FIXED POINT ITERATION

Open Methods

- The open methods are based on formulas that require only a single starting value of x or two starting values that do not necessarily bracket the root.

- Open methods sometimes diverge or move away from the true root as the computation progress.

FIXED POINT ITERATION

In this method , the equation $f(x) = 0$ is rewritten in the form

$$x = g(x) \quad (1)$$

and iterative procedure is adopted using the relation

$$x_{n+1} = g(x_n) \qquad for \quad n = 0,1,2\ldots \qquad (2)$$

Where, a new approximation to root x_{n+1} , is found using the previous approximation x_n (x_0 denotes the initial guess) .

The procedure is repeated until a convergence criterion satisfied . For example

$$|x_{n+1} - x_n| \leq \varepsilon_1 \qquad and/or \qquad |f(x_{n+1})| \leq \varepsilon_2 \qquad (3)$$

Definition:

Geometrically the fixed point of a function $y_2 = g(x)$ are the points of intersection of $y_2 = g(x)$ *and* $y_1 = x$. See figs below.

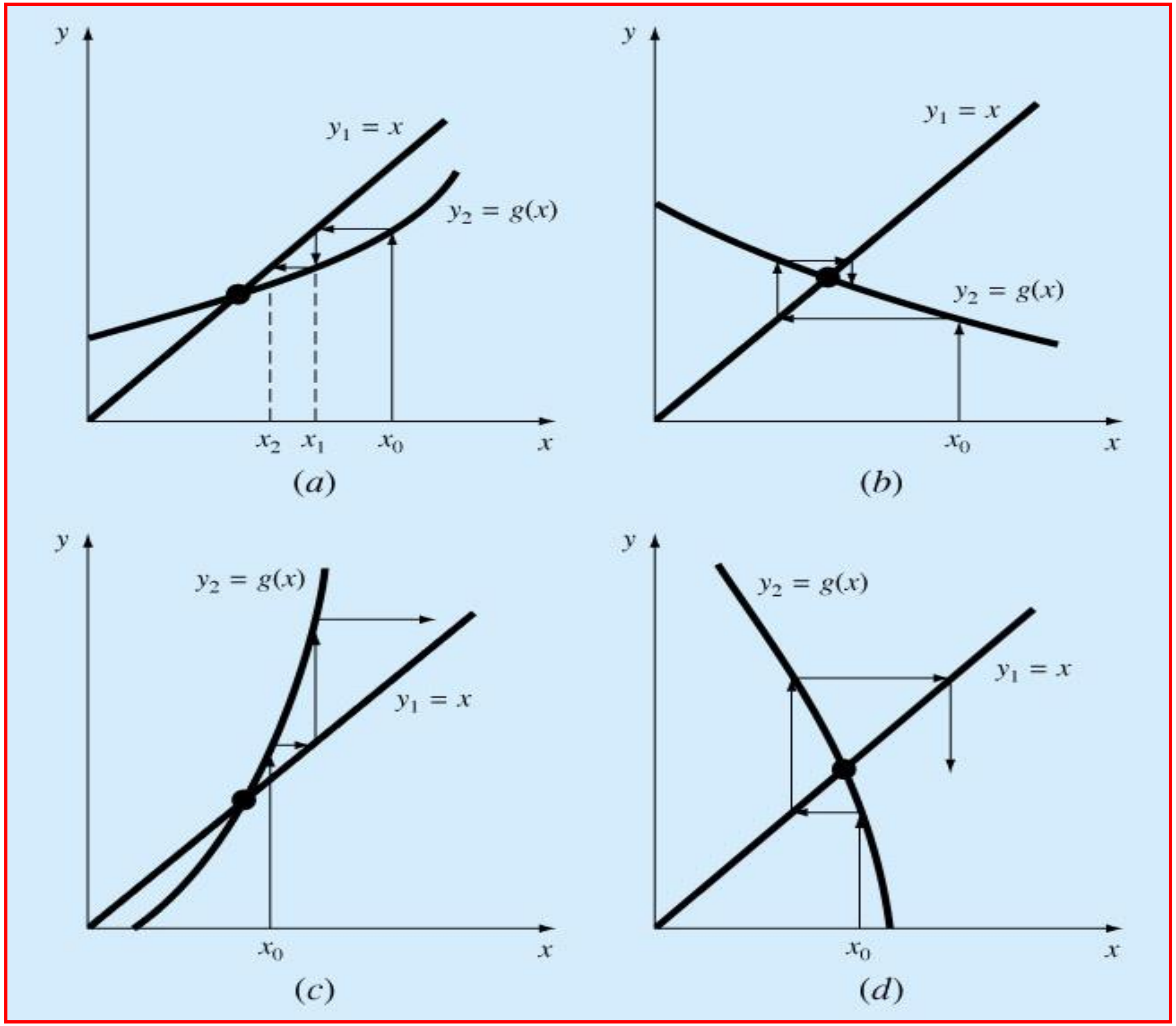


Fig. (a) and (b) convergent fixed point iteration and Fig. (c) and (d) divergent fixed point iteration

Definition:

The iteration $x_{n+1} = g(x_n)\ \ for\ \ n = 0,1,2...$ is called fixed point.

