THE SOLUTION OF NONLINEAR EQUATIONS $f(x) = 0$

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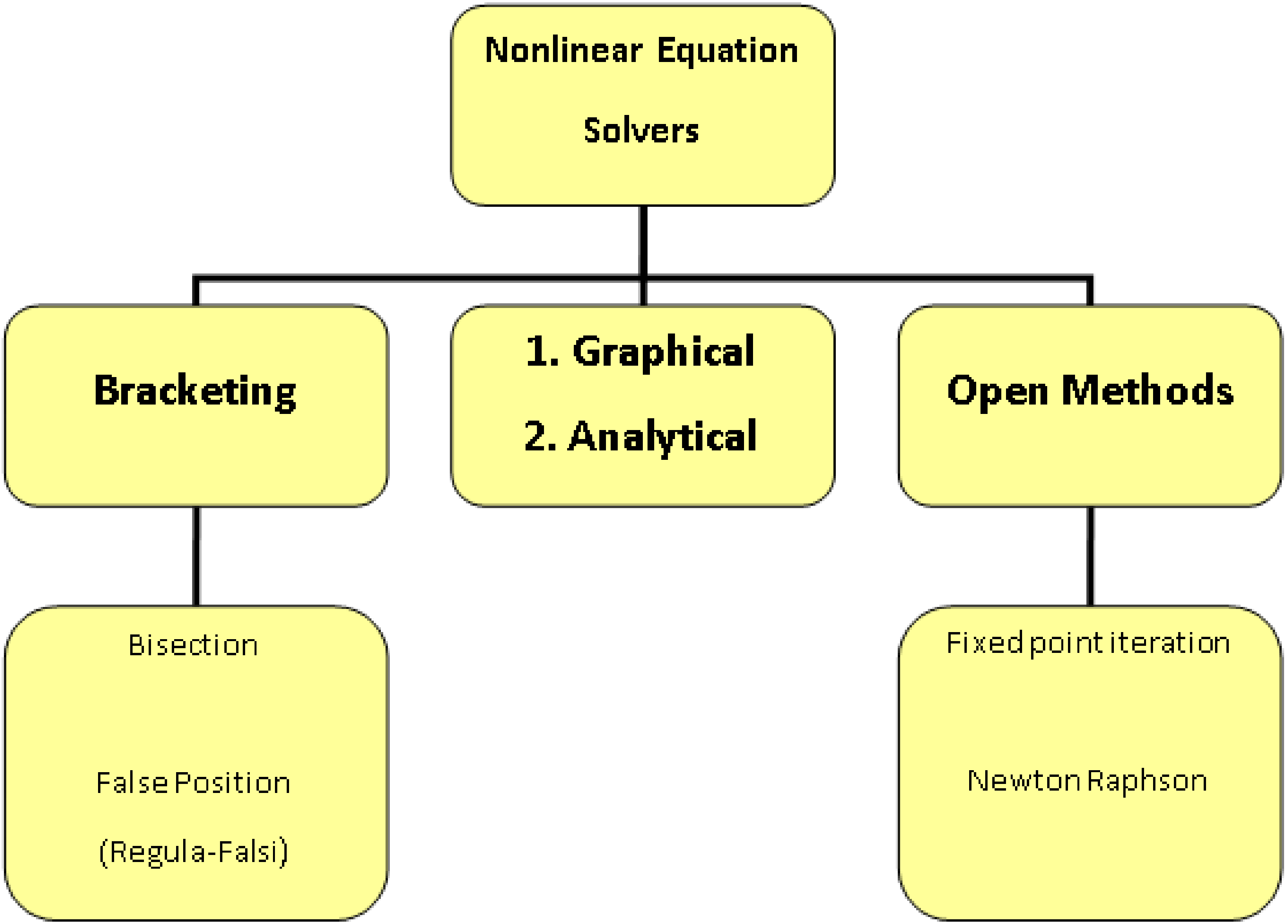


- The root of an equation is the value of x that $f(x) = 0$.

- Roots are called the zeros of equation.

- There are many functions for which the root cannot be determined so easily.

Methods for Determining Roots

Noncomputer methods.

1. Graphical method
2. Analytical method

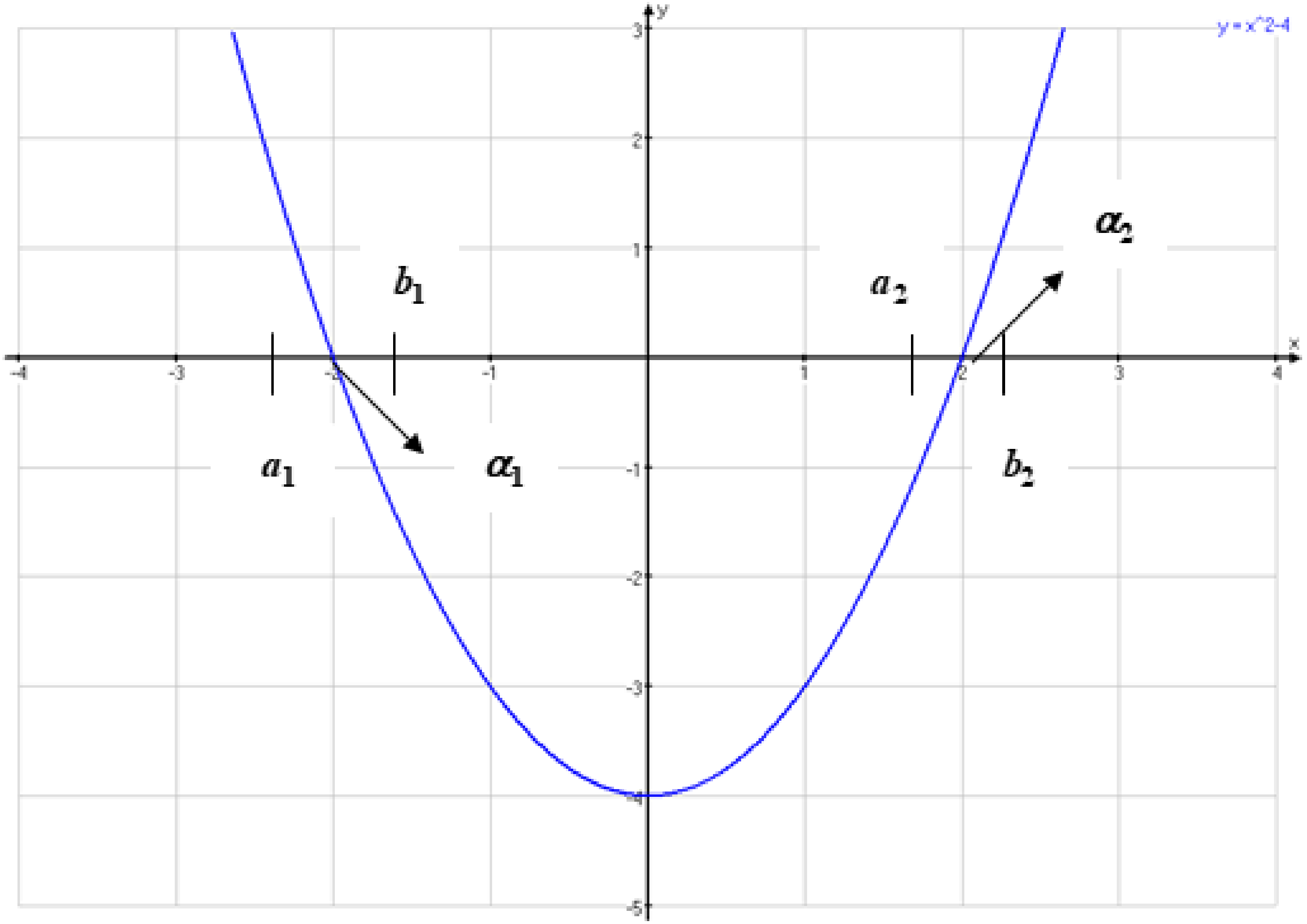
Computer methods.

1. Bisection method
2. False Position method
3. Fixed point iteration
4. Newton’s method
5. Secant Method

NONCOMPUTER METHODS

Separation of roots

To separate the roots of $f(x) = 0$ is to divide the whole domain of permissible values into intervals in each of which there is only one root.

