THE SOLUTION OF LINEAR SYSTEMS AX=B

LU METHOD

DIRECT METHOD

TRIANGULAR FACTORIZATION

Definition :

The non-singular matrix A has a triangular factorization if it can be expressed as the product of a lower-triangular matrix L and an upper-triangular matrix U .

$$A = LU \qquad (1)$$

In the matrix form, this is written as (consider 3x3 matrix)

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \underbrace{\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}}_{\mathbf{L}} \underbrace{\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}}_{\mathbf{U}}$$

(**Note:** Assume value of $u_{ii} = 1 \quad or \quad l_{ii} = 1$)

For the solution of linear system using LU decomposition method is as follows

$$\begin{gathered} Ax = b \\ (LU)x = b \end{gathered} \qquad (2)$$

Solution of (2) is as follows

- Solve using forward substitution $Ly = b$
- Solve using backward substitution $Ux = y$

$\textbf{\textit{Note}}$: $If \;\; A = A^T \;\; (A \;\; is \;\; symmetric) \;\; then \;\; LU = LL^T$

CROUT DECOMPOSITION

Given $Ax = b$, if $A \neq A^T$ then we can decompose A as follows

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

L **U**

Then, find elements of LU matrices

$$\boxed{l_{11} = a_{11}}$$

$$l_{11}u_{12} = a_{12} \quad \Rightarrow \boxed{u_{12} = \frac{a_{12}}{l_{11}}}$$

$$l_{11}u_{13} = a_{13} \quad \Rightarrow \boxed{u_{13} = \frac{a_{13}}{l_{11}}}$$

$$\boxed{l_{21} = a_{21}}$$

$$l_{21}u_{12} + l_{22} = a_{22} \quad \Rightarrow \boxed{l_{22} = a_{22} - l_{21}u_{12}}$$

$$l_{21}u_{13} + l_{22}u_{23} = a_{23} \quad \Rightarrow \boxed{u_{23} = \frac{a_{23} - l_{21}u_{13}}{l_{22}}}$$

$$\boxed{l_{31} = a_{31}}$$

$$l_{31}u_{12} + l_{32} = a_{32} \quad \Rightarrow \boxed{l_{32} = a_{32} - l_{31}u_{13}}$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = a_{33} \quad \Rightarrow \boxed{l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}}$$

Example: Solve

$$\begin{aligned} 2x_1 - 5x_2 + x_3 &= 12 \\ -x_1 + 3x_2 - x_3 &= -8 \\ 3x_1 - 4x_2 + 2x_3 &= 16 \end{aligned}$$

using LU decomposition method

Solution:

$$Ax = b \qquad \Rightarrow \begin{pmatrix} 2 & -5 & 1 \\ -1 & 3 & -1 \\ 3 & -4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ -8 \\ 16 \end{pmatrix}$$

Find LU matrices

$$\begin{pmatrix} 2 & -5 & 1 \\ -1 & 3 & -1 \\ 3 & -4 & 2 \end{pmatrix} = \underset{\mathbf{L}}{\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}} \underset{\mathbf{U}}{\begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}}$$

$$\boxed{l_{11} = 2}$$

$$l_{11}u_{12} = -5 \qquad \Rightarrow \boxed{u_{12} = \frac{-5}{2}}$$

$$l_{11}u_{13} = 1 \qquad \Rightarrow \boxed{u_{13} = \frac{1}{2}}$$

$$\boxed{l_{21} = -1}$$

$$l_{21}u_{12} + l_{22} = 3 \qquad \Rightarrow \boxed{l_{22} = 3 - \frac{5}{2} = \frac{1}{2}}$$

$$l_{21}u_{13} + l_{22}u_{23} = -1 \qquad \Rightarrow \boxed{u_{23} = \frac{-1 - \left(\left(-1\right) \times \frac{1}{2}\right)}{\frac{1}{2}} = -1}$$

$$\boxed{l_{31} = 3}$$

$$l_{31}u_{12} + l_{32} = -4 \qquad \Rightarrow \boxed{l_{32} = -4 - \left(3 \times \frac{-5}{2}\right) = \frac{7}{2}}$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 2 \qquad \Rightarrow \boxed{l_{33} = 2 - \frac{3}{2} - \frac{7}{2} = 4}$$

