# BABY STEPS TOWARDS ÉTALE COHMOLOGY

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Contents



# 1. Scheme Topics

## 1.1. **Projective Schemes.**

1.1.1. *Alternative Construction.* Some authors construct a projective scheme in the way done in [Hart77]; meaning that you do something analogous to the construction of an affine scheme. You define a Zariski topology on projective space where you're working with a graded ring, say, $S = \bigoplus_{d\geq 0} S_d$ (a direct sum of abelian groups) such that $S_d \cdot S_e \subset S_{d+e}$, and so on (you've probably already seen this before)—we will come back this construction soon. However there's is another way which requires something we need to talk about: gluing! With scheme theory we're allotted a lot flexibility as we're able to form new schemes out of of old ones by gluing. This contrast with our 'old' theory of varieties is much more optimistic as with classical varieties often enough gluing two varieties fails to be a variety.

**Proposition 1.1.** Let $\{U_i\}$ be an open covering of a topological space $X$, suppose $\mathscr{F}_i$ is a sheaf on $U_i$ and $f_{ij}\colon \mathscr{F}_i|_{U_i\cap U_j} \to \mathscr{F}_j|_{U_i\cap U_j}$ are isomorphisms satisfying the cocycle conditions $f_{ij} = \mathrm{id}$ and $f_{ik} = f_{jk} \circ f_{ij}$ for all $i, j, k$. Then there exists a unique sheaf $\mathscr{F}$ on $X$ and isomorphisms $g_i\colon \mathscr{F}|_{U_i} \to \mathscr{F}_i$ such that $g_j = f_{ij} \circ g_i$ on $U_i \cap U_j$.

*Proof.* We construct such a sheaf $\mathscr{F}$ given gluing data. Let $\mathscr{F}$ be given by $\mathscr{F}(U) = \bigsqcup_i \mathscr{F}_i(U\cap U_i)/\sim$ where $s_i \sim s_j$ if and only if $s_i|_V = s_j|_V$ where $V := U \cap (U_i \cap U_j)$ for any pair $i, j$. The purpose for the cocycle conditions are to ensure that $\sim$ does indeed define an equivalence relation. It follows from our hypotheses that $\mathscr{F}$ is a sheaf, and the projection maps $\mathscr{F} \to \mathscr{F}_i$ induce morphisms $g_i\colon \mathscr{F}_i|_U \to \mathscr{F}_i$ that are isomorphisms by construction, and lastly $\mathscr{F}$ is thus uniquely determined by the data of $\mathscr{F}_i$. □

**Proposition 1.2.** Let $X_i$ be a collection of $S$-schemes, and for each pair of indices $(i, j)$, an open subscheme $X_{ij}$ (ref. §2.2) and an isomorphism of $S$-schemes $f_{ij}\colon X_{ij} \to X_{ji}$ subject to the following:

(i) $f_{ii} = \mathrm{id}_{X_i}$;

(ii) $f_{ij} = f_{ji}^{-1}$ and $f_{ij}(X_{ij} \cap X_{ik}) = X_{ji} \cap X_{jk}$;
(iii) $f_{ik} = f_{jk} \circ f_{ij}$ on $X_{ij} \cap X_{ik}$.

Then there exists an $S$-scheme $X$, unique up to isomorphism, with a covering of open $S$-subschemes isomorphic to $\{X_i\}$ such that the maps $f_{ij}$ on $X_{ij}$ correspond to the identity map in the images of $X_{ij}$ in $X$.

*Proof.* We write $X = \bigsqcup_i X_i/\sim$ where the relation is given by $x_i \sim x_j$ if and only if $x_i \in X_i$ and $x_j \in X_j$ such that $f(x_{ij}) = x_j$. By hypothesis, the three conditions (i)-(iii) give us that this relation is indeed an equivalence relation. Now we have $g_i \colon X_i \to X$ being an inclusion and $g_i = g_j \circ f_{ij}$ on $X_i \cap X_j$ and by Proposition 1.1 we can glue along ${g_i}_* \mathscr{O}_{X_i} = U_i$ and $\{f_{ij}\}$, which gives us $(X_i, \mathscr{O}_X)$ being a ringed space. Yet for $P \in X_i$ we have $\mathscr{O}_{X_i,P} \simeq \mathscr{O}_{X,P}$, and so we have indeed have that $(X, \mathscr{O}_X)$ is local. For affine opens $U_{ij} \subset X_i$, the image $\{U_i\}$ of $X$ form an open affine covering of $X$, making $X$ a scheme. Lastly, for the maps $g_i \colon U_i \to X_i$ we obtain isomorphisms, for which $\pi_i \colon X_i \to S$ gives us a family of morphisms that we can glue $\pi_i \circ g_i^{-1}$ that yield a map $X \to S$. Now we are done and uniqueness follows from construction of $X$. □

**Example 1.1.** We construct $\mathbf{P}^1$ by gluing: Let $X = \operatorname{Spec} A[\frac{x_0}{x_1}]$ and $Y = \operatorname{Spec} A[\frac{x_1}{x_0}]$ (as subrings of $A[x_0, x_1, \frac{1}{x_0}, \frac{1}{x_1}]$). There is a clear map $\pi \colon X \to Y$ induced by the morphism at level of rings given by $\frac{x_0}{x_1} \mapsto \frac{x_1}{x_0}$. We glue along distinguished open sets $D(\frac{x_0}{x_1}) \subset X$ and $D(\frac{x_1}{x_0}) \subset Y$ and the morphism $A[\frac{x_0}{x_1}]_{x_0/x_1} \to A[\frac{x_1}{x_0}]_{x_1/x_0}$. By Proposition 1.2, we obtain a new scheme which we denote by $\mathbf{P}^1_A$ and call it the *projective line* over $A$.

This construction of $\mathbf{P}^1$ with $n = 1$ can be generalized to the case of higher values of $n$, but this becomes almost immediately gross to work with, so we revert back to the construction that's analogous to the construction of $\operatorname{Spec} R$ as a scheme for some ring $R$.

1.1.2. *Canonical Construction.* Let $S$ be graded ring, that is, $S$ can decomposed into a direct summand of abelian groups $S = \bigoplus_{d \geq 0} S_d$, such that $S_d \cdot S_e \subset S_{d+e}$; an element in $S_d$ is said to be *homogenous of degree* $d$. Note that $S_0$ is a subring of $S$ as $S_0 \cdot S_0 \subset S_0$ is closed by multiplication and it takes some effort to show that $1 \in S_0$, and moreover each $S_d$ is an $S_0$-module.

An element $x \in S$ can be written as a unique sum $\sum_{d \in \mathbf{Z}_{\geq 0}} x_d$ with $x_d \in S_d$ being called *homogeneous components*, and $x$ is called a *homogeneous element* of $S$, and the elements of $S_n$, specifically, are said to be *homogeneous* of degree $n$ where we write $\deg x = n$ when $x \in S_n$. An ideal of $I$ of $S$ is called *homogeneous* if it can be generated by homogeneous elements.

Lastly, we define the ideal $S_+ := \bigoplus_{d>0} S_d$ and it is called the *irrelevant ideal* of $S$. Our usual examples of graded rings are $k[x]$, as we can decompose $k[x] = \bigoplus_{n \in \mathbf{Z}} kx^n$ where $kx^n = 0$ if $n < 0$ and $k[x] = \cdots 0 \oplus 0 \oplus k \oplus kx \oplus kx^2 \oplus kx^3 \oplus \cdots$ [1], and also $k[x, y]$ as we can write $k[x, y] = \bigoplus_{(i,j) \in \mathbf{Z} \times \mathbf{Z}} kx^i y^j$ where $kx^i y^j = 0$ if $i, j < 0$. Note that the irrelevant ideal of $k[x_0, \ldots, x_n]$ is $(x_0, \ldots, x_n)$. An ideal $\mathfrak{a}$ of $S$ is said to be *homogeneous* if every homogeneous component of each element of $\mathfrak{a}$ is in $\mathfrak{a}$.

A homomorphism between graded rings $\varphi \colon R \to S$ is a ring homomorphism $\varphi(R_n) \subset S_n$ for all $n$. We can form the category of graded rings $\mathsf{GrRings}$ with graded rings as objects and morphisms as graded ring morphism.

We've talked about when we have a graded ring $S = \bigoplus_{d \geq 0} S_d$, but we need to consider when $d \in \mathbf{Z}$. Let $S$ be such that $S = \bigoplus_{n \in \mathbf{Z}} S_n$; these kinds of graded rings are often called $\mathbf{Z}$-*graded rings*, and they in particular appear when talking about localizations of graded rings. Let $T$ be a multiplicative system of $S$ whose elements are all homogeneous. We define a grading on $T^{-1}S$ for $t \in T$ and $g$ a homogeneous element of $S$ by letting $\deg g/t = \deg g - \deg t$; that is,

$$(T^{-1}S)_n = \left\{ \frac{f}{t} \in T^{-1}S \colon f \in S_n, t \in T, \text{ and } \deg f - \deg t = n \right\},$$

and the localized ring $T^{-1}S$ decomposes as the direct sum $T^{-1}S = \bigoplus_{n \in \mathbf{Z}} (T^{-1}S)_n$ making $T^{-1}S$ a $\mathbf{Z}$-graded ring.

[1] You can rewrite this decomposition into principal ideals of $k[x]$ and $k[x] = \cdots 0 \oplus 0 \oplus (k) \oplus (x) \oplus (x^2) \oplus (x^3) \oplus \cdots$ so it is especially clear that $(x^n)(x^m) \subset (x^{n+m})$.

Finally, we give a topology to a graded ring in a similar way done to a typical ring. Take $S$ a graded ring, and denote $\operatorname{Proj} S$ to be the set of homogeneous ideals of $S$ that don't contain the irrelevant ideal $S_+$. Now for a homogeneous ideal $\mathfrak{a}$ of $S$, we define the subset $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Proj} S \colon \mathfrak{p} \supseteq \mathfrak{a}\}$, and we endow $\operatorname{Proj} S$ with the *Zariski topology* by declaring these to be closed set. Similar identities hold like in the (affine) Zariski topology such as $V(\sum \mathfrak{a}_i) = V(\mathfrak{a}_i)$, $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$, and $V(\sqrt{\mathfrak{a}}) = V(\mathfrak{a})$ where $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{a}_i$ are homogeneous ideals of $S$. Furthermore, we define for any homogeneous $f \in S_+$ the set $D_+(f) := \{\mathfrak{p} \in \operatorname{Proj} S \colon f \notin \mathfrak{p}\} = \operatorname{Proj} S \setminus V((f))$, where $(f)$ is the homogeneous ideal generated by $f$ in $S$. These sets are open and such sets form a base for the topology on $\operatorname{Proj} S$. We define a structure sheaf $\mathscr{O}_{\operatorname{Proj} S}$ on $\operatorname{Proj} S$ (p. 76, [Hart77]) with the following properties: (i) For $\mathfrak{p} \in \operatorname{Proj} S$, $\mathscr{O}_{\operatorname{Proj} S, \mathfrak{p}} \simeq S_{(\mathfrak{p})}$, where $S_{(\mathfrak{p})}$ denotes the elements of degree zero in the localization of $S$ with respect to the multiplicative set $T = S \setminus \mathfrak{p}$; (ii) For a given homogeneous $f \in S_+$, $\mathscr{O}_{\operatorname{Proj} S}(D_+(f)) \simeq S_{(f)}$, where $S_{(f)}$ is the ring of degree zero elements in the localization of $S_f$.

Referring back to the initial part of this section, we define $\mathbf{P}^n_k = \operatorname{Proj} k[x_0, \dots, x_n]$ to be *projective space* over $k$. Lastly a *projective variety* over $k$ is a scheme $\operatorname{Proj} S$, where $S = k[x_0, \dots, x_n]/\mathfrak{I}$ where $\mathfrak{I}$ is a homogeneous ideal (corresponding to a family of homogeneous polynomial equations $f_i(x_0, \dots, x_n)$ with $f_i$ being homogeneous).

In a specific case of $S = A[t]$ where $A$ is a ring and the grading has $\deg t = 1$ and $\deg x = 0$ for all $x \in A$. The irrelevant ideal of $A[t]$ is $(t)$, making no homogeneous ideal of $\operatorname{Proj} S$ contain $S_+ = (t)$, and $D_+(t)$ is all of $\operatorname{Proj} S$. Now $D_+(t) \simeq \operatorname{Spec} S_{(t)} = \operatorname{Spec} A[t]_{(t)} := \operatorname{Spec}((A[t]_t)_0)$ and as $A[t]_{(t)} = A$, we get that $\operatorname{Proj} A[t] = \operatorname{Spec} A$.

## 1.2. Noetherian & Dimension.

1.2.1. *Noetherian.* In later sections, we will be almost exclusively be working with schemes that are *noetherian*; this section is meant to serve as a refresher and also fix some notation that will be relevant.

**Definition 1.1.** A scheme $X$ is said to be be **locally noetherian** if it can be covered by open affine subsets $\operatorname{Spec} A_i$, where each $A_i$ is a noetherian ring.[2] The scheme $X$ is **noetherian** if it is locally noetherian and quasi-compact; that is, $X$ can be covered by a finite number of open affine subsets $\operatorname{Spec} A_i$ where each $A_i$ is a noetherian ring.

**Example 1.2.**

- An affine scheme $\operatorname{Spec} A$ where $A$ is noetherian is a noetherian scheme.
- For a morphism of schemes $f \colon X \to S$ where $S$ is noetherian and $f$ is of finite type we have X being noetherian as well. In particular, any variety of noetherian (see §1.3).

**Definition 1.2.** A scheme $X$ is **connected** if the underlying topological space of $X$ is not a disjoint union of two open subsets. A scheme $X$ is **irreducible** if the underlying topological of space is not a union of two proper closed subsets, and $X$ is **reducible** if $X$ is not irreducible. The scheme $X$ is **reduced** if for every open $U \subset X$, the ring $\mathscr{O}_X(U)$ has no nilpotent elements. Lastly $X$ is **integral** if for every open $U \subset X$, the ring $\mathscr{O}_X(U)$ is an integral domain.

**Example 1.3.** The affine scheme $\operatorname{Spec} k[x, y]/(xy)$ of $\mathbf{A}^2_k$ is reducible. The affine scheme $\operatorname{Spec} k[x, y]/(y^2 - x^2(x + 1))$ is irreducible.

**Theorem 1.1.** The ideal $\sqrt{(0)} \subset \mathrm{R}$ is prime if and only if $\operatorname{Spec} R$ is irreducible. In particular, if $R$ is an integral domain, then $\operatorname{Spec} R$ is irreducible.

**Lemma 1.1.** Let $X$ be an irreducible scheme. Then there is a unique point $\eta \in X$ with $\overline{\eta} = X$. (Such a point $\eta \in X$ is called the *generic point* of X.)

*Proof.* Recall that the closure of a set has many formulation in general topology; two of these being that $\overline{\eta}$ is equivalent to the following: (1) The closure $\overline{\eta}$ is the smallest closed set contain $\eta$, (2) The closure $\overline{\eta}$ is the intersection of all closed sets containing $\eta$. Let $\mathfrak{q}$ and $\mathfrak{p}$ be two points in

[2]The ring $A_i$ is noetherian, according to the ring theoretic definition, if for any non empty ascending chain of ideas is stationary, or, alternatively, if any non empty family of ideals has a maximal element. A result in ring theory worth noting is that a ring $R$ is noetherian if any ideal of $R$ is finitely generated.

$\operatorname{Spec} A$ with $\mathfrak{q} \in \overline{\mathfrak{p}}$ (i.e. $\mathfrak{p} \subset \mathfrak{q}$). Now if $X = \operatorname{Spec} A$ is affine then $\eta = \sqrt{(0)}$ works, and is unique. In generality, if we have $V = \operatorname{Spec} B, U = \operatorname{Spec} A \subset X$ are affine open subsets of $X$ with generic points $\eta_A$ and $\eta_B$. Observe that any nontrivial open set of $X$ is dense, so then $\overline{\eta_A} = X = \operatorname{Spec} A$. It remains to check uniqueness. Note that if $\operatorname{Spec} B \subset \operatorname{Spec} A$, then $\eta_B = \eta_A$ (as hinted in the second sentence). Yet, more generally, if $\operatorname{Spec} R \subset \operatorname{Spec} A \cap \operatorname{Spec} B \subset X$ then $\eta_A = \eta_B = \eta_R$. □

In the case where $X$ is integral, then $\eta \in \operatorname{Spec} A \subset X$ is simply the point corresponding to the trivial ideal $(0) \subset A$, and the local ring $\mathscr{O}_{X,\eta} := k(X)$ is called the *function field* of $X$. As an example, consider $X = \operatorname{Spec} \mathbf{Z}$, which clearly integral. Then $\eta \leftrightsquigarrow (0) \subset \mathbf{Z}$ and $\mathscr{O}_{\operatorname{Spec} \mathbf{Z},(0)} = \mathbf{Q} = k(X)$.

1.2.2. *Dimension of a scheme.*

**Definition 1.3.** Let $X$ be a scheme. The **dimension** of $X$, denoted $\dim X$, is the supremum (possibly $+\infty$) of all integers $n$ such that there exists a chain

$$Y_0 \subset Y_1 \subset \cdots \subset Y_n$$

of distinct irreducible closed subsets of $X$.

**Example 1.4.**

(i) For a ring $A$, $\dim \operatorname{Spec} A[x_1, \ldots, x_n] = \dim \mathbf{A}_A^n = \dim A + n$.
(ii) The dimension of $\operatorname{Spec} \mathbf{Z}$ is one.
(iii) As $k$ is a field, then $\operatorname{Spec} k$ has just the trivial prime ideal, and so $\operatorname{Spec} k$ is topologically just a point. The dimension of $\operatorname{Spec} k$ is zero, and $\dim \operatorname{Spec} k[\epsilon] = 0$ as well.

From commutative algebra, we define a *height* of a prime ideal $\mathfrak{p}$ of a ring $A$ to be the greatest length $t \in \mathbf{Z}_{\geq 0}$ of a chain of distinct prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_t = \mathfrak{p},$$

and we denote the height of $\mathfrak{p}$ by $\operatorname{ht} \mathfrak{p} = t$. Furthermore, for the ring $A$, we define its *Krull dimension* to be the $\sup\{\operatorname{ht} \mathfrak{p} \colon \mathfrak{p} \in \operatorname{Spec} A\}$. In particular, the height of a prime ideal $\mathfrak{q}$ for some ring $A$ is the dimension of the localized ring $R_{\mathfrak{q}}$.

**Theorem 1.2** (Nagata)**.** Let $A$ be a ring and $\mathfrak{p}$ be a prime ideal of $A$. Then each of the following are sufficient suppositions for $\dim A = \operatorname{ht}(p) + \dim A/\mathfrak{p}$ to be true:

(a) If $A$ is an integral domain that is a finitely generated algebra over a field.
(b) If $A$ is a noetherian local ring with maximal ideal $\mathfrak{m}$ and $q \in \mathfrak{m}$ is not a zero-divisor then $\dim A/(q) = \dim A - 1$.
(c) The ring $A$ is a Cohen-Macaulay local ring.
(d) If $A$ is a ring of finite dimension in which all maximal chains of prime ideals have the same height.

**Example 1.5.** If $k$ is a field, then $k[x_1, \ldots, x_n]$ satisfies condition $(a)$ and any quotient of $k[x_1, \ldots, x_n]/I$ indeed does as well. The ring $\mathbf{Z}$ and $\mathbf{Z}_\ell$ satisfy condition $(d)$. We leave it up to the reader to establish examples of $(b)$ and $(c)$—some of our stated examples satisfy these conditions as well though.

