NONLINEAR SYSTEMS

ITERATION FOR NONLINEAR SYSTEMS

Many problems ın engineering and science require the solution of a system of nonlinear equations.
Consider a system of two nonlinear equations.

$$f(x,y) = 0$$
$$g(x,y) = 0$$

The problem can be stated as follows

Given the continuous functions $f(x,y)$ and $g(x,y)$, find the values $x = x^*$ and $y = y^*$ such that $f(x^*,y^*) = 0$ and $g(x^*,y^*) = 0$

FIXED POINT ITERATION FOR NONLINEAR SYSTEMS

FIXED POINT ITERATION

Given

$$f_1(x,y) = 0$$
$$f_2(x,y) = 0 \qquad (1)$$

for applying the method of iteration the system (1) is reduced to the form

$$x = g_1(x,y)$$
$$y = g_2(x,y) \qquad (2)$$

The algorithm of solution is given by the formulas

$$x_{n+1} = g_1(x_n, y_n)$$
$$y_{n+1} = g_2(x_n, y_n) \qquad (3)$$

Where x_0, y_0 is some initial approximation.

Theorem :

Let is some closed neighbourhood R, $(a \leq x \leq A \ , \ b \leq y \leq B)$ there be one and only one solution $x = \xi \ , \ \ y = \eta$ for the system (2) if,

1. The functions $g_1(x, y) \ \ and \ \ g_2(x, y)$ are defined and continuously differentiable in R.
2. The initial approximations $x_0, \ y_0$ and all successive approximations $x_n, \ y_n \ \ (n = 1,2, \dots)$ belong to R.
3. The following inequalities are fulfilled in R

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| < 1$$

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| < 1 \qquad (4) \qquad \text{Or} \quad \|J(G(x, y))\| < 1$$

Then the process of successive approximations (3) converges to the solution

$$x = \xi \ , \ \ y = \eta$$

of the system.

Constructing iterative functions for the system

Consider,

$$g_1(x,y) = x + \alpha f_1(x,y) + \beta f_2(x,y)$$
$$g_2(x,y) = y + \gamma f_1(x,y) + \delta f_2(x,y)$$

where $\alpha\delta \neq \beta\gamma$

Find the coefficients $\alpha,\ \beta,\ \gamma\ \ and\ \delta$ as approximate solutions of the following system of equations at (x_0, y_0) .

$$\frac{\partial g_1}{\partial x} = 0 \quad \Rightarrow \quad 1 + \alpha \frac{\partial f_1}{\partial x} + \beta \frac{\partial f_2}{\partial x} = 0$$
$$\frac{\partial g_1}{\partial y} = 0 \quad \Rightarrow \quad \alpha \frac{\partial f_1}{\partial y} + \beta \frac{\partial f_2}{\partial y} = 0$$
$$\frac{\partial g_2}{\partial x} = 0 \quad \Rightarrow \quad \gamma \frac{\partial f_1}{\partial x} + \delta \frac{\partial f_2}{\partial x} = 0$$
$$\frac{\partial g_2}{\partial y} = 0 \quad \Rightarrow \quad 1 + \gamma \frac{\partial f_1}{\partial y} + \delta \frac{\partial f_2}{\partial y} = 0$$

Example:

Perform 2 iterations of a convergent fixed point method to approximate the root of the following system.

$$\begin{aligned} f_1(x,y) &= \sqrt{x+y} + xy - 2.3 = 0 \\ f_2(x,y) &= x^3 - y^3 - 9xy = 0 \end{aligned} \qquad (1)$$

$$by \;\; taking \;\; (x_0, y_0) = (1.92, 0.44)$$

Solution:

Construct iterative function $g_1(x,y)$ *and* $g_2(x,y)$ where ,

$$\begin{aligned} g_1(x,y) &= x + \alpha(\sqrt{x+y} + xy - 2.3) + \beta(x^3 - y^3 - 9xy) \\ g_2(x,y) &= y + \gamma(\sqrt{x+y} + xy - 2.3) + \delta(x^3 - y^3 - 9xy) \end{aligned} \qquad (2)$$

$$\frac{\partial g_1}{\partial x} = 0 \quad \Rightarrow \quad 1 + \alpha\left(\frac{1}{2\sqrt{x+y}} + y\right) + \beta(3x^2 - 9y) = 0 \qquad \ldots.(3)$$

$$\frac{\partial g_1}{\partial y} = 0 \quad \Rightarrow \quad \alpha\left(\frac{1}{2\sqrt{x+y}} + x\right) + \beta(-3y^2 - 9x) = 0 \qquad \ldots..(4)$$