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1/2 & 0 \\ 3 & 7/2 & 4 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & -5/2 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Solve

$$Ly = b \quad \Rightarrow \quad \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1/2 & 0 \\ 3 & 7/2 & 4 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 12 \\ -8 \\ 16 \end{pmatrix} \downarrow$$

$$2y_1 = 12 \qquad \Rightarrow y_1 = 6$$

$$-y_1 + \frac{1}{2}y_2 = -8 \qquad \Rightarrow y_2 = -4$$

$$3y_1 + \frac{7}{2}y_2 + 4y_3 = 16 \qquad \Rightarrow y_3 = 3$$

Solve

$$Ux = y \quad \Rightarrow \quad \begin{pmatrix} 1 & -5/2 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 3 \end{pmatrix} \quad \uparrow$$

$$x_3 = 3$$

$$x_2 - x_3 = -4 \qquad \Rightarrow x_2 = -1$$

$$x_1 - \frac{5}{2}x_2 + \frac{1}{2}x_3 = 6 \qquad \Rightarrow x_1 = 2$$

$$Solution\ \ set$$

$$\{(x_1, x_2, x_3)\} = \{(2, -1, 3)\}$$

Example: Solve the following system of the form by using LU-Decomposition.

$$4x_1 - x_2 + x_3 = 6$$
$$-x_1 + 6x_2 + x_3 = -6$$
$$x_1 + x_2 + 8x_3 = 8$$

Solution:

$$Ax = b \quad \Rightarrow \quad \begin{bmatrix} 4 & -1 & 1 \\ -1 & 6 & 1 \\ 1 & 1 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 8 \end{bmatrix}$$

$$Ax = B$$
$$LUx = B$$
$$Ux = Y \qquad LY = B$$

Find LU matrices

$$\begin{bmatrix} 4 & -1 & 1 \\ -1 & 6 & 1 \\ 1 & 1 & 8 \end{bmatrix} = \underbrace{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{U}}$$

$$\boxed{l_{11} = 4}$$

$$l_{11}u_{12} = -1, \boxed{u_{12} = -\frac{1}{4}}$$

$$l_{11}u_{13} = 1, \boxed{u_{13} = \frac{1}{4}}$$

$$\boxed{l_{21} = -1}$$

$$l_{21}u_{12} + l_{22} = 6, \qquad \frac{1}{4} + l_{22} = 6, \qquad \boxed{l_{22} = \frac{23}{4}}$$

$$l_{21}u_{13} + l_{22}u_{23} = 1, \qquad -\frac{1}{4} + \frac{23}{4}u_{23} = 1, \qquad \boxed{u_{23} = \frac{5}{23}}$$

$$\boxed{l_{31} = 1,}$$

$$l_{31}u_{12} + l_{32} = 1, \qquad -\frac{1}{4} + l_{32} = 1, \boxed{l_{32} = \frac{5}{4}}$$

$$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 8, \qquad \frac{1}{4} + \frac{5}{4}\left(\frac{5}{23}\right) + l_{33} = 8, \qquad \boxed{l_{33} = \frac{172}{23}}$$

$$\begin{bmatrix} 4 & -1 & 1 \\ -1 & 6 & 1 \\ 1 & 1 & 8 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & 1 \\ -1 & 6 & 1 \\ 1 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ -1 & \dfrac{23}{4} & 0 \\ 1 & \dfrac{5}{4} & \dfrac{172}{23} \end{bmatrix} \begin{bmatrix} 1 & -\dfrac{1}{4} & \dfrac{1}{4} \\ 0 & 1 & \dfrac{5}{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$LY = B \quad \Longrightarrow \quad \begin{bmatrix} 4 & 0 & 0 \\ -1 & \frac{23}{4} & 0 \\ 1 & \frac{5}{4} & \frac{172}{23} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 8 \end{bmatrix} \downarrow$$

$$4y_1 = 6, \qquad y_1 = \frac{3}{2}$$

$$-y_1 + \frac{23}{4}y_2 = -6, \qquad \frac{3}{2} + \frac{23}{4}y_2 = -6\,, \quad y_2 = -\frac{18}{23}$$

$$y_1 + \frac{5}{4}y_2 + \frac{172}{23}y_3 = 8, \qquad \frac{3}{2} + \frac{5}{4}\left(-\frac{18}{23}\right) + \frac{172}{23}y_3 = 8, \qquad y_3 = 1$$

