Derived Functors and Derived Categories

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1 Derived Hom

Injective resolutions

Let $\mathcal{A}$ be an abelian category with enough injectives, and F be a left-exact functor. The following is an important theorem which we should always think to utilize:

THEOREM 1.1. If $\mathcal{A}$ is an abelian category with enough injectives, then every $A \in \mathcal{A}$ has an injective resolution.

That is, given $M \in \mathcal{A}$, there is a long exact sequence

$$0 \to A \xrightarrow{\varepsilon} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \cdots, \tag{1}$$

where each I^i with $i \geq 0$ is an injective object in $\mathcal{A}$, $I^i = 0$ for all $i < 0$, and ε is an injection. Recall that an object $J \in \mathcal{A}$ is called *injective* for every injection $A \hookrightarrow B$ and every map $A \to J$, there exists a map $B \to J$ making the following diagram commute

$$\begin{tikzcd} J & \\ A \arrow[u] \arrow[r, hook] & B \arrow[ul, dashed] \end{tikzcd}$$

Notice that if we consider the contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-,I)$ where if $f: A \to B$ in $\mathcal{A}$ we map this to $\mathrm{Hom}_{\mathcal{A}}(f,I) := f^*\colon \mathrm{Hom}_{\mathcal{A}}(B,I) \to (A,I)$ with $f^*(g) = g \circ f$, then I will be an injective object of $\mathcal{A}$ if and only if $\mathrm{Hom}_{\mathcal{A}}(-,I)$ gives an exact functor (i.e. it will preserve surjectivity). This follows as $\mathrm{Hom}_{\mathcal{A}}(-,I)$ is a left-exact functor, and so if we have that

$$0 \to M \xrightarrow{g} N \xrightarrow{h} L \to 0$$

is exact, then all we check is that $\mathrm{Hom}_{\mathcal{A}}(N,L) \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{A}}(M,L)$ is surjective and $\ker g^* = \mathrm{Im}\, h^*$; this is easily checked to be.

Now, going back to the long exact sequence of (1), we have $A \cong \ker d^0$ as $\ker d^0 = \operatorname{Im} \varepsilon$ and we have an isomorphism $A \cong \operatorname{Im} \varepsilon$ via $x \mapsto \varepsilon(x)$ (obviously surjective and injectivity follows from ε being injective), and thereby we deduce an important fact, which we will prove soon. Given the long exact sequence of (1), we have two induced cochain complexes

$$\begin{tikzcd}
M[0]\colon & 0 \arrow[r] & M \arrow[r] \arrow[d, "\varepsilon"] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
I^\bullet\colon & 0 \arrow[r, "d^{-1}"] & I^0 \arrow[r, "d^0"] & I^1 \arrow[r, "d^1"] & I^2 \arrow[r, "d^2"] & \cdots
\end{tikzcd}$$

which share the same cohomologies. This is the fact we wanted to show, i.e. that $M[0] \to I^\bullet$ is a quasi-isomorphism. We can compute the cohomologie of the cochain complex $M[0]$ (indexing the objects starting at zero from the left and so on), quickly as it is just:

$$H^i(M[0]) = \begin{cases} M & i = 1, \\ 0 & i \neq 0 \end{cases}$$

Similarly, you can quickly check that $H^1(I^\bullet) = \ker d_0 / \operatorname{Im} d^{-1} = \ker d_0 \cong M$ and $H^i(I^\bullet) = 0$ for every i not being one, so we have $H^i(I^\bullet) = 0$. Thereby we have our claimed quasi-isomorphism.

Going back to talking about our left exact functor F, we can apply this to the cochain complex $I^\bullet$ to obtain a left exact sequence

$$F(I^\bullet)\colon\ 0 \xrightarrow{F(d^{-1})} F(I^0) \xrightarrow{F(d^0)} F(I^1) \xrightarrow{F(d^1)} F(I^2) \to \cdots$$

and compute the cohomologie of this at the i-th spot, we define this object as $R^iF(M)$. That is, $R^iF(M) := H^i(F(I^\bullet)) = \ker F(d^i)/\operatorname{Im} F(d^{i-1})$. Inspecting, the left exact sequence up to $F(I^1)$, we can see that $R^0F(M) = \ker F(d^0)/\operatorname{Im} F(d^{-1}) = F(\ker d^0) \cong F(M)$ as F is left exact if and only if it preserves kernels (and all functors preserve isomorphisms).

