Lagrange Interpolation Polynomial, Newton polynomials

INTERPOLATION AND POLYNOMIAL APPROXIMATION

Interpolation Methods

Why would we be interested in interpolation methods?

- Interpolation method are the basis for other procedures that we will deal with:
    - Numerical differentiation
    - Numerical integration
    - Solution of ODE (ordinary differential equations) and PDE (partial differential equations)

Why would we be interested in interpolation methods?

- These methods demonstrate some important theory about polynomials and the accuracy of numerical methods.
- The interpolation of polynomials serve as an excellent introduction to some techniques for drawing smooth curves.

Interpolation uses the data to approximate a function, which will fit all of the data points. All of the data is used to approximate the values of the function inside the bounds of the data.

Interpolation is a process of finding a formula (often a polynomial) whose graph will pass through a given set of points (x, y).

Interpolation: Problem Specification

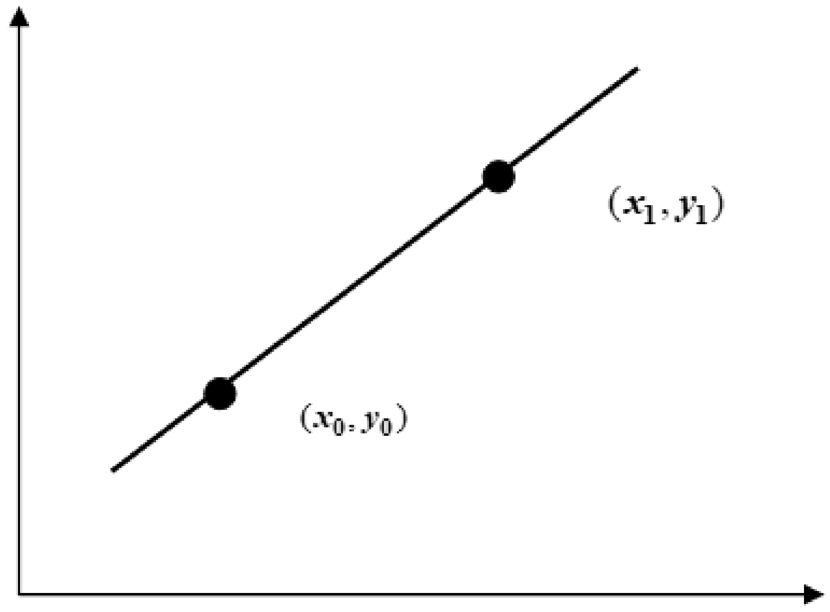
Input: A set S of data $f(x_0, y_0) \cong (x_1, y_1) \cong \sim\sim\sim \cong (x_\mathrm{n}, y_\mathrm{n}) = g(x_\mathrm{i}, y_\mathrm{i}) \in R^2$

Output: A function ||( ) which interpolates the given set of data, i.e. $\|(x_,) = y_i$, for $0 \leq i \leq n$

Note: In general, the interpolating function ||( ) NOT unique.

LAGRANGE INTERPOLATION

Linear interpolation uses a line segment that passes two points



The slope between (x_0, y_0) $\quad and \quad$ (x_1, y_1)

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

from the point slope formula $y = y_0 + m(x - x_0)$ can be rearranged as

$$y = y_0 + (y_1 - y_0)\left(\frac{x-x_0}{x_1-x_0}\right) \qquad (1)$$

When formula (1) is expanded , the result is a polynomial of degree ≤ 1. Evaluating at $x_0 \quad and \quad x_1$ produces $y_0 \quad and \quad y_1$, respectively.

$$\begin{aligned} P(x_0) &= y_0 \\ P(x_1) &= y_1 \end{aligned} \qquad (2)$$

Lagrange used slightly different method to find this polynomial as follows

$$y = P_1(x) = y_0 \frac{(x-x_1)}{(x_0-x_1)} + y_1 \frac{(x-x_0)}{(x_1-x_0)} \qquad (3)$$

