An Introduction to Hilbert Spaces

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1 Introduction

Hilbert space is a mathematical concept that describes a vector space equipped with an inner product. It is widely used in quantum mechanics and other areas and provides a way to solve problems in infinite-dimensional spaces. The space is complete and has many important applications in mathematics and physics.

In this essay, I give an introduction to the theory of Hilbert spaces including the inner product and orthogonality. And I prove the Riesz–Fréchet representation theorem, which establishes a connection between its continuous dual space and a Hilbert space. The last chapter briefly introduces bounded operators and prove two related theorems.

2 Groundwork

2.1 Inner product space

Definition 2.1. An inner product is a function and it has the following properties:

If W is a complex vector space:

$$\langle \cdot \ , \ \cdot \rangle : W \times W \to \mathbb{C}$$

s.t. $\forall w_0, w_1, w_2 \in W$ and $\gamma, \beta \in \mathbb{C}$,

(i) $\langle w_0, \gamma w_1 + \beta w_2 \rangle = \gamma \langle w_0, w_1 \rangle + \beta \langle w_0, w_2 \rangle$;
$\langle \gamma w_0 + \beta w_1, w_2 \rangle = \overline{\gamma} \langle w_0, w_2 \rangle + \overline{\beta} \langle w_1, w_2 \rangle$;

(ii) $\langle w_0, w_1 \rangle = \overline{\langle w_1, w_0 \rangle}$; (Conjugate symmetry)

(iii) $\langle w_0, w_0 \rangle \geq 0$ with equality only for $w_0 = 0$.

Definition 2.2. An inner product space is a vector space together with an inner product. (i.e. $(W, \langle \cdot \ , \ \cdot \rangle)$ defines an inner product space.)

Example 1. $(C([0,1])\ , \langle p, q \rangle = \int_0^1 p(x)\overline{q(x)}\, dx)$ is an inner product space, where $C([0,1])$ is the space of complex-valued continuous functions on [0,1]. [4, p. 5]

2.2 Normed space

Definition 2.3. W is a real(or complex) vector space . A *norm* on W is a map $||\cdot||$: $W \to \mathbb{R}$ such that:

(i) $||w|| > 0$ if $w \neq 0$; And $||w|| = 0$ if $w = 0$

(ii) $||\alpha w|| = |\alpha| ||w||$ $\forall \alpha \in \mathbb{R}$ or $\mathbb{C}$ and $w \in W$;

(iii) $||w_1 + w_2|| \leq ||w_1|| + ||w_2|| \; \forall w_1, w_2 \in W$.

An *inner product norm* on W is a function from W to $\mathbb{R}$ such that $||w|| = \sqrt{\langle w, w \rangle} \; \forall w \in W$.

Definition 2.4. A normed space is a vector space together with a norm operation $\| \cdot \|$. i.e. $(W, \| \cdot \|)$ is a normed space.

Theorem 2.1 (Cauchy-Schwartz inequality with inner product)**.** *If V is an inner product space then for all $a, b \in V$*

$$|\langle a, b \rangle| \leq \|a\| \|b\|$$

$[5, p.113 - 114]$

Proof. suppose that $a - rb \neq 0 \; \forall r \in \mathbb{C}$. Then,

$$\begin{aligned} \langle a - rb, a - rb \rangle &= \langle a, a - rb \rangle - a \langle b, a - rb \rangle \\ &= \langle a, a \rangle - \overline{r} \langle a, b \rangle - r \langle b, a \rangle + \|r\|^2 \langle b, b \rangle \end{aligned}$$

Let $r = \frac{\langle a, b \rangle}{\langle b, b \rangle}$, then

$$\begin{aligned} \langle a - rb, a - rb \rangle &= \langle a, a \rangle - \frac{\langle b, a \rangle}{\langle b, b \rangle} \langle a, b \rangle - \frac{\langle a, b \rangle}{\langle b, b \rangle} \langle b, a \rangle + \frac{\langle a, b \rangle \langle b, a \rangle}{\langle b, b \rangle^2} \langle b, b \rangle \\ &= \langle a, a \rangle - \frac{\langle a, b \rangle \langle b, a \rangle}{\langle b, b \rangle^2} \langle b, b \rangle \end{aligned}$$

