

HOMOTOPY SEMINAR: NOTES

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0. CATEGORY THEORY

0.1. Categories. The theory of categories should be thought of as a reformulation/language for a birdseye view of mathematics. Perhaps too harsh of a statement, but the subject is too dry if the sake is to understand only categories and not to fervently apply it to well-known categories to see how such things blossom in their respective contexts.

Definition 0.1. A morphism **category** \mathcal{C} consists of a collection of **objects**, denoted as $\text{Ob}(\mathcal{C})$, a collection of **morphisms** given any $A, B \in \text{Ob}(\mathcal{C})$, written $\text{Hom}_{\mathcal{C}}(A, B)$, and a function for any $A, B, C \in \text{Ob} \mathcal{C}$

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(A, B) \times \text{Hom}_{\mathcal{C}}(B, C) &\rightarrow \text{Hom}_{\mathcal{C}}(A, C) \\ (f, g) &\mapsto g \circ f, \end{aligned}$$

satisfying the following properties:

- (i) For any $A \in \text{Ob}$, an identity map $1_A \in \text{Hom}_{\mathcal{C}}(A, A)$ such that $1_A \circ f = f$ and $g \circ 1_A = g$ for all $f \in \text{Hom}_{\mathcal{C}}(B, A)$ and $g \in \text{Hom}_{\mathcal{C}}(A, B)$.
- (ii) For all $f \in \text{Hom}_{\mathcal{C}}(A, B)$, $g \in \text{Hom}_{\mathcal{C}}(B, C)$, and $h \in \text{Hom}_{\mathcal{C}}(C, D)$, we have $h \circ (g \circ f) = (h \circ g) \circ f$
- (iii) For all $f \in \text{Hom}_{\mathcal{C}}(A, B)$, we have $f \circ 1_A = f = 1_B \circ f$.

Example 0.1.

- (i) The category of groups, Grps , where the objects are groups and have as morphisms between them group homomorphisms. Similarly, the category of abelian groups, Ab , and the category of rings, Rings , that has objects rings and morphisms are ring homomorphisms.
- (ii) The category of topological spaces, Top , where the objects are topological spaces and the morphisms are continuous maps.
- (iii) The category of sets, Sets , which has as objects sets and morphisms are usual functions.
- (iv) For a ring A , the category of A -modules, Mod_A , that has as objects A -modules and morphisms are A -linear maps. If $A = k$ is a field, then this is the category of k -vector spaces, Vec_k .
- (v) The category of pure Hodge structures for a given weight n , pHod_n , where the objects are pure Hodge structures of weight n and the morphisms are homomorphisms of the underlying abelian groups.

Definition 0.2. A category \mathcal{C} is a **locally small** if $\text{Hom}_{\mathcal{C}}(A, B)$ forms a set for $A, B \in \text{Ob}(\mathcal{C})$. Furthermore, \mathcal{C} is **small** if \mathcal{C} is locally small and any object A in \mathcal{C} forms a set.

Example 0.2.

- (i) All the categories from Example 0.1 are locally small.
- (ii) The category Cat that has as objects categories and morphisms as *functors* (we will see what this is soon) is not locally small.

Definition 0.3. A morphism $f: A \rightarrow B$ in a category \mathcal{C} is an **isomorphism** if there exists a morphism $g: B \rightarrow A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$. We write $A \simeq B$ if there exists an isomorphism between A and B . An **automorphism on** A is a morphism $A \rightarrow A$ that is an isomorphism.

Definition 0.4. A category \mathcal{C} where every morphism is an isomorphism is a **groupoid**.

Exercise 0.1. Let \mathcal{C} be a locally small category. If $A \in \mathcal{C}$, then show that the invertible morphisms of $\text{Hom}_{\mathcal{C}}(A, A)$ form a group, which is denoted $\text{Aut}(A)$. Furthermore, show that if $X \simeq Y$ in \mathcal{C} , then $\text{Aut}(X) \simeq \text{Aut}(Y)$. (This is a nice *invariant* which is extremely relevant in Galois theory: That is, if we know that $\text{Aut}(X) \not\simeq \text{Aut}(Y)$, e.g. if they don't have the same size, then we can conclude that $X \not\simeq Y$.)

