Yulian Kraynyak
HW7

Question 3, A: Exercise 8.2.2 | b

(b) $f(n) = n^3 + 3n^2 + 4$. Prove that $f = \Theta(n^3)$.

$$f = O(n^3)$$

Proof:

Let $c = 8$ and $n_0 = 1$. For $1 \leq n, n \leq n^3$, so:

$$n^3 \leq n^3$$
$$3n^2 \leq 3n^3$$
$$4 \leq 4n^3$$

After adding the inequalities the resulting expression is:

$$n^3 + 3n^3 + 4n^3 = 8n^3$$

Therefore, $f = O(n^3)$ because $n^3 + 3n^2 + 4 \leq 8n^3$ for $n \geq 1$.

$$f = \Omega(n^3)$$

Proof:

Let $c = 1$ and $n_0 = 1$. If we drop every term except for n^3 the resulting expression is smaller:

$$n^3 + 3n^2 + 4 \geq n^3$$

Therefore, $f = \Omega(n^3)$ because $n^3 + 3n^2 + 4 \geq n^3$ for $n \geq 1$.

Question 3, B: Exercise 8.3.5 | a - e

(a) Describe in English how the sequence of numbers is changed by the algorithm. (Hint: try the algorithm out on a small list of positive and negative numbers with $p = 0$)

> The algorithm rearranges the sequence of numbers by using p as the threshold, arranging numbers less than p to the left of numbers greater than p.

(b) What is the total number of times that the lines `i := i + 1` or `j := j - 1` are executed on a sequence of length n? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the two lines are executed.

> The number of times each individual line is executed depends on the values (negative and positive when $p = 0$) of the numbers in the sequence. `i := i + 1` is triggered on negative numbers and `j := j - 1` is triggered on positive numbers.
>
> **Example:**
> Input: `-1, 4, -2, 9, 8, -3, 2`
> Output: `-1, -3, -2, 9, 8, 4, 2`
>
> Execution count for `i := i + 1` : 3
> Execution count for `j := j - 1` : 4
>
> Assuming $p = 0$:
> To maximize `i := i + 1` executions, all the numbers in the input have to be negative.
> To maximize `j := j - 1` executions, all the numbers in the input have to be positive.

(c) What is the total number of times that the swap operation is executed? Does your answer depend on the actual values of the numbers in the sequence or just the length of the sequence? If so, describe the inputs that maximize and minimize the number of times the swap is executed.

> The total number of swaps is the number of numbers greater than p in the input sequence.
>
> The answer depends on the **actual values** (less than or greater than p) of the numbers in the sequence, not just the length of the sequence.
>
> Assuming $p = 0$:
> To maximize swap executions: input a sequence with all positive numbers.
> To minimize swap executions: input a sequence with all negative numbers.

(d) Give an asymptotic lower bound for the time complexity of the algorithm. Is it important to consider the worst-case input in determining an asymptotic lower bound (using Ω) on the time complexity of the algorithm? (Hint: argue that the number of swaps is at most the number of times that i is incremented or j is decremented).

> $\Omega(n)$

(e) Give a matching upper bound (using O-notation) for the time complexity of the algorithm.

> $O(n)$

Question 4, A: Exercise 5.1.1 | b, c

Consider the following definitions for sets of characters:

Digits $= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
Letters $= \{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$
Special characters $= \{*, \&, \$, \#\}$

(b) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters.

Let S be the set of special characters, D the set of digits, and L the set of letters. The 3 sets are mutually disjoint, so the total number of characters is:

$$|\, S \cup D \cup L \,| = |\, S \,| + |\, D \,| + |\, L \,| = 4 + 10 + 26 = 40$$

$$Length\ of\ 7 = |\, S \cup D \cup L \,|^7 = 40^7$$
$$Length\ of\ 8 = |\, S \cup D \cup L \,|^8 = 40^8$$
$$Length\ of\ 9 = |\, S \cup D \cup L \,|^9 = 40^9$$

Total number of passwords:
$$40^7 + 40^8 + 40^9 = 268{,}861{,}440{,}000{,}000$$

(c) Strings of length 7, 8, or 9. Characters can be special characters, digits, or letters. The first character cannot be a letter.

Let S be the set of special characters, D the set of digits, and L the set of letters.