As in the case of $\mathbf{Z}$, we know that its maximal ideals are the same as prime ideals as $\mathbf{Z}$ is a principal ideal domain, and $\operatorname{Spec} \mathbf{Z}$ looks like a "line" where there are countably many points of the form $(p)$ where $p$ is prime in $\mathbf{Z}$ and the prime ideal $(0)$ is the generic point on the line. The height of any prime ideal $(p)$ is given by the chain $(0) \subset (p)$, and as $(p)$ is maximal then 1 must be the height of $(p)$. Thus we can see that $\dim \mathbf{Z} = 1$. More generally, for any principal idea domain that is not a field, the Krull dimension is 1. Recall that $k$ is a field if and only if $k[x]$ is a principal ideal domain, and so $\dim k[x] = 1$ (and the Krull dimension of a field $k$ itself is 0 as the trivial ideal $(0)$ is only prime ideal of its spectrum).

**Proposition 1.3.** For an affine scheme $\operatorname{Spec} A$ for a ring $A$, the dimension of $\operatorname{Spec} A$ is the same as the Krull dimension of $A$.

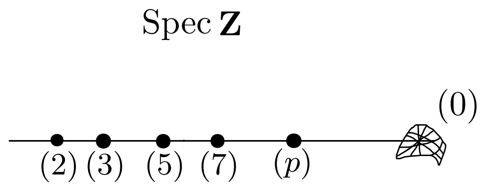


FIGURE 1. A doodle of Spec $\mathbf{Z}$, which took the author over an hour to make latex useable. We have all closed points of Spec $\mathbf{Z}$ (that is, primes $(p)$ with $p > 0$) on the line with the additional *generic point* (0); the reason to draw (0) as it is (although there is really is no 'correct way to depict it [Vakil22]) is because geometrically there's more to it than the other points ... geometrically all points are contained in (0) as we have the inclusion reversing bijection.

*Proof.* The closed irreducible subsets of $\operatorname{Spec} A$ are sets of the form $V(\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of $A$. As we have incidence-reversing bijections between prime ideals of $A$ and irreducible closed subsets of $\operatorname{Spec} A$, then the claim clearly follows. □

**Proposition 1.4.** Let $X$ be a topological space.

(i) If $Y \subset X$ is a subspace of $X$, then $\dim Y \leq \dim X$. In particular, if $X$ is irreducible with finite dimension and $Y$ is properly contained in $X$, then $\dim Y < \dim X$.

(ii) If $X$ is covered by open (or closed) sets $U_i$, then $\dim X = \sup_i \dim U_i$.

(iii) If $\dim X$ is finite, and $Y \subseteq X$ is a closed irreducible subset such that $\dim Y = \dim X$, then $X = Y$.

(iv) For a scheme $X$, $\dim X = \sup_{x \in X} \dim \mathscr{O}_{X,x}$.

**Definition 1.4.** Let $X$ be a scheme and $Y$ an irreducible closed subset of $X$. The **codimension** $\operatorname{codim}(Y, X)$ of $Y$ in $X$ is the supremum of all integers $n$ such that there exists a chain of distinct closed subsets

$$Y = Y_0 \subset Y_1 \subset \cdots \subset Y_n$$

**Example 1.6.** For $A = \mathbf{Z}$, codimension of an irreducible closed subset $V(\mathfrak{p}) \subset \operatorname{Spec} A$ is the height of the prime ideal $\mathfrak{p}$. Then for the irreducible closed subset $V((2))$ we have $\operatorname{codim}(V((2)), \operatorname{Spec} \mathbf{Z}) = 1$. More generally, for this example, $\operatorname{codim}(V(\mathfrak{p})), \operatorname{Spec} \mathbf{Z}) = 1$ for $\mathfrak{p} > 0$, and for $\mathfrak{p} = 0$, $\operatorname{codim}(V(\mathfrak{p}), \operatorname{Spec} \mathbf{Z}) = 0$ as $\operatorname{ht}((0)) = 0$.

**Proposition 1.5.** For an affine scheme $X = \operatorname{Spec} A$ and irreducible closed subset $V(p)$ of $X$ we have $\operatorname{ht}(p) = \operatorname{codim}(V(p), \operatorname{Spec} A)$.

*Proof.* Let $\mathfrak{p}$ be a prime ideal of $\operatorname{Spec} A$. Consider a chain $V(\mathfrak{p}) = Z_0 \subset Z_1 \subset \cdots \subset Z_n$, for which $V(\mathfrak{p}_i) = Z_i$ for some prime ideals $\mathfrak{p}_i$ of $A$ for $i \geq 0$. As we have a reverse bijection between we get a chain $\mathfrak{p}_n \subset \mathfrak{p}_{n-1} \subset \cdots \mathfrak{p}_1 \subset \mathfrak{p}_0 = \mathfrak{p}$. □

**Proposition 1.6.** Let $X$ be a scheme. Let $x \in X$ be a point and set $Z = \overline{\{x\}}$. Then $\dim \mathscr{O}_{X,x} = \operatorname{codim}(Z, X)$.

### 1.3. Types of Subschemes and Immersions.

**Definition 1.5.** An **open subscheme** of a scheme $X$ is a scheme $U$, where $U \subseteq X$ is an open subset, and whose structure sheaf $\mathscr{O}_U$ is isomorphic to the restriction sheaf $\mathscr{O}_X|_U$ of $X$. An **open immersion** is a morphism $f \colon X \to Y$ that induces an isomorphism of $X$ with an open subscheme of $Y$.

Alternatively, although this is the definition provided in [Hart77], we can perhaps choose a more *immediate* definition. Let $X$ be a scheme and let $U \subseteq X$ be an open set. Then $(U, \mathscr{O}_X|_U)$ is a scheme which we call an *open subscheme* of $X$. Then the natural morphism $\ell \colon U \to X$, where $\ell^\sharp \colon \mathscr{O}_X \to \ell_* \mathscr{O}_X|_U$ is called an *open immersion.* Recall that the pullback $\ell^\sharp$ here means that for any open set $V \subseteq X$, we have $\ell_* \mathscr{O}_X|_U(V)$ corresponding to $\mathscr{O}_X|_U(\ell^{-1}(V))$. In particular, recall that the open sets $D(f)$ of $\operatorname{Spec} A$, where $f \in A$ and $A$ is a ring, form a basis for the topology, and furthermore $D(f)$ is quasi compact (which means that $\operatorname{Spec} A$ is quasi-compact as $D(1) = \operatorname{Spec} A$).

Additionally, the open subset $D(f) \subseteq \operatorname{Spec} A$ can be canonically be identified to the spectrum $\operatorname{Spec} A_f$ (which we think of being a directly corresponding open subscheme of $\operatorname{Spec} A$). More formally, $(D(f), \mathscr{O}_{\operatorname{Spec} A}|_{D(f)}) \simeq (\operatorname{Spec} A_f, \mathscr{O}_{\operatorname{Spec} A_f})$. Thus we essentially classify open immersions to be (locally) of the form $\operatorname{Spec} A_f \to \operatorname{Spec} A$.

While this notion of open subschemes and open immersions is rather intuitive, the definition of what a closed subscheme and closed immersions will be comparably more difficult due to the fact that we have to define the locally ringed space structure on a closed set, for which there is no canonical choice.

**Definition 1.6.** A **closed immersion** is a morphism $f\colon Y \to X$ of schemes such that $f$ induces a homeomorphism of $|Y|$ onto a closed subset of $|X|$, and furthermore the induced map $f^\sharp\colon \mathscr{O}_X \to f_*\mathscr{O}_Y$ of sheaves on $X$ is surjective. A **closed subscheme** of a scheme X is an equivalence class of closed immersions, where we say $f\colon Y \to X$ and $f'\colon Y' \to X$ are equivalent if there is an isomorphism $i\colon Y' \to Y$ such that $f' = f \circ i$.

**Lemma 1.2** ([SP], Tag 00E5)**.** Let $R$ be a ring. Let $I$ be an ideal of $R$. The map $R \to R/I$ induces via the functoriality of Spec a homeomorphism

$$\operatorname{Spec}(R/I) \to V(I) \subseteq \operatorname{Spec}(R).$$

The inverse is given by $\mathfrak{p} \mapsto \mathfrak{p}/I$.

**Example 1.7.** Let $A$ be a ring and let $J$ be an ideal of $A$. Then the natural map $\pi\colon A \to A/J$ induces a map of schemes $\pi^\sharp = \operatorname{Spec}(\pi)\colon \operatorname{Spec} A/J \to \operatorname{Spec} A$ which is a closed immersion.[3] The map $\pi^\sharp$ induces a homeomorphism of $\operatorname{Spec} A/J$ onto $V(J)$ of $\operatorname{Spec} A$—the correspondence theorem in commutative algebra gives us that the prime ideals of $A/J$ correspond to prime ideals of $A$ that contain $J$, i.e. $V(J)$ (in fact, it is not too hard to show that if $\mathfrak{q}$ is any ideal of $A$, then $A/\pi^{-1}(\mathfrak{q}) \simeq (A/J)/\mathfrak{q}$). Furthermore, the map of structure sheaves $\mathscr{O}_{\operatorname{Spec} A} \to \pi_*^\sharp \mathscr{O}_{\operatorname{Spec} A/J}$ is surjective as it is surjective on stalks (on stalks, we have localizations of $A$ and $A/J$, respectively). In general, if $\mathfrak{J}$ is any ideal of $A$, then we produce a closed subscheme on a closed set $V(\mathfrak{J})$ of $\operatorname{Spec} A$. To quote directly, "In particular, every closed subset [$\operatorname{Spec} A/\mathfrak{J}$] of $\operatorname{Spec} A$ has many closed subscheme structures, corresponding to all the ideals [$\mathfrak{J}$] for which [$V(\mathfrak{J}) = \operatorname{Spec} A/\mathfrak{J}$]. In fact, every closed subscheme structure on a closed subset [$\operatorname{Spec} A/\mathfrak{J}$] of an affine scheme [$\operatorname{Spec} A$] arises from an ideal in this way." ([Hart77], 85)

In accordance with this example, a definition in line with this is the following definition will be the one as given in [Vakil22], and the one that we will stick with as the author believes it to be frankly a better definition.[4] However, we need to introduce the notion of an *affine morphism* beforehand.

**Definition 1.7.** A morphism $f\colon X \to Y$ is **affine** if for every affine open set $U$ of $Y$, then $f^{-1}(U)$ (interpreted as an open subscheme of $X$) is an affine scheme. The map $f$ is said to be **finite** if it is affine and if for each affine open set $\operatorname{Spec} B$ of $Y$, $f^{-1}(\operatorname{Spec} B)$ is the sprectrum of a finite $B$-algebra.

**Definition 1.8** ([Vakil22], 9.1.1)**.** A morphism $\pi\colon X \to Y$ is a **closed embedding** (or **closed immersion**) if it is an affine morphism, and for every affine open subset $\operatorname{Spec} B \subset Y$, with $\pi^{-1}(\operatorname{Spec} B) \simeq \operatorname{Spec} A$, the map $B \to A$ is surjective (i.e., of the form $B \to B/I$ our desired local model). We often say that $X$ is a **closed subscheme** of $Y$. A closed embedding is the same thing as an isomorphism with a closed subscheme.

**Theorem 1.3.** Let $X$ and $Y$ be two schemes over $S$ with morphisms $f\colon X \to S$ and $g\colon Y \to S$. Then the fibred product

[3] Recall that that if $\varphi\colon R \to S$ is a map of ring then we have induced map of schemes where $\operatorname{Spec}(\varphi)\colon \operatorname{Spec} S \to \operatorname{Spec} R$ is given by $\operatorname{Spec}(\varphi)\colon \mathfrak{p} \mapsto \varphi^{-1}(\mathfrak{p})$.

[4] It is given as a hard exercise in [Vakil22] to show that Definition 1.6 and that Definition 1.8 in fact are equivalent. In II.II.5 [Hart77], we later on get a description that is close to this one in form a corollary that says given any affine scheme $\operatorname{Spec} A$ there is a $1-1$ correspondence between ideals $\mathfrak{a}$ of $A$ and closed subschemes $Y$ of $\operatorname{Spec} A$, given by $\mathfrak{a} \mapsto \operatorname{Spec} A/\mathfrak{a}$ in $X$; in particular, every closed subscheme of $\operatorname{Spec} A$ is affine.

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ X \times_S Y \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & S \\ & X \arrow[ur, "g"'] & \end{tikzcd}$$

exists in the category of schemes, and is unique up to isomorphism.

This statement is easy for the case of affine schemes as once you reverse the arrows you can quickly recognize that the resulting fibred product is constructed by tensoring, but for the general case it gets quickly complicated. The idea is to work locally and then start patching, that is, for $f: X \to S$ and $g: Y \to S$ we write $S = \bigcup \operatorname{Spec} T_i$ and then start taking preimages and begin to glue with a large amount of gluing data. For clarity, and for no hand waving, the reader should consult Theorem 3.3. in II.II.3 of [Hart77] or Theorem 10.1.1 in III.10.1 of [Vakil22].

**Corollary 1.1.** Let $X$ and $Y$ be $S$-schemes. Let $S = \bigcup_{i\in I} U_i$ be any affine open covering of $S$. For each $i \in I$, let $f^{-1}(U_i) = \bigcup_{j\in J_i} V_j$ be an affine open covering of $f^{-1}(U_i)$ and $g^{-1}(U_i) = \bigcup_{k\in K_i} W_k$ be an affine open covering of $g^{-1}(U_i)$. Then the following is an affine open covering for $X \times_S Y$:

$$X \times_S Y = \bigcup_{i\in I} \bigcup_{j\in J_i, k\in K_i} V_j \times_{U_i} W_k.$$

**Example 1.8.** Let $k$ be a field. For $k \to k[x]$ we have an induced morphism of schemes $\mathbf{A}^1_k \to \operatorname{Spec} k$ which gives $\mathbf{A}^1_k \times_{\operatorname{Spec} k} \mathbf{A}^1_k \simeq \operatorname{Spec}(k[x] \otimes_k k[y]) \simeq \operatorname{Spec} k[x, y] = \mathbf{A}^2_k$. However, the underlying topological space of $\mathbf{A}^2_k = \operatorname{Spec} k[x, y]$ is not the same as the product of the topological space $\mathbf{A}^1_k$ with itself since $\mathbf{A}^2_k$ has more points.

**Lemma 1.3.** Let $X, Y$, and $Z$ be three schemes over a base scheme $S$.

(a) $X \times_S S \simeq X$;
(b) $X \times_S Y \simeq Y \times_S X$;
(c) $(X \times_S Y) \times_S Z \simeq X \times_S (Y \times_S Z)$.

*Proof.* These follow from the uniqueness of the fibre product. □

**Definition 1.9.** Let $X$ be a scheme over $S$. Let $S' \to S$ be a morphism of schemes. The **base change** of $X$ is the scheme $X_{S'} = S' \times_S X$ over $S'$. If $f: X \to Y$ is morphism of schemes over $S$ and $S' \to S$ is another morphism of schemes, the **base change of** $f$ is the induced morphism $f' = id_{S'} \times_{\mathrm{id}_S} f: X_{S'} \to Y_{S'}$.

**Example 1.9.** Let $X = \operatorname{Spec} \mathbf{Q}[x, y]/(x^2 + y^2 - 1)$ and $Y = \operatorname{Spec} \mathbf{Q}[x, y]/(x^2 + y^2 + 1)$. If $S = \mathbf{Q}(i)$, then the base change admits $X_S \simeq Y_S$, but yet $X$ and $Y$ are not isomorphic themselves as $X$ admits a $\mathbf{Q}$-algebra homomorphism while $Y$ doesn't.

**Proposition 1.7.** Let $f: X \to Y$ be a closed immersion.

(i) If $g: Y \to Z$ is a closed immersion, then $g \circ f$ is a closed immersion.
(ii) For any $Y \to Y'$, the induced map $X \times_Y Y' \to Y'$ is also a closed immersion.
(iii) $f: X \to Y$ is affine local on the target; that is, for any affine open $V \subset Y$, the restriction $f^{-1}(V) \to V$ is also a closed immersion, and if there is an affine open cover $\{V_i\}$ of $Y$ for which $f^{-1}(V_i) \to V_i$ is a closed immersion then $f$ is a closed immersion.

These sorts of characteristics of morphisms as described in the preceding proposition are common, i.e. a morphism that is *stable* under composition, base change, and is affine local on target. We ascribe the following definition to capture these classes of morphisms.

**Definition 1.10.** Let $\mathscr{P}$ be a family of morphism of schemes. We say $\mathscr{P}$ is a **stable class** if the following hold:

(i) If $\pi: X \to Y$ and $\rho: Y \to Z$ are in $\mathscr{P}$, then so is their composition, i.e. $\rho \circ \pi: X \to Z$ is a morphism in $\mathscr{P}$.

(ii) If $\pi\colon X \to Y$ is in $\mathscr{P}$, and $Y \to Y'$ is a morphism of schemes, then the induced map $X \times_Y Y' \to Y'$ is also in $\mathscr{P}$.
(iii) If $\pi\colon X \to Y$ is in $\mathscr{P}$, then for any open set $V \subset Y$, the restriction $\pi^{-1}(V) \to V$ is in $\mathscr{P}$. And for a given morphism $\pi\colon X \to Y$ of schemes, if there exists an open cover $\{V_i\}$ of $Y$ for which each restricted morphism $\pi^{-1}(V_i) \to V_i$ is in $\mathscr{P}$ then the morphism $\pi\colon X \to Y$ is also in $\mathscr{P}$.

**Example 1.10.**

(1) Immersions (whether closed or open) form a stable class.
(2) Quasi-compact morphisms form a stable class.
(3) Affine and finite morphisms form a stable
(4) (locally) of finite type and (locally) of finite presentation morphisms form a stable class
(5) Proper morphisms form a stable class.

**Definition 1.11.** Let $f\colon X \to Y$ be a morphism of schemes. The **diagonal morphism** is the unique morphism $\Delta_{X/Y}\colon X \to X \times_Y X$ whose composition with both projection maps $p_1, p_2 : X \times_Y X \to X$ is the identity map of $X \to X$. We say that the morphism $f$ is **separated** if the diagonal morphism $\Delta_{X/Y}$ is a closed immersion. In that case we also say $X$ is **separated over** $Y$. A scheme $X$ is separated if it is separated over $\operatorname{Spec} \mathbf{Z}$.

$$\begin{tikzcd}
 & & X \arrow[dr, "f"] & \\
X \arrow[r, dashed, "\Delta"] \arrow[urr, bend left] \arrow[drr, bend right] & X \times_Y X \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & Y \\
 & & X \arrow[ur, "f"'] &
\end{tikzcd}$$

**Lemma 1.4.** Let $f\colon X \to Y$ be a morphism of affine schemes, then $f$ is separated.

*Proof.* Write $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then $X \times_Y X = \operatorname{Spec}(A \otimes_B A)$, and the diagonal morphism corresponds to $A \otimes_B A \to A$, where $a \otimes b \mapsto ab$. This map is clearly surjective, and we have that $A \simeq A \otimes_B A/J$ for some ideal $J$ of $A \otimes_B A$. Hence $\Delta_{X/Y}$ is a closed immersion, and $f$ is thus separated. ◻

**Definition 1.12** (Varieties)**.** Let $X/k$ be a scheme over an algebraically closed field. An **affine variety** is an integral scheme of finite type over $k$, i.e. $X = \operatorname{Spec} A$ where $A$ is a finitely generated $k$-algebra with no zero divisors. A **prevariety** $X/k$ is an irreducible scheme of finite type over $k$ which has a finite affine covering consisting of affine varieties. Furthermore, a prevariety becomes a **variety** if $X/k$ is integral and separated of finite type over $k$.