Note that $\langle a, a \rangle - \frac{\langle a, b \rangle \langle b, a \rangle}{\langle b, b \rangle^2} \langle b, b \rangle > 0$. That is,

$$\langle a, a \rangle \langle b, b \rangle - \langle a, b \rangle \overline{\langle a, b \rangle} > 0$$

So we can conclude that $|\langle a, b \rangle|^2 < \langle a, a \rangle \langle b, b \rangle$ and hence $|\langle a, b \rangle| \leq \|a\| \|b\|$ for $a - rb \neq 0$.

And we claim that the equality holds when there exists $r \in \mathbb{C}$ such that $b = ra$, since

$$\begin{aligned} |\langle a, b \rangle|^2 &= |\langle a, rb \rangle|^2 \\ &= |\overline{r} \langle a, a \rangle|^2 \\ &= |\overline{r}|^2 \langle a, a \rangle|^2 \\ &= |r|^2 \langle a, a \rangle^2 \\ &= \langle a, a \rangle r \overline{r} \langle a, a \rangle \\ &= \langle a, a \rangle \langle ra, ra \rangle \\ &= \langle a, a \rangle \langle b, b \rangle \end{aligned}$$

□

2.3 Metric and metric spaces

Definition 2.5 (Metric and metric space)**.** Let $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$. Then a **metric** d on a set V is a mapping: $V \times V \to \mathbb{R}_+$ satisfying the following properties: $\forall v_0, v_1, v_2 \in V$

(i) $d(v_0, v_1) = d(v_1, v_0)$;

(ii) $d(v_0, v_2) \leq d(v_0, v_1) + d(v_1, v_2)$;

(iii) $d(v_0, v_1) = 0$ if and only if $v_0 = v_1$.

A set V together with d as defined above gives us a **metric space** (V, d).

Example 2. Let $X := \mathbb{R}^2$ and $d(a, b) = |a_1 - b_1| + |a_2 - b_2|$ where $a = (a_1, a_2), b = (b_1, b_2)$. The properties (i) and (iii) are trivial. To prove the second property of a metric, suppose that $c = (c_1, c_2)$, then:

$$d(a, c) = |a_1 - c_1| + |a_2 - c_2| \leq |a_1 - b_1| + |b_1 - c_1| + |a_2 - b_2| + |b_2 - c_2| \leq d(a, b)$$

for all $a, b, c \in \mathbb{R}^2$.

2.4 Completeness

Definition 2.6 (Cauchy sequence)**.** If (V, d) is a metric space. $(a_n) \subset V$ is a Cauchy sequence if $\forall \epsilon > 0$, $\exists N > 0$ s.t.

$$d(a_i, a_j) < \epsilon$$

$\forall i, j \leq N$

Definition 2.7 (Completeness)**.** A metric space (V, d) is complete if any Cauchy sequence in V has a limit that is also in V.

3 Hilbert spaces

Definition 3.1 (Hilbert space)**.** A Hilbert space is a complete inner product space.

Example 3. $\ell^2 = \{x = (x_1, x_2, x_3, ...) : x_k \in \mathbb{C}, \sum_{k=1}^{\infty} |x_k|^2 < \infty\}$ with $\langle y, x \rangle = \sum_{i=1}^{\infty} \overline{y_k} x_k$ is a Hilbert space

Example 4 (Not Hilbert space)**.** The space $\mathbb{C}[a, b]$ with $\langle g, f \rangle = \int_a^b \overline{g(x)} f(x)\, dx$ is not a Hilbert space;

4 Orthogonal and orthonormal system

4.1 Orthonormal basis

Definition 4.1 (Orthogonal)**.** p, q in a Hilbert space $\mathcal{H}$ are *orthogonal* if $\langle p, q\rangle = 0$. We also write this as $f \perp g$.