EXT FUNCTOR

Given some objects $A, B \in \mathcal{A} = \mathsf{Mod}_k$, we call a short exact sequence

$$0 \to A \to C \to B \to 0$$

an *extension* of B by A. For example, $C = A \oplus B$ always provides an extension. Another example is the case of $A = \mathbf{Z}$ and $B = \mathbf{Q}/\mathbf{Z}$ and $k = \mathbf{Z}$, i.e. considering these as abelian groups. We have an extension of $\mathbf{Q}/\mathbf{Z}$ by $\mathbf{Z}$ as the following sequence of abelian groups is exact

$$0 \to \mathbf{Z} \xrightarrow{i} \mathbf{Q} \xrightarrow{\pi} \mathbf{Q}/\mathbf{Z} \to 0$$

We should keep this example in mind. Now, in general, if $\mathcal{E}$ is an abelian category, and fix some $L \in \mathcal{E}$, then $h_L(-)\colon = \operatorname{Hom}_{\mathcal{E}}(L, -)$ defines a left exact functor from $\mathcal{E} \to \mathsf{Ab}$, where given $f\colon X \to Y$ in $\mathcal{E}$, we have the induced map $h_L(f) := f^*\colon \operatorname{Hom}_{\mathcal{E}}(L, X) \to \operatorname{Hom}_{\mathcal{E}}(L, Y)$ via $f^*\colon g \mapsto f \circ g$. We can form the right derived functor of $\operatorname{Hom}_{\mathcal{E}}(L, -)$ by way of defining $R^i \operatorname{Hom}_{\mathcal{E}}(L, -) := \operatorname{Ext}^i_{\mathcal{E}}(L, -)$. If we took R to be a ring, and fix L as an R-module, then $h_L\colon \mathsf{Mod}_R \to \mathsf{Ab}$ defines a left exact functor, and we thereby can form its right derived functors

$$R^i h_L(-) = R^i \operatorname{Hom}_R(L, -) := \operatorname{Ext}^i_R(L, -).$$

So, $\mathrm{Ext}^i_R(L,-)(N) := \mathrm{Ext}^i_R(L,N)$ means that we take an injective resolution of N firstly, and then apply $\mathrm{Hom}(L,-)$ to the truncated sequence and talk about cohomology in that situation. We note here that, in general, for an abelian category and with enough injectives, $\mathrm{Ext}^i(L,-)\colon \mathcal{A} \to \mathsf{Ab}$, for fixed $L \in \mathcal{A}$, is an additive and covariant for $i \geq 0$. The use of the phrase "Ext" in the right derived functors of $\mathrm{Hom}_R(L,-)$ is a reference to the word "extension". This reference alludes to the fact that what we should be thinking of the derived functors of Hom as measuring how many inequivalent extensions there are (whereby we define equivalence by isomorphisms of short exact sequence that are the identity). That is, two extensions of B by A, say, $0 \to A \to C \to B \to 0$ and $0 \to A \to C' \to B \to 0$, are equivalent if there exists a map $\varphi\colon C \to C'$ which makes the following diagram commute

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "1_A"] & C \arrow[r] \arrow[d, "\varphi"] & B \arrow[r] \arrow[d, "1_B"] & 0 \\ 0 \arrow[r] & A \arrow[r] & C' \arrow[r] & B \arrow[r] & 0 \end{tikzcd} \tag{2}$$

Note that the 5-Lemma induces the fact that φ must actually be an isomorphism, and thus there exists $\varphi^{-1}\colon C' \to C$ which proves that the relation is symmetric. The relation is obviously symmetric and transitive (by use of compositions). Now we define $\mathfrak{ext}(B,A)$ to be the set of of equivalence classes of extensions; we should remark here that this is actually more like $\mathfrak{ext}^1(B,A) := \mathfrak{ext}(B,A)$ as we can generate higher order terms by working with larger extensions of B by A, i.e. more terms are going to fit between the connecting map from $A \to B$ which are still exact at every spot and we define the equivalence relation similarly. The collection $\mathfrak{ext}(B,A)$ has the somewhat "trivial" element $0 \to A \to A \oplus B \to B \to 0$ in it, and thereby we denote it by 0.

Recall that given a short exact sequence (or extension of B by A, if you prefer this)

$$0 \to A \xrightarrow{j} C \xrightarrow{p} B \to 0$$

is *split* if there exists a homomorphism $i\colon B \to C$ such that $p \circ i = \mathrm{id}_B$.