Each term on the right hand side of (3) involves linear factor . Hence the sum is a polynomial of degree ≤ 1
The quotients in (3) are denoted by

$$L_{1,0}(x) = \frac{(x-x_1)}{(x_0-x_1)} \quad , \quad L_{1,1}(x) = \frac{(x-x_0)}{(x_1-x_0)} \qquad (4)$$

which are called the Lagrange coefficients polynomials based on the nodes $x_0 \ and \ x_1$. Using this notation , (3) can be written in summation form

$$P_1(x) = \sum_{k=0}^{1} y_k L_{1,k} = y_0\, L_{1,0} + y_1 L_{1,1} \qquad (5)$$

The generalization of (5) is the construction of a polynomial of degree at most that passes through the points $(x_0, y_0), (x_1, y_1), \ldots, (x_N, y_N)$ and has the form

$$P_N(x) = \sum_{k=0}^{N} y_k L_{N,k}\,(x) \qquad (6)$$

where

$$L_{N,k}(x) = \frac{\prod_{\substack{j=0 \\ j\neq k}}^{N} (x - x_j)}{\prod_{\substack{j=0 \\ j\neq k}}^{N} (x_k - x_j)} \qquad (7)$$

Using (6) and (7) , find 1st , 2nd, and 3rd Lagrange Interpolation polynomial.

- ❑ 1st order Lagrange interpolation polynomial

$$P_1(x) = \sum_{k=0}^{1} y_k L_{1,k} = y_0\, L_{1,0} + y_1 L_{1,1}$$

where

$$L_{1,0}(x) = \frac{(x - x_1)}{(x_0 - x_1)} \quad , \quad L_{1,1}(x) = \frac{(x - x_0)}{(x_1 - x_0)}$$

- 2nd order Lagrange interpolation polynomial

$$P_2(x) = \sum_{k=0}^{2} y_k L_{2,k} = y_0\, L_{2,0} + y_1 L_{2,1} + y_2 L_{2,2}$$

where

$$L_{2,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \quad , \quad L_{2,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \quad , \quad L_{2,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

- [ ] 3rd order Lagrange interpolation polynomial

$$P_3(x) = \sum_{k=0}^{3} y_k L_{3,k} = y_0\, L_{3,0} + y_1 L_{3,1} + y_2 L_{3,2} + y_3 L_{3,3}$$

where

$$L_{3,0}(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \quad , \quad L_{3,1}(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$L_{3,2}(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \quad , \quad L_{3,3}(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

ERROR TERMS AND ERROR BOUNDS

Theorem: Assume that $f \in C^{N+1}[a,b]$ *and that* $x_0, x_1, \ldots, x_N \in [a,b]$ *are* $N+1$ *nodes.* If $x \in [a,b]$ then

$$f(x) = P_N(x) + E_N(x) \quad (1)$$

where $P_N(x)$ polynomial that can be approximate $f(x)$

$$f(x) \approx P_N(x) = \sum_{k=0}^{N} y_k L_{N,k}(x) \qquad (2)$$

The error term $E_N(x)$ has the form

$$E_N(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_N)}{(N+1)!} f^{(N+1)}(c) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\textit{for some value of } c = c(x) \textit{ that lies in the interval } [a,b]$$

$$|E_N(x)| \le \frac{|(x-x_0)(x-x_1)\ldots(x-x_N)|}{(N+1)!} M \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$M = \max_{x_0 \le x \le x_N} \left| f^{(N+1)}(c) \right|$$

Theorem: (Error bounds for Lagrange Interpolation, equally space nodes)
Assume that $f(x)$ defined on $[a, b]$, which contains equally space nodes $x_k = x_0 + hk$. Additionally, assume that $f(x)$ and the derivatives of $f(x)$ up to the N+1 , are continuous and bounded on the special subinterval $[x_0, x_1], [x_1, x_2]$ *and* $[x_2, x_3]$ respectively ; that is

$$M = \left|f^{(N+1)}(c)\right| \leq M_{N+1} \quad for \quad x_0 \leq x \leq x_N \quad N = 1,2,3$$

The error terms corresponding the cases N=1,2,3 have the following useful bounds on their magnitude.

$$|E_1(x)| \leq \frac{h^2 M_2}{8} \quad valid \ \ for \ \ [x_0, x_1]$$

$$|E_2(x)| \leq \frac{h^3 M_3}{9\sqrt{3}} \quad valid \ \ for \ \ [x_0, x_2]$$

$$|E_3(x)| \leq \frac{h^4 M_4}{24} \quad valid \ \ for \ \ [x_0, x_3]$$

Example: The function $y = \sin x$ is tabulated as

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
y	0	0.707	1

Using lagrange interpolation polynomial, find its value at $x = \frac{\pi}{6}$ estimate the error.