We will use the following statement in theorem 4.6:

The **orthogonal** of A is:

$$A^{\perp} = \{f \in \mathcal{H} | \langle a, f\rangle = 0, \forall a \in A\}$$

Definition 4.2 (Orthonormal System)**.** A set $S = \{e_\alpha\}$ in $\mathcal{H} \setminus \{0\}$ is called an *orthogonal system* if $e_\alpha \perp e_\beta$ when $\alpha \neq \beta$ and $\|e_\alpha\| = 1$.

Definition 4.3 (Orthonormal basis)**.** An orthonormal system S is an *orthonormal basis* if

$$y = \sum_{j=1}^{\infty} \alpha_j y_j$$

$\forall y \in \mathcal{H}$, where $\alpha_j \in \mathbb{C}$ are uniquely determined and $y_j's \in S$ are distinct.

Theorem 4.1 (Pythagorean Theorem with Inner product)**.** *Consider the Hilbert space* $(\mathcal{H}, \langle \cdot\ ,\ \cdot\rangle)$ *and* $\|.\|$ *is the norm on it. If* $\{f_1, f_2, ..., f_n\} \subset \mathcal{H}$ *is an orthogonal set then* $\| \sum_{i=1}^n f_i\|^2 = \sum_{i=1}^n \|f_i\|^2$

Proof. The inner product is defined by

$$\langle f_i, f_j\rangle = \begin{cases} \langle f_i, f_j\rangle & \text{if i =j,} \\ 0 & \text{if } else. \end{cases}$$

$$\begin{aligned} \|\sum_{i=1}^n f_i\|^2 &= \langle \sum_{i=1}^n f_i, \sum_{j=1}^n f_j\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n \langle f_i, f_j\rangle \\ &= \sum_{i=1}^n \|f_i\|^2. \end{aligned}$$

□

Theorem 4.2. *If* $e_1, e_2, ..., e_n$ *is an orthonormal basis of a Hilbert space* $\mathcal{H}$ *and* $\gamma_1, ..., \gamma_m$ *are complex numbers and* $x \in \mathcal{H}$*. Then for* $c_i = \langle x, e_i \rangle$

$$||x - \sum_{i=1}^{m} \gamma_i e_i||^2 = ||x||^2 - \sum_{i=1}^{m} |\gamma_i - c_i|^2 - \sum_{i=1}^{m} |c_i|^2$$

Proof. By Pythagorean Theorem 4.1, we have

$$\langle \sum \gamma_i e_i, \sum \gamma_i e_i \rangle = \sum \gamma_i \overline{\gamma_i}$$

and then:

$$\begin{aligned}
\|x - \sum \gamma_i e_i\|^2 &= \langle x - \sum \gamma_i e_i, x - \sum \gamma_i e_i \rangle \\
&= \langle x, x \rangle - \sum \gamma_i \langle e_i, x \rangle - \sum \overline{\gamma_i} \langle x, e_i \rangle + \sum \gamma_i \overline{\gamma_i} \\
&= \|x\|^2 - \sum \gamma_i \overline{c_i} - \sum \overline{\gamma_i} c_i + \sum \gamma_i \overline{\gamma_i} \\
&= \|x\|^2 + \sum (\gamma_i - c_i)(\overline{\gamma_i} - \overline{c_i}) - \sum c_i \overline{c_i} \\
&= \|x\|^2 + \sum |\gamma_i - c_i|^2 - \sum |c_i|^2
\end{aligned}$$

□

We can fix x and e_i so that $\sum \gamma_i e_i$ can represent $lin\{e_1, ..., e_m\}$. And note that $\langle x, e_i \rangle = c_i$ are all fixed , when $\gamma_i = c_i$ for all i, we can obtain the minimum of $||x - \sum_{i=1}^{m} \gamma_i e_i||^2$. Now we are able to derive the following important theorem.

