximal ideals as their residue fields are isomorphic to the field $\mathbb{Q}(\sqrt{2})$. □

Exercise 3.8 (3.2.I)**.** Omitted.

Exercise 3.9. [3.2.J] Suppose A is a ring, and I an ideal of A. Let $\varphi\colon A \to A/I$. Show that φ^{-1} gives an inclusion-preserving bijection between prime ideals of A/I and prime ideals of A containing I. Thus we can picture $\operatorname{Spec} A/I$ as a subset of $\operatorname{Spec} A$.

Proof. (Slogan: It is an important fact that the prime ideals of A/I are in bijection with the prime ideals of A containing I.) We map

$$\{\text{prime ideals of } A/I\} \leftrightarrow \{\text{prime ideals of } A \text{ containing } I\}$$
$$J \mapsto \varphi^{-1}(J)$$
$$\varphi(J) \leftarrow J$$

Let $\mathfrak{p}$ be a prime ideal of A/I. Then $\varphi^{-1}(\mathfrak{p}) \subset A$ is nonempty as $0 \in \varphi^{-1}(\mathfrak{p})$ and in particular, $x \in I$ implies $\varphi(x) \in \mathfrak{p}$, trivially as $\varphi(x) = 0 \in \mathfrak{p}$, so I is contained in $\varphi^{-1}(\mathfrak{p})$. Now we show that $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of A. Let $x, y \in \varphi^{-1}(\mathfrak{p})$ and $r \in A$. So $\varphi(x), \varphi(y) \in \varphi^{-1}(\mathfrak{p})$ which implies that $r\varphi(x) - \varphi(y) = \varphi(rx) - \varphi(y) = \varphi(rx - y) \in \mathfrak{p}$ so $rx - y \in \varphi^{-1}(\mathfrak{p})$, and thus $\varphi^{-1}(\mathfrak{p})$ is an ideal of A. Now let $xy \in \varphi^{-1}(\mathfrak{p})$. Then $\varphi(xy) = \varphi(x)\varphi(y) \in \mathfrak{p}$, as $\mathfrak{p}$ is prime, then, without loss of generality, we have $\varphi(x) \in \mathfrak{p}$ so $x \in \varphi^{-1}(\mathfrak{p})$, and hence $\varphi^{-1}(\mathfrak{p})$ is a prime ideal. For the opposite direction of the bijection, let $\mathfrak{q}$ be a prime ideal of A containing I. Then given $\varphi(x), \varphi(y) \in \varphi(\mathfrak{q})$ with $x, y \in A$ and $\overline{r} \in A/I$, we have $\overline{r}\varphi(x) + \varphi(y) = \varphi(rx + y) \in \mathfrak{q}$ as $\mathfrak{q}$ is an ideal. Let $st \in \varphi(\mathfrak{q})$, i.e. $st = \varphi(q)$ for $q \in \mathfrak{q}$. Then as φ is a surjection, then $s = \varphi(a)$ and $\varphi(b) = t$ for some $a, b \in A$. This implies $\varphi(q) = st = \varphi(a)\varphi(b) = \varphi(ab)$ and so $ab \in \mathfrak{q}$ which implies, without loss of generality, $a \in \mathfrak{q}$. Hence $s \in \varphi(\mathfrak{p})$, so $\varphi(\mathfrak{p})$ is a prime ideal of A/I. The inclusion preserving aspects are just set theory facts. Lastly, its an annoying but easy set theory argument to show that $\varphi \circ \varphi^{-1} = \mathrm{id}$ and $\varphi^{-1} \circ \varphi = \mathrm{id}$ so we omit that here. □

Exercise 3.10. [3.2.K] Suppose S is a multiplicative subset of A. Describe an order-preserving bijection of the prime ideals of $S^{-1}A$ with the prime ideals of A that don't meet the multiplicative set S.