$$\frac{\partial g_2}{\partial x} = 0 \quad \Rightarrow \quad \gamma\left(\frac{1}{2\sqrt{x+y}} + y\right) + \delta(3x^2 - 9y) = 0 \qquad \ldots\ldots(5)$$

$$\frac{\partial g_2}{\partial y} = 0 \quad \Rightarrow \quad 1 + \gamma\left(\frac{1}{2\sqrt{x+y}} + x\right) + \delta(-3y^2 - 9x) = 0 \qquad \ldots\ldots.(6)$$

where,

$(x_0, y_0) = (1.92, 0.44)$ *substitute in* $(3), (4), (5)$ *and* (6)

from (3) and (4)

$$\begin{array}{ll} 17.86 & \times\,(0.765\alpha + 7.0992\beta = -1) \\ 7.0992 & \times\,(2.245\alpha - \ \ 17.86\beta \ \ = \ \ \ 0) \\ \_\_ + & \overline{} \\ & 29.6\alpha = -17.86 \qquad \Rightarrow \alpha = -0.603 \\ & \qquad\qquad\qquad\qquad\quad \beta \ = -0.0757 \end{array}$$

from (5) and (6)

$$\begin{array}{ll} 17.86 & \times\,(0.765\gamma + 7.0992\delta = \ \ \ 0) \\ 7.0992 & \times\,(2.245\gamma - \ \ 17.86\delta \ \ = \ -1) \\ \_\_ + & \overline{} \\ & 29.6\gamma = -7.0992 \qquad \Rightarrow \gamma = -0.239 \\ & \qquad\qquad\qquad\qquad\quad \delta \ = 0.0258 \end{array}$$

then,

$$g_1(x,y) = x - 0.603(\sqrt{x+y} + xy - 2.3) - 0.0757(x^3 - y^3 - 9xy)$$
$$g_2(x,y) = y - 0.239(\sqrt{x+y} + xy - 2.3) + 0.0258(x^3 - y^3 - 9xy)$$

$$such\ \ that\ \ \|J(G(x,y))\| < 1$$

$$where$$
$$x_{n+1} = g_1(x_n, y_n)$$
$$y_{n+1} = g_2(x_n, y_n)$$

Iteration 1: $(x_0, y_0) = (1.92,0.44)$

$$x_1 = 1.92 - 0.603\left(\sqrt{1.92 + 0.44} + (1.92 \times 0.44) - 2.3\right) - 0.0757\left(1.92^3 - 0.44^3 - 9(1.92 \times 0.44)\right) = 1.917$$
$$y_1 = 0.44 - 0.239\left(\sqrt{1.92 + 0.44} + (1.92 \times 0.44) - 2.3\right) + 0.0258\left(1.92^3 - 0.44^3 - 9(1.92 \times 0.44)\right) = 0.404$$

Iteration 2: $(x_1, y_1) = (1.917,0.404)$

$$x_2 = 1.917 - 0.603\left(\sqrt{1.917 + 0.404} + (1.917 \times 0.404) - 2.3\right) - 0.0757\left(1.917^3 - 0.404^3 - 9(1.917 \times 0.404)\right) = 1.91757$$
$$y_2 = 0.404 - 0.239\left(\sqrt{1.910 + 0.404} + (1.917 \times 0.404) - 2.3\right) + 0.0258\left(1.917^3 - 0.404^3 - 9(1.917 \times 0.404)\right) = 0.4047$$

Check error $\varepsilon = \|f_i(x_2, y_2)\|_\infty = \max\{|f_1(x_2, y_2), ||f_2(x_2, y_2)|\}$

$$|f_1(1.91757,0.4047)| = 5.9818 \times 10^{-5}$$
$$|f_2(1.91757,0.4047)| = 4 \times 10^{-4}$$

$$\varepsilon = \max\{5.9818 \times 10^{-5}, 4 \times 10^{-4}\} = 4 \times 10^{-4}$$

Example: (fixed point system)

Construct the iterative functions to approximate the root of

$$f_1(x,y) = \cos(x+5y) - (x+y^2) + 2.46 = 0$$
$$(1)$$
$$f_2(x,y) = y^2(2-x) - x^3 = 0$$

using fixed point iteration. Perform two iterations with $(x_0, y_0) = (0.9, 1.5)$

