EASTERN MEDITERRANEAN UNIVERSITY

FACULTY OF ARTS AND SCIENCES

Name Surname :......................... **Student No** :...................**Sign** :...........

Math 373 Numerical Analysis Final Exam

Q1	Q2	Q3	Q4	TOT

Spring Semester 2020-2021 24 June 2021 Duration: 80 minutes

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1. Derive the numerical differentiation formula using Taylor Series expansion

$$f'(x) = \frac{3f_0 - 4f_{-1} + f_{-2}}{2h}$$

by obtaining the error and give the optimum stepsize h that minimizes the error

$$-4 / f(x-h) = f(x) - h f'(x) + \frac{h^2 f''(x)}{2!} - \frac{h^3 f'''(x)}{3!} + \cdots$$

$$f(x-2h) = f(x) - 2h f'(x) + \frac{4h^2 f''(x)}{2!} - \frac{8h^3 f'''(x)}{3!} + \cdots$$

$+$

$$-4f_{-1} + f_{-2} = -3f(x) + 2h f'(x) - \frac{5h^3 f'''(x)}{3!}$$

$$f'(x) = \frac{3f_0 - 4f_{-1} + f_{-2}}{2h} + \frac{4h^2 f'''(x)}{12} \quad 15$$

$$E(f,h) = E_{round}(f,h) + E_{trunc}(f,h) = \frac{3e - 4e + e}{2h} + \frac{h^2 M}{3}$$

$$|E(f,h)| \le \frac{8\varepsilon}{2h} + \frac{h^2 M_3}{3} = \frac{4\varepsilon}{h} + \frac{h^2 M_3}{3} \quad 5$$

$$\phi'(h) = \frac{-4\varepsilon}{h^2} + \frac{2h M_3}{3} = 0 \Rightarrow h = \left(\frac{6\varepsilon}{M_3}\right)^{1/3} \quad 5$$

2. Solve the initial value problem

$$y' = ty^{-3} - \frac{yt^{-1}}{2}$$

using Heun's method over $[1, 1.4]$ with h=0.1 and y(1)=2 to obtain approximation for the solution of the I.V.P at y(1.2)

$P_0 = y_0 + hf(t_0, y_0) = 2 + 0.1 f(1,2) \approx 1.912$ 6

$y_1 = y_0 + \frac{h}{2}\left[f(t_0, y_0) + f(t_1, P_0)\right] = 2 + \frac{0.1}{2}\left[f(1,2) + f(1.1, 1.912)\right] = 1.9206$ 6

$P_1 = y_1 + h\, f(t_1, y_1) = 1.9206 + 0.1 f(1.1, 1.9206) = 1.8494$ 6

$y_2 = y_1 + \frac{h}{2}\left[f(t_1, y_1) + f(t_2, P_1)\right] = 1.9206 + \frac{0.1}{2}\left[f(1.1, 1.9206) + f(1.2, 1.8494)\right]$

$y(1.2) \approx y_2 = 1.8559$ 7

Question3. (25 pts.)

a) Find the curve fit $y = Cxe^{-Dx}$, using the data linearization for the given points.

x_k	y_k
0.2	0.72
0.3	1.38
0.4	2.24
0.5	4.26

b) Determine RMS Error $E_2(f)$.

a) $y = Cxe^{-Dx}$

$\frac{y}{x} = Ce^{-Dx}$ 5

$\ln\left(\frac{y}{x}\right) = \ln C - Dx$

$y^* = Ax + B$ where $y^* = \ln\left(\frac{y}{x}\right)$, $A = -D$, $B = \ln C$

x_k	y_k	y_k/x_k	$y_k^* = \ln\left(\frac{y_k}{x_k}\right)$	x_k^2	$x_k y_k^*$	$f(x_k)$	$e_k = \lvert f(x_k) - y_k \rvert$	e_k^2
0.2	0.72	3.6	1.2809	0.04	0.25618	0.6987	0.0213	0.00045
0.3	1.38	4.6	1.5261	0.09	0.45783	1.3841	0.0041	0.000017
0.4	2.24	5.6	1.7228	0.16	0.68912	2.4371	0.01971	0.039
0.5	4.26	8.52	2.1424	0.25	1.0712	4.0232	0.2368	0.056
1.4	8.6		6.6722	0.54	2.47433			0.0955
			1	1	1	1	1	1