Proof. Write $i\colon A \to S^{-1}A$ where $i(a) = a/1$. [2] Then

$$\{\text{prime ideals of } \mathfrak{p} \text{ of } S^{-1}A\} \leftrightarrow \{\text{prime ideals of } A, \mathfrak{p} \text{ with } \mathfrak{p} \cap S = \varnothing\}$$
$$\mathfrak{p} \mapsto i^{-1}(\mathfrak{p})$$
$$S^{-1}\mathfrak{p} \leftarrow \mathfrak{p}$$

Given a prime ideal $\mathfrak{p}$ of $S^{-1}A$, then $i^{-1}(\mathfrak{p})$ is a prime ideal of A as it is the preimage of a homomorphism, and it is disjoint from S as if not then we would choose some $q \in i^{-1}(\mathfrak{p}) \cap S$ so that $q/1 \in \mathfrak{p}$ and $q \in S$ which would imply that $1 \in \mathfrak{p} = S^{-1}A$ and get a contradiction. For the opposite direction, let $\mathfrak{p}$ be a prime ideal of A with $\mathfrak{p} \cap S = \varnothing$. Let $x/m \cdot y/n = xy/mn \in S^{-1}\mathfrak{p}$; that is, there exists $z \in \mathfrak{p}$, $s \in S$ such that $xy/mn = z/s$ so there exists $t \in S$ such that $(xys - mnz)t = 0$ (or equivalent $xyst = (mnt)z$ is in $\mathfrak{p}$). As such, $xyst \in \mathfrak{p}$ implies that $x, y, s,$ or t is in $\mathfrak{p}$, but as $s, t \in S$ and we assume $S \cap \mathfrak{p} = \varnothing$, then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$, so $x/m \in S^{-1}\mathfrak{p}$ or $y/n \in S^{-1}\mathfrak{p}$. Hence $S^{-1}\mathfrak{p}$ is a prime ideal of $S^{-1}A$. Similar to the last problem, the inclusion preserving aspects are just set theory facts. Lastly, its an annoying but easy set theory argument to show that the inverse of the maps, just like the last problem. □

Exercise 3.11 (3.2.L)**.**

$$(\mathbb{C}[x,y]/(xy))_x \xrightarrow{\sim} \mathbb{C}[x]_x$$

Show that these two rings are isomorphic.

Proof. There is a clear intuition between these two rings.

[2]Recall that given a ring A and a subset $S \subset A$ such that S is multiplicatively closed and $0 \notin S$ but $1_A \in S$, then we write

$$S^{-1}A = \left\{ \frac{a}{s} : a \in A, s \in S \right\} / =$$

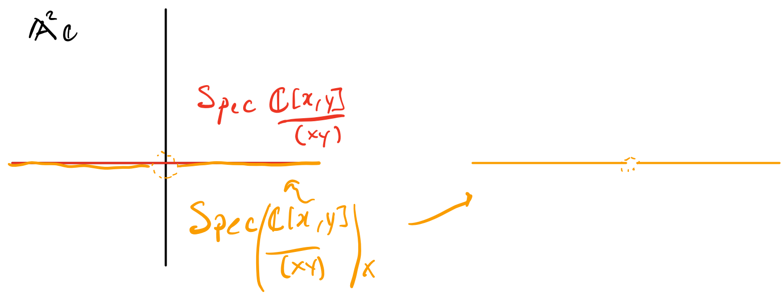

FIGURE 1. The right side certainly looks like $\operatorname{Spec} \mathbb{C}[x]_x = \mathbb{A}^1_{\mathbb{C}} - \{0\}$.

Consider the following exact sequence

$$0 \to (xy) \to \mathbb{C}[x,y] \to \mathbb{C}[x,y]/(xy) \to 0$$

Then as localization is an exact functor, and we localize at $S = \{1, x, x^2, \ldots\}$, which gives us another exact sequence

$$0 \to (xy)_x \to \mathbb{C}[x,y]_x \to (\mathbb{C}[x,y]/(xy))_x \to 0$$

Now $(xy)_x = \{\frac{xyf(x,y)}{x^k} : f(x,y) \in \mathbb{C}[x,y], k \in \mathbb{Z}_{\geq 0}\}$ and, for $k > 1$, $xyf(x,y)/x^k$ and $yf(x,y)/x^{k-1}$ are related as

