HOMOTOPY SEMINAR: NOTES

AARON ANDREWS AND JUAN SERRATOS

Contents

0. Category Theory

0.1. **Categories.** The theory of categories should be thought of as a reformulation/language for a birdseye view of mathematics. Perhaps too harsh of a statement, but the subject is too dry if the sake is to understand only categories and not to fervently apply it to well-known categories to see how such things blossom in their respective contexts.

Definition 0.1. A morphism **category** $\mathscr{C}$ consists of a collection of **objects**, denoted as $\mathrm{Ob}(\mathscr{C})$, a collection of **morphisms** given any $A, B \in \mathrm{Ob}(\mathscr{C})$, written $\mathrm{Hom}_{\mathscr{C}}(A, B)$, and a function for any $A, B, C \in \mathrm{Ob}\,\mathscr{C}$

$$\mathrm{Hom}_{\mathscr{C}}(A,B) \times \mathrm{Hom}_{\mathscr{C}}(B,C) \to \mathrm{Hom}_{\mathscr{C}}(A,C)$$
$$(f,g) \mapsto g \circ f,$$

satisfying the following properties:

(i) For any $A \in \mathrm{Ob}$, an identity map $1_A \in \mathrm{Hom}_{\mathscr{C}}(A,A)$ such that $1_A \circ f = f$ and $g \circ 1_A = g$ for all $f \in \mathrm{Hom}_{\mathscr{C}}(B,A)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(A,B)$.
(ii) For all $f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$, $g \in \mathrm{Hom}_{\mathscr{C}}(B,C)$, and $h \in \mathrm{Hom}_{\mathscr{C}}(C,D)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
(iii) For all $f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$, we have $f \circ 1_A = f = 1_B \circ f$.

Example 0.1.

(i) The category of groups, Grps, where the objects are groups and have as morphisms between them group homomorphisms. Similarly, the category of abelian groups, Ab, and the category of rings, Rings, that has objects rings and morphisms are ring homomorphisms.
(ii) The category of topological spaces, Top, where the objects are topological spaces and the morphisms are continuous maps.
(iii) The category of sets, Sets, which has as objects sets and morphisms are usual functions.
(iv) For a ring A, the category of A-modules, Mod_A, that has as objects A-modules and morphisms are A-linear maps. If $A = k$ is a field, then this is the category of k-vector spaces, Vec_k.
(v) The category of pure Hodge structures for a given weight n, $\mathsf{pHod_n}$, where the objects are pure Hodge structures of weight n and the morphisms are homomorphisms of the underlying abelian groups.

Definition 0.2. A category $\mathscr{C}$ is a **locally small** if $\mathrm{Hom}_{\mathscr{C}}(A,B)$ forms a set for $A, B \in \mathrm{Ob}(\mathscr{C})$. Furthermore, $\mathscr{C}$ is **small** if $\mathscr{C}$ is locally small and any object A in $\mathscr{C}$ forms a set.

Example 0.2.

(i) All the categories from Example 0.1 are locally small.
(ii) The category Cat that has as objects categories and morphisms as *functors* (we will see what this is soon) is not locally small.

Definition 0.3. A morphism $f\colon A \to B$ in a category $\mathscr{C}$ is an **isomorphism** if there exists a morphism $g\colon B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. We write $A \simeq B$ if there exists an isomorphism between A and B. An **automorphism on** A is a morphism $A \to A$ that is an isomorphism.

Definition 0.4. A category $\mathscr{C}$ where every morphism is an isomorphism is a **groupoid**.

Exercise 0.1. Let $\mathscr{C}$ be a locally small category. If $A \in \mathscr{C}$, then show that the invertible morphisms of $\mathrm{Hom}_{\mathscr{C}}(A, A)$ form a group, which is denoted $\mathrm{Aut}(A)$. Furthermore, show that if $X \simeq Y$ in $\mathscr{C}$, then $\mathrm{Aut}(X) \simeq \mathrm{Aut}(Y)$. (This is a nice *invariant* which is extremely relevant in Galois theory: That is, if we know that $\mathrm{Aut}(X) \not\simeq \mathrm{Aut}(Y)$, e.g. if they don't have the same size, then we can conclude that $X \not\simeq Y$.)