Solution:

where

$$g_1(x,y) = x + \alpha[\cos(x+5y) - (x+y^2) + 2.46] + \beta[y^2(2-x) - x^3] \quad \ldots\ldots\ldots(2)$$
$$g_2(x,y) = y + \gamma[\cos(x+5y) - (x+y^2) + 2.46] + \delta[y^2(2-x) - x^3] \quad \ldots\ldots\ldots(3)$$

$$\frac{\partial g_1}{\partial x} = 0 \quad \Rightarrow \quad 1 + \alpha[-\sin(x+5y) - 1] + \beta[-y^2 - 3x^2] = 0 \qquad \ldots\ldots\ldots(4)$$

$$\frac{\partial g_1}{\partial y} = 0 \quad \Rightarrow \quad \alpha[-5\sin(x+5y) - 2y] + \beta[4y - 2xy] = 0 \qquad \ldots\ldots\ldots\ldots(5)$$

$$\frac{\partial g_2}{\partial x} = 0 \quad \Rightarrow \quad \gamma[-\sin(x+5y) - 1] + \delta[-y^2 - 3x^2] = 0 \qquad \ldots\ldots\ldots(6)$$

$$\frac{\partial g_2}{\partial y} = 0 \quad \Rightarrow \quad 1 + \gamma[-5\sin(x+5y) - 2y] + \delta[4y - 2xy] = 0 \qquad \ldots\ldots\ldots\ldots(7)$$

substitute

$(x_0, y_0) = (0.9, 1.5)$ *in* $(4), (5), (6)$ *and* (7) *to find* α, β, γ *and* δ *(exercise)*

Where $\alpha = 0.082,\ \beta = 0.1811,\ \gamma = 0.1166$ *and* $\delta = -0.0461$ *then*

$$x_{n+1} = x_n + 0.082[\cos(x_n + 5y_n) - (x_n + y_n^2) + 2.46] + 0.1811[y_n^2(2 - x_n) - x_n^3]$$

$$y_{n+1} = y_n + 0.1166[\cos(x_n + 5y_n) - (x_n + y_n^2) + 2.46] - 0.0461[y_n^2(2 - x_n) - x_n^3]$$

$$Start \quad (x_0, y_0) = (0.9, 1.5) \;\; to \;\; find$$
$$(x_1, y_1) = (1.117, 1.278)$$
$$(x_2, y_2) = (1.129, 1.281) \quad (exercise) \;\; also \;\; calculate \;\; \|f_i(1.129, 1.281)\|_\infty \;\; for \;\; i = 1,2$$

EXERCİSE: Solve the system

$$(x-1)^3 - y = 0$$
$$(x-1)^2 + \left(y - \frac{1}{2}\right)^2 = 4$$

$with \quad (x_0, y_0) = (2.3, 1.9)$ by using fixed point method . Perform 2 iteration.

Example: Construct the iterative functions $g_1(x, y)$ and $g_2(x, y)$ to approximate to root of

$$f_1(x, y) = 2x^3 - 12x - y - 1 = 0$$

$$f_2(x, y) = 3y^2 - 6y - x - 3 = 0$$

Using Fixed Point Iteration. Perform two iterations with $(x_0, y_0) = (2.5, 2.5)$

Solution:

Now, we construct the iterative functions,

$$g_1(x, y) = x + \alpha f_1(x, y) + \beta f_2(x, y) \qquad g_1(x, y) = x + \alpha(2x^3 - 12x - y - 1) + \beta(3y^2 - 6y - x - 3)$$

$$g_2(x, y) = y + \gamma f_1(x, y) + \delta f_2(x, y) \qquad g_2(x, y) = y + \gamma(2x^3 - 12x - y - 1) + \delta(3y^2 - 6y - x - 3)$$

$$\frac{\partial g_1}{\partial x} = 1 + \alpha(6x^2 - 12) - \beta \qquad \frac{\partial g_1}{\partial y} = -\alpha + \beta(6y - 6)$$

$$\frac{\partial g_2}{\partial x} = \gamma(6x^2 - 12) - \delta \qquad \frac{\partial g_2}{\partial y} = 1 - \gamma + \delta(6y - 6)$$