Theorem 4.3. *If* $e_1, e_2, ..., e_n$ *is an orthonormal basis of a Hilbert space* $\mathcal{H}$ *and* $y \in \mathcal{H}$*. The closest point* z *of* $lin\{e_1, e_2, ..., e_n\}$ *to* y *is*

$$z = \sum_{i=1}^{n} \langle y, e_i \rangle e_i$$

and the distance $d = \|y - z\|$ *has the property:*

$$d^2 = \|y\|^2 - \sum_{i=1}^{n} |\langle y, e_i \rangle|^2$$

4.2 Bessel's inequality

Theorem 4.4 (Bessel's inequality)**.** *If an orthonormal sequence* $(e_i)_{i\in\mathbb{N}}$ *is in a Hilbert space* $\mathcal{H}$*, we have* $\|x\|^2 \geq \sum_{j=1}^{\infty} |\langle x, e_j\rangle|^2 \ \forall x \in \mathcal{H}$

Proof. Let $y_m = \sum_{j=1}^{m}\langle x, e_j\rangle e_j$ for some integer m.

According to the Theorem 4.3, we have

$$\|x - y_m\|^2 = \|x\|^2 - \sum_{j=1}^{m} |\langle x, e_j\rangle|^2$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{m} |\langle x, e_j\rangle|^2 &= \|x\|^2 - \|x - y_m\|^2 \\ &\leq \|x\|^2\end{aligned}$$

Let m be infinity then we are done. □

Theorem 4.5. *A Hilbert space* $\mathcal{H}$ *has an orthonormal basis* $\{x_1, x_2, ..., x_m, ...\}$*. And* $k_1, k_2, ..., k_m, ...$ *are scalars such that* $(k_1, k_2, ..., k_m, ...) \in \ell^2$*.*

$$g = \sum_{i=1}^{\infty} k_i x_i \tag{1}$$

converges in $\mathcal{H}$*.*

And we also have: $\|g\|^2 = \sum_{i=1}^{\infty} |k_i|^2$

Proof. Let $S_n = \sum_{i=1}^{n} k_i x_i$, and the series $\sum_{i=1}^{\infty} k_i x_i$ is convergent if and only if the sequence (S_n) is convergent, then it is sufficient to show that (S_n) is Cauchy.

$$\begin{aligned}\|S_{M+G} - S_M\|^2 &= \|\sum_{m=1}^{M+G} k_m x_m - \sum_{m=1}^{M} k_m x_m\|^2 \\ &= \|\sum_{m=M+1}^{M+G} k_m x_m\|^2 && (*) \\ &= \sum_{m=M+1}^{M+G} |k_m|\|x_n\|^2 && (**) \\ &= \sum_{m=M+1}^{M+G} |k_m|^2\end{aligned}$$

And $\sum_{m=M+1}^{M+G} |k_m|^2$ converges uniformly to 0 since $(k_1, k_2, ..., k_m, ...) \in \ell^2$. (S_n) is convergent then we can conclude that the series $\sum_{i=1}^{\infty} k_i x_i$ is convergent □

4.3 Orthogonal Decomposition Theorem and Riesz Representation Theorem

Theorem 4.6 (Orthogonal decomposition)**.** *A Hilbert space* $\mathcal{H} = \mathcal{A} \oplus \mathcal{A}^{\perp}$ *(direct sum of spaces) if* $\mathcal{A}$ *is a closed subspace of* $\mathcal{H}$*. That is, for any* $a \in \mathcal{H}$*,* $a = b + c$ *where* $b \in \mathcal{A}$ *and* $c \in \mathcal{A}^{\perp}$*. [2, p. 6]*

Proof. let $a \in \mathcal{H}$, and $\delta = inf\{\|a-b\| : b \in \mathcal{A}\}$ and set $\{b_n\}$ to be a sequence in $\mathcal{A}$ s.t. $\|a - b_n\| \to \delta$. And $\forall b_n, b_m \in \mathcal{A}$, $\frac{1}{2}(b_n + b_m) \in \mathcal{A}$

$$2(\|b_n - a\|^2 + \|b_m - a\|^2) = \|b_n - b_m\|^2 + \|b_n + b_m - 2a\|^2$$

because $\frac{1}{2}(b_n + b_m) \in \mathcal{A}$,

$$\begin{aligned}\|b_n - b_m\|^2 &= 2\|b_n - a\|^2 + 2\|b_m - a\|^2 - 4\|\frac{1}{2}(b_n + b_m) - a\|^2 \\ &\leq 2\|b_n - a\|^2 + 2\|b_m - a\|^2 - 4\delta^2\end{aligned}$$