Exercise 0.2. Understand the following statement: A *group* is a groupoid $\mathscr{G}$ with a single object. (That is, if $x \in \mathscr{G}$ is the single object of the groupoid, then what is $\mathrm{Hom}_{\mathscr{G}}(x, x) = \mathrm{Aut}_{\mathscr{G}}(x)$.)

Definition 0.5. A **subcategory** $\mathscr{D}$ of a cateogry $\mathscr{C}$ is a category that has some of its objects and morphisms of $\mathscr{C}$ such that the morphisms of $\mathscr{D}$ include the identity morphisms of the objects of $\mathscr{D}$, and are closed under composition. A subcategory $\mathscr{D}$ of $\mathscr{C}$ is a **full subcategory** if $\mathrm{Hom}_{\mathscr{D}}(A, B) = \mathrm{Hom}_{\mathscr{C}}(A, B)$ for all $A, B \in \mathscr{D}$.

We've already seen quite a lot of full subcategories. Namely, the category of abelian groups is a full category of the category of groups, and for a field k the category Vec_k forms a full subcategory of the category of k-modules. Lastly, for those that know about the theory of schemes, the category of affine schemes forms a full subcategory of the category of schemes.

0.2. **Functors.** Lets start from the motivating standpoint of the category of Grps. For two groups $(G, *)$ and $(H, \cdot)$, which may have distinct binary operations on them, a group homomorphism is a set-theoretic map $\varphi\colon G \to H$ such that $\varphi(a * b) = \varphi(a) \cdot \varphi(b)$, i.e. information from the group G gets transferred over nicely to the group H by way of (loosely) preservation of binary operations. In a similar way of preservation of binary operations for groups, we define a "map" from a category to another category that preserves the operations from the original category to the category we're mapping to; this is what a *functor* is.

Definition 0.6. A **covariant functor** (resp. **contravariant functor**) $F\colon \mathscr{C} \to \mathscr{H}$ is a map that associates every object $X \in \mathscr{C}$ to an object $F(X) \in \mathscr{H}$ and every morphism $g\colon X \to Y$ in $\mathscr{C}$ is associated to a map $F(g)\colon F(X) \to F(Y)$ in $\mathscr{H}$ (resp. $F(g)\colon F(Y) \to F(X)$), which satisfies $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, and for a composition of maps $g\colon X \to Y$ and $h\colon Y \to Z$ in $\mathscr{C}$ we have $F(h \circ g) = F(h) \circ F(g)$ (resp. $F(h \circ g) = F(g) \circ F(h)$).

Example 0.3.

(i) Let A be a ring and fix and an A-module M. The assignment $\mathrm{Hom}_A(M, -)\colon \mathsf{Mod}_A \to \mathsf{Mod}_A$ via $N \mapsto \mathrm{Hom}_A(M, N)$ defines a covariant functor, while $\mathrm{Hom}_A(-, M)\colon \mathsf{Mod}_A \to \mathsf{Mod}_A$ via $N \mapsto \mathrm{Hom}_A(N, M)$ defines a contravariant functor. For A-linear maps $f\colon N \to L$, the map $\mathrm{Hom}_A(M, f) := f_*\colon \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, L)$ is given by $f_*(g) = f \circ g$, while for the contravariant version we have $\mathrm{Hom}_A(f, M) := f^*\colon \mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(L, M)$ given by $f^*(g) = g \circ f$.
(ii) We have a contravariant functor $\mathrm{Spec}(-)\colon \mathsf{Ring} \to \mathsf{Top}$ given by $A \mapsto \mathrm{Spec}\, A$, where the right hand side is the spectrum of the ring considered with the Zariski topology. For a given map of rings $\varphi\colon A \to B$, we have $\mathrm{Spec}(\varphi)\colon \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ where $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$.
(iii) Let X be a topological space. A continuous map $f\colon [0, 1] \to X$ is called a *path* from $f(0)$ to $f(1)$, we denote f^{-1} to be the map from $f(1)$ to $f(0)$ where $f^{-1}(t) = f(1-t)$, and $i_{x_0}\colon [0, 1] \to X$ be the constant map to $x_0 \in X$. We define an equivalence relation on X by saying $x_0 \sim x_1$ if and only if there exists a path from x_0 to x_1, and we write $\pi_0(X) = X/\sim$ with this equivalence relation. Then $\pi_0\colon \mathsf{Top}_* \to \mathsf{Set}$ defines a covariant functor.
(iv) The *forgetful functor* is a functor that essentially drops some of the added object and map structures: The forgetful functor $F\colon \mathsf{Ring} \to \mathsf{Ab}$ is the functor that takes a ring $A \in \mathsf{Ring}$ and maps it to $F(A) = A$ but now considered only as an abelian group, i.e. viewing a ring A as a triple $(A, +, \cdot)$ our functor views A as the double $F(A) = A = (A, +)$.

