

# FUNDAMENTALS OF QUANTITATIVE MODELING

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*Module 2: deterministic models and optimization*



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## Module 2 content

- Linear models
- Growth and decay in discrete time
- Growth and decay in continuous time
- Classical optimization

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## Deterministic models

- There are no random/uncertain components (inputs and/or outputs) in these models
- If the inputs to the model are the same then the outputs will always be the same
- The downside of deterministic models: it is hard to assess uncertainty in the outputs

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## Linear models

- Recall the formula definition of a straight line:  $y = mx + b$
- Characterization of a line: the slope is constant
- The change in  $y$  for a one-unit change in  $x$  is the same, **regardless** of the value of  $x$
- In practice you should ask if the constant slope assertion is realistic
- If it is not realistic, then a straight line model is probably not the way to go

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## A linear cost function

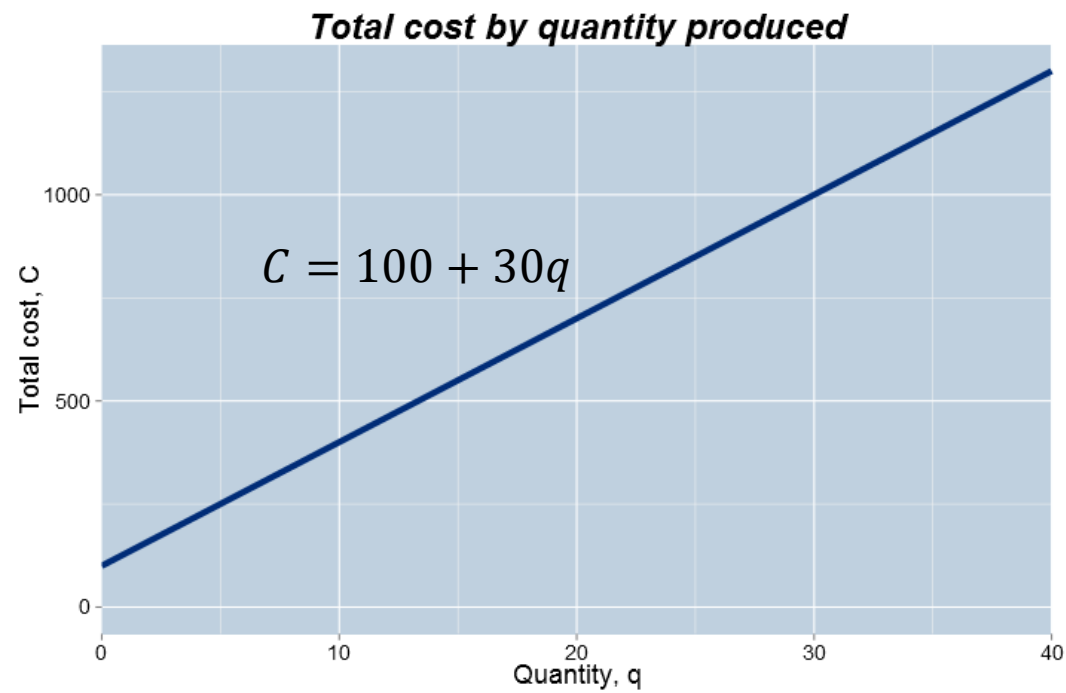
- Call the number of units produced  $q$ , and the total cost of producing  $q$  units  $C$
- Define:

$$C = 100 + 30q$$

- Calculate some illustrative values:

$q$	$C$
0	100
10	400
20	700

## Graphing the cost function



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## Interpretation

- The two coefficients in the line are the intercept and slope:  $b$  and  $m$  in general, 100 and 30 in this particular instance
- $b$ : the total cost of producing 0 units
  - A better interpretation: that part of total cost that doesn't depend on the quantity produced: the **fixed** cost
- $m$ : the slope of the line
  - The change in total cost for an incremental unit of production: the **variable** cost