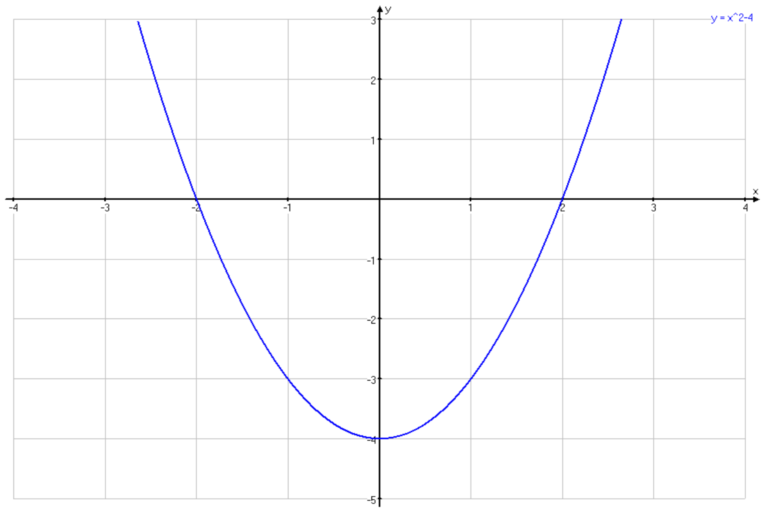


For the solution of the roots, we will use,

1. Graphical method
2. Analytical method

GRAPHICAL METHOD OF SEPARATING ROOTS

1ST TECHNIQUE: It is easy to separate the roots if the graph of the function is constructed

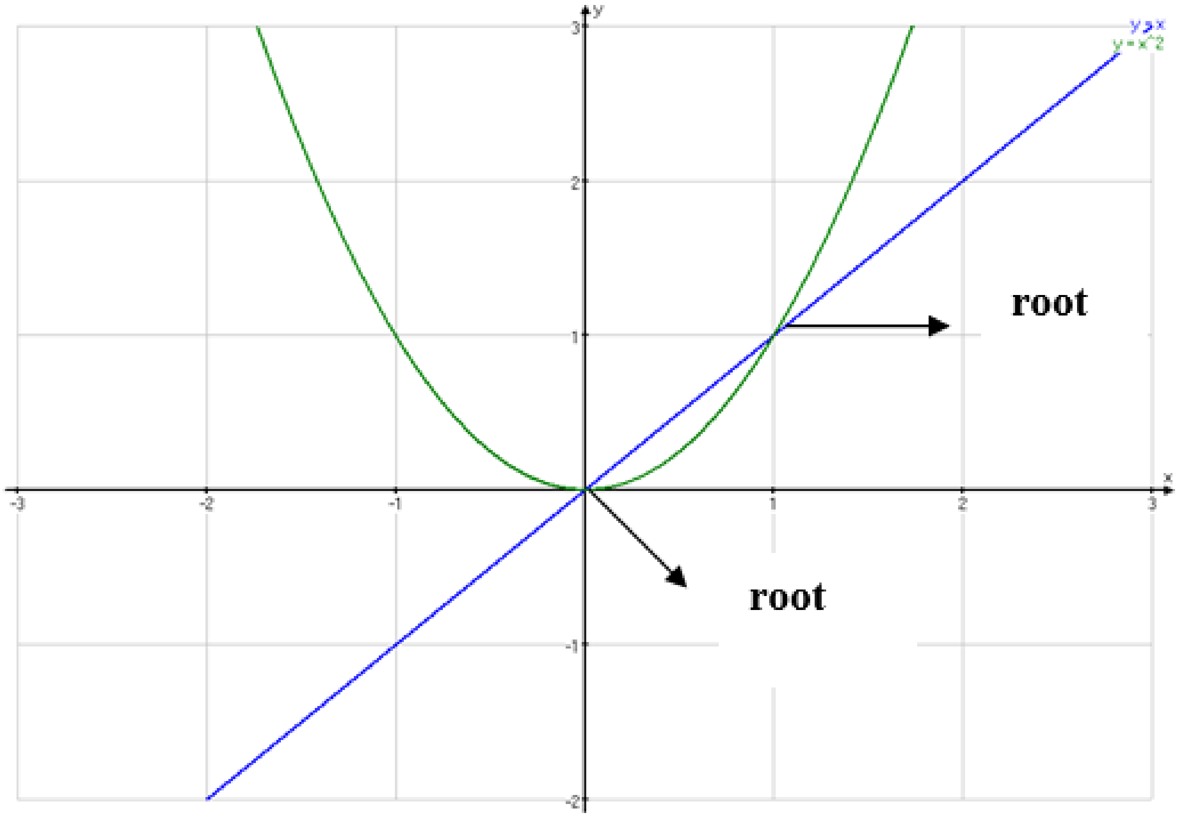


Where α_1 and α_2 are the roots of $y = f(x) = 0.$

2nd TECHNIQUE : Given $y = f(x)$ and $f(x) = r(x) - s(x) = 0$, we can write

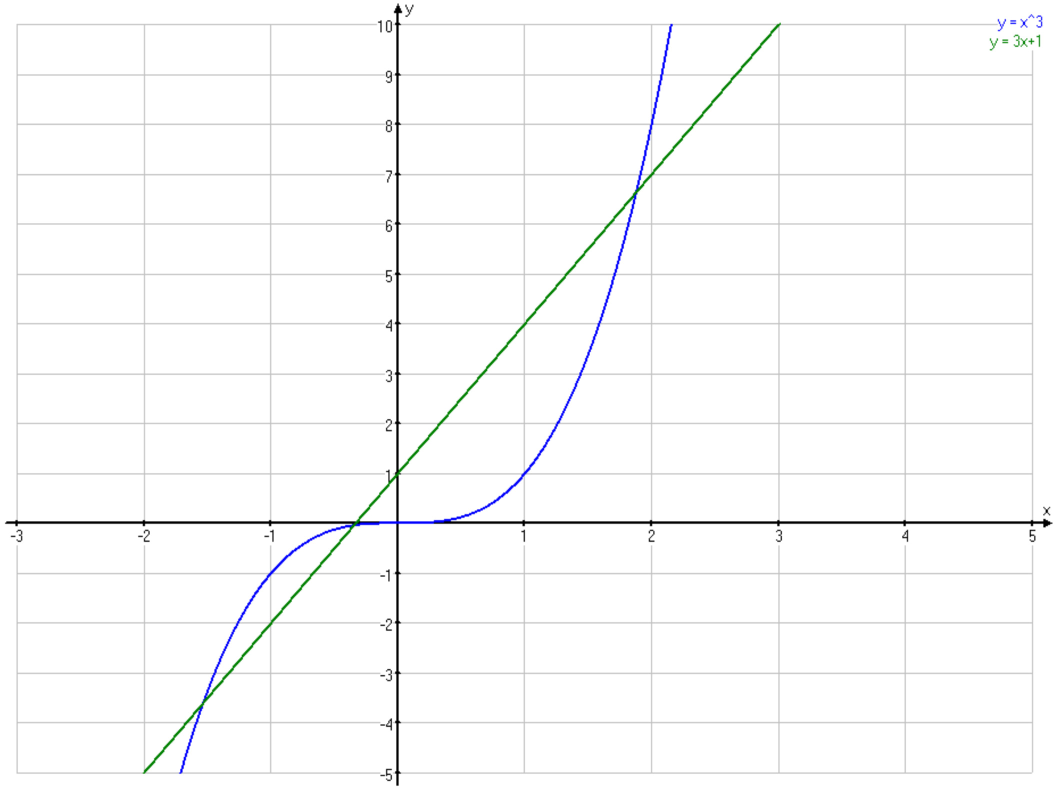
$$f(x) = r(x) - s(x) = 0 \quad \Rightarrow \quad r(x) = s(x)$$

Then sketch the graph of $y = r(x)\ and\ y = s(x)$, the points of intersection of the graph of these functions are the roots of the given equation as follows



Example: Use graphical method to find the interval(s) in which the roots of the equation $x^3 - 3x - 1 = 0$ are included.

where , $x^3 = 3x + 1$ *let* $g(x) = x^3$ *and* $s(x) = 3x + 1$



Roots lies on $x_0 \in [-2,-1]$, $x_1 \in [-1,0]$ *and* $x_2 \in [1,2]$

NOTE: Graphical methods is not very precise . It makes it possible to roughly determine the interval.

