THE SOLUTION OF LINEAR SYSTEMS
AX=B

ITERATIVE METHODS

Given

$$\begin{array}{cccccc}
a_{11}x_1 + & a_{12}x_2 + & a_{13}x_3 + & \ldots & + a_{1n}x_n & = b_1 \\
a_{21}x_1 + & a_{22}x_2 + & a_{23}x_3 + & \ldots & + a_{2n}x_n & = b_2 \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
. & . & . & . & . & . \\
a_{n1}x_1 + & a_{n2}x_2 + & a_{n3}x_3 + & \ldots & + a_{nn}x_n & = b_n
\end{array} \qquad (1)$$

Matrix vector equation of (1),

$$Ax = b \qquad (2)$$

From one point iterative method for nonlinear equations, necessary and sufficient condition for convergence

$$|g'(x)| < 1$$

Similar for linear system (1), from (1)

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \ldots - a_{1n}x_n) = g_1(x_2, x_3, \ldots, x_n)$$
$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \ldots - a_{2n}x_n) = g_2(x_1, x_3, \ldots, x_n)$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \tag{3}$$
$$x_n = \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_1 - a_{n3}x_3 - \ldots - a_{nn-1}x_{n-1}) = g_n(x_1, x_2, \ldots, x_{n-1})$$

for convergence

$$\left|\frac{\partial g_1}{\partial x_1}\right| + \left|\frac{\partial g_1}{\partial x_2}\right| + \left|\frac{\partial g_1}{\partial x_3}\right| + \ldots + \left|\frac{\partial g_1}{\partial x_n}\right| < 1$$
$$\left|\frac{\partial g_2}{\partial x_1}\right| + \left|\frac{\partial g_2}{\partial x_2}\right| + \left|\frac{\partial g_2}{\partial x_3}\right| + \ldots + \left|\frac{\partial g_2}{\partial x_n}\right| < 1$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \tag{4}$$
$$\left|\frac{\partial g_n}{\partial x_1}\right| + \left|\frac{\partial g_n}{\partial x_2}\right| + \left|\frac{\partial g_n}{\partial x_3}\right| + \ldots + \left|\frac{\partial g_n}{\partial x_{n-1}}\right| + \left|\frac{\partial g_n}{\partial x_n}\right| < 1$$

from (4)

$$\begin{aligned}
|0| + \left|\frac{-a_{12}}{a_{11}}\right| + \left|\frac{-a_{13}}{a_{11}}\right| + \ldots + \left|\frac{-a_{1n}}{a_{11}}\right| &< 1 \\
\left|\frac{-a_{21}}{a_{22}}\right| + |0| + \left|\frac{-a_{23}}{a_{22}}\right| + \ldots + \left|\frac{-a_{2n}}{a_{22}}\right| &< 1 \\
&\vdots \\
\left|\frac{-a_{n1}}{a_{nn}}\right| + \left|\frac{-a_{n2}}{a_{nn}}\right| + \left|\frac{-a_{n3}}{a_{nn}}\right| + \ldots + \left|\frac{-a_{nn-1}}{a_{nn}}\right| + |0| &< 1
\end{aligned} \quad (5)$$

from (5)

$$\begin{aligned}
|a_{12}| + |a_{13}| + \ldots + |a_{1n}| &< |a_{11}| \\
|a_{21}| + |a_{23}| + \ldots + |a_{2n}| &< |a_{22}| \\
&\vdots \\
|a_{n1}| + |a_{n3}| + \ldots + |a_{nn-1}| &< |a_{\mathrm{nn}}|
\end{aligned} \quad (6)$$

Then , necessary condition for convergence for linear system using iterative method is

$$\sum_{i=1(i\neq j)}^{n} |a_{ij}| \leq |a_{ii}|$$

Diagonally dominant

$$\sum_{i=1(i\neq j)}^{n} |a_{ij}| < |a_{ii}|$$

Strictly Diagonally dominant

JACOBI METHOD

JACOBI METHOD

Given linear system

$$Ax = b \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$where,$$

$$A = L + D + U \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$where,$$

$$L = lower\ triangular\ matrix$$

$$U = upper\ \ triangular\ matrix$$

$$D = diagonal\ \ matrix$$

Substitute (2) in (1)

$$(L + D + U)x = b \quad \Rightarrow (L + U)x + Dx = b$$

$$Dx = b - (L + U)x \quad \Rightarrow x = D^{-1}(b - (L + U)x)$$

Jacobi Method : $\boxed{x_{n+1} = D^{-1}(b - (L+U)x_n)}$ (3)