**Definition 1.13.** Let $f\colon X \to Y$ be a morphism of schemes, and let $y \in Y$ be a point. Then we define the **fibre** of the morphism $f$ over the point $y$ to be the scheme $X_y = X \times_Y \operatorname{Spec} \kappa(y)$, where the canonical map $\operatorname{Spec} \kappa(y) \to Y$ is given by composition $\operatorname{Spec} \kappa(y) \to \operatorname{Spec} \mathscr{O}_{Y,y} \to Y$.

**Remark 1.1.** If we have a morphism of schemes $\pi\colon X \to S$, it is best to think the fibre $X_s$ of a point $s \in S$ as $\pi^{-1}(s)$ since $\pi^{-1}(s)$ is in fact homeomorphic to $X_s$. We give $\pi^{-1}(s)$ the structure of a $\kappa(s)$-scheme in our definition.

**Lemma 1.5.** Let $f\colon X \to Y$ be a morphism of schemes and let $y \in Y$. Then $X_y = X \times_Y \operatorname{Spec} \kappa(y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.

**Example 1.11.** Let $k = \overline{k}$. Consider the mapping $f\colon k[x] \to k[x,y,z]/(yz-x)$ given by composing with the inclusion $k[x] \to k[x,y,z]$ along with the surjection $k[x,y,z] \to k[x,y,z]/(yz-x)$. Then, geometrically, we have induced map of spectra $\operatorname{Spec} k[x,y,z]/(yz-x) \to \mathbf{A}^1_k$; let $X = \operatorname{Spec} k[x,y,z]/(yz-x)$. If we take some point $p \in \mathbf{A}^1_k$, then inspecting the fibre gives $X \times_{\mathbf{A}^1_k} k[x]/(x-p) \simeq \operatorname{Spec}[y,z]/(yz-p)$

We need the following lemma to prove the following propoistion, however, this lemma does require some work by talking about *integral* ring morphisms (that is, a ring map $A \to B$ where

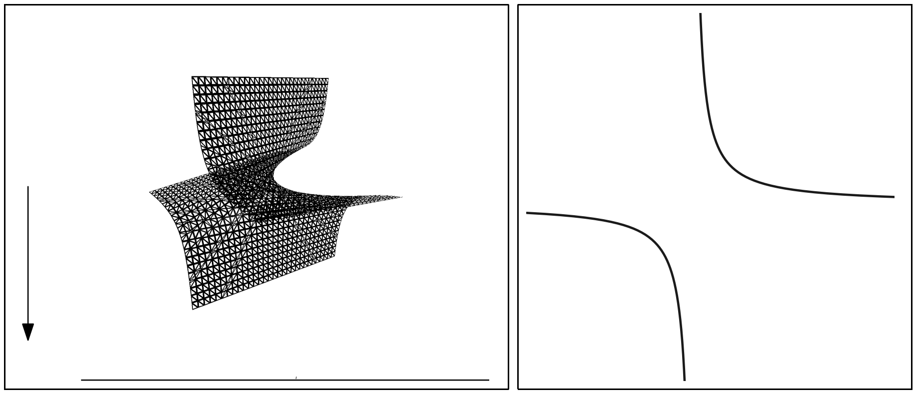

FIGURE 2. Depiction of the morphism $X \to \mathbf{A}^1_k$ on the left and $f^{-1}(p) \simeq \operatorname{Spec}[y,z]/(yz-p)$, with $p \neq 0$, on the right. If $p = 0$, then we have just $\operatorname{Spec}[y,z]/(yz)$ which is the union of the $z$ and $y$-axis.

every element of $B$ is the zero of a monic polynomial with coefficients in $A$) and the Going Up and Lying Over theorems; to learn about this we refer the reader to III.8.2. in [Vakil22]. But, in short, the Lying Over theorem geometrically means that if $A \to B$ is integral, then $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective—this translates as this as Lying Over says that given a integral morphism $A \to B$ then for any prime ideal $\mathfrak{q}$ of $A$, there is a prime ideal $\mathfrak{p}$ of $B$ such that $\mathfrak{p} \cap A = \mathfrak{q}$. A morphism of schemes $\pi\colon X \to Y$ is *integral* if $\pi$ is affine, and for any affine open $\operatorname{Spec} B \subset Y$ with $\pi^{-1}(\operatorname{Spec} B) = \operatorname{Spec} A$ the induced map $B \to A$ is an integral ring map.

**Lemma 1.6.** Let $A \neq 0$ be a finitely generated $k$-algebra, and $d \geq 0$ be an integer. Then $\dim A = d$ if and only if there exists a finite injective $k$-algebra homomorphism $k[x_1, \ldots, x_n] \to A$.

**Proposition 1.8.** For two integral schemes of finite type over $k$,

$$\dim X \times_{\operatorname{Spec} k} Y = \dim X + \dim Y.$$

*Proof.* Let both $(U_i)_i$ and $(V_i)_i$ be open affine coverings of and $X$ and $Y$. The product $(U_i \times V_j)_{i,j}$ is an open affine covering of $X \times_{\operatorname{Spec} k} Y$ by Corollary 1.1. Furthermore, by Proposition 1.4 (ii) we can let $X$ and $Y$ be both affine; let $\ell = \dim X$ and $k = \dim Y$. By Noether normalization, we have an induced finite injective homomorphism $k[x_1, \ldots, x_\ell] \to \Gamma(X, \mathscr{O}_X)$ and $k[x_{\ell+1}, \ldots, x_{\ell+k}] \to \Gamma(Y, \mathscr{O}_Y)$. Finally we tensor and get a map

$$k[x_1, \ldots, x_{\ell+k}] \to \Gamma(X, \mathscr{O}_X) \otimes_k \Gamma(Y, \mathscr{O}_Y) = \Gamma(X \times_{\operatorname{Spec} k} Y, \mathscr{O}_{X \times Y}),$$

which is finite and injective, and we can conclude the proposition after applying Lemma 1.6. $\square$

**Example 1.12.** Let $X = \operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_m) \subset \mathbf{A}^m_k$ and $Y = \operatorname{Spec} k[y_1, \ldots, y_t]/(g_1, \ldots, g_r) \subset \mathbf{A}^t_k$ be varieties. Then $X \times_{\operatorname{Spec} k} Y = \operatorname{Spec}(\Gamma(X, \mathscr{O}_X) \otimes_k \Gamma(Y, \mathscr{O}_Y))$ which is a variety of $\mathbf{A}^{n+t}_k$ that is cut out by $f_1, \ldots f_m, g_1, \ldots, g_r$. Lastly, $\dim(X \times_{\operatorname{Spec} k} Y) = \dim(X) + \dim(Y)$.

**Example 1.13.** For a scheme $X/\mathbf{Z}$, there is a so-called *generic fibre* $X_{\mathbf{Q}}$ over $\mathbf{Q}$ and *special fibres* $X_p$ (conceived of as being 'reduction modulo $p$') over each each finite field $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$. Lets see why these special notions are merited. A great question to ask is what are the prime ideals of $\mathbf{Z}[x]$ (or points of its spectra). As we have the natural inclusion $\mathbf{Z} \to \mathbf{Z}[x]$, then we have an induced map of schemes $\pi\colon \operatorname{Spec} \mathbf{Z}[x] \to \operatorname{Spec} \mathbf{Z}$. To establish the prime ideals of $\operatorname{Spec} \mathbf{Z}[x]$ we will look at the fibers of this map. We do this by pulling this back to $\operatorname{Spec} \kappa([\mathfrak{p}]) \to \operatorname{Spec} \mathbf{Z}$. We have to establish two cases: if $[\mathfrak{p}] = [0]$ or $[\mathfrak{p}] \neq [0]$. For $[p] = [0]$, we have $\operatorname{Spec} \mathbf{Q} \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} \mathbf{Z}[x] = \operatorname{Spec}(\mathbf{Z}[x] \otimes_{\mathbf{Z}} \mathbf{Q}) = \operatorname{Spec} \mathbf{Q}[x]$. When $[p] \neq [0]$, we have $\operatorname{Spec} \mathbf{F}_p \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} \mathbf{Z}[x] = \operatorname{Spec}(\mathbf{F}_p \otimes_{\mathbf{Z}} \mathbf{Z}[x]) = \operatorname{Spec} \mathbf{F}_p[x]$. Thus we have an association to the points in $\operatorname{Spec} \mathbf{Z}[x]$: the prime ideals are in bijection with $\operatorname{Spec} \mathbf{F}_p[x]$ and $\operatorname{Spec} \mathbf{Q}[x]$. Lastly, this means that $\dim \operatorname{Spec} \mathbf{Z}[x] = 2$ by comparing Krull dimensions'. There is a thought given by Mumford that we should picture schemes such as $\operatorname{Spec} \mathbf{Z}[x]$ as a union of affine lines $\mathbf{A}_{\mathbf{Q}}$ and $\mathbf{A}_{\mathbf{F}_p}$; this is made clear in his famous depiction of $\operatorname{Spec} \mathbf{Z}[x]$:

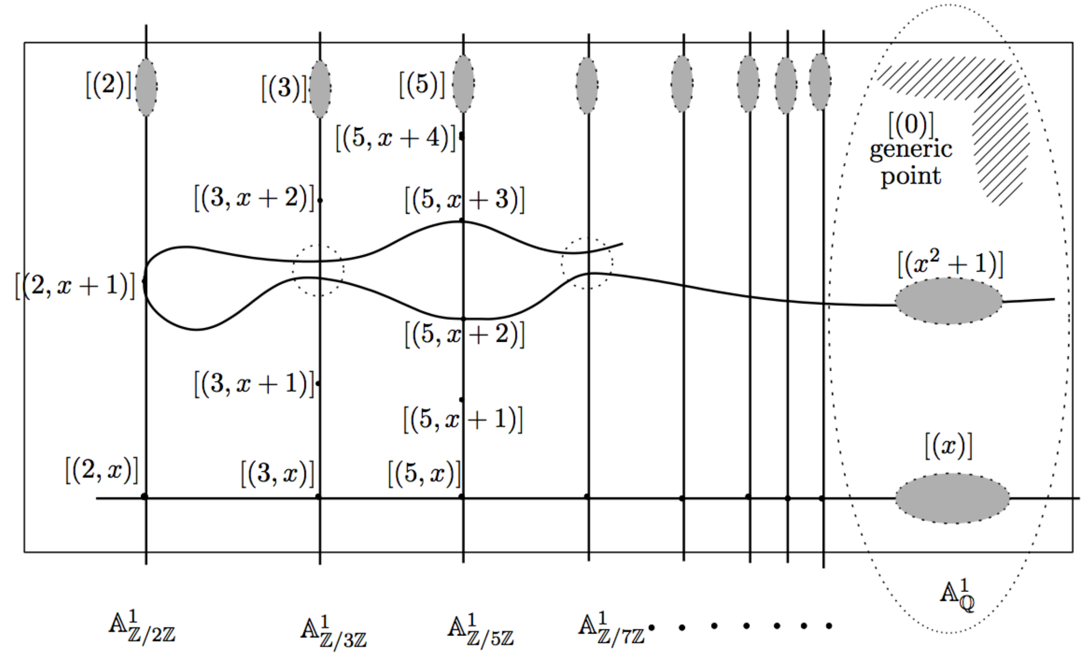


FIGURE 3. [Mumf]

This general technique is very powerful and another interesting case that is analogous to the previous example is figuring out $\operatorname{Spec}\mathbf{Z}_\ell[x]$ where this denotes the spectrum of the ring of polynomials with $\ell$-adic integer coefficients. Essentially we know that $\mathbf{Z}_\ell$ is a DVR and thus $\operatorname{Spec}\mathbf{Z}_\ell$ has only two prime ideals, the ideals $(0)$ and $(\ell)$. We can embed $\mathbf{Z}_\ell \to \mathbf{Z}_\ell[x]$ and get an induced map of spectra $\pi\colon \operatorname{Spec}\mathbf{Z}[x]_\ell \to \operatorname{Spec}\mathbf{Z}_\ell$, and only have two options for taking preimages of two points in $\operatorname{Spec}\mathbf{Z}_\ell$. It amounts to checking $\operatorname{Spec}\kappa([0])\times_{\mathbf{Z}_\ell}\operatorname{Spec}\mathbf{Z}[x] = \operatorname{Spec}(\mathbf{Q}_\ell \otimes_{\mathbf{Z}_\ell}\mathbf{Z}[x]) = \operatorname{Spec}\mathbf{Q}_\ell[x]$, and similarly $\operatorname{Spec}\kappa[(\ell)]\times_{\mathbf{Z}_\ell}\operatorname{Spec}\mathbf{Z}[x] = \operatorname{Spec}(\mathbf{Z}/\ell\mathbf{Z}\otimes_{\mathbf{Z}_\ell}\mathbf{Z}[x]) = \operatorname{Spec}\mathbf{Z}/\ell\mathbf{Z}[x]$. Hence the prime ideals of $\operatorname{Spec}\mathbf{Z}_\ell[x]$ are in bijection with the prime ideals of $\mathbf{Z}/\ell\mathbf{Z}[x] \simeq \mathbf{F}_\ell[x]$ and $\mathbf{Q}_\ell[x]$. We leave it to the reader to further establish what $\operatorname{Spec}\mathbf{Z}/\ell\mathbf{Z}[x]$ and $\operatorname{Spec}\mathbf{Q}_\ell[x]$ actually are.

**Definition 1.14.** Let $k$ be a field. A **curve** is a variety of dimension 1 over $k$.

**Example 1.14.** Both $\mathbf{A}^1_k$ and $\mathbf{P}^1_k$ over a base field $k$ are curves. Moreover, the spectra of $k[x,y]/(f(x,y))$ is a curve if and only $f(x,y)$ is an irreducible polynomial in $k[x,y]$; this is clear as $\operatorname{Spec} k[x,y]/(f(x,y)$ is variety and the dimension of $k[x,y]/(f(x,y))$ is 1 since this is a field as $(f(x,y))$ is irreducible and so maximal. Now take your favorite elliptic and Fermat curves, say, $y^2 = x^3 - x$ and $y^2 = x^3 - 2$, then you have two curves $\operatorname{Spec}\mathbf{Q}[x,y]/(y^2 - x^3 - x)$ and $\operatorname{Spec}\mathbf{Q}[x,y]/(y^2 - x^3 + x)$. Lastly, note that as $p(x) = x^2 - 2$ is irreducible over $\mathbf{Q}$, but not over $\mathbf{R}$, so you have a curve in one case, yet not once you extend the field.

1.4. **Interlude on the Functor of Points.** The main interest of algebraic geometry is to further understand how polynomial equations relate to geometry, and learning the fruitful mathematics around it. In learning scheme theory, often enough it is unclear how we should think of things geometrically; although the dictionary for an algebraic geometer may consist of the following relations:

$$\text{maximal ideals of } A \longleftrightarrow \text{closed points of } \operatorname{Spec} A \tag{1}$$

$$\text{prime ideals of } A \longleftrightarrow \text{irreducible closed subsets of } \operatorname{Spec} A \tag{2}$$

$$\text{radical ideals of } A \longleftrightarrow \text{closed subsets of } \operatorname{Spec} A \tag{3}$$

$$\text{ideals of } A \leftrightsquigarrow \text{closed subschemes of } \operatorname{Spec} A \tag{4}$$

We can go back to this initial perspective very quickly with scheme theory by taking the *functor of points* point of view; in fact, Grothendieck sought for algebraic geometry to be based on this in later years. Let $X$ be a subvariety in $\mathbf{A}^n_k = \operatorname{Spec} k[x_1, \dots, x_n]$ that is cut out by the family of polynomials

$$\begin{aligned} f_1(a_1, \dots, a_n) &= 0 \\ f_2(a_1, \dots, a_n) &= 0 \\ &\vdots \\ f_m(a_1, \dots, a_n) &= 0. \end{aligned}$$

Thus we're specifically looking at $R = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ (i.e. $R$ is a finitely presented $k$-algebra) and $X$ itself is the affine $k$-variety $\operatorname{Spec} R = \mathbf{A}^n_R$. Then we say that *$k$-rational point*, or simply a *$k$-point*, on $X$ is an $n$-tuple $a = (a_1, \dots, a_n)$ in $k^n$ such that $f_1, \dots, f_m$ vanish on $a$. Furthermore, for the set of $k$-points on $X$ is in bijection between the set $\operatorname{Hom}_{\mathsf{CAlg}_k}(R, k)$ and thus as well in bijection with $\operatorname{Hom}_{\mathsf{Sch}/k}(\operatorname{Spec} k, X)$. In part, Grothendieck sought to push the view of considering all ideals of a given ring, rather than just prime ideals. This is notably achieved by associating a functor of points to an affine scheme. For simplicity, we will consider more affine schemes for this discussion. For the space $\mathbf{A}^n_{\mathbf{Z}} = \operatorname{Spec} \mathbf{Z}[x_1, \dots, x_n]$, an $R$-point is a morphism $g\colon \operatorname{Spec} R \to \operatorname{Spec} \mathbf{Z}[x_1, \dots, x_n]$, which induces a morphism $\mathbf{Z}[x_1, \dots, x_n] \to R$, and determines/is determined by an $n$-tuple of elements of $R$ where $(a_1, \dots, a_n) = (g^*(x_1), \dots, g^*(x_n))$. Thereby we get $\mathbf{A}^n_{\mathbf{Z}}(R) = R^n$. Extending this to work done at the beginning of this paragraph, take $X = \operatorname{Spec}(\mathbf{Z}[x_1, \dots, x_n]/(f_1, \dots, f_m))$, then the set of $R$-points of $X$ can be found by considering a morphism $g\colon \operatorname{Spec} R \to X$ which is determined by $(a_1, \dots, a_n) = (g^*(x_1), \dots, g^*(x_n))$ such that $h \mapsto h(a_1, \dots, a_n)$ determined a morphism $\mathbf{Z}[x_1, \dots, x_n]/(f_1, \dots, f_m) \to R$ (i.e. the elements of $R^n$ such that $(a_1, \dots, a_n)$ vanish for all $f_j$ of the ideal $(f_1, \dots f_m)$).

**Remark 1.2.** For any ring $A$ and $R$ an $A$-algebra. If $(f_1, \dots, f_m)$ is an ideal of $A[x_1, \dots, x_n]$ and $X = \operatorname{Spec}(A[x_1, \dots, x_n]/(f_1, \dots, f_m))$, then the set of $R$-points $\operatorname{Hom}_{\mathbf{Sch}/A}(\operatorname{Spec} R, X)$ consists of $n$-tuples $(a_1, \dots, a_n) \in R^n$ such that $f_1(a_1, \dots, a_n) = \cdots = f_m(a_1, \dots, a_n) = 0$.

**Example 1.15.** For the affine scheme $X = \operatorname{Spec}(\mathbf{Z}[x_1, x_2]/(x^2 + y^2 - 1)$, the set $X(\mathbf{R})$ is simply the unit circle in $\mathbf{R}^2$, while $X(\mathbf{Q})$ is the set of pairs $(a/c, b/c)$ in $\mathbf{Q}^2$ such that $a^2 + b^2 = c^2$.

