Tutte's Characterization of TUMs
A demonstration by A M Gerards

Cojocaru Andrei, Pavel Nicolae

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Tutte's Theorem

Definition

Let A be a 0, 1-matrix. Then the following are equivalent:

1. A has a totally unimodular signing
2. A cannot be transformed to

$$M(F_7) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

by applying (repeatedly) the following operations:

$$\begin{aligned} &\textit{deleting rows or columns} \\ &\textit{permuting rows or columns} \\ &\textit{taking the transposed matrix} \\ &\textit{pivoting over } GF(2) \end{aligned} \tag{1}$$

Tutte's Theorem, more definitions

Definition

A matrix is *totally unimodular* if all of its subdeterminants are 0, 1 or -1. (In particular, all entries of a totally unimodular matrix are 0, 1 or -1).

Definition

Signing a 0, 1-matrix means replacing some of the 1s by -1s.

Definition

Pivoting a matrix A on an entry $\epsilon = \pm 1$ means replacing

$$A = \begin{pmatrix} \epsilon & y^T \\ x & D \end{pmatrix} \text{ by } B = \begin{pmatrix} -\epsilon & y^T \\ x & D - \epsilon x y^T \end{pmatrix} \tag{2}$$

Preliminaries

Lemma 1

Let G be a connected bipartite graph (with no parallel edges). If deleting any two nodes in the same partition yields a disconnected graph, then G is a path or a circuit.

Lemma 2

Let A be a square 0, 1, -1 matrix. If G(A) is a circuit, then A is totally unimodular if and only if the number of -1s in A is congruent to n modulo 2.

Lemma 3

Let M_1 and M_2 be two totally unimodular matrices. If $M_1 \equiv M_2(mod 2)$, then M_1 can be obtained from M_2 by multiplying some rows and columns with -1.
$(M_1 \equiv M_2(mod 2) : \exists P$ so that $P^T M_1 P = M_2)$.

Pivoting properties

Some useful properties of the pivoting operation (2):

$$\begin{aligned} &\text{(i) Pivoting B on -}\epsilon\text{ yields A} \\ &\text{(ii) If A is square, then det(A)} = \pm\, det(D - \epsilon x y^T) \\ &\text{(iii) If A is totally unimodular, then B is totally unimodular} \\ &\text{(iv) If G(A) is connected, then G(B) is connected} \\ &\text{(G(B) disconnected plus (i)} \implies \text{G(A) disconnected)} \end{aligned} \tag{3}$$

Proof of Tutte's theorem, $\Longrightarrow$

Proof.

The first three operations of (1) obviously don't change the matrix having an unimodular signing or not.

Property (3) (iii) of pivoting means pivoting does not change the matrix having an unimodular signing either.

Since $M(F_7)$ has no totally unimodular signing, (i) $\implies$ (ii). $\square$

Proof of Tutte's theorem, $\Longleftarrow$

Let A be a 0, 1 matrix, satisfying (ii) but with no totally unimodular signing. We may assume that each proper sumbatrix of A has a totally unimodular signing, so the bipartite graph G(A) is connected.

G(A) is not a path or circuit, or A would have a totally unimodular signing. Hence by Lemma 1, A or A^T is equal (up to permutation of columns) to [x|y|N], where x and y are two column vectors and where G(N) is connected. (x and y correspond to the nodes we eliminate and still get a connected graph).

By the initial assumption, both [x|N] and [y|N] have a totally unimodular signing, and by *Lemma 3* they can be chosen so that N is signed in the same way in both cases.

Hence A or A^T has a signing A'=[x'|y'|N']

$$\begin{aligned} &\text{(i) G(N') is connected} \\ &\text{(ii) Both [x'|N'] and [y'|N'] are totally unimodular} \end{aligned} \tag{4}$$

Proof of Tutte's theorem, $\Longleftarrow$ (2)

Claim

We can assume that the matrix [x'|y'] has a submatrix of the form

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Proof Sketch

Pivoting A' on entries in N' does not change property (4) so pivot in such a way that the smallest submatrix with its determinant not 0, 1 or -1 is as small as possible. This submatrix has to be of the form of the claim, and a submatrix of [x'|y'].
The two rows corresponding to the submatrix have a path in the corresponding graph, thus:

Proof of Tutte's theorem, $\Longleftarrow$ (3)

A has a submatrix of the form:

$$\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & \dots & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & \dots & 0 & 1 \\
 & & 1 & \underline{1} & & & & \\
 & & & 1 & \underline{1} & & 0 & \\
* & * & & & 1 & \ddots & & \\
 & & & 0 & & \ddots & \underline{1} & \\
 & & & & & & 1 & 1
\end{pmatrix}$$

Proof of Tutte's theorem, $\Longleftarrow$ (4)

By pivoting on the underlined elements, deleting the rows and columns containing them, multiplying some rows/columns by -1 (and exchanging x' and y' if necessary), we obtain a submatrix of the form:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ a & b & 1 & 1 \end{pmatrix}$$

It's still the case that deleting columns x' or y', the remaining matrix is still totally unimodular, so a $=$ 1 and b $=$ 0. In this case, A can be transformed to $M(F_7)$, which contradicts our assumption.

Original Article

A Short Proof of Tutte's Characterization of Totally Unimodular Matrices
by A. M. H. Gerards
Tilburg University, The Netherlands