LEMMA 1.2. An extension of B by A splits if and only if it is equivalent to the trivial extension.

PROOF. Let $0 \to A \xrightarrow{j} C \xrightarrow{p} B \to 0$ be a split extension with $i\colon B \to C$ such that $p \circ i = \mathrm{id}_B$. Define the map $\varphi\colon A \oplus B \to C$ by $\varphi(a,b) = j(a) + i(b)$. For the backwards direction, if we had a map from $\varphi\colon C \xrightarrow{\sim} A \oplus B$, then the map $s(c) = \varphi(0,c)$ gives a splitting. □

PROPOSITION 1.3. An R-module P (resp. I) is projective (resp. injective) if and only if $\mathfrak{ext}(P,A)$ (resp. $\mathfrak{ext}(A,I)$) is trivial for any R-module A.

PROOF. It is well known that if $0 \to A \to B \to P \to 0$ is exact with P projective, then it splits in Mod_R, so P is an extension of A by definition and as it splits then it is equivalent to the trivial extension by Lemma 1.2. For the backwards direction, we assume that $\mathfrak{ext}(P,A)$ consists of only the trivial splitting. As P is an R-module, then it is isomorphic to a quotient F/K with F free. Thereby we have an exact sequence $0 \to K \to F \to P \to 0$, as this sequence must split, then $F \cong P \oplus K$, so P is projective. [1] □

[1] An R-module is projective if and only if there exists an R-module K such that $P \oplus K$ is free.

Now, one can read more about this construction in [Rot79], in Chapter 7, which establishes the main result that there is an isomorphism between $\mathfrak{ext}(B,A) \cong \operatorname{Ext}^1_R(B,A)$ (considered as groups, where the binary operation of $\mathfrak{ext}(B,A)$ is something called *Baer multiplication*), where the $\operatorname{Ext}^1_R(B,A)$ corresponds to the derived functor perspective, which preserves the trivial element. In fact, this extends for higher order $\mathfrak{ext}(-,-)$ groups, i.e. $\mathfrak{ext}^n(B,A) \cong \operatorname{Ext}^n(B,A)$.

Given an injective object M of $\mathcal{A}$, we can consider a trivial resolution of M, namely, $T^\bullet\colon\ 0 \to M \to M \to 0 \to \cdots$ so that $R^iF(M)$ will be trivial except at the zeroth spot; this is obvious as $H^i(F(T^\bullet)) = 0$ for $i > 0$. Objects $J \in \mathcal{A}$ satisfying $R^iF(J) = 0$ for all $i > 0$ are called *acyclic*.

To go off topic, but to establish a useful fact, we should define what a "divisible" abelian group is. That is, an abelian group M is *divisible* if for all $x \in M$ and every $m \in \mathbf{N} := \mathbf{Z}_{>0}$ there exists $y \in M$ such that $ny = x$.

THEOREM 1.4 ([STA23], TAG 01D7). An abelian group J is an injective object in the category of abelian groups if and only if J is divisible.

COROLLARY 1.5. Let I be an abelian group with the property that for every $m \in \mathbf{N}$, multiplication by m on I is surjective. Then I is an injective object in Ab. In particular, $\operatorname{Ext}^i_{\mathbf{Z}}(H,I) = 0$, with $i \geq 1$, for all abelian groups H.

PROOF. As $\cdot m\colon I \to I$ is surjective, then for all $x \in I$, we can find y such that $my = x$. We conclude that I is a divisible abelian group, and thus I is injective in Ab. The last sentence of the statement follows as injective objects are acyclic. $\square$

In fact, any field of characteristic zero is a divisible, so injective in Ab. Although I've avoided a lot of the technical details by not proving Theorem 1.4, we can conclude using Corollary 1.5 that $\mathbf{Q}/\mathbf{Z}$ is an injective object in Ab. So, if H is an arbitrary abelian group, then $\operatorname{Ext}^1_{\mathbf{Z}}(H,\mathbf{Q}/\mathbf{Z}) \cong \mathfrak{ext}(H,\mathbf{Q}/\mathbf{Z}) = 0$, i.e. if we wanted to fill in the diagram

$$0 \to \mathbf{Q}/\mathbf{Z} \to * \to H \to 0$$

so that we get exact sequence, then it must be the trivial one, i.e. $\mathbf{Q}/\mathbf{Z} \oplus H$, up to equivalence. We can see that $\mathbf{Q}/\mathbf{Z}$ is not projective by Proposition 1.3: We've noted that

