Discussion of Fat Tail Models in Volatility Estimation

BU MSMFT, Fall 2022

Preface

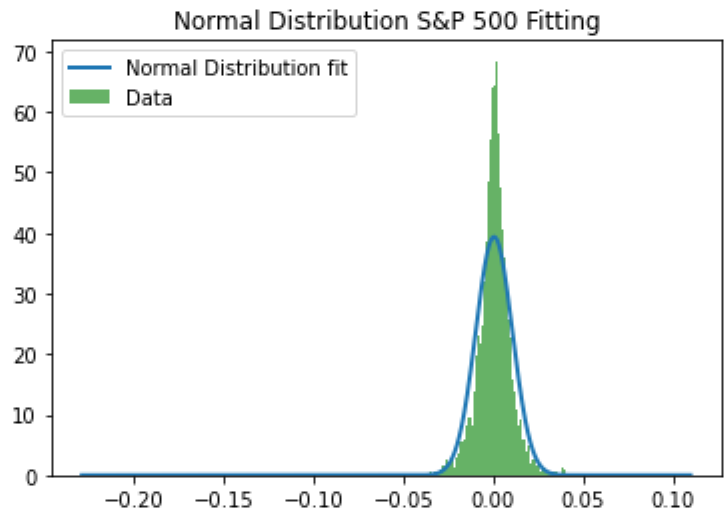
For the past few decades, some financial crises have driven the whole world economy subsequently stagnated and declined sharply. It caused many institutions to have cash-flow problems, leading them to bankruptcy. One of the significant reasons is that the existing models they use are unable to help them capture extreme events in the real world with agility and predict future volatility. So, we consider other models and distributions that might do a better job than the normal distribution model which is frequently utilized in the finance industry.

Normality Check

To confirm our conjecture, we first visualize the real dataset, check whether the histogram graph of real market data fits the normal distribution probability density function. And then we use the formal statistical test, the Shapiro-Wilk test.

First, we collect data from 1957 to 2022 from Yahoo Finance and clean the data by eliminating some lost data points. For the histogram, we convert simple returns to log returns. Then we create a histogram to visualize returns, and plot normal probability density function with data mean and data standard deviation.

From the following graph, we can see S&P 500 daily returns do not perform as normal distribution. First, as we can see, many extreme values of the dataset can not be covered by normal probability density function line. Also, the tails of both sides are fatter, compared to the bell-shaped symmetric density function of normal distribution. More than that, since the 2008 financial crisis generates many negative outlier data points, the graph performs heavier on the left tail than the right tail.



In order to be more intuitively reflective, we also perform the Shaprio-Wallis test. The output of the test is that the test statistic is 0.888 and the corresponding p-value is 0.000. Since the p-value is less than the confidence level, 0.05, we reject the null hypothesis of the Shapiro-Wilk test that data is normality distributed. That means, we can conclude that the daily return of S&P 500 from 1957 to 2022 is not normally distributed, and it shows fat-tails.

Extreme Scenario Modeling

Through data visualization, we have concluded some key features of stock returns. One feature is that extreme values occur more frequently than they are supposed to according to a normal distribution. We define this feature as fat tails. The other important feature is the appearance of a tall peak. It indicates that the vast majority of stock returns show around the mean value.

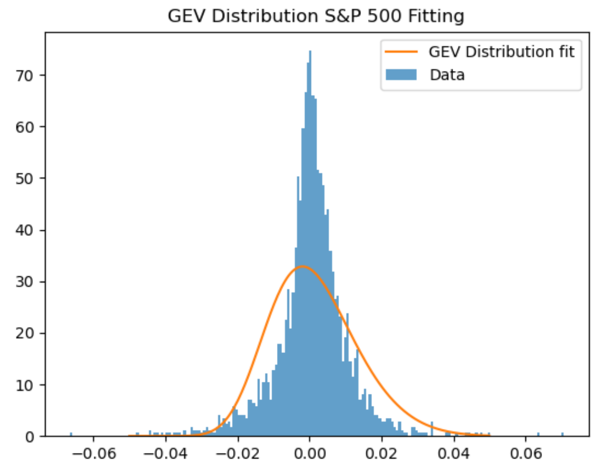
From previous work, we have confirmed the existence of fat tails in real financial data. In this way, the normal distribution is not accurate enough to simulate the stock returns. So we are supposed to find other kinds of distributions to do the simulation. Those distributions have to take fat tails into consideration, meanwhile, they should match other features of stock returns like the tall peak. As a result, we think of extreme scenario modeling.

We choose the Generalized Extreme Value distribution as the specific modeling method, which is short for GEV distribution. The main reason why we choose this distribution is the extreme value theorem. Under this theorem, GEV distribution can better simulate the maximum of independent and identical random variable sequences, so we might gain more accurate results using GEV distribution when extreme scenarios show more frequently.

As for simulation, we take several trials in order to get the most fitting result. We first tried a long period of time for 50 years and obtained an insignificant result. To solve this problem, we adjusted the time slot year by year. Through this process, we found the most suitable time period, from 03/04/2009 to 03/04/2019.