Exercise 0.2. Understand the following statement: A *group* is a groupoid \mathcal{G} with a single object. (That is, if $x \in \mathcal{G}$ is the single object of the groupoid, then what is $\text{Hom}_{\mathcal{G}}(x, x) = \text{Aut}_{\mathcal{G}}(x)$.)

Definition 0.5. A **subcategory** \mathcal{D} of a category \mathcal{C} is a category that has some of its objects and morphisms of \mathcal{C} such that the morphisms of \mathcal{D} include the identity morphisms of the objects of \mathcal{D} , and are closed under composition. A subcategory \mathcal{D} of \mathcal{C} is a **full subcategory** if $\text{Hom}_{\mathcal{D}}(A, B) = \text{Hom}_{\mathcal{C}}(A, B)$ for all $A, B \in \mathcal{D}$.

We've already seen quite a lot of full subcategories. Namely, the category of abelian groups is a full category of the category of groups, and for a field k the category Vec_k forms a full subcategory of the category of k -modules. Lastly, for those that know about the theory of schemes, the category of affine schemes forms a full subcategory of the category of schemes.

0.2. Functors. Lets start from the motivating standpoint of the category of Grps . For two groups $(G, *)$ and (H, \cdot) , which may have distinct binary operations on them, a group homomorphism is a set-theoretic map $\varphi: G \rightarrow H$ such that $\varphi(a * b) = \varphi(a) \cdot \varphi(b)$, i.e. information from the group G gets transferred over nicely to the group H by way of (loosely) preservation of binary operations. In a similar way of preservation of binary operations for groups, we define a "map" from a category to another category that preserves the operations from the original category to the category we're mapping to; this is what a *functor* is.

Definition 0.6. A **covariant functor** (resp. **contravariant functor**) $F: \mathcal{C} \rightarrow \mathcal{H}$ is a map that associates every object $X \in \mathcal{C}$ to an object $F(X) \in \mathcal{H}$ and every morphism $g: X \rightarrow Y$ in \mathcal{C} is associated to a map $F(g): F(X) \rightarrow F(Y)$ in \mathcal{H} (resp. $F(g): F(Y) \rightarrow F(X)$), which satisfies $F(\text{id}_X) = \text{id}_{F(X)}$, and for a composition of maps $g: X \rightarrow Y$ and $h: Y \rightarrow Z$ in \mathcal{C} we have $F(h \circ g) = F(h) \circ F(g)$ (resp. $F(h \circ g) = F(g) \circ F(h)$).

Example 0.3.

- (i) Let A be a ring and fix an A -module M . The assignment $\text{Hom}_A(M, -): \text{Mod}_A \rightarrow \text{Mod}_A$ via $N \mapsto \text{Hom}_A(M, N)$ defines a covariant functor, while $\text{Hom}_A(-, M): \text{Mod}_A \rightarrow \text{Mod}_A$ via $N \mapsto \text{Hom}_A(N, M)$ defines a contravariant functor. For A -linear maps $f: N \rightarrow L$, the map $\text{Hom}_A(M, f) := f_*: \text{Hom}_A(M, N) \rightarrow \text{Hom}_A(M, L)$ is given by $f_*(g) = f \circ g$, while for the contravariant version we have $\text{Hom}_A(f, M) := f^*: \text{Hom}_A(N, M) \rightarrow \text{Hom}_A(L, M)$ given by $f^*(g) = g \circ f$.
- (ii) We have a contravariant functor $\text{Spec}(-): \text{Ring} \rightarrow \text{Top}$ given by $A \mapsto \text{Spec } A$, where the right hand side is the spectrum of the ring considered with the Zariski topology. For a given map of rings $\varphi: A \rightarrow B$, we have $\text{Spec}(\varphi): \text{Spec } B \rightarrow \text{Spec } A$ where $\mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$.
- (iii) Let X be a topological space. A continuous map $f: [0, 1] \rightarrow X$ is called a *path* from $f(0)$ to $f(1)$, we denote f^{-1} to be the map from $f(1)$ to $f(0)$ where $f^{-1}(t) = f(1 - t)$, and $i_{x_0}: [0, 1] \rightarrow X$ be the constant map to $x_0 \in X$. We define an equivalence relation on X by saying $x_0 \sim x_1$ if and only if there exists a path from x_0 to x_1 , and we write $\pi_0(X) = X / \sim$ with this equivalence relation. Then $\pi_0: \text{Top}_* \rightarrow \text{Set}$ defines a covariant functor.