The 3 sets are mutually disjoint, so the total number of possible characters for the first character is:

$$|\, S \cup D \,| = |\, S \,| + |\, D \,| = 4 + 10 = 14$$

And total possible characters for the rest following the first character is:

$$|\, S \cup D \cup L \,| = |\, S \,| + |\, D \,| + |\, L \,| = 4 + 10 + 26 = 40$$

$$Length\ of\ 7 = |\, S \cup D \,| * |\, S \cup D \cup L \,|^6 = 14 \quad * 40^6$$
$$Length\ of\ 8 = |\, S \cup D \,| * |\, S \cup D \cup L \,|^7 = 14 \quad * 40^7$$
$$Length\ of\ 9 = |\, S \cup D \,| * |\, S \cup D \cup L \,|^8 = 14 \quad * 40^8$$

Total number of passwords:
$$14 * (40^6 + 40^7 + 40^8) = 94{,}101{,}504{,}000{,}000$$

Question 4, B: Exercise 5.3.2 | a

(a) How many strings are there over the set $\{a, b, c\}$ that have length 10 in which no two consecutive characters are the same? For example, the string *"abcbcbabcb"* would count and the strings *"abbbcbabcb"* and *"aacbcbabcb"* would not count.

The first character can be any of the three characters a, b, c, but the following nine characters can only be one of two characters since they cannot match the previous character.

$$3 * 2^9 = 1{,}536$$

Question 4, C: Exercise 5.3.3 | b, c

License plate numbers in a certain state consist of seven characters. The first character is a digit (0 - 9). The next four characters are capital letters (A - Z) and the last two characters are digits. Therefore, a license plate number in this state can be any string of the form:

$$Digit - Letter - Letter - Letter - Letter - Digit - Digit$$

(b) How many license plate numbers are possible if no digit appears more than once?

Let L be the set of capital letters (A - Z) and D be the set of digits (0 - 9).

$$|\, L \,| = 26$$
$$|\, D \,| = 10$$

There are 10 possible digits for the first value, 26 possible letters for each of the following four values, 9 possible digits for the sixth value, and 8 possible digits for the seventh value.

So the total number of possible license plates is:

$$10 * 26^4 * 9 * 8 = 329{,}022{,}720$$

(c) How many license plate numbers are possible if no digit or letter appears more than once?

Let L be the set of capital letters (A - Z) and D be the set of digits (0 - 9).

$$|\, L \,| = 26$$
$$|\, D \,| = 10$$

There are 10 possible digits for the first value, 26 possible letters for the second value, 25 possible letters for the third value, 24 possible letters for the fourth value, 23 possible letters for the fifth value, 9 possible digits for the sixth value, and 8 possible digits for the seventh value.

So the total number of possible license plate numbers is:

$$10 * 26 * 25 * 24 * 23 * 9 * 8 = 258{,}336{,}000$$

Question 4, D: Exercise 5.2.3 | a, b

Let $B = \{0, 1\}$. B^n is the set of binary strings with n bits. Define the set E_n to be the set of binary strings with n bits that have an even number of 1's. Note that zero is an even number, so a string with zero 1's (i.e., a string that is all 0's) has an even number of 1's.

(a) Show a bijection between B^9 and E_{10}. Explain why your function is a bijection.

First of all, B^9 and E_{10} have the same cardinality. We divide 2^{10} by 2 because we are only concerned with the binary strings with an even number of 1’s:

$$| B^9 | = 2^9 = 512$$
$$| E_{10} | = 2^{10}/2 = 512$$

If x is a string with an even number of 1’s, we add a 0 to it, and if x has an odd number of 1’s, we add a 1, making it an even number of 1’s.

Thus our bijection is:

$$f(x) = \begin{cases} x + 0 & \text{if } x \in E(9) \\ x + 1 & \text{if } n \notin E(9) \end{cases}$$

(b) What is $| E_{10} |$?

$$2^{10}/2 = 512$$

Question 5, A: Exercise 5.4.2 | a, b

At a certain university in the U.S., all phone numbers are 7-digits long and start with either 824 or 825.

(a) How many different phone numbers are possible?

Let $P = \{824, 825\}$ and D be the set of digits 0 to 9.

$$| P | = 2$$
$$| D | = 10$$

There are only two possible prefixes to pick for the first three digits, leaving four remaining digits which can be any digit from 0 to 9, so the total number of phone numbers is:

$$2 * 10^4 = 20{,}000$$

(b) How many different phone numbers are there in which the last four digits are all different?

Let $P = \{824, 825\}$ and D be the set of digits 0 to 9.

$$| P | = 2$$
$$| D | = 10$$

There are only two possible prefixes to pick from for the first three digits. Given that the last four digits can't repeat, there are 10 digits to choose from for the 4th digit, 9 for the 5th digit, 8 for the 6th digit and 7 for the 7th digit.

$$2 * 10 * 9 * 8 * 7 = 10{,}080$$

Question 5, B: Exercise 5.5.3 | a - g

How many 10-bit strings are there subject to each of the following restrictions?