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$$

is an exact sequence, so after applying $\operatorname{Hom}_{\mathbf{Z}}(\mathbf{Q}/\mathbf{Z},-)$, and if we assume $\mathbf{Q}/\mathbf{Z}$ is projective, we have the exact sequence

$$0 \to \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Q}/\mathbf{Z},\mathbf{Z}) \to \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Q}/\mathbf{Z},\mathbf{Q}) \to \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Q}/\mathbf{Z},\mathbf{Q}/\mathbf{Z}) \to 0$$

and there are in fact a lot more zeros than at first glance.

LEMMA 1.6. If M is a torsion A-module and N is torsion free A-module, then $\operatorname{Hom}_A(M,N)$ is trivial.

PROOF. This is easy. Suppose that $\varphi : M \to N$ is an A-linear map. For any $x \in M$, we can find nonzero $f \in A$ such that $xf = 0$, which implies that $\varphi(x) =: s \in N$ produces a torsion element as $sf = \varphi(s)f = \varphi(sf) = 0$ which implies that $s = \varphi(x) = 0$ as f is nonzero and N is torsion free. Thus φ must be the trivial map. $\square$

COROLLARY 1.7. If A is a Dedekind domain, M a torsion module, and N flat, then $\mathrm{Hom}_A(M,N)$ is trivial. In particular, if A is a PID this holds.

PROOF. Over a Dedekind domain we have free if and only if torsion free, and any PID is Dedekind. □

Note that we might not have that $\mathrm{Hom}_A(N,M)$ being trivial; for example, project $\mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$ via $\pi : \mathbf{Q} \to \mathbf{Q}/\mathbf{Z}$, so $\pi \in \mathrm{Hom}(\mathbf{Q}, \mathbf{Q}/\mathbf{Z})$ is not trivial . Going back to our exact sequence as mentioned before the lemma, we have the simplification:

$$0 \to 0 \to 0 \to \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Q}/\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \to 0$$

but this is absurd as then $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Q}/\mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \cong (0)$, but $\mathrm{id}_{\mathbf{Q}/\mathbf{Z}}$ is a distinct element from the zero map in $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Q}/\mathbf{Z}, \mathbf{Q}/\mathbf{Z})$. Hence $\mathbf{Q}/\mathbf{Z}$ is not a projective object in Ab. Note that, although intuitively clear, we cannot have that $\mathrm{Ext}^i_R(M,N) \cong \mathrm{Ext}^i_R(N,M)$ in general (although this works for the derived form of $- \otimes M$), but from the work we've done we can conclude this since $\mathbf{Q}/\mathbf{Z}$ is not projective, i.e. there exists a $\mathbf{Z}$-module (i.e. abelian group) M with $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Q}/\mathbf{Z}, M) \neq (0) \ncong (0) = \mathrm{Ext}^1_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$. This implies that if we wanted to build the derived contravariant version of Hom, we would need to distinguish ourselves from the covariant version.

Before we end of this section, we should mention that there are actually many left exact functors one could think of. One of the most prominent ones in algebraic geometry is the *global sections functor*. That is, given a scheme $(X, \mathscr{O}_X)$, we may consider an exact sequence of $\mathscr{O}_X$-modules, say,

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

and after we apply the (left exact) functor $\Gamma(X,-)$, which sends an $\mathscr{O}_X$-module $\mathscr{F}$ to $\Gamma(X,\mathscr{F}) := \mathscr{F}(X)$, we have the chain complex

$$0 \to \Gamma(X,\mathscr{F}) \to \Gamma(X,\mathscr{G}) \to \Gamma(X,\mathscr{H})$$

Now it is a theorem that the category of $\mathscr{O}_X$-modules is abelian and has enough injectives, and thereby allowing using to consider injective resolutions. Given some $\mathscr{O}_X$-module $\mathscr{F}$, we take an injective resolution $0 \to \mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \mathscr{I}^2 \to \cdots$ Then we delete $\mathscr{F}$ and apply $\Gamma(X,-)$ to get a chain complex

$$0 \to \Gamma(X,\mathscr{I}^0) \to \Gamma(X,\mathscr{I}^1) \to \cdots$$

Now we define the *sheaf cohomology functors* $H^i(X,\mathscr{F})$ as the right derived functors of $R^i\Gamma(\mathscr{F})$, i.e.