(v) Let G be a group. For elements $g, h \in G$ we define the *commutator* of g and h by $[g, h] := g^{-1}h^{-1}gh \in G$, and we define the *commutator subgroup* $D(G)$, often also called the *derived subgroup* and notated by $[G, G]$, as the subgroup of G generated by all commutators, o.e.

$$D(G) = [G, G] = \langle [g, h] \colon g, h \in G \rangle.$$

A good exercise here is the following: Let G be a group and denote $D(G)$ as its derived subgroup. Let N be a subgroup of G. Then prove that the N is normal in G and G/N is an abelian group if and only if $D(G) \subset N$. Now, relevant to the example we want to make, we have a functor $(-)^{\mathrm{ab}} \colon \mathsf{Grps} \to \mathsf{Ab}$, often called the *abelianization*, where $G \mapsto G/D(G)$ and the morphisms are the natural quotient ones.

We check here that Example 0.3 (i) does indeed define functors... well we check that the contravariant one does and leave the covariant one as an exercise. Let A be a ring and fix an A-module M. Let $N \in \mathsf{Mod}_A$. Then $\mathrm{Hom}_A(N, M)$ does indeed define an object of Mod_A as $\mathrm{Hom}_A(N, M)$ is a module with structure given pointwise, i.e. $(f + g)(x) := f(x) + g(x)$, and we define a map $\cdot \colon A \times \mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(N, M)$ where $(r, f) \mapsto rf$; these satisfy the module axioms and makes $\mathrm{Hom}_A(N, M)$ an A-module. Now let $f \colon N \to L$ in Mod_A. Then $f^* \colon \mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(L, M)$ is such that $f^*(g) = g \circ f$ which defines an A-linear map of $\mathrm{Hom}_A(L, M)$ as

$$M \xrightarrow{g} N \xrightarrow{f} L,$$

and lastly $\mathrm{id}_N^* \colon \mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(N, M)$ gives $\mathrm{id}_N^*(f) = f \circ \mathrm{id}_N = f$. Hence we have $\mathrm{Hom}_A(-, M)$ defines a contravariant functor as claimed.

Lemma 0.1. Let $F \colon \mathscr{C} \to \mathscr{D}$ be a covariant (resp. contravariant) functor. If $X \simeq Y$ in $\mathscr{C}$, then $F(X) \simeq F(Y)$.

Proof. Let $f \colon X \to Y$ be an isomorphism in $\mathscr{C}$, and as such there exists $g \colon Y \to X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$. As F is a covariant functor, then $F(f) \circ F(g) = 1_{F(Y)}$ and $F(g) \circ F(f) = 1_{F(X)}$, and thus we conclude $F(X) \simeq F(Y)$. To get a diagrammatic point of view of the statement, look at the following commuting diagrams:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "1_X"'] & Y \arrow[d, "g"] \arrow[dr, "1_Y"] & & & F(X) \arrow[r, "F(f)"] \arrow[dr, "1_{F(X)}"'] & F(Y) \arrow[d, "F(g)"] \arrow[dr, "1_{F(Y)}"] & \\
& X \arrow[r, "f"] & Y & \overset{F}{\Longrightarrow} & & F(X) \arrow[r, "F(f)"] & F(Y)
\end{tikzcd}$$

□

Example 0.4 (Extravagant). The functor $\mathcal{M}_g \colon \mathsf{Sch} \to \mathsf{Groupoid}$ that sends a scheme X to the set of isomorphism classes of flat families of genus g curves over X.

