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Stat assignment -4.

Q1 $X_1, X_2, X_3, \ldots X_n \overset{iid}{\sim} N(\mu, 1)$.

$\because \quad P(X_i = x_i) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x_i-\mu)^2/2\sigma^2}$

$\therefore \Rightarrow \quad P(X_i = x_i) = \dfrac{1}{\sqrt{2\pi}} e^{-(x_i-\mu)^2/2}$

. . . . . (since $\sigma^2 = 1 \Rightarrow \sigma = 1$)

$\therefore$ likelihood $\Rightarrow L(\mu) = \prod_{i=1}^{n} P(X_i = \cancel{x_i})$

$\therefore L(\mu) = \prod_{1\le i\le n} \dfrac{1}{\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2}}$

we want to maximize $L(\mu)$ $\therefore$ $L'(\mu) = 0$ $\cancel{L''(\mu)<0}$

& $L''(\mu) < 0$,

must be true.

$\therefore$ also considering $\log(L)$ makes calculations easier,

since we want to check for $L'(\mu) = 0$ & $L''(\mu) < 0$

$\log(L(\mu))$ ~~log(L'(\mu))~~ preserves these properties.

$\therefore \quad \log(L(\mu)) = n\log\left(\dfrac{1}{\sqrt{2\pi}}\right) - \sum_{1}^{n}\left(\dfrac{(x_i-\mu)^2}{2}\right)$

taking derivative $\Rightarrow$

$$\frac{L'(\mu)}{L(\mu)} = 2\sum_{i=1}^{n}\left(\frac{(x_i-\mu)^2}{2}\right) = \sum_{1}^{n}(x_i-\mu)$$

$\Rightarrow \quad L'(\mu) = L(\mu)\sum_{1}^{n}(x_i-\mu)$

since we want $L'(\mu) = 0$ & $L(\mu) \neq 0 \Rightarrow$

$\sum_{1}^{n}(x_i-\mu) = 0$ for some μ.

$\therefore \quad \sum_1^n (x_i) - n \cdot \mu_0 = 0 \quad \Rightarrow \mu_0 = \frac{\sum_1^n x_i}{n}$

$\Rightarrow$ when $\mu_0 = \frac{\sum_1^n x_i}{n} \Rightarrow L'(\mu_0) = 0$

(This is mean).

lets now check for $L''(\mu_0)$

$L'(\mu) = L(\mu) \sum_1^n (x_i - \mu)$

$\Rightarrow L''(\mu) = L'(\mu) \sum_1^n (x_i - \mu) + L(\mu) \cdot \sum_1^n (-1)$

now for $\mu = \mu_0$

$L'(\mu_0) \cdot \left(\sum_1^n (x_i - \mu_0)\right) = 0$ as $L'(\mu_0) = 0$.

$\therefore \quad L''(\mu_0) = L(\mu_0) \cdot (-n)$

now since $L(\mu) \geq 0 \quad \forall \mu$. ~~now since~~

we have $L(\mu_0) \geq 0 \Rightarrow -n\, L(\mu_0) \leq 0$

$\therefore$ L is max at $\mu_0 = \frac{\sum_1^n x_i}{n}$

$\therefore$ this μ_0 is our MLE.

2) $X_1, X_2, \ldots X_n \overset{i.i.d.}{\sim} N(3, \sigma^2)$

$\therefore \quad P(X_i = x_i) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-(x_i - \mu)^2/2\sigma^2}$

$\Rightarrow \quad P(X_i = x_i) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-(x_i - 3)^2/2\sigma^2}$

$\therefore \quad L(\sigma^2) = \prod_{i=1}^n P(X_i = x_i) = \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} e^{-(x_i - 3)^2/2\sigma^2}$

Applying similar logic as before we get;

$\log(L(\sigma^2)) = -n \log\sigma - n\log(\sqrt{2\pi}) - \sum_1^n \left(\frac{(x_i - 3)^2}{2\sigma^2}\right)$

$= -\frac{n}{2} \log\sigma^2 - n\log(\sqrt{2\pi}) - \sum_1^n \left(\frac{(x_i - 3)^2}{2\sigma^2}\right)$

$\therefore \quad \frac{L'(\sigma^2)}{L(\sigma^2)} = \frac{-n}{2\sigma^2} - 0 + \sum_1^n \left(\frac{(x_i-3)^2}{2\sigma^4}\right)$

$\therefore \quad L'(\sigma^2) = L(\sigma^2) \cdot \left(\frac{-n}{2\sigma^2} + \sum_1^n \left(\frac{(x_i-3)^2}{2\sigma^4}\right)\right)$

$\therefore$ for $L(\sigma^2)$ to be max. at σ_0 $L'(\sigma_0^2) = 0$ & $L''(\sigma_0) < 0$

$\therefore \quad L'(\sigma_0^2) = 0 \Rightarrow L(\sigma_0^2) \cdot \left(\frac{-n}{2\sigma_0^2} + \sum_1^n \left(\frac{(x_i-3)^2}{2\sigma_0^4}\right)\right) = 0$

since $L(\sigma_0^2) \geq 0$

$$\frac{-n}{2\sigma_0^2} + \sum_1^n \frac{(x_i-3)^2}{2\sigma_0^4} = 0$$

$$\therefore \left(\frac{-n}{2\sigma_0^2} + \sum_1^n \left(\frac{(x_i-3)^2}{2\sigma_0^4}\right)\right) 2\sigma_0^4 = (0)\, 2\sigma_0^4$$

$$\Rightarrow -n\sigma_0^2 + \sum_1^n (x_i-3)^2 = 0$$

$$\Rightarrow \sigma_0^2 = \frac{\sum_1^n (x_i-3)^2}{n} \quad (\text{standard deviation})$$

lets check for $L''(\sigma^2)$.