$$H^i(X,\mathscr{F}) := R^i\Gamma(\mathscr{F}) = \frac{\ker \Gamma(X,d^i)}{\mathrm{Im}\,\Gamma(X,d^{i-1})} = \frac{\ker\{\Gamma(X,\mathscr{I}^i) \to \Gamma(X,\mathscr{I}^{i+1})\}}{\mathrm{Im}\{\Gamma(X,\mathscr{I}^{i-1}) \to \Gamma(X,\mathscr{I}^i)\}}.$$

DERIVED $\mathrm{Hom}(-,M)$

As motivated by the last thing we stated in the last section, we want to develop the theory of right derived functor of $\mathrm{Hom}_A(-,M)$ for an A-module M. This is actually very similar to the derived form of $\mathrm{Hom}_A(M,-)$, but essentially we just need to switch arrow direction, as $\mathrm{Hom}_A(-,M)$ is contravariant.

Recall that an A-module P is *projective* if given an A-linear map $f\colon P \to M$ and a surjective map $h\colon N \to M$, then there exists a lift $f'\colon N \to P$ which makes the following diagram commute

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ N \arrow[ur, dashed, "\exists f'"] \arrow[r, "h"] & M \arrow[r] & 0 \end{tikzcd}$$

Equivalently, there are many alternative definitions of a projective module:

LEMMA 1.8. Let R be a ring. Let P be an R-module. The following are equivalent

(1) P is projective.
(2) There exists an R-module K such that $P \oplus K$ is free.
(3) There exists R-modules F and K, with F free, with $P = F \oplus K$.
(4) $\mathrm{Hom}_R(P,-)\colon \mathsf{Mod}_R \to \mathsf{Mod}_R$ is an exact functor.
(5) Every A-linear map onto P has a section.
(6) Every exact sequence $0 \to M \to P \to 0$ in Mod_R splits.

In similar fashion as to talking about injective resolutions, we have *projective resolutions* of objects in $\mathcal{A}$. That is, a *projective resolution* of M in $\mathcal{A}$ is an exact sequence

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \to 0$$

where each of the objects P_i are projective, $P_i = 0$ for all $i > 0$, and π is a surjection. Using a similar yoga as done from the first section, we take a projective resolution $\cdots \xrightarrow{d^2} P_1 \xrightarrow{d^1} P_0 \to M \to 0$, we truncate this sequence to delete M and arrived at a chain complex

$$P^\bullet\colon\ \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_{-1}} 0$$

We apply the left exact contravariant functor $T(-)$

$$0 \to T(P_0) \xrightarrow{T(d_{-1})} T(P_1) \xrightarrow{T(d_1)} T(P_2) \xrightarrow{T(d_2)} \to \cdots$$

and we compute the (co)homology groups of this complex to get

$$\begin{aligned} R^i T(M) = \frac{\ker\{T(P_i) \to T(P_{i+1})\}}{\mathrm{Im}\{T(P_{i-1}) \to T(P_i)\}} &= \ker T(d_i)/\operatorname{Im} T(d_{i-1}) \\ &= T(\ker d_i)/\operatorname{Im} T(d_{i-1}) \end{aligned}$$

We once again have that $R^0F(M) \cong F(M)$, i.e. $R^0F \cong F$. Lastly, we should mention that given an exact sequence

$$0 \to M \to M' \to M'' \to 0$$

this induced a long exact sequence

$$0 \to F(M'') \to F(M') \to F(M) \to R^1F(M'') \to R^1F(M') \to R^1F(M) \to R^2F(M'') \to \cdots$$

Thereby applying this to $\mathrm{Hom}_R(-,M)$ the left exact contravariant functor, where M is an R-module, to define $\mathrm{Ext}^i_R(N,M)$, we first choose a projective resolution of N, i.e. $\cdots \to P_1 \to P_0 \to N \to 0$, then we apply $\mathrm{Hom}_R(-,M)$ to the truncated resolution so that we obtain the following chain complex

$$0 \to \mathrm{Hom}_R(P_0,M) \to \mathrm{Hom}_R(P_1,M) \to \cdots$$

We define $\mathrm{Ext}^i_R(N,M)$ as the i-th (co)homology group by

$$\mathrm{Ext}^i_R(N,M) = \frac{\ker\{\mathrm{Hom}_R(P_i,M) \to \mathrm{Hom}_R(P_{i+1},M)\}}{\mathrm{Im}\{\mathrm{Hom}_R(P_{i-1},M) \to \mathrm{Hom}_R(P_i,M)\}}$$