Consider 3x3 linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad (4)$$

Let matrix A is diagonally dominant, then from (3)

$$\boxed{\begin{aligned} x_1^{n+1} &= \frac{1}{a_{11}}\left(b_1 - (a_{12}x_2^n + a_{13}x_3^n)\right) \\ x_2^{n+1} &= \frac{1}{a_{22}}\left(b_2 - (a_{21}x_1^n + a_{23}x_3^n)\right) \\ x_3^{n+1} &= \frac{1}{a_{33}}\left(b_3 - (a_{31}x_1^n + a_{32}x_2^n)\right) \end{aligned}}$$

$$Start \ \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right)$$

Example: Solve the following system using Jacobi Method with starting point $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = (0,0,0)$ and check $\left\|R^{(2)}\right\|_\infty$

$$\begin{aligned} 3x_1 - \quad x_2 + \quad x_3 &= 1 \\ 3x_1 + 3x_2 + 7x_3 &= 4 \\ 3x_1 + 6x_2 + 2x_3 &= 0 \end{aligned} \qquad (1)$$

$$\underbrace{\begin{pmatrix} 3 & -1 & 1 \\ 3 & 3 & 7 \\ 3 & 6 & 2 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}}_{\mathbf{b}}$$

Check diagonally dominancy

$$|-1| + |1| < |3| \qquad \Rightarrow 2 < 3 \quad (ok)$$

$$|3| + |7| < |3| \qquad \Rightarrow 10 < 3 \quad (no) \qquad \Rightarrow \qquad \text{Above system is not diagonally dominant}$$

$$|3| + |6| < |2| \qquad \Rightarrow \quad 9 < 2 \quad (no)$$

Interchange equation (2) and equation (3) in (1) then

$$\begin{pmatrix} 3 & -1 & 1 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}$$

$$|-1| + |1| < |3| \quad \Rightarrow 2 < 3 \quad (ok)$$
$$|3| + |2| < |6| \quad \Rightarrow 5 < 6 \quad (ok)$$
$$|3| + |3| < |7| \quad \Rightarrow \quad 6 < 7 \quad (ok)$$

Now , system is diagonally dominant

Apply Jacobi Method

$$x_1^{(n+1)} = \frac{1}{3}\left(1 - \left(-x_2^{(n)} + x_3^{(n)}\right)\right)$$
$$x_2^{(n+1)} = \frac{1}{6}\left(0 - \left(3x_1^{(n)} + 2x_3^{(n)}\right)\right)$$
$$x_3^{(n+1)} = \frac{1}{7}\left(4 - \left(3x_1^{(n)} + 3x_2^{(n)}\right)\right)$$

$$Start \ \left(x_1^{(0)}, x_2^{(0)}, x_2^{(0)}\right) = (0,0,0)$$

Iteration 1:

$$x_1^{(1)} = \frac{1}{3}\left(1 - \left(-x_2^{(0)} + x_3^{(0)}\right)\right) = \frac{1}{3}(1 - (0 + 0)) = 0.333$$

$$x_2^{(1)} = \frac{1}{6}\left(0 - \left(3x_1^{(0)} + 2x_3^{(0)}\right)\right) = \frac{1}{6}\left(0 - \left(0 + 2(0)\right)\right) = 0$$

$$x_3^{(1)} = \frac{1}{7}\left(4 - \left(3x_1^{(0)} + 3x_2^{(0)}\right)\right) = \frac{1}{7}\left(4 - \left(3(0) + 3(0)\right)\right) = 0.572$$

$$\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\right) = (0.333, 0, 0.572)$$

Iteration 2:

$$x_1^{(2)} = \frac{1}{3}\left(1 - \left(-x_2^{(1)} + x_3^{(1)}\right)\right) = \frac{1}{3}(1 - (0 + 0.572)) = 0.142$$

$$x_2^{(2)} = \frac{1}{6}\left(0 - \left(3x_1^{(1)} + 2x_3^{(1)}\right)\right) = \frac{1}{6}\left(0 - \left(3(0.333) + 2(0.572)\right)\right) = -0.357$$

$$x_3^{(2)} = \frac{1}{7}\left(4 - \left(3x_1^{(1)} + 3x_2^{(1)}\right)\right) = \frac{1}{7}\left(4 - \left(3(0.333) + 3(0)\right)\right) = 0.428$$

$$\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}\right) = (0.142, -0.357, 0.428)$$