- (iv) The *forgetful functor* is a functor that essentially drops some of the added object and map structures: The forgetful functor $F: \text{Ring} \rightarrow \text{Ab}$ is the functor that takes a ring $A \in \text{Ring}$ and maps it to $F(A) = A$ but now considered only as an abelian group, i.e. viewing a ring A as a triple $(A, +, \cdot)$ our functor views A as the double $F(A) = A = (A, +)$.
- (v) Let G be a group. For elements $g, h \in G$ we define the *commutator* of g and h by $[g, h] := g^{-1}h^{-1}gh \in G$, and we define the *commutator subgroup* $D(G)$, often also called the *derived subgroup* and notated by $[G, G]$, as the subgroup of G generated by all commutators, o.e.

$$D(G) = [G, G] = \langle [g, h] : g, h \in G \rangle.$$

A good exercise here is the following: Let G be a group and denote $D(G)$ as its derived subgroup. Let N be a subgroup of G . Then prove that the N is normal in G and G/N is an abelian group if and only if $D(G) \subset N$. Now, relevant to the example we want to make, we have a functor $(-)^{\text{ab}}: \text{Grps} \rightarrow \text{Ab}$, often called the *abelianization*, where $G \mapsto G/D(G)$ and the morphisms are the natural quotient ones.

We check here that Example 0.3 (i) does indeed define functors... well we check that the contravariant one does and leave the covariant one as an exercise. Let A be a ring and fix an A -module M . Let $N \in \text{Mod}_A$. Then $\text{Hom}_A(N, M)$ does indeed define an object of Mod_A as $\text{Hom}_A(N, M)$ is a module with structure given pointwise, i.e. $(f + g)(x) := f(x) + g(x)$, and we define a map $\cdot : A \times \text{Hom}_A(N, M) \rightarrow \text{Hom}_A(N, M)$ where $(r, f) \mapsto rf$; these satisfy the module axioms and makes $\text{Hom}_A(N, M)$ an A -module. Now let $f: N \rightarrow L$ in Mod_A . Then $f^*: \text{Hom}_A(N, M) \rightarrow \text{Hom}_A(L, M)$ is such that $f^*(g) = g \circ f$ which defines an A -linear map of $\text{Hom}_A(L, M)$ as

$$M \xrightarrow{g} N \xrightarrow{f} L,$$

and lastly $\text{id}_N^*: \text{Hom}_A(N, M) \rightarrow \text{Hom}_A(N, M)$ gives $\text{id}_N^*(f) = f \circ \text{id}_N = f$. Hence we have $\text{Hom}_A(-, M)$ defines a contravariant functor as claimed.

Lemma 0.1. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a covariant (resp. contravariant) functor. If $X \simeq Y$ in \mathcal{C} , then $F(X) \simeq F(Y)$.

Proof. Let $f: X \rightarrow Y$ be an isomorphism in \mathcal{C} , and as such there exists $g: Y \rightarrow X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$. As F is a covariant functor, then $F(f) \circ F(g) = 1_{F(Y)}$ and $F(g) \circ F(f) = 1_{F(X)}$, and thus we conclude $F(X) \simeq F(Y)$. To get a diagrammatic point of view of the statement, look at the following commuting diagrams:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow 1_X & \downarrow g \\ & & X \xrightarrow{f} Y \end{array} \quad \xRightarrow{F} \quad \begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ & \searrow 1_{F(X)} & \downarrow F(g) \\ & & F(X) \xrightarrow{F(f)} F(Y) \end{array}$$

□

Definition 0.7. A morphism $f: X \rightarrow Y$ in a category \mathcal{C} is an **epimorphism** if for all objects Z in \mathcal{C} and all morphisms $g_1, g_2: Y \rightarrow Z$ we have $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$. A morphism $f: X \rightarrow Y$ in \mathcal{C} is a **monomorphism** if for all objects Z and $g_1, g_2: Z \rightarrow X$ we have that $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

Lemma 0.2. Let $f: X \rightarrow Y$ and $h: Y \rightarrow Z$ be monomorphisms (resp. epimorphisms), then their composition $h \circ f: X \rightarrow Z$ is a monomorphism (resp. epimorphism).