EXAMPLE 1.9. Now that we're done with all that, we should do an example! We want to compute $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z},\mathbf{Z})$, and so we begin with a projective resolution and then apply $\mathrm{Hom}_{\mathbf{Z}}(-,\mathbf{Z})$ to it. We choose the projective $0 \to \mathbf{Z} \xrightarrow{\cdot m} \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z} \to 0$, which induces a long exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z},\mathbf{Z}) \to \mathbf{Z} \xrightarrow{\cdot m} \mathbf{Z} \to \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/n\mathbf{Z},\mathbf{Z}) \to 0$$

and as $\mathbf{Z}/n\mathbf{Z}$ is a torsion $\mathbf{Z}$-module this makes $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/n\mathbf{Z},\mathbf{Z}) = 0$ by Lemma 1.2, so

$$0 \to \mathbf{Z} \xrightarrow{\cdot m} \mathbf{Z} \to \mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/n\mathbf{Z},\mathbf{Z}) \to 0$$

is exact and therefore $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/n\mathbf{Z},\mathbf{Z}) \cong \mathbf{Z}/n\mathbf{Z}$.

By way of our definition of our left exact functor F having been *derived* to form $R^iF(-)$ and our definition of $R^iF(-)$ not making explicit reference to the chosen injective resolution, we should most certainly (for this to make any sense) expect that our derived functor R^iF doesn't depend on such different resolutions, i.e. if $I^\bullet$ and $J^\bullet$ are each injective resolutions of, say, $M \in \mathcal{A}$, then we should have $H^i(F(I^\bullet)) \cong H^i(F(J^\bullet))$. This is indeed true! But requires quite a lot of work.

2 Derived $- \otimes_A -$

In this section, we build up the derived tensor product (considering the tensor product as a functor between category of modules). In the last section, we talked strictly about functor which are left exact, but this is not the case for the tensor product. That is, given an A-module M, the covariant functor $- \otimes M \colon \mathsf{Mod}_A \to \mathsf{Mod}_A$ is right exact. To showcase a nice proof of this fact, we recall the following fact:

THEOREM 2.1 (TENSOR-HOM ADJUNCTION). For A a commutative ring and A-modules M, N and L, we have

$$\mathrm{Hom}(N \otimes_A M, L) \cong \mathrm{Hom}(N, \mathrm{Hom}_A(M,L)), \tag{3}$$

Now, suppose we're given $0 \to N \to J \to L \to 0$ an exact sequence of A-modules. Then we can apply, for any A-module P, $\mathrm{Hom}(-,P)$ to get a left exact sequence

$$0 \to \mathrm{Hom}(L,P) \to \mathrm{Hom}(J,P) \to \mathrm{Hom}(N,P)$$

and we lastly apply $\mathrm{Hom}(M,-)$ to get left exact sequence

$$0 \to \mathrm{Hom}(M,\mathrm{Hom}(L,P)) \to \mathrm{Hom}(M,\mathrm{Hom}(J,P)) \to \mathrm{Hom}(M,\mathrm{Hom}(N,P)).$$

Using Theorem 2.2, we have that

$$0 \to \mathrm{Hom}(M \otimes_A L, P) \to \mathrm{Hom}(M \otimes_A J, P) \to \mathrm{Hom}(M \otimes_R, N, P)$$

and this left exact sequence is the result of applying $\mathrm{Hom}(-P)$, so

$$M \otimes_A N \to M \otimes_A J \to M \otimes_A L \to 0$$

is right exact. As you can expect, we want to talk about an analog to the derived functor of a functor which is left exact. By showing that $-\otimes_A M$ is right exact, we should note that the tensor product will essentially not preserve kernels (in particular, right exact functors preserve cokernels but not kernels), so it doesn't preserve injectivity. For example, for the inclusion map $i\colon \mathbf{Z} \to \mathbf{Q}$, and $M = \mathbf{Z}/2\mathbf{Z}$, then $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}/2\mathbf{Z} = (0)$ as we can write $a \otimes b = (a/2) \otimes 2b = a/2 \otimes 0 = 0$. As $\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/2\mathbf{Z} \cong \mathbf{Z}/2\mathbf{Z}$, then the induced map $i \otimes \mathrm{id}_M \colon \mathbf{Z}/2\mathbf{Z} \to (0)$ cannot possibly be injective.

