# NUMERICAL DIFFERENTIAL EQUATIONS

# SOLUTION OF DIFFERENTIAL EQUATIONS

The central concern of this chapter is approximating the solution to an initial value problem for a differential equation.

## INITIAL VALUE PROBLEM

**Definition:** A solution to an initial value problem (I.V.P)

$$y' = f(t, y) \quad with \quad y(t_0) = y_0 \qquad \text{.....................................}(1)$$

on an interval $[t_0, b]$ is a differentiable function $y = y(t)$ such that

$$y(t_0) = y_0 \qquad and \qquad y'(t) = f(t, y(t)) \qquad for \quad all \quad t \in |t_0, b| \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Notice that the solution curve $y = y(t)$ must pass through the initial point $(t_0, y_0)$

# EULER'S METHOD

Euler's method is one of the simplest and also earliest technique developed for the solution of the ordinary differential equations.

## Derivation of Method

Euler's method can be derived in several ways, the derivation is based on Taylor's series expansion is considered here. Assume that $y(t)$ , $y'(t)$ $and$ $y''(t)$ are continuous and Taylor's theorem to expand $y(t)$ about $t = t_0$. For each value of $t$ , there exist a value $c_1$ that lies between $t_0$ and $t$ so that

$$y(t) = y(t_0) + (t - t_0)y'(t_0) + \frac{(t - t_0)^2}{2!}y''(c_1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

When , $y'(t_0) = f(t_0, y(t_0))$ $and$ $h = t_1 - t_0$ are substituted in (3) , the result is an expression for $y(t_1)$.

$$y(t_1) = y(t_0) + (t_1 - t_0)y'(t_0) + \frac{(t_1 - t_0)^2}{2!}y''(c_1)$$

$$= y(t_0) + hf(t_0, y(t_0)) + \frac{h^2}{2!}y''(c_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

If the step size h is chosen small enough , then we neglect the second order term (involving $h^2$ ) and get

$$y_1 = y_0 + hf(t_0, y_0) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

The process is repeated and generates a sequence of points that approximate the solution curve $y = y(t)$. The general step for Euler method is

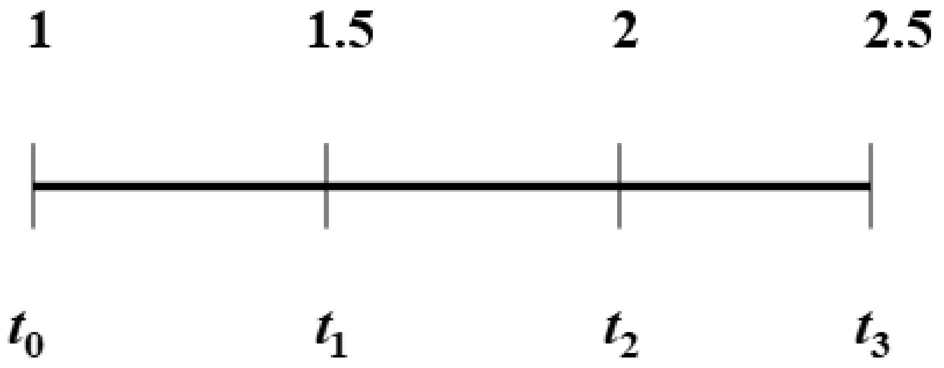
$$\boxed{y_{k+1} = y_k + hf(t_k, y_k)} \quad for\ \ k = 0,1,2,\ldots,M-1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$where \quad t_{k+1} = t_k + h$$