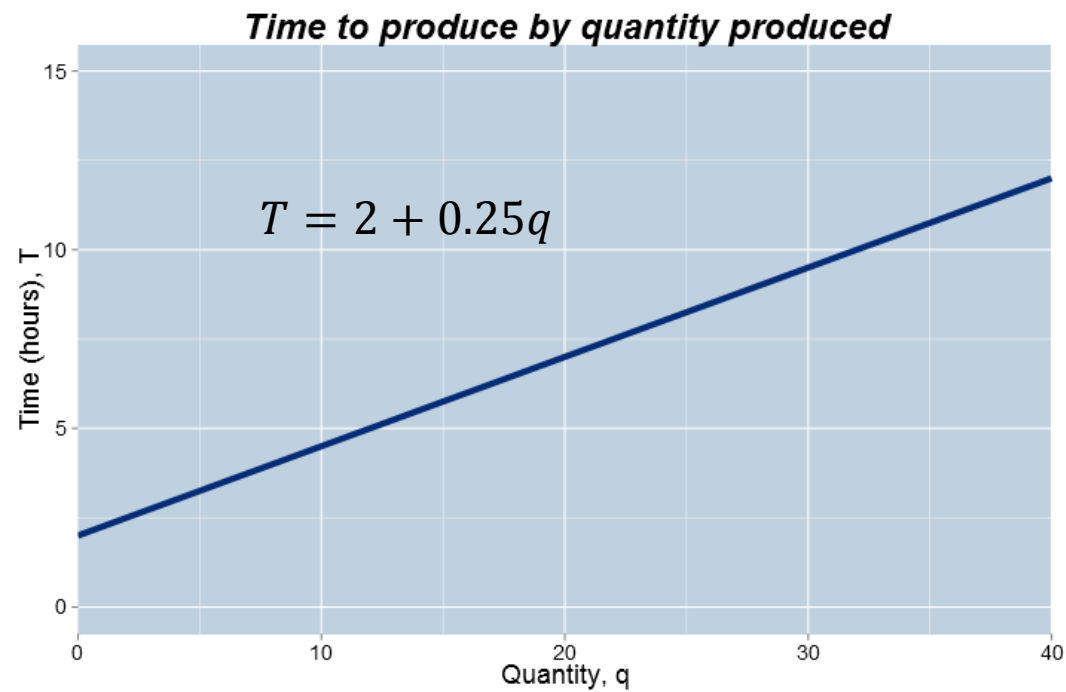
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## Example with a “time-to-produce” function

- It takes 2 hours to set up a production run, and each incremental unit produced always takes an additional 15 minutes (0.25 hours)
- Call  $T$  the time to produce  $q$  units, then
$$T = 2 + 0.25q$$
- Interpretation
  - $b$ : the **setup time** (2 hours)
  - $m$ : the **work rate** (15 minutes per additional item)



## Graphing the time-to-produce function



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## Linear programming

- One of the key uses of linear models is in ***Linear Programming (LP)***, which is a technique to solve certain ***optimization*** problems
- These models incorporate ***constraints*** to make them more realistic
- Linear programming problems can be solved with add-ins for common spreadsheet programs

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## Growth in discrete time

- Growth is a fundamental business concept
  - The number of customers at time  $t$
  - The revenue in quarter  $q$
  - The value of an investment at some time  $t$  in the future
- Sometimes a linear model may be appropriate for a growth process
- But an alternative to a ***linear (additive) growth*** model is a ***proportionate*** one
- Proportionate growth: a constant percent increase (decrease) from one period to the next

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## Simple interest

- Start off with \$100 (***principal***) and at the end of every year earn 10% ***simple interest*** on the initial \$100
- Simple interest means that interest is only earned on the principal investment

Year	0	1	2	3	4	10
Value	100	110	120	130	140	200

- Every year the investment grows by the same amount (\$10)