We should find $\alpha, \beta, \gamma, \delta$ from the $J(g_1, g_2)_{(x_0, y_0)} = 0$

$$J(g_1, g_2) = \begin{pmatrix} 1 + \alpha(6x^2 - 12) - \beta & -\alpha + \beta(6y - 6) \\ \gamma(6x^2 - 12) - \delta & 1 - \gamma + \delta(6y - 6) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$J(g_1, g_2)_{(2.5, 2.5)} = \begin{pmatrix} 1 + \alpha(6(2.5)^2 - 12) - \beta & -\alpha + \beta(6(2.5) - 6) \\ \gamma(6(2.5)^2 - 12) - \delta & 1 - \gamma + \delta(6(2.5) - 6) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$J(g_1, g_2)_{(2.5,2.5)} = \begin{pmatrix} 1 + 25.5\alpha - \beta & -\alpha + 9\beta \\ 25.5\gamma - \delta & 1 - \gamma + 9\delta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$1 + 25.5\alpha - \beta = 0, -\alpha + 9\beta = 0, 25.5\gamma - \delta = 0, 1 - \gamma + 9\delta = 0$$

$$1 + 25.5\alpha - \beta = 0$$
$$-\alpha + 9\beta = 0$$

$$9 + 228.5\alpha = 0, \alpha = -0.0394$$

$$-0.0394 + 9\beta = 0, \beta = 0.00438$$

$$25.5\gamma - \delta = 0$$
$$1 - \gamma + 9\delta = 0$$

$$1 + 228.5\gamma = 0, \gamma = -0.00438$$

$$25.5(-0.00438) - \delta = 0, \delta = -0.11169$$

Substitute $\alpha, \beta, \gamma, \delta$ in $g_1(x, y)$ and $g_2(x, y)$

$$g_1(x, y) = x - 0.0394(2x^3 - 12x - y - 1) + 0.00438(3y^2 - 6y - x - 3)$$

$$g_2(x, y) = y - 0.00438(2x^3 - 12x - y - 1) - 0.11169(3y^2 - 6y - x - 3)$$

The iteration is

$$x_{n+1} = g_1(x_n, y_n)$$
$$y_{n+1} = g_2(x_n, y_n)$$

$x_0 = 2.5, y_0 = 2.5$

Iteration1.

$$x_1 = g_1(2.5,2.5) = 2.5 - 0.0394(2(2.5)^3 - 12(2.5) - (2.5) - 1) + 0.00438(3(2.5)^2 - 6(2.5) - (2.5) - 3) = 2.58099$$
$$y_1 = g_2(2.5,2.5) = 2.5 - 0.00438(2(2.5)^3 - 12(2.5) - (2.5) - 1) - 0.11169(3(2.5)^2 - 6(2.5) - (2.5) - 3) = 2.79401$$

$$f_1(x_1, y_1) = 2(2.58099)^3 - 12(2.58099) - 2.79401 - 1 = -0.37931$$

$$f_2(x_1, y_1) = 3(2.79401)^2 - 6(2.79401) - 2.58099 - 3 = 1.07443$$

$$\varepsilon = \left\| \begin{matrix} f_1(x_1, y_1) \\ f_2(x_1, y_1) \end{matrix} \right\|_\infty = \max \begin{Bmatrix} |f_1(x_1, y_1)| \\ |f_2(x_1, y_1)| \end{Bmatrix} = 1.07443 \qquad \left\| \begin{matrix} \Delta x \\ \Delta y \end{matrix} \right\|_\infty = \left\| \begin{matrix} x_1 - x_0 \\ y_1 - y_0 \end{matrix} \right\| = \left\| \begin{matrix} |0.08099| \\ |0.29401| \end{matrix} \right\| = 0.29401$$

Iteration2.

$$x_2 = g_1(x_1, y_1) = g_1(2.58099, 2.79401) = 2.60064$$
$$y_2 = g_2(x_1, y_1) = g_2(2.58099, 2.79401) = 2.46265$$

$$f_1(x_2, y_2) = 2(2.60064)^3 - 12(2.60064) - 2.46265 - 1 = 0.50764$$
$$f_2(x_2, y_2) = 3(2.46265)^2 - 6(2.46265) - 2.60064 - 3 = -2.18261$$

$$\varepsilon = \left\| \begin{matrix} f_1(x_2, y_2) \\ f_2(x_2, y_2) \end{matrix} \right\|_\infty = 2.18261 \qquad \left\| \begin{matrix} \Delta x \\ \Delta y \end{matrix} \right\|_\infty = \left\| \begin{matrix} x_2 - x_1 \\ y_2 - y_1 \end{matrix} \right\| = \left\| \begin{matrix} |0.01965| \\ |0.33136| \end{matrix} \right\| = 0.33136$$

NEWTON METHOD FOR NONLINEAR SYSTEMS

Newton's Method for Nonlinear Systems

Consider the system of nonlinear equations

$$\begin{aligned} u &= f_1(x, y) \\ v &= f_2(x, y) \end{aligned} \qquad (1)$$

Which can be considered a transformation from the xy- plane into uv-plane with starting point (x_0, y_0) where image is the point (u_0, v_0) .