Thus, $\quad y_1 = \frac{3}{2},\ \ y_2 = -\frac{18}{23},\ y_3 = 1$

Then,

$$Ux = Y \quad \Rightarrow \quad \begin{bmatrix} 1 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & \frac{5}{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -18/23 \\ 1 \end{bmatrix} \uparrow$$

$$x_3 = 1,$$

$$x_2 + \frac{5}{23}x_3 = -\frac{18}{23}, \qquad x_2 = -1$$

$$x_1 - \frac{1}{4}x_2 + \frac{1}{4}x_3 = \frac{3}{2}, \qquad x_1 = 1$$

Thus, the solution of this system from the LU-Decomposition is $\{(x_1, x_2, x_3)\} = \{(1, -1, 1)\}$

CHOLESKY DECOMPOSITION

CHOLESKY DECOMPOSITION

Given $Ax = b$, if $A = A^T$ ($A\ is\ symmetric$) and A is positive definite then we can decompose as follows

Positive definite for 3x3 matrix:

$$If\ |a_{11}| > 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0\ and\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} > 0\ then\ matrix\ is\ positive\ definite$$

$$A = LL^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \underbrace{\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}}_{L^T} \quad where,\ \ a_{21} = a_{12},\ a_{31} = a_{13},\ a_{32} = a_{23}$$

$$l_{11}^2 = a_{11} \quad \Rightarrow \boxed{l_{11} = \sqrt{a_{11}}} \quad , \quad l_{11}l_{21} = a_{12} \quad \Rightarrow \boxed{l_{21} = \frac{a_{12}}{l_{11}}}$$

$$l_{11}l_{31} = a_{13} \quad \Rightarrow \boxed{l_{31} = \frac{a_{13}}{l_{11}}} \quad , \quad l_{21}^2 + l_{22}^2 = a_{22} \quad \Rightarrow \boxed{l_{22} = \sqrt{a_{22} - l_{21}^2}}$$

$$l_{21}l_{31} + l_{32}l_{22} = a_{23} \quad \Rightarrow \boxed{l_{32} = \frac{a_{23} - l_{21}l_{31}}{l_{22}}} \quad , \quad l_{31}^2 + l_{32}^2 + l_{33}^2 = a_{33} \quad \Rightarrow \boxed{l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2}}$$

Example : Solve the system of equations

$$4x - y = 8$$
$$-x + 4y - z = -3$$
$$-y + 4z = 4$$

by using $LL^T \quad or \quad LU$ decomposition

Solution: Matrix form of given linear system is Type equation here.

$$\underset{A}{\begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}} \underset{x}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{b}{\begin{pmatrix} 8 \\ -3 \\ 4 \end{pmatrix}}$$

where $A = A^T \quad (A \;\; is \;\; symmetric) \;\; and$

$$|4| > 0, \qquad \begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} = 16 - 1 = 15 > 0, \qquad and \qquad \begin{vmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{vmatrix} = (4)(15) - 4 = 56 > 0$$

positive definite , so we can $use \quad LL^T$

$$\begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{pmatrix}$$

$$l_{11}^2 = 4 \quad \Rightarrow \boxed{l_{11} = 2} \quad , \quad l_{11} l_{21} = -1 \quad \Rightarrow \boxed{l_{21} = \frac{-1}{2}}$$

$$l_{11} l_{31} = 0 \quad \Rightarrow \boxed{l_{31} = 0} \quad , \quad l_{21}^2 + l_{22}^2 = 4 \quad \Rightarrow \boxed{l_{22} = \sqrt{4 - \frac{1}{4}} = \frac{\sqrt{15}}{2}}$$

$$l_{21} l_{31} + l_{32} l_{22} = -1 \quad \Rightarrow \boxed{l_{32} = \frac{-2}{\sqrt{15}}} \quad , \quad l_{31}^2 + l_{32}^2 + l_{33}^2 = 4 \quad \Rightarrow \boxed{l_{33} = \sqrt{4 - 0 - \frac{4}{15}} = \sqrt{\frac{56}{15}}}$$