Check $\left\|R^{(2)}\right\|_\infty$

$$\left\|R_1^{(2)}\right\|_\infty = \left\|b - Ax^{(2)}\right\|_\infty = 1 - \left(3x_1^{(2)} - x_2^{(2)} + x_3^{(2)}\right) = -0.229$$

$$\left\|R_2^{(2)}\right\|_\infty = \left\|b - Ax^{(2)}\right\|_\infty = 0 - \left(3x_1^{(2)} + 6x_2^{(2)} + 2x_3^{(2)}\right) = -0.86$$

$$\left\|R_3^{(2)}\right\|_\infty = \left\|b - Ax^{(2)}\right\|_\infty = 4 - \left(3x_1^{(2)} + 3x_2^{(2)} + 7x_3^{(2)}\right) = 1.649$$

$$\left\|R^{(2)}\right\|_\infty = \max\left\{\begin{matrix} |-0.229| \\ |-0.86| \\ |1.649| \end{matrix}\right\} = 1.649$$

GAUSS-SEIDEL METHOD

GAUSS-SEIDEL METHOD

Given linear system

$$Ax = b \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(1)$$

$$where,$$

$$A = L + D + U \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(2)$$

Substitute (2) in (1)

$$(L + D + U)x = b \quad \Rightarrow (L + U)x + Dx = b$$

$$Dx = b - (L + U)x \quad \Rightarrow x = D^{-1}(b - Lx - Ux)$$

Gauss- Seidel Method : $\boxed{x_{n+1} = D^{-1}(b - Lx_{n+1} - Ux_n)}$ (3)

Consider 3x3 linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_1 + a_{33}x_3 &= b_3 \end{aligned} \quad \textbf{(4)}$$

$$\boxed{\begin{aligned} x_1^{n+1} &= \frac{1}{a_{11}}\left(b_1 - (a_{12}x_2^n + a_{13}x_3^n)\right) \\ x_2^{n+1} &= \frac{1}{a_{22}}\left(b_2 - (a_{21}x_1^{n+1} + a_{23}x_3^n)\right) \\ x_3^{n+1} &= \frac{1}{a_{33}}\left(b_3 - (a_{31}x_1^{n+1} + a_{32}x_2^{n+1})\right) \end{aligned}}$$

$$Start \quad \left(x_1^{(0)}, x_2^{(0)}, x_2^{(0)}\right)$$

Note: Error calculation for both methods, use

$$\|R\|_\infty = \|b - Ax\|_\infty = \max\|b - Ax\|$$

Example: Solve the following linear system using Gauss-Seidel method with $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = (0,0,0)$ for 2 iteration only, and evaluate $\left\|R^{(2)}\right\|_\infty$.

$$\begin{aligned} 4x_1 - \quad x_2 + \quad x_3 &= 7 \\ 4x_1 - 8x_2 + \quad x_3 &= -21 \\ -2x_1 + x_2 + 5x_3 &= 19 \end{aligned}$$

$$\underbrace{\begin{pmatrix} 4 & -1 & 1 \\ 4 & -8 & 1 \\ -2 & 1 & 5 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -21 \\ 19 \end{pmatrix}$$

Check diagonally dominancy

$$\begin{aligned} &|-1| + |1| < |4| \quad &\Rightarrow 2 < 4 \quad (ok) \\ &|4| + |1| < |-8| \quad &\Rightarrow 5 < 8 \quad (ok) \\ &|-2| + |1| < |5| \quad &\Rightarrow 3 < 5 \quad (ok) \end{aligned}$$

Now , system is diagonally dominant

Gauss-Seidel Method

$$x_1^{(n+1)} = \frac{1}{4}\left(7 - \left(-x_2^{(n)} + x_3^{(n)}\right)\right)$$

$$x_2^{(n+1)} = \frac{-1}{8}\left(-21 - \left(4x_1^{(n+1)} + x_3^{(n)}\right)\right)$$

$$x_3^{(n+1)} = \frac{1}{5}\left(19 - \left(-2x_1^{(n+1)} + x_2^{(n+1)}\right)\right)$$

$$Start \ \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = (0,0,0)$$

Iteration 1:

$$x_1^{(1)} = \frac{1}{4}\left(7 - \left(-x_2^{(0)} + x_3^{(0)}\right)\right) = \frac{1}{4}\left(7 - (-0 + 0)\right) = 1.75$$

$$x_2^{(1)} = \frac{-1}{8}\left(-21 - \left(4x_1^{(1)} + x_3^{(0)}\right)\right) = \frac{-1}{8}\left(-21 - (4(1.75) + 0)\right) = 3.5$$

$$x_3^{(1)} = \frac{1}{5}\left(19 - \left(-2x_1^{(1)} + x_2^{(1)}\right)\right) = \frac{1}{5}\left(19 - (-2(1.75) + 3.5)\right) = 3.8$$

$$\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\right) = (1.75,3.5,3.8)$$