If both $f_1(x, y)$ and $f_2(x, y)$ have continuous partial derivatives then,

$$u - u_0 \cong \frac{\partial f_1(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f_1(x_0, y_0)}{\partial y}(y - y_0)$$

$$v - v_0 \cong \frac{\partial f_2(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f_2(x_0, y_0)}{\partial y}(y - y_0)$$

Then the Jacobian Matrix $J(x_0, y_0)$ is used , this relationship is easier to visualize.

$$\begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1(x_0,y_0)}{\partial x} & \frac{\partial f_1(x_0,y_0)}{\partial y} \\ \frac{\partial f_2(x_0,y_0)}{\partial x} & \frac{\partial f_2(x_0,y_0)}{\partial y} \end{pmatrix} \begin{pmatrix} (x - x_0) \\ (y - y_0) \end{pmatrix} \qquad (2)$$

If the system in (1) is written as a vector $V = F(x)$ the Jacobian $J(x, y)$ is the two dimensional analogy of the derivative (2) can be written as

$$\Delta F \cong J(x_0, y_0)\Delta x \qquad (3)$$

We now use (3) to derive Newton's Method in two dimensions.
Consider the system (1) with set equal to zero.

$$\begin{aligned} 0 &= f_1(p, q) \\ 0 &= f_2(p, q) \end{aligned} \qquad (4)$$

Suppose that (p, q) is a solution of (4) .
To develop Newton's Method for solving (4) , we need to consider small changes in the functions near the point (p_0, q_0).

$$u - u_0 = \Delta u \qquad \Delta p = x - p_0$$
$$v - v_0 = \Delta v \qquad \Delta q = y - q_0 \qquad (5)$$

Set $(x, y) = (p, q)$ in (1) and use (4) to see that $(u, v) = (0,0)$. Hence the changes in the dependent variable are

$$u - u_0 = f_1(p, q) - f_1(p_0, q_0) = 0 - f_1(p_0, q_0)$$
$$v - v_0 = f_2(p, q) - f_2(p_0, q_0) = 0 - f_2(p_0, q_0) \qquad (6)$$

Use the result of (6) in (2) to get the linear transformation

$$\begin{pmatrix} \dfrac{\partial f_1(p_0,q_0)}{\partial x} & \dfrac{\partial f_1(p_0,q_0)}{\partial y} \\ \dfrac{\partial f_2(p_0,q_0)}{\partial x} & \dfrac{\partial f_2(p_0,q_0)}{\partial y} \end{pmatrix} \begin{pmatrix} \Delta p \\ \Delta q \end{pmatrix} = - \begin{pmatrix} f_1(p_0,q_0) \\ f_2(p_0,q_0) \end{pmatrix}$$

$$J(p_0,q_0) \qquad\qquad \Delta P \qquad\qquad -\Delta F$$

Then,

$$J(p_0,q_0)\Delta P = -\Delta F \quad solve \;\; \Delta P, \quad \boxed{\Delta P = -[J(p_0,q_0)]^{-1}\Delta F}$$

OUTLINE OF NEWTON'S METHOD

Suppose that P_k as been obtained

Step 1: Evaluate the function

$$F(P_k) = \begin{pmatrix} f_1(p_k, q_k) \\ f_2(p_k, q_k) \end{pmatrix}$$

Step 2: Evaluate the Jacobian Matrix

$$\begin{pmatrix} \dfrac{\partial f_1(p_k, q_k)}{\partial x} & \dfrac{\partial f_1(p_k, q_k)}{\partial y} \\ \dfrac{\partial f_2(p_k, q_k)}{\partial x} & \dfrac{\partial f_2(p_k, q_k)}{\partial y} \end{pmatrix}$$

Step 3: Solve

$$J(P_k)\Delta P = -\Delta F(P_k) \quad for \quad \Delta P$$

Step 4: Compute the next point

$$P_{k+1} = P_k + \Delta P$$

Now , repeat the process

Example: Use starting point $(p_0, q_0) = (3.795, 4.594)$ for the Newton's method to solve the nonlinear systems

$$x^2 - \frac{2}{3}y^2 - \frac{1}{3} = 0$$

$$\frac{x^2}{2} - x + y - 8 = 0$$

Compute (p_1, q_1) *and* (p_2, q_2)