Theorem:

Let g be a continuously differentiable function which maps the interval I into itself ($I = [a, b]$) . Thus

$$x \in I \quad \Rightarrow \quad g(x) \in I$$

$$Suppose\ \ further\ \ that$$

$$|g'(x)| < 1 \quad , \quad x \in I$$

Then

a) g has a unique fixed point in I, α say

b) for any choice $x_0 \in I$, the sequence $x_{n+1} = g(x_n)$ converges to α .

- The sequence $x_0, x_1, x_2, x_3, \ldots$ will converge to a root of the equation $x = g(x)$ provided a suitable starting value x_0 is chosen and $-1 < g'(x) < 1$. That is

- Fixed point iteration converges if $|g'(x)| < 1$.

Example: Use fixed point iteration to find the positive root of

$$e^x - 1 - 2x = 0 \quad with\ an \quad accuracy \quad \varepsilon = 10^{-2}$$

Solution:

First locate the root(s)

$$f'(x) = e^x - 2 = 0 \qquad \Rightarrow \quad e^x = 2 \qquad \Rightarrow \quad x = \ln 2 \approx 0.69 \quad (take \quad x = 1)$$

	$-\infty$.	-4	-3	-2	-1.5	0	1	2	3	$\ldots\infty$
$f(x)=e^{x}-1-2x$						0	----	+	+	+

Positive root lies in [1,2]
Find g(x)

$$e^{x}-1-2x=0 \quad \Rightarrow x=\frac{1}{2}(e^{x}-1) \quad where \quad g(x)=\frac{1}{2}(e^{x}-1\Big)$$

a)

$$g'(x)=\frac{1}{2}e^{x} \quad and \quad |g'(x)|>1 \quad in \;\; [1,2] \quad not \;\; satisfy \;\; theorem$$

$$e^x - 1 - 2x = 0 \quad \Rightarrow e^x = 2x + 1 \quad \Rightarrow x = \ln(2x + 1\,) \;\; where \quad g(x) = \ln(2x + 1\,)$$

b)

$$g'(x) = \frac{2}{2x+1} \quad and \quad |g'(x)| < 1 \quad in \;\; [1,2] \qquad satisfy \quad theorem$$

Then fixed point iteration

$$x_{n+1} = g(x_n) \quad \Rightarrow \quad x_{n+1} = \ln(2x_n + 1)$$

Choose any point between [1,2], let choose starting point $x_0 = 1.5$ and start iteration

Iteration1: $x_1 = \ln(2x_0 + 1) = \ln(2(1.5) + 1) = 1.38629$

$$Successive\ \ error: \ |x_1 - x_0| = |1.38629 - 1.5| = 0.114$$
$$residual\ \ \ error: \ |f(x_1)| = |f(1.38629) = 0.227|$$

Iteration2: $x_2 = \ln(2x_1 + 1) = \ln(2(1.38629) + 1) = 1.32775$

$$Successive\ \ error: \ |x_2 - x_1| = |1.32775 - 1.38629| = 0.0585$$
$$residual\ \ \ error: \ |f(x_2)| = |f(1.32775) = 0.117|$$

Iteration3: $x_3 = \ln(2x_2 + 1) = \ln(2(1.32775) + 1) = 1.29623$

$$Successive\ \ error: \ |x_3 - x_2| = |1.29623 - 1.32775| = 0.0315$$
$$residual\ \ \ error: \ |f(x_3)| = |f(1.29623)| = 0.063$$

Iteration4: $x_4 = \ln(2x_3 + 1) = \ln(2(1.29623) + 1) = 1.27883$

$$Successive\ \ error:\ \ |x_4 - x_3| = |1.27883 - 1.29623| = 0.017$$
$$residual\quad error:\ \ |f(x_4)| = |f(1.27883)| = 0.034$$

Iteration5: $x_5 = \ln(2x_4 + 1) = \ln(2(1.27883) + 1) = 1.2691$

$$Successive\ \ error:\ \ |x_5 - x_3| = |1.2691 - 1.27883| = 0.0097 < \varepsilon = 10^{-2}\ \ (STOP)$$
$$residual\quad error:\ \ |f(x_5)| = |f(1.2691)| = 0.019\ \ (DO\ \ NOT\ \ STOP\ \ if\ \ \text{Residual}\ \ error\ \ used)$$