**Example 1.16** (Fermat)**.** For $X = \operatorname{Spec}(\mathbf{Z}[x, y]/(x^n + y^n - z^n))$, we have $X(\mathbf{Q}) = \varnothing$ for $n \geq 3$.

**Proposition 1.9.** Let $X$ be a scheme and $k$ a field. Then to give a mapping of schemes $\operatorname{Spec} k \to X$ is to give a point $x \in X$ and an embedding $\kappa(x) \to k$.

*Proof.* Given a map of schemes $f\colon \operatorname{Spec} k \to X$, we have the data of a morphism of underlying topological spaces $\operatorname{Spec} k \to X$ and a morphism of sheaves $f^\sharp\colon \mathscr{O}_X \to f_*\mathscr{O}_{\operatorname{Spec} k}$. As $k$ is a field, then topologically $\operatorname{Spec} k = \{*\}$ is just a point, which corresponds to the zero ideal of $k$. Thus $f\colon \{*\} \to X$ gives a point of $X$, say, $P \in X$. On the level of stalks, we have $f^\sharp_P\colon \mathscr{O}_{X,P} \to \mathscr{O}_{\operatorname{Spec} k, \{*\}} = k$, that is, $f^\sharp_P(\mathfrak{m}_P) = (0)$ where $\mathfrak{m}_P$ denotes the unique maximal ideal of the local ring $\mathscr{O}_{X,P}$. Therefore we have that the mapping of local rings $f^\sharp_P$ factors through the quotient $\mathscr{O}_{X,P}/\mathfrak{m}_P = \kappa(P)$. Lastly, if we have a ring map of fields then it is necessarily an inclusion and thus we have an inclusion $\kappa(P) \to k$. For the other direction, lets take a point $x \in X$ with an inclusion $\kappa(x) \to k$. We define a map $f$ from $\operatorname{Spec} k$ to $X$ by taking the point $\{*\}$ in the topological space $\operatorname{Spec} k$ to the point $x$ in $X$. Moreover, for our mapping of sheaves we define $\mathscr{O}_X \to f_*\mathscr{O}_{\operatorname{Spec} k}$ to be determined by any open set $U \subseteq X$ such that $x \in U$ or $x \notin U$ as follows: for $x \in U$, we write our morphism of sheaves as the composition $\mathscr{O}_X(U) \to \mathscr{O}_{X,x} \to \kappa(x) \to k = f_*\mathscr{O}_{\operatorname{Spec} k, x}$, and for $x \notin U$ then the target and the mapping are both zero. □

We make the following definition for schemes.

**Definition 1.15.** Let $X$ be an $S$-scheme. If $T$ is an $S$-scheme, then the **set of $T$-points on $X$** is $X(T) = \operatorname{Hom}_S(T, X)$.

As hinted at above, for any scheme $Y$, we have $\mathbf{A}^n(Y) = \operatorname{Hom}_{\mathbf{Sch}}(Y, \mathbf{A}^n) = \Gamma(Y, \mathscr{O}_Y)^n$. What this is meant to say is that $\mathbf{A}^n$ *represents* the functor for which takes a scheme $Y$ to $n$-tuples of global sections of $\Gamma(Y, \mathscr{O}_Y)$.

Now if we have $T \to T'$ being a morphism of $S$-schemes, then for any map $T' \to X$ we have a map $T \to S$ given by composition, and thus this provides a map of sets $X(T') \to X(T)$. This determines a functor: the *functor of points* of an $S$-scheme $X$ is the contravariant functor $X(-)\colon \mathbf{Sch}/X \to \mathbf{Sets}$ where $T \mapsto X(T) = \mathrm{Hom}_S(T, X)$. In general this type of functor is given the name 'representable'; if $\mathscr{C}$ is a category, then a functor $F\colon \mathscr{C} \to \mathbf{Sets}$ is *representable* if there is some object $X \in \mathscr{C}$ such that $F$ is isomorphic to $\mathrm{Hom}(X, -)$, and we denote the functor $\mathrm{Hom}(X, -)$ by $h_X$.

**Lemma 1.7** (Yoneda)**.** Let $X$ and $Y$ be $S$-schemes. Then the map

$$\mathrm{Hom}_S(X, Y) \to \mathrm{Hom}(h_X, h_Y)$$

where $\psi \mapsto h(\psi)$ is a bijection, i.e. for every morphism of $S$-schemes we get a natural transformation between $h_X$ and $h_Y$.

In the world of algebraic geometry,
There is a concept that holds great mystery:
The functor of points, so elegant and grand,
A tool that helps us understand
The geometric shapes that we see.
.
With its power to map and transform,
The functor of points helps us to inform
Our understanding of algebraic varieties
And the algebraic structures they embody.
.
From the curves and surfaces of the plane,
To the higher-dimensional spaces that remain
Beyond our grasp, the functor of points
Guides us through the intricate joints
Of algebraic geometry's domain.
.
So let us celebrate this powerful tool,
And thank the mathematicians who made it cool,
For with the functor of points on our side,
We can unlock the secrets of algebraic geometry with pride.

FIGURE 4. Poem about functor of points produced by ChatGPT.

1.5. **Quasi-Coherent and Coherent Sheaves.** The canonical construction of a scheme from [Hart77] begins with bare bones description where we attach a sheaf of rings to the Zariski topology on $X = \operatorname{Spec} A$ by defining, for any open subset $U \subseteq X$, $\mathscr{O}_X(U)$ to be the set (with a ring structure) $s\colon U \to \bigsqcup_{\mathfrak{p}\in U} A_{\mathfrak{p}}$ such that the point $[\mathfrak{p}]$ associated to the prime ideal $\mathfrak{p}$ is in $U$ and $s[\mathfrak{p}] \in A_{\mathfrak{p}}$ with $s$ being locally a fraction. Now we make an analogous construction to the case of an $A$-module $M$, where we define a sheaf of modules $\widetilde{M}$ on $\operatorname{Spec} A$. We do this as follows: Suppose $M$ is an $A$-module. For any open set $U$ of $X = \operatorname{Spec} A$ we define the group $\widetilde{M}(U)$ to be the set of functions $s\colon U \to \bigsqcup_{\mathfrak{p}\in U} M_{\mathfrak{p}}$ such that the point $[\mathfrak{p}]$ associated to the prime ideal $\mathfrak{p}$ is in $U$ and $s[\mathfrak{p}] \in A_{\mathfrak{p}}$ with $s$ being locally a fraction with, i.e. for each $[\mathfrak{p}] \in U$ there is an open neighborhood $[\mathfrak{p}] \in V \subseteq U$ such that we have elements $m \in M$ and $f \in A$, such that for each $[\mathfrak{q}] \in V, f \notin [\mathfrak{q}]$, and $s(\mathfrak{q}) = m/f$ in $M_{\mathfrak{q}}$. We call $\widetilde{M}$ the *sheaf associated* to $M$ on $\operatorname{Spec} A$.

**Proposition 1.10** ([Hart77], II.5, 5.1)**.** Let $A$ be a ring, let $M$ be an $A$-module, and let $\widetilde{M}$ be the sheaf on $X = \operatorname{Spec} A$ associated to $M$. Then:

(a) $\widetilde{M}$ is an $\mathscr{O}_X$-module;
(b) for each $\mathfrak{p} \in X$, the stalk $(\widetilde{M})_{\mathfrak{p}}$ of the sheaf $\widetilde{M}$ at $\mathfrak{p}$ is isomorphic to the localized module $M_{\mathfrak{p}} \simeq A_{\mathfrak{p}} \otimes_A M$;

(c) for any $f \in A$, the $A_f$-module $\widetilde{M}(D(f))$ is isomorphic to the localized module $M_f \simeq A_f \otimes_A M$;
(d) in particular, $\Gamma(X, \widetilde{M}) = M$.

**Lemma 1.8.** Let $X = \operatorname{Spec} A$. For any two $A$-modules $M$ and $N$ there is a 1:1 bijection

$$\{\text{morphisms of } A\text{-modules } M \to N\} \longleftrightarrow \{\text{morphisms of sheaves } \widetilde{M} \to \widetilde{N}\}.$$

In particular, we have an exact, fully faithful functor from $\mathsf{Mod}_A$ to $\mathsf{Mod}_{\mathscr{O}_X}$ given by $M \mapsto \widetilde{M}$.

*Proof.* If we have an $A$-module morphism $M \to N$, we can localize at some $P \in X$ so we have a map $M_P \to N_P$, and thus we have a morphism of sheaf of modules associated to $X$ by Proposition 1.10. Similarly, given a morphism of sheaves $\widetilde{M} \to \widetilde{N}$ we apply the global sections functor which gives an $A$-module homomorphism by Proposition 1.10. Lastly, these two functorial operations that are inverse to each other and therefore we're done and have shown that the functor is fully faithful.

Exactness of $M \mapsto \widetilde{M}$ follows from $0 \to N_1 \to N_2 \to N_3 \to 0$ being an exact sequence of $A$-modules if and only if for all prime ideals $\mathfrak{p}$ of $A$ the sequence $0 \to (N_1)_\mathfrak{p} \to (N_2)_\mathfrak{p} \to (N_3)_\mathfrak{p} \to 0$ and so $0 \to (\widetilde{N_1})_\mathfrak{p} \to (\widetilde{N_2})_\mathfrak{p} \to (\widetilde{N_3})_\mathfrak{p} \to 0$ is an exact sequence on stalks by the previous paragraph. □

**Definition 1.16.** Let $(X, \mathscr{O}_X)$ be a scheme. A sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$ is **quasicoherent** if $X$ can be covered by open affine subsets $V_i = \operatorname{Spec} A_i$ such that for each $i$ there is an $A_i$-module $M_i$ with $\mathscr{F}|_{V_i} \simeq \widetilde{M}_i$. Equivalently, $\mathscr{F}$ is quasicoherent if for every affine open subset $\operatorname{Spec} A \subset X$ there is an $A$-module $M$ such that $\mathscr{O}_X|_{\operatorname{Spec} A} \simeq \widetilde{M}$. We say that $\mathscr{F}$ is **coherent** if furthermore each $M_i$ can be taken to be a finitely generated $A_i$-module.

We form the category $\mathsf{QCoh}_{\mathscr{O}_X}$ and $\mathsf{Coh}_{\mathscr{O}_X}$ for the category of quasicoherent and coherent $\mathscr{O}_X$-modules, respectively, and taking the morphisms in both categories to be morphisms of $\mathscr{O}_X$-modules.

We introduce (or recall) some particularly important sheaves of $\mathscr{O}_X$-modules: The *tensor product* of two $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$ to be the sheaf associated to the presheaf $U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}(U)$, which we will simply denote by $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$. Moreover, if $U \subseteq X$ is an open set and $\mathscr{F}$ and $\mathscr{G}$ are two $\mathscr{O}_X$-modules, then the presheaf given by $U \mapsto \operatorname{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf that is denoted by $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$.[5] Note that for $U \subseteq X$ open, $\mathscr{F}|_U$ is an $\mathscr{O}_X|_U$-module, and $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is indeed itself an $\mathscr{O}_X$-module.

**Corollary 1.2.** Let $X = \operatorname{Spec} A$. There is an equivalence of categories between $\mathsf{QCoh}_{\mathscr{O}_X}$ and $\mathsf{Mod}_A$ given by $\Gamma(X, -)$ and $M \mapsto \widetilde{M}$.

*Proof.* This is easy after Lemma 1.8 since on $\operatorname{Spec} A$ a quasi-coherent sheaf $\mathscr{F}$ is of the form $\widetilde{M}$ and $\Gamma(M, \mathscr{F}) = M$. □

**Proposition 1.11.** Let $X$ be a scheme.

(1) The kernel, cokernel, and image of a map between quasi-coherent $\mathscr{O}_X$-modules are quasi-coherent.
(2) If $\mathscr{F}$ and $\mathscr{E}$ are quasi-coherent $\mathscr{O}_X$-modules, then $\mathscr{F} \oplus \mathscr{E}$ and $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{E}$ are quasi-coherent.

*Proof.* The kernel, cokernel, image, direct sum, and tensor product commute with $M \mapsto \widetilde{M}$; note here that for any $\mathscr{O}_X$-module $\mathscr{S}$ we have $\operatorname{Hom}_{\operatorname{Spec} A}(\widetilde{M}, \mathscr{S}) \simeq \operatorname{Hom}_A(M, \Gamma(\operatorname{Spec} A, \mathscr{S}))$, which demonstrates that tensoring commutes with $M \mapsto \widetilde{M}$. □

**Corollary 1.3.** The category $\mathsf{QCoh}_{\mathscr{O}_X}$ is an abelian category.

**Theorem 1.4.** For a scheme $X$, the category $\mathsf{QCoh}_{\mathscr{O}_X}$ has enough injectives.

*Proof.* This is long and difficult so we refer the reader to [Sta18, Tag 077K] and therefore we win. □

[5] The notation may be a little confusing so we should additionally make a clear remark that when we're given an open set $U$ of $X$ then $\mathscr{F}|_U$ is the restriction sheaf down to that open set, and so $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}) \neq \operatorname{Hom}(\mathscr{F}(U), \mathscr{G}(U))$; that is, we're looking at morphisms of $\mathscr{O}_X$-modules for the sheaf $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$.

**Example 1.17.** Let $A$ be a ring and $\mathfrak{q}$ be an ideal of $A$. Then $\operatorname{Spec}(A/\mathfrak{q}) \subseteq \operatorname{Spec} A$ can be considered as a the closed subscheme of $\operatorname{Spec} A$. Let $\rho\colon \operatorname{Spec}(A/\mathfrak{q}) \to \operatorname{Spec} A$ be the inclusion morphism; the sheaf $\rho_*\mathscr{O}_{\operatorname{Spec}(A/\mathfrak{q})}$ is a coherent $\mathscr{O}_{\operatorname{Spec} A}$-module, where $\rho_*\mathscr{O}_{\operatorname{Spec}(A/\mathfrak{q})} \simeq \widetilde{(A/\mathfrak{q})}$. More generally, given any scheme $X$, its corresponding structure sheaf $\mathscr{O}_X$ is quasi-coherent since any scheme is covered by affine opens $\mathcal{U} = \{\operatorname{Spec} A_i \colon i \in I\}$ and the restriction to any affine open $\operatorname{Spec} A_\alpha \in \mathcal{U}$ with the structure sheaf gives $\mathscr{O}_X|_{\operatorname{Spec} A_\alpha} \simeq \widetilde{A_\alpha}$ .

We say an $\mathscr{O}_X$-module $\mathscr{F}$ is *free of rank* $r$ if $\mathscr{F}$ is isomorphic to the direct sum of $r$ copies of $\mathscr{O}_X$-module; for shorthand, we write $\mathscr{F} \simeq \mathscr{O}_X^{\oplus r} := \mathscr{O}_X \oplus \mathscr{O}_X \oplus \cdots \oplus \mathscr{O}_X$ $r$ many times. Additionally, $\mathscr{F}$ is *locally free of rank* $r$ if there exists an open affine cover $\mathfrak{U} = \{U_i\}$ of $X$ such that each restriction $\mathscr{F}|_{U_i}$ is isomorphic to $\mathscr{O}_X|_{U_i}^{\oplus r}$. Moreover, if we have the special case of an $\mathscr{O}_X$-module $\mathscr{L}$ being locally free of rank $r = 1$, then we say that $\mathscr{L}$ is an *invertible* $\mathscr{O}_X$-module (or, as some do, $\mathscr{L}$ is also sometimes called a *line bundle*). Note that an $\mathscr{O}_X$-module $\mathscr{L}$ that is locally of rank $r$ is quasi-coherent, since we have an $\mathscr{O}_X$-module $\mathscr{L}$ that is locally free of rank $r$ then we have at any open affine cover $\mathfrak{U} = \{U_i = \operatorname{Spec} A_i \colon i \in I\}$ this gives $\mathscr{L}|_{\operatorname{Spec} A_i} \simeq \mathscr{O}_X|_{\operatorname{Spec} A_i}^{\oplus r} \simeq \widetilde{A_i^{\oplus r}} = \widetilde{A_i}^{\oplus r}$.

For $\pi\colon X \to Y$ is a morphism of schemes we have a functor $\pi_*\colon \mathsf{Mod}_{\mathscr{O}_X} \to \mathsf{Mod}_{\mathscr{O}_Y}$ where if $\mathscr{F}$ is an $\mathscr{O}_X$-module we define $\pi_*\mathscr{F}\colon U \mapsto \mathscr{F}(\pi^{-1}(U))$ and $\pi_*\mathscr{F}$ is a $\pi_*\mathscr{O}_X$-module, and we call this functor a *pushforward* or *direct image* of $\mathscr{F}$.

Similarly, kind of, for $f\colon X \to Y$ a map of schemes, if $\mathscr{G}$ is a sheaf on $Y$, we let $f^{-1}(\mathscr{G})$ denote the sheaffication of the presheaf $U \mapsto \varinjlim_{f(U)\subset V} \mathscr{G}(V)$. Observe that if we wanted to define the inverse image in the way you'd expect where $f^{-1}(\mathscr{G}(U)) = \mathscr{G}(f(U))$, for $U \subset X$ open, this will not always work as $f(U)$ is not necessarily open. What we've done is approximated by using $f^{-1}(\mathscr{G}) : U \mapsto \varinjlim_{f(U)\subset V} \mathscr{G}(V)$, yet this presheaf is not a sheaf so we still had to sheaffify it. Note that if we had the inclusion $\iota\colon \{y\} \to Y$ where $y \in Y$, then $\iota^{-1}\mathscr{G}(\{y\}) = \varinjlim_{y \in V} \mathscr{G}(V) = \mathscr{G}_y$, the stalk of $\mathscr{G}$ at the point $y \in Y$. Yet there is another fault to this sheaf as we would not necessarily get a sheaf of $\mathscr{O}_X$-modules given that $\mathscr{G}$ is an $\mathscr{O}_Y$-module. To put a band aid on this we define for a sheaf of $\mathscr{O}_Y$-modules $\mathscr{G}$ it's *pullback* or *inverse image* by $f^*\mathscr{G} = f^{-1}\mathscr{G} \otimes_{f^{-1}\mathscr{O}_Y} \mathscr{O}_X$, which gives us a functor $f^*\colon \mathsf{Mod}_{\mathscr{O}_Y} \to \mathsf{Mod}_{\mathscr{O}_X}$. We remark here that for a given point $x \in X$, we have $(f^{-1}\mathscr{G})_x \simeq \mathscr{G}_{f(x)}$, which gives us that $f^{-1}$ is an exact functor and that $f^*(\mathscr{O}_Y) = \mathscr{O}_X$. And also if $\mathscr{E}$ quasi-coherent on $Y$, then $f^*\mathscr{E}$ is quasi-coherent, which is something we cannot say about $f^*\mathscr{E}$ sadly. The main relationship between $f^*$ and $f_*$ is that they're adjoint to one another, meaning that for $f\colon X \to Y$ and $\mathscr{F}$ and $\mathscr{O}_X$-module and $\mathscr{G}$ and $\mathscr{O}_Y$-module we have

$$\operatorname{Hom}_{\mathsf{Mod}_{\mathscr{O}_X}}(f^*\mathscr{G}, \mathscr{F}) \simeq \operatorname{Hom}_{\mathsf{Mod}_{\mathscr{O}_Y}}(\mathscr{G}, f_*\mathscr{F}).$$

Lastly, a *sheaf of ideals* on a scheme $X$ is a sheaf of modules $\mathscr{I}$ such that for any open set $U$ of $X$, $\mathscr{I}(U)$ is an ideal of $\mathscr{O}_X(U)$. Furthermore, if $\pi\colon Y \to X$ is a closed immersion, then we have a surjection on sheaves $\pi^\sharp\colon \mathscr{O}_X \to \pi_*\mathscr{O}_Y$, and the sheaf $\pi_*\mathscr{O}_Y$ is quasi-coherent—this follows from the following proposition—for which we define the *the kernel sheaf* $\mathcal{I}_{Y/X}$ to be the kernel of $\pi^\sharp$. (Note that $\mathscr{O}_X \to \pi_*\mathscr{O}_Y \simeq \mathscr{O}_X/\mathcal{I}_{Y/X}$.)