Solution:

$$Given \quad x_0 = 0,\ x_1 = \frac{\pi}{4}\ , x_2 = \frac{\pi}{2}\ , y_0 = 0,\ y_1 = 0.707\ , y_2 = 1$$

$$Find\ P_2(x),\ \ P_2(\frac{\pi}{6})\ \ and\ \ |E_2(x)|$$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
y	0	0.707	1

$$P_2(x) = \sum_{k=0}^{2} y_k L_{2,k} = y_0\, L_{2,0} + y_1 L_{2,1} + y_2 L_{2,2} = 0 + y_1 L_{2,1} + y_2 L_{2,2} \qquad (where, \quad y_0 = 0 \;\; then\; y_0 L_{2,0} \; = 0)$$

$$L_{2,1}(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x - 0)\left(x - \frac{\pi}{2}\right)}{\left(\frac{\pi}{4} - 0\right)\left(\frac{\pi}{4} - \frac{\pi}{2}\right)} = \frac{-16}{\pi^2}\left(x^2 - \frac{\pi x}{2}\right)$$

$$L_{2,2}(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x - 0)\left(x - \frac{\pi}{4}\right)}{\left(\frac{\pi}{2} - 0\right)\left(\frac{\pi}{2} - \frac{\pi}{4}\right)} = \frac{8}{\pi^2}\left(x^2 - \frac{\pi x}{4}\right)$$

$$P_2(x) = 0.707\left[\frac{-16}{\pi^2}\left(x^2 - \frac{\pi x}{2}\right)\right] + 1\left[\frac{8}{\pi^2}\left(x^2 - \frac{\pi x}{4}\right)\right] = -1.146x^2 + 1.8x + 0.81x^2 - 0.636x$$

$$\boxed{P_2(x) = -0.336x^2 + 1.164x}$$

$$P_2\left(\frac{\pi}{6}\right) = -0.336\left(\frac{\pi}{6}\right)^2 + 1.164\left(\frac{\pi}{6}\right) = 0.517 \quad \Rightarrow \boxed{P_2\left(\frac{\pi}{6}\right) = 0.517}$$

$$|E_2(x)| \le \frac{h^3 M_3}{9\sqrt{3}} \quad where \quad h = \frac{\pi}{4} \quad \Rightarrow$$

$x_k = x_0 + hk \Rightarrow x_2 = x_0 + 2h$

$\frac{\pi}{2} = 0 + 2h$

$\boxed{h = \pi/4}$

$$Find\ \ M_3\ ,\ \left|f^{(3)}(x)\right| \le M_3\ \ in\ \ \left[0, \frac{\pi}{2}\right]$$

$$f'(x) = \cos x\ \ , f''(x) = -\sin x\ \ ,\ f'''(x) = -\cos x$$

$$|f'''(x)| = |-\cos x| \quad maximum\ value\ of f'''(x)\ in \left[0, \frac{\pi}{2}\right] \quad M_3 = |-\cos 0| = 1 \quad then$$

$$\boxed{|E_2(x)| \le \frac{h^3 M_3}{9\sqrt{3}} = \frac{\left(\frac{\pi}{4}\right)^3 \times 1}{9\sqrt{3}} = 0.03}$$

Example: Find the approximation function at $x_0 = 0.3$ of the following data. Using

a. Two different definitions for first order Lagrange Polynomial
b. A second order Lagrange Polynomial.
c. Third order Lagrange Polynomial.
d. Discuss the error in using the exact polynomial and the Error Formula.

$x_0 = 0.3$ →

x	f(x)
0.1	0.981
0.2	0.928
0.4	0.744
0.5	0.625
0.6	0.496

Solution:

a. Two different definition for the first order Lagrange Polynomial.