Solution:

where,

$$f_1(x,y) = x^2 - \frac{2}{3}y^2 - \frac{1}{3} \qquad \frac{\partial f_1}{\partial x} = 2x \quad , \quad \frac{\partial f_1}{\partial y} = -\frac{4}{3}y$$

$$f_2(x,y) = \frac{x^2}{2} - x + y - 8 \qquad \frac{\partial f_2}{\partial x} = x - 1 \quad , \quad \frac{\partial f_2}{\partial y} = 1$$

Iteration 1:

$$F(3.795,4.594) = \begin{pmatrix} 3.795^2 - \frac{2}{3}4.594^2 - \frac{1}{3} \\ \frac{3.795^2}{2} - 3.795 + 4.594 - 8 \end{pmatrix} = \begin{pmatrix} -1.19 \times 10^{-3} \\ 1.25 \times 10^{-5} \end{pmatrix}$$

$$J(3.795,4.594) = \begin{pmatrix} 2(3.795) & \frac{-4}{3}(4.594) \\ 3.795 - 1 & 1 \end{pmatrix} = \begin{pmatrix} 7.59 & -6.125 \\ 2.795 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 7.59 & -6.125 \\ 2.795 & 1 \end{pmatrix} \begin{pmatrix} \Delta p \\ \Delta q \end{pmatrix} = -\begin{pmatrix} -1.19 \times 10^{-3} \\ 1.25 \times 10^{-5} \end{pmatrix}$$

$$7.59\Delta p - 6.125\Delta q = 1.19 \times 10^{-3}$$
$$2.795\Delta p + \Delta q = -1.25 \times 10^{-5} \qquad (exercise)$$

$$where, \qquad \Delta p = 4.50613 \times 10^{-5} \;, \quad \Delta q = -1.38446 \times 10^{-4}$$

$$\Delta p = p_1 - p_0 \quad \Rightarrow \quad p_1 = \Delta p + p_0 = 4.50613 \times 10^{-5} + 3.795 \approx 3.7951$$

$$\Delta q = q_1 - q_0 \quad \Rightarrow \quad q_1 = \Delta q + q_0 = -1.38446 \times 10^{-4} + 4.594 \approx 4.59386$$

$$(p_1, q_1) = (3.7951, 4.59386)$$

$$\left\| \begin{matrix} \Delta p_1 \\ \Delta q_1 \end{matrix} \right\| = \max \left\{ \begin{matrix} |4.50613 \times 10^{-5}| \\ |-1.38446 \times 10^{-4}| \end{matrix} \right\} = 1.38446 \times 10^{-4}$$

Iteration 2:

$$F(3.7951, 4.59386) = \begin{pmatrix} 3.7951^2 - \frac{2}{3} 4.59386^2 - \frac{1}{3} \\ \frac{3.7951^2}{2} - 3.7951 + 4.59386 - 8 \end{pmatrix} = \begin{pmatrix} 2.01 \times 10^{-3} \\ -1.079 \times 10^{-4} \end{pmatrix}$$

$$J(3.7951, 4.59386) = \begin{pmatrix} 2(3.7951) & \frac{-4}{3}(4.59386) \\ 3.7951 - 1 & 1 \end{pmatrix} = \begin{pmatrix} 7.5902 & -6.1248 \\ 2.7951 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 7.5902 & -6.1248 \\ 2.7951 & 1 \end{pmatrix}\begin{pmatrix} \Delta p \\ \Delta q \end{pmatrix} = -\begin{pmatrix} 2.01 \times 10^{-3} \\ -1.079 \times 10^{-4} \end{pmatrix}$$

$$7.5902\Delta p - 6.1248\Delta q = -2.01 \times 10^{-3}$$
$$2.7951\Delta p + \Delta q = 1.079 \times 10^{-4} \qquad (exercise)$$

$$\Delta p = p_2 - p_1 \quad \Rightarrow \quad p_2 = \Delta p + p_1 = 3.795$$
$$\Delta q = q_2 - q_1 \quad \Rightarrow \quad q_2 = \Delta q + q_1 = 4.5935$$
$$(p_2, q_2) = (3.795, 4.5935)$$

$$\left\| \begin{matrix} \Delta p_2 \\ \Delta q_2 \end{matrix} \right\| = \max\begin{Bmatrix} |-0.0001| \\ |-0.00036| \end{Bmatrix} = 0.00036$$

