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HW8

Question 7, A: Exercise 6.1.5 | b - d

A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

(b) What is the probability that the hand is a three of a kind? A three of a kind has 3 cards of the same rank. The other two cards do not have the same rank as each other and do not have the same rank as the three with the same rank. For example, {4♠, 4♦, 4♣, J♠, 8♥} is a three of a kind.

$$p(E) = \frac{|\,E\,|}{|\,S\,|}$$

First we select a rank, $C(13,1)$, and the suits of the three-of-a-kind, $C(4,3)$. Then we select two other ranks (not including that of the three-of-a-kind), $C(12,2)$, and their suits, $C(4,1) * C(4,1)$. So the probability that a five-card hand is a three-of-a-kind is:

$$|\,E\,| = C(13,1) * C(4,3) * C(12,2) * C(4,1)^2 = 54{,}912$$
$$|\,S\,| = C(52,5) = 2{,}598{,}960$$
$$p(E) = \frac{54{,}912}{2{,}598{,}960} \approx 0.021128$$

(c) What is the probability that all 5 cards have the same suit?

$$p(E) = \frac{|\,E\,|}{|\,S\,|}$$

First, we select a suit, $C(4,1)$, and then five cards, $C(13,5)$. So the probability that all five cards have the same suit is:

$$|\,E\,| = C(4,1) * C(13,5) = 4 * 1287 = 5{,}148$$
$$|\,S\,| = C(52,5) = 2{,}598{,}960$$
$$p(E) = \frac{5{,}148}{2{,}598{,}960} \approx 0.00198$$

(d) What is the probability that the hand is a two of a kind? A two of a kind has two cards of the same rank (called the pair). Among the remaining three cards, not in the pair, no two have the same rank and none of them have the same rank as the pair. For example, {4♠, 4♦, J♠, K♣, 8♥} is a two of a kind.

$$p(E) = \frac{|E|}{|S|}$$

First, we select the rank of the two-of-a-kind, $C(13, 1)$, and the cards' suits, $C(4, 2)$. Then we select the rank of the remaining three cards, making sure no rank repeats, $C(12, 1) * C(11, 1) * C(10,1)$, as well as their suits, $C(4, 1) * C(4, 1) * C(4, 1)$. So the probability of a two-of-a-kind with no rank repeating for the other three cards is:

$$|E| = C(13, 1) * C(4, 2) * C(12, 3) * C(4, 1)^3 = 1{,}098{,}240$$
$$|S| = C(52, 5) = 2{,}598{,}960$$

$$p(E) = \frac{1{,}098{,}240}{2{,}598{,}960} \approx 0.42257$$

Question 7, B: Exercise 6.2.4 | a - d

A 5-card hand is dealt from a perfectly shuffled deck of playing cards. What is the probability of each of the following events?

(a) The hand has at least one club.

$$E = Hand\ has\ at\ least\ one\ club$$
$$\underline{E} = Hand\ does\ not\ have\ a\ club$$

First, we find the probability that a hand does not have a club by selecting a five cards, not including clubs (13 clubs in a deck of 52):

$$|\ \underline{E}\ | = C(39,5)$$
$$p(\underline{E}) = \frac{C(39,5)}{C(52,5)} = \frac{575{,}757}{2{,}598{,}960} = 0.2215336134$$

Then, we take the complement to find the probability that the five-card hand has at least one club:

$$p(E) = 1 - \frac{575{,}757}{2{,}598{,}960} = 1 - 0.2215336134 \approx 0.7785$$

(b) The hand has at least two cards with the same rank.

$$E = Hand\ has\ at\ least\ two\ cards\ of\ the\ same\ rank$$
$$\underline{E} = Hand\ does\ not\ have\ any\ cards\ of\ the\ same\ rank$$