1. Through $[0.2, 0.4] \rightarrow [0.928, 0.744]$

The formula of the first order Lagrange Polynomial is

$$P_1(x) = y = \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)} y_1$$

with $x_0 = 0.2,\ x_1 = 0.4,\ y_0 = 0.928,\ y_1 = 0.744$

$$P_1(x) = y = \frac{(0.4 - x)}{(0.4 - 0.2)}(0.928) + \frac{(x - 0.2)}{(0.4 - 0.2)}(0.744) = \left(\frac{-0.928 + 0.744}{0.2}\right)x + \left(\frac{(0.4)(0.928) - (0.2)(0.744)}{0.2}\right)$$

$$P_1(x) = y = -0.92x + 1.112$$

2. Through $[0.1,0.4] \rightarrow [0.981,0.744]$

$$x_0 = 0.1,\ x_1 = 0.4,\ y_0 = 0.981,\ y_1 = 0.744$$

$$Q_1(x) = y = \frac{(0.4 - x)}{(0.4 - 0.1)}(0.981) + \frac{(x - 0.1)}{(0.4 - 0.1)}(0.744) = \left(\frac{-0.981 + 0.744}{0.3}\right)x + \left(\frac{(0.4)(0.981) - (0.1)(0.744)}{0.3}\right)$$

$$Q_1(x) = y = -0.79x + 1.06$$

b. Second order Lagrange Polynomial ($P_2(x)$)

$$x_0 = 0.1,\ x_1 = 0.2,\ x_2 = 0.4$$

$$y_0 = 0.981,\ y_1 = 0.928,\ y_2 = 0.744$$

$$L_{2,0}(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x - 0.2)(x - 0.4)}{(0.1 - 0.2)(0.1 - 0.4)} = \frac{x^2 - 0.6x + 0.08}{0.03}$$

$$L_{2,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0.1)(x-0.4)}{(0.2-0.1)(0.2-0.4)} = -\left(\frac{x^2-0.5x+0.04}{0.02}\right)$$

$$L_{2,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0.1)(x-0.2)}{(0.4-0.1)(0.4-0.2)} = \frac{x^2-0.3x+0.02}{0.06}$$

$$P_2(x) = L_{2,0}(x)y_0 + L_{2,1}(x)y_1 + L_{2,2}(x)y_2$$

$$= \left(\frac{x^2-0.6x+0.08}{0.03}\right)(0.981) - \left(\frac{x^2-0.5x+0.04}{0.02}\right)(0.928) + \left(\frac{x^2-0.3x+0.02}{0.06}\right)(0.744)$$

$$= \left(\frac{0.981}{0.03} - \frac{0.928}{0.02} + \frac{0.744}{0.06}\right)x^2 + \left(\frac{(-0.6)(0.981)}{0.03} + \frac{(0.5)(0.928)}{0.02} - \frac{(0.3)(0.744)}{0.06}\right)x + \left(\frac{(0.08)(0.981)}{0.03} - \frac{(0.04)(0.928)}{0.02} + \frac{(0.02)(0.744)}{0.06}\right)$$

$$P_2(x) = -1.3x^2 - 0.14x + 1.008$$

c. Third order Lagrange Polynomial ($P_3(x)$) .

$$x_0 = 0.1,\ x_1 = 0.2,\ x_2 = 0.4,\ x_3 = 0.5$$

$$y_0 = 0.981,\ y_1 = 0.928,\ y_2 = 0.744,\ y_3 = 0.625$$

$$L_{3,0}(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} = \frac{(x - 0.2)(x - 0.4)(x - 0.5)}{(0.1 - 0.2)(0.1 - 0.4)(0.1 - 0.5)} = -\left(\frac{x^3 - 1.1x^2 + 0.38x - 0.04}{0.012}\right)$$

$$L_{3,1}(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} = \frac{(x - 0.1)(x - 0.4)(x - 0.5)}{(0.2 - 0.1)(0.2 - 0.4)(0.2 - 0.5)} = \frac{x^3 - x^2 + 0.29x - 0.02}{0.006}$$

$$L_{3,2}(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} = \frac{(x - 0.1)(x - 0.2)(x - 0.5)}{(0.4 - 0.1)(0.4 - 0.2)(0.4 - 0.5)} = -\left(\frac{x^3 - 0.8x^2 + 0.17x - 0.01}{0.006}\right)$$

$$L_{3,3}(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} = \frac{(x - 0.1)(x - 0.2)(x - 0.4)}{(0.5 - 0.1)(0.5 - 0.2)(0.5 - 0.4)} = \frac{x^3 - 0.7x^2 + 0.14x - 0.008}{0.012}$$