$$xyf(x,y)x^{k-1} - yf(x,y)x^k = x^k y f(x,y) - yx^k f(x,y) = 0,$$

and we conclude $(xy)_x = (y)_x$. Consider the map defined by $\pi \colon \mathbb{C}[x,y]_x \to \mathbb{C}[x]_x$ by $f(x,y)/x^k \mapsto f(x,0)/x^k$, which is well defined and a homomorphisms of rings. Let $f(x)/x^k \in \mathbb{C}[x]_x$. Then define $f^*(x,y) = f(x) + y$, which of course gives $f^*(x,0) = f(x) + 0 = f(x)$ and this implies π is surjective. Now we claim that $\ker \pi = (y)_x$. Let $h(x,y) \in \ker \pi$, then $h(x,y) = g(x,y)/x^k$ with $\pi(h(x,y)) = 0/1$, so $g(x,0)/x^k = 0/1$, so $g(x,0) = 0$ in $\mathbb{C}[x]$ and this implies that $y \mid g(x,y)$. Thus $\ker \pi \subset (y)_x$. Now given $h(x,y)y/x^k \in (y)_x$, then $\pi(h(x,y)y/x^k) = 0$. Therefore $\ker \pi = (y)_x$, and we conclude that $\mathbb{C}[x,y]_x/(y)_x \simeq \mathbb{C}[x]_x$. By our exact sequence above, $(\mathbb{C}[x,y]/(xy))_x \simeq \mathbb{C}[x,y]_x/(xy)_x = \mathbb{C}[x,y]_x/(y)_x$, and so $\mathbb{C}[x]_x \simeq (\mathbb{C}[x,y]/(xy))_x$. □

Exercise 3.12 (3.2.M)**.** If $\varphi \colon B \to A$ is a map of rings, and $\mathfrak{p}$ is a prime ideal of A, show that $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of B.

Proof. We've actually already done this in some previous exercises, but we contain all the details here. Let $\mathfrak{p}$ be a prime ideal of A. Now $\varphi^{-1}(\mathfrak{p})$ is not empty as $\varphi(0) = 0 \in \mathfrak{p}$. Let $x, y \in \varphi^{-1}(\mathfrak{p})$ and $r \in B$. Then $\varphi(rx - y) = \varphi(r)\varphi(x) - \varphi(y) \in \mathfrak{p}$, so $rx - y \in \varphi^{-1}(\mathfrak{p})$, which makes this an ideal. Now let $ab \in \varphi^{-1}(\mathfrak{p})$, then $\varphi(ab) = \varphi(a)\varphi(b) \in \mathfrak{p}$, so, without loss of generality, $\varphi(a) \in \mathfrak{p}$, so $a \in \varphi^{-1}(\mathfrak{p})$. Therefore $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of B. □

Exercise 3.13 (3.2.N)**.** Let B be a ring.

(a) Suppose $I \subset B$ is an ideal. Show that the map $\operatorname{Spec} B/I \to \operatorname{Spec} B$ is the inclusion of §3.2.7.
(b) Suppose $S \subset B$ is a multiplicative set. Show that the map $\operatorname{Spec} S^{-1}B \to \operatorname{Spec} B$ is the inclusion of §3.2.8.

Proof. (a) Let $I \subset B$ an ideal. The map $\operatorname{Spec} \pi \colon : \operatorname{Spec} B/I \to \operatorname{Spec} B$, $\mathfrak{p} \mapsto \pi^{-1}(\mathfrak{p})$, where $\pi \colon B \to B/I$ is the projection, is the map in 3.2.7.

(b) For $S \subset B$ is a multiplicative set, with canonical map $i \colon B \to S^{-1}B$, $i(x) = x/1$, so that $\operatorname{Spec} i \colon \operatorname{Spec} S^{-1}B \to \operatorname{Spec} B$ is given by $\mathfrak{p} \mapsto i^{-1}(\mathfrak{p})$. This is the map of 3.2.8. □

Exercise 3.14. Ring elements that have a power that is 0 are called nilpotents.

(a) Show that if I is an ideal of nilpotents, then the inclusion $\operatorname{Spec} B/I \to \operatorname{Spec} B$ of Exercise 3.2.J. (Exercise 3.9) is a bijection. Thus nilpotents don't affect the underlying set.

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where $a_1/s_1 = a_2/s_2$ if and only if there exists $t \in S$ such that $t(a_1 s_2 - a_2 s_1) = 0$.

(b) Show that the nilpotents of a ring B form an ideal. This ideal is called the nilradical, and is denoted $\mathfrak{N} = \mathfrak{N}(B)$.

Proof. (a) I proved (although it doesn't read like it) Exercise 3.15 first so that my initial idea of the proof of the statement would follow quickly. (I really enjoyed the technique Ravi gave to prove the reverse inclusion of Exercise 3.15.) Let I be a nilpotent ideal of $\operatorname{Spec} B$. Then we have that $I \subset \mathfrak{N}(B)$, but by Exercise 3.15, this implies I is contained in all prime ideals of $\operatorname{Spec} B$, so we have the bijection

$$\operatorname{Spec} B/I \leftrightarrow \{\mathfrak{p} \in \operatorname{Spec} B \colon I \subset \mathfrak{p}\} = \operatorname{Spec} B,$$

for I an ideal of nilpotents.