We can produce derived functors for $-\otimes_A M$, whereby the motivation of (or at least the *equivalence* to the derived functor, which will see later on, is to look at torsion elements). In the cases that $-\otimes_A M$ that does provide an exact functor, we call these special A-modules *flat*.

Now we build up the Tor functor. Take a projective resolution an A-module M, say, $\cdots \to P_1 \to P_0 \to M \to 0$. We apply $-\otimes_A L$, for an A-module L, to the truncated/deleted resolution of M, which provides the chain complex

$$\cdots \to P_1 \otimes_A L \to P_0 \otimes_A L \to 0.$$

We define $\mathrm{Tor}_A^i(M,L) = \ker\{P_i \otimes_A L \to P_{i-1} \otimes_A L\}/\operatorname{Im}\{P_{i+1} \otimes_A L \to P_i \otimes_A L\}$, i.e. it is the i-th cohomology group of the truncated sequence after applying $-\otimes_A L$. We compute in general $\mathrm{Tor}_A^0(M,L)$ quite quickly: Given the projective resolution $\cdots \to P_1 \to P_0 \xrightarrow{\pi} M \to 0$, we can easily see that $M \cong \operatorname{coker} d_1 = P_0/\operatorname{Im} d_1$, so

$$\begin{aligned}\mathrm{Tor}_A^0 = P \otimes_A L/\operatorname{Im}\{P_1 \otimes_A L \to P_0 \otimes_A L\} &= P_0 \otimes_A L/\operatorname{Im}(d_1 \otimes_A L)\\ &= \operatorname{coker}(d_1 \otimes_A L) \cong M \otimes_A L,\end{aligned}$$

as right exact functors preserve cokernels. So, the general yoga is that if $\mathcal{A}$ is an abelian category with enough projective, then we take a projective resolution of M, and form the truncated resolution of M and denote it by $P_\bullet$, and lastly we define the left derived functors of a right exact functor F by

$$L_iF(M) = H_i(F(P_\bullet)) = \ker F(d_i)/\operatorname{Im} F(d_{i+1}).$$

Using this sort reformulation, makes some arguments easier to comprehend without being so lost in notation, in my opinion. If P is a projective A-module, then we can see easily as well that $\mathrm{Tor}_A^i(P,M)$ will vanish for all $i \geq 0$. We have a trivial resolution of P by $\to \cdots 0 \to P \to P \to 0$, so $0 \to F(P) \to F(P) \to 0$ forms $F(P_\bullet)$, and $H_i(F(P_\bullet)) = \ker F(d_i)/\operatorname{Im} F(d_{i+1}) = 0$ for $i \geq 1$. Thus, if P is a projective A-module, then $\mathrm{Tor}_A^i(P,M) = 0$ for all A-modules M.

THEOREM 2.2. Suppose that $0 \to N' \to N \to N'' \to 0$ is a sequence of A-modules. Then there exists a long exact sequence

$$\cdots \to \mathrm{Tor}_A^i(M,N') \to \mathrm{Tor}_A^i(M,N) \to \mathrm{Tor}_A^i(M,N'') \to \mathrm{Tor}_A^{i-1}(M,N') \to \cdots$$

for every A-module M.

PROPOSITION 2.3. If N is flat, then $\mathrm{Tor}_A^i(M,N) = 0$ for all $i \geq 1$.

PROOF. This is obvious, as if we take a projective resolution of M, say, $\cdots \to P_1 \to P_0 \to M \to 0$. Then $\cdots \to P_1 \otimes_A N \to P_0 \otimes_A N \to 0$ is an exact sequence, so $\mathrm{Tor}_A^i(M,N) = 0$ for all $i \geq 1$. $\square$

In fact the converse holds, and it alludes to a much deeper fact:

THEOREM 2.4 ([STA23], TAG 00M5). Let A be a ring. Let M be an A-module. The following are equivalent:

(1) The module M is flat over A.
(2) For all $i > 0$ the functor $\mathrm{Tor}_A^i(M,-)$ is zero.
(3) The functor $\mathrm{Tor}_A^1(M,-)$ is zero.
(4) For all ideals $I \subset A$ we have $\mathrm{Tor}_A^1(M,A/I) = 0$.
(5) For all finitely generated ideals $I \subset A$ we have $\mathrm{Tor}_A^1(M,A/I) = 0$.