$$P_3(x) = L_{3,0}(x)y_0 + L_{3,1}(x)y_1 + L_{3,2}(x)y_2 + L_{3,3}(x)y_3$$

$$= -\left(\frac{x^3 - 1.1x^2 + 0.38x - 0.04}{0.012}\right)(0.981) - \left(\frac{x^3 - x^2 + 0.29x - 0.02}{0.006}\right)(0.928) - \left(\frac{x^3 - 0.8x^2 + 0.17x - 0.01}{0.006}\right)(0.744) + \left(\frac{x^3 - 0.7x^2 + 0.14x - 0.008}{0.012}\right)(0.625)$$

$$= \left(-\frac{0.981}{0.012} - \frac{0.928}{0.006} - \frac{0.744}{0.006} + \frac{0.625}{0.012}\right)x^3 + \left(\frac{(1.1)(0.981)}{0.012} - \frac{(0.928)}{0.006} + \frac{(0.8)(0.744)}{0.006} - \frac{(0.7)(0.625)}{0.012}\right)x^2 +$$

$$+\left(\frac{(-0.38)(0.981)}{0.012} + \frac{(0.29)(0.928)}{0.006} - \frac{(0.17)(0.744)}{0.006} + \frac{(0.14)(0.625)}{0.012}\right)x +$$

$$+\left(\frac{(0.04)(0.981)}{0.012} - \frac{(0.02)(0.928)}{0.006} + \frac{(0.01)(0.744)}{0.006} - \frac{(0.008)(0.625)}{0.012}\right)$$

$$P_3(x) = x^3 - 2x^2 + 1$$

Now, we use another set points to find $Q_3(x)$.

$$x_0 = 0.2,\ x_1 = 0.4,\ x_2 = 0.5,\ x_3 = 0.6$$

$$y_0 = 0.928,\ y_1 = 0.744,\ y_2 = 0.625,\ y_3 = 0.496$$

$$L_{3,0}(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} = \frac{(x - 0.4)(x - 0.5)(x - 0.6)}{(0.2 - 0.4)(0.2 - 0.5)(0.2 - 0.6)} = -\left(\frac{x^3 - 1.5x^2 + 0.74x - 0.12}{0.024}\right)$$

$$L_{3,1}(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} = \frac{(x - 0.2)(x - 0.5)(x - 0.6)}{(0.4 - 0.2)(0.4 - 0.5)(0.4 - 0.6)} = \frac{x^3 - 1.3x^2 + 0.52x - 0.06}{0.004}$$

$$L_{3,2}(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} = \frac{(x - 0.2)(x - 0.4)(x - 0.6)}{(0.5 - 0.2)(0.5 - 0.4)(0.5 - 0.6)} = -\left(\frac{x^3 - 1.2x^2 + 0.44x - 0.048}{0.003}\right)$$

$$L_{3,3}(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} = \frac{(x - 0.2)(x - 0.4)(x - 0.5)}{(0.6 - 0.2)(0.6 - 0.4)(0.6 - 0.5)} = -\left(\frac{x^3 - 1.1x^2 + 0.38x - 0.04}{0.008}\right)$$

$$Q_3(x) = L_{3,0}(x)y_0 + L_{3,1}(x)y_1 + L_{3,2}(x)y_2 + L_{3,3}(x)y_3$$

$$= -\left(\frac{x^3 - 1.5x^2 + 0.74x - 0.12}{0.024}\right)(0.928) + \frac{x^3 - 1.3x^2 + 0.52x - 0.06}{0.004}(0.744) - \left(\frac{x^3 - 1.2x^2 + 0.44x - 0.048}{0.003}\right)(0.625) - \left(\frac{x^3 - 1.1x^2 + 0.38x - 0.04}{0.008}\right)(0.496)$$

$$= \left(-\frac{0.928}{0.024} + \frac{0.744}{0.004} - \frac{0.625}{0.003} + \frac{0.496}{0.008}\right)x^3 + \left(\frac{(-1.5)(0.928)}{0.024} - \frac{(1.3)(0.744)}{0.004} + \frac{(1.2)(0.625)}{0.003} - \frac{(1.1)(0.496)}{0.008}\right)x^2 +$$

$$+\left(\frac{(-0.74)(0.928)}{0.024} + \frac{(0.52)(0.744)}{0.004} - \frac{(0.44)(0.625)}{0.003} + \frac{(0.38)(0.496)}{0.008}\right)x +$$