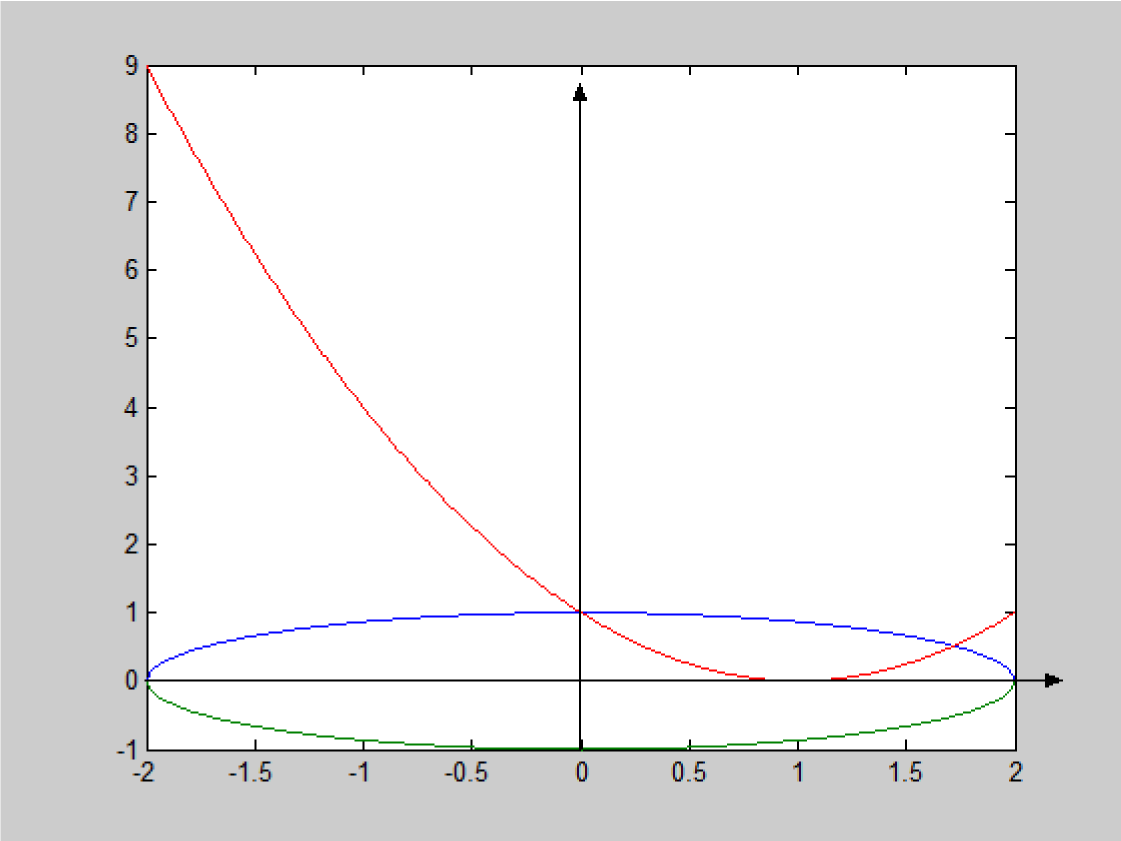
Example: Approximate to root of

$$f_1(x,y) = x^2 + 4y^2 - 4 = 0$$

$$f_2(x,y) = x^2 - 2x - y + 1 = 0$$

Using Newton Method. Perform three iterations with $(x_0, y_0) = (1.5,0.5)$.

Solution: When we sketch the graph of this system from the Matlab, we can see $(x_0, y_0) = (1.5,0.5)$.



We start with $(x_0, y_0) = (1.5, 0.5)$

Iteration1.

$$F(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} x^2 + 4y^2 - 4 \\ x^2 - 2x - y + 1 \end{bmatrix}$$

$$F(1.5,0.5) = \begin{bmatrix} f_1(1.5,0.5) \\ f_2(1.5,0.5) \end{bmatrix} = \begin{bmatrix} -0.75 \\ -0.25 \end{bmatrix}$$

$$J(f_1, f_2) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 8y \\ 2x - 2 & -1 \end{bmatrix}$$

$$J(f_1, f_2)_{(1.5,0.5)} = \begin{bmatrix} 2(1.5) & 8(0.5) \\ 2(1.5) - 2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$$

$$J(f_1, f_2)_{(1.5,0.5)} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -F(1.5,0.5)$$

$$\begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}$$

Now, we solve this system from any way. We use Inverse Matrix Method.