Iteration6: $x_6 = \ln(2x_5 + 1) = \ln(2(1.2691) + 1) = 1.26361$

$$Successive\ \ error:\ \ |x_6 - x_5| = |1.26361 - 1.2691| = 5.49 \times 10^{-3} < \varepsilon = 10^{-2}$$
$$residual\quad error:\ \ |f(x_6)| = |f(1.26361)| = 0.0109\ \leq\ \varepsilon = 10^{-2}$$

Approximate root $x \approx 1.2691$ (when successive error is used)
Approximate root $x \approx 1.26361$ (when residual error is used)

Example: Let $f(x) = x^2 - 2x - 1$

If we write $f(x) = r(x) - h(x)$, where $r(x) = x^2 \; and \; h(x) = 2x + 1$ which of the following iterative methods will converge to the positive root

1. $h(x_{n+1}) = r(x_n)$

2. $h(x_n) = r(x_{n+1})$

Also define the region through which the iterative method is convergent, apply 3 steps of the iterative method.

Solution:

First locate the roots

	$-\infty$.	-4	-3	-2	-1.5	0	1	2	3	...∞
$f(x) = x^2 - 2x - 1$						---	----	---	+	+

Positive root lie in [2,3] because $f(2)f(3) < 0$

a) $2x + 1 = x^2 \Rightarrow x = \frac{x^2-1}{2}$ therefore

$$g(x) = \frac{x^2 - 1}{2} \quad and \quad g'(x) = x \quad where \quad |g'(x)| = |x| > 1 \quad in \quad [2,3]$$

That is iteration not converge to fixed point

b) $2x + 1 = x^2 \quad \Rightarrow x = \sqrt{2x + 1}$

$$g(x) = \sqrt{2x + 1} \quad \Rightarrow g'(x) = \frac{1}{\sqrt{2x + 1}} \quad where\ |g'(x)| = \left|\frac{1}{\sqrt{2x + 1}}\right| < 1 \quad in \quad [2,3]$$

That is iteration converges to fixed point.

Fixed point iteration is $\quad x_{n+1} = \sqrt{2x_n + 1} \qquad$ choose $\ x_0 = 2.5$

Iteration1: $\quad x_1 = \sqrt{2x_0 + 1} \quad \Rightarrow x_1 = \sqrt{2(2.5) + 1} = 2.4494$

Iteration2: $\quad x_2 = \sqrt{2x_1 + 1} \quad \Rightarrow x_2 = \sqrt{2(2.4494) + 1} = 2.42878$

Iteration3: $x_3 = \sqrt{2x_2 + 1} \quad \Rightarrow x_3 = \sqrt{2(2.42878) + 1} = 2.42024$

After 3 iteration

$$x \approx 2.42024 \qquad with$$
$$successive\ error: \ |x_3 - x_2| = 8.35 \times 10^{-3}$$
$$residual\ error \quad : \ |f(x_3)| = 0.017$$

Example: Use fixed point iteration to find the positive root of

$$f(x) = x^3 - 3x - 2 \;\; with\ an\ \ accuracy\ \ \varepsilon = 10^{-3}$$

Solution:

First locate the root(s)

$f(x) = x^3 - 3x - 2, \;\; D: (-\infty, \infty) \qquad f'(x) = 3x^2 - 3 = 0, \;\; \text{x} = \pm 1$

x	$-\infty \ldots$	-1.5	-0.5	0	1.5	3
$f(x) = x^3 - 3x - 2$	$-$	$-$	$-$	$-$	$-$	$+$

The root lies on [1.5,3].

Find g(x)

$a)\quad x^3 - 3x - 2 = 0$

$$x^3 - 2 = 3x$$

$$x = \frac{x^3 - 2}{3}$$

$$g(x) = \frac{x^3-2}{3},\ g'(x) = x^2$$

$$g(x) = \frac{x^3 - 2}{3}\quad and\quad |g'(x)| > 1\quad in\ \ [1.5,3]\quad not\quad satisfy\quad theorem$$

b) $x^3 - 3x - 2 = 0$

$$x^3 = 3x + 2$$

$$x = (3x + 2)^{1/3}$$

$$g(x) = (3x + 2)^{1/3},\ g'(x) = (3x + 2)^{-2/3}$$

$$g(x) = (3x + 2)^{1/3} and\quad |g'(x)| < 1\quad in\ \ [1.5,3]\quad satisfy\quad theorem$$

