Probability and Statistics

Unit I: Basic Probability

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Contents

Basic Probability

Definition (Sample space:)

Let S denote the set of all possible outcomes of an experiment. S is called as the sample space of the experiment.

Definition (Event:)

An **event** is a subset of a sample space S.

Definition (Probability of A:)

For each event A of the sample space S we suppose that a number P(A), called the probability of A, is defined and is such that

1. $0 \leq P(A) \leq 1$
2. $P(A) = 1$

Properties of Probability

1. $$P(\overline{A}) = 1 - P(A)$$

2. $$P(\phi) = 0$$

3. $$\text{If} \qquad A \subseteq B,$$
$$\text{then} \qquad P(A) \leq P(B)$$
.

4. $$\text{If} \quad A = A_1 \cup A_2 \cup \ldots \cup A_n$$
, where A_1, A_2,...,A_n are mutually exclusive events, then
$$P(A) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

Problem

If two dice are rolled, what is the probability that the sum of the upturned faces will equal 7?

Solution: Two dice are rolled. Sample space is

S=$\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),$
$(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),$
$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),$
$(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

A: event that the sum of the upturned faces will equal 7.

$$A = \{(6,1),(1,6),(2,5),(5,2),(4,3),(3,4)\}$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

Problem[02] A fair coin is tossed 4 times. find the respective probabilities.

1. More heads than tails are obtained.
2. Tails occur on the even numbered tosses.

Solution: A fair coin is tossed four times. Sample space
S=$\{HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTTH, TTHH, HTHT, THTH, THHT, TTTH, TTHT, THTT, HTTT, TTTT\}$

A: event that there are more heads than tails are obtained
A=$\{HHHH, HHHT, HHTH, HTHH, THHH\}$ $n(A) = 5$

$$P(A) = \frac{n(A)}{n(S)} = \frac{5}{16}$$

B:event that tails occur on the even numbered tosses.
B=$\{HTHT, TTTT, HTTT, TTTH\}$ $n(B) = 4$

$$P(B) = \frac{n(B)}{n(S)} = \frac{4}{16} = \frac{1}{4}$$

Problem[03] If 3 balls are "randomly drawn" from a bowl containing 6 white and 5 black balls, what is the probability that one of the balls is white and the other two black?

Solution: White balls: 06, black balls: 05

Total balls: 11

No. of ways of choosing 1 white ball $= {}^{6}C_1 = 6$

No. of ways of choosing 2 black balls $= {}^{5}C_2 = 10$

No. of ways of choosing 1 white and two black balls $= {}^{11}C_3 = 165$

$$P(1 \text{ white } \& \text{ 2 black balls}) = \frac{{}^{6}C_1 . {}^{5}C_2}{{}^{11}C_3}$$

$$= \frac{6 \times 10}{165} = \frac{4}{11}$$

Problem[04] A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

Solution: Men: 06, Women : 09

Total balls: 15

No. of ways of selecting 3 men from 6 $={}^6C_3$=20

No. of ways of selecting 2 men from 9 $={}^9C_2$=36

No. of ways of selecting 5 members $={}^{15}C_5$=3003

$$P(3 \text{ men } \& \text{ } 2 \text{ women}) = \frac{{}^6C_3 . {}^9C_2}{{}^{15}C_5}$$

$$= \frac{20 \times 36}{3003} = \frac{240}{1001}$$

Conditional probability

A and B are two events.

Probability of event A, given that B has occured is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Probability of event B, given that A has occured is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

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Conditional probability

Two events A and B are such that P[A∩B] = 0.15, P[A∪B] = 0.65, and $P[A|B] = 0.5$. Find $P[B|A]$.

Solution:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$0.5 = \frac{0.15}{P(B)}$$

$$P(B) = \frac{.15}{.5} = .3$$

We know,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$0.65 = P(A) + 0.3 - 0.15$$

$$P(A) = 0.5$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(B|A) = \frac{0.15}{0.5} = .3$$

Examples

Sachin rolls two dice and tells you that there is at least one 6. What is the probability that the sum is at least 9?