The adjusting process shows that moving the data from 2008 out of the data set results in a better simulation effect. One possible reason is that the financial crisis in 2008 hit the market. Under these circumstances, we cannot find a reasonable distribution to fit the real world.

The graph below shows the comparison of real data and the related GEV distribution from 2009 to 2019. The fitting result is neither better nor worse.



Quantitatively, we estimate the 3 key parameters in GEV distribution. The result is as follows.

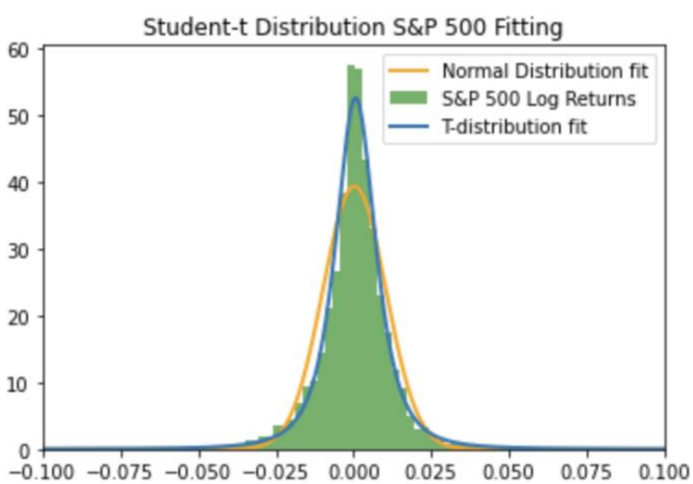
Parameter / Estimate	Shape (ξ)	Location (μ)	Scale (σ)
Estimated Value	0.14869	-0.00368	0.01133

Whole Data Modeling

We further examine the distribution to fit the data after acquiring the essential parameters of the data. We moved to another parametric approach based on an assumption that represents a model for the entire distribution, the whole data modeling. We choose the distribution which is frequently used to explain data sets with broader tails than usual, Student's t distribution as a model for asset returns.

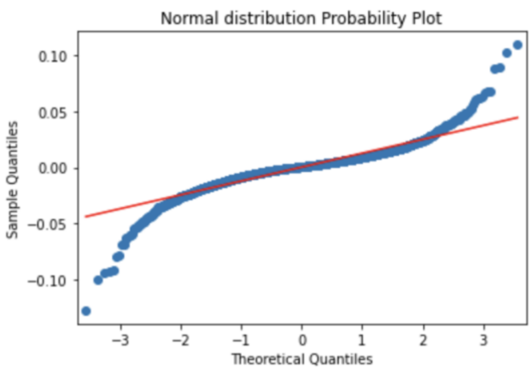
Unlike the normal distribution, t-distribution is more peaked around the center and has fatter tails. T-distribution takes three parameters as input. Besides mean and standard deviation, the degree of freedom can be interpreted as a measure of the extent of departure from the normal distribution. By varying the degree of freedom, one can model the tails of the distribution given different properties of given assets.

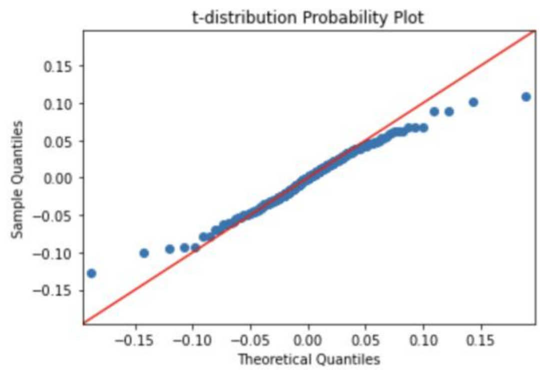
The following graph showed how the t-distribution fit the log return for the 22 year data set. From the graph of the density of the model, we can see that the S&P 500 log returns fit the t distribution plot much better than the normal distribution.



Parameter / Estimate	DoF	Location (μ)	Scale (σ)
Estimated Value	2.50519	0.00068	0.00689

From the Q-Q plot, we can see that t distribution is a much better model for fitting the log returns of a financial asset. This is because the t-distribution has heavier tails so it is more likely to fit the extreme values of the data sets. In contrast to the normal distribution, which is overly optimistic in the long scope, the t distribution better captured the risk cross time.





Volatility Forecast Effect of t-distribution

From the previous discussion, t-distribution has an outstanding performance when handling extreme events, which means that the forecasts on extreme volatility with the application of t-distribution will outweigh that of normal distribution.