Then fixed point iteration

$$x_{n+1} = g(x_n) \quad \Rightarrow \quad x_{n+1} = (3x_n + 2)^{\frac{1}{3}}$$

Choose any point between [1.5,3], let choose starting point $x_0 = 2.5$ and start iteration

$$x_0 = 2.5$$

Iteration1: $x_1 = g(x_0) = (3x_0 + 2)^{1/3} = (3(2.5) + 2)^{1/3} = 2.11791$

$$|x_1 - x_0| = |2.11791 - 2.5| = 0.38209 > 10^{-3}$$

Iteration2: $x_2 = g(x_1) = (3x_1 + 2)^{1/3} = (3(2.11791) + 2)^{1/3} = 2.02905$

$$|x_2 - x_1| = |2.02905 - 2.11791| = 0.08886 > 10^{-3}$$

Iteration3: $x_3 = g(x_2) = (3x_2 + 2)^{1/3} = (3(2.02905) + 2)^{1/3} = 2.00724$

$$|x_3 - x_2| = |2.00724 - 2.02905| = 0.002181 > 10^{-3}$$

Iteration4: $x_4 = g(x_3) = (3x_3 + 2)^{1/3} = (3(2.00724) + 2)^{1/3} = 2.00181$

$$|x_4 - x_3| = |2.00181 - 2.00724| = 5.43 \times 10^{-3} < 10^{-3} \quad (STOP)$$

n	x_n	$f(x_n)$	$\left\|x_{n+1}-x_n\right\|$
0	2.5	$f(2.5)=6.125$	---------
1	2.11791	$f(2.11791)=1.14625$	$\left\|x_1-x_0\right\|=\left\|2.11791-2.5\right\|=0.38209$
2	2.02905	$f(2.02905)=0.29219$	$\left\|x_2-x_1\right\|=\left\|2.02905-2.11791\right\|=0.08886$
3	2.00724	$f(2.00724)=0.06548$	$\left\|x_3-x_2\right\|=\left\|2.00724-2.02905\right\|=0.002181$
4	2.00181	$f(2.00181)=0.01631$	$\left\|x_4-x_3\right\|=\left\|2.00181-2.00724\right\|=5.43\times10^{-3}$

Approximate root $x \approx 2.00181$ for successive error

NEWTON METHOD

NEWTON METHOD (NEWTON – RAPHSON METHOD)

To obtain an iteration with rapid convergence to the solution of the equation $f(x) = 0$. We seek a rearrangement of that equation satisfying $x = g(x) \leftrightarrow f(x) = 0$ with an additional property $g(s) = 0$.

DERIVATION OF THE METHOD

- Most widely used method.
- Based on Taylor series expansion:

$$f(x_{n+1}) = f(x_n) + f'(x_n)\Delta x + f''(x_n)\frac{\Delta x^2}{2!} + O\Delta x^3$$

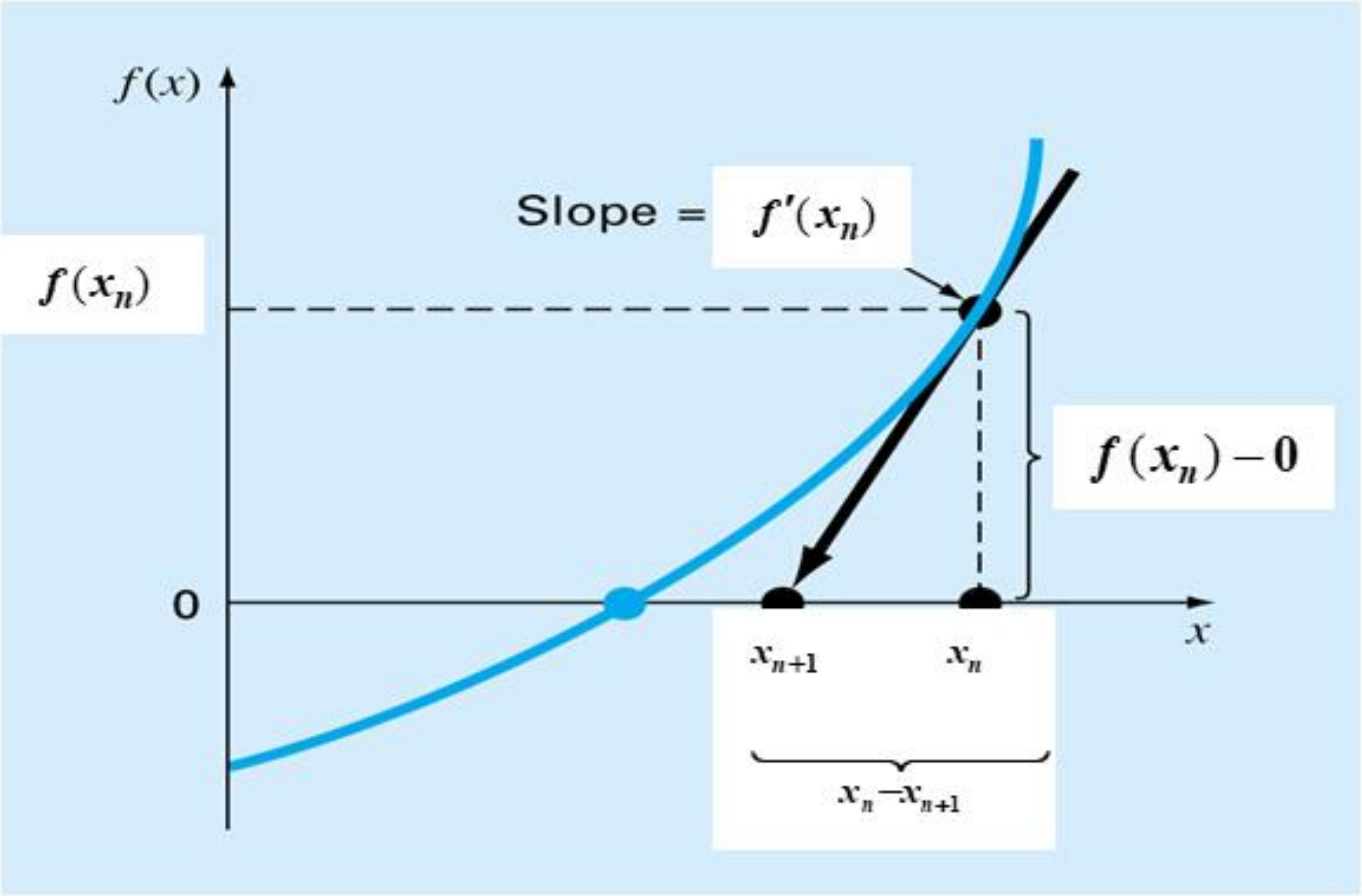
$$\text{The root is the value of } x_{n+1} \text{ when } f(x_{n+1}) = 0$$