**EXAMPLE:** Use Euler's method to solve approximately the initial value problem

$$y' = 1 + \frac{y}{t} \quad over \quad [1,2.5] \quad with \quad y(1) = 1$$

Using the step size $h = 0.5$




$y_{k+1} = y_k + hf(t_k, y_k)$ and $y' = f(t,y) = 1 + \frac{y}{t}$

$$y_1 = y_0 + hf(t_0, y_0) = 1 + 0.5\left[1 + \frac{1}{1}\right] = 2$$

$$y_1 = 2 = y(1.5)$$

$$y_2 = y_1 + hf(t_1, y_1) = 2 + 0.5\left[1 + \frac{2}{1.5}\right] = 3.1666667$$

$$y_2 = 3.1666667 = y(2)$$

$$y_3 = y_2 + hf(t_2, y_2) = 3.1666667 + 0.5\left[1 + \frac{3.1666667}{2}\right] = 4.45833333$$

$$y_3 = 4.45833333 = y(2.5)$$

**Exact solution :** $y(t) = t(1 + \ln t)$

| t | Approx. Solution | Exact solution | $\varepsilon = \left\| y_{ex} - y_{app} \right\|$ |
|---|---|---|---|
| 1.0 | 1.00000000 | 1.00000000 | |
| 1.5 | 2.00000000 | 2.10819766 | 0.108198 |
| 2.0 | 3.16666667 | 3.38629436 | 0.219628 |
| 2.5 | 4.45833333 | 4.79072683 | 0.332393 |

## EXERCISE

Use Euler's method to solve approximately the I.V.P

$$y' = \frac{t}{y} \quad , \quad 0 \leq t \leq 3 \quad , \quad y(0) = 1$$

Using the step size h=0.5

Exact Solution: $y(t) = \sqrt{t^2 + 1}$

and find the error for each time step.

# HEUN'S METHOD

Consider I.V.P

$$y' = f(t, y) \quad over \quad [a, b] \quad with \quad y(t_0) = y_0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Use fundamental theorem of calculus to obtain the solution point $(t_1, y_1)$, that is integrate $y'(t)$ over

$$\int_{t_0}^{t_1} f(t, y(t))\, dt = \int_{t_0}^{t_1} y'(t)\, dt = y(t_1) - y(t_0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (2)

$$y(t_1) = y(t_0) + \int_{t_0}^{t_1} f(t, y(t))\, dt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Use Trapezoidal rule for (3), then

$$y(t_1) = y(t_0) + \frac{h}{2}[f(t_0, y(t_0) + f(t_1, y(t_1)))] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$where \quad y(t_1) = y_0 + hf(t_0, y_0) \quad then$$

$$y(t_1) = y(t_0) + \frac{h}{2}[f(t_0, y(t_0) + f(t_1, y_0 + hf(t_0, y_0)))] \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

which is called Heun's Method.

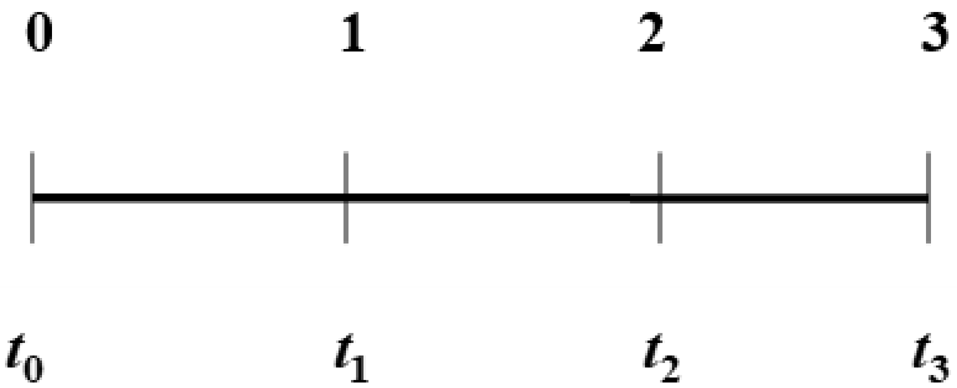
The general step for Heun's Method is

$$\boxed{\begin{aligned} p_{k+1} &= y_k + hf(t_k, y_k) \\ &\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6) \\ y_{k+1} &= y_k + \frac{h}{2}[f(t_k, y_k) + f(t_{k+1}, p_{k+1})] \end{aligned}}$$