(a) No restrictions.

$$2^{10} = 1{,}024$$

(b) The string starts with 001.

$$2^7 = 128$$

(c) The string starts with 001 or 10.

$$2^7 + 2^8 = 128 + 256 = 384$$

(d) The first two bits are the same as the last two bits.

$$2^8 = 256$$

(e) The string has exactly six 0's.

$$C(10,6) = \frac{10!}{6!\,(10-6)!} = 210$$

(f) The string has exactly six 0's and the first bit is 1.

$$C(9,6) = \frac{9!}{6!\,(9-6)!} = 84$$

(g) There is exactly one 1 in the first half and exactly three 1's in the second half.

$$C(5,1) * C(5,3) = 5 * \frac{5!}{3!\,(5-3)!} = 5 * 10 = 50$$

Question 5, C: Exercise 5.5.5 | a

(a) There are 30 boys and 35 girls that try out for a chorus. The choir director will select 10 girls and 10 boys from the children trying out. How many ways are there for the choir director to make his selection?

$$C(30,10) * C(35,10) = \frac{30!}{10!\,(30-10)!} * \frac{30!}{10!\,(30-10)!} = 30{,}045{,}015 * 183{,}579{,}396$$

Question 5, D: Exercise 5.5.8 | c - f

This question refers to a standard deck of playing cards. If you are unfamiliar with playing cards, there is an explanation in "Probability of an event" section under the heading "Standard playing cards." A five-card hand is just a subset of 5 cards from a deck of 52 cards.

(c) How many five-card hands are made entirely of hearts and diamonds?

A deck of 52 cards contains 13 cards that are hearts, 13 cards that are diamonds. Together, there are 26 possible cards that can make a five-card hand of hearts and diamonds.

$$C(26,5) = \frac{26!}{5!\,(26-5)!} = 65{,}780$$

(d) How many five-card hands have four cards of the same rank?

Choose the rank of the four-of-a-rank: $C(13,1) = 13$
Choose the suits of the four-of-a-rank: $C(4,4) = 1$
Choose the rank of the fifth card (different rank): $C(12,1) = 12$
Choose the suit of the fifth card: $C(4,1) = 4$

$$C(13,1) * C(4,4) * C(12,1) * C(4,1) = 13 * 1 * 12 * 4 = 624$$

(e) A "full house" is a five-card hand that has two cards of the same rank and three cards of the same rank. For example, {queen of hearts, queen of spades, 8 of diamonds, 8 of spades, 8 of clubs}. How many five-card hands contain a full house?

Choose the rank of the two-of-a-rank: $C(13,1) = 13$
Choose the suits of the two-of-a-rank: $C(4,2) = \frac{4!}{2!(4-2)!} = 6$
Choose the rank of the three-of-a-rank (different rank): $C(12,1) = 12$
Choose the suit of the three-of-a-rank: $C(4,3) = \frac{4!}{3!(4-3)!} = 4$

$$C(13,1) * C(4,2) * C(12,1) * C(4,3) = 13 * 6 * 12 * 4 = 3{,}744$$

(f) How many five-card hands do not have any two cards of the same rank?

There are $C(13, 5)$ ways to choose five different ranks of cards. There are 4 suits to choose from for each rank. So the total number of five-card hands that don't have any two cards of the same rank is:

$$C(13, 5) * 4^5 = \frac{13!}{5!\,(13-5)!} * 1{,}024 = 1{,}287 * 1{,}024 = 1{,}317{,}888$$

Question 5, E: Exercise 5.6.6 | a, b

A country has two political parties, the Demonstrators and the Repudiators. Suppose that the national senate consists of 100 members, 44 of which are Demonstrators and 56 of which are Repudiators.

(a) How many ways are there to select a committee of 10 senate members with the same number of Demonstrators and Repudiators?

Choose Demonstrators: $C(44,5) = \frac{44!}{5!(44-5)!} = 1{,}086{,}008$
Choose Repudiators: $C(56,5) = \frac{56!}{5!(56-5)!} = 3{,}819{,}816$

$$C(44,5) * C(56,5) = 1{,}086{,}008 * 3{,}819{,}816 = 414{,}834{,}421{,}848$$

(b) Suppose that each party must select a speaker and a vice speaker. How many ways are there for the two speakers and two vice speakers to be selected?