$$+\left(\frac{(0.12)(0.928)}{0.024} - \frac{(0.06)(0.744)}{0.004} + \frac{(0.048)(0.625)}{0.003} - \frac{(0.04)(0.496)}{0.008}\right)$$

$$Q_3(x) = x^3 - 2x^2 + 1$$

d. So, the exact function $f(x) = x^3 - 2x^2 + 1$

$$f(0.3) = (0.3)^3 - 2(0.3)^2 + 1 = 0.847$$

$$P_1(0.3) = y = -0.92(0.3) + 1.112 = 0.836$$

$$Q_1(0.3) = y = -0.79(0.3) + 1.06 = 0.823$$

Error in Lagrange Polynomial,

$$E_N(x) = \frac{M_{N+1}}{(N+1)!}(x - x_0)(x - x_1) \dots (x - x_N)$$ where $M_{N+1} = f^{N+1}(c)$ and $c \in [x_0, x_N]$

$$|E_N(x)| \leq \frac{M_{N+1}}{(N+1)!}|(x - x_0)(x - x_1) \dots (x - x_N)|$$ where $M_{N+1} = \max_{x_0 < x < x_N} |f^{N+1}(x)|$

So, Error for $P_1(x)$ is $E_1(x)$

$f(x) = x^3 - 2x^2 + 1 \qquad f'(x) = 3x^2 - 4x \quad f''(x) = 6x - 4 \qquad f'''(x) = 6$

$$E_1(x) = \frac{f''(c)}{2!}(x - x_0)(x - x_1) \qquad \textit{where } c \in [0.2, 0.4]$$

Since, $f''(x)$ is a line function, then we should check the value of $f''(x)$ at the end points of $[0.2, 0.4]$.

$$|f''(0.2)| = |6(0.2) - 4| = 2.8$$

$$|f''(0.4)| = |6(0.4) - 4| = 1.6$$

$$|E_1(0.3)| \le \frac{2.8}{2!}|(0.3 - 0.2)(0.3 - 0.4)| = 0.014 \qquad |f(0.3) - P_1(0.3)| = |0.847 - 0.836| = 0.011$$

Error for $Q_1(x)$ is $E_1(x) = \frac{f''(c)}{2!}(x - x_0)(x - x_1)$ where $c \in [0.1,0.4]$

$|f''(0.1)| = |6(0.1) - 4| = 3.4$

$|f''(0.4)| = |6(0.4) - 4| = 1.6$

$|E_1(0.3)| \leq \frac{3.4}{2!}|(0.3 - 0.1)(0.3 - 0.4)| = 0.034$ $\quad |f(0.3) - Q_1(0.3)| = |0.847 - 0.823| = 0.024$

Error for $P_2(x)$ is $E_2(x) = \frac{f'''(c)}{3!}(x - x_0)(x - x_1)(x - x_2)$ where $c \in [0.1,0.4]$

$P_2(0.3) = -1.3(0.3)^2 - 0.14(0.3) + 1.008 = 0.849$

$|E_2(0.3)| \leq \frac{6}{3!}|(0.3 - 0.1)(0.3 - 0.2)(0.3 - 0.4)| = 0.002$ $\quad |f(0.3) - P_2(0.3)| = |0.847 - 0.849| = 0.002$

Error for $P_3(x)$ is zero, since $f^{1v}(x)$ is zero .

Exercise1.

Find Lagrange interpolation polynomial which agrees with the following data and obtain a bound for the errors in estimating the function $f(x) = \sin\left(\frac{\pi x}{2}\right)$.

x	0	0.5	1
y	0	0.707	1

Exercise2.

Consider $y = \cos x \;\; over \;\; [0,1.2]$. Determine the error bounds for the Lagrange polynomials $P_1(x),\; P_2(x) \;\; and \;\; P_3(x)$.

Exercise3.

a) Find a second degree polynomial approximating the function $f(x) = \sqrt{x} + \ln x$, and passing from the points $\left(1, f(1)\right)$, $\left(2, f(2)\right)$ and $(4, f(4))$.

b) Approximate $f(2.5)$ using the polynomial in part (a)

c) use the error formula

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x - x_0)(x - x_1)(x - x_2)\ldots(x - x_N)$$

To estimate the error at x=2.5

Exercise4.