0.3. **Abelian Categories.**

1. Introduction: ∞-categories and related prerequsities

1.1. **(Pre)Sheaves.**

A recurring term you will forever hear is that of a *presheaf* and a *sheaf*, where the latter is an extension of a presheaf that satisfies some extra properties. The reason this is heard so often is that the concept isn't actually a very hard one (it is also very general), and will be heard in many concepts (especially in the algebraic contexts).

Definition 1.1. Let $\mathscr{C}$ be a category. A **presheaf over** $\mathscr{C}$ is a functor of the form $\mathscr{F} \colon \mathscr{C}^{\mathrm{op}} \to \mathsf{Sets}$.

If one prefers, the equivalent definition is that $\mathscr{F} \colon \mathscr{C} \to \mathsf{Sets}$ is a *presheaf* if $\mathscr{F}$ is a contravariant functor (in Definition 1.1, it is instead defined to be covariant but we preemptively flip the arrows when we consider the opposite category). Additionally, in our definition we explicitly use the category of sets, but it is often that we change this category to something like the category of abelian groups, or rings, and so on. We say that $\mathscr{F} \colon \mathscr{C} \to \mathsf{Ab}$ is an *abelian-valued presheaf* for an extra level of clarity, although it is often clear from context. Given a presheaf $\mathscr{F}$ over $\mathscr{C}$ and an object $U \in \mathscr{C}$, we call the elements of $\mathscr{F}(U)$ *sections.* A morphism of presheaves $\eta \colon \mathscr{F} \to \mathscr{L}$, where $\mathscr{F}$ and $\mathscr{L}$ are both presheaves over $\mathscr{C}$, is simply a natural transformation. Lastly, we can build a category out of presheaves: it's a functor category $\mathsf{Fun}(\mathscr{C}^{\mathrm{op}}, \mathsf{Sets}) := \mathsf{PSh}(\mathscr{C})$, i.e it's the category that has objects as presheaves $\mathscr{F}: \mathscr{C} \to \mathsf{Sets}$ and has as morphisms natural transformations between presheaves.

In algebraic geometry, one often fixes a topological space X and considers the category X_{open}, where the objects are open subsets of X and, for any given two open subsets of X, say, U and V, the morphism is either the inclusion map $i\colon U \to V$ if $U \subset V$ or no morphism at all if neither is contained in the other. The theory of schemes is built out of considering sheaves $\mathscr{O}_X : X_{\text{open}} \to \mathsf{Rings}$ with a lot more structure.

Definition 1.2. The **Yoneda embedding** is the functor $h\colon \mathscr{C} \to \mathsf{PSh}(\mathscr{C})$ where $h(A) := h_A = \mathrm{Hom}_{\mathscr{C}}(-, A)$ for an object $A \in \mathscr{C}$.

Exercise 1.1. Verify that $\mathrm{Hom}_{\mathscr{C}}(-, A)$ is a presheaf.

Theorem 1.1 (Yoneda Lemma). For any presheaf $\mathscr{F}$ over $\mathscr{C}$, there is a natural bijection of the form

$$\mathrm{Hom}_{\mathsf{PSh}(\mathscr{C})}(h_A, \mathscr{F}) \xrightarrow{\sim} \mathscr{F}(A)$$

...[To be completed]

1.2. **The category of simplicial sets.** Consider the category whose objects are are finite subsets $[n] = \{0, 1, 2, \ldots, n\} = \{i \in \mathbf{Z}\colon 1 \leq i \leq n\}$, for $n \geq 0$, which inherit the natural ordering of integers, and has morphisms $f\colon [n] \to [m]$ that are (non strict) order-perserving maps, i.e. functions $f\colon [n] \to [m]$ with the property that $a < b$ in $[n]$ implies $f(a) \leq f(b)$ in $[m]$. We denote this category by Δ and is called the *simplicial category*, which is a small category by construction.s