**Proposition 1.12.** Let $\pi\colon Y \to X$ be the inclusion of a closed subscheme.

(a) If $\mathscr{F}$ is a quasi-coherent sheaf on $Y$, then $\pi_*\mathscr{F}$ is quasi-coherent on $X$.
(b) There is an exact sequence

$$0 \to \mathcal{I}_{Y/X} \to \mathscr{O}_X \to \pi_*\mathscr{O}_Y \to 0$$

*Proof.* We may reduce to the case of affines. Let $X = \operatorname{Spec} A$, and thus $Y = \operatorname{Spec} A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of $A$. Let $\mathscr{F}$ be quasi-coherent on $Y$. Then for every open affine $U = \operatorname{Spec} T$ of $Y$, there is a $T$-module $M$ such that $\mathscr{F}|_U \simeq \widetilde{M}$. Then $\pi_*\mathscr{F}$ is a sheaf of $\mathscr{O}_X$-modules given by $U \mapsto \mathscr{F}(\pi^{-1}(U))$. As $\pi$ is an affine morphism, then $\pi^{-1}(U)$ is an affine open of $Y$ and thus $\mathscr{F}|_{\pi^{-1}(U)} \simeq \widetilde{L}$ for some $T$-module. Hence $\pi_*\mathscr{F}$ is quasi-coherent on $X$.

(b) It remains to check that $\mathscr{O}_X \to \pi_*\mathscr{O}_Y$ is surjective. We may reduce this to checking with affine $X = \operatorname{Spec} A$. This forces $Y$ to be the form $Y = \operatorname{Spec} A/I$ for some ideal $I \subset A$. But $\mathscr{O}_X$ and $\pi_*\mathscr{O}_Y$ can be recognized to be $\widetilde{A}$ and $\widetilde{A/I}$, respectively. Applying Lemma 1.8, the claim follows as

$A \to A/I$ is surjective and $\mathcal{I}_{Y/X}$ can be recognized as $\widetilde{I}$, so we have the exact sequence of $A$-modules $0 \to I \to A \to A/I \to 0$. □

We've focused a lot on affine schemes throughout this section, yet there is of course another side to this story when we consider projective objects that we will talk a little about. Let $S$ be a graded ring, i.e. $S = \bigoplus_{d\geq 0} S_d$. A *graded $S$-module* $M$ is an $S$-module $M$ such that $M = \bigoplus_{d\in\mathbf{Z}} M_d$ with $S_d \cdot M_e \subset M_{d+e}$. For a graded $S$-module $M$ we define the *twisted* $S$-module $M(k)$ for $k \in \mathbf{Z}$ with $M(k)_d = M_{d+k}$. Similar to how we started this section, we have the following:

**Theorem 1.5** (II.II.5.11, [Hart77])**.** Let $S$ be a graded ring and $M$ a graded $S$-module. Let $X = \operatorname{Proj} S$.

(a) For any $\mathfrak{p} \in X$, the stalk $(\widetilde{M})_{\mathfrak{p}} = M_{\mathfrak{p}}$.
(b) For any homogeneous $f \in S_+$, we have $\widetilde{M}|_{D_+(f)} \simeq \widetilde{M_{(f)}}$.
(c) $\widetilde{M}$ is a quasi-coherent $\mathscr{O}_X$-module. If $S$ is noetherian and $M$ is finitely generated, then $\widetilde{M}$ is coherent.

**Definition 1.17.** Let $S$ be a graded ring and write $X = \operatorname{Proj} S$. For any $n \in \mathbf{Z}$, we define sheaf $\mathscr{O}_X(n) = \widetilde{S(n)}$. For $n = 1$, we call $\mathscr{O}_X(1)$ the *twisting sheaf* of Serre. For $\mathscr{F}$, a sheaf of $\mathscr{O}_X$-modules $\mathscr{F}$, we write $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n)$.

For the rest of this subsection we will take $R$ to be graded ring such that $R$ is finitely generated by $R_1$ as an $R_0$-algebra, e.g. $R = A[x_0, \dots, x_n]$ where $A = R_0$.

**Proposition 1.13.** Let $X = \operatorname{Proj} R$. The sheaf $\mathscr{O}_X(n)$ is invertible.

*Proof.* Let $f \in R_1$. It's sufficient to that the restriction of $\mathscr{O}_X(n)$ to $D_+(f)$ is invertible. As $\mathscr{O}_X(n) = \widetilde{R(n)}$, and by Proposition 2.5(b) of II.II.2 in [Hart77], we have $\mathscr{O}_X(n)$ to $D_+(f)$ is isomorphic to $\widetilde{R(n)}_{(f)}$ on $\operatorname{Spec} R_{(f)}$. Now $R_{(f)}$ consists of elements of degree zero in $R_f$ while $R(n)_{(f)}$ has elements of degree $n$ in $R_f$ by definition, but $R_{(f)}$ and $R(n)_{(f)}$ are isomorphic via $s \mapsto f^{-n}s$ (note that $f$ is invertible in $S_f$ for any $n \in \mathbf{Z}$). As $R$ is finitely generated by $R_1$ as an $R_0$ algebra, we have $\operatorname{Proj} R$ covered by open sets $D_+(f)$ with $f \in R_1$. Therefore $\mathscr{O}_X(n)$ is invertible. □

1.6. **The Picard Group.** Recall from the previous section we describe a locally free $\mathscr{O}_X$-module that is of rank one as being 'invertible'. The reasoning for this kind of terminology is that we can form a group of invertible $\mathscr{O}_X$-modules under tensor product over $\mathscr{O}_X$ (for multiplication) based on isomorphism classes of invertible $\mathscr{O}_X$-modules.

We should go through the process of showing that this does indeed form a group. Firstly, let $\mathscr{L}$ and $\mathscr{E}$ be two invertible $\mathscr{O}_X$-modules. Then each point $P$ of $X$ has a neighborhood $P \in U$ such that $\mathscr{E}|_U \simeq \mathscr{O}_X$ and $\mathscr{L}|_U \simeq \mathscr{O}_X$; whereby $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{L} \simeq \mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{O}_X = \mathscr{O}_X$ once restricted down to the neighborhood $U$ of $P$. We've shown that if $\mathscr{E}$ and $\mathscr{L}$ are invertible, then so is their tensor product. Secondly, $\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{E} \simeq \mathscr{E} \simeq \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_X$, making $\mathscr{O}_X$ a unit element. Now we check that $\mathscr{E}^\vee = \mathcal{H}om(\mathscr{E}, \mathscr{O}_X)$ is invertible. As both $\mathscr{E}$ and $\mathscr{O}_X$ are invertible, then for an affine open $U$ of $X$, $\mathscr{E}|_U \simeq \mathscr{O}_X$ and $\mathscr{O}_X|_U \simeq \mathscr{O}_X$, which gives that $\mathcal{H}om(\mathscr{E}_U, \mathscr{O}_X|_U) \simeq \mathcal{H}om(\mathscr{O}_X, \mathscr{O}_X) \simeq \mathscr{O}_X$; hence $\mathscr{E}^\vee$ in invertible. Lastly, $\mathscr{E} \otimes \mathscr{E}^\vee \to \mathscr{O}_X$ restricting once again to a suitable $U$ gives us that this is indeed an isomorphism. Therefore we have such a group as described where $\mathscr{O}_X$ is our unit and for any invertible sheaf $\mathscr{E}$ its inverse, up to isomorphism, is given by $\mathscr{E}^\vee$ (called the *dual sheaf* of $\mathscr{E}$). We call this group the *Picard group* of X, which is denoted as Pic(X).

1.7. **Definition and Examples of Group Schemes.**

**Definition 1.18.** Let $S$ be a scheme. A **group scheme** over $S$ is an $S$-scheme $G$ along with the following data for morphisms:

(i) $m\colon G \times_S G \to G$, called the *multiplication map*;
(ii) $e\colon S \to G$, called the *unit section*; and
(iii) $i\colon G \to G$, called the *inverse*.

that fit into the following commutative diagrams:

(a)

$$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "(m,\mathrm{id})"] \arrow[d, "(\mathrm{id},m)"] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"] & G \end{tikzcd}$$

(b)

$$\begin{tikzcd} G \times_S S \arrow[r, "\sim"] \arrow[dr, "(\mathrm{id},e)"'] & G \arrow[r, "\mathrm{id}"] & G \\ & G \times_S G \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} S \times_S G \arrow[r, "\sim"] \arrow[dr, "(e,\mathrm{id})"'] & G \arrow[r, "\mathrm{id}"] & G \\ & G \times_S G \arrow[ur, "m"'] & \end{tikzcd}$$

(c)

$$\begin{tikzcd} G \arrow[r, shift left, "(i,\mathrm{id})"] \arrow[r, shift right, "(\mathrm{id},i)"'] \arrow[d] & G \times_S G \arrow[d, "m"] \\ S \arrow[r, "e"] & G \end{tikzcd}$$

**Definition 1.19.** Let $G$ and $H$ be two group schemes over a scheme $S$. A morphism $G \to H$ is called a **homomorphism** if for any $S$-scheme $T$ the map $G(T) \to H(T)$ is a group homomorphism. The **kernel** of a homomorphism $\phi\colon G \to H$, denoted by $\ker\phi$, is the group scheme given by for any $S$-scheme $T$ we have $\ker\phi(T)$ being the kernel of the induced group homomorphism $G(T) \to H(T)$.

**Example 1.18.** For any group scheme $G$ and $n \in \mathbf{Z}_{>0}$, we have a homomorphism (or endomorphism) $G \to G$ given by $g \mapsto g^n$; this is called the *multiplication of* $G$ *by* $n$, denoted shortly by $[n]_G$.

**Lemma 1.9.** Given a scheme $S$, an $S$-scheme $G$ is a group scheme if and only the set $G(T)$ for any $S$-scheme $T$ carries a functorial group structure.

**Example 1.19.**

(i) For an affine scheme, $\operatorname{Spec} A$ over $k$, we define *comultiplication* $\mu\colon A \to A \otimes_k A$, *counit* $\epsilon\colon A \to k$, and *coinverse* $\iota\colon A \to A$ as induced from the $k$-algebra structure of $A$.

(ii) The *additive group* over $k$ is a scheme $\mathbf{G}_a = \operatorname{Spec} k[t]$ with $\mathbf{G}_a(R) = R$ for any $k$-algebra $R$. The comultiplication, coinverse, and counit, respectively, are given by $\mu(t) = t \otimes 1 + 1 \otimes t$, $\epsilon(t) = 0$, and $\iota(t) = -t$.

(ii) The *multiplicative group* over $k$ is a scheme $\mathbf{G}_m = \operatorname{Spec} k\left[t, \frac{1}{t}\right]$ with $\mathbf{G}_m(R) = R^\times$ for any $k$-algebra $R$. The comultiplication, coinverse, and counit, respectively, are given by $\mu(t) = t \otimes t$, $\epsilon(t) = 1$, and $\iota(t) = t^{-1}$.

(iii) The *$n$-th roots of unity* over $k$ is a scheme $\mu_n = \operatorname{Spec} k[t]/(t^n - 1)$ with $\mu_n(R) = \{r \in R\colon r^n = 1\}$. The morphisms are as in (ii) above except for coinverse, which is instead given by $\iota(t) = t^{n-1}$. The group scheme $\mu_n$ is regarded as a so-called *closed subgroup scheme* of $\mathbf{G}_m$ defined by the surjective morphism $k\left[t, \frac{1}{t}\right] \to k[t]/(t^n - 1)$

(iv) Let $k$ have characteristic $p$, then we define a group scheme $\alpha_p = \operatorname{Spec} k[t]/(t^p)$ with $\alpha_p(R) = \{x \in R\colon x^p = 0\}$. For the surjective morphism $k[t] \to k[t]/(t^p)$, we regard $\alpha_p$ as a closed subgroup scheme of $\mathbf{G}_a$. The morphisms on $\alpha_p$ are the same as given in (ii) above.

**Definition 1.20.** Let $G = \operatorname{Spec} A$ be an affine group scheme over a noetherian ring $R$. The affine group scheme $G$ is a (commutative) **finite flat group scheme of order** $n$ if it satisfies the following two properties:

(a) $G$ is locally free of rank $n$ over $R$, i.e. $R \to A$ is locally free $R$-algebra of rank $n$.

(b) $G$ is commutative given that the following diagram commutes, where $\rho\colon (x, y) \mapsto (y, x)$ and $m$ is multiplication of $G$

$$\begin{tikzcd} G \times_R G \arrow[rr, "\rho"] \arrow[dr, "m"'] & & G \times_R G \arrow[dl, "m"] \\ & G & \end{tikzcd}$$

In light of this definition, we can observe that an affine group scheme $G = \operatorname{Spec} A$ over a noetherian ring $R$ satisfies Definition 1.20 (a) if and only if the structure map $G \to \operatorname{Spec} R$ if finite flat. Similarly, $G$ satisfies Definition 1.20 $(b)$ if and only if $G(S)$ is a commutative for all $R$-algebras $S$.

**Example 1.20.** Two examples are that the $n$-th roots of unity group scheme $\mu_n$ is a finite flat group scheme of order $n$ and the group scheme $\alpha_p$ is also a finite flat group scheme of order $p$, as defined in Example 1.19.

## 2. Differentials

2.1. **Construction.** Let $S$ be an $R$-algebra and $M$ be a $S$-module. Then we say an $R$-*derivation* of $S$ into $M$ is an $R$-linear map $d\colon S \to M$ satisfying the product rule, also called the *Leibnitz rule*: $d(ss') = sds' + s'ds$, and that $dr = 0$ for all $r \in R$.

In fact, if we have that the product rule holds true, it's simple to see that an $A$-derivation $d$ is $R$-linear if and only if it vanishes on all elements of the form $r \cdot 1$ with $a \in A$: If $d$ is $R$-linear, then $d(r\cdot 1) = rd(1) = 0$ as $d(1\cdot 1) = d(1^2) = 1d(1)+1d(1) = 2d(1)$ so $2d(1)-d(1^2) = d(1)(2-1) = d(1) = 0$; if $d$ vanishes on $A$, then the product rule gives us that $d(rs) = sd(r) + rd(s) = rd(s)$. We denote the set of these derivations by $\operatorname{Der}_R(S, M) = \{d\colon S \to M \colon d \text{ is an } R\text{-derivation of } S \text{ into } M\}$.

**Definition 2.1.** Let $S$ be an $R$-algebra. The **module of Kähler differentials** or **module of relative differential forms** of $S$ over $R$ is a $S$-module $\Omega^1_{S/R}$ endowed with an $R$-derivation $d\colon S \to \Omega^1_{S/R}$ having the following universal property. For any $S$-module $M$ and for any $R$-derivation $d'\colon S \to M$, there exists a unique morphism of $S$-modules $\phi\colon \Omega^1_{S/R} \to M$ such that $d' = \phi \circ d$.

$$\begin{tikzcd} S \arrow[r, "d'"] \arrow[d, "d"'] & M \\ \Omega^1_{S/R} \arrow[ur, dashed, "\phi"] & \end{tikzcd}$$

**Proposition 2.1.** The module of relative differential forms $(\Omega^1_{B/A}, d)$ exists and is unique up to isomorphism.

*Proof.* Let $F$ be the free $B$-module generated by the symbols $db$, for $b \in B$. Let $E$ be the submodule of $F$ that is generated by the elements of the form (1) $d(b + b') - db - db'$ with $b, b' \in B$, (2) $d(bb') - bdb' - b'db$ for $b, b' \in B$, and (3) $da$ for $a \in A$. We set $\Omega^1_{B/A} = F/E$ and $d\colon B \to \Omega^1_{B/A}$ which sends $b \mapsto db$ in $\Omega^1_{B/A}$. Verifying that $\Omega^1_{B/A}$ is unique up to isomorphism is standard and is omitted here. □

**Corollary 2.1.** $\Omega_{B/A}$ is generated as a $B$-module by $d(B)$.

There is also a universality property attached to $\Omega_{S/R}$ given by if we have any $R$-derivation $f : S \to M$ it then factors uniquely through the corresponding universal derivation by composition; this is essentially the assertion that

$$\begin{aligned} \operatorname{Hom}_S(\Omega_{S/R}, M) &\simeq \operatorname{Der}_R(S, M) \\ f &\mapsto f \circ d. \end{aligned}$$

**Example 2.1.** Let $S$ be a ring and $S = R[x_1, \dots, x_n]$, then $\Omega_{S/R}$ is a free $R$-module generated by $\{dx_1, \dots, dx_n\}$ and the universal derivation $d\colon S \to \Omega^1_{S/R}$ is given by

$$df = \sum_{i=1} \frac{\partial f}{\partial x_i} dx_i$$

where $\frac{\partial f}{\partial x_i}$ denotes the formal partial derivative of $f$ with respect to $x_i$.

## 2.2. Properties of $\Omega_{B/A}$ and important exact sequences.

**Lemma 2.1.** Let $R$ be a commutative ring. Then:

(i) For any multiplicative subset $S$ of $R$, $\Omega_{S^{-1}R/R} = 0$

(ii) If $S$ is an $R$-algebra where $R \to S$ is surjective, then $\Omega_{S/R} = 0$. In particular, for any ideal $I$ of $R$, $\Omega^1_{(R/I)/R} = 0$.

*Proof.* (i) For any $t \in S^{-1}R$ there is some $s \in S$ such that $st \in R$. As $d\colon S^{-1}R \to \Omega_{S^{-1}R/R}$ is an $R$-derivation then $dr = 0$ for all $r \in R$ and so $0 = d(st) = sdt + tds = sdt$ as $S \subset R$. Now this gives $0 = s(dt)$ and as $s$ is nonzero then $dt = 0$.

(ii) Let $\varphi\colon R \to S$ be a surjection. The for any $s \in S$, we have $\varphi(r) = s$ for some $r \in R$, and thus $ds = d\varphi(r) = d(1 \cdot r) = rd(1) = 0$. Moreover, as for any ideal $I$ of $R$ we have a natural surjection $\pi\colon R \to R/I$ then $\Omega_{(R/I)/I} = 0$. □

If we're given the following commutative square of ring homomorphisms,

$$(\star)\qquad \begin{tikzcd} B \arrow[r, "\varphi"] & B' \\ A \arrow[u, "\alpha"] \arrow[r, "\psi"'] & A' \arrow[u, "\beta"'] \end{tikzcd}$$

then it follows from the construction of the universal derivation that we have the following induced commuting square

$$\begin{tikzcd} \Omega_{S/R} \arrow[r] & \Omega_{S'/R'} \\ S \arrow[u, "d"] \arrow[r, "\varphi"'] & S' \arrow[u, "d'"'] \end{tikzcd}$$

The map from $\Omega_{B/A} \to \Omega_{B'/A'}$ is given by $fdg \in \Omega_{B/A}$ to $\varphi(f)d\varphi(g)$ in $\Omega_{B'/A'}$.