ANALYTICAL METHOD OF SEPARATING OF ROOTS

Theorem : Assume we want to find a root of equation $f(x) = 0$. Assume that $f: R \to R$ and f is continuous. Let $[a, b] \subset R$ be such that

$$f(a)f(b) < 0$$

Then, by the intermediate value theorem, there exists $\bar{x} \in [a, b]$ such that $f(\bar{x}) = 0$.

Now , we can recommend the following sequence of operations to separate the roots using the analytic method.

1. Find the critical values of $f(x)$, that is $f'(x) = 0$.
2. Compute a table of signs of the function $f(x)$ setting α equal to
   - a) The critical values of the derivative or the values close them
   - b) The boundaries of the interval $[a, b]$
3. Determine the intervals at the endpoints of which the function assumes values of opposite signs. These intervals contain only one root each in its interior.

Example: Separate the roots of $f(x) = 2^x - 5x - 3 = 0$ using the analytical method.

Solution:

$$f(x) = 2^x - 5x - 3 \quad , \; D: (-\infty, \infty)$$

$$f'(x) = 2^x \ln 2 - 5 = 0 \quad \Rightarrow \quad 2^x = \frac{5}{\ln 2} \quad \Rightarrow \ln(2^x) = \ln\left(\frac{5}{\ln 2}\right)$$

$$x \ln(2) = \ln\left(\frac{5}{\ln 2}\right) \quad \Rightarrow x = \frac{\ln 5 - \ln\ln 2}{\ln 2} \approx 2.85 \qquad \left(2.85 \;\; close\ to \;\; 3\right)$$

$f(x) = 2^x - 5x - 3$	$-\infty\ldots$	-2	-1	0	1	2	3	4	5	$\ldots\infty$
	+	+	+	-	-	-	-	-	+	+

The roots of the equation are in the interval (-1,0) and (4,5).

Example: Separate the roots of the equation $x^3 + 3x^2 - 3 = 0$.

$$f(x) = x^3 + 3x^2 - 3, D: (-\infty, \infty)$$

$$f'(x) = 3x^2 + 6x = 0 \quad then \quad 3x(x+2) = 0$$
$$x = 0 \quad and \quad x = -2$$

x	$-\infty\ldots$	-3	-2	-1	0	1	2	3	$\ldots\infty$
$f(x) = x^3 + 3x^2 - 3$	-	-	+	-	-	+	+	+	+

Roots lie on (-3,-2) , (-2,-1) and (0,1)

COMPUTER METHODS

BRACKETING METHODS (Or, two point methods for finding roots)

- Bracketing methods consider the fact that a function typically changes sign in the vicinity of the root.

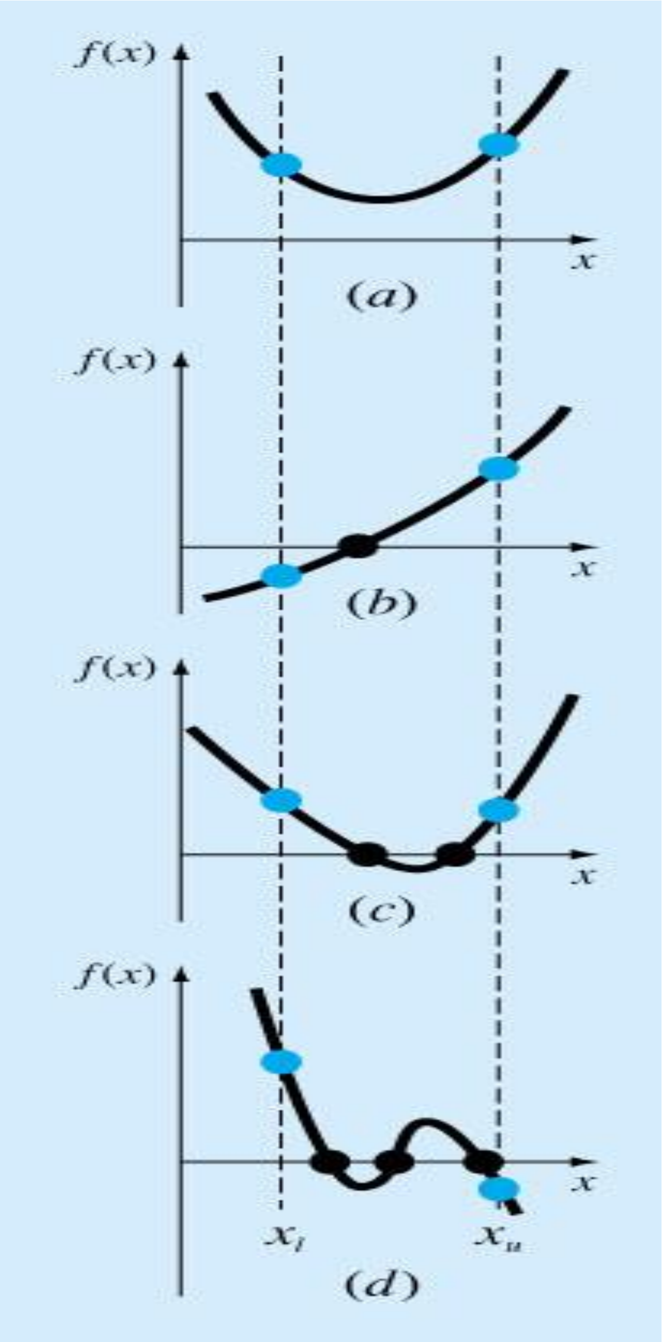
- In this process two initial guesses for the root are required.

- These two initial guesses are called x_{min} (lower) $(or\ x = a)$ and x_{max} (upper) $(or\ x = b)$.

Definition:

Assume that $f(x)$ is a continuous function . Any number r for which $f(r) = 0$ is called a root of the equation $f(x) = 0$. Also we say that r is a zero of the function $f(x)$.

For example ,



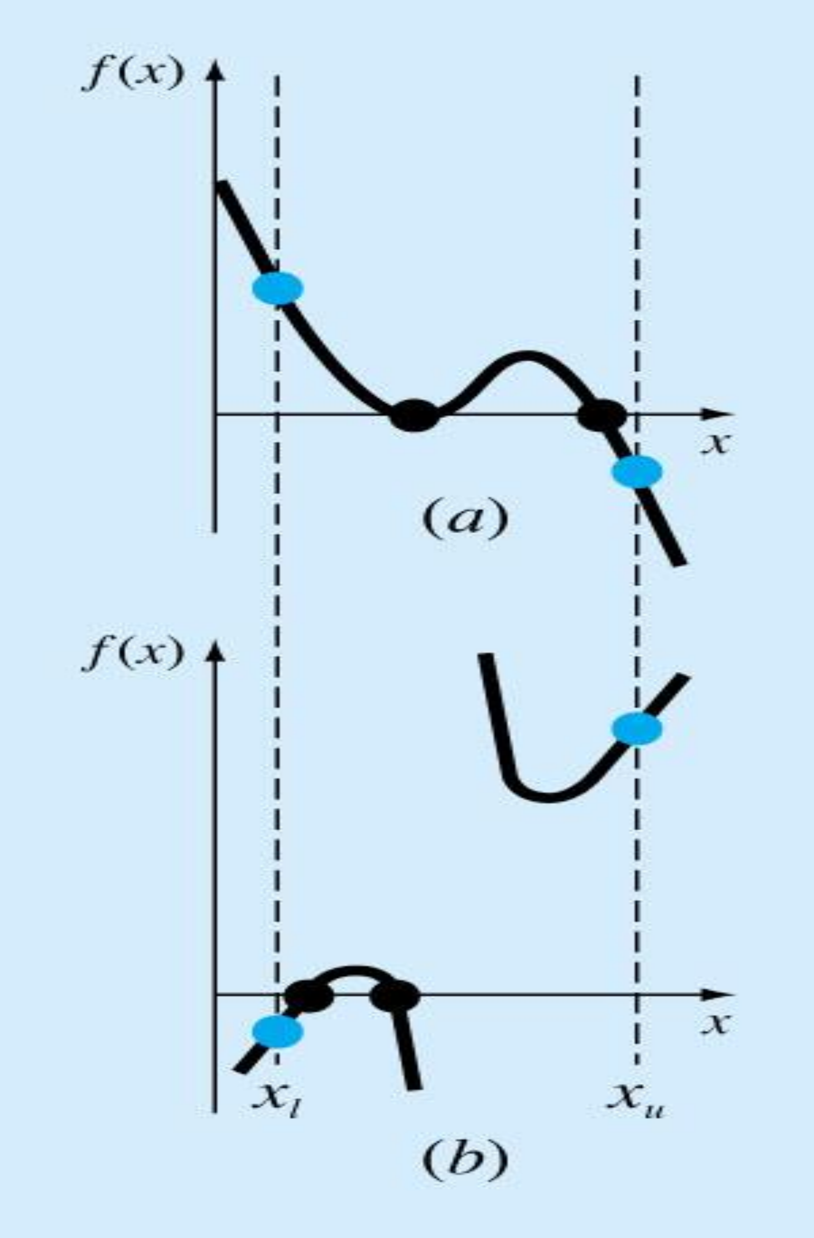
No answer (No root)