$$\text{Rearranging,}$$

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n)$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{Newton–Raphson formula}$$

ANOTHER DERIVATION OF NEWTON METHOD



Slope of the tangent line

$$m = \frac{f(x_n) - 0}{x_n - x_{n+1}} \quad where \quad m = f'(x_n) \quad \Rightarrow \quad f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}}$$

$$\Rightarrow x_n - x_{n+1} = \frac{f(x_n)}{f'(x_n)} \quad and \quad Newton \quad method \quad \boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

For Newton-Raphson method $$g(x) = x - \frac{f(x)}{f'(x)} \quad and \quad g'(x) = \frac{f(x)f''(x)}{\left(f'(x)\right)^2}$$

For convergence $$|g'(x)| = \left|\frac{f(x)f''(x)}{\left(f'(x)\right)^2}\right| < 1 \quad for \;\; all \;\; x \in I$$

Example: Show that Newton's İterative Method to find the nth root of the number C is given by

$$x_{i+1} = \frac{1}{n}\left[(n-1)x_i + \frac{C}{x_i^{n-1}}\right]$$

Apply 3 iterations of the above form to find the approximation to the root of the following equation

$$f(x) = x^3 - 161$$

What is the accuracy of the last iterative value of x?

Solution

$$where \quad f(x) = x^n - C \quad then \quad f'(x) = nx^{n-1}$$

$$Newton\ \ Method: \ x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{(x_i^n - C)}{nx_i^{n-1}} = \frac{1}{n}\left[\frac{nx_i^n}{x_i^{n-1}} - \frac{x_i^n}{x_i^{n-1}} + \frac{C}{x_i^{n-1}}\right]$$

$$\Rightarrow x_{i+1} = \frac{1}{n}\left[(n-1)x_i + \frac{C}{x_i^{n-1}}\right]$$

$$Where \quad f(x) = x^3 - 161 \quad locate\ \ the\ \ root$$

$$(exercise)\ \ root\ lie\ in\ \ [5,6]$$

$$choose\ x_0 = 5.5 \quad where\ \ n = 3 \quad and \quad C = 161$$

ITERATION 1: $x_1 = \frac{1}{3}\left[2(5.5) + \frac{161}{5.5^2}\right] = 5.44$

ITERATION 2: $x_2 = \frac{1}{3}\left[2(5.44) + \frac{161}{5.44^2}\right] = 5.4396$

ITERATION 3: $x_3 = \frac{1}{3}\left[2(5.4396) + \frac{161}{5.44396^2}\right] = 5.44011$

$after\ 3\ \ iteration \quad x \approx 5.44011$

$\varepsilon = |f(5.44011)| = 4.497 \times 10^{-4}$

Example : Use Newton Method to find the upper root of the following equation with an accuracy $\varepsilon < 10^{-2}$

$$x^3 - 7x + 3 = 0$$

Solution:

First locate the root(s)

$$f(x) = x^3 - 7x + 3 \quad then \quad f'(x) = 3x^2 - 7 = 0 \quad \Rightarrow x = \pm\sqrt{\frac{7}{3}} \approx \pm 1.5275$$

$$take \quad x = \pm 1.5 \quad which \quad is \quad close \quad to \quad \pm 1.5275$$

	$-\infty$	-4	-3	-2	-1.5	0	1	1.5	2	3	$\ldots\infty$
$f(x) = x^3 - 7x + 3$	---	---	---	+	+	+	----	---	----	+	+

Roots lie in (-3,-2) , (0,1) and (2,3)
upper root $\alpha \in (2,3)$

We need use the following conditions to choose starting point of this iteration.