**Lemma 2.2** ([SP], Tag 00RR)**.** In the diagram ($\star$), suppose $B \to B'$ is surjective with kernel $I \subset B$. Then $\Omega_{B/A} \to \Omega_{B'/A'}$ is surjective with kernel generated as a $B$-module by the elements $ds$, where $s \in B$ such that that $\varphi(s) \in \beta(A')$. (This includes in particular the elements $d(i)$, where $i \in I$.)

**Proposition 2.2.** Let $A \to B \to C$ be ring maps. Then there is a canonical exact sequence

$$C \otimes_B \Omega_{B/A} \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

*Proof.* For the ring maps $A \to B \to C$, we can fit this into the following diagram

$$\begin{tikzcd} C \arrow[r] & C \\ A \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

Applying Lemma 2.2 to the identity map $C \to C$, as it is surjective, we can conclude that

$$\Omega_{C/A} \to \Omega_{C/B} \to 0$$

is exact at $\Omega_{C/B}$ as $\Omega_{C/A} \to \Omega_{C/B}$ is surjective where $dc \mapsto dc$. We have more relations with $\Omega_{C/B}$ however as $db = 0$ for $b \in B$. On the other hand, if we look at the diagram

$$\begin{tikzcd} B \arrow[r] & C \\ A \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

there is a map from $\Omega_{B/A} \to \Omega_{C/A}$, but this is a map of $B$-modules and so if we want to get a map of $C$-modules then we need to tensor up to $C$. So the preceding commutative square gives us the following sequence

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$$

where $\Omega_{B/A} \otimes_B C$ is a $C$-module. This sequence is indeed exact as $\Omega_{C/B} \otimes_B C$ is generated as a $C$-module by $db \otimes 1$ where $b \in B$ and $\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \colon db \otimes 1 \mapsto db$ are the elements in $\ker\{\Omega_{C/A} \to \Omega_{C/B}\}$ by Lemma 2.2. □

**Proposition 2.3.** Let $B$ be an $A$-algebra. Then

(a) For any other $A$-module $A'$, there is a natural isomorphism of $A' \otimes_A B$-modules

$$\Omega^1_{(A'\otimes_A B)/A'} \simeq \Omega^1_{B/A} \otimes_B (A' \otimes_A B)$$

(b) If $S$ is a multiplicative set of $B$, then

$$\Omega_{B/A} \otimes_B B[S^{-1}] \to \Omega_{B[S^{-1}]/A}$$

is an isomorphism.

(c) Suppose $C = B/I$ for an ideal $I \subset B$. Then there is an exact sequence of $C$-modules

$$I/I^2 \xrightarrow{f} \Omega^1_{B/A} \otimes_B C \xrightarrow{d\varphi} \Omega^1_{C/A} \to 0$$

where $f \colon x + I^2 \mapsto dx \otimes 1$ and $\varphi$ is the quotient map.

*Proof.* (a) We proceed by obtaining another satisfactory description of the universal property of kahler differentials, and thus obtain an isomorphism. The universal property of $\Omega_{B/A}$ gives us the derivation $d \colon B \to \Omega_{B/A}$ and this induces an $A'$-linear map $d \otimes \mathrm{id}_{A'} \colon A' \to \Omega_{B/A} \otimes_A A' = \Omega_{B/A} \otimes_B (A' \otimes_A B)$ which is a derivation. Let $\rho \colon B \to A' \otimes_A B$ be the canonical map $x \mapsto x \otimes 1$. Now given some $A'$-derivation into an $(A' \otimes_A B)$-module $\theta \colon A' \otimes_A B \to M$, the mapping $\theta \circ \rho$ from $S$ to $M$ will be an $R$-derivation, and it factor as $f \circ d$ by the universal property since we have an induced $B$-bilinear map $f \colon \Omega_{B/A} \to M$. Hence $f \otimes \mathrm{id}_{R'} \colon \Omega_{B/A} \otimes_A A' \to M \otimes_B (A' \otimes_A B) = M$ gives us the satisfactory factorization of $\theta$.

(b) Note $\Omega_{B/A} \otimes_B B[S^{-1}] = \Omega_{B/A}[S^{-1}]$. Then by Proposition 2.2, we have that

$$\Omega_{B/A}[S^{-1}] \to \Omega_{B[S^{-1}]/A} \to \Omega_{B[S^{-1}]/B} \to 0,$$

but $\Omega_{B[S^{-1}]/B} = 0$ since if we take $b \in B$ and choose $s \in S$ such that $sb \in B$ then $sdb = d(sb) = 0$ which means that $db = 0$ as $s$ is invertible in $B$. The map $\Omega_{B/A}[S^{-1}] \to \Omega_{B[S^{-1}]/A}$ is given by $\frac{db}{s} \mapsto \frac{1}{s} \cdot d(\frac{b}{1})$ which is an injection. So then we have

$$0 \to \Omega_{B/A}[S^{-1}] \to \Omega_{B[S^{-1}]/A} \to 0,$$

which does indeed induce the isomorphism we wanted.

(c) After applying the Yondea Lemma, the isomorphism between Der and Hom, and unwinding some definitions, it suffices to show that

$$0 \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M) \to \mathrm{Hom}_C(I/I^2, M)$$

is an exact sequence of $C$-modules $M$. Firstly we have $\mathrm{Hom}_C(I/I^2, M) = \mathrm{Hom}_{B/I}(I/I^2, M) \simeq \mathrm{Hom}_B(I, M)$ per the universal property of quotients. The morphism $\mathrm{Der}_A(B, M) \to \mathrm{Hom}_B(I, M)$ sends a map $d \colon B \to M$ to $d|_I \colon I \to M$, and lastly the quotient of this is map is $\mathrm{Der}_A(B/I, M)$ and thus the sequence is exact and we win. □

**Corollary 2.2.** Let $R \to S$ be a ring map and $I$ the kernel of $S \otimes_R S \xrightarrow{\mu} S$, where $\mu$ denotes the multiplication map. Then there is a canonical isomorphism of $S$-modules

$$\begin{aligned} I/I^2 &\xrightarrow{\sim} \Omega_{S/R} \\ f &\mapsto f \otimes 1 - 1 \otimes f. \end{aligned}$$

*Proof.* Consider the following commutative diagram

$$\begin{tikzcd} S \otimes_R S \arrow[r, "\mu"] & S \\ S \arrow[u] \arrow[r] & S \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

This gives us the following short exact sequence

$$0 \to I/I^2 \to \Omega_{S\otimes_R S/S} \otimes_{S\otimes_R S} S \to \Omega_{S/S} \to 0,$$

but $\Omega_{S/S} = 0$ because of surjectivity, so this simplifies to

$$0 \to I/I^2 \to \Omega_{S\otimes_R S/S} \otimes_{S\otimes_R S} S \to 0.$$

Now by part (a) we have that $\Omega_{S\otimes_R S/S} = \Omega_{S/R} \otimes_S (S \otimes_R S)$ as $S \to S \otimes_R S$ is the base change of $R \to S$, and moreover we compute $\Omega_{S\otimes_R S/S} \otimes_{S\otimes_R S} S$, for which the statement proposition follows immediately:

$$\Omega_{S\otimes_R S/S} \otimes_{S\otimes_R S} S = \Omega_{S/R} \otimes_S (S \otimes_R S) = \Omega_{S/R}.$$

□

**Corollary 2.3.** If $R = R'$ and $S \to S'$ is split surjective with kernel $I$ ($\star$). Then

$$\begin{aligned} I/I^2 &\to \Omega_{S/R} \otimes_S S' \to \Omega_{S'/R} \to 0 \\ f + I^2 &\mapsto df \otimes 1. \end{aligned}$$

is a split short exact sequence.

*Proof.* Let $\alpha\colon S \to S'$ be surjective, and let $\beta\colon S' \to S$ be the splitting. Then $S = I \oplus \beta(S')$ and $\beta$ yields a right inverse (also called a section) of $\Omega_{S/R} \otimes_S S' \to \Omega_{S'/R}$ and we conclude that this map is split surjective. Consider the map $D\colon S \to I/I^2$, $x \mapsto x - \beta(\alpha(x))$. Then we claim that $D$ is an $R$-derivation. We see this since: $D(xy) = xy - \beta(\alpha(xy)) = x(y - \beta(y)) + \beta(\alpha(y))(x - \beta(\alpha(x))$ and $\beta(\alpha(y)) - y \in I$, which gives us that $\beta(\alpha(y))(x - \beta(\alpha(x))) = y(x - \beta(\alpha(x)))$; therefore $D(xy) = x(y - \beta(y)) + y(x - \beta(\alpha(x)))$ and $D$ is an $R$-derivation. Now $xD(s) = 0$ with $x \in I$ and $s \in S$. By the universal property we have that $D$ gives us a map $\tau\colon \Omega_{S/R} \otimes_S S' \to I/I^2$, which gives injectivity of the left hand maps. □

**Example 2.2.** Let $R$ be a ring and consider $R[x_1, \dots, x_n]$. Then $R \to R[x_1, \dots, x_n] \to R$ is split by evaluation of $R[x_1, \dots, x_n]$ at zero. The kernel of this surjection is given by $I = (x_1, \dots, x_n)$, and $I/I^2$ is free of rank $n$ with basis images of $x_i$. By our previous work, we have that $I/I^2 \simeq \Omega_{R[x_1,\dots,x_n]/R}$, and so, in particular, we can conclude that $\Omega_{R[x_1,\dots,x_n]/R}$ is free of rank $n$ with basis $dx_1, \dots, dx_n$.

**Lemma 2.3.** If $S$ is a finitely presented $R$-algebra, then $\Omega_{S/R}$ is a finitely presented $S$-module.

*Proof.* Write $S = R[x_1, \dots, x_n]/(f_1, \dots, f_m)$. Then we get a surjective map $R[x_1, \dots, x_n] \to S$ with kernel $I = (f_1, \dots, f_m)$. This now induces an exact sequence

$$\begin{aligned} I/I^2 &\to \Omega_{R[x_1,\dots,x_n]/R} \otimes_{R[x_1,\dots,x_n]} S \to \Omega_{S/R} \to 0 \\ f &\mapsto \left(\sum_i \frac{\partial f}{\partial x_i} dx_i\right) \otimes 1, \end{aligned}$$

and $\Omega_{R[x_1,\dots,x_n]/R} \otimes_{R[x_1,\dots,x_n]} S$ is free of rank $n$ and $I/I^2$ is a finitely generated $S$-module by the images of $f_1, \dots, f_m$. This implies that $\Omega_{S/R}$ is finitely presented. □

**Definition 2.2.** Let $A \to B$ be a ring homomorphism. The **sheaf of Kähler differentials** for $\pi\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is the sheaf $\Omega^1_{\operatorname{Spec} B/\operatorname{Spec} A} = \widetilde{\Omega^1_{B/A}}$ on $\operatorname{Spec} B$.

## 3. Étale morphisms

3.1. **Unramified map and Separable Extensions.** A morphism of ring maps $R \to S$ is said to be of *finite type* there is a surjection of $R$-algebras $R[x_1, \dots, x_n] \to S$, and also a ring map $R \to S$ is said to be of *finite presentation* if $S$ is isomorphic to a quotient of $R[x_1, \dots, x_n]$. In line with this, we make the following definition for schemes.

**Definition 3.1.** Let $f\colon X \to Y$ be a morphism of schemes, let $x \in X$, and let $\ell = f(x)$. Then we say that $f$ is **locally of finite presentation (resp. finite type)** at $x$ if there exists an affine open neighborhoods $U = \operatorname{Spec} A \subset X$ of $x$ and $V = \operatorname{Spec} B \subset Y$ of $\ell$ with $f(U) \subset V$ such that the induced ring map $\mathscr{O}_Y(V) \to \mathscr{O}_X(U)$ is of finite presentation (resp. finite type). We say that $f$ is locally of finite presentation if $f$ is locally of finite presentation at every $x \in X$.

**Example 3.1.** Any open immersion is locally of finite presentation, and an open immersion is of finite presentation if and only if it is quasi-compact.

Any closed immersion is of finite type since if we have $\pi \colon X \to Y$ being a closed immersion then for any point $y \in Y$, and an affine open $V = \operatorname{Spec} A$ of $Y$, then $\pi^{-1}(V) \to V$ is also a closed immersion and thus have an induced surjective map $A \to A/I$ where $A/I$ is indeed a finitely generated $A$-algebra.

**Example 3.2.** For a ring $R$, the open immersion $\operatorname{Spec} R_f \to \operatorname{Spec} R$ is a morphism of finite type as $R_f$ is generated as an $R$ algebra by $1/f$ where $f \in R$. However, the localization at a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ of $R$ does not induce a morphism of schemes that is finite type in general as $A_\mathfrak{p}$ is not a finitely generated $A$-algebra. In particular, $\operatorname{Spec}(\mathscr{O}_{X,P}) \to X$ is not of finite type where $P \in X$. More generally, a morphism of affine schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is of finite type if $B$ is a finitely generated $A$-algebra. So, using this, if we let $K$ be a number field (that is, a finite field extension of $\mathbf{Q}$, e.g. $\mathbf{Q}(\sqrt{2})$) then the corresponding ring of algebraic integers $O_K$ is indeed a finitely generated $\mathbf{Z}$-module; thus we have that $\operatorname{Spec} O_K \to \operatorname{Spec} \mathbf{Z}$ is of finite type. Relevant to latter discussion: let $p$ be prime, $\zeta$ is a $p$th root of unity and consider the corresponding number field $\mathbb{Q}(\zeta)$ (often called the *cyclotomic field*). Then the integral basis for $O_{\mathbb{Q}(\zeta)} = \mathbf{Z}[\zeta]$is given by $(1, \zeta, \zeta^2, \dots, \zeta^{p-2})$ and it is of rank $[\mathbf{Q}(\zeta) : \mathbf{Q}]$.

**Proposition 3.1.** Let $X/S$ be a scheme over $S$ with structure morphism $f \colon X \to S$. Then $X/S$ is of locally finite type (resp. finite type/resp. affine/resp. finite) if $X/S$ is of locally finite type (resp. finite type/resp.affine/resp. finite) for one affine covering of $S$.

**Definition 3.2.** A local homomorphism $f \colon A \to B$ of local rings is **unramified** if $f(\mathfrak{m}_A)B = \mathfrak{m}_B$ and the map on residue fields $\kappa(\mathfrak{m}_A) = A/\mathfrak{m}_A \to \kappa(\mathfrak{m}_B) = B/\mathfrak{m}_B$ is a finite separable extension.

**Discussion 3.1.** A nonzero polynomial $f$ in $k[x]$ where $k$ is a field is said to be *separable* if $(f, f') = (1)$, i.e. $f$ and its (formal) derivative generate the unit ideal of $k[x]$; otherwise $f$ is *inseparable*. Let $L/K$ be an algebraic field extension. (In the case we talk about finite field extensions, we mean $K$ is subfield of $L$ such that the degree of $L/K$ is finite, i.e. $[L \colon K] < \infty$, where $[L \colon K] = \dim_K L$ as a $K$-vector space.) An element $\alpha \in L$ is said to be *separable over* $K$ if $\alpha$ is algebraic over $K$ for a separable polynomial in $K[x]$ [6] Furthermore, the field $L$ is called *separable* over $K$ if every $x \in L$ is separable; otherwise $L$ is said to be *inseparable*. And, lastly, a field $K$ is called *perfect* if all its finite field extensions are separable.

**Proposition 3.2.** All finite fields, algebraically closed fields, and all fields of characteristic zero are perfect.

**Lemma 3.1.** If $f \in k[x]$ is an irreducible polynomial for a field $k$, then $f$ is inseparable if and only if $f' = 0$.

*Proof.* Let $f(x) \in k[x]$ such that it is irreducible. Suppose that $f$ is inseparable; then $\gcd(f(x), f'(x))$ is a nontrivial divisor for $f(x)$ and $f'(x)$. Thus $\gcd(f(x), f'(x)) = \deg f(x)$ as we have $f$ being irreducible. So $\deg f'(x) < \deg f(x) = \deg(\gcd((f(x), f'(x)))$, so $\gcd(f(x), f'(x))$ can't divide $f'(x)$ unless if we had $f'(x) = 0$. Now as $f$ is irreducible, then $\deg f(x) > 0$ and $f(x)$ is not zero nor a unit of $k[x]$. Let $f(x)' = 0$. This gives us $\gcd(f(x), f'(x)) = f$ which is not in $k^\times$, and $f$ is inseparable. □

In the case that we do not have a perfect field, and we would like to check whether or not we have a separable field extension on our hands, it's arduous work to verify that every element of the extension is indeed separable. The following two theorems (that are difficult) will be of practical use as it will allow us to check sufficiently on a set of field generators for an extension.

[6]Some define, very directly, $\alpha \in L$ to be *separable* if the the minimal (necessarily irreducible) polynomial of $\alpha$ over $K$ is a separable polynomial (i.e. it posses no repeated roots in a field extension, or, the most easiest definition to work with, its formal derivative is not zero). The meaning of minimal polynomial is that if we let $\alpha \in L$ then $\alpha$ has a minimal polynomial when $f(\alpha) = 0$ (i.e. $\alpha$ is algebraic over $K$) for some non-zero polynomial $f(x) \in F[x]$ that is defined as being the monic polynomial of least degree of all polynomials in $F[x]$ for which $\alpha$ is a root. For example, if $L = \mathbf{R}$ and $K = \mathbf{Q}$, then $\alpha = \sqrt{2} \in L$ has the minimal polynomial $a(x) = x^2 - 2$. In the discussion above, our $\alpha$ which is algebraic for some separable polynomial in $K[x]$ gives us that its minimal polynomial in necessarily separable.

**Theorem 3.1** (Primitive Element Theorem)**.** Every finite separable extension of $K$ has the form $K(\alpha)$ for some $\alpha \in K(\alpha)$ that is separable over $K$.

**Corollary 3.1.** Let $L/K$ be a finite field extension. Then $L/K$ is separable if and only $L \simeq K[x]/(f(x))$ for some monic irreducible separable polynomial $f \in K[x]$.

*Proof.* This is immediate. Let $L/K$ be a finite separable field extension. Then $L = K(\alpha)$ for some separable $\alpha$. Take $f$ to be the minimal polynomial for which $\alpha$ is algebraic over, and let $\alpha$ be the image of $x$ in $k[x]/(f)$. □

As with Definition 3.2, we need for the residue field of $A$ at its local ring $\mathfrak{m}_A$ to be such that $\kappa(\mathfrak{m}_B) \simeq \kappa(\mathfrak{m}_A)[x]/(f(x))$, where $f$ is separable (i.e. $f(x)$ is irreducible and $f'(x) \neq 0$). This discussion of separable extensions is particularly important to our latter notion of *étale*, and how separable field extensions play with the module of relative differentials.