Nice case (one root)

Oops!! (two roots!!)

Three roots

(Might work for a while!!)



Two roots

(Might work for a while!!)

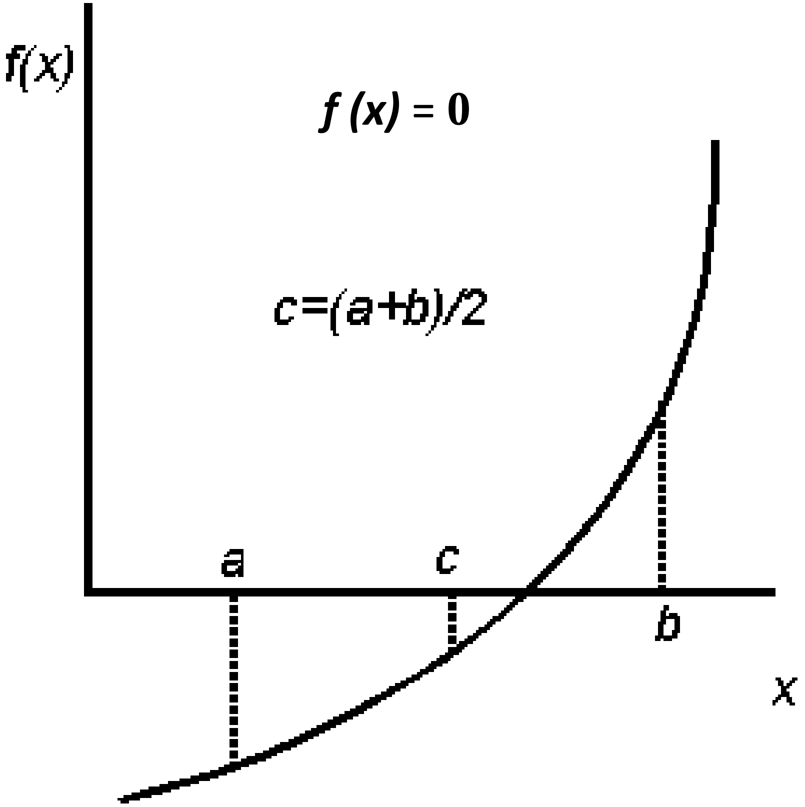
Discontinuous function.

Need special method

BISECTION METHOD

BISECTION METHOD

In this section, we develop our first bracketing method for finding a zeros of continuous function.



BISECTION METHOD

Algorithm

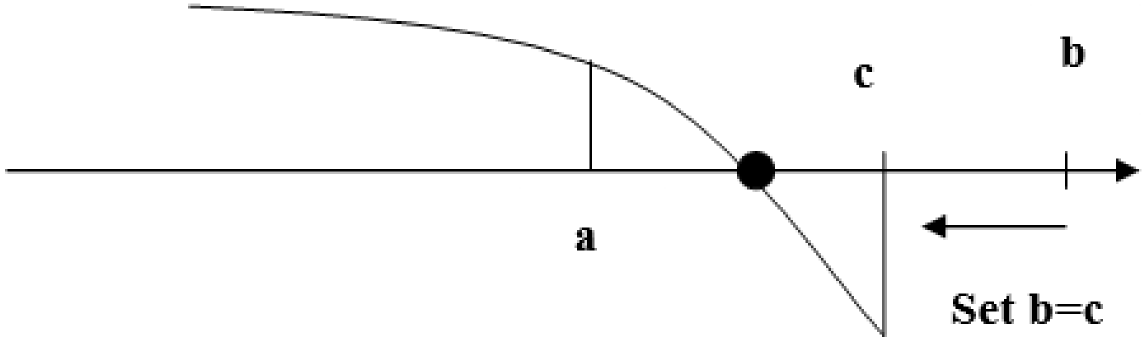
Step1: The decision step for this process of interval halving is first to choose the midpoint

$$c = \frac{a+b}{2} \quad (1)$$

where, $f(a)f(b) < 0$ and then to analyse the three possibilities that might arise

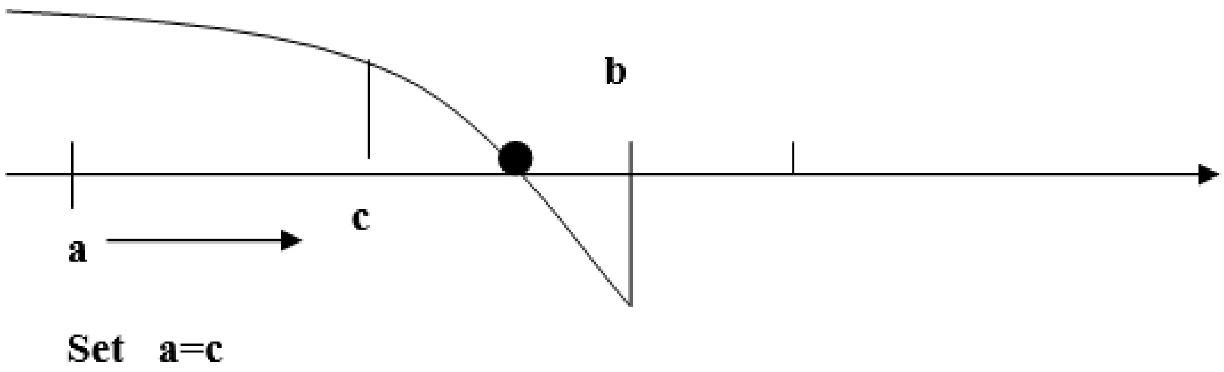
Step 2:

a. If $f(a)$ and $f(c)$ have opposite signs ($f(a)f(c) < 0$) a zero lies in $[a, c]$.



Therefore, set b=c and return equation (1).