Solution: Sachin rolls two dice. Sample space

S=$\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),$
$(2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),$
$(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),$
$(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

A: event that there is atleast one 6

A=$\{(1,6),(2,6),(3,6),(4,6),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

$n(A) = 11$

Solution

B: event that the sum is atleast 9.
B={(4,5),(5,5),(3,6),(6,3),(4,6),(6,4),(5,5),(5,6),(6,5),(6,6)}
$n(B) = 10$

$A \cap B = \{(3,6),(4,6),(5,6),(6,3),(6,4),(6,5),(6,6)\}$
$n(A \cap B) = 7$
$P(A \cap B) = \frac{7}{36}$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(B|A) = \frac{7}{11}$$

A set of events is said to be independent if the occurrence of any one of them does not depend on the occurrence or non-occurrence of the others.

When 2 events A and B are independent, it is obvious from the definition that P(B|A) = P(B).

If the events A and B are independent, the product theorem takes the form

$$P(A \cap B) = P(A) \times P(B)$$

Example

Two fair dice are thrown independently. Three events A, B, and C are defined as follows:

1. Odd face with the first die.
2. Odd face with second die.
3. Sum of the numbers in the 2 dice is odd.

Are the events A, B and C mutually independent?

Solution: Two dice are thrown independently . Sample space is

$$S=\{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),\\ (2,4),(2,5),(2,6),(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6),(5,1),(5,2),(5,3),\\ (5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

A:event odd face with the first die.

A={(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)}

$$P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$

B: event that odd face with second die.

B={(1, 1), (1, 3), (1, 5), (2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5) (4, 1), (4, 3), (4, 5), (5, 1), (5, 3), (5, 5), (6, 1), (6, 3), (6, 5)}

$$P(B) = \frac{n(B)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$

$A \cap B = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{9}{36} = \frac{1}{4}$$

Solution

C:event that sum of the numbers in the 2 dice is odd.
C=$\{(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),(3,2),(3,4),(3,6),$
$(4,1),(4,3),(4,5),(5,2),(5,4),(5,6),(6,1),(6,3),(6,5)\}$

$$P(C) = \frac{n(C)}{n(S)} = \frac{18}{36} = \frac{1}{2}$$

Similarly,

$$P(B \cap C) = P(B).P(C)$$

B and C are independent.
and

$$P(C \cap A) = P(C).P(A)$$

C and A are independent.

$$A \cap B \cap C = \phi$$

$$P(A \cap B \cap C) = 0$$

$$P(A).P(B).P(C) = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8}$$

$$P(A \cap B \cap C) \neq P(A).P(B).P(C)$$

Therefore, A, B and C are not mutually independent.

Question

Two events A and B are such that
$P(A) = \frac{1}{3}$,$P(B) = \frac{1}{4}$, $P(A \cup B) = \frac{1}{2}$, *Find* $P[B|A]$ *and* $P[A|\overline{B}]$.

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$P(A \cap B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{2}$

$$P(A \cap B) = \frac{1}{12}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$P(B|A) = \frac{\frac{1}{12}}{\frac{1}{3}} = \frac{1}{4}$$

$$P(A|\overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})}$$

$$P(\overline{B}) = 1 - P(B) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(A \cap \overline{B}) = P(A) - P(A \cap B) = \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$$

$$P(A|\overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{4}$$

$$\boxed{P(A|\overline{B}) = \frac{1}{4}}$$

Question

A box contains 4 bad and 6 good tubes. Two are drawn out from the box one after other. One of them is tested and found to be good. What is the probability that the other one is also good? [Ans $\frac{5}{9}$]

Let A = one of the tubes drawn is good and
B = the other tube is good.
P(both tubes drawn are good)$=P(A \cap B) = \frac{^6C_2}{^{10}C_2} = \frac{1}{3}$
Knowing that one tube is good, the conditional probability that the other tube is also good is required, i.e., $P(B|A)$ is required. By definition,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{3}}{\frac{6}{10}} = \frac{5}{9}$$

Question

From 6 positive and 8 negative numbers, 4 numbers are chosen at random (without replacement) and multiplied. What is the probability that the product is positive? [ANS $\frac{505}{1001}$]

Solution If the product is to be positive, all the 4 numbers must be positive or all the 4 must be negative or 2 of them must be positive and the other 2 must be negative.