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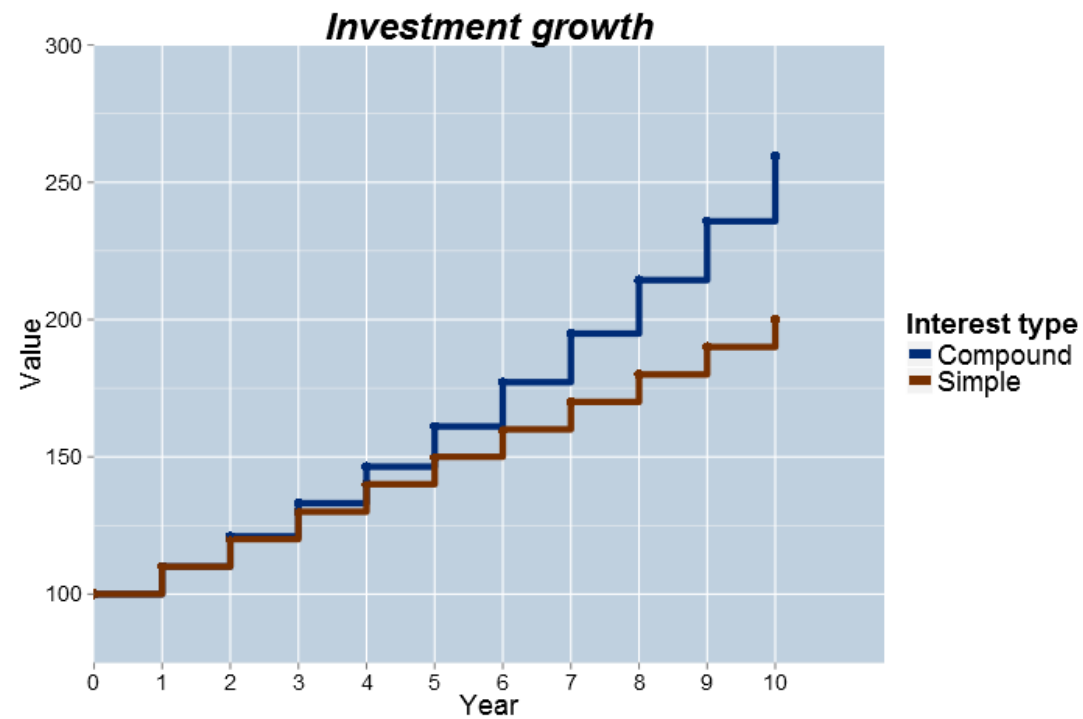
## Compound interest

- Start off with \$100 (*principal*) and at the end of every year earn 10% *compound interest* on the initial \$100
- Compound interest means that the interest itself earns interest in subsequent years

Year	0	1	2	3	4	10
Value	100.00	110.00	121.00	133.10	146.41	259.37

- Notice that the growth is no longer the same absolute amount each year, but it is the same proportionate (relative) amount (10%)

## The growth of the two investments



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## Constant proportionate growth

- Denote the initial amount as  $P_0$
- Denote the constant proportionate growth factor by  $\theta$  (for example 10% growth means multiply by 1.1 at the end of each time period)
- The growth progression is

Time	0	1	2	3	t
Amount, $P_t$	$P_0$	$P_0 \theta$	$P_0 \theta^2$	$P_0 \theta^3$	$P_0 \theta^t$

- $\theta > 1$  means the process is growing
- $\theta < 1$  means the process is declining/decaying
- This type of progression is called a ***geometric progression*** or ***geometric series***

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## Example

- An Indian Ocean nation caught 200,000 tonnes of fish this year
- The catch is projected to fall by a constant 5% factor each year for the next 10 years
- How many fish are predicted to be caught 5 years from now?
- Including this year, what is the total expected catch over the next 5 years?





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## The constant multiplier

- For the catch to fall by 5% each year, means that the multiplier is  $\theta = 0.95$
- In general, if the process is changing by R% in each time period, then the multiplier is

$$\theta = 1 + \frac{R}{100}$$

- Example:
  - if the increase is 5% then  $\theta = 1.05$
  - If the decrease is 5% then  $\theta = 0.95$