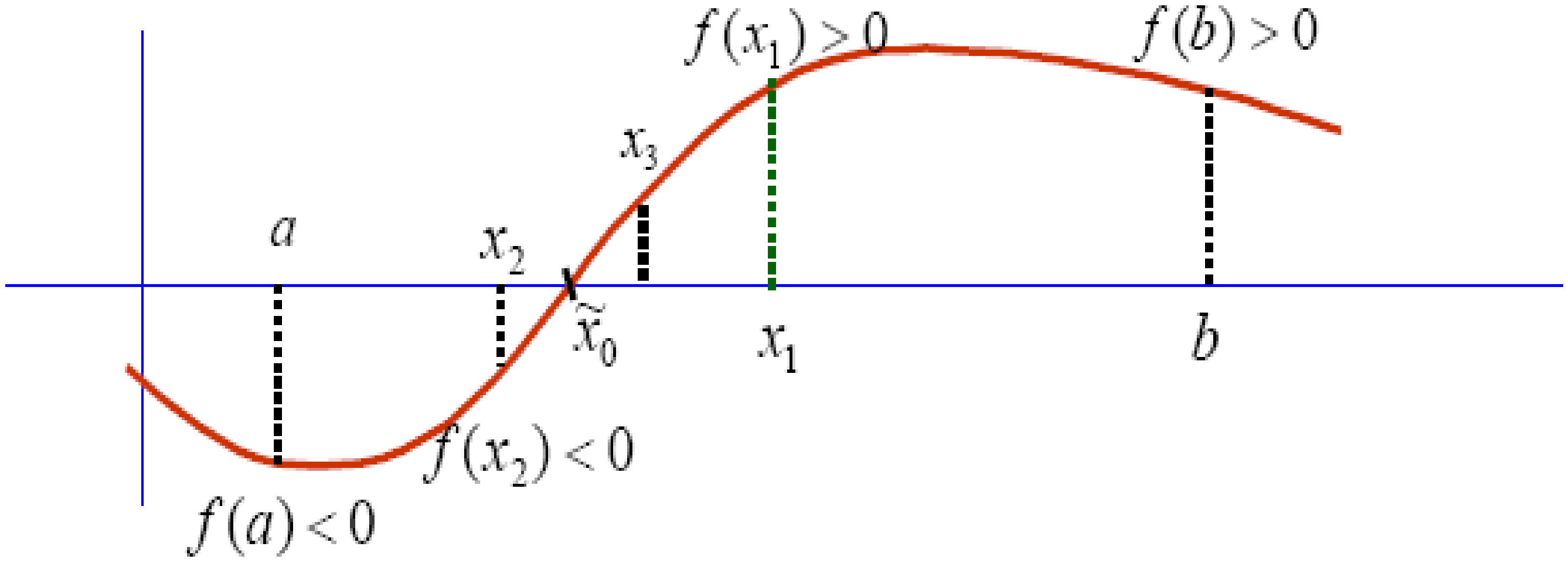
b. If $f(b)$ and $f(c)$ have opposite signs ($f(b)f(c) < 0$) a zero lies in $[c, b]$
($or\ f(a)f(c) > 0$)



Therefore, set $a = c$ and return equation (1).

c. If $f(c) = 0$, then the zero is c.

Interval halving (bisection) is an iterative procedure. The solution is not obtained directly by a single calculation. The iteration are continued until the size of an interval decreases below a prescribed tolerance ε_1, that is $|b_i - a_i| \leq \varepsilon_1$, or the value of $f(x)$ decreases a prescribed tolerance ε_2 , that is $|f(c_i)| \leq \varepsilon_2$, or both.



Example: Let $f(x) = x^3 + x^2 - 3x - 3$

a) Find the location of the root(s).
b) Find an approximation for the positive root using Bisection method with accuracy $\varepsilon < 1 \times 10^{-2}$

Solution:

a) where $f(x) = x^3 + x^2 - 3x - 3$, then the roots are

$$f'(x) = 0 \quad \Rightarrow \quad 3x^2 + 2x - 3 = 0$$

$$x_1 = -2 + \frac{\sqrt{40}}{6} \approx 0.72 \ (take\ x_1 = 1) \quad and \quad x_1 = -2 - \frac{\sqrt{40}}{6} \approx -1.38 \ (take\ x_2 = -1.5)$$

	$-\infty \ldots$	-4	-3	-2	-1.5	0	1	2	3	$\ldots \infty$
x^3+x^2-3x-3	—	—	—	—	+	—	—	+	+	+

Roots lies in (-2, -1.5) , (-1.5, 0) and (1,2) , positive root lie in (1,2)

b) Use bisection method to approximate positive root

a	c=(a+b)/2	b	f(a)	f(c)	f(a) f(c)	$\varepsilon_1 = \lvert b-a \rvert$	$\varepsilon_2 = \lvert f(c) \rvert$
1	1.5	2	-4	-1.875	>0	1	1.875
1.5	1.75	2	-1.875	0.171	<0	0.5	0.171
1.5	1.625	1.75	-1.875	-0.943359	>0	0.25	0.943359
1.625	1.6875	1.75	-0.943359	-0.409424	>0	0.125	0.409424
1.6875	1.71875	1.75	-0.409424	-0.124786	>0	0.0625	0.124786
1.71875	1.734375	1.75	-0.124786	0.02203	<0	0.03125	0.02203
1.71875	1.726563	1.734375	-0.124786	-0.051755	>0	0.015625	0.051755
1.726563	1.730469	1.734375	-0.051755	-0.014957	>0	0.007813	0.014957
1.730469	1.732422	1.734375	-0.014957	-0.003513	>0	0.003906	0.003513

$$x \approx 1.730469 \quad if \quad \varepsilon = |b-a| = 0.007813 \;\; is\ used$$
$$or$$
$$x \approx 1.732422 \quad if \quad \varepsilon = |f(c)| = 0.003513 \;\; is\ used$$

Theorem (Bisection Theorem)

Assume that $f \epsilon [a, b]$ and that there exist a number r$\epsilon[a, b]$ such that $f(r) = 0$. If $f(a)$ and $f(b)$ have opposite signs, and $\{c_n\}_{n=0}^{\infty}$represents the sequence of midpoints generated by the bisection process

$$a_0 \leq a_1 \leq \ldots a_n \leq \ldots \leq r \leq \ldots \leq b_n \leq \ldots \leq b_1 \leq b_0$$

and

$$[a_{n+1}, b_{n+1}] = [a_n, c_n] \quad and \quad [a_{n+1}, b_{n+1}] = [c_n, b_n]$$

then

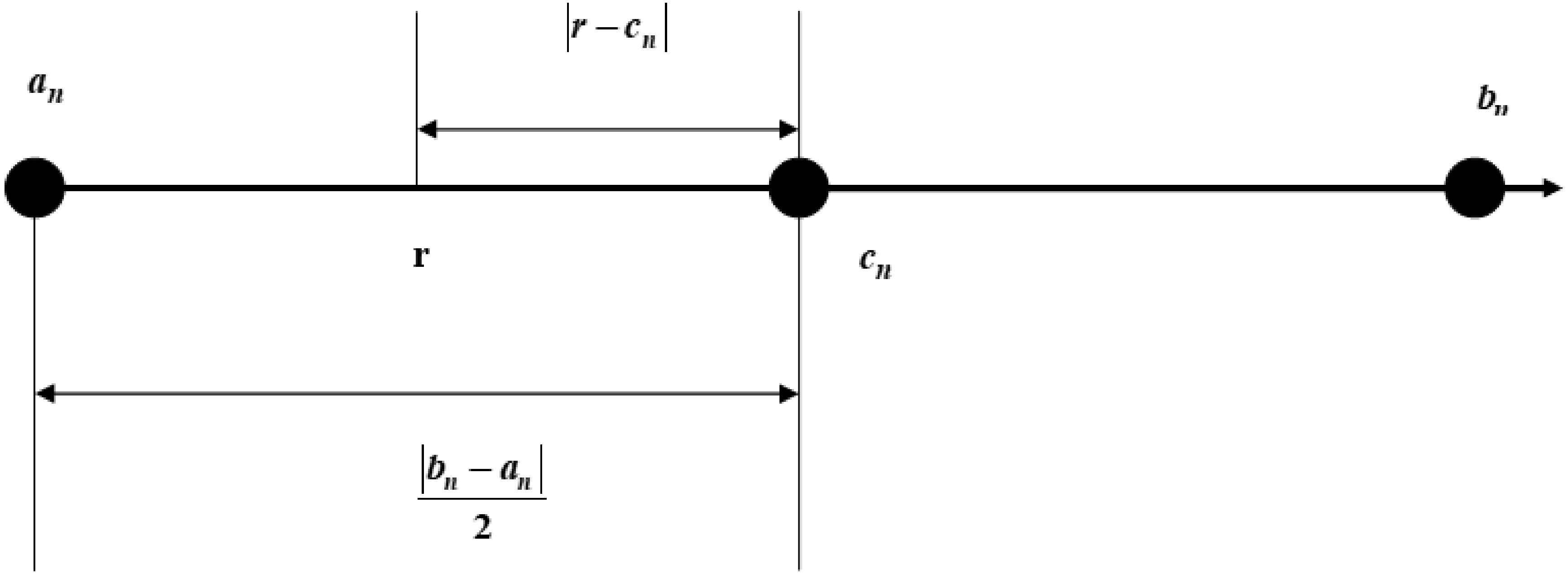
$$|r - c_n| \leq \frac{b-a}{2^{n+1}} \qquad for \quad n = 0,1, \ldots \qquad (2)$$

And therefore the sequence $\{c_n\}_{n=0}^{\infty}$ converges to the zero $x = r$; that is

$$\lim_{n\to\infty} c_n = r \qquad (3)$$

Proof :

Since both zero r and the midpoint c_n lie in the interval $[a_n, b_n]$, the distance between c_n and r cannot be greater than half the width of this interval (see fig. below).