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -1/2 & \sqrt{15}/2 & 0 \\ 0 & -2/\sqrt{15} & \sqrt{56/15} \end{pmatrix}$$

$$Solve \quad Ly = b$$

$$\begin{pmatrix} 2 & 0 & 0 \\ -1/_2 & \sqrt{15}/_2 & 0 \\ 0 & -2/\sqrt{15} & \sqrt{56/\quad 15} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ 4 \end{pmatrix}$$

$$y_1 = 4 \;, \quad y_2 = \frac{-2}{\sqrt{15}} \;\; and \;\; y_3 = \sqrt{\frac{56}{15}} \;\; (exercise \Bigg)$$

$$Solve \quad L^T x = y$$

$$\begin{pmatrix} 2 & -1/_2 & 0 \\ 0 & \sqrt{15}/_2 & -2/\sqrt{15} \\ 0 & 0 & \sqrt{56/\quad 15} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{-2}{\sqrt{15}} \\ \sqrt{\frac{56}{15}} \end{pmatrix}$$

$$x = 2 \;, \quad y = 0 \;\; and \;\; z = 1 \;\; (exercise)$$

EXERCISE

Q1) Solve the following linear equations using LU decomposition for the matrix A such that $u_{ii} = 2 \quad for \quad i = 1,2,3$

$$\begin{pmatrix} 6 & 10 & 0 \\ 12 & 26 & 4 \\ 0 & 9 & 12 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Q2) Find an LU matrix decomposition for the following matrix provided that

$$l_{ii} = 0.5 \quad for \quad i = 1,2,3 \quad where$$

$$A = \begin{pmatrix} 0.5 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 4 & 7 \end{pmatrix}$$

Q3) Solve the following set of linear equations using LU decomposition method.

$$
\begin{aligned}
4x_1 - 2x_2 &= 4 \\
-2x_1 + 4x_2 - 2x_3 &= 2 \\
-2x_2 + 4x_3 &= 1
\end{aligned}
$$

Q4) Solve the following set of linear equations using LU decomposition method.

$$
\begin{aligned}
x + 3z &= 5 \\
3x + y + 6z &= 12 \\
-5x + 2y - z &= 8
\end{aligned}
$$

Q2) Find an LU matrix decomposition for the following matrix provided that

$$l_{ii} = 0.5 \ \ for \ \ i = 1,2,3 \quad where$$

$$A = \begin{pmatrix} 0.5 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 4 & 7 \end{pmatrix}$$

$$L = \begin{pmatrix} 0.5 & 0 & 0 \\ 1 & 0.5 & 0 \\ 2 & 0 & 0.5 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix}$$

when we chose $\boxed{l_{32}=1}$ we get $\boxed{u_{33}=-6}$

$$A = L.U \Rightarrow \begin{pmatrix} 0.5 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 4 & 7 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0 \\ l_{21} & 0.5 & 0 \\ l_{31} & l_{32} & 0.5 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$0.5\,u_{11} = 0.5 \quad \boxed{u_{11}=1}$

$0.5\,u_{12} = 1 \quad \boxed{u_{12}=2}$

$0.5\,u_{13} = 2 \quad \boxed{u_{13}=4}$

$l_{21}u_{11} = 1 \quad \boxed{l_{21}=1}$

$l_{21}u_{12} + 0.5u_{22} = 2$

$(1)\cancel{(2)} + 0.5\,u_{22} = \cancel{2}$

$\boxed{u_{22}=0}$

$l_{21}u_{13} + 0.5u_{23} = 5$

$(1)(4) + 0.5\,u_{23} = 5$

$\boxed{u_{23}=2}$

$l_{31}u_{11} = 2 \quad \boxed{l_{31}=2}$

$l_{31}u_{12} + l_{32}u_{22} = 4$

$(2)(2) + l_{32}(0) = 4 \Rightarrow l_{32} \in \mathbb{R}$

Choose $\boxed{l_{32}=0}$

$l_{31}u_{13} + l_{32}u_{23} + 0.5u_{33} = 7$

$(2)(4) + (0)(2) + 0.5\,u_{33} = 7$

$\boxed{u_{33}=-2}$