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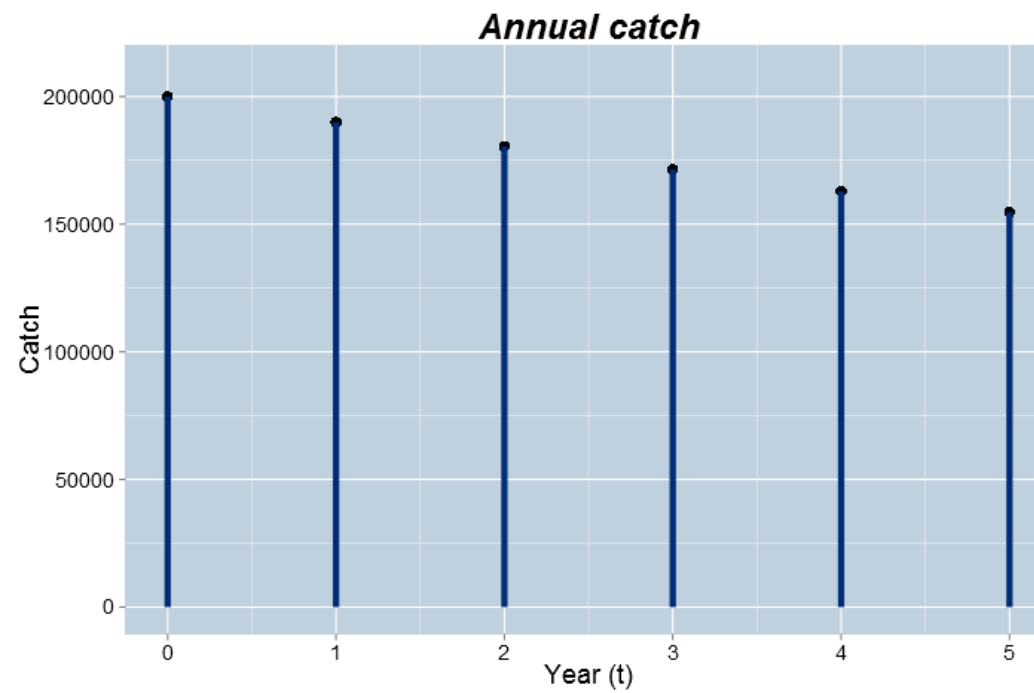
## Working out the fish catch

- With  $P_0 = 200,000$  and  $\theta = 0.95$ , in 5 years the catch will be  $200,000 \times 0.95^5 = 154,756$
- Over the next 5 years:
- We get the total catch summing over the years in question

Year	Catch
0	200,000
1	190,000
2	180,500
3	171,475
4	162,901.25
5	154,756.1875
Total	1,059,632

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## Graphing the annual fish catch



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## The sum of the geometric series

- If we denote the sum up to time  $t$  as  $S_t$ , then

$$S_t = P_0 \frac{1 - \theta^{t+1}}{1 - \theta}$$

- With the fisheries example:

$$S_5 = 200,000 \frac{1 - 0.95^{(5+1)}}{1 - 0.95} = 1,059,632$$

- The mathematical formulation of the model can sometimes provide more direct answers than a spreadsheet

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## Present and future value

- If there is no inflation and the prevailing interest rate is 4%, then which of the following options would you prefer?
  - \$1000 today or \$1500 in ten years?
- Either look to see how much \$1000 will be worth in ten years, or
- Calculate how much you would have to invest today to get \$1500 ten years from now
- The latter approach relies on the concept of ***present value***
  - the expected current value of an income stream

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## The present value calculation

- We know that  $P_t = P_0 \theta^t$  and making  $P_0$  the subject of the formula means that  $P_0 = P_t \theta^{-t}$
- Therefore \$1500 in ten years time in a 4% interest rate environment is worth  $1500 (1 + 0.04)^{-10}$  in today's money
- This equals \$1013.346, which is more than \$1000, so you should prefer the second investment of \$1500 received in ten years
- This straightforward proportionate increase model allows for a simple discounting formula

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## Uses of present value

- A primary use is in discounting investments to the present time
- An ***annuity*** is a schedule of fixed payments over a specified and finite time period
- The present value of an annuity is the ***sum*** of the present values of each separate payment
- Present value is also used in ***lifetime customer value*** calculations