(b) This ideal is nonzero as $\mathfrak{N}(B)$ contains 0. Let $x, y \in \mathfrak{N}(B)$. Then $x^m = 0$ and $y^n = 0$ for $m, n > 0$. So, $(x-y)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^{m+n-k} y^k = 0$, and $x - y \in \mathfrak{N}(B)$. Lastly, given $r \in B$, we have $(rx)^m = r^m x^m = 0$, so $rx \in \mathfrak{N}(B)$. Thus $\mathfrak{N}(B)$ is an ideal. □

Exercise 3.15 (3.2.S)**.**

Theorem 3.1. The nilradical $\mathfrak{N}(A)$ is the intersection of all prime ideals of A. Geometrically: a function of $\operatorname{Spec} A$ vanishes at every point if and only if it is nilpotent. Let I be an ideal of nilpotents, i.e. $I^k = 0$ for some $k > 0$.

Proof. We want to show that

$$\mathfrak{N}(A) = \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p}.$$

For the forward inclusion, let $x \in \mathfrak{N}(A)$. Then, $x^m = 0$ for some $m > 0$ integer, which implies that $x^m = 0 \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} A$, and so $xx^{m-1} \in \mathfrak{p}$; we conclude that $x \in \mathfrak{p}$ using the fact this is a prime ideal. For the backwards direction, we approach this by contrapositive. That is, given some $x \notin \mathfrak{N}(A)$, we want to find a prime ideal $\mathfrak{p}$ of $\operatorname{Spec} A$ such that $x \notin \mathfrak{p}$. (Note that x cannot be zero as $0 \in \mathfrak{N}(A)$.) Consider the multiplicative set $\{1, x, x^2, \dots\} = S$; we localize A at this set, to get $A_x := S^{-1}A$, which is not trivial as $x \neq 0$. By Zorn's lemma, there exists some maximal ideal $\mathfrak{m} \subset A_x$, and all maximal ideals are prime, so in particular $\operatorname{Spec} A_x \neq \varnothing$. Using the bijection between prime ideals of Exercise 3.10, we send $\mathfrak{m} \mapsto i^{-1}(\mathfrak{m}) := \mathfrak{p}$, where $i \colon A \to A_x$ is the canonical map, and we have that $\mathfrak{p} \cap S = \varnothing$, i.e. $x \notin \mathfrak{p}$. So we have have found a prime ideal of A which doesn't contain x, and thus by contrapositive we conclude that $\cap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} \subset \mathfrak{N}(A)$. Hence we have the equality of sets and we conclude the statement of the theorem. □

3.3. **Visualizing schemes I: generic points.**

3.4. **The underlying topological space of an affine scheme.**

Exercise 3.16 (3.4.A)**.** Check that the x-axis is contained in $V(xy, yz)$. (The x-axis is defined by $y = z = 0$, and the y-axis and z-axis are defined analogously.)

Proof. My best understanding of the word contained in the statement of the exercise is that the x-axis is an "element" of $V(xy, yz) = \{\mathfrak{p} \in \operatorname{Spec} A \colon (xy, yz) \subset \mathfrak{p}\}$ as this consists of a collection of prime ideals of $\mathbb{C}[x, y, z]$. The ideal $(y, z) \subset \mathbb{C}[x, y, z]$ is prime by the fact that the map $\mathbb{C}[x, y, z] \to \mathbb{C}[x]$ is surjective by $f(x, y, z) \mapsto f(x, 0, 0)$. Lastly, clearly $(xy, yz) \subset (y, z)$. □

Exercise 3.17 (3.4.B)**.** Show that if S is the ideal generated by (S), then $V(S) = V((S))$.