Proof. We prove the epimorphism case and the other one follows by nearly the exact same argument. Write $q = h \circ f$, and let L be any object in \mathcal{C} with $g_1, g_2: Z \rightarrow L$. Then $g_1 \circ q = g_2 \circ q$ is the same as $g_1 \circ (h \circ f) = g_2 \circ (h \circ f)$ and $(g_1 \circ h) \circ f = (g_2 \circ h) \circ f$ as f is epi and implies $g_1 \circ h = g_2 \circ h$ implies $g_1 = g_2$ as h is epi as well. □

The above definition attempts to generalize the situation in Sets, i.e. a map is surjective if and only if it has a right inverse, and a map is injective if and only if it has a left inverse. In some ways, we should see monomorphisms as usual injections and epimorphisms as surjections, but we caution here that in a lot of categories we cannot treat these things as being the same. Even in the category of Grps, we don't have the

case that the notions coincide. But we do in the case of Mod_A for some ring A . Lastly, we should note that being both a monomorphism and an epimorphism will not be an isomorphism in a lot of categories. For example, in Ring , the embedding $\mathbb{Z} \rightarrow \mathbb{Q}$ is both epi and mono, but of course not an isomorphism.

Proposition 0.1. In Mod_A ,

- (i) A morphism is injective if and only if it is a monomorphism.
- (ii) A morphism is surjective if and only if it is an epimorphism.

Definition 0.8. An object Y in a category \mathcal{C} is **initial** if $\text{Hom}_{\mathcal{C}}(Y, Z) = \{*\}$ for all $Z \in \mathcal{C}$; an object K in \mathcal{C} is **terminal** if $\text{Hom}_{\mathcal{C}}(Z, K) = \{*\}$ for all $Z \in \mathcal{C}$. If Y is both initial and final, then it is called a **zero object**.

1. INTRODUCTION: ∞ -CATEGORIES AND RELATED PREREQUISITES

1.1. (Pre)Sheaves. A recurring term you will forever hear is that of a *presheaf* and a *sheaf*, where the latter is an extension of a presheaf that satisfies some extra properties. The reason this is heard so often is that the concept isn't actually a very hard one (it is also very general), and will be heard in many concepts (especially in the algebraic contexts).

Definition 1.1. Let \mathcal{C} be a category. A **presheaf over \mathcal{C}** is a functor of the form $\mathcal{F}: \mathcal{C}^{\text{op}} \rightarrow \text{Sets}$.

If one prefers, the equivalent definition is that $\mathcal{F}: \mathcal{C} \rightarrow \text{Sets}$ is a *presheaf* if \mathcal{F} is a contravariant functor (in Definition 1.1, it is instead defined to be covariant but we preemptively flip the arrows when we consider the opposite category). Additionally, in our definition we explicitly use the category of sets, but it is often that we change this category to something like the category of abelian groups, or rings, and so on. We say that $\mathcal{F}: \mathcal{C} \rightarrow \text{Ab}$ is an *abelian-valued presheaf* for an extra level of clarity, although it is often clear from context. Given a presheaf \mathcal{F} over \mathcal{C} and an object $U \in \mathcal{C}$, we call the elements of $\mathcal{F}(U)$ *sections*. A morphism of presheaves $\eta: \mathcal{F} \rightarrow \mathcal{L}$, where \mathcal{F} and \mathcal{L} are both presheaves over \mathcal{C} , is simply a natural transformation. Lastly, we can build a category out of presheaves: it's a functor category $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Sets}) := \text{PSh}(\mathcal{C})$, i.e. it's the category that has objects as presheaves $\mathcal{F}: \mathcal{C} \rightarrow \text{Sets}$ and has as morphisms natural transformations between presheaves.

In algebraic geometry, one often fixes a topological space X and considers the category X_{open} , where the objects are open subsets of X and, for any given two open subsets of X , say, U and V , the morphism is either the inclusion map $i: U \rightarrow V$ if $U \subset V$ or no morphism at all if neither is contained in the other. The theory of schemes is built out of considering sheaves $\mathcal{O}_X: X_{\text{open}} \rightarrow \text{Rings}$ with a lot more structure.