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## Continuous compounding

- The compounding period for an investment can be yearly, monthly, weekly, daily etc.
- As the compounding period gets shorter and shorter, in the limit, the process is said to be ***continuously compounded***
- If a principal amount  $P_0$  is continuously compounded at a nominal annual interest rate of  $R\%$ , then at year  $t$ ,  
$$P_t = P_0 e^{rt} \text{ where } r = R/100$$



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## Continuous compounding

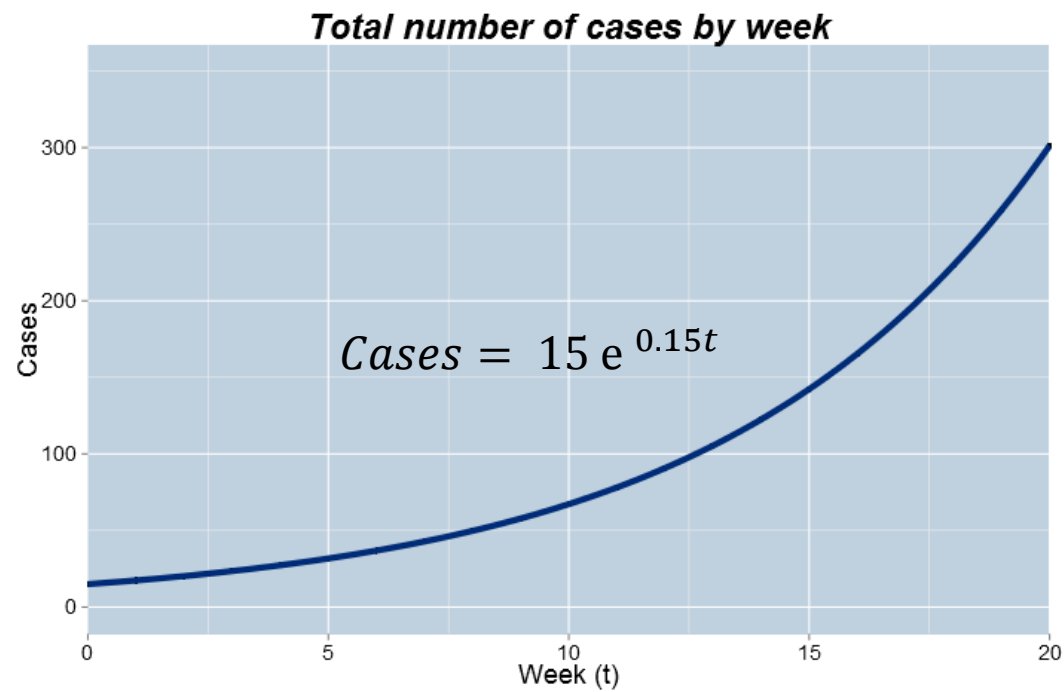
- In this model,  $t$  can take on any value in an interval, whereas in the discrete model  $t$  could only take on distinct values
- \$1000 ***continuously compounded*** at a nominal annual interest rate of 4% is worth  $1000 e^{0.04} = 1040.80$  after one year
- Note that this is a little more than if the 4% was earned at the very end of the time period, in which case you would have exactly \$1040 at the end of the year

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## Modeling an epidemic

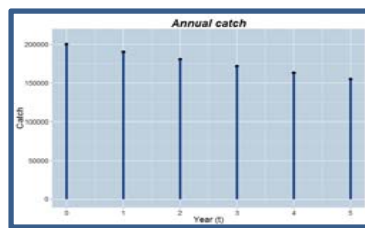
- The model  $P_t = P_0 e^{rt}$  doesn't just describe money growing
- This model is called exponential growth or decay depending on whether  $r$  is positive or negative respectively
- A continuous time model for the initial stages of an epidemic states that the number of cases at week  $t$  is  $15 e^{0.15t}$
- Halfway through week 7, how many cases do you expect?