$$\left(\sum x_k^2\right) A + \left(\sum x_k\right) B = \sum x_k y_k^*$$

$$\left(\sum x_k\right) A + NB = \sum y_k^*$$

$-4 /\quad 0.54A + 1.4B = 2.47433$

$1.4 /\quad 1.4A + 4B = 6.6722$

$-0.2A = -0.55624 \Rightarrow A = 2.7812 \quad \boxed{D = -2.7812}$ 2

$0.54A + 1.4B = 2.47433$

$0.54(2.7812) + 1.4B = 2.47433$

$$\boxed{B = 0.69463} \quad (2)$$

$B = \ln C \Rightarrow C = e^{B} = e^{0.69463} = 2.002968$ $\boxed{C = 2.002968}$ (2)

Curve Fit : $y = C \times e^{-Dx}$

$$y = 2.002968 \times e^{2.7812x} \quad (3)$$

b) RMS Error

$$E_2(f) = \left[\frac{1}{N} \Sigma |e_k|^2\right]^{1/2}$$

$$= \left[\frac{1}{4}(0.0955)\right]^{1/2}$$

$$= 0.1545 \quad (5)$$

Question4. (25 pts.)

a) Determine the number M and the interval width h so that the Composite Simpson Rule for 2M subintervals can be used to compute the given integral with an accuracy of 3×10^{-6}.

$$\int_2^3 \frac{1}{5-x}\,dx$$

b) Evaluate $\int_2^3 \frac{1}{5-x}\,dx$ using Composite Simpson Rule with an error bound by 3×10^{-6}.

a) $[a,b] = [2,3]$, $f(x) = \frac{1}{5-x} = (5-x)^{-1}$

$f'(x) = (5-x)^{-2}$

$f''(x) = 2(5-x)^{-3}$

$f'''(x) = 6(5-x)^{-4}$

$f^{(4)}(x) = 24(5-x)^{-5} = \frac{24}{(5-x)^5}$ 3

$\max\limits_{2 \le x \le 3} \{ f^{(4)}(c) \} = \max \left\{ \frac{24}{(5-2)^5}, \frac{24}{(5-3)^5} \right\} = \frac{24}{32} = 0.75$ 3

$|E_s(f,h)| \le \frac{|b-a|\,|f^{(4)}(c)|\,h^4}{180} \le 3 \times 10^{-6}$ 2

$\Rightarrow \frac{0.75 h^4}{180} \le 3 \times 10^{-6}$

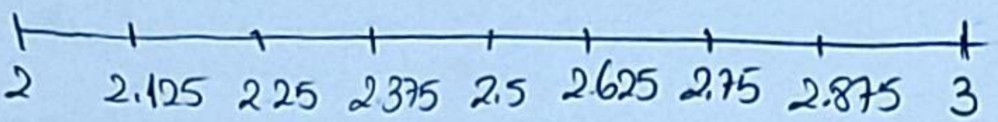
$h^4 \le \frac{(3 \times 10^{-6}) \times 180}{0.0988} = 7.2 \times 10^{-4}$

$\boxed{h \approx 0.1638}$ 4

$$h = \frac{b-a}{2M} \Rightarrow M = \frac{b-a}{2h} = \frac{3-2}{2(0.1638)} = 3.0525$$

M should be even, $\boxed{M=4}$ 4

b) when $M=4 \Rightarrow h = \frac{b-a}{2M} = \frac{1}{8} = 0.125$, $f(x) = \frac{1}{5-x}$



$$S(f,h) = \frac{h}{3}\left[f(2)+f(3)\right] + \frac{2h}{3}\left[f(2.25)+f(2.5)+f(2.75)\right]$$
$$+ \frac{4h}{3}\left[f(2.125)+f(2.375)+f(2.625)+f(2.875)\right]$$ 3

$$= \frac{0.125}{3}\left[0.333+0.5\right] + \frac{0.25}{3}\left[0.364+0.4+0.444\right]$$
$$+ \frac{0.5}{3}\left[0.348+0.381+0.421+0.471\right]$$

$$= 0.40554$$ 4

Exact Value:

$$\int_2^3 \frac{1}{5-x}\,dx = -\ln(5-x)\Big|_2^3 = \ln 3 - \ln 2 = 0.4055$$