Thus,

$$|r - c_n| \le \frac{b_n - a_n}{2} \qquad for \;\; all \quad n \qquad (4)$$

Observe that the successive interval width form the pattern

$$b_1 - a_1 = \frac{b_0 - a_0}{2}$$

$$b_2 - a_2 = \frac{b_1 - a_1}{2} = \frac{b_0 - a_0}{2^2}$$

then, $$b_n - a_n = \frac{b_0 - a_0}{2^n} \qquad (5)$$

Combining (4) and (5) result in

$$|r - c_n| \leq \frac{b_0 - a_0}{2^{n+1}} \qquad for \ \ all \quad n$$

- The number N of repeated bisection needed to guarantee that the Nth midpoint c_n is an approximation to a zero and has an error less than the pre assigned value ε is

$$N = \operatorname{int}\left(\frac{\ln(b-a) - \ln\varepsilon}{\ln 2}\right) \qquad (6)$$

Proof: Use

$$\frac{|b - a|}{2^{n+1}} < \varepsilon \qquad \text{then}$$

$$\ln\left(\frac{|b-a|}{2^{n+1}}\right) < \ln\varepsilon \qquad \Rightarrow \ln(b-a) - (N+1)\ln 2 < \ln\varepsilon$$

$$N > \frac{\ln(b-a) - \ln\varepsilon}{\ln 2} - 1$$

Therefore , the smallest value of N is

$$N = \text{int}\left(\frac{\ln(b-a) - \ln\varepsilon}{\ln 2}\right)$$

Example: The bisection method is used to find a zeros of f(x) in the interval [2,7]. How many times must this interval be bisected to guarantee that the approximation c_n has an accuracy of 5×10^{-9} .

Given $a = 2 \;\; , \; b = 7 \quad and \quad \varepsilon = 5 \times 10^{-9}$

$$N = \mathrm{int}\left(\frac{\ln(7-2) - \ln(5 \times 10^{-9})}{\ln 2}\right) = \mathrm{int}(29.83) \qquad take \quad N = 30$$

If the given interval 30 times bisected we obtain an accuracy 5×10^{-9}

- **Advantage:**

  A global method: it always converges no matter how far you start from the actual root.

- **Disadvantage:**

  It cannot be used to find roots when the function is tangent is the axis and does not pass through the axis.

  For example: $f(x) = x^2$

  It converges slowly compared with other methods.

Example: Let $f(x) = 2\sqrt{x} + 2x - 5$

a) Find the location of the root(s).

b) Find an approximation for the root using Bisection method by 6 times.

Solution:

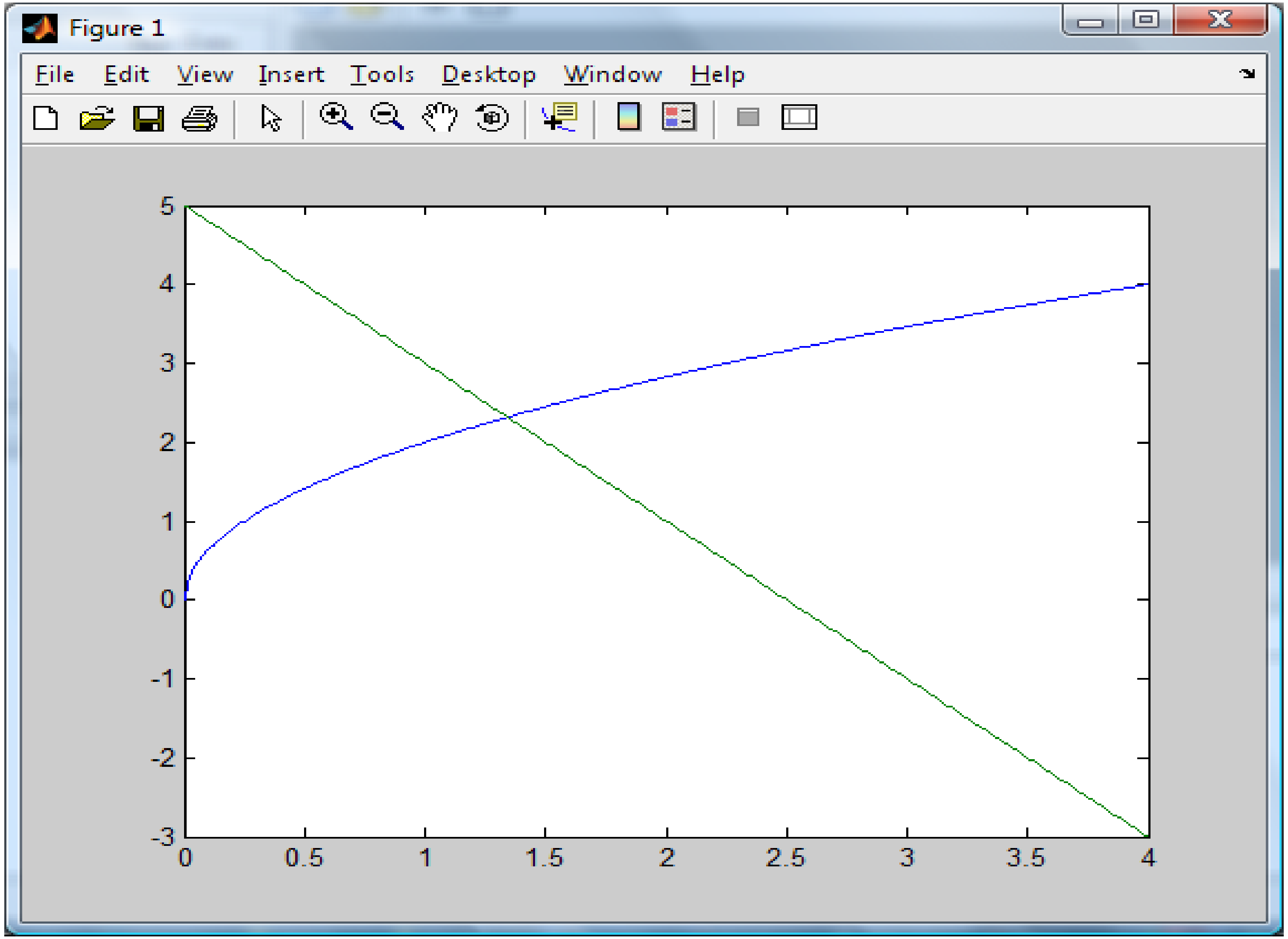
a) We can use two methods to find the location of the root(s).

1. Graphical Method
2. Analytical Method

1. Graphical Method

$$f(x) = 2\sqrt{x} + 2x - 5 = 0$$

$$y_1 = 2\sqrt{x} \qquad \text{and} \qquad y_2 = 5 - 2x$$



Roots lies on $x_0 \in [1,2]$, where $f(1) = -1 < 0,\ f(2) = 1.81 > 0 \quad f(1)f(2) < 0$

2. Analytical Method

$$f(x) = 2\sqrt{x} + 2x - 5 = 0 \qquad f'(x) = \frac{1}{\sqrt{x}} + 2 = 0 \qquad x = \frac{1}{4}$$

x	0	$\frac{1}{8}$	$\frac{1}{4}$	1	$\frac{3}{2}$	2
Sign of $f(x)$	-	-	-	-	+	+

Roots lies on [1,2] or [1,3/2]

b) Now we can find an approximation root of $f(x) = 2\sqrt{x} + 2x - 5$ on the interval [1,2] by using Bisection Method