Iteration 2:

$$x_1^{(2)} = \frac{1}{4}\left(7 - \left(-x_2^{(1)} + x_3^{(1)}\right)\right) = \frac{1}{4}\left(7 - (-3.5 + 3.8)\right) = 1.675$$

$$x_2^{(2)} = \frac{-1}{8}\left(-21 - \left(4x_1^{(2)} + x_3^{(1)}\right)\right) = \frac{-1}{8}\left(-21 - (4(1.675) + 3.8)\right) = 3.93$$

$$x_3^{(2)} = \frac{1}{5}\left(19 - \left(-2x_1^{(2)} + x_2^{(2)}\right)\right) = \frac{1}{5}\left(19 - (-2(1.675) + 3.93)\right) = 3.684$$

$$\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}\right) = (1.675,3.93,3.684)$$

Check $\left\|R^{(2)}\right\|_\infty$

$$R_1^{(2)} = 7 - \left(4x_1^{(2)} - x_2^{(2)} + x_3^{(2)}\right) = 0.546$$

$$R_2^{(2)} = -21 - \left(4x_1^{(2)} - 8x_2^{(2)} + x_3^{(2)}\right) = 0.056$$

$$R_3^{(2)} = 19 - \left(-2x_1^{(2)} + x_2^{(2)} + 5x_3^{(2)}\right) = 0$$

$$\left\|R^{(2)}\right\|_\infty = \max\begin{Bmatrix} |0.546| \\ |0.056| \\ 0 \end{Bmatrix} = 0.546$$

Example: Solve the following system using Jacobi and Gauss-Seidel Method with starting point $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = (0,0,0)$and check $\|R^{(2)}\|_\infty$.

$$x_1 + 2x_2 + 4x_3 = 3$$
$$-2x_1 + x_3 = -1$$
$$x_1 - 3x_2 + x_3 = 5$$

Solution:

The matrix form of this system is

$$\begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & 1 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$$

$$|2| + |4| < |1| \Rightarrow 6 < 1 \quad (no)$$
$$|-2| + |1| < |0| \Rightarrow 3 < 0 \quad (no)$$
$$|1| + |-3| < |1| \Rightarrow 4 < 1 \quad (no)$$

Above system is not diagonally dominant

Interchange equation (1) and equation (2) in A

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 2 & 4 \\ 1 & -3 & 1 \end{bmatrix}$$

$$\begin{aligned} &|0| + |1| < |-2| && \Rightarrow 1 < 2 \ \ (ok) \\ &|1| + |4| < |1| && \Rightarrow 5 < 1 \ \ (\text{no}) \\ &|1| + |-3| < |1| && \Rightarrow 4 < 1 \ \ (no) \end{aligned}$$

Above system is not diagonally dominant

Interchange equation (2) and equation (3) in A

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 1 & -3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$\begin{aligned} &|0| + |1| < |-2| && \Rightarrow 1 < 2 \ \ (ok) \\ &|1| + |1| < |-3| && \Rightarrow 2 < 3 \ \ (\text{ok}) \\ &|1| + |2| < |4| && \Rightarrow 3 < 4 \ \ (ok) \end{aligned}$$

Now , system is diagonally dominant

$$\begin{aligned} -2x_1 + x_3 &= -1 \\ x_1 - 3x_2 + x_3 &= 5 \\ x_1 + 2x_2 + 4x_3 &= 3 \end{aligned}$$