Proof. The ideal (S) is the smallest ideal containing S. If $\mathfrak{q} \in V(S)$, the $S \subset \mathfrak{q}$, and as (S) is the smallest ideal containing S, then $S \subset (S) \subset \mathfrak{q}$, so $\mathfrak{q} \in V((S))$. For the reverse inclusion, given $\mathfrak{p} \in V((S))$, then $(S) \subset \mathfrak{p}$ and clearly $S \subset (S) \subset \mathfrak{p}$ once again as (S) is the smallest ideal containing S. □

Exercise 3.18 (3.4.C)**.**

(a) Show that $\varnothing$ and $\operatorname{Spec} A$ are both open subsets of $\operatorname{Spec} A$.
(b) If I_i is a collection of ideals (as i runs over some index set), show that $\cap_i V(I_i) = V(\sum_i I_i)$. Hence the union of any collection of open sets is open.
(c) Show that $V(I_1) \cup V(I_2) = V(I_1 I_2)$.

Proof. (a) $\varnothing = V((1))$ and $\operatorname{Spec} A = V(\varnothing)$. (b) Let $\{I_i\}_i$ be a collection of ideals. Given $\mathfrak{p} \in \operatorname{Spec} A$ such that $I_i \subset \mathfrak{p}$ for all $i \in I$, we conclude that $I_i \subset \sum_i I_i \subset \mathfrak{p}$ as this summand is the smallest ideal containing all I_i. The backwards inclusion falls from essentially the same reason. (c) Let $\mathfrak{p} \in V(I_1)$, without loss of generality, i.e. $I_1 \subset \mathfrak{p}$ and $\mathfrak{p} \in \operatorname{Spec} A$. Suppose that $I_1 I_2 \not\subset \mathfrak{p}$. Then there exists $y \in I_1 I_2$, where $y = \sum_{i=1}^n s_i t_i$ with $s_i \in I_1$ and $t_i \in I_2$, such that $y \notin \mathfrak{p}$. But as $y = s_1(t_1) + \cdots + s_n(t_n)$ and we consider t_i as elements of A, then $y \in I_1$ and we get a contradiction. (To reinforce the statement of the theorem we give another proof (a slogan I've heard, and like): as $I_1 \subset \mathfrak{p}$ and $I_1 I_2 \subset I_1 \cap I_2 \subset I_1 \subset \mathfrak{p}$ we're done.) For the last inclusion, let $\mathfrak{p} \in \operatorname{Spec} A$ be such that $I_1 I_2 \subset \mathfrak{p}$. Suppose that I_1 and I_2 are both not contained in $\mathfrak{p}$. Then we can find $x \in I_1$ and $y \in I_2$ with $x, y \notin \mathfrak{p}$. But $xy \in I_1 I_2 \subset \mathfrak{p}$ so either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ as $\mathfrak{p}$ is prime, and thus we have a contradiction. ☐

Exercise 3.19 (3.4.D)**.** If $I \subset A$ is an ideal, then define its radical by

$$\sqrt{I} := \{r \in A \colon r^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

For example, the nilradical $\mathfrak{N}$ is simply $\sqrt{(0)}$. Show that $\sqrt{I}$ is an ideal. Show that $V(\sqrt{I}) = V(I)$. We say an ideal is radical if it equals its own radical [i.e. if $\sqrt{I} = I$]. Show that $\sqrt{\sqrt{I}} = \sqrt{I}$, and that prime ideals are radical.

A consequence is that as $(I \cap J)^2 \subset IJ \subset I \cap J$, then $V(I \cap J) = V(IJ)$.

Proof. Let $I \subset A$ be an ideal. Clearly $\sqrt{I}$ is nonempty as $0 \in \sqrt{I}$, and given $x, y \in \sqrt{I}$, we have $x - y \in \sqrt{I}$ using the usual binomial argument and recognizing that each term is in I, and given $r \in A$ we have $rx \in \sqrt{I}$ as $(rx)^m = r^m x^m \in I$. For the second statement of the exercise, fix $\mathfrak{p} \in \operatorname{Spec} A$. If $\sqrt{I} \subset \mathfrak{p}$, then $I \subset \mathfrak{p}$ as $I \subset \sqrt{I}$, obviously. If $I \subset \mathfrak{p}$, then take $y \in \sqrt{I}$, which implies that $y^m \in I$ for $m > 0$ so $y^m \in \mathfrak{p}$ and hence $y \in \mathfrak{p}$ as $\mathfrak{p}$ is prime; we conclude $\sqrt{I} \subset \mathfrak{p}$. For the last statement, let $x \in \sqrt{\sqrt{I}}$. Then $x \in A$ such that $x^m \in \sqrt{I}$ with $m > 0$ so $(x^m)^k \in I$ with $k > 0$ and thus $x^{mk} \in I$ so $x \in \sqrt{I}$. For the opposite inclusion, given $x \in \sqrt{I}$, we have $x^k \in I$ for $k > 0$ and as $I \subset \sqrt{I}$ we're done. Now we show that prime ideals are radical. This is pretty trivial: one is inclusion is obvious and given $x \in \sqrt{\mathfrak{p}}$ then $x^m \in \mathfrak{p}$ so $x \in \mathfrak{p}$ as $\mathfrak{p}$ is prime. ☐