First, we find the probability that a hand does not have any cards of the same rank by selecting five different ranks, $C(13,5)$, and their suits, $C(4,1) * C(4,1) * C(4,1) * C(4,1) * C(4,1)$. So the probability that a five-card hand does not have any cards of the same rank is:

$$|\ \underline{E}\ | = C(13,5) * 4^5$$
$$p(\underline{E}) = \frac{C(13,5) * 4^5)}{C(52,5)} = \frac{1{,}317{,}888}{2{,}598{,}960} = 0.5070828331$$

Then, we take the complement to find that the probability of a five-card hand having at least two cards of the same rank is:

$$p(E) = 1 - 0.5070828331 \approx 0.4929$$

(c) The hand has exactly one club or exactly one spade.

$$C = Hand\ has\ exactly\ one\ club$$
$$S = Hand\ has\ exactly\ one\ spade$$

First, we find how many five-card hands have exactly one club and how many hands have exactly one spade:

$$|\ C\ | = C(13,1) * C(39,4) = 1{,}069{,}263$$
$$|\ S\ | = C(13,1) * C(39,4) = 1{,}069{,}263$$

Then, we find the probability that a five-card hand has exactly one club and exactly one spade:

$$p(C \cup S) = P(C) + P(S) = \frac{C(13,1) * C(39,4)}{C(52,5)} + \frac{C(13,1) * C(39,4)}{C(52,5)}$$
$$= \frac{1{,}069{,}263}{2{,}598{,}960} + \frac{1{,}069{,}263}{2{,}598{,}960} \approx 0.8228$$

(d) The hand has at least one club or at least one spade.

$$C = Hand\ has\ at\ least\ one\ club$$
$$S = Hand\ has\ at\ least\ one\ spade$$
$$C \cup S = Hand\ has\ at\ least\ one\ club\ or\ one\ spade$$
$$\underline{C \cap S} = Hand\ does\ not\ have\ clubs\ or\ spades$$

$$|\ \underline{C \cap S}\ | = C(26,5)$$
$$p(C \cup S) = 1 - \frac{C(26,5)}{C(52,5)} = 1 - \frac{65{,}780}{2{,}598{,}960} = 1 - 0.02531012405 \approx 0.9747$$

Question 8, A: Exercise 6.3.2 | a - e

The letters $\{a, b, c, d, e, f, g\}$ are put in a random order. Each permutation is equally likely. Define the following events:

A: The letter b falls in the middle (with three before it and three after it)
B: The letter c appears to the right of b, although c is not necessarily immediately to the right of b. For example, "$agbdcef$" would be an outcome in this event.
C: The letters "def" occur together in that order (e.g. "$gdefbca$")

(a) Calculate the probability of each individual event. That is, calculate $p(A)$, $p(B)$, and $p(C)$.

6 possible spots for every other letter other than b, so:

$$p(A) = \frac{6!}{7!} = \frac{1}{7} \approx 0.1429$$

$$p(B) = \frac{7!}{2 * 7!} = \frac{1}{2} = 0.5$$

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5 ways for "def" to occur together, in order, so:

$$p(C) = \frac{5!}{7!} = \frac{1}{42} \approx 0.0238$$

(b) What is $p(A|C)$?

$$p(A|C) = \frac{p(A \cap C)}{p(C)}$$

There are only two possible spots for "def" if b is in the middle. For each spot, there are 3! Possible arrangements of the rest of the letters, so:

$$p(A \cap C) = \frac{2 * 3!}{7!} = \frac{1}{420}$$
$$p(C) = \frac{1}{42}$$
$$p(A|C) = \frac{\frac{1}{420}}{\frac{1}{42}} = \frac{1}{10}$$

(c) What is $p(B|C)$?

$$p(B|C) = \frac{p(B \cap C)}{p(C)}$$
$$p(B \cap C) = \frac{1}{7!} * \frac{5!}{2} = \frac{1}{84}$$
$$p(C) = \frac{1}{42}$$
$$p(B|C) = \frac{\frac{1}{84}}{\frac{1}{42}} = \frac{1}{2}$$