Apply Jacobi Method

$$x_1^{(n+1)} = \frac{1}{2}\left(1 + x_3^{(n)}\right)$$
$$x_2^{(n+1)} = -\frac{1}{3}\left(5 - \left(x_1^{(n)} + x_3^{(n)}\right)\right)$$
$$x_3^{(n+1)} = \frac{1}{4}\left(3 - \left(x_1^{(n)} + 2x_2^{(n)}\right)\right)$$

$$Start\ \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = (0,0,0)$$

$$-2x_1 + x_3 = -1$$
$$x_1 - 3x_2 + x_3 = 5$$
$$x_1 + 2x_2 + 4x_3 = 3$$

Iteration 1:

$$x_1^{(1)} = \frac{1}{2}\left(1 + x_3^{(0)}\right) = \frac{1}{2}(1+0) = 0.5$$

$$x_2^{(1)} = -\frac{1}{3}\left(5 - \left(x_1^{(0)} + x_3^{(0)}\right)\right) = -\frac{1}{3}\left(5 - (0+0)\right) = -1.6667$$

$$\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\right) = (0.5, -1.6667, 0.75)$$

$$x_3^{(1)} = \frac{1}{4}\left(3 - \left(x_1^{(0)} + 2x_2^{(0)}\right)\right) = \frac{1}{4}\left(3 - (0+0)\right) = 0.75$$

$$\left\|R^{(1)}\right\|_\infty = \left\|Ax^{(1)} - b\right\|_\infty = \left\|\begin{pmatrix} -2 & 0 & 1 \\ 1 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 0.5 \\ -1.6667 \\ 0.75 \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}\right\|_\infty = \left\|\begin{matrix} |0.75| \\ |1.2501| \\ |-2.8334| \end{matrix}\right\|_\infty = 2.8334$$

$$\left\|\begin{matrix} \left|x_1^{(1)} - x_1^{(0)}\right| \\ \left|x_2^{(1)} - x_2^{(0)}\right| \\ \left|x_3^{(1)} - x_3^{(0)}\right| \end{matrix}\right\| = \left\|\begin{matrix} 0.5 \\ 1.6667 \\ 0.75 \end{matrix}\right\|_\infty = 1.6667$$

Iteration 2:

$$x_1^{(2)} = \frac{1}{2}\left(1 + x_3^{(1)}\right) = \frac{1}{2}(1 + 0.75) = 0.875$$

$$x_2^{(2)} = -\frac{1}{3}\left(5 - \left(x_1^{(1)} + x_3^{(1)}\right)\right) = -\frac{1}{3}\left(5 - (0.5 + 0.75)\right) = -1.25$$

$$\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}\right) = (0.875, -1.25, 1.45835)$$

$$x_3^{(2)} = \frac{1}{4}\left(3 - \left(x_1^{(1)} + 2x_2^{(1)}\right)\right) = \frac{1}{4}\left(3 - (0.5 + 2(-1.667)))\right) = 1.45835$$

$$\left\|R^{(2)}\right\|_\infty = \left\|Ax^{(2)} - b\right\|_\infty = \left\|\begin{pmatrix} -2 & 0 & 1 \\ 1 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 0.875 \\ -1.25 \\ 1.45835 \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}\right\|_\infty = \left\|\begin{matrix} |0.70835| \\ |1.08335| \\ |1.2084| \end{matrix}\right\|_\infty = 1.2084$$

$$\left\|\begin{matrix} \left|x_1^{(2)} - x_1^{(1)}\right| \\ \left|x_2^{(2)} - x_2^{(1)}\right| \\ \left|x_3^{(2)} - x_3^{(1)}\right| \end{matrix}\right\| = \left\|\begin{matrix} 0.375 \\ 0.4167 \\ 0.70835 \end{matrix}\right\|_\infty = 0.70835$$

Apply Gauss-Seidel Method

$$x_1^{(n+1)} = \frac{1}{2}\left(1 + x_3^{(n)}\right)$$

$$x_2^{(n+1)} = -\frac{1}{3}\left(5 - \left(x_1^{(n+1)} + x_3^{(n)}\right)\right)$$

$$x_3^{(n+1)} = \frac{1}{4}\left(3 - \left(x_1^{(n+1)} + 2x_2^{(n+1)}\right)\right)$$

$$-2x_1 + x_3 = -1$$
$$x_1 - 3x_2 + x_3 = 5$$
$$x_1 + 2x_2 + 4x_3 = 3$$

$$Start \;\; \left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = (0,0,0)$$