Choose speaker from Demonstrators: $C(44,1) = 44$
Choose vice speaker from Demonstrators: $C(43,1) = 43$
Choose speaker from Repudiators: $C(56,1) = 56$
Choose vice speaker from Repudiators: $C(55,1) = 55$

$$C(44,1) * C(43,1) * C(56,1) * C(55,1) = 44 * 43 * 56 * 55 = 5{,}827{,}360$$

Question 6, A: Exercise 5.7.2 | a, b

A 5-card hand is drawn from a deck of standard playing cards.

(a) How many 5-card hands have at least one club?

In a deck of 52 cards, there are $C(52, 5)$ ways to pick a five-card hand. There are $C(39, 5)$ ways to pick a five-card hand without a club (52 minus 13 cards which are clubs). So, by taking the complement, the total number of five-card hands which have at least one club is:

$$C(52, 5) - C(39, 5) = \frac{52!}{5!\,(52-5)!} - \frac{39!}{5!\,(39-5)!} = 2{,}598{,}960 - 575{,}757 = 2{,}023{,}203$$

(b) How many 5-card hands have at least two cards with the same rank?

In a deck of 52 cards, there are $C(13, 5)$ ways to choose five different ranks of cards. Then there are four different suits to choose from, for each rank. So the total number of five-card hands **without** any two cards of the same rank is $C(13, 5) * 4^5$.

To find the total number of five-card hands with at least two cards of the same rank we take the complement:

$$C(52, 5) - C(13, 5) * 4^5 = 2{,}598{,}960 - 1{,}287 * 1{,}024 = 1{,}281{,}072$$

Question 6, B: Exercise 5.8.4 | a, b

20 different comic books will be distributed to five kids.

(a) How many ways are there to distribute the comic books if there are no restrictions on how many go to each kid (other than the fact that all 20 will be given out)?

Each book can be given to one of five kids, and there are 20 books. So the total number of ways to distribute all of the comic books to five kids is:

$5 * 5 * 5 * 5. .. \textit{upto 20 times} = 5^{20} = 95{,}367{,}431{,}640{,}625$

(b) How many ways are there to distribute the comic books if they are divided evenly so that 4 go to each kid?

There are twenty books in total, five kids, and four possible books each kid can get. So the total number of ways to evenly distribute the comic books is:

$$\frac{20!}{4!\,4!\,4!\,4!\,4!} = \frac{20!}{4!^5} = 305{,}540{,}235{,}000$$

Question 7:

How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

(a) 4

Let X be the set of 5 elements and Y be a set of 4 elements.

A function is one-to-one if each element in the domain X maps to exactly one element in the target Y. The domain X has 5 elements while the target Y has 4, so it's impossible for the function to be one-to-one, since an element in X will have to be mapped to two elements in Y or not be mapped at all.

So the total number of one-to-one functions is: **zero**

(b) 5

Let X be the set of 5 elements and Y be a set of 5 elements.

A function is one-to-one if each element in the domain X maps to exactly one element in the target Y.

The first element in X can be mapped to 5 possible elements in Y.
The second element in X can be mapped to 4 possible elements in Y.
The third element in X can be mapped to 3 possible elements in Y.
The fourth element in X can be mapped to 2 possible elements in Y.
The fifth element in X can be mapped to 1 remaining element in Y.

So the total number of one-to-one functions is:

$$5 * 4 * 3 * 2 * 1 = 120$$

(c) 6

Let X be the set of 5 elements and Y be a set of 6 elements.

A function is one-to-one if each element in the domain X maps to exactly one element in the target Y.

The first element in X can be mapped to 6 possible elements in Y.
The second element in X can be mapped to 5 possible elements in Y.
The third element in X can be mapped to 4 possible elements in Y.
The fourth element in X can be mapped to 3 possible elements in Y.
The fifth element in X can be mapped to 2 possible elements in Y.

So the total number of one-to-one functions is:

$$6 * 5 * 4 * 3 * 2 = 720$$

(d) 7

Let X be the set of 5 elements and Y be a set of 7 elements.

A function is one-to-one if each element in the domain Y maps to exactly one element in the target Y.

The first element in X can be mapped to 7 possible elements in Y.
The second element in X can be mapped to 6 possible elements in Y.
The third element in X can be mapped to 5 possible elements in Y.
The fourth element in X can be mapped to 4 possible elements in Y.
The fifth element in X can be mapped to 3 possible elements in Y.

So the total number of one-to-one functions is:

$$7 * 6 * 5 * 4 * 3 = 2520$$