(d) What is $p(A|B)$?

$$p(A|B) = \frac{p(A \cap B)}{p(B)}$$
$$p(A \cap B) = \frac{1}{7!} * \frac{6!}{2} = \frac{1}{14}$$
$$p(B) = \frac{1}{2}$$
$$p(A|B) = \frac{\frac{1}{14}}{\frac{1}{2}} = \frac{1}{7}$$

(e) Which pairs of events among$A$, B, and C are independent?

$$P(A \cap B) = \frac{1}{14}$$
$$P(A) * P(B) = \frac{1}{7} * \frac{1}{2} = \frac{1}{14}$$
A and B are independent

$$P(A \cap C) = \frac{1}{420}$$
$$P(A) * P(C) = \frac{1}{7} * \frac{1}{42} = \frac{1}{249}$$
A and C are **NOT** independent

$P(B \cap C) = \frac{1}{84}$

$$P(B) * P(C) = \frac{1}{2} * \frac{1}{42} = \frac{1}{84}$$
B and C are independent

Question 8, B: Exercise 6.3.6 | b, c

A biased coin is flipped 10 times. In a single flip of the coin, the probability of heads is 1/3 and the probability of tails is 2/3. The outcomes of the coin flips are mutually independent. What is the probability of each event?

(b) The first 5 flips come up heads. The last 5 flips come up tails.

> There is a $\frac{1}{3}$ probability of flipping a head, and $\frac{2}{3}$ probability of flipping a tail. So the probability of flipping five heads followed by five tails is:
>
> $$(\frac{1}{3})^5 * (\frac{2}{3})^5 \approx 0.0005419$$

(c) The first flip comes up heads. The rest of the flips come up tails.

> There is a $\frac{1}{3}$ probability of flipping a head, and a $\frac{2}{3}$ probability of flipping a tial. So the probability of getting a head on the first flip, followed by tails on the next nine flips is:
>
> $$(\frac{1}{3}) * (\frac{2}{3})^9 \approx 0.0086709$$

Question 8, C: Exercise 6.4.2 | a

(a) Assume that you have two dice, one of which is fair, and the other is biased toward landing on six, so that 0.25 of the time it lands on six, and 0.15 of the time it lands on each of 1, 2, 3, 4 and 5. You choose a die at random, and roll it six times, getting the values 4, 3, 6, 6, 5, 5. What is the probability that the die you chose is the fair die? The outcomes of the rolls are mutually independent.

$$F = Fair\ die\ selected$$
$$\underline{F} = Loaded\ die\ selected$$
$$X = The\ die\ comes\ out\ to\ \{4, 3, 6, 6, 5, 5\}$$

$$p(F) = 0.5$$
$$p(\underline{F}) = 0.5$$
$$p(X|F) = \frac{1}{6^6} = \frac{1}{46{,}656} = 0.00002143347051$$
$$p(X|\underline{F}) = 0.15 * 0.15 * 0.25 * 0.25 * 0.15 * 0.15 = 0.000031640625$$

$$\begin{aligned} p(F|X) &= \frac{p(X|F) * p(F)}{p(X|F) * p(F) + p(X|\underline{F}) * p(\underline{F})} \\ &= \frac{0.00002143347051 * 0.5}{(0.00002143347051 * 0.5) + (0.000031640625 * 0.5)} \\ &= \frac{0.00001071673525}{0.00001071673525 + 0.0000158203125} \\ &\approx 0.404 \end{aligned}$$

Question 9, A: Exercise 6.5.2 | a, b

A hand of 5 cards is dealt from a perfectly shuffled deck of playing cards. Let the random variable A denote the number of aces in the hand.

(a) What is the range of A?