PROOF. We prove the first three equivalences. (1) $\Rightarrow$ (2) Proposition 2.3. (2) $\Rightarrow$ (3) Immediate. (3) $\Rightarrow$ (1) Let $0 \to N \to T \to L \to 0$ be an exact sequence of A-modules. By Theorem 2.2, we get the long exact sequence

$$\cdots \to \mathrm{Tor}_A^1(M,N) \to \mathrm{Tor}_A^1(M,T) \to \mathrm{Tor}_A^1(M,L) \to M \otimes_A N \to M \otimes_A T \to M \otimes_A L \to 0$$

so as $\mathrm{Tor}_A^1(M,-)$ vanishes, then we get $0 \to M \otimes_A N \to M \otimes_A T \to M \otimes_A L \to 0$ as an exact sequence, and hence M is flat over A. $\square$

EXAMPLE 2.5. For any $\mathbf{Z}$-module B, there are nice identifications for $\mathrm{Tor}_{\mathbf{Z}}^i(\mathbf{Z}/p\mathbf{Z},B)$. We can use the projective resolution of $\mathbf{Z}/p\mathbf{Z}$,

$$0 \to \mathbf{Z} \xrightarrow{\cdot p} \mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0$$

to get $0 \to \mathbf{Z} \otimes_{\mathbf{Z}} B \to \mathbf{Z} \otimes_{\mathbf{Z}} B \to 0$ which is the same as $0 \to B \xrightarrow{\cdot p} B \to 0$, so $\mathrm{Ext}_{\mathbf{Z}}^1(\mathbf{Z}/p\mathbf{Z},B) = \{x \in B \colon xp = 0\} :=_p B$ and $\mathrm{Ext}_{\mathbf{Z}}^0(\mathbf{Z}/p\mathbf{Z},B) = \ker\{B \to 0\}/\mathrm{Im}\{B \xrightarrow{\cdot p} B\} = B/pB \cong B \otimes_{\mathbf{Z}} \mathbf{Z}/p\mathbf{Z}$. Furthermore, you can see that $\mathrm{Ext}_{\mathbf{Z}}^i(\mathbf{Z}/p\mathbf{Z},B) = 0$ for all $i \geq 2$.

LEMMA 2.6. If $A = \mathbf{Z}$, then $\mathrm{Tor}_{\mathbf{Z}}^i(M,N) = 0$ for all $i \geq 2$.

PROOF. For $\mathbf{Z}$-modules (i.e. abelian group), every projective A-module is free. Given an arbitrary $\mathbf{Z}$-module M, we can take the free abelian group $F = \mathbf{Z}\langle M \rangle$ by the elements of M and get a corresponding epi map $p \colon F \to M$. The kernel of p is also free, as it is a subgroup of a free abelian groups (specific property of $A = \mathbf{Z}$). Thus $0 \to K \to F \to M \to 0$ is a projective (in fact free) resolution of M. Now $P_i = 0$ and $H_i(P_\bullet \otimes_{\mathbf{Z}} N) = 0$ for $i \geq 2$. $\square$

LEMMA 2.7. The rational numbers, $\mathbf{Q}$, are flat over $\mathbf{Z}$-modules. In particular any localization of $\mathbf{Z}$ is flat.

PROOF. Let $0 \to N \to T \to L \to 0$ be an exact sequence of $\mathbf{Z}$-modules and $S \subset \mathbf{Z}$ a multiplicative subset. As localization is an exact functor, then $0 \to S^{-1}N \to S^{-1}T \to S^{-1}L \to 0$ is exact. There is a natural isomorphism: for each $\mathbf{Z}$-module X we have $S^{-1}X \cong X \otimes_{\mathbf{Z}} S^{-1}\mathbf{Z}$, so $0 \to N \otimes_A S^{-1}\mathbf{Z} \to T \otimes_{\mathbf{Z}} S^{-1}\mathbf{Z} \to L \otimes_{\mathbf{Z}} S^{-1}\mathbf{Z} \to 0$ is exact, and thereby $S^{-1}\mathbf{Z}$ is flat. The rational numbers are flat as $\mathbf{Q}$ is identified with $S^{-1}\mathbf{Z}$ with $S = \mathbf{Z} \setminus \{0\}$. $\square$

Bibliography

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