**Proposition 3.3.** Suppose $L/K$ is a finite extension. If $L/K$ is separable, then $\Omega^1_{L/K} = 0$.

*Proof.* Suppose $L$ be a finite separable extension of $K$. As per Lemma 3.1, we have that $L \simeq K[x]/(f(x))$, for which $(f(x), f'(x)) = K[x]$ but this additionally means that $f(x)$ has derivative not zero (as mentioned in Discussion 3.1). Now $d\colon K[x]/(f(x)) \simeq L \to \Omega_{L/K}$ and $0 = d(f(x)) = f'(x)dx$, meaning that $dx = 0$. As $\Omega_{L/K}$ is generated by $dx$, this shows that $\Omega_{L/K} = 0$. □

**Definition 3.3.** A morphism $f\colon X \to Y$ of schemes is **unramified at** $P \in X$ if $f$ is locally of finite type, the induced map on local rings $f^\sharp\colon \mathscr{O}_{Y,f(P)} \to \mathscr{O}_{X,P}$ is such that $f^\sharp(\mathfrak{m}_{Y,f(P)})\mathscr{O}_{X,P} = \mathfrak{m}_{X,P}$, and the extension of residue fields $\kappa(P)/\kappa(f(P))$ is finite and separable, where $\kappa(P) = \mathscr{O}_{X,P}/\mathfrak{m}_{X,P}$ and $\kappa(f(P)) = \mathscr{O}_{Y,f(P)}/\mathfrak{m}_{Y,f(P)}$. We say that $f$ is **unramified** if it is unramified at all $P \in X$.

### 3.2. Flat maps.

**Definition 3.4.** An $A$-module $M$ is **flat (over** $A$**)** if the functor $M \otimes_A -\colon \mathsf{Mod}_A \to \mathsf{Mod}_\mathsf{A}$ is exact. An $A$-module $M$ is **faithfully flat (over** $A$**)** if every complex of a $A$-modules $N_1 \to N_2 \to N_3$ is exact if and only if $M \otimes_A N_1 \to M \otimes_A N_2 \to M \otimes_A N_3$ is exact.

We call a morphism of rings $A \to B$ **flat** if it makes $B$ into a flat $A$-module. Additionally, a ring morphism $A \to B$ is **faithfully flat** if it makes $B$ into a faithfully flat $B$-module.

**Example 3.3.** Free modules and projective modules are each flat.

**Lemma 3.2.** An $A$-module $M$ is faithfully flat if and only if for all $A$-module maps $\alpha\colon N \to N'$ we have $\alpha = 0$ if and only of $\mathrm{id}_M \otimes \alpha = 0$.

*Proof.* The forward direction is straightforward. As $\alpha\colon N \to N'$ then $0 \to \ker(\alpha) \to N \to N'$ is exact if and only $0 \to \ker(\alpha) \otimes_A M \to N \otimes_A M \to N' \otimes_A M$ is exact and thus this yields the result as $M \otimes_A \ker(\alpha) \simeq \ker(\mathrm{id}_M \otimes \alpha)$. For the backwards direction, let $N_1 \to N_2 \to N_3$ be some complex of $A$-modules. Then $M \otimes_A N_1 \to M \otimes_A N_2 \to M \otimes_A N_3$ is exact. Let $x \in \ker(N_2 \to N_3)$ of the original complex, and define the map $\alpha\colon A \to N_2/\operatorname{im} N_1$ where $\alpha\colon r \mapsto rx + \operatorname{im} N_1$. We get that $\alpha \otimes \mathrm{id}_M = 0$ as $M$ is assumed to be flat. Lastly, $\alpha = 0$, and as $x \in \operatorname{im}(N_1 \to N_2)$ then we can conclude the backward statement. □

**Proposition 3.4.** Let $M$ be a flat $A$-module. The following are equivalent:

(i) $M$ is faithfully flat.
(ii) For every $A$-module $N \neq 0$, $M \otimes_A N \neq 0$.
(iii) For all $\mathfrak{p} \in \operatorname{Spec} A$, $M \otimes_A \kappa(\mathfrak{p}) \neq 0$.
(iv) For all maximal ideals $\mathfrak{m}$ of $A$, $M \otimes_A \kappa(\mathfrak{m}) = M/\mathfrak{m}M \neq 0$.

*Proof.* $(i) \Rightarrow (ii)$ Take $M$ to be faithfully flat and $N$ be a nonzero $A$-module. For the identity map $\mathrm{id}_N\colon N \to N$, which is a nonzero map, gives us another nonzero map $M \otimes_A N \to M \otimes_A N$ with $M \otimes_A N \neq 0$ by Lemma 3.2. The implications from $(ii) - (iv)$ follow from this implication. $(iv) \Rightarrow (i)$ Let $N \neq 0$ and take a nonzero element $n \in N$. Define the map $f\colon A \to M$ by $a \mapsto an$, and write $I = \ker f$. We have an injective map $A/I \to N$. Let $\mathfrak{m}$ be a maximal ideal of $A$ where $\mathfrak{m}$ lies above $I$; we also have a surjective map $A/I \to A/\mathfrak{m}$. As $M$ is flat, then we have an injection $A/I \otimes_A M \to N \otimes_A M$ and also another surjective morphism $A/I \otimes_A M \to A/\mathfrak{m} \otimes_A M = M/\mathfrak{m}M$.

By hypothesis, $M/\mathfrak{m}M \neq 0$ and as $A/I \otimes_A M \to M/\mathfrak{m}M$ is surjective, $A/I \otimes_A M \neq 0$. Lastly, because $A/I \otimes_A M \to N \otimes_A M$ is injective we have that $N \otimes_A M \neq 0$. □

**Proposition 3.5.** Flat morphisms form a stable class.

**Theorem 3.2.** Let $A \to B$ be a ring map, and let $N$ be a $B$-module. Then, $N$ is a flat $A$-module (with the obvious structure) if and only if for all maximal ideals $\mathfrak{n}$ of B.

**Definition 3.5.** A morphism of schemes $f : X \to Y$ is a said to be **flat at** $P \in X$ if $\mathscr{O}_{X,P}$ is flat as an $\mathscr{O}_{Y,f(P)}$-module. The map $f : X \to Y$ is said to be **flat** if it is flat at every point of $X$. And $f : X \to Y$ of schemes if called **faithfully flat** if $f$ is flat and surjective (on topological spaces).

**Example 3.4.** An open immersion is flat. An open immersion is locally of the form $\operatorname{Spec} A_f \to \operatorname{Spec} A$ for some ring $A$ and $f \in A$, and $A_f$ is flat as for a multiplicative set $S \subseteq A$ the localization $S^{-1}A$ is a flat module.

We talked a bit about quasi-coherent sheaves and the pullback functor in §1.5, which a quick discussion of flatness is useful. Let $\pi : X \to Y$ be a map of schemes. Then we have a induced map $\pi^* : \mathsf{QCoh}_{\mathscr{O}_Y} \to \mathsf{QCoh}_{\mathscr{O}_X}$, which is only right exact. But once $\pi : X \to Y$ is flat, then indeed $\pi^*$ is an exact functor; this should somewhat be expected as, for affine schemes, $\operatorname{Spec} B \to \operatorname{Spec} A$ the pullback aligns with $- \otimes_A B : \mathsf{Mod}_A \to \mathsf{Mod}_B$.

**Theorem 3.3.** Let $f : A \to B$ be a morphism of rings. The following are equivalent:

(i) $f$ is faithfully flat.
(ii) $f$ is flat, and the induced map on spectra $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective
(iii) $f$ is injective, and $\operatorname{coker} f$ is flat as an $A$-module.

**Lemma 3.3.** Let $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ be local ring and $M \neq 0$ finitely generated $B$-module. Then, for a morphism of local rings $\varphi : (A, \mathfrak{m}) \to (B, \mathfrak{n})$, the module $M$ is faithfully flat if and only if $M$ is flat.

*Proof.* The reverse direction is clear as a faithfully flat module is flat by definition. For the other direction, take $M$ to be flat. To show that $M$ is faithfully flat we use the criterion of Proposition 3.4 (iv). As $\varphi$ is a local morphism, then $M \otimes_A A/\mathfrak{m} = M/\varphi(\mathfrak{m})M = M/\mathfrak{n}M \neq 0$, by Nakayama's lemma. □

If we take $f : A \to B$ to be a flat ring map where $\mathfrak{p} \subset B$ is a prime ideal, then we have an induced flat map $A_{f^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$. As $B_\mathfrak{p}$ is of course a finitely generated $B_\mathfrak{p}$-module then by the previous lemma we get that $B_\mathfrak{p}$ is a flat $A_{f^{-1}(\mathfrak{p})}$-module. Therefore $A_{f^{-1}(\mathfrak{p})} \to B_\mathfrak{p}$ is in fact faithfully flat.

As an important side note this the notion of flat morphisms for schemes does in fact play nicely with affine schemes, and what I mean by this is that $A \to B$ is flat if and only if $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat. If we assume for $A \to B$ to be flat, and take a prime ideal $\mathfrak{q} \in \operatorname{Spec} B$, then $A_{f^{-1}(\mathfrak{q})} \to B_\mathfrak{q}$ is flat, so $\operatorname{Spec} B \to \operatorname{Spec} A$ is a flat map. For the other direction, if we assume $\operatorname{Spec} B \to \operatorname{Spec} A$ to be flat, then, by definition, we have that $A_{f^{-1}(\mathfrak{q})} \to B_\mathfrak{q}$ is flat for any prime ideal $\mathfrak{q} \in \operatorname{Spec} B$. Also, $A \to A_{f^{-1}(\mathfrak{q})}$ is flat, which gives that $A \to B_\mathfrak{q}$ is flat as well. By Theorem 3.2, $A \to B$ is flat, and therefore we win. Thus, as a consequence, if $A \to B$ is faithfully flat, then obviously $f$ is flat so the map on spectra $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat, and $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective by Therorem 3.3. By definition, we have that $\operatorname{Spec} B \to \operatorname{Spec} A$ is faithfully flat. Hence if $A \to B$ is faithfully flat then $\operatorname{Spec} B \to \operatorname{Spec} A$ is faithfully flat.

**Remark 3.1** (Milne)**.** It suffices to check the unramified and flatness condition for closed points of $P$ of $X$.

**Theorem 3.4.** Let $X$ and $Y$ be irreducible varieties, and $f : X \to Y$ faithfully flat. Then

$$\dim X_y = \dim X - \dim Y. \tag{5}$$

### 3.3. **Being flat and unramified.**

**Definition 3.6.** A morphism of schemes $f: X \to Y$ is called **étale at** $P \in X$ if $f$ is unramified and flat at $P$. It is called **étale** if it is étale at every point $P \in X$.

**Example 3.5.** Open immersions are étale. A closed immersion is unramified but is not étale as the morphism $A/I \to A$ is not flat.

**Theorem 3.5.** Let $f: X \to Y$ be a morphism locally of finite type. The following are equivalent:

(i) $f$ is unramified at $x$.
(ii) $(\Omega_{X/Y})_x = 0$.
(iii) There is an open neighborhood $U \subseteq X$ of $x$ such that the diagonal morphism $\Delta_{X/Y}: X \to X \times_Y X$ restricts to an open immersion $\Delta_{X/Y}|_U : U \to X \times_Y X$.

*Proof.* $(i) \implies (ii)$ This is a local condition so we may assume $X = \operatorname{Spec} B$, and $Y = \operatorname{Spec} A$. Take $x = \mathfrak{q}$ and $y = \mathfrak{p}$ as prime ideals for $B$ and $A$ respectively. As we're assuming $f$ is unramified at $\mathfrak{q}$, then $\mathfrak{p}B_\mathfrak{q} = \mathfrak{q}B_\mathfrak{q}$ which gives $B_\mathfrak{q} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p}) = B_\mathfrak{q} \otimes_{A_\mathfrak{p}} (A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}) \simeq B_\mathfrak{q}/\mathfrak{p}B_\mathfrak{q} = \kappa(\mathfrak{q})$. Now, $(\Omega_{X/Y})_x \simeq \Omega_{B_\mathfrak{q}/A_\mathfrak{p}}$, and $\Omega_{B_\mathfrak{q}/A_\mathfrak{p}} \otimes_{B_\mathfrak{q}} \kappa(\mathfrak{p}) \simeq \Omega_{(B_\mathfrak{q} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p}))/\kappa(\mathfrak{p})} = \Omega_{\kappa(\mathfrak{q})/\kappa(\mathfrak{p})}$. The extension of fields $\kappa(\mathfrak{q})/\kappa(\mathfrak{p})$ is finite and separable, and so by Proposition 3.3 we have $\Omega_{\kappa(\mathfrak{q})/\kappa(\mathfrak{p})} = 0$. Lastly, as $f$ is of finite type then $\Omega_{B_\mathfrak{q}/A_\mathfrak{p}}$ is finitely generated, which gives, by Nakayama's lemma, that $\Omega_{B_\mathfrak{q}/A_\mathfrak{p}} = 0$. The tech behind proving the rest of equivalences is arduous work that would frankly wouldn't bring more insight to the reader from reading this specific paper, and thus we refer to the reader to look at [Milne13]. $\square$

**Proposition 3.6.** Étale morphisms form a stable class.

**Example 3.6.** Recall that $\mathbf{G}_m = \mathbf{A}^1_k \backslash \{0\}$ where we pick a base field $k$ for $\mathbf{G}_m$. Then $\mathbf{G}_m \backslash \{1\} \to \mathbf{G}_m$ where $t \mapsto t^2$ and $\operatorname{ch} k \neq 2$ is étale.

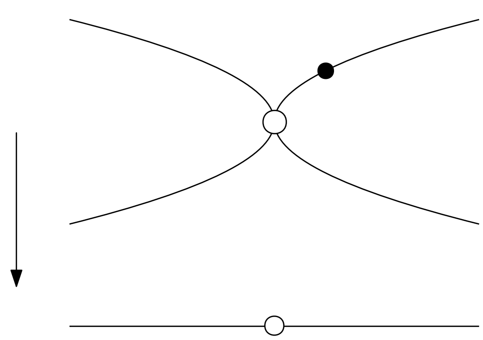

FIGURE 5. The map $t \mapsto t^2$. As at every point there are two square roots and we've deleted one of the preimagess of one the image then the above is our depiction of the mapping. In fact, this is an important example of an étale morphism not being finite (as it is not proper, however it is an étale surjection).

**Example 3.7.** The morphism $\mathbf{G}_m \to \mathbf{G}_m$ where $t \mapsto t^n$ and $n$ is invertible for the underlying scheme is étale.

**Example 3.8.** (i) The mapping $\mathbf{A}^1_k \to \mathbf{A}^1_k$ where $t \mapsto t^2$ is not étale. From the previous example, we can see that it is ramified at zero, and indeed we can compute the relative kahler differentials to get $\Omega_{\mathbf{A}^1_k/\mathbf{A}^1_k} = \frac{k[t]dt}{d(t^2)} = \frac{k[t]}{2tdt}$ and $2tdt$ doesn't generate the module.

(ii) The map $\mathbf{A}^1_k \to \operatorname{Spec} k[x,y]/(y^2 - x^3)$ where $t \mapsto (t^2, t^3)$ is not étale as it is indeed ramified at zero and is furthermore not flat. Similarly, $\mathbf{A}^1_k \to \operatorname{Spec} k[x,y]/(y^2 - x^2 - x^3)$ where $t \mapsto (t^2 - 1, t^3 - t)$ is not étale

**Theorem 3.6.** A morphism of schemes $f: X \to Y$ is étale if and only for each point $x \in X$ there exists open affine neighborhoods $U = \operatorname{Spec} R$ of $x$ and $V = \operatorname{Spec} S$ of $y = f(x)$ such that for some $n \in \mathbf{Z}_{\geq 0}$,

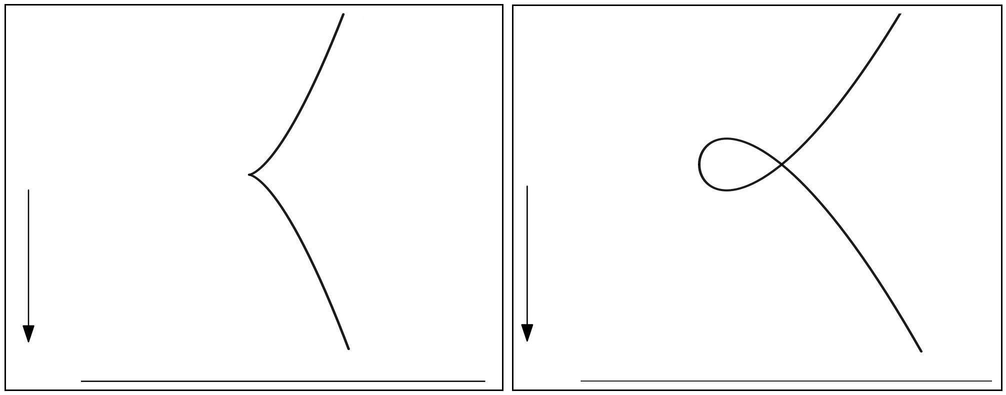

FIGURE 6. The image to the left and right depict the morphism from $\mathbf{A}_k^1 \to \operatorname{Spec} k[x,y]/(y^2 - x^3)$ and $\mathbf{A}_k^1 \to \operatorname{Spec} k[x,y]/(y^2 - x^2 - x^3)$ respectively.

$$R = S[T_1, \ldots, T_n]/(f_1, \ldots, f_n) \text{ and } \det \left[ \frac{\partial f_i}{\partial T_i} \right]_{i,j} \in R^\times$$

**Example 3.9.** The map $\operatorname{Spec} \mathbf{Z}[T]/(T \pm 1) \to \operatorname{Spec} \mathbf{Z}[T]$ is étale as calculating the determinant for $f(T) = T \pm 1$ results with simply $\pm 1$ which are of course both units in $\mathbf{Z}[T]/(T \pm 1)$. We note here that checking whether or not you have a unit on your hands after calculating the Jacobian for rings such as $R = S[T_1, \ldots, T_n]/(f_1, \ldots, f_n)$ is a difficult process in a lot of instances, yet to check for a specific point as given in the theorem you may just evaluate the result at the point in question. More generally, for a ring $A$, $A[T]/(T^r - a)$ is étale over $A$ if and only if $ra$ is invertible in $A$.

**Theorem 3.7.** Étale morphisms are open, that is, if $f: X \to Y$ is étale and $U$ is an open subset of $X$ then $f(U) \subset Y$ is open.

## 4. Sites and Sheaves

We will describe the basic and fundamental aspects of what Grothendieck topologies (and sites) throughout this section, and give a reason as to why they are actually desirable. In part, the motivation for the construction of these things are so that we can *free* the theory of sheaves from topological spaces in some sense; we need not rely strictly on working within the of open sets for some topological space, $X_{\text{open}}$, to get the full power of the theory developed surrounding (pre)sheaves of the form $\mathscr{F}: X_{\text{open}} \to \mathscr{C}$ (where we are often interested in $\mathscr{C}$ being **Ab** or **CRing**, and some others). It was Grothendieck who of course pioneered this in, for example, his 1957 Tohôku paper where he developed a sheaf theory (and a cohomology theory for sheaves) without needing to rely on topological spaces in the general sense.

As an aside to note, topology initially started out with working metric spaces, such as **R** and **C** with their corresponding norms, but the movement from *working away* from metric spaces and to topological spaces was the idea of abstracting away from the metric to subsets of the metric space (decreed to be open sets), as that was the more essential part of the construction of metric spaces. Then we can see a Grothendieck topology as an abstraction away from open sets, to a theory based on *coverings*, which is what Grothendieck saw as the essential part of topological spaces.