Exercise 3.20 (3.4.E)**.** If $I_1, \ldots, I_n$ are ideals of A, show that $\sqrt{\cap_{i=1}^n I_i} = \bigcap_{i=1}^n \sqrt{I_i}$.

Proof. Let $x \in \sqrt{\cap_{i=1}^n I_i}$. Then $x \in A$ such that $x^m \in \cap_{i=1}^n I_i$ for $m > 0$, and so x^m is in I_i for all i, i.e. $x \in \sqrt{I_i}$ for all i, and hence $x \in \bigcap_{i=1}^n \sqrt{I_i}$. For the reverse, let $x \in \cap_{i=1}^n \sqrt{I_i}$. Then $x^{m_i} \in I_i$ for all i and $m_i > 0$, so $x \in \sqrt{\cap_{i=1}^n I_i}$. ☐

Exercise 3.21. [3.4.F] Show that $\sqrt{I}$ is the intersection of all the prime ideals containing I.

Proof. Using Exercise 3.15, we can apply the statement to A/I for any ring A and an ideal I of A. ☐

Exercise 3.22 (3.4.G)**.** Describe the topological space $\mathbb{A}_k^1$.

Proof. As k is a field, then $k[x]$ is a PID. We have that $\mathbb{A}_k^1 - \varnothing = \mathbb{A}_k^1$ and $\mathbb{A}_k^1 - \operatorname{Spec} k[x] = \varnothing$ are closed sets of $\mathbb{A}_k^1$. Moreover, given (f) an ideal of $k[x]$, we consider $V((f))$. Now, as $V((f)) = \{\mathfrak{p} \in \mathbb{A}_k^1 \colon (f) \subset \mathfrak{p}\}$, and given such a $\mathfrak{p}$ it is also principal, say, $\mathfrak{p} = (p(x))$. Then $(f) \subset (p(x))$, and as $(p(x))$ is also maximal then $p(x)$ is irreducible. Lastly, note that $p(x) \mid f(x)$, so as $f(x)$ has only finitely many irreducible factors, then $V((f))$ will be the set of irreducible factors of f if f is reducible. If f is irreducible, then $V((f)) = \{(f)\}$. So the topology of $\mathbb{A}_k^1$ consists of the closed sets: $\mathbb{A}_k^1$, the empty set, and finite sets (which contain maximal elements). ☐

Exercise 3.23 (3.4.H)**.** By showing that closed sets pull back to closed sets, show that π is a continuous map. Interpret Spec as a contravariant functor Rings $\to$ Top.

Proof. Given $\phi \colon B \to A$, we have an induced $\pi \colon \operatorname{Spec} A \to \operatorname{Spec} B$. Let $S \subset \operatorname{Spec} B$. Then its an easy verification that $\pi^{-1}(V(S)) = V(\phi(S))$, i.e. given a closed set, the preimage is also closed. Therefore π is a continuous map. We can clearly interpret Spec as a contravariant functor as given $\phi \colon B \to A$ a map of rings then $\operatorname{Spec} \phi := \pi \colon \operatorname{Spec} A \to \operatorname{Spec} B$ is a continuous map. (The other properties of a contravariant functor follow easily.) ☐

Exercise 3.24 (3.4.I)**.** Suppose that $I, S \subset B$ are an ideal and multiplicative subset respectively.

(a) Show that $\operatorname{Spec} B/I$ is naturally a *closed* subset of $\operatorname{Spec} B$. If $S = \{1, f, f^2, \dots\}$ $(f \in B)$, show that $\operatorname{Spec} S^{-1}B$ is naturally an *open* subset of $\operatorname{Spec} B$. Show that for arbitrary S, $\operatorname{Spec} S^{-1}B$ need not be open or closed.

(b) Show that the Zariski topology on $\operatorname{Spec} B/I$ (resp. $\operatorname{Spec} S^{-1}B$) is the subspace topology induced by inclusion in $\operatorname{Spec} B$.

