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HW11

Question 5:

(a) Use mathematical induction to prove that for any positive integer n, 3 divides $n^3 + 2n$ (leaving no remainder).

Hint: you may want to use the formula: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Proof.
By induction on n.

Base case:
For the base case, we need to prove true for $n = 1$; 3 evenly divides $P(1)$.

$$P(1) = (1)^3 + 2(1) = 3$$

3 evenly divides 3. ✓

Therefore, $P(1)$ is true.

Inductive case:
Assume true for $n = k$; 3 evenly divides $P(k)$:

$P(k) = k^3 + 2k$ is evenly divisible by 3

Which can be expressed as:

$k^3 + 2k = 3m$, for some integer m

We will prove true for $n = k + 1$; 3 evenly divides P(k+1):

$$P(k+1) = (k+1)^3 + 2(k+1)$$

$$= k^3 + 3k^2(1) + 3k(1)^2 + (1)^3 + 2k + 2$$

$$= k^3 + 3k^2 + 3k + 2k + 3$$

We substitute $k^3 + 2k$ with the inductive hypothesis, $3m$:

$$= 3m + 3k^2 + 3k + 3$$

$$= 3(m + k^2 + k + 1)$$

Since m and k are integers, $3(m + k^2 + k + 1)$ can be expressed as 3 times an integer j, or $3(j)$ which is also an integer.

3 evenly divides $3(j)$, therefore 3 evenly divides $(k + 1)^3 + 2(k + 1)$ ✓

Therefore, $P(k + 1)$ is true.

(b) Use strong induction to prove that any positive integer $n (n \geq 2)$ can be written as a product of primes.

Proof.
By strong induction on n.

Base case:
For the base case, we need to prove true for $n = 2$.

Since 2 is a prime number, it already is a product of one prime number: 2. ✓

Therefore, $P(2)$ is true.

Inductive case:
Assume that for $k \geq 2$, any integer j in the range from 2 through k can be expressed as a product of prime numbers. We will show that $k + 1$ can be expressed as a product of prime numbers.

If $k + 1$ is prime, then it can be expressed as the product of itself and 1. If $k + 1$ is not prime, then it is a product of two other integers, a, b, where $a \geq 2$ and $b \geq 2$, which are prime numbers:

$$k + 1 = a \cdot b$$

Which can be expressed as:

$$a = \frac{(k+1)}{b}$$

$$b = \frac{(k+1)}{a}$$

Since $a \geq 2$ and $b \geq 2$:

$$a = \frac{k+1}{b} < k + 1$$

$$b = \frac{(k+1)}{a} < k + 1$$

Therefore, $k + 1$ can be expressed as the product of primes.

Question 6, A: Exercise 7.4.1 | a - g

Define $P(n)$ to be the assertion that:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Verify that $P(3)$ is true.

$$\sum_{j=1}^{3} j^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$$

$$\frac{3(3+1)(2 \cdot 3 + 1)}{6} = \frac{3 \cdot 4 \cdot 7}{6} = 14$$

$$14 = 14 \checkmark$$

Therefore, $P(3)$ is true.

(b) Express $P(k)$.

$$P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

(c) Express $P(k+1)$.

$$P(k+1) = \sum_{j=1}^{k+1} j^2 = \frac{(k+1) \cdot ((k+1)+1) \cdot (2(k+1)+1)}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

(d) In an inductive proof that for every positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

what must be proven in the base case?

For the base case, we need to prove true for $n = 1$.

$$\sum_{j=1}^{1} j^2 = 1^2 = 1$$

$$\frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{6}{6} = 1$$

$$1 = 1 \checkmark$$

Therefore, $P(1)$ is true.

(e) In an inductive proof that for every positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

what must be proven in the inductive step?

In the inductive step, we must prove true for $n = k + 1$.

$$P(k+1) = \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

(f) What would be the inductive hypothesis in the inductive step from your previous answer?

$$P(k+1) = \sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

(g) Prove by induction that for any positive integer n,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof.
By induction on n.

Base case:
For the base case, we need to prove true for $n = 1$.

$$\sum_{j=1}^{1} j^2 = 1^2 = 1$$

$$\frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{6}{6} = 1$$

$$1 = 1 \checkmark$$

Therefore, $P(1)$ is true.

Inductive case:
Assume true for $n = k$:

$$P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

We will prove true for $n = k + 1$:

$$P(k+1) = \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$P(k+1) = \sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{(k+1)(k(2k+1)+6(k+1))}{6}$$

$$= \frac{(k+1)(2k^2+k)+6k+6}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Therefore, $P(k+1)$ is true.

Question 6, B: Exercise 7.4.3 | c

Prove each of the following statements using mathematical induction.

(Hint: You may want to use the following fact: $\frac{1}{(k+1)^2} \leq \frac{1}{k(k+1)}$)

(c) Prove that for $n \geq 1$,

$$\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$$

Proof.
By induction on n.

Base case:
For the base case, we need to prove true for $n \geq 1$.

$$\sum_{j=1}^{1} \frac{1}{j^2} = \frac{1}{1^2} = 1$$

$$2 - \frac{1}{1} = 1$$

$$1 \geq 1 \checkmark$$

Therefore, $P(1)$ is true.

Inductive case:
We assume true for $n = k$:

$$\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$$

We will prove true for $k \geq 1$:

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$$

$$\sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{k(k+1)}$$

$$\leq 2 - \frac{k+1}{k(k+1)} + \frac{1}{k(k+1)}$$

$$\leq 2 - \frac{k+1}{k(k+1)}$$

$$\leq 2 - \frac{1}{k}$$

$$2 - \frac{1}{k} \leq 2 - \frac{1}{k+1}$$

✓

Therefore, $P(k \geq 1)$ is true.

Question 6, C: Exercise 7.5.1 | a

Prove each of the following statements using mathematical induction.

(a) Prove that for any positive integer n, 4 evenly divides $3^{2n} - 1$.

Proof.
By induction on n.

Base case:
For the base case, we need to prove true for $n = 1$; 4 evenly divides $P(1)$.

$$P(1) = 3^{2(1)} - 1 = 8$$

4 evenly divides 8. ✓

Therefore, $P(1)$ is true.

Inductive case:
We assume true for $n = k$, 4 evenly divides $P(k)$:

$$P(k) = 3^{2k} - 1 \text{ is evenly divisible by 4}$$

Which can be expressed as:

$$3^{2k} - 1 = 4m,$$
$$3^{2k} = 4m + 1, \text{ for some integer } m.$$

We will prove true for $n = k + 1$; 4 evenly divides $P(k+1)$:

$$P(k+1) = 3^{2(k+1)} - 1$$

$$= 3^{2k+2} - 1$$

$$= 9 \cdot 3^{2k} - 1$$

We substitute 3^{2k} with the inductive hypothesis, $4m + 1$:

$$= 9 \cdot (4m + 1) - 1$$

$$= 36m + 8$$

$$= 4(9m + 2)$$

Since m and k are integers, $4(9m + 2)$ can be expressed as 4 times an integer j, or $4(j)$, which is also an integer.

4 evenly divides $4(j)$, therefore 4 evenly divides $3^{2(k+1)} - 1$. ✓

Therefore, $P(k+1)$ is true.