1. $$x_0 = \frac{a+b}{2} = \frac{1+2}{2} = 1.5$$

x	1	1.5	2
Sign of $f(x)$	-	+	+

2. Now the new interval is [1, 1.5]

$$x_0 = \frac{a+b}{2} = \frac{1+1.5}{2} = 1.25$$

x	1	1.25	1.5
Sign of $f(x)$	-	-	+

3. The new interval is [1.25, 1.5]

$$x_0 = \frac{a+b}{2} = \frac{1.25+1.5}{2} = 1.375$$

x	1.25	1.375	1.5
Sign of $f(x)$	-	+	+

4. The new interval is [1.25, 1.375]

$$x_0 = \frac{a+b}{2} = \frac{1.25+1.375}{2} = 1.3125$$

x	1.25	1.3125	1.375
Sign of $f(x)$	-	-	+

5. The new interval is [1.3125, 1.375]

$$x_0 = \frac{a+b}{2} = \frac{1.3125+1.375}{2} = 1.34375$$

x	1.3125	1.34375	1.375
Sign of $f(x)$	-	+	+

6. The new interval is [1.3125, 1.34375]

x	1.3125	1.328125	1.34375
Sign of $f(x)$	-	-	+

$$x_0 = \frac{a+b}{2} = \frac{1.3125+1.34375}{2} = 1.328125$$

Bisection Method for $f(x) = 2\sqrt{x} + 2x - 5$ to find approximation root for 6 steps

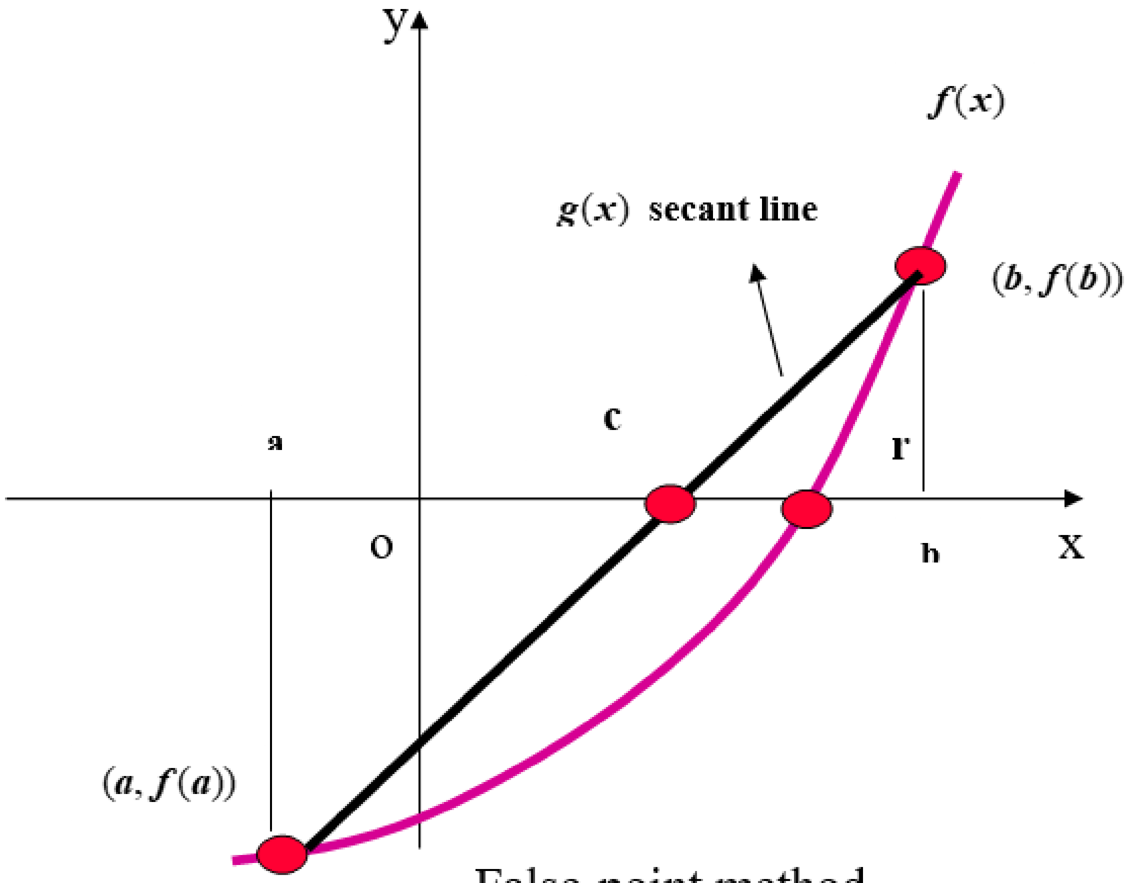
n	a	b	$x_0 = \frac{a+b}{2}$	$f(x_0)$	$\lvert b-a \rvert$	$\lvert r-x_0 \rvert < \frac{b-a}{2^{n+1}}$
0	1	2	1.5	0.4495	1	0.5
1	1	1.5	1.25	-0.2639	0.5	0.25
2	1.25	1.5	1.375	0.0952	0.25	0.125
3	1.25	1.375	1.3125	-0.0837	0.125	0.0625
4	1.3125	1.375	1.34375	0.0059	0.0625	0.03125
5	1.3125	1.34375	1.328125	-0.03886	0.03125	0.015625
6	1.328125	1.34375	1.3359375	-0.01647	0.015625	0.0078125

FALSE POSITION METHOD

FALSE POSITION METHOD (REGULA FALSI)

Another popular algorithm is the method of false position method. It was developed because the bisection method converges at a fairly slow speed.

In the false position method , the nonlinear function f(x) is assumed to be linear function g(x) in the interval (a,b) as follows



False-point method

False-Position Method

To find the value of c, we write down two versions of the slope m of the line $L = g(x)$

$$m = \frac{f(b)-f(a)}{b-a} \qquad (1)$$

Where, the points $(a, f(a))$ and $(b, f(b))$ are used , and

$$m = \frac{0-f(b)}{c-b} \qquad (2)$$

Where, the points $(c, 0)$ and $(b, f(b))$ are used . Equating the slopes (1) and (2), we have

$$\frac{f(b)-f(a)}{b-a} = \frac{0-f(b)}{c-b}$$ which easily solved for c to get

$$c = b - \frac{f(b)(b-a)}{f(b)-f(a)} \qquad (3)$$

The three possibilities are the same as before (bisection method)

1. If $f(a)f(c) < 0 \quad the \;\; root \;\; lies \;\; in \;\; [a, c]$

2. If $f(a)f(c) > 0 \quad the \;\; root \;\; lies \;\; in \;\; [c, b]$

3.. If $f(c) = 0 \quad then \;\; x = c$

Example : Use False Position Method to approximate $\sqrt{5}$ to 3 decimal places.
$(i.e\ \ 0.5 \times 10^{-3})$

Solution:

First find $f(x)$ where

$$x = \sqrt{5} \quad \Rightarrow \quad x^2 = 5 \quad \Rightarrow \quad x^2 - 5 = 0 \quad that\ is \quad f(x) = x^2 - 5$$

Use $f(x) = x^2 - 5$ to locate the root(s)

$$f'(x) = 2x = 0 \quad \Rightarrow \quad x = 0$$

	$-\infty \ldots$	-3	-2	-1	0	1	2	3	4	$\ldots \infty$
$x^2 - 5$	+	+	—	—	—	—	—	+	+	+

$\sqrt{5}$ locate in [2,3]

a	$c = b - \frac{f(b)(b-a)}{f(b)-f(a)}$	b	f(a)	f(c)	f(a)f(c)	$\varepsilon = \lvert f(c) \rvert$
2	2.2	3	-1	-0.16	>0	0.16
2.2	2.2307	3	-0.16	-0.023	>0	0.023
2.2307	2.235	3	-0.023	-4.775×10^{-3}	>0	4.775×10^{-3}
2.235	2.2359	3	-4.775×10^{-3}	-7.012×10^{-4}	>0	7.012×10^{-4}

Approximation of the root $x = \sqrt{5} \approx 2.2359$ with accuracy 7.012×10^{-4}

Exercises:

Exercise1. Given $lnx - x^2 + 2x = 0$

Solve above equation, using

a) Bisection Method

b) False Position Method

correct up to at least 2 decimal places i.e $\varepsilon = 0.5 \times 10^{-2}$

c) Compare the two methods according to the number of iterations performed.