There are 4 aces and the number of aces in a five-card hand can range from 0 to 4.

$$\{0, 1, 2, 3, 4\}$$

(b) Give the distribution over the random variable A.

$$(0, \frac{C(48,5)}{C(52,5)}), (1, 4 * \frac{C(48,4)}{C(52,5)}), (2, C(4,2) * \frac{C(48,3)}{C(52,5)}), (3, 4 * \frac{C(48,2)}{C(52,5)}), (4, \frac{C(48,1)}{C(52,5)})$$

Question 9, B: Exercise 6.6.1 | a

(a) Two student council representatives are chosen at random from a group of 7 girls and 3 boys. Let G be the random variable denoting the number of girls chosen. What is $E[G]$?

The range of G can be 0 girls, 1 girl, or 2 girls. So:

$$G = \{0, 1, 2\}$$

Number of ways to choose representatives from total girls and boys:

$$C(10, 2) = 45$$

Number of ways to choose representatives from boys:

$$C(3, 2) = 3$$

Number of ways to choose representatives from girls:

$$C(7, 2) = 21$$

$$E[G] = (0 * \frac{3}{45}) + (1 * \frac{21}{45}) + (2 * \frac{21}{45}) = 1.4$$

Question 9, C: Exercise 6.6.4 | a, b

(a) A fair die is rolled once. Let X be the random variable that denotes the square of the number that shows up on the die. For example, if the die comes up 5, then $X = 25$. What is $E[X]$?

$$X = \{1, 4, 9, 16, 25, 36\}$$
$$E[X] = (1 * \frac{1}{6}) + (4 * \frac{1}{6}) + (9 * \frac{1}{6}) + (16 * \frac{1}{6}) + (25 * \frac{1}{6}) + (36 * \frac{1}{6}) \approx 15.1667$$

(b) A fair coin is tossed three times. Let Y be the random variable that denotes the square of the number of heads. For example, in the outcome HTH, there are two heads and $Y = 4$. What is $E[Y]$?

Possible outcomes:

$HHH, Y = 9$	$TTT, Y = 0$
$HHT, Y = 4$	$TTH, Y = 1$
$HTT, Y = 1$	$THH, Y = 4$
$HTH, Y = 4$	$THT, Y = 1$

$$Y = \{0, 1, 4, 9\}$$
$$E[Y] = (0 * \frac{1}{8}) + (1 * \frac{3}{8}) + (4 * \frac{3}{8}) + (9 * \frac{1}{8}) = 3$$

Question 9, D: Exercise 6.7.4 | a

(a) A class of 10 students hang up their coats when they arrive at school. Just before recess, the teacher hands one coat selected at random to each child. What is the expected number of children who get his or her own coat?

Let C = total number of coats
Let C_i = a random variable that is 1 if the i^{th} coat belongs to the correct child, 0 if not.

$$C = C_1 + C_2 + C_3 + C_4 + C_5 + C_6 + C_7 + C_8 + C_9 + C_{10}$$

$$p(C_1 = 1) = \frac{1}{10}$$
$$E[C_1] = 1 * \frac{1}{10} + 0 * \frac{9}{10} = \frac{1}{10}$$

All the $E[C_i]$ are the same, so:

$$E[C] = E[C_1] + E[C_2] + E[C_3] + E[C_4] + E[C_5] + E[C_6] + E[C_7] + E[C_8] + E[C_9] + E[C_{10}]$$
$$E[C] = 10 * (\frac{1}{10}) = 1$$

Question 10, A: Exercise 6.8.1 | a - d

The probability that a circuit board produced by a particular manufacturer has a defect is 1%. You can assume that errors are independent, so the event that one circuit board has a defect is independent of whether a different circuit board has a defect.

(a) What is the probability that out of 100 circuit boards made exactly 2 have defects?

$$b(k;\ n, p) = C(n, k) * p^k q^{n-k}$$
$$= C(100, 98) * {\frac{99}{100}}^{98} * {\frac{1}{100}}^{2} = 4{,}950 * 0.3734642805 * 0.0001 \approx 0.1849$$

(b) What is the probability that out of 100 circuit boards made at least 2 have defects?