No. of ways of choosing 4 positive numbers = $6C_4 = 15$.

No. of ways of choosing 4 negative numbers = $8C_4 = 70$.

No. of ways of choosing 2 positive and 2 negative numbers

$$= 6C_2 \times 8C_2 = 420.$$

Total no. of ways of choosing 4 numbers from all the 14 numbers

$$= 14C_4 = 1001.$$

P(the product is positive)

$$= \frac{\text{No. of ways by which the product is positve}}{\text{Total no. of ways}}$$

$$= \frac{15 + 70 + 420}{1001} = \frac{505}{1001}$$

Question

A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. Two articles are chosen from the lot at random (without replacement). Find the probability that

1. *Both are good.* [Ans $\frac{3}{8}$]
2. *Both have major defects.*[Ans $\frac{1}{120}$]
3. *At least 1 is good.*[Ans $\frac{7}{8}$]
4. *At most 1 is good.*[Ans $\frac{5}{8}$]
5. *Exactly I is good* [Ans $\frac{1}{2}$]
6. *Neither has major defects.* [Ans $\frac{91}{120}$]
7. *Neither is good.* [Ans $\frac{1}{8}$]

Solution Although the articles may be drawn one after the other, we can consider that both articles are drawn simultaneously, as they are drawn without replacement.

(i) $$P(\text{both are good}) = \frac{\text{No. of ways drawing 2 good articles}}{\text{Total no. of ways of drawing 2 articles}}$$

$$= \frac{10C_2}{16C_2} = \frac{3}{8}$$

(ii) P(both have major defects)

$$= \frac{\text{No. of ways of drawing 2 articles with major defects}}{\text{Total no. of ways}}$$

$$= \frac{2C_2}{16C_2} = \frac{1}{120}$$

(iii) P(at least 1 is good) = P(exactly 1 is good or both are good)

$= P$(exactly 1 is good and 1 is bad or both are good)

$$= \frac{10C_1 \times 6C_1 + 10C_2}{16C_2} = \frac{7}{8}$$

(iii) $P(\text{at least 1 is good}) = P(\text{exactly 1 is good or both are good})$

$$= P(\text{exactly 1 is good and 1 is bad or both are good})$$

$$= \frac{10C_1 \times 6C_1 + 10C_2}{16C_2} = \frac{7}{8}$$

(iv) $P(\text{atmost 1 is good}) = P(\text{none is good or 1 is good and 1 is bad})$

$$= \frac{10C_0 \times 6C_2 + 10C_1 \times 6C_1}{16C_2} = \frac{5}{8}$$

(v) $P(\text{exactly 1 is good}) = P(\text{1 is good and 1 is bad})$

$$= \frac{10C_1 \times 6C_1}{16C_2} = \frac{1}{2}$$

(vi) $P(\text{neither has major defects})$

$$= P(\text{both are non-major defective articles})$$

$$= \frac{14C_2}{16C_2} = \frac{91}{120}$$

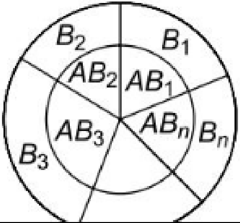
(vii) $P(\text{neither is good}) = P(\text{both are defective})$

$$= \frac{6C_2}{16C_2} = \frac{1}{8}$$

Total Probability

Let $\{B_1, B_2, ..., B_n\}$ be a partition of the sample space S, and suppose each one of the events B_1 , B_2,... ,B_n, has nonzero probability of occurrence. Let A be any event. Then

$$P(A) = P(B_1)P(A|B_1) + P(B_2)P(A|B_2) + ... + P(B_n)P(A|B_n)$$



Question

Ankit is planning to pick up a friend at the airport. He has figured out that the plane is late 80% of the time when it rains, but only 30% of the time when it does not rain. If the weather forecast that morning calls for a 40% chance of rain, what is the probability that the plane will be late?

