1) Let $f(x)=2^x$

a) Construct the divided difference table based on the nodes $x_0=0, x_1=1, x_2=2, x_3=3$.

b) Write down the Newton polynomial $P_3(x)$

c) Evaluate $f(2.5)$ by using $P_3(x)$.

SOLUTION

a)

x_k	$f[x_k]$	$f[,]$	$f[,,]$	$f[,,,]$
0	**1=a_0**			
1	**2**	**1=a_1**		
2	**4**	**2**	**0.5=a_2**	
3	**8**	**4**	**1**	**0.1666=a_5**

b)

$$P_3(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+a_2(x-x_0)(x-x_1)(x-x_1)$$

$$=1+1(x-0)+0.5(x-0)(x-1)+0.1666(x-0)(x-1)(x-2)$$
$$=1+x+0.5x^2-0.5x+0.1666x^3-0.4998x^2+0.3332x$$

$$P_3(x)=0.1666x^3+0.0002x^2+0.8332x+1$$

$c)\ P_3(2.5)=5.687375$

2)

a) ~~Use differentiation of the Lagrange Polynomial~~ to derive the formula using Taylor expan.

$$f'(x) = \frac{-3f(x)+4f(x+h)-f(x+2h)}{2h}$$

b) If $E_{trunc}(f,h) = \frac{h^2}{3}f'''(x)$, evaluate the derivative at the point $x = 1$ using the above formula for the function $f(x) = x - 4x^2$ such that the rounding error should not exceed $\varepsilon = 10^{-4}$ and the total error bounded by 10^{-2} and verify the result.

SOLUTION

a) Start Lagrange interpolation polynomial for f(t) based on the 3 points x_0, x_1 and x_2.

$$f(t) = f_0\frac{(t-x_1)(t-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1\frac{(t-x_0)(t-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2\frac{(t-x_0)(t-x_1)}{(x_2-x_0)(x_2-x_1)}$$

differentiate w.r.t t

$$f'(t) = f_0\frac{(t-x_1)+(t-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1\frac{(t-x_0)+(t-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2\frac{(t-x_0)+(t-x_1)}{(x_2-x_0)(x_2-x_1)}$$

then substitute of x_0 *instead of t*

$$f'(x_0) = f_0\frac{(x_0-x_1)+(x_0-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1\frac{(x_0-x_0)+(x_0-x_2)}{(x_1-x_0)(x_1-x_2)} + f_2\frac{(x_0-x_0)+(x_0-x_1)}{(x_2-x_0)(x_2-x_1)}$$

where $x_i - x_j = (i-j)h$ *then*

$$f'(x_0) = f_0\frac{(-h)+(-2h)}{(-f)(-2h)} + f_1\frac{(0)+(-2h)}{(h)(-h)} + f_2\frac{(0)+(-h)}{(2h)(h)} \Rightarrow f'(x_0) = \frac{-3f_0+4f_1-f_2}{2h}$$

$$b)\ E_T = \frac{h^2}{3}f'''(x) \ \ and \ \ E_r = \frac{-3e_0+4e_1-e_2}{2h} \Rightarrow E(f,h) = E_r + E_T = \frac{-3e_0+4e_1-e_2}{2h} + \frac{h^2}{3}f'''(x)$$

$$|E(f,h)| \le \left|\frac{-3e_0+4e_1-e_2}{2h}\right| + \left|\frac{h^2}{3}f'''(x)\right| \ \ where\ |e_k| \le \varepsilon \ \ and \ \ f'''(x) = 0 \ \ then$$

$$|E(f,h)| \le \frac{|-3e_0|+|4e_1|+|-e_2|}{2h} + \frac{h^2}{3}(0) \le \frac{3\varepsilon+4\varepsilon+\varepsilon}{2h} \le 10^{-2} \ \ where\ \varepsilon = 10^{-4}$$

$$\frac{4(10^{-4})}{h} \le 10^{-2} \Rightarrow h = \frac{4(10^{-4})}{10^{-2}} \Rightarrow h = 0.04$$

$$f'(1) = \frac{-3f(1)+4f(1+0.04)-f(1+2(0.04))}{2(0.04)} = \frac{-3(-3)+4(-.2864)-(-3.5856)}{0.08} = -7$$

$$Exact\ value: f' = 1 - 8x \Rightarrow f'(1) = -7$$

3) Compute $\frac{25}{2}\int_0^1 f(x)f'(x)\,dx$ where $f(x)=x^2+2$, using the Composite Trapezoidal rule, your results should be accurate to $\varepsilon=5\times10^{-1}$.