First, we find the probability of zero defects:

$$b(k;\ n, p) = C(n, k) * p^k q^{n-k}$$
$$= C(100, 100) * {\frac{99}{100}}^{100} * {\frac{1}{100}}^{0} = 1 *\ 0.3660323413 * 1 = 0.3660323413$$

Next, we find the probability of one defect:

$$b(k;\ n, p) = C(n, k) * p^k q^{n-k}$$
$$= C(100, 99) * {\frac{99}{100}}^{99} * {\frac{1}{100}}^{1} = 100 * 0.3697296376 * 0.01 = 0.3697296376$$

Then, we find the probability of at least 2 defects:

$$= 1 - (Probability\ of\ zero\ defects\ +\ Probability\ of\ one\ defect)$$
$$= 1 - (0.3660323413 + 0.3697296376) \approx 0.2642$$

(c) What is the expected number of circuit boards with defects out of the 100 made?

$$D = Circuit\ boards\ with\ defects\ out\ of\ the\ 100\ made$$
$$E[D] = n * p = 100 * \frac{1}{100} = 1$$

(d) Now suppose that the circuit boards are made in batches of two. Either both circuit boards in a batch have a defect or they are both free of defects. The probability that a batch has a defect is 1%. What is the probability that out of 100 circuit boards (50 batches) at least 2 have defects? What is the expected number of circuit boards with defects out of the 100 made? How do your answers compared to the situation in which each circuit board is made separately?

Probability that out of 100 circuit boards (50 batches) at least 2 have defects:

First, we find the probability of zero defects out of 50 batches:

$$b(k;\ n,p) = C(n,k) * p^k q^{n-k}$$
$$= C(50,50) * \frac{99}{100}^{50} * \frac{1}{100}^{0} = 1 * 0.6050060671 * 1 = 0.6050060671$$

Next, we find the probability of one defect out of 50 batches:

$$b(k;\ n,p) = C(n,k) * p^k q^{n-k}$$
$$= C(50,49) * \frac{99}{100}^{49} * \frac{1}{100}^{1} = 50 * 0.6111172395 * 0.01 = 0.3055586198$$

Then, we find the probability of at least two defects:

$$= 1 - (Probability\ of\ zero\ defects + Probability\ of\ one\ defect)$$
$$= 1 - (0.6050060671 + 0.3055586198) \approx 0.0894$$

Expected number of defects out of the 100 made:

$$D = Circuit\ boards\ with\ defects\ out\ of\ the\ 100\ made\ (50\ batches)$$
$$E[D] = n * p = 50 * 0.01 = 0.5$$

Compared to section (c) where each board is made separately, the probability of at least two defects is greatly reduced, and the number of expected defects per 100 is halved when done in batches of 50.

Question 10, B: Exercise 6.8.3 | b

A gambler has a coin which is either fair (equal probability heads or tails) or is biased with a probability of heads equal to 0.3. Without knowing which coin he is using, you ask him to flip the coin 10 times. If the number of heads is at least 4, you conclude that the coin is fair. If the number of heads is less than 4, you conclude that the coin is biased.

(b) What is the probability that you reach an incorrect conclusion if the coin is biased?

First, we find the probability that there are less than 4 heads:

$$p(x < 4) = p(0) + p(1) + p(2) + p(3)$$
$$= (C(10,0) * 0.3^0 * 0.7^{10}) + (C(10,1) * 0.3^1 * 0.7^9) + (C(10,2) * 0.3^2 * 0.7^8) + (C(10,3) * 0.3^3 * 0.7^7)$$
$$= (1 * 1 * 0.0282475249) + (10 * 0.3 * 0.040353607) + (45 * 0.09 * 0.05764801) + (120 * 0.027 * 0.0823543)$$
$$= 0.6496107184$$

Then, we find the probability that there are at least 4 heads:

$$p(x \geq 4) = 1 - 0.6496107184 \approx 0.35$$