**Example:** Use Heun's method to solve I.V.P

$$y' = \frac{t - y}{2} \quad on \quad [0,3] \quad with \quad y(0) = 1$$

$$where \quad h = 1$$




$$y_0 = 1 \quad and \quad f(t, y) = \frac{t - y}{2}$$

- $$p_1 = y_0 + hf(t_0, y_0) = 1 + 1\left[\frac{0-1}{2}\right] = 0.5$$
- $$y_1 = y_0 + \frac{h}{2}[f(t_0, y_0) + f(t_1, p_1)] = 1 + \frac{1}{2}\left[-\frac{1}{2} + \left(\frac{1-0.5}{2}\right)\right] = 0.875$$

$$\boxed{y_1 = 0.875}$$

- $$p_2 = y_1 + hf(t_1, y_1) = 0.875 + 1\left[\frac{1-0.875}{2}\right] = 0.9375$$
- $$y_2 = y_1 + \frac{h}{2}[f(t_1, y_1) + f(t_2, p_2)] = 0.875 + \frac{1}{2}\left[\frac{1-0.875}{2} + \left(\frac{2-0.0.9375}{2}\right)\right] = 1.171875$$

$$\boxed{y_2 = 1.171875}$$

- $$p_3 = y_2 + hf(t_2, y_2) = 1.171875 + 1\left[\frac{2-1.171875}{2}\right] = 1.5859375$$
- $$y_3 = y_2 + \frac{h}{2}[f(t_2, y_2) + f(t_3, p_3)] = 1.171875 + \frac{1}{2}\left[\frac{2-1.171875}{2} + \left(\frac{3-1.5859375}{2}\right)\right] = 1.732422$$

$$\boxed{y_3 = 1.732422}$$

$$y' = \frac{t-y}{2}$$

**Exact:** $y(t) = 3e^{\frac{-t}{2}} - 2 + t$

$$y(3) = 3e^{\frac{-3}{2}} - 2 + 3 = 1.66939048$$

**Error :** $\varepsilon = |y_{ex} - y_{app}| = |1.66939048 - 1.732422| = 0.063$

**Exercise1 :** Use Heun's method to find approximations to the initial value problem

$$y' = y - t \ , \ \ y(0) = 1.5 \quad on \ \ [0,0.5] \quad where \ \ h = 0.25$$

**Exercise2 :** Given the following initial value problem $y' = 2ty^2 \ , \ \ y(0) = 1$

to approximate y(0.3), use Euler formula and the following given formula

$$y(t+h) = y(t) + hf\left(t + \frac{h}{2}, y(t) + \frac{h}{2}f(t,y)\right)$$

with h=0.1 and compare the result from above two formulas with exact solution given by $y = \dfrac{1}{1-t^2}$

# RUNGE KUTTA METHODS

## FOURTH ORDER RUNGE KUTTA METHOD (RK4)

The fourth order Runge Kutta method (RK4) simulates the accuracy of the Taylor series of order N=4. The method is based on computing $y_{k+1}$ as follows

$$y_{k+1} = y_k + w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where, $k_1$ , $k_2$ , $k_3$ $and$ $k_4$ have the form

$$\begin{gathered} k_1 = hf(t_k, y_k) \\ k_2 = hf(t_k + a_1 h, y_k + b_1 k_1) \\ k_3 = hf(t_k + a_2 h, y_k + b_2 k_1 + b_3 k_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2) \\ k_4 = hf(t_k + a_3 h, y_k + b_4 k_1 + b_5 k_2 + b_6 k_3) \end{gathered}$$

where,

$$a_1 = \frac{1}{2} \quad , a_2 = \frac{1}{2} \quad , a_3 = 1$$

$$b_1 = \frac{1}{2} \quad , b_2 = 0 \quad , b_3 = \frac{1}{2} \quad , b_4 = 0 \quad , b_5 = 0 \quad , b_6 = 1 \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