$L''(\sigma^2) = L(\sigma^2) \cdot \left(\frac{n}{2\sigma^4} - \sum_1^n \left(\frac{(x_i-3)^2}{\sigma^6}\right)\right) + L'(\sigma^2)\left(\frac{-n}{2\sigma^2} + \sum_1^n \frac{(x_i-3)^2}{2\sigma^4}\right)$

how since $L'(\sigma_0^2) = 0 \Rightarrow L''(\sigma_0^2) = L(\sigma_0^2)\left(\frac{n}{2\sigma^4} - \sum_1^n \frac{(x_i-3)^2}{\sigma_0^6}\right)$

now since $\sigma_0^2 = \frac{\sum_1^n (x_i-3)^2}{n} \Rightarrow \sum_1^n (x_i-3)^2 = n\sigma_0^2$

$$\Rightarrow \therefore L''(\sigma_0^2) = L(\sigma_0^2)\left(\frac{n}{2\sigma_0^4} - \frac{n}{\sigma_0^4}\right)$$

$$= L(\sigma_0^2)\left(\frac{-n}{2\sigma_0^4}\right)$$

since $L(\sigma_0^2) \geq 0$

$\Rightarrow \frac{-n}{2\sigma_0^4} L(\sigma_0^2) \leq 0 \rightarrow L''(\sigma^2) \leq 0$

$\therefore L(\sigma_0^2)$ is max at $\sigma^2 = \sigma_0^2 = \frac{\sum (x_i-3)^2}{n}$ which is our required M.L.E.

Q3 $X_1, X_2, \dots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

from first question we know that if σ^2 is constant, then M.L.E of μ is mean. i.e. $\mu_0 = \dfrac{\sum_1^n x_i}{n}$

from question 2. we know that if μ is constant, then M.L.E of σ^2 is standard deviation, i.e. $\sigma_0^2 = \dfrac{\sum_1^n (x_i - \mu_0)^2}{n}$

we can fix μ in here as it's ind. of σ^2 $\Rightarrow$
if we estimate σ^2 by fixing $\mu = \mu_0$,

$\therefore$ we get $\sigma_0^2 = \dfrac{\sum_1^n (x_i - \mu_0)^2}{n}$.

$\therefore$ the required M.L.E is $\mu = \mu_0 = \sum_1^n x_i / n$

& $\sigma_0^2 = \dfrac{\sum_1^n (x_i - \mu_0)^2}{n}$

Q4 $X_1, X_2 \dots X_n \overset{i.i.d.}{\sim}$ R.V. (random variable) s.t.

$P(X_i = x_i) = \lambda e^{-\lambda x_i}$, $\lambda > 0$

$\therefore$ likelihood function;

$$L(\lambda) = \prod_{i=1}^n P(X_i = x_i)$$

$$= \prod_{i=1}^n \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \cdot \sum_1^n x_i}$$

now applying same logic as before we take, for MLE,

$L'(\lambda_0) = 0$ & $L''(\lambda_0) < 0$.

$$\therefore \quad L'(\lambda) = \lambda^n \cdot e^{-\lambda \sum_1^n x_i} \cdot -\sum_1^n x_i + n\lambda^{n-1} \cdot e^{-\lambda \cdot \sum_1^n x_i}$$

$$= \lambda^{n-1} \cdot e^{-\lambda \cdot \sum_1^n x_i} \cdot \left(-\lambda \cdot \left(\sum_{i=1}^n x_i\right) + n\right)$$

$\therefore$ we know $L'(\lambda_0) = 0 \Rightarrow$

$$-\lambda_0 \sum_1^n x_i + n = 0$$

$$\Rightarrow \quad \lambda_0 = \frac{n}{\sum_1^n x_i} = \frac{1}{\left(\frac{\sum_1^n x_i}{n}\right)}$$

(this is reciprocal of mean)

lets check for second derivative test.

$$L'(\lambda) = \lambda^n e^{-\lambda \sum_1^n x_i} \cdot \left(-\sum_1^n x_i\right) + n\lambda^{n-1} e^{-\lambda \sum_1^n x_i}$$

$$\therefore \quad L''(\lambda) = \lambda^{n-1}\left(\lambda e^{-\lambda \sum_1^n x_i}\left(\sum_1^n x_i\right)^2 - e^{-\lambda \sum_1^n x_i} \sum_1^n x_i + n e^{-\lambda \sum_1^n x_i}\left(-\sum_1^n x_i\right)\right) + (n-1)\lambda^{n-2}\left(\lambda e^{-\lambda \sum_1^n x_i}\left(-\sum_1^n x_i\right) + n e^{-\lambda \sum_1^n x_i}\right)$$

$$L''\left(\frac{1}{\bar{x}}\right) = \left(\frac{1}{\bar{x}}\right)^{n-1}\left[\cancel{e^{-n}\left(\sum^n (x_i)\right) n} - e^{-n}\left(\sum_1^n x_i\right) + \cancel{n e^{-n}\left(-\sum_1^n x_i\right)}\right] + (n-1)\left(\frac{1}{\bar{x}}\right)^{n-2}\left(\cancel{-ne^{-n}} + \cancel{ne^{-n}}\right)$$

$$= -\left(\frac{1}{\bar{x}}\right)^{n-1} \cdot e^{-n} \cdot \sum_1^n x_i \quad < 0$$

QED