Definition 1.2. The **Yoneda embedding** is the functor $h: \mathcal{C} \rightarrow \text{PSh}(\mathcal{C})$ where $h(A) := h_A = \text{Hom}_{\mathcal{C}}(-, A)$ for an object $A \in \mathcal{C}$.

Exercise 1.1. Verify that $\text{Hom}_{\mathcal{C}}(-, A)$ is a presheaf.

Theorem 1.1 (Yoneda Lemma). For any presheaf \mathcal{F} over \mathcal{C} , there is a natural bijection of the form

$$\text{Hom}_{\text{PSh}(\mathcal{C})}(h_A, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}(A)$$

...[To be completed]

1.2. The category of simplicial sets. Consider the category whose objects are finite subsets $[n] = \{0, 1, 2, \dots, n\} = \{i \in \mathbb{Z}: 1 \leq i \leq n\}$, for $n \geq 0$, which inherit the natural ordering of integers, and has morphisms $f: [n] \rightarrow [m]$ that are (non strict) order-perserving maps, i.e. functions $f: [n] \rightarrow [m]$ with the property that $a < b$ in $[n]$ implies $f(a) \leq f(b)$ in $[m]$. We denote this category by Δ and is called the *simplicial category*, which is a small category by construction.

Definition 1.3. A **simplicial set** is a presheaf over Δ . We denote $\text{Set}_{\Delta} = \text{PSh}(\Delta)$ to be the *category of simplicial sets*.

Given $n \geq 0$, we write $\Delta^n = \text{Hom}_{\Delta}(-, [n])$ and call it the *standard n simplex*, that is, Δ^n is the simplicial set represented by $[n] \in \Delta$. So, we write $(\Delta^n)_m = \text{Hom}_{\Delta}([m], [n])$. For a simplicial set $X: \Delta \rightarrow \text{Sets}$ and an integer $n \geq 0$, we write $X_n = X([n]) \simeq \text{Hom}_{\text{Set}_{\Delta}}(\Delta^n, X)$, by Theorem 1.1, for the set of *n -simplices* of X . Lastly, a *simplex* of X is an element of X_n for some nonnegative integer n .

We define, for $0 \leq i \leq n$, the pair of following maps:

$$\begin{aligned} \partial_i : [n-1] &\rightarrow [n], & \partial_i(j) &= \begin{cases} j & j < i, \\ j+1 & i \leq j \end{cases} \\ \sigma_i : [n+1] &\rightarrow [n], & \sigma_i(j) &= \begin{cases} j & j \leq i, \\ j-1 & i < j \end{cases} \end{aligned}$$

The map $\partial_i : [n-1] \rightarrow [n]$ is called the i -th *face map*, which is injective, and $\sigma_i : [n+1] \rightarrow [n]$ is called the i -th *degeneracy map*, which is surjective.

Lemma 1.1. The following relations hold:

$$\begin{aligned} \partial_j \circ \partial_i &= \partial_i \circ \partial_{j-1}, & i < j, \\ \sigma_j \circ \sigma_i &= \sigma_i \circ \sigma_{j+1}, & i \leq j, \text{ and} \\ \sigma_j \circ \partial_i &= \begin{cases} \partial_i \circ \sigma_{j-1}, & i < j, \\ 1_{[n]}, & i \in \{j, j+1\}, \\ \partial_{i-1} \circ \sigma_j, & i > j+1. \end{cases} \end{aligned}$$