Iteration 1:

$$x_1^{(1)} = \frac{1}{2}\left(1 + x_3^{(0)}\right) = \frac{1}{2}(1+0) = 0.5$$

$$x_2^{(1)} = -\frac{1}{3}\left(5 - \left(x_1^{(1)} + x_3^{(0)}\right)\right) = -\frac{1}{3}(5 - (0.5 + 0)) = -1.5$$

$$\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}\right) = (0.5, -1.5, , 1.375)$$

$$x_3^{(1)} = \frac{1}{4}\left(3 - \left(x_1^{(1)} + 2x_2^{(1)}\right)\right) = \frac{1}{4}(3 - (0.5 + (2)(-1.5))) = 1.375$$

$$\left\|R^{(1)}\right\|_\infty = \left\|Ax^{(1)} - b\right\|_\infty = \left\|\begin{pmatrix} -2 & 0 & 1 \\ 1 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 0.5 \\ -1.5 \\ 1.375 \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}\right\|_\infty = \left\|\begin{matrix} |1.375| \\ |1.375| \\ |0| \end{matrix}\right\|_\infty = 1.375$$

$$\left\|\begin{matrix} \left|x_1^{(1)} - x_1^{(0)}\right| \\ \left|x_2^{(1)} - x_2^{(0)}\right| \\ \left|x_3^{(1)} - x_3^{(0)}\right| \end{matrix}\right\| = \left\|\begin{matrix} 0.5 \\ 1.5 \\ 1.375 \end{matrix}\right\|_\infty = 1.5$$

Iteration 2:

$$x_1^{(2)} = \frac{1}{2}\left(1 + x_3^{(1)}\right) = \frac{1}{2}(1 + 1.375) = 1.1875$$

$$x_2^{(2)} = -\frac{1}{3}\left(5 - \left(x_1^{(2)} + x_3^{(1)}\right)\right) = -\frac{1}{3}(5 - (1.1875 + 1.375)) = -0.8125$$

$$x_3^{(2)} = \frac{1}{4}\left(3 - \left(x_1^{(2)} + 2x_2^{(2)}\right)\right) = \frac{1}{4}(3 - (1.1875 + 2(-0.8125)))) = 0.859375$$

$$\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}\right) = (1.1875, -0.8125, 0.859375)$$

$$\left\|R^{(2)}\right\|_\infty = \left\|Ax^{(2)} - b\right\|_\infty = \left\|\begin{pmatrix} -2 & 0 & 1 \\ 1 & -3 & 1 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 1.1875 \\ -0.8125 \\ 0.859375 \end{pmatrix} - \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}\right\|_\infty = \left\|\begin{matrix} |-0.515625| \\ |-0.515625| \\ |0| \end{matrix}\right\|_\infty = 0.515625$$

$$\left\|\begin{matrix} \left|x_1^{(2)} - x_1^{(1)}\right| \\ \left|x_2^{(2)} - x_2^{(1)}\right| \\ \left|x_3^{(2)} - x_3^{(1)}\right| \end{matrix}\right\| = \left\|\begin{matrix} 0.6875 \\ 0.6875 \\ 0.515625 \end{matrix}\right\|_\infty = 0.6875$$

So, Gauss-Seidel Method is better than Jacobi Method.

EXERCISES

Q1) Perform 2 iterations of the Gauss Seidel method with starting point $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = (0,0,0)$ for solving system of linear equations.

$$\begin{aligned} 4x - z &= 2 \\ 5y + z &= 2 \\ x + 2z &= 5 \end{aligned}$$

Q2) Perform 2 iterations of the Gauss Seidel method with starting point $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}\right) = (0,0,0,0)$ for solving system of linear equations.

$$\begin{aligned} x_1 \quad + 3x_3 &= 4 \\ 2x_2 + 5x_4 &= 6 \\ 4x_1 + 3x_3 &= 7 \\ 6x_2 + 3x_4 &= 8 \end{aligned}$$

Q3) Solve the following linear system of equations using Gauss-Seidel Method with starting point $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}\right) = (0,0,0,0)$ (first check for diagonally dominancy, then perform 2 iterations)

$$\begin{bmatrix} -1 & 0 & 3 & -1 \\ 0 & 3 & -1 & -1 \\ 2 & 0 & 0 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 2 \\ 5 \end{bmatrix}$$

Q4) Given

$$\begin{aligned} x_2 + 3x_3 &= 11 \\ 4x_1 - x_2 + x_3 &= 5 \\ 3x_1 - 4x_2 &= -5 \end{aligned}$$

Perform 2 suitable convergent iterations of the Jacobi Method for this system with starting point $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = (0,0,0)$ and evaluate $\left\|R^{(2)}\right\|_\infty$.