1. If $f(a)f''(x) > 0$ on the interval of root [a,b] then starting point is $x_0 = a$

2. If $f(b)f''(x) > 0$ on the interval of root [a,b] then starting point is $x_0 = \mathrm{b}$

$$f(x) = x^3 - 7x + 3, \qquad f'(x) = 3x^2 - 7, \qquad f''(x) = 6x$$

$$f(2)f''(x) = (-3)(6x) = -18x < 0 \text{ on } [2,3]$$

$$f(3)f''(x) = (9)(6x) = 54x > 0 \text{ on } [2,3]$$

So, starting point is $x_0 = 3$

Newton iteration for given function $x_{n+1} = x_n - \frac{x_n{}^3 - 7x_n + 3}{3x_n{}^2 - 7}$,

$$x_1 = x_0 - \frac{x_0{}^3 - 7x_0 + 3}{3x_0{}^2 - 7} = 3 - \frac{3^3 - 7 \times 3 + 3}{3 \times 3^2 - 7} = 2.55$$

$$x_2 = x_1 - \frac{x_1{}^3 - 7x_1 + 3}{3x_1{}^2 - 7} = 2.55 - \frac{2.55^3 - 7 \times 2.55 + 3}{3 \times 2.55^2 - 7} = 2.411573$$

$$x_3 = x_2 - \frac{x_2{}^3 - 7x_2 + 3}{3x_2{}^2 - 7} = 2.41.. - \frac{2.41..^3 - 7 \times 2.41.. + 3}{3 \times 2.41..^2 - 7} = 2.397795$$

$$residual\ \ error: \ |f(2.397795)| = 1.367595 \times 10^{-3} < 10^{-2}$$

$$Therefore, \ \ x \approx 2.397795$$

Example : Use Newton Method to find the upper root of the following equation with an accuracy $\varepsilon < 10^{-2}$

$$x^3 - 3x - 2 = 0$$

Solution:

First locate the root(s)

$$f(x) = x^3 - 3x - 2 = 0 \quad then \quad f'(x) = 3x^2 - 3 = 0 \quad \Rightarrow x = \pm 1$$

x	$-\infty\ldots$	-1.5	-0.5	0	1.5	3
$f(x) = x^3 - 3x - 2$	$-$	$-$	$-$	$-$	$-$	$+$

Root lie in (1.5,3) , $\alpha \in (1.5,3)$

We need use the following conditions to choose starting point of this iteration.

1. If $f(a)f''(x) > 0$ on the interval of root [a,b] then starting point is $x_0 = a$

2. If $f(b)f''(x) > 0$ on the interval of root [a,b] then starting point is $x_0 = \mathrm{b}$

$$f(x) = x^3 - 3x - 2, \qquad f'(x) = 3x^2 - 3, \qquad f''(x) = 6x$$

$$f(1.5)f''(x) = (-3.125)(6x) = -18.75x < 0 \ \text{ on } [1.5,3]$$

$$f(3)f''(x) = (16)(6x) = 96x > 0 \ \text{ on } [1.5,3]$$

So, starting point is $x_0 = 3$

Newton Iteration is $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$x_0 = 3$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{16}{24} = 2.3333,\ |x_1 - x_0| = |2.3333 - 3| = 0.6667$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.3333 - \frac{f(2.3333)}{f'(2.3333)} = 2.0556,\ |x_2 - x_1| = |2.0556 - 2.3333| = 0.2777$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.0556 - \frac{f(2.0556)}{f'(2.0556)} = 2.0019,\ \ |x_3 - x_2| = |2.0019 - 2.0556| = 0.0537$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.0019 - \frac{f(2.0019)}{f'(2.0019)} = 2,\ \ |x_4 - x_3| = |2 - 2.0019| = 0.0019 < 10^{-2},\ STOP\ here.$$

$$Therefore,\quad x = 2$$

Newton Method when starting point is $x_0 = 3$

n	x_n	$f(x_n)$	$\lvert x_{n+1} - x_n \rvert$
0	3	$f(3) = 16$	----------
1	2.3333	$f(2.3333) = 3.7033$	$\lvert x_1 - x_0 \rvert = \lvert 2.3333 - 3 \rvert = 0.6667$
2	2.0556	$f(2.0556) = 0.5191$	$\lvert x_2 - x_1 \rvert = \lvert 2.0556 - 2.3333 \rvert = 0.2777$
3	2.0019	$f(2.0019) = 0.0171$	$\lvert x_3 - x_2 \rvert = \lvert 2.0019 - 2.0556 \rvert = 0.0537$
4	2	$f(2) = 0$	$\lvert x_4 - x_3 \rvert = \lvert 2 - 2.0019 \rvert = 0.0019$