$\boxed{\text{From now on we will assume that our schemes are noetherian.}}$ [7]

### 4.1. Basics of Sites.

We will closely follow *sites* as they are presented in the Stacks Project (SP, [SP]). Without so much of the formality, in a category $\mathcal{S}$, we call a family of morphisms with a *fixed* target (say, the fixed target was $V \in \mathcal{S}$) a collection $\mathfrak{U} = \{\varphi_i : U_i \to V\}_{i \in I}$ that satisfies some nice properties: (i) an object $V \in \mathcal{S}$ (as described before), (ii) a set $I$ (possibly empty), and for all $i \in I$, a morphism $\varphi_i : U_i \to V$ of $\mathcal{S}$ with target $V$.

[7] The reader is invited to attempt to generalize statements that are made that do not require the noetherian hypothesis for our schemes.

**Definition 4.1** ([SP], Tag 00VH)**.** A **site** consists of a category $\mathcal{S}$ and a set $\mathrm{Cov}(\mathcal{S})$ consisting of families of morphisms with fixed target called **coverings**, such that:

(1) (isomorphism) if $\varphi\colon V \to U$ is an isomorphism in $\mathcal{S}$, then $\{\varphi\colon V \to U\}$ is a covering,
(2) (locality) if $\{\varphi_i\colon V_i \to V\}_{i\in I}$ is a covering and for each $i \in I$ we are given a covering $\{\psi_{ij}\colon U_{ij} \to V_i\}_{j\in J_i}$, then $\{\varphi_i \circ \psi_{ij}\colon U_{ij} \to V\}_{(i,j)\in\prod_{i\in I}\{i\}\times I_i}$ is also a covering, and
(3) (base change) if $\{V_i \to V\}_{i\in I}$ is a covering and $U \to V$ is a morphism in $\mathcal{S}$, then:
   (i) for all $i \in I$ the fibre product $V_i \times_V U$ exists in $\mathcal{S}$, and
   (ii) $\{V_i \times_V U \to V\}_{i\in I}$ is a covering.

When we refer to some site $(\mathcal{S}, \mathrm{Cov}(\mathcal{S}))$, we will abuse notation and just refer to it's underlying category.

As according to the base change properties of a site, we can make note of the fact that we require fibre products to exists so to work with, in a way, similar to that of intersection of sets. Recall that if $\mathcal{S} = \mathbf{Sets}$ and $U, V \subseteq W$ in **Sets** then their fibre product $U \times_W V$ is simply the intersection of $U$ and $V$, i.e. $U \times_W V = U \cap V$. This fibre product is the one associated to the following example of a site.

**Example 4.1.** (The Zariski Site). Let $X$ be a topological space. Consider the category of open sets in $X$, denoted as $X_{\mathrm{open}}$, where for any $U, V \in X_{\mathrm{open}}$,

$$\mathrm{Hom}_{X_{\mathrm{open}}}(U,V) = \begin{cases} \{i\}, & \text{if } U \subseteq V\text{, and } i\colon U \to V \text{ is the inclusion} \\ \emptyset & \text{otherwise.} \end{cases}$$

Now let $\mathrm{Cov}(W)$ be the collection of families $\{W_i \to W\}_{i\in I}$ such that $\bigcup_i W_i = W$, i.e. $\{W_i\}_{i\in I}$ form an open covering of $W$. Then $\mathrm{Cov}(W)$ is a Grothendieck topology on $\mathcal{S}$, and we call $\mathrm{Cov}(W)$ the *classical Grothendieck topology.* Furthermore, if we let $X$ be a scheme, then the site associated to the underlying topological space $|X|$, is called the *(small) Zariski site*, denoted by $X_{\mathrm{Zar}}$.

**Example 4.2.** (étale, fppf, and fpqc sites ). Let $X$ be a scheme.

We define $X_{ét}$, which is called the *small étale site* on $X$, to be the full subcategory of $\mathsf{Et}/X$ whose objects are étale morphisms $V \to X$, i.e. consisting of schemes étale over $X$, and we call a collection of (surjective étale) morphisms $\{\varphi_i\colon V_i \to V\}_{i\in I}$ an open covering (i.e. to be in $\mathrm{Cov}(V)$) if $\bigcup_i \varphi_i(V_i) = V$ as topological spaces. As a quick example, $\mathbf{A}^1 \setminus \{0\}$ (over $\mathbf{C}$) to $\mathbf{A}^1 \setminus \{0\}$ where $x \mapsto x^2$ is an étale cover. As an aside, these $X$-morphisms between the objects of $X_{ét}$ are also étale.

The *big étale site* is given by the category $\mathsf{Sch}/X$ where a covering of a scheme $X$ is a collection of surjective étale morphisms $\{\varphi_i\colon U_i \to X\}$ of $\mathsf{Sch}/X$ such that $\bigcup_i \varphi_i(U_i) = X$. We denote the big étale site by $X_{Ét}$.

To define the *big fppf site* $X_{\mathrm{fppf}}$ (resp. the *big fpqc site*, $X_{\mathrm{fpqc}}$), on $\mathsf{Sch}/X$ we call an open covering a family $\{\varphi_i\colon V_i \to V\}_{i\in I}$ of $X$-morphisms such that $\bigsqcup V_i \to V$ is fppf (resp. fpqc). The reason for these abbreviations are that fppf and fpqc are French for *fidèlement plat de présentation finie* and *fidèlement plat quasi-compact* respectively.

As shown how we described the small and big étale sites, it is clear to see how we would describe the small fppf and fpqc sites, however, we omit a description since they're generally not nice to work with; a main flaw is the failure of the morphisms between the objects of the small sites not being themselves fppf or fpqc in either case—all $k$-varieties are fppf over $\operatorname{Spec} k$, yet a $k$-morphism between two $k$-varieties is not necessarily flat, and there's also a similar fault with fpqc.

4.2. **(Pre)Sheaves as sites.**

**Definition 4.2.** A **presheaf (of abelian groups)** $\mathscr{F}$ on a site $\mathcal{S}$ is a contravariant functor $\mathscr{F}\colon \mathcal{S} \to \mathbf{Ab}$. An element in $\mathscr{F}(U)$, where $U \in \mathcal{S}$, is called a **section of $\mathscr{F}$ over** $U$.

**Example 4.3.** Let $M$ be an abelian group. Then we define the *constant presheaf* $M$ on a site $\mathcal{S}$ as the contravariant functor $\underline{M}$ such that $\underline{M}(U) = M$ for all $U \in \mathcal{S}$ and for all morphisms of $\mathcal{S}$ to the identity morphism $M \to M$.

**Definition 4.3.** Let $A, B, C$ be sets, and $f\colon A \to B$, $g\colon B \to C$, $h\colon B \to C$ be functions. Then the sequence

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C$$

is said to be **exact** if $f$ is injective, and $f(A)$ equals the *equalizer* $\{b \in B\colon g(b) = h(b)\}$ of $g$ and $h$.

**Definition 4.4.** Let $\mathscr{F}$ be a presheaf on a site $\mathcal{S}$. Then $\mathscr{F}$ is a **sheaf** if

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \times_U U_j) \tag{6}$$

is exact for all open covering $\{U_i \to U\}$. Notice that the arrows in the right of the exact sequence correspond to the projections from $U_i \times_U U_j \to U_i$ and $U_i \times_U U_j \to U_j$. A **morphism of sheaves** $\mathscr{F} \to \mathscr{G}$ is a morphism at the level of presheaves.

We will call a sheaf on $X_{ét}$ an *étale sheaf.* Notice that if we have an étale sheaf then it is also a sheaf on the Zariski site, which is what we mostly commonly think of sheaf as being from the beginning and up 'till now parts of this paper. It's often a difficult task to check whether or not you have a sheaf on your hands, but there is a useful criterion that we should give.

**Theorem 4.1.** Let $\mathscr{F}$ be a presheaf on $X_{ét}$. Then $\mathscr{F}$ is an étale sheaf if and only if $\mathscr{F}$ is a sheaf on the Zariski site and for any covering $V \to U$ of affine étale $X$-schemes, the following is an equalizer diagram

$$\mathscr{F}(U) \longrightarrow \mathscr{F}(V) \rightrightarrows \mathscr{F}(V \times_U V).$$

That is, it suffices to check on affine étale coverings consisting of only one map.

**Example 4.4.** Let $V \to X$ be an étale morphism, and write $\Gamma(V, \mathscr{O}_V) = \mathscr{O}_{X_{ét}}(V)$. This is indeed a Zariski sheaf, and so it remains to use the Theorem 4.1 criterion to check that this is an étale sheaf. Let $U_1 = \operatorname{Spec} A \to U_2 = \operatorname{Spec} B$ be a morphism of affine étale $X$-schemes, then it remains to check that

$$A \to B \rightrightarrows B \otimes_A B \tag{7}$$

is an equalizer diagram. The maps $B \rightrightarrows B \otimes_A B$ are the maps $b \mapsto b \otimes 1$ and $b \mapsto 1 \otimes b$. It's simple to see that having an equalizer digram of (7) corresponds to having an exact sequence $0 \to A \to B \xrightarrow{b \mapsto b\otimes 1 - 1\otimes b} B \otimes_A B$, which is an exact sequence as $\operatorname{Spec} A \to \operatorname{Spec} B$ is faithfully flat by Theorem 3.3.

**Example 4.5.** Let $X$ be an $S$-scheme, and consider the presheaf $U \mapsto \operatorname{Hom}(U, X)$, which is a sheaf on $X_{Zar}$. For affine $X = \operatorname{Spec} T$, the exactness of $X(A) = \operatorname{Hom}(T, A) \to X(B) = \operatorname{Hom}(T, B) \rightrightarrows X(B \otimes_A B) = \operatorname{Hom}(T, B \otimes_A B)$ is true because of the work done from the previous example. Therefore we have that $U \mapsto \operatorname{Hom}(U, X)$ is an étale sheaf.

**Theorem 4.2** (Grothendieck)**.** Let $F$ be a representable functor $\mathsf{Sch}/S \to \mathsf{Set}$. Then $F$ is a sheaf in the fpqc topology.

### 4.3. Étale cohomology definition.

**Theorem 4.3** ([SP], Section 03NT)**.** The category of abelian sheaves on a site is an abelian category which has enough injectives.

Given this theorem, we define *étale cohomology* in terms of a derived functor description on a site. Fix a scheme $X$. Then as we have $\Gamma(X, -)\colon \mathsf{AbShv}_{X_{ét}} \to \mathsf{Ab}$, the global section functor, that is once again left exact, we build right derived functors $R^i\Gamma$, which are additive functors from $\mathsf{AbShv}_{X_{ét}}$ to $\mathsf{Ab}$. For an abelian étale sheaf $\mathscr{F}$, choose an injective resolution $\mathscr{I}^\bullet$ of $\mathscr{F}$, for which we write $R^i\Gamma(X_{ét}, \mathscr{F}) = h^i(\mathscr{F}(\mathscr{I}^\bullet))$. These right derived functors have the following properties:

- There is a natural isomorphism $\Gamma(X_{ét}, -) \simeq R^0\Gamma(X_{ét}, -)$.

- If $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is an exact sequence of abelian étale sheaves on $X_{\acute{e}t}$, then there exists natural connecting homomorphisms $\delta^i \colon H^i_{\acute{e}t}(X, \mathscr{H}) \to H^{i+1}(X_{\acute{e}t}, \mathscr{F})$ , giving the long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r, "\delta^{i-1}"] & H^i_{\acute{e}t}(X,\mathscr{F}) \arrow[r] & H^i_{\acute{e}t}(X,\mathscr{G}) \arrow[r] & H^i_{\acute{e}t}(X,\mathscr{H}) \arrow[dlll, "\delta^i"'] \\
H^{i+1}_{\acute{e}t}(X,\mathscr{F}) \arrow[r] & H^{i+1}_{\acute{e}t}(X,\mathscr{G}) \arrow[r] & H^{i+1}_{\acute{e}t}(X,\mathscr{H}) \arrow[dll, "\delta^{i+1}"] & & \\
\cdots & & & &
\end{tikzcd}$$

Étale cohomology is focused most $R^i\Gamma(X_{\acute{e}t}, \underline{\mathbf{Z}/\ell^n\mathbf{Z}}) = H^i_{\acute{e}t}(X, \underline{\mathbf{Z}/\ell^n\mathbf{Z}})$, where this constant étale sheaf provides the niceness needed to provide something analogous to singular cohomology. It actually takes quite some work to show that $\underline{\mathbf{Z}/\ell^n\mathbf{Z}}$ is in fact an étale sheaf: this is worked out by using the previous section and getting a representation for $\underline{\mathbf{Z}/\ell^n\mathbf{Z}}$, namely, this (pre)sheaf is represented by $(\mathbf{Z}/\ell^n\mathbf{Z}) \times X$[8] and $\underline{\mathbf{Z}/\ell^n\mathbf{Z}}(U) = \mathrm{Hom}_{\mathsf{Top}}(U^{\mathsf{Top}}, \mathbf{Z}/\ell^n\mathbf{Z})$. Importantly, we should caution here that the constant sheaf description is different as to how things work in the Zariski site; the constant sheaf here instead associates to every open set and the stated constant object to $d$ many copies of the itself where $d$ is the number of components.

### 4.4. $\ell$-adic cohomology definition.

**Definition 4.5.** The $i$-th $\ell$-adic cohomology of $X$, where $X$ is a scheme of finite type over an algebraically closed field $k$ of characteristic $p \geq 0$, where $\ell \neq p$, is defined to be

$$H^i{}_{\text{ét}}(X, \mathbf{Z}_\ell) = \varprojlim_m H^i{}_{\text{ét}}(X, \mathbf{Z}/\ell^n\mathbf{Z}),$$

which has inherited structure of a $\mathbf{Z}_\ell$-module, where $\mathbf{Z}_\ell = \varprojlim \mathbf{Z}/\ell^n\mathbf{Z}$ denotes the $\ell$-adic integers, so we may also extend coefficients to the quotient field of $\mathbf{Z}_\ell$, (that is, extend to $\mathbf{Q}_\ell$), as follows

$$H^i{}_{\text{ét}}(X, \mathbf{Q}_\ell) := H^i{}_{\text{ét}}(X, \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

## 5. The Zeta function (& the Weil Conjectures)

**5.1. Zeta functions.** Formally speaking, the *Riemann zeta function* is the meromorphic continuation of the holomorphic function defined for $s \in \mathbf{C}$ with $\mathfrak{Re}(s) > 1$ defined by $\zeta(s) := \sum_{n\geq} \frac{1}{n^s}$. It is famously known what Euler did to $\zeta(s)$; he used the unique factorization property of positive integers and a rewriting using facts about infinite geometric series which allows to write $\zeta(s)$ as the so-called *Euler product*

$$\zeta(s) = \sum_{n\geq} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}.$$

Now we can further this description of the zeta function using our language of schemes by the following:

$$\begin{aligned}
\zeta(s) &= \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \\
&= \prod_{\mathfrak{m} \in \mathrm{Specm}(\mathbf{Z})} \frac{1}{1-(|\mathbf{Z}/\mathfrak{m}|)^{-s}} \\
&= \prod_{\substack{\text{closed points} \\ x \in \mathrm{Spec}\,\mathbf{Z}}} \frac{1}{1-(|\kappa(x)|)^{-s}}
\end{aligned}$$

**Definition 5.1.** Let $X$ be a scheme of finite type over the finite field $k = \mathbf{F}_q$. For every $m \geq 1$, we write $N_m = |X(\mathbf{F}_{q^m})|$. The **local Weil zeta function** of $X$ is the formal power series

$$Z(X;t) = \exp\left(\sum_{m\geq 1} \frac{N_m}{m} t^m\right) \in \mathbf{Q}[[t]].$$

[8]meaning that we're taking the disjoint of $X$ $\ell^n$ many times.

**Example 5.1** (Some Computations)**.**

- Let $X = \mathbf{A}^n_{\mathbf{F}_q}$. For $m \geq 1$, it's clear to see that we have $X(\mathbf{F}_{q^m}) = (\mathbf{F}_{q^m})^n$ and which gives that $|X(\mathbf{F}_{q^m})| = q^{nm}$. Using the fact that $\log(1+t) = \sum_{m\geq 1} \frac{(-1)^{m+1}t^m}{m}$, we then get
$$Z(X,t) = \exp\left(\sum_{m\geq 1} \frac{q^{nm}}{m} t^m\right) = \exp(-\log(1-q^n t)) = \frac{1}{1-q^n t}$$
- Let $X = \mathbf{P}^n_{\mathbf{F}_q}$. Then as we can decompose $X = \mathbf{A}^n_{\mathbf{F}_q} \cup \mathbf{P}^{n-1}_{\mathbf{F}_q}$, and using what we just worked out for $\mathbf{A}^n_{\mathbf{F}_q}$, we get $N_m(\mathbf{P}^n_{\mathbf{F}_q}) = 1 + q^m + q^{2m} + \cdots + q^{nm}$ for $m \geq 1$ and $Z(X,t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^n t)}$.
- Let $Y$ be a variety over $\mathbf{F}_q$, then $N_m(Y \times \mathbf{A}^1_{\mathbf{F}_q}) = q^m N_m(Y)$ and
$$Z(Y \times \mathbf{A}^1_{\mathbf{F}_q}, t) = \exp\left(\sum_{m\geq 1} \frac{q^m N_m(Y)}{m} t^m\right) = \exp\left(\sum_{m\geq 1} N_m(Y)\frac{(qt)^m}{m}\right) = Z(Y, qt).$$
More generally, $Z(Y \times \mathbf{A}^n_{\mathbf{F}_q}, t) = Z(Y, q^n t)$.

5.2. **The Weil Conjectures.** In a sentence, the Weil conjectures give us data about the number of points on varieties over finite fields.

5.2.1. *Statements of Weil Conjectures.*

**Theorem 5.1.** Let $X$ be a smooth projective variety of dimension $n$ over $k = \mathbf{F}_q$. Let $Z(X,t)$ be the local Weil zeta function of $X$. Then

(i) (Rationality) The function $Z(X,t)$ is rational over $t$, i.e. a quotient of polynomials with rational coefficients.

(ii) (Riemann Hypothesis) We can write $Z(X,t)$ as
$$Z(X,t) = \frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)}$$
where $P_0(t) = 1$, $P_{2n}(t) = 1 - q^n t$, and for each $1 \leq i \leq 2n-1$, $P_i(t)$ is a polynomial with integer coefficients, where $P_i(t) = \prod(1 - \alpha_{ij} t)$ and $\alpha_{ij}$ are algebraic integers with $|a_{ij}| = q^{i/2}$

(iii) (Functional Equation) $Z(X,t)$ satisfies the function equation
$$Z\left(X, \frac{1}{q^d t}\right) = \pm q^{\frac{d\chi}{2}} t^{\chi} Z(X,t)$$
where $\chi = \sum_i (-1)^i \beta_i$ for $\beta_i = \deg P_i(t)$

(iv) (Betti numbers) Omitted. See page 451 of [Hart77].[9]

We've in fact basically already proven the rational statement as well as the Riemann hypothesis for $\mathbf{P}^1_{\mathbf{F}_q}$ (and basically for some other cases) by using the computations of Example 5.1: the rationality statement is clear and the Riemann hypothesis requires the observation that $P_1(t) = 1$, and we're done. The remaining statements are considerably more difficult (attempting to prove these requires more knowledge of things that we haven't covered throughout this paper; in particular, some *duality* results).

[9]The reason we omit this is that we haven't spoken about an important concept which the Betti numbers conjecture is centered around—we haven't said anything about analytification of a 'nice' scheme.

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