## Graphing the epidemic

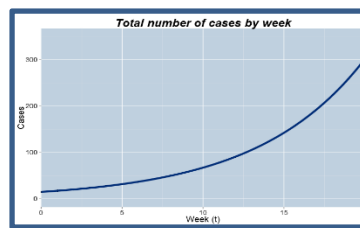


## Calculating the expected number of cases

- Halfway through week 7, means  $t = 7.5$
- $15 e^{0.15 \times 7.5} = 46.2 \approx \mathbf{46 \text{ cases}}$
- Interpretation of the 0.15 coefficient:
  - There is an approximate 15% weekly growth rate in cases
- Continuous models allow calculation at any value of  $t$ , not just a set of discrete values



Discrete model



Continuous model

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## Using a model for optimization

- A common modeling objective is to perform a subsequent optimization
- The objective of the optimization is to find the value of an input that maximizes/minimizes an output
- Example: find the price at which profit is maximized

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## Demand model

- Consider the demand model:

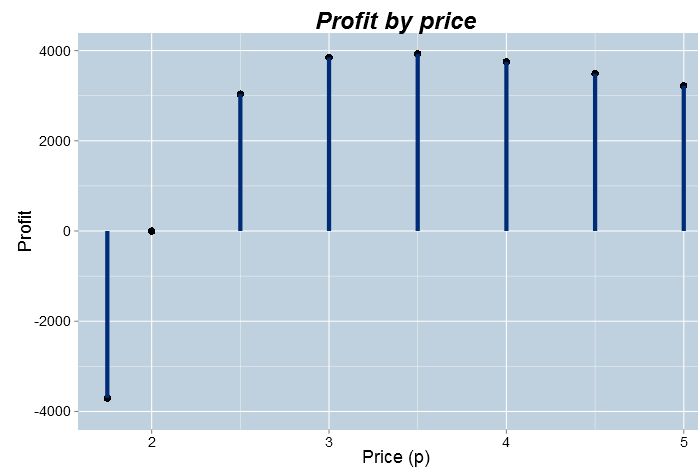
$$\text{Quantity} = 60,000 \text{ Price}^{-2.5}$$

- If the price of production is constant at  $c = 2$  for each unit, then at what price is profit maximized?
- Profit = Revenue – Cost
- Revenue = Price  $\times$  Quantity =  $p \times q$
- Profit =  $p q - c q = q (p - c)$   
$$= 60,000 p^{-2.5} (p - 2)$$
- Goal: Choose  $p$  to maximize this equation

## Brute force approach

- Choose different values of  $p$  and plot profit

Price	Profit
1.75	-3702.509
2.0	0.000
2.5	3035.787
3.0	3849.002
3.5	3927.104
4.0	3750.000
4.5	3491.885
5.0	3219.938



The optimal price is somewhere between 3 and 4

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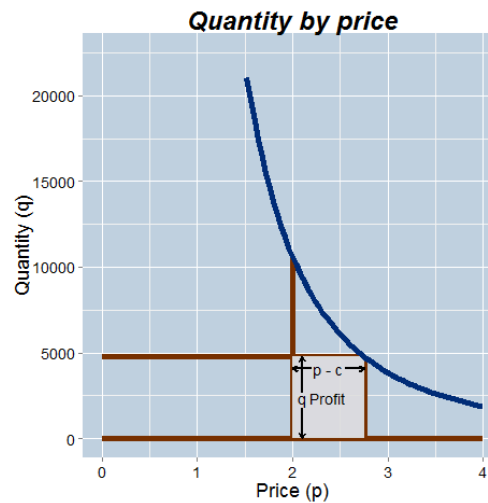
## Calculus approach

- Profit is maximized when the **derivative** (rate of change) of profit with respect to price equals 0
- Through **calculus** one obtains the optimal value of price as  $p_{opt} = \frac{c}{1+b}$ , where  $c$  is the production cost and  $b$  is the exponent in the power function
- In this example  $c = 2$  and  $b = -2.5$ . This gives
$$p_{opt} = \frac{2 \times -2.5}{(1 - 2.5)} \approx 3.33$$
- The value  $(-b)$  is known as **the price elasticity of demand**



## Visualizing the calculus solution

- The area of the gray shaded box is the profit and the objective is to find the value of price for which the area of the box is largest



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## Summary

- Linear models
- Growth and decay in discrete time: geometric series models
- Growth and decay in continuous time: exponential growth models
- Present and future value
- Use classical optimization (calculus) to gain additional value from the model