Proof. (a) We have that $\operatorname{Spec} B/I$ is naturally a closed subset of $\operatorname{Spec} B$ as it can be identified with $V(I) = \{\mathfrak{p} \in \operatorname{Spec} B : I \subset \mathfrak{p}\}$. Recall that an open set in this situation is open if and only if $Y = \operatorname{Spec} B - D$ is closed for closed D. For $S = \{1, f, f^2, \dots\}$, we pick $V((f)) = D$ as our closed set we get that $\operatorname{Spec} B - D = Y = \operatorname{Spec} S^{-1}B$. We show that $\operatorname{Spec} \mathbb{Q}$ cannot be an open Zariski subset of $\operatorname{Spec} \mathbb{Z}$. Note that $\operatorname{Spec} S^{-1}\mathbb{Z} = \operatorname{Spec} \mathbb{Q} = (0)$ where $S = \mathbb{Z} - \{0\}$. If we assume that $\operatorname{Spec} \mathbb{Q}$ is open in $\operatorname{Spec} \mathbb{Z}$, then we can find an ideal J of $\mathbb{Z}$ such that $\operatorname{Spec} \mathbb{Q} = (0) = \operatorname{Spec} \mathbb{Z} - V(J)$. This means that $V(J)$ essentially gets rid of all prime ideals except (0), i.e. $V(J) = \{(p) \in \operatorname{Spec} \mathbb{Z} : p > 0 \text{ prime}\}$, but then J is contained in all prime ideals (p) with $p > 0$ but this is empty so $J = \varnothing$, and we get a contradiction in the equation $\operatorname{Spec} \mathbb{Q} = (0) = \operatorname{Spec} \mathbb{Z} - V(\varnothing) = \operatorname{Spec} \mathbb{Z} - \operatorname{Spec} \mathbb{Z} = \varnothing$. Thus $\operatorname{Spec} \mathbb{Q}$ cannot be an open set in the Zariski topology of $\operatorname{Spec} \mathbb{Z}$. For the opposite case, pick $B = \mathbb{Z}_p$ and $S = \{1, p, p^2, \dots\}$, then $S^{-1}\mathbb{Z}_p = \mathbb{Q}_p$, the p-adic numbers. We claim that $\operatorname{Spec} \mathbb{Q}_p$ is not closed in $\operatorname{Spec} \mathbb{Z}_p$. If it was, then $\operatorname{Spec} \mathbb{Q}_p = V(I)$ for some ideal $I \subset \mathbb{Z}_p$. As $\mathbb{Z}_p$ is a PID, then $I = p^m\mathbb{Z}_p$ for $m \geq 1$ (if $m = 1$, then $p\mathbb{Z}_p$ is the unique maximal/prime ideal of $\mathbb{Z}_p$). As $\mathbb{Q}_p$ is a field, then $\operatorname{Spec} \mathbb{Q}_p = (0) = V(p^m\mathbb{Z}_p)$. We get a contradiction for $m = 1$, but for $m = 2$, we have $V(p^2\mathbb{Z}_p) = p\mathbb{Z}_p$, and so on. □

Exercise 3.25 (3.4.J)**.** Suppose $I \subset B$ is an ideal. Show that f vanishes on $V(I)$ if and only if $f \in \sqrt{I}$.

Proof. $(\Rightarrow)$ If f vanishes on $V(I)$, then $f \in \mathfrak{p}$ for all prime ideals of B with $I \subset \mathfrak{p}$. Then by Exercise 3.21, $\sqrt{I}$ is the intersection of all the prime ideals containing I, so $f \in \sqrt{I}$. $(\Leftarrow)$ If $f \in \sqrt{I}$, then $f \equiv 0 \pmod{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ that contain I by Exercise 3.21. So, f vanishes on $V(I)$. □

Exercise 3.26 (3.4.K)**.** Describe the topological space $\operatorname{Spec} k[x]_{(x)}$.

Proof. By an earlier exercise, we know that $X = \operatorname{Spec} k[x]_{(x)}$ consists of just (0) and (x). Now X and $\varnothing$ are each closed and open, (0) is open, and (x) is closed. □

Exercise 3.27 (3.5.A)**.** Show that the distinguished open sets form a base for the (Zariski) topology. (Hint: Given a subset $S \subset A$, show that the complement of $V(S)$ is $\cup_{f \in S} D(f)$.)

Proof. Let $S \subset A$. □

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