Newton Method with starting point $x_0 = 2.5$

n	x_n	$f(x_n)$	$\|x_{n+1} - x_n\|$
0	2.5	$f(3) = 6.125$	-----------
1	2.1111	$f(2.1111) = 1.0753$	$\|x_1 - x_0\| = \|2.1111 - 2.5\| = 0.3889$
2	2.0074	$f(2.0074) = 0.0669$	$\|x_2 - x_1\| = \|2.0556 - 2.3333\| = 0.1037$
3	2	$f(2) = 0$	$\|x_3 - x_2\| = \|2 - 2.0074\| = 0.0074$

Fixed Point Iteration with starting point $x_0 = 2.5$

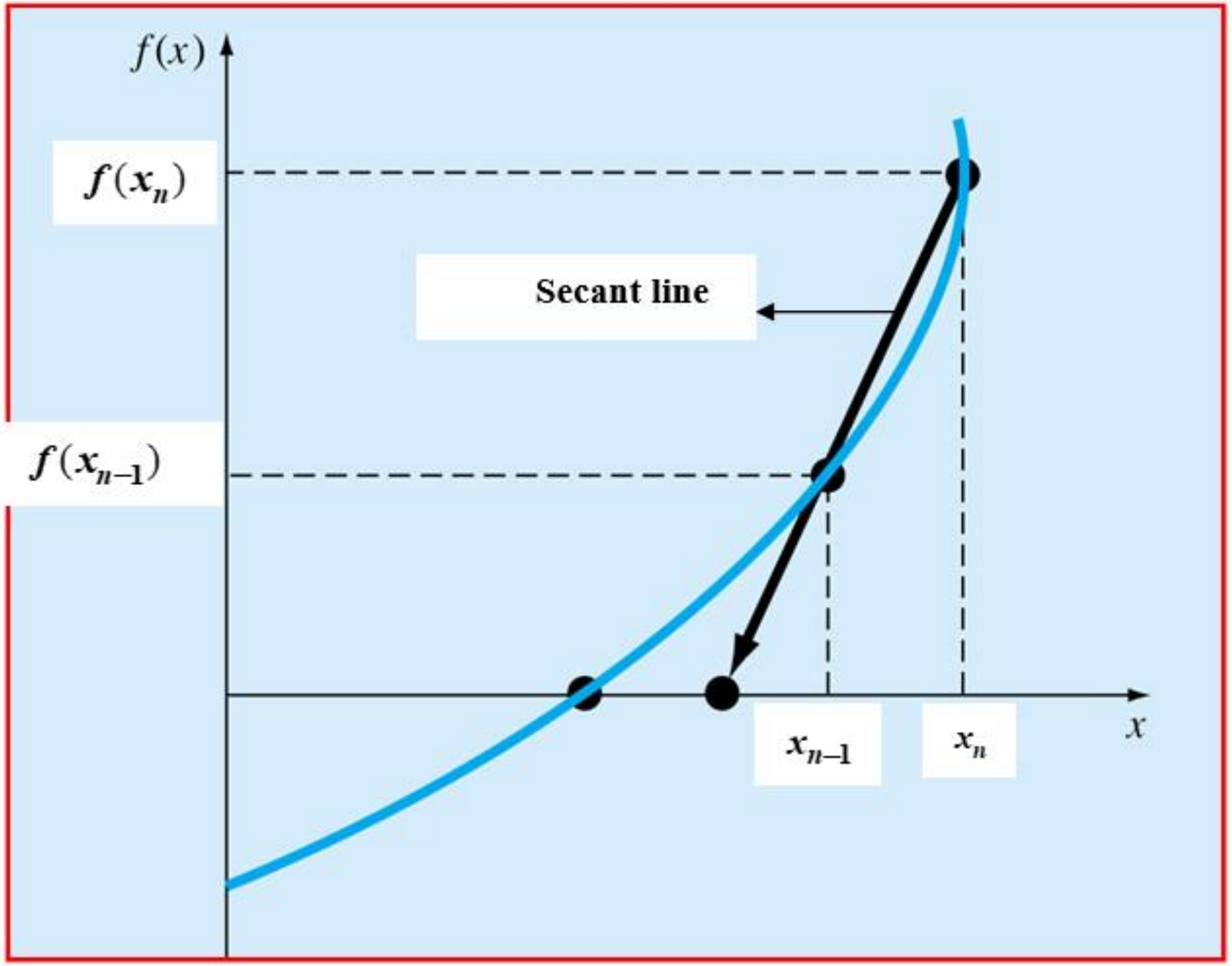
n	x_n	$f(x_n)$	$\|x_{n+1} - x_n\|$
0	2.5	$f(2.5) = 6.125$	---------
1	2.11791	$f(2.11791) = 1.14625$	$\|x_1 - x_0\| = \|2.11791 - 2.5\| = 0.38209$
2	2.02905	$f(2.02905) = 0.29219$	$\|x_2 - x_1\| = \|2.02905 - 2.11791\| = 0.08886$
3	2.00724	$f(2.00724) = 0.06548$	$\|x_3 - x_2\| = \|2.00724 - 2.02905\| = 0.002181$
4	2.00181	$f(2.00181) = 0.01631$	$\|x_4 - x_3\| = \|2.00181 - 2.00724\| = 5.43 \times 10^{-3}$