Proof. These are quite annoying to explicitly compute, so we do the first and leave it up to you to do the rest (or at least convince yourself of the rest). Let $i < j$. Then $\partial_j(\partial_i(k)) = k$ if $k < i$ or $k+1$ if $i \leq k$. If $k < i$, then $\partial_j(\partial_j(k)) = \partial_j(k) = k$, and $\partial_i \circ \partial_{j-1}(k) = \partial_i(k)$, as $k < i < j$ implies $k < j-1$, and $\partial_i(k) = k$. If $i \leq k$, then $\partial_j(\partial_i(k)) = \partial_j(k+1)$: (1.) If $i = k$ and $k+1 < j$, then $\partial_j(k+1) = k+1$ and this aligned with $\partial_i(\partial_{j-1}(k)) = \partial_i(k) = k+1$, and if $i = k$ with $j \leq k+1$, then $\partial_j(k+1) = k+2$ and this is aligned with $\partial_i(\partial_{j-1}(k)) = \partial_i(k+1) = k+2$; (2.) If $i < k$ and $k+1 < j$, then $\partial_j(k+1) = k+1$ and we have $\partial_i(\partial_{j-1}(k)) = \partial_i(k) = k+1$, and if $i < k$ and $j \leq k+2$, then $\partial_j(k+1) = k+2$ and $\partial_i(\partial_{j-1}(k)) = \partial_i(k+1) = k+2$. \square

Lemma 1.2. Every morphism $f \neq 1_{[n]} \in \text{Hom}_\Delta([n], [m])$ can be written as a composition

$$f = \partial_{i_1} \circ \cdots \circ \partial_{i_r} \circ \sigma_{j_1} \circ \cdots \circ \sigma_{j_s}$$

with $0 \leq i_r < \cdots < i_1 \leq m$ and $0 \leq j_1 < \cdots < j_s < n$, where $m = n - s + r$. This decomposition is unique.

SECTION ω : HOMOTOPY THEORY AND HOMOLOGICAL ALGEBRA

Throughout this section, we will develop the notion of a *derived category*, but to remain real to the subject, we develop the definition of *derived categories*. Namely, given a "nice" category \mathcal{A} , we can associate its derived category, denoted $\mathbf{D}(\mathcal{A})$.

We hear this story quite frequently: We want to be able to do homological algebra in better and better settings. The abstraction usually begins with developing homological algebra in Ab or Mod_A for some given ring A , but then we abstract the main source of homological properties that are specific to these categories to develop an abstracted theory that instead of working in the categories just described, we work in an *abelian category*. Although I may be mistaken, the next jump of a more ideal setting to do homological algebra is in derived settings instead now. To develop the framework and definition is the goal of this section.

1.3. Abelian Categories.

1.3.1. Additive categories. As remarked in the introduction to this section, we take Mod_A for a ring A to be our motivation towards a notion of an abelian categories, and we define an *additive category* as a preliminary step, but these categories are in themselves interesting.

So, in the category Mod_A , given some A -modules M and N , the set of maps $\text{Hom}(M, N)$ forms an abelian group: we define a binary operation of addition on this set as we take some $f, g \in \text{Hom}(M, N)$ and define $(f+g)(x) := f(x)+g(x)$, which does indeed define abelian group. Furthermore, we have \mathbb{Z} -bilinear property with $\text{Hom}(M, N)$, that is, we have the following:

$$\begin{aligned} (f_1 + f_2) \circ g &= (f_1 \circ g) + (f_2 \circ g) \\ f \circ (g_1 + g_2) &= (f \circ g_1) + (f \circ g_2). \end{aligned}$$

We're essentially stating that we have that composition of maps in Mod_A distributes over addition. Additionally, the category Mod_A has a zero object, namely, the trivial module (0) , where given any A -module B , we have $\text{Hom}((0), B)$ consists of only a zero map where $0 \mapsto 0$ and it is terminal as $\text{Hom}(B, (0))$ consists of a zero map with $x \mapsto 0$ for all $x \in B$. Lastly, given any two A -modules M and N we have a product between them, i.e. we have another A -module $M \times N$. We make the following definition to generalize these properties.

Definition 1.4. A category \mathcal{A} is **additive** if it satisfies the following properties:

- (i) For $A, B \in \mathcal{A}$, $\text{Hom}_{\mathcal{A}}(A, B)$ is an abelian group, and composition of morphisms are \mathbb{Z} -bilinear, that is, the $\text{Hom}_{\mathcal{A}}(Y, Z) \times \text{Hom}_{\mathcal{A}}(X, Y) \rightarrow \text{Hom}_{\mathcal{A}}(X, Z)$ with $(g, f) \mapsto g \circ f$ are \mathbb{Z} -bilinear.
- (ii) There exists a zero object in \mathcal{A} , denoted 0 .
- (iii) For any $X, Y \in \mathcal{A}$, the categorical product $X \times Y$ exists in \mathcal{A} .