SOLUTION

$$f(x)=x^2+2 \Rightarrow F(x)=\frac{25}{2}f(x)f'(x)=\frac{25}{2}(2x)\left(x^2+2\right) \Rightarrow F(x)=25x^3+50x$$

$$\textit{Evaluate } \frac{25}{2}\int_0^1 f(x)f'(x)dx=\int_0^1\left(25x^3+50x\right)dx$$

$$\left|E_T(f,h)\right| \le \left|\frac{-(b-a)F^{(2)}(c)h^2}{12}\right| \le 0.5, \qquad F^{(2)}(x)=150x$$

$$\left|E_T(f,h)\right| \le \frac{Kh^2(1)}{12} \le 0.5, \quad \textit{where } K=\max_{0\le x\le 1}\left|150x\right|=150$$

$$h^2 \le \frac{12\times0.5}{150} \Rightarrow h=0.2$$

$$\updownarrow \_\_\_ \updownarrow \_\_\_ \updownarrow \_\_\_ \updownarrow \_\_\_ \updownarrow \_\_\_ \updownarrow$$

0 0.2 0.4 0.6 0.8 1.0

$$\mathrm{M}=\frac{b-a}{h}=\frac{1-0}{0.2}=5$$

$$T(f,h)=\frac{h}{2}\left[F(0)+F(1)\right]+h\sum_{k=1}^{4}F(x_k)=\frac{0.2}{2}\left[F(0)+F(1)\right]+0.2\left[F(0.2)+F(0.4)+F(0.6)+F(0.8)\right]$$

$$\textit{Where } F(x)=25x^3+50x$$

$$T(f,h)=\frac{0.2}{2}\left[0+75\right]+0.2\left[10.2+21.6+35.4+52.8\right]=31.5$$

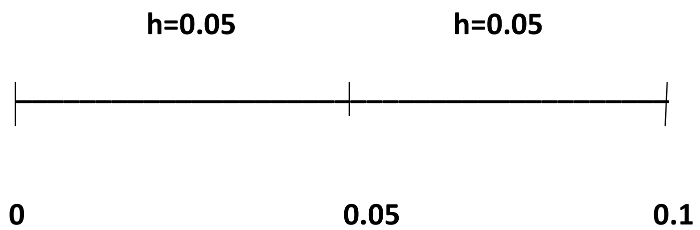
$$\textit{EXACT}: \int_0^1\left(25x^3+50x\right)dx=\frac{25}{4}x^4\Big]_0^1+\frac{50}{2}x^2\Big]_0^1=\frac{25}{4}+\frac{50}{2}=31.25$$

4) Given the following differential equation

$$y' = 3y + 3t \quad , \quad y(0) = 1$$

with the exact solution $y(t) = \frac{4}{3}e^{3t} - t - \frac{1}{3}$. Use Heun's Method to find of the solution at $t = 0.1$ with h=0.05 and compare the result with the exact solution at $t = 0.1$.

SOLUTION



$$f(t,y) = 3y + 3t \quad \text{where } y(0) = 1$$

$$*k = 0 \quad \text{from } t_0 = 0 \Rightarrow t_1 = 0.05, \quad y_0 = 1$$

$$P : p_1 = y_0 + hf(t_0, y_0) = 1 + 0.05\big(3(0) + 3(1)\big) = 1.15$$

$$C : y_1 = y_0 + \frac{h}{2}\left[f(t_0, y_0) + f(t_1, p_1)\right] = 1 + 0.025\Big[\big(3(0) + 3(1)\big) + \big(3(0.05) + 3(1.15)\big)\Big] = 1.165$$

$$*k = 1 \quad \text{from } t_1 = 0.05 \Rightarrow t_2 = 0.1, \quad y_1 = 1.165$$

$$P : p_2 = y_1 + hf(t_1, y_1) = 1.165 + 0.05\big(3(0.05) + 3(1.165)\big) = 1.34725$$

$$C : y_2 = y_1 + \frac{h}{2}\left[f(t_1, y_1) + f(t_2, p_2)\right] = 1.165 + 0.025\Big[\big(3(0.05) + 3(1.165)\big) + \big(3(0.1) + 3(1.34725)\big)\Big] = 1.36446$$

$$Exact : y(0.1) = \frac{4}{3}e^{3(0.1)} - 0.1 - \frac{1}{3} = 1.36647841$$