So, Newton Method is faster than Fixed Point Iteration to find root of the function x=2

SECANT METHOD

SECANT METHOD

When the derivative function $f'(x)$, is unavailable or prohibitively costly to evaluate, an alternative to Newton's Method is required. The preferred alternative is the Secant Method.



Newton formula :

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

From above figure , slope of the secant line is

$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

substitute in Newton method

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}} \quad SECANT METHOD$$

- Notice that this is very similar to the false position method in form
- Still requires two initial estimates
- But it doesn't bracket the root at all times - there is no sign test

Example: Find the common point between $f_1(x) = x^3 \quad and \quad f_2(x) = x - 3$ using secant method for 3 iterations . What is the achieved accuracy for the last iterative value.

Solution: Root lie in [-2,-1] **(EXERCISE)**

Choose $x_0 = -1.8 \quad and \quad x_1 = -1.2$

Iteration 1: $$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = -1.6232$$

Iteration 2: $$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)} = -1.6918$$

Iteration 3: $$x_4 = x_3 - \frac{f(x_3)(x_3 - x_2)}{f(x_3) - f(x_2)} = -1.67095$$

With error $\varepsilon = |f(-1.67096)| = 5.45 \times 10^{-3}$

Example: Find the approximation to the root by using Secant Method for the following function.

$$f(x) = xe^{1-x}$$

Solution:

From the Analytical Method

The function $f(x) = xe^{1-x}$ and the derivative of the function is

$$f'(x) = e^{1-x} - xe^{1-x} = 0$$

the critical point x=1.

x	-1	$\frac{1}{2}$	1
Sign of $f(x)$	-	+	+

$\nearrow$

There exist a root on the interval $\left[-1, \frac{1}{2}\right]$

Now, we use Secant Method for the function $f(x) = xe^{1-x}$

with the interval . We should use $x_{n+1} = x_n - f(x_n)\dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \quad \text{n} = 1, 2, 3, 4, \ldots$

Let $x_0 = -0.5, x_1 = 0.2 \in [-1,0.5]$

$$x_0 = -0.5, x_1 = 0.2, f(-0.5) = -2.240844, f(0.2) = 0.445108$$

$$x_2 = 0.2 - \frac{f(0.2)(0.2-(-0.5))}{f(0.2)-f(-0.5)} = 0.083998, \qquad f(0.083998) = 0.209934$$

$$x_3 = 0.083998 - \frac{f(x_2)(x_2-x_1)}{f(x_2)-f(x_1)} = -0.019554, f(-0.019554) = -0.054203$$

$$x_4 = -0.019554 - \frac{f(x_3)(x_3-x_2)}{f(x_3)-f(x_2)} = 0.001696, f(0.001696) = 0.004602$$

The results for 5 steps of Secant Method for the function $f(x) = xe^{1-x}$ is shown below.

Secant Method

k	x_k	$f(x_k)$	$x_{k+1} - x_k$
0	-0.5	-2.240844	0.7
1	0.2	0.445108	-0.116002
2	0.083998	0.209934	-0.103552
3	-0.019554	-0.054203	0.02125
4	0.001696	0.004602	-

Exercises:

Q1) Construct the convergent fixed point iteration to find the lowest root of the following equation with an accuracy $\varepsilon < 10^{-2}$

$$\ln x - x^2 + 7x - 8 = 0$$

Q2) Use the method of fixed point to solve the equation

$$e^x + x - 2 = 0$$

with an accuracy $\varepsilon = 0.005$

Q3) Find the smallest root of the function $f(x) = sinx - cosx$ over $[0,2\pi]$

using Newton's method with an accuracy $\varepsilon = 0.001$.

Q4) Estimate $\sqrt{3}$ using Newton's method with an accuracy $\varepsilon \leq 0.001$

Q5) Find an approximation to the root of $e^{x} - (x - 1) = 0$, using False position method for 3 steps .

Q6) Given $e^{x} + \frac{2}{3}x - 2 = 0$

a) Separate the roots using analytical method.

b) Approximate the largest root of the above equation with an accuracy $\varepsilon \leq 0.01$ of using

i) Fixed point iteration

ii) Newton's Method

comments the result