A functor $F: \mathcal{A} \rightarrow \mathcal{B}$ of additive categories is **additive** if the induced map $\text{Hom}_{\mathcal{A}}(A, B) \rightarrow \text{Hom}_{\mathcal{B}}(F(A), F(B))$ is a homomorphism of abelian groups for any $A, B \in \mathcal{A}$.

Example 1.1.

- The category of rings is not additive as the initial object of Ring is \mathbb{Z} which is not terminal as 0 is the terminal object in this category. (There are some other issues with this example being additive, but we just give one failure of it.)
- The category of modules over a ring A is additive, and thereby it follows that $\text{Ab} = \text{Mod}_{\mathbb{Z}}$ and $\text{Vec}_k = \text{Mod}_k$ for a field k are also additive categories.
- Consider the functor $F_N: \text{Ab} \rightarrow \text{Ab}$ where we have that N is a (normal) subgroup of G and map it to G/N and given $f: G \rightarrow H$ we have $F(f) = \bar{f}: G/N \rightarrow H/N$ and $\bar{f}: x + N \mapsto f(x) + N$. Then this describes an additive functor as $\text{Hom}(G, H) \rightarrow \text{Hom}(G/N, H/N)$ is described by the map $\pi(f + g) = \overline{f + g} = \bar{f} + \bar{g}$.

1.4. Abelian Categories and (co)Kernels. Starting in a similar fashion as to how the last section went, we're motivated by abstracting essentially one more concept from the category of modules. To define the co(homology) of a given complex, we need the notion of a kernel and image in Mod_A , i.e. if we have a (short) complex of A -modules

$$X^\bullet: 0 \rightarrow A^{i-1} \xrightarrow{f^{i-1}} A^i \xrightarrow{f^i} A^{i+1} \rightarrow 0$$

we define the i -th cohomology as $H^i(X^\bullet) = \ker f^i / \text{im } f^{i-1}$. (The homology group is defined in an analogous way where the maps are in descending order instead of the ascending as we've defined for the cohomology). As such, we abstract the notion of kernel and cokernel to abstract the notion of cohomology to other categories.

Definition 1.5. Let \mathcal{A} be a category with a 0-object. A **kernel** of a morphism $f: B \rightarrow C$ is a map $i: A \rightarrow B$ such that $f \circ i = 0$, and that is universal with respect to this property. The kernel is written $\ker f \rightarrow B$. As a diagram:

$$\begin{array}{ccccc} & Z & & & \\ & \downarrow \exists! & \searrow 0 & & \\ A & \xrightarrow{i} & B & \xrightarrow{f} & C \\ & \searrow & & \nearrow 0 & \end{array}$$

A **cokernel** of a morphism $f: B \rightarrow C$ is a map $j: B \rightarrow A$ such that $j \circ f = 0$, and is universal with respect to this property. The cokernel is written $B \rightarrow \text{coker } f$.

Lets look at the category of modules. We know that for the kernel of an A -linear map $f: A \rightarrow B$ we write $\ker f = \{x \in A: \varphi(x) = 0\}$ and $\text{coker } f = B / \text{im } f$, but we should actually check that these satisfy the properties of Definition 1.5. Clearly given some A -linear map $f: A \rightarrow B$ we have a map $i: \ker f \rightarrow A$ as this is just a subset of A , and $f \circ i(x) = f(x) = 0$ as $x \in \ker f$. Now suppose we have some other pair (j, Z) of Mod_A such that there is a map $j: Z \rightarrow A$ such that $f \circ j = 0$. Lastly, there must exist a unique map $Z \rightarrow A$

as $f \circ j = 0$ which means that it factors through the usual kernel of f , i.e. as $f(j(x)) = 0$ for all $x \in Z$ and $j(x) \in A$ then $j(x) \in \ker f$.

Exercise 1.2. Verify that $\operatorname{coker} \varphi = B / \operatorname{im} \varphi$ for an A -linear map $\varphi: A \rightarrow B$ satisfies the properties of Definition 1.5.

Definition 1.6. An **abelian category** is an additive category where

- (i) every morphism admits a kernel and a cokernel, and
- (ii) every monomorphism is a kernel and every epimorphism is a cokernel.

REFERENCES

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