Q3) Solve the following linear system of equations using Gauss-Seidel Method with starting point $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}\right) = (0,0,0,0)$(first check for diagonally dominancy, then perform 2 iterations)

$$\begin{bmatrix} -1 & 0 & 3 & -1 \\ 0 & 3 & -1 & -1 \\ 2 & 0 & 0 & -1 \\ 0 & 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 2 \\ 5 \end{bmatrix}$$

It is not diagonally dominant

Since $|3| + |-1| \not< |-1|$

$|2| + |-1| \not< |0|$

$$\begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 3 & -1 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 5 \\ 5 \end{pmatrix}$$

$|-1| < |2|$ ✓

$|-1| + |-1| < |3|$ ✓

$|-1| + |-1| < |3|$ ✓

$|-1| < |4|$ ✓

It is Diagonally Dominant.

Gauss Seidel.

$$x_1^{(n+1)} = \frac{1}{2}\left[2 + x_4^{(n)}\right]$$

$$x_2^{(n+1)} = \frac{1}{3}\left[-2 + x_3^{(n)} + x_4^{(n)}\right]$$

$$x_3^{(n+1)} = \frac{1}{3}\left[5 + x_1^{(n+1)} + x_4^{(n)}\right]$$

$$x_4^{(n+1)} = \frac{1}{4}\left[5 + x_3^{(n+1)}\right]$$

Starting point $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}\right) = (0,0,0,0)$

Iteration 1

$$x_1^{(1)} = \frac{1}{2}\left[2+0\right] = 1$$

$$x_2^{(1)} = \frac{1}{3}\left[-2+0+0\right] = \frac{-2}{3} = -0.6667$$

$$x_3^{(1)} = \frac{1}{3}\left[5+1+0\right] = 2$$

$$x_4^{(1)} = \frac{1}{4}\left[5+2\right] = \frac{7}{4} = 1.75$$

$$\left(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)}\right) = (1, -0.6667, 2, 1.75)$$

Residual Error:
$$\|R^{(1)}\|_\infty = \left\| \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 3 & -1 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -0.6667 \\ 2 \\ 1.75 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 5 \\ 5 \end{pmatrix} \right\|_\infty = \left\| \begin{matrix} |-1.75| \\ |-3.75| \\ |-1.75| \\ |0| \end{matrix} \right\|_\infty = 3.75$$

Successive Error:
$$\left\| \begin{matrix} x_1^{(1)} - x_1^{(0)} \\ x_2^{(1)} - x_2^{(0)} \\ x_3^{(1)} - x_3^{(0)} \\ x_4^{(1)} - x_4^{(0)} \end{matrix} \right\|_\infty = \left\| \begin{matrix} |1| \\ |-0.6667| \\ |2| \\ |1.75| \end{matrix} \right\|_\infty = 2$$

Iteration 2

$$x_1^{(2)} = \frac{1}{2}\left[2 + x_4^{(1)}\right] = \frac{1}{2}\left[2 + 1.75\right] = 1.875$$

$$x_2^{(2)} = \frac{1}{3}\left[-2 + x_3^{(1)} + x_4^{(1)}\right] = \frac{1}{3}\left[\cancel{-2} + \cancel{2} + 1.75\right] = 0.583$$

$$x_3^{(2)} = \frac{1}{3}\left[5 + x_1^{(2)} + x_4^{(1)}\right] = \frac{1}{3}\left[5 + 1.875 + 1.75\right] = 2.875$$

$$x_4^{(2)} = \frac{1}{4}\left[5 + x_3^{(2)}\right] = \frac{1}{4}\left[5 + 2.875\right] = 1.96875$$

$$\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, x_4^{(2)}\right) = \left(1.875, 0.583, 2.875, 1.96875\right)$$

Residual Error:

$$\|R^{(2)}\|_\infty = \left\| \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 3 & -1 & -1 \\ -1 & 0 & 3 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1.875 \\ 0.583 \\ 2.875 \\ 1.96875 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 5 \\ 5 \end{pmatrix} \right\|_\infty = \left\| \begin{matrix} |-0.2187| \\ |-1.0947| \\ |-0.2187| \\ |0| \end{matrix} \right\|_\infty = 1.0947$$

Successive Error:

$$\left\| \begin{matrix} x_1^{(2)} - x_1^{(1)} \\ x_2^{(2)} - x_2^{(1)} \\ x_3^{(2)} - x_3^{(1)} \\ x_4^{(2)} - x_4^{(1)} \end{matrix} \right\|_\infty = \left\| \begin{matrix} |1.875 - 1| \\ |0.583 + 0.667| \\ |2.875 - 2| \\ |1.96875 - 1.75| \end{matrix} \right\|_\infty = \left\| \begin{matrix} |0.875| \\ |1.2497| \\ |0.875| \\ |0.21875| \end{matrix} \right\|_\infty = 1.2497$$

Q4) Given

Not diag. dom.