1. Approach and Preparation

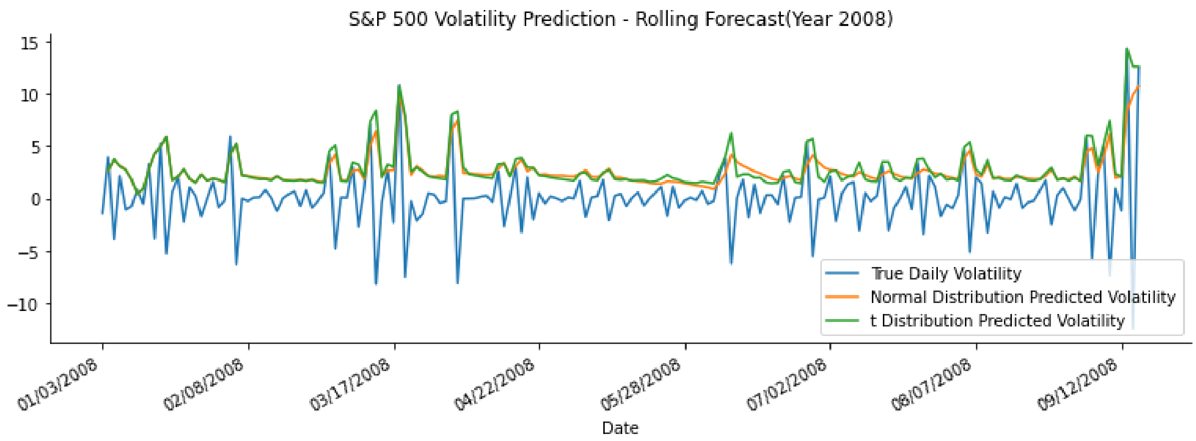
The generalized AutoRegressive Conditional Heteroskedasticity (GARCH) model is the primary method to build a volatility forecast model with t-distribution. The GARCH model is an effective time series model to make predictions in terms of Value-at-Risk(VAR), maximum expected loss, etc.

Volatility is believed to be autocorrelated, which is the requirement of the GARCH model, but no constant or white noise over time. To confirm the assumption, a stationary test and white noise test were performed. Then, to make regression on historical volatility data, the order of AR and MA are identified according to the ACF and PCF Plots.

2. Model Building

To better visualize and evaluate the forecasting result of t-distribution, we apply both t-distribution, which fits real financial data better, and normal distribution, which is considered as a prerequisite of financial model, on the GARCH model. The data is from January 2008 to October 2008, a representative time period since the financial market showed high volatility.

Rolling window approach is also applied to build the GARCH model. Rather than robustly divide the historical data set into two parts: train set and test set, rolling windows will be more accurate. Our team uses existing historical data for model fitting, forecast 1-day ahead, and do these steps repeatedly as time goes forward.



	accuracy	dist	std
normal-distribution	**32%**	**40%**	**1.56**
t-distribution	**39%**	**60%**	**2.08**

In terms of accuracy in forecasting volatility trends, the t-distribution model is 22% higher than the normal-distribution model. For 60% of the time, the distance between the t-distribution model forecast result and the true value is narrower. The standard deviation of the t-distribution model is 2.08, which is 33% higher than the normal-distribution model.

Application of t-distribution in Terms of Tail Risk Management

Noted that investors hedge against tail risk so that they can enhance returns over the long term. There are many options that investors can use to hedge against tail risk.

We analyze a number of tail risk hedging strategies, grouping them into four categories: long volatility, low volatility equity investing (stock selection), trend following, and equity exposure management. Strategies in the long volatility category are VIX and variance. Long volatility strategies are a natural hedge for equity, as stock market declines are often accompanied by steep volatility. A low-volatility equity investment strategy is a portfolio of negative beta stocks that benefits from yield anomalies or stock pick ability. Trend following strategies have a payoff profile of a lookback straddle that deliver higher performance during pronounced uptrends and downtrends in the market. The fourth and final category, risk management, includes strategies to limit equity risk.

Volatility forecast plays an important role in these four strategies, which means that t-distribution can be vital to ultimate those strategies.

To be more specific, there is a famous risk hedging model that involves volatility calculation is risk parity model. Risk parity model evaluates the impact on portfolio performance of the different asset allocations by calculating the risk contribution of each asset to the whole portfolio. Under this circumstance, t-distribution can be used as an effective tool to forecast the risk of each asset, and it can help to achieve the target tail risk reduction.

A more accurate forecast on volatility can significantly improve portfolio performance during tail risk events. T-distribution can be a breaking point to address the problem of protecting against extreme events. And a new direction that can help improve long-term performance, even for diversified investors looking to derive a premium from risky assets.

Conclusion

After the 2008 financial crisis, it seems that the existence of fat tails in the literature abounds in models that can account for fat tails, but asset returns should take into account all possible stylized facts, related to statistics. The best approach is to choose an extended model that includes the normal distribution as a special case and then test its performance on the real data. The more theoretical question of whether the chosen model is the best one among other models when it comes to better performance in common financial calculations and fits the daily percentage returns of the S&P 500. We approached our goal using Whole Data Modelling of the t-distribution, as well as extreme scenario models of the Generalized Extreme Value distribution (GEV). We then used the GARCH model to build the volatility forecast model with t-distribution. After visualizing and evaluating the forecasting result of t-distribution, we applied normal distribution on the forecasting model too to make a comparison between these two distributions. The result led our project's proposal student t distribution is better for financial calculations so that the investment choices take the expectation of more severe events into account.

Reference

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