Solution:

R:Event it rains, L: Plane is late

$P(R) = 40\% = 0.40$, $P(L|R) = 80\% = 0.80, P(L|\overline{R}) = 30\% = 0.30$

$$P(L) = P(R).P(L|R) + P(\overline{R}).P(L|\overline{R})$$

$$= (0.4)(0.8) + (0.6)(0.30)$$

$$P(L) = 0.50$$

Question

There are 3 true coins and 1 false coin with head on both sides. A coin is chosen at random and tossed 4 times. If head occurs all the 4 times, what is the probability that 4 heads occurs?

Solution $P(T) = P(\text{the coin is a true coin}) = \dfrac{3}{4}$

$P(F) = P(\text{the coin is a false coin}) = \dfrac{1}{4}$

Let A = Event of getting all heads in 4 tosses

Then $P(A/T) = \dfrac{1}{2} \times \dfrac{1}{2} \times \dfrac{1}{2} \times \dfrac{1}{2} = \dfrac{1}{16}$ and $P(A/F) = 1$.

By Bayes' theorem,

$$P(F/A) = \frac{P(F) \times P(A/F)}{P(F) \times P(A/F) + P(T) \times P(A/T)}$$

$$= \frac{\frac{1}{4} \times 1}{\frac{1}{4} \times 1 + \frac{3}{4} \times \frac{1}{16}} = \frac{16}{19}$$

Baye' s Theorem

Baye' s Theorem

If $\{B_1, B_2, ..., B_n\}$ be a set of exhaustive and mutually exclusive events associated with a random experiment and A is another event associated with (or caused by) B_i , then

$$P(B_i|A) = \frac{P(B_i) \times P(A|B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A|B_i)}$$

Question

In a bolt factory, machines A, B and C produce 25, 35 and 40% of the total output, respectively. Of their outputs, 5, 4 and 2%, respectively, are defective bolts. If a bolt is chosen at random from the combined output, what is the probability that it is defective? If a bolt chosen at random is found to be defective, what is the probability that it was produced by B?

$\left[\textbf{SOLVED IN CLASSROOM}\right]$

Home work problem

Question

A box contains 2000 components of which 5% are defective. A second box contains 500 components of which 40% are defective. Two other boxes contain 1000 components, each with 10% defective components. We select at random one of the above boxes and remove from it at random a single component.

1. *What is the probability that the component is defective?*
2. *Finding that the selected component is defective, what is the probability that it was drawn from box 2?*

Question

A bag contains 5 balls and it is not known how many of them are white. Two balls are drawn at random from the bag and they are noted to be white. What is the chance that all the balls in the bag are white

Since 2 white balls have been drawn out,
the bag must have contained 2, 3, 4 or 5 white balls.
Let B_2 = Event of the bag containing 2 white balls,
B_3 = Event of the bag containing 3 white balls,
B_4 = Event of the bag containing 4 white balls and
B_5 = Event of the bag containing 5 white balls.

Let A = Event of drawing 2 white balls.
$P(A|B_2) = \frac{^2C_2}{^5C_2} = \frac{1}{10}$, $P(A|B_3) = \frac{^3C_2}{^5C_2} = \frac{3}{10}$
$P(A|B_4) = \frac{^4C_2}{^5C_2} = \frac{3}{5}$, $P(A|B_5) = \frac{^5C_2}{^5C_2} = \frac{1}{1} = 1$
Since the number of white balls in the bag is not known, B_i' s are equally likely.
$P(B_2) = P(B_3) = P(B_4) = P(B_5) = \frac{1}{4}$
By Baye' s theorem

$$P(B_5|A) = \frac{P(B_5) \times P(A|B_5)}{\sum P(B_i) \times P(A|B_i)} = \frac{\frac{1}{4} \times 1}{\frac{1}{4} \times \left(\frac{1}{10} + \frac{3}{10} + \frac{3}{5} + 1\right)} = \frac{1}{2}$$

References

- T. Veerarajan, Probability, Statistics and Random Processes, Tata McGraw-Hill 2003, 3 rd edition, 2008.

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