Use lagrange interpolation to approximate y(0) for the following data.

x	2	3	4
y	5	4	2

NEWTON POLYNOMIALS

If the Lagrange interpolation are used , there is no constructive relationship between $P_{N-1}(x)\ \ and\ \ P_N(x)$. Each polynomial has to be constructed individually and the work required to compute the higher degree polynomials involves many computations.
We take the new approach and construct Newton's polynomials that have the recursive pattern.

$$P_1(x) = a_0 + a_1(x - x_0) \tag{1}$$

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) = P_1(x) + a_2(x - x_0)(x - x_1) \tag{2}$$

$$\begin{aligned} P_3(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \\ &= P_2(x) + a_3(x - x_0)(x - x_1)(x - x_3) \end{aligned} \tag{3}$$

$$\vdots$$

$$P_N(x) = P_{N-1}(x) + a_N(x - x_0)(x - x_1)(x - x_3)\ldots(x - x_{N-1}) \tag{4}$$

The polynomial (4) is said to be Newton Polynomial with N centers $x_0, x_1, \ldots, x_{N-1}$. It involves sums of products of linear factors up to

$$a_N(x - x_0)(x - x_1)(x - x_3)\ldots(x - x_{N-1})$$

So $P_N(x)$ will simply to be an ordinary polynomials of degree $\leq N$.

The Newton interpolation uses a divided difference method. The technique allows one to add additional points easily

$$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots$$

For given set of data points: $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2)

If three data points are available, the estimate is improved by introducing some curvature into the line connecting the points.

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

A simple procedure can be used to determine the values of the coefficients

$$x = x_0 \qquad a_0 = f(x_0)$$

$$x = x_1 \qquad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$x = x_2 \qquad a_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

General Form of Newton's Interpolating Polynomials

$$P_N(x) = f(x_0) + (x - x_0)f[x_1, x_0] + (x - x_0)(x - x_1)f[x_2, x_1, x_0]$$
$$+ \cdots + (x - x_0)(x - x_1)\cdots(x - x_{N-1})f[x_N, x_{N-1}, \cdots, x_0]$$

$$a_0 = f(x_0)$$
$$a_1 = f[x_1, x_0]$$
$$a_2 = f[x_2, x_1, x_0]$$
$$\vdots$$
$$a_N = f[x_N, x_{N-1}, \cdots, x_1, x_0]$$

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j}$$

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k}$$
$$\vdots$$
$$f[x_N, x_{N-1}, \ldots, x_1, x_0] = \frac{f[x_N, x_{N-1}, \ldots, x_1] - f[x_{N-1}, x_{N-2}, \ldots, x_0]}{x_N - x_0}$$

and use to construct the divided differences in the following table

x_k	$f[x_k]$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ,\ ]$
x_0	$f(x_0)$				
x_1	$f(x_1)$	$f[x_1, x_0]$			
x_2	$f(x_2)$	$f[x_2, x_1]$	$f[x_2, x_1, x_0]$		
x_3	$f(x_3)$	$f[x_3, x_2]$	$f[x_3, x_2, x_1]$	$f[x_3, x_2, x_1, x_0]$	
x_4	$f(x_4)$	$f[x_4, x_3]$	$f[x_4, x_3, x_2]$	$f[x_4, x_3, x_2, x_1]$	$f[x_4, x_3, x_2, x_1, x_0]$

Theorem: Newton Polynomial

Suppose that $x_0, x_1, \ldots, x_{N-1}$ are N+1 distinct numbers in $[a, b]$. There exist a unique polynomial $P_N(x)$ of degree at most N with the property that

$$f(x_j) = P_N(x_j) \quad for \quad j = 0,1, \ldots, N$$

The Newton form polynomial is

$$P_N(x) = P_{N-1}(x) + a_N(x - x_0)(x - x_1)(x - x_3)\ldots(x - x_{N-1})$$

where $\quad a_k = f[x_k, x_{k-1}, \ldots, x_1, x_0] \quad for \quad k = 0,1, \ldots, N \quad$ and the Newton approximation

$$f(x) = P_N(x) + E_N(x)$$

then

$$E_N(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_N)}{(N + 1)!} f^{(N+1)}(c\Big)$$