Notice that $\lim_{n\to\infty} \lim_{m\to\infty} 2\|b_n - a\|^2 + 2\|b_m - a\| - 4\delta^2 = 0$. Then the sequence $\{b_n\}$ is Cauchy.

We denote the limit by $b = \lim_{n\to\infty} b_n$ and $c = a - b$. Then $b \in \mathcal{A}$ because $\mathcal{A}$ is closed and $\|a - b\| = \delta$. Now we need to show that $c \in \mathcal{A}^{\perp}$. Suppose that $t \in \mathcal{A}$, and W.L.O.G we assume that $\langle c, t\rangle$ is real.

We define a function

$$f(u) = \|c + ut\|^2 = \|c\|^2 + 2u\langle c, t\rangle + u^2\|t\|^2$$

is real for $u \in \mathbb{R}$ and its minimum is δ^2 at $u = 0$ since $c + ut = a - (b - ut)$ and $b - ut \in \mathcal{A}$. Then $f(u)$ achieves the minimum at $2\langle c, t\rangle = 0$, so $c \in \mathcal{A}$ $\square$

Definition 4.4 (Dual space)**.** For any vector space H over a field F, the **dual space** H^* is defined as the set of all linear maps ϕ: $H \to F$. [3]

Theorem 4.7 (Riesz representation theorem)**.** $\mathcal{H}^*$ *is a dual space of the Hilbert space* $\mathcal{H}$ *and* $f \in \mathcal{H}^*$*, then* $\exists$ *a unique* $b \in \mathcal{H}$ *s.t.* $f(a) = \langle a, b\rangle$ $\forall$ $a \in \mathcal{H}$

Proof. (i) If $f(a) = \langle a, b\rangle = \langle a, b'\rangle$ for all $a \in \mathcal{H}$. Let $a = b - b'$, we have $\|b - b'\|^2 = 0$. Therefore $b = b'$.

(ii) (existence)

If $f(a) \equiv 0$, then we let $b = 0 \implies f(a) = \langle a, b\rangle$.

If $f(a)$ is not 0. Let $V = \{a \in \mathcal{H} : f(a) = 0\}$.

Since $f(a)$ is a bounded linear map, V is a closed subset of $\mathcal{H}$. By Theorem 4.4, we will have $V^{\perp} \neq \{0\}$. This is because if $V^{\perp} = \{0\}$

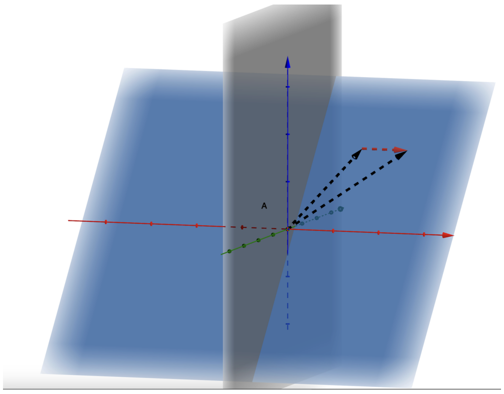

Figure 1: Orthogonal decomposition in Hilbert space

$\forall a \in \mathcal{H}$, then we have $a = b + c$, $b \in V$, $c \in V^{\perp} \implies f(a) = f(b) = 0$ $\forall a \in V$, which is a contradiction.

Now we set c to be in $V^{\perp}$ and $\|c\| = 1$. Let $u := f(a)c - f(c)a$, we have $u \in V$ since f is a linear function.