$$w_1 = \frac{1}{6} \quad , w_2 = \frac{1}{3} \quad , w_3 = \frac{1}{3} \quad and \quad w_4 = \frac{1}{6}$$

then,

$$y_{k+1} = y_k + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4$$

$$\boxed{y_{k+1} = y_k + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

and

$$k_1 = hf(t_k, y_k)$$

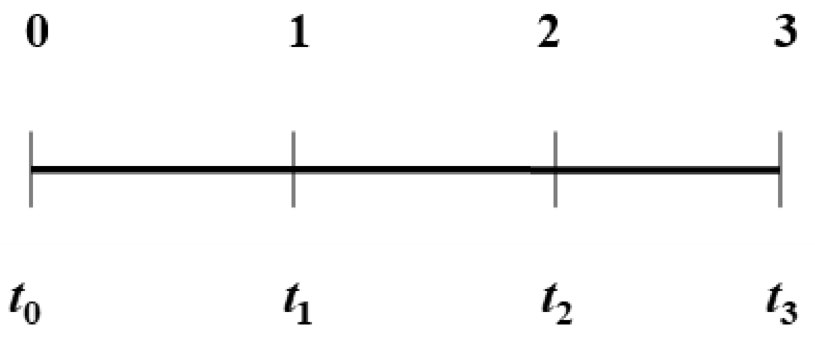
$$k_2 = hf(t_k + \frac{h}{2}, y_k + \frac{k_1}{2})$$

$$k_3 = hf(t_k + \frac{h}{2}, y_k + \frac{k_2}{2}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$k_4 = hf(t_k + h, y_k + k_3)$$

**Example:** Use the RK4 method solve the I.V.P

$$y' = \frac{t-y}{2} \quad on \quad [0,3] \quad with \quad y(0) = 1$$
$$where \quad h = 1$$




$$y_0 = 1 \quad and \quad f(t,y) = \frac{t-y}{2}$$

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(t_0, y_0) = 1f(0,1) = \frac{0-1}{2} = -0.5$$

$$k_2 = hf(t_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 1f(0 + 0.5, 1 + \left(\frac{-0.5}{2}\right)) = f(0.5, 0.75) = \frac{0.5 - 0.75}{2} = -0.125$$

$$k_3 = hf(t_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 1f(0 + 0.5, 1 + \left(\frac{-0.125}{2}\right)) = f(0.5, 0.9375) = \frac{0.5 - 0.9375}{2} = -0.21875$$

$$k_4 = hf(t_0 + h, y_0 + k_3) = 1f(0 + 1, 1 + (-0.21875)) = f(1, 0.78125) = \frac{1 - 0.78125}{2} = 0.109375$$

$$y_1 = 1 + \frac{1}{6}[-0.5 + 2(-0.125) + 2(-0.21875) + 0.109375] = 0.8203125$$

$$\boxed{y_1 = 0.8203125}$$

exercise , find $y_2$ $and$ $y_3$

# SECOND ORDER RUNGE KUTTA METHOD (RK2)

Second order version Runge Kutta method is

$$y_{k+1} = y_k + h(a_1k_1 + a_2k_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$where,$$

$$k_1 = f(t_k, y_k)$$

$$k_2 = f(t_k + p_1h, y_k + q_{11}k_1h) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$where,\; a_1 + a_2 = 1, \quad a_2p_1 = \frac{1}{2}, \quad a_2q_{11} = \frac{1}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

We have three equations and four unknowns , we must assume a value of one of the unknowns to determine the other three. Suppose that we specify a value for $a_2$ . Then equation (3) can be solved simultaneously for

$$a_1 = 1 - a_2, \quad p_1 = q_{11} = \frac{1}{2a_2}$$

We can choose an infinite number of values for $a_2$, we present three of the most commonly used.