EXAMPLE: Let $f(x) = x^3 - 4x$. Construct the divided difference table based on the nodes

$$x_0 = 1\ , x_1 = 2\ , x_2 = 3\ , x_3 = 4\ , x_4 = 5\ , x_5 = 6$$

Find the Newton polynomials $P_3(x)$ base on the above nodes.

x_k	$f[x_k]$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$	$f[\,,\,,\,,\,,\,]$
$x_0 = 1$	$f(1) = -3$				
$x_1 = 2$	$f(2) = 0$	$\frac{0-(-3)}{2-1} = 3$			
$x_2 = 3$	$f(3) = 15$	$\frac{15-0}{3-2} = 15$	$\frac{15-3}{3-1} = 6$		
$x_3 = 4$	$f(4) = 48$	$\frac{48-15}{4-3} = 33$	$\frac{33-15}{4-2} = 9$	$\frac{9-6}{4-1} = 1$	
$x_4 = 5$	$f(5) = 105$	$\frac{105-48}{5-4} = 57$	$\frac{57-33}{5-3} = 12$	$\frac{12-9}{5-2} = 1$	$\frac{1-1}{5-1} = 0$
$x_5 = 6$	$f(6) = 192$	$\frac{192-105}{6-5} = 87$	$\frac{87-57}{6-4} = 15$	$\frac{15-12}{6-3} = 1$	$\frac{1-1}{6-2} = 0$

where $a_0 = -3$, $a_1 = 3$, $a_2 = 6$, $a_3 = 1$, $a_4 = 0$

$$P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$P_3(x) = -3 + 3(x - 1) + 6(x - 1)(x - 2) + 1(x - 1)(x - 2)(x - 3)$$

$$then \;\; add \;\; like \;\; terms$$

$$P_3(x) = x^3 - 4x$$

Example: For the function $y = f(x)$, given the following table, use the divided difference to determine highest possible order and use the Newton's polynomial to determine the value of x^* if $f(x^*) = \frac{11}{3}$.

x_k	$f[x_k]$	$f[\,.\,]$	$f[\,.\,.\,]$
$x_0 = 0$	$f(0) = 1$ $= a_0$		
$x_1 = 1$	$f(1) = 2$	$\frac{2-1}{1-0} = 1 = a_1$	
$x_2 = 3$	$f(3) = 6$	$\frac{6-2}{3-1} = 2$	$\frac{2-1}{3-0} = \frac{1}{3} = a_2$

Highest possible order 2nd order

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

$$P_2(x) = 1 + 1(x - 0) + \frac{1}{3}(x - 0)(x - 1)$$

$$then \;\; add \;\; like \;\; terms$$

$$P_2(x) = \frac{1}{3}x^2 + \frac{2}{3}x + 1 \qquad\qquad find \;\; x^* \quad where \;\; f(\,x^*) = \frac{11}{3}$$

$$\frac{1}{3}x^2 + \frac{2}{3}x + 1 \;\; = \frac{11}{3} \qquad \Rightarrow \;\; \frac{1}{3}x^2 + \frac{2}{3}x - \frac{8}{3} = 0 \quad \Rightarrow \;\; x^2 + 2x - 8 = 0$$

$$(x + 4)(x - 2) = 0 \quad then \;\; x^* = -4 \quad and \quad x^* = 2$$

$$x^* = 2 \in [0,3] \quad , \qquad x^* = -4 \notin [0,3]$$

Exercise1: Use the Newton divided-difference formula to compute the cubic interpolating polynomial $P_3(x)$ for the function $f(x) = \sin(\frac{\pi x}{2})$ using interpolation points

$$x_0 = 0\,,\ x_1 = 1\,,\ x_2 = 2 \quad and \quad x_3 = 3$$

What is an upper bound for the error at x=0.5 ?

Exercise2: Given the table of the function $f(x) = 2^x$

x	0	1	2	3
f(x)	1	2	4	8

a) Write down the Newton polynomials $P_1(x)$, $P_2(x)$, $P_3(x)$.
b) Evaluate f(2.5) by using $P_3(x)$.

Exercise3: For the function $y = f(x)$ given the following table, use the divided differences to determine highest possible order and use Newton's polynomial to determine the value of x^* if $f(x^*) = 0$

x	0	1	2
y	0	-3	-4