Definition 1.3. A **simplicial set** is a presheaf over Δ. We denote $\mathsf{Set}_\Delta = \mathsf{PSh}(\Delta)$ to be the *category of simplicial sets.*

Given $n \geq 0$, we write $\Delta^n = \mathrm{Hom}_\Delta(-, [n])$ and call it the *standard* n *simplex*, that is, Δ^n is the simplicial set represented by $[n] \in \Delta$. So, we write $(\Delta^n)_m = \mathrm{Hom}_\Delta([m], [n])$. For a simplicial set $X\colon \Delta \to \mathsf{Sets}$ and an integer $n \geq 0$, we write $X_n = X([n]) \simeq \mathrm{Hom}_{\mathsf{Sets}_\Delta}(\Delta^n, X)$, by Theorem 1.1, for the set of n*-simplices* of X. Lastly, a *simplex* of X is an element of X_n for some nonnegative integer n.

We define, for $0 \leq i \leq n$, the pair of following maps:

$$\partial_i\colon [n-1] \to [n], \qquad \partial_i(j) = \begin{cases} j & j < i, \\ j+1 & i \leq j \end{cases}$$

$$\sigma_i : [n+1] \to [n], \qquad \sigma_i(j) = \begin{cases} j & j \leq i, \\ j-1 & i \leq j \end{cases}$$

The map $\partial_i\colon [n-1] \to [n]$ is called the i-th *face map*, which is injective, and $\sigma_i\colon [n+1] \to [n]$ is called the i-th *degeneracy map*, which is surjective.

Lemma 1.1. The following relations hold:

$$\begin{aligned} \partial_j \circ \partial_i &= \partial_i \circ \partial_{j-1}, \quad && i < j, \\ \sigma_j \circ \sigma_i &= \sigma_i \circ \sigma_{j+1}, \quad && i \leq j, \text{ and} \\ \sigma_j \circ \partial_i &= \begin{cases} \partial_i \circ \sigma_{j-1}, & i < j, \\ 1_{[n]}, & i \in \{j, j+1\}, \\ \partial_{i-1} \circ \sigma_j, & i > j+1. \end{cases} \end{aligned}$$

Proof. These are quite annoying to explicitly compute, so we do the first and leave it up to you to do the rest (or at least convince yourself of the rest). Let $i < j$. Then $\partial_j(\partial_i(k)) = k$ if $k < i$ or $k+1$ if $i \leq k$. If $k < i$, then $\partial_j(\partial_j(k)) = \partial_j(k) = k$, and $\partial_i \circ \partial_{j-1}(k) = \partial_i(k)$, as $k < i < j$ implies $k < j-1$, and $\partial_i(k) = k$. If $i \leq k$, then $\partial_j(\partial_i(k)) = \partial_j(k+1)$: (1.) If $i = k$ and $k+1 < j$, then $\partial_j(k+1) = k+1$ and this aligned with $\partial_i(\partial_{j-1}(k)) = \partial_i(k) = k+1$, and if $i = k$ with $j \leq k+1$, then $\partial_j(k+1) = k+2$ and this is aligned with $\partial_i(\partial_{j-1}(k)) = \partial_i(k+1) = k+2$; (2.) If $i < k$ and $k+1 < j$, then $\partial_j(k+1) = k+1$ and we have $\partial_i(\partial_{j-1}(k)) = \partial_i(k) = k+1$, and if $i < k$ and $j \leq k+2$, then $\partial_j(k+1) = k+2$ and $\partial_i(\partial_{j-1}(k)) = \partial_i(k+1) = k+2$. $\square$

Lemma 1.2. Every morphism $f \neq 1_{[n]} \in \mathrm{Hom}_\Delta([n], [m])$ can be written as a composition

$$f = \partial_{i_1} \circ \cdots \partial_{i_r} \circ \sigma_{j_1} \circ \cdots \circ \sigma_{j_s}$$

with $0 \leq i_r < \cdots < i_1 \leq m$ and $0 \leq j_1 < \cdots < j_s < n$, where $m = n - s + r$. This decomposition is unique.

References

[Lur12] Jacob Lurie. Higher topos theory. ***Annals of Mathematics Studies***, **170**, 2012.

Email address: `jserrato@usc.edu`