So $0 = \langle u, c\rangle = f(a) - \langle a, \overline{f(c)}c\rangle$. Let $b = \langle \overline{f(c)}, c\rangle$, we have $f(a) = \langle a, b\rangle$

$\square$

5 Bounded operators

Definition 5.1 (Bounded operator)**.** Suppose that V and W are normed spaces. T is a bounded operator if it satisfies the following inequality:

$$\|Tv_0\| \leq C\|v_0\|$$

$\forall v_0 \in V$ and C is some arbitrary positive constant.

Theorem 5.1. *Suppose that V and W are normed spaces and T: $V \to W$ is a linear operator. The following statements are equivalent.*

(i) T is continuous at any point

(ii) T is bounded

[6, $p.62 - 63$]

Proof. Suppose that (i) is true, then T is continuous at 0.

$\exists \delta > 0$ s.t. $\|v\| < \delta \implies \|Tv\| < 1$

We notice that $\forall v \in V$ s.t. $\|v\| \leq 1$ then we have $\|\frac{\delta v}{2}\| < \delta \implies \|T(\frac{\delta v}{2})\| < 1$. Then $\|Tv\| < \frac{2}{\delta}$, and therefore (ii) is true.

Next, suppose that (ii) is true, and let

$$S = sup\{\|Tv\| : \|v\| \leq 1, v \in V\}$$

Given any $v_0, v_1 \in V$ where $v_0 \neq v_1$, let $u = \frac{v_0 - v_1}{\|v_0 - v_1\|}$, so $\|u\| = 1$ and

$$\|T(\frac{v_0 - v_1}{\|v_0 - v_1\|})\| \leq S$$

Therefore, $\|T(v_0) - T(v_1)\| \leq S\|v_0 - v_1\|$ implies that T is a continuous operator □

Notation: $\mathcal{L}(V, W)$ is the space of all continuous linear operators from V to W.

Theorem 5.2. *Let V, W be Hilbert spaces and T be a linear operator in $\mathcal{L}(V, W)$. Then $\exists T^* \in \mathcal{L}(V, W)$ s.t.*

$$\langle Tv, w \rangle = \langle v, T^* w \rangle \tag{2}$$

$\forall v \in V$ and $w \in W$

Proof. Since $v \mapsto \langle Tv, w \rangle$ $\forall w \in W$ is a continuous linear mapping on V. Then $\exists$ a unique $x \in V$ s.t $\langle Tv, w \rangle = \langle v, x \rangle$ $\forall v \in V$ by Theorem 4.7.

We set $T^* w$ to be x. So the equation (2) is true where T^* is a mapping from W to V. We claim that T^* is a linear mapping, and this is because $\forall w, x \in W$ and $\alpha, \beta \in \mathbb{C}$: $\forall v \in V$ we have

$$\begin{aligned}
\langle v, T^*(\alpha w + \beta x) \rangle &= \langle Tv, \alpha w + \beta x \rangle \\
&= \overline{\alpha} \langle Tv, w \rangle + \overline{\beta} \langle Tv, x \rangle \\
&= \overline{\alpha} \langle v, T^* w \rangle + \overline{\beta} \langle v, T^* x \rangle \\
&= \langle v, \alpha T^* w + \beta T^* x \rangle
\end{aligned}$$

By the property of inner products, we have

$$T^*(\alpha w + \beta x) = \alpha T^* w + \beta T^* x$$

which proves the linearity of T^*.

Now we need to show the boundedness of T^*. $\forall w \in W$ we have

$$\begin{aligned}
\|T^* w\|^2 &= \langle T^* w, T^* w \rangle \\
&= \langle TT^* w, w \rangle \\
&\leq \|TT^* w\| \|w\|
\end{aligned}$$

and $\|TT^* w\| \leq \|T\| \|T^* w\| \implies \|T^* w\|^2 \leq \|T\| \|T^* w\| \|w\|$ If $\|T^* w\| > 0$ we then have $\|T^* w\| \leq \|T\| \|w\|$. If $\|T^* w\| = 0$ then equality holds.

Therefore we can conclude that T^* is bounded. □

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