Exercise2. Given $e^x + x - 2 = 0$

Solve above equation, using

a) Bisection Method

b) False Position Method

correct up to at least 2 decimal places i.e $\varepsilon = 0.5 \times 10^{-2}$

c) Compare the two methods according to the number of iterations performed.

NOTE:

1. To compare between two methods you should start by the same interval for both of them.

2. You must use same error calculation for both of methods.

3. The termination criteria used in the Bisection Method ($\varepsilon \leq |b_n - a_n|$) is not useful for the False Position method and may result in infinite loop. The best choice is to use $\varepsilon \leq |f(c_n)|$ to compare the two methods.

Example: Let $f(x) = 2\sqrt{x} + 2x - 5$

a) Find the location of the root(s).

b) Find an approximation for the root using False Position method by 6 times.

Solution:

a) We can use two methods to find the location of the root(s).

1. Graphical Method
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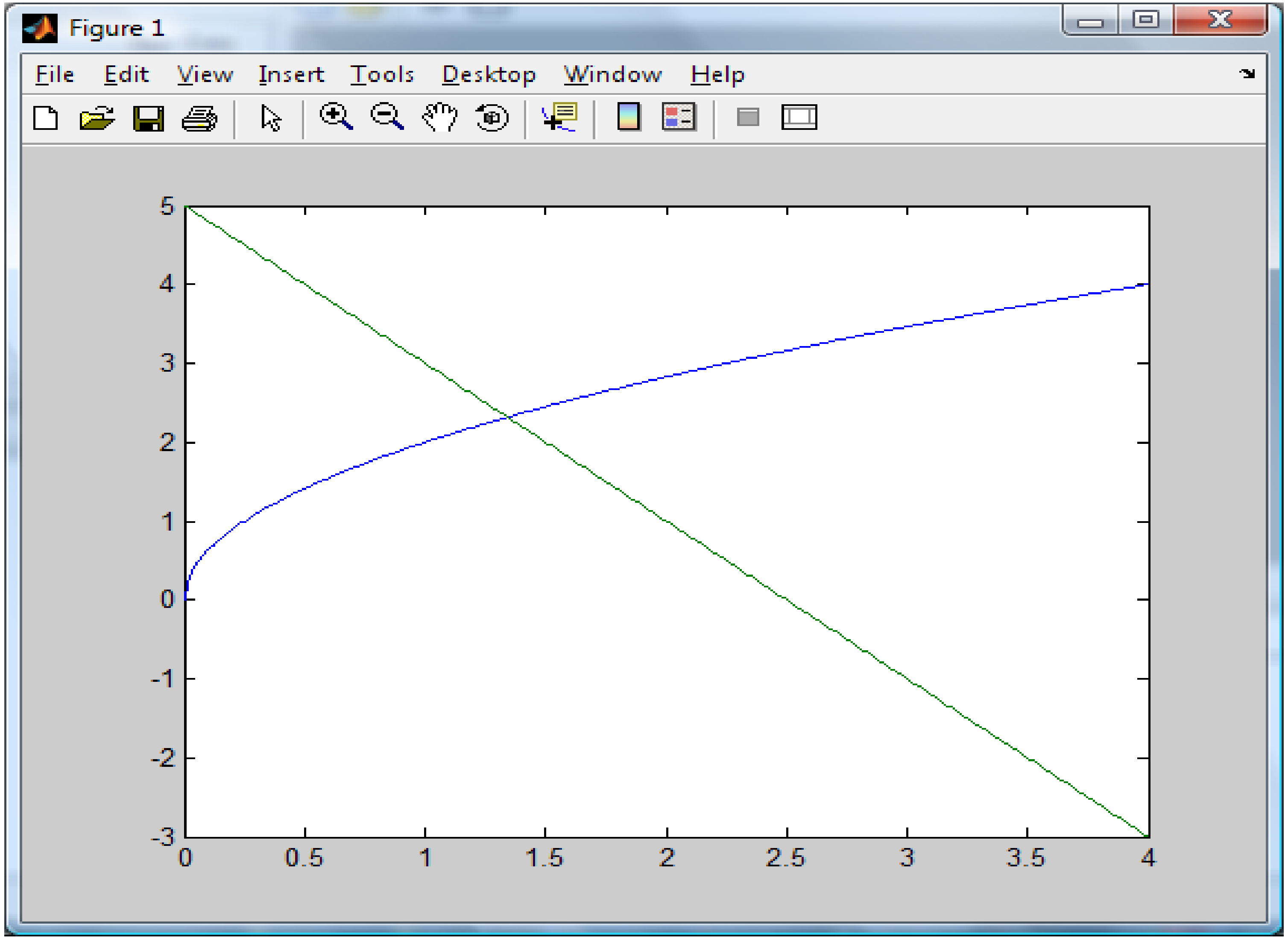
$y_1 = 2\sqrt{x}$ and $y_2 = 5 - 2x$



Roots lies on $x_0 \in [1,2]$, where $f(1) = -1 < 0,\ f(2) = 1.81 > 0 \quad f(1)f(2) < 0$

2. Analytical Method

$$f(x) = 2\sqrt{x} + 2x - 5 = 0 \qquad f'(x) = \frac{1}{\sqrt{x}} + 2 = 0 \qquad x = \frac{1}{4}$$

x	0	$\frac{1}{8}$	$\frac{1}{4}$	1	$\frac{3}{2}$	2
Sign of $f(x)$	-	-	-	-	+	+

Roots lies on [1,2] or [1,3/2]

b) Now we can find an approximation root of $f(x) = 2\sqrt{x} + 2x - 5$ on the interval [1,2] by using False Position Method

$$x_i = b - f(b)\frac{b-a}{f(b)-f(a)}, \quad i = 1, 2, 3, 4, \ldots$$

1. $a = 1, b = 2, f(1) = -1, f(2) = 1.828427$

$$x_1 = 2 - (1.828427)\frac{2-1}{1.828427+1} = 1.353553$$

x	1	1.353553	2
Sign of $f(x)$	-	+	+

2. $a = 1, b = 1.353553, f(1) = -1, f(1.353553) = 0.033952$

$$x_2 = 1.353553 - (0.033952)\frac{1.353553-1}{0.033952+1} = 1.341943$$

x	1	1.341943	1.353553
Sign of $f(x)$	-	+	+

3. $a = 1, b = 1.341943, f(1) = -1, f(1.341943) = 0.000731$

x	1	1.341693	1.341943
Sign of $f(x)$	-	+	+

$$x_3 = 1.341943 - (0.000731)\frac{1.341943 - 1}{0.000731 + 1} = 1.341693$$

4. $a = 1, b = 1.341693, f(1) = -1, f(1.341943) = 0.000015$

x	1	1.3416879	1.341693
Sign of $f(x)$	-	+	+

$$x_4 = 1.341693 - (0.000015)\frac{1.341693 - 1}{0.000015 + 1} = 1.3416879$$

5. $a = 1, b = 1.3416879, f(1) = -1, f(1.3416879) = 0.00000845$

$$x_5 = 1.3416879 - (0.000000845)\frac{1.3416879 - 1}{0.000000845 + 1}$$
$$= 1.3416876$$

x	1	1.3416876	1.3416879
Sign of $f(x)$	-	-	+

6. $a = 1.3416876, b = 1.3416879, f(1.3416876) = -0.0000000138, f(1.3416879) = 0.00000845$

$$x_6 = 1.3416879 - (-1.38 \times 10^{-8})\frac{1.3416879 - 1.3416876}{8.45 \times 10^{-7} + 1.38 \times 10^{-8}} = 1.341687905$$

x	1.3416876	1.341687905	1.3416879
Sign of $f(x)$	-	+	+

False Position Method for $f(x) = 2\sqrt{x} + 2x - 5$ to find approximation root for 6 steps

n	x_k	x_{k+1}	$x_k = b - f(b)\dfrac{b-a}{f(b)-f(a)}$	$f(x_k)$	$\lvert x_{k+1} - x_k \rvert$
0	1	2	1.353553	0.033952	1
1	1	1.353553	1.341943	0.000731	0.353553
2	1	1.341943	1.341693	0.000015	0.341943
3	1	1.341693	1.3416879	0.000000845	0.341693
4	1	1.3416879	1.3416876	-0.0000000138	0.3416879
5	1.3416876	1.3416879	1.341687905	0.0000008595	0.0000003