$R_1 \Leftrightarrow R_2$

$R_2 \Leftrightarrow R_3$

$$x_2 + 3x_3 = 11$$
$$4x_1 - x_2 + x_3 = 5$$
$$3x_1 - 4x_2 = -5$$

$4x_1 - x_2 + x_3 = 5$

$3x_1 - 4x_2 = -5$

$x_2 + 3x_3 = 11$

$|-1| + |1| < |4|$ ✓

$|3| < |-4|$ ✓

$|1| < |3|$ ✓

Diagonally Dominant.

Perform 2 suitable convergent iterations of the Jacobi Method for this system with starting point $\left(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}\right) = (0,0,0)$ and evaluate $\left\|R^{(2)}\right\|_\infty$.

Matrix Form: $\begin{pmatrix} 4 & -1 & 1 \\ 3 & -4 & 0 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 11 \end{pmatrix}$ It is Diagonally dominant.

Jacobi Meth:

$$x_1^{(n+1)} = \frac{1}{4}\left[5 + x_2^{(n)} - x_3^{(n)}\right]$$

$$x_2^{(n+1)} = -\frac{1}{4}\left[-5 - 3x_1^{(n)}\right]$$

$$x_3^{(n+1)} = \frac{1}{3}\left[11 - x_2^{(n)}\right]$$

Starting point $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}) = (0,0,0)$

Iteration 1

$$x_1^{(1)} = \frac{1}{4}\left[5 + x_2^{(0)} - x_3^{(0)}\right] = \frac{5}{4} = 1.25$$

$$x_2^{(1)} = -\frac{1}{4}\left[-5 - 3(x_1^{(0)})\right] = \frac{5}{4} = 1.25$$

$$x_3^{(1)} = \frac{1}{3}\left[11 - x_2^{(0)}\right] = \frac{11}{3} = 3.6667$$

$$(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}) = (1.25, 1.25, 3.6667)$$

Residual Error : $$\|R^{(1)}\|_\infty = \left\| \begin{pmatrix} 4 & -1 & 1 \\ 3 & -4 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1.25 \\ 1.25 \\ 3.6667 \end{pmatrix} - \begin{pmatrix} 5 \\ -5 \\ 11 \end{pmatrix} \right\|_\infty = \left\| \begin{matrix} |2.4167| \\ |3.75| \\ |1.2501| \end{matrix} \right\|_\infty = 3.75$$

Successive Error : $$\| x^{(n+1)} - x^{(n)} \|_\infty = 3.6667$$

Iteration 2

$$x_1^{(2)} = \frac{1}{4}\left[5 + x_2^{(1)} - x_3^{(1)}\right] = \frac{1}{4}\left[5 + 1.25 - 3.6667\right] = 0.6458$$

$$x_2^{(2)} = -\frac{1}{4}\left[-5 - 3\,x_1^{(1)}\right] = -\frac{1}{4}\left[-5 - 3(1.25)\right] = 2.1875$$

$$x_3^{(2)} = \frac{1}{3}\left[11 - x_2^{(1)}\right] = \frac{1}{3}\left[11 - 1.25\right] = 3.25$$

$$\left(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}\right) = (0.6458, 2.1875, 3.25)$$

Residual Error: $\|R^{(2)}\|_\infty = \left\| \begin{pmatrix} 4 & -1 & 1 \\ 3 & -4 & 0 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0.6458 \\ 2.1875 \\ 3.25 \end{pmatrix} - \begin{pmatrix} 5 \\ -5 \\ 11 \end{pmatrix} \right\|_\infty = \left\| \begin{matrix} |-1.3543| \\ |-1.8126| \\ |0.9375| \end{matrix} \right\|_\infty = 1.8126$

Successive Error: $\left\| \begin{matrix} |0.6458 - 1.25| \\ |2.1875 - 1.25| \\ |3.25 - 3.6667| \end{matrix} \right\| = \left\| \begin{matrix} |-0.6042| \\ |0.9375| \\ |-0.4167| \end{matrix} \right\|_\infty = 0.9375$