**1. HEUN'S METHOD** ($a_2 = \frac{1}{2}$) Single corrector

$$y_{k+1} = y_k + \frac{h}{2}(k_1 + k_2)$$
$$where,$$
$$k_1 = f(t_k, y_k)$$
$$k_2 = f(t_k + h, y_k + k_1 h)$$

**2. THE MIDPOINT METHOD** ($a_2 = 1$)

$a_1 = 0, \quad p_1 = q_{11} = \frac{1}{2}$ then

$$y_{k+1} = y_k + k_2 h$$
$$where,$$
$$k_1 = f(t_k, y_k)$$
$$k_2 = f\left(t_k + \frac{h}{2}, y_k + \frac{1}{2}k_1 h\right)$$

## ROLSTON'S METHOD $(a_2 = \frac{2}{3})$

$$if \quad a_2 = \frac{2}{3} \quad , \quad a_1 = \frac{1}{3} \quad and \quad p_1 = q_{11} = \frac{3}{4}$$

$$y_{k+1} = y_k + h\left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)$$

$$where,$$

$$k_1 = f(t_k, y_k)$$

$$k_2 = f\left(t_k + \frac{3}{4}h, y_k + \frac{3}{4}k_1 h\right)$$

**Example(E.Q) :** Find the value of the solution at the point x=0.4 of

$$y' = 1 + t - y^2 \quad , \quad y(0) = 1$$

defined over the interval [0,1] with h=0.2 using the second order Runge Kutta formula

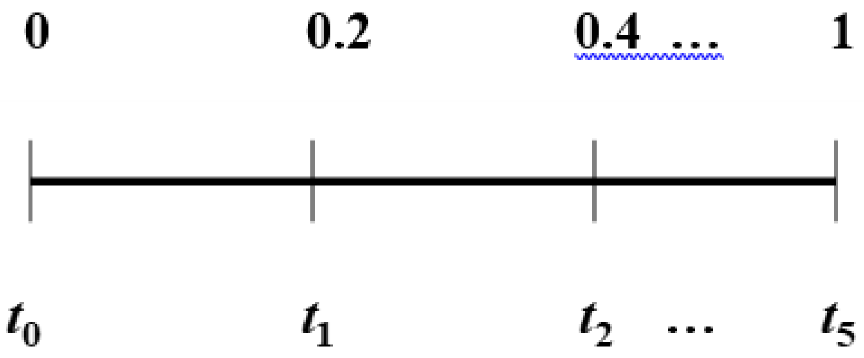
$$y_{k+1} = y_k + h\left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)$$

$$where,$$

$$k_1 = f(t_k, y_k)$$

$$k_2 = f(t_k + \frac{3}{4}h, y_k + \frac{3}{4}k_1 h\Big)$$

**Solution:**



$$y_1 = y_0 + h\left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right) = 1 + 0.2\left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)$$

$$where,$$

$$k_1 = f(t_0, y_0) = f(0,1) = 1 + 0 - 1 = 0$$

$$k_2 = f(t_0 + \frac{3}{4}0.2, y_0 + \frac{3}{4}0(0.2)) = f(0.15,1) = 1 + 0.15 - 1 = 0.15$$

$$y(0.2) = y_1 = 1 + 0.2\left(\frac{1}{3}0 + \frac{2}{3}0.15\right) = 1.02$$

$$\boxed{y(0.2) = y_1 = 1.02}$$

$$y_2 = y_1 + h\left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right) = 1.02 + 0.2\left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)$$

$$where,$$

$$k_1 = f(t_1, y_1) = f(0.2,1.02) = 1 + 0.2 - 1.02^2 = 0.159$$

$$k_2 = f(t_1 + \frac{3}{4}0.2, y_1 + \frac{3}{4}(0.159)(0.2)) = f(0.35,1.043) = 1 + 0.35 - 1.043^2 = 0.26$$

$$y(0.4) = y_2 = 1.02 + 0.2\left(\frac{1}{3}0.159 + \frac{2}{3}0.26\right) = 1.06531$$

$$\boxed{y(0.4) = y_2 =}\ 1.06531$$