When we check the determinant of $J(f_1, f_2)$ $\det(J) = \begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} = -3 - 4 = -7 \neq 0$

so the inverse of this matrix exists.

$$J^{-1}(f_1, f_2)_{(1.5,0.5)} = -\frac{1}{7}\begin{bmatrix} -1 & -4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0.14286 & 0.57143 \\ 0.14286 & -0.42857 \end{bmatrix}$$

Then $\begin{bmatrix} \Delta x_1 \\ \Delta y_1 \end{bmatrix} = J^{-1}(f_1, f_2)_{(1.5,0.5)}\big(-F(1.5,0.5)\big) = \begin{bmatrix} 0.14286 & 0.57143 \\ 0.14286 & -0.42857 \end{bmatrix}\begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0.2500025 \\ 0.0000025 \end{bmatrix}$

$$x_1 = x_0 + \Delta x_1 = 1.5 + 0.2500025 = 1.7500025$$
$$y_1 = y_0 + \Delta y_1 = 0.5 + 0.0000025 = 0.5000025$$

$$\left\|\begin{matrix} \Delta x_1 \\ \Delta y_1 \end{matrix}\right\| = \max\left\{\begin{matrix} |0.2500025| \\ |0.0000025| \end{matrix}\right\} = 0.2500025$$

Iteration2.

$$F(x_1, y_1) = F(1.7500025, 0.5000025) = \begin{bmatrix} 0.062519 \\ 0.062501 \end{bmatrix}$$

$$J(f_1, f_2)_{(1.7500025, 0.5000025)} = \begin{bmatrix} 3.500005 & 4.00002 \\ 1.500005 & -1 \end{bmatrix} \qquad \det(J) = \begin{vmatrix} 3.500005 & 4.00002 \\ 1.500005 & -1 \end{vmatrix} = -9.500055$$

$$J(f_1, f_2)_{(1.7500025, 0.5000025)} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -F(1.7500025, 0.5000025)$$

$$J^{-1}(f_1, f_2) = \begin{bmatrix} 0.105263 & 0.421052 \\ 0.15894 & -0.368419 \end{bmatrix}$$

$$\begin{bmatrix} \Delta x_2 \\ \Delta y_2 \end{bmatrix} = J^{-1}(f_1, f_2)\big(-F(1.7500025, 0.5000025)\big) = \begin{bmatrix} 0.105263 & 0.421052 \\ 0.15894 & -0.368419 \end{bmatrix} \begin{bmatrix} -0.062519 \\ -0.062501 \end{bmatrix} = \begin{bmatrix} -0.032897 \\ 0.013155 \end{bmatrix}$$

$$x_2 = x_1 + \Delta x_2 = 1.7500025 - 0.032897 = 1.7171055$$
$$y_2 = y_1 + \Delta y_2 = 0.5000025 + 0.013155 = 0.5131575$$

$$\left\| \begin{matrix} \Delta x_2 \\ \Delta y_2 \end{matrix} \right\| = \max \begin{Bmatrix} |-0.032897| \\ |0.013155| \end{Bmatrix} = 0.032897$$

Iteration3.

$$F(x_2, y_2) = F(1.7171055,0.5131575) = \begin{bmatrix} 0.001774 \\ 0.001083 \end{bmatrix}$$

$$J(f_1, f_2)_{(1.7171055,0.5131575)} = \begin{bmatrix} 3.434211 & 4.10526 \\ 1.434211 & -1 \end{bmatrix} \qquad \det(J) = \begin{vmatrix} 3.434211 & 4.10526 \\ 1.434211 & -1 \end{vmatrix} = -9.32202$$

$$J(f_1, f_2)_{\ (1.7171055,0.5131575)} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = -F(1.7171055,0.5131575)$$

$$J^{-1}(f_1, f_2) = \begin{bmatrix} 0.107273 & 0.440383 \\ 0.153852 & -0.368398 \end{bmatrix}$$

$$\begin{bmatrix} \Delta x_3 \\ \Delta y_3 \end{bmatrix} = J^{-1}(f_1, f_2)\big(-F(1.7171055,0.5131575)\big) = \begin{bmatrix} 0.107273 & 0.440383 \\ 0.153852 & -0.368398 \end{bmatrix} \begin{bmatrix} 0.001774 \\ 0.001083 \end{bmatrix} = \begin{bmatrix} -0.000667 \\ 0.000126 \end{bmatrix}$$

$$x_3 = x_2 + \Delta x_3 = 1.7171055 - 0.000667 = 1.7164385$$
$$y_3 = y_2 + \Delta y_3 = 0.5131575 + 0.000126 = 0.5132835$$

$$\left\| \begin{matrix} \Delta x_3 \\ \Delta y_3 \end{matrix} \right\| = \max \begin{Bmatrix} |-0.000667| \\ |0.000126| \end{Bmatrix} = 0.000667$$