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HW5

Question 3, A: Exercise 4.1.3, sections b, c

Which of the following are functions from R to R? If f is a function, give its range.

(b) $f(x) = 1/(x^2 - 4)$

Not a function. If $x = 2$ or $x = -2$, then $f(x)$ is undefined.

(c) $f(x) = \sqrt{x^2}$

$f(x)$ is a function. Its range is $x \geq 0$, for all positive real numbers.

Question 3, B: Exercise 4.1.5, sections b, d, h, i, l

Express the range of each function using roster notation.

(b) Let $A = \{2, 3, 4, 5\}$. $f: A \rightarrow \mathbb{Z}$ such that $f(x) = x^2$.

$$\{4, 9, 16, 25\}$$

(d) $f: \{0, 1\}^5 \rightarrow \mathbb{Z}$. For $x \in \{0, 1\}^5, f(x)$ is the number of 1's that occur in x.
For example$f(01101) = 3$, because there are three 1's in the string "01101".

$$\{0, 1, 2, 3, 4, 5\}$$

(h) Let $A = \{1, 2, 3\}$. $f: A \times A \rightarrow \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (y, x)$.

$$A \times A = \{1, 2, 3\} \times \{1, 2, 3\}$$
$$A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

$$\{(1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2), (1, 3), (2, 3), (3, 3)\}$$

(i) Let $A = \{1, 2, 3\}$. $f: A \times A \rightarrow \mathbb{Z} \times \mathbb{Z}$, where $f(x, y) = (x, y + 1)$.

$$A \times A = \{1, 2, 3\} \times \{1, 2, 3\}$$
$$A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

$$\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$$

(l) Let $A = \{1, 2, 3\}$. $f: P(A) \rightarrow P(A)$. For $X \subseteq A$, $f(X) = X - \{1\}$.

$$P(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

$$\{\{\}, \{2\}, \{3\}, \{2, 3\}\}$$

Question 4, Part 1, A: Exercise 4.2.2, sections c, g, k

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

(c) $h: Z \rightarrow Z, h(x) = x^3$

> **One-to-one**, but <u>not</u> onto.
>
> For example, there is no $x \in Z$ such that $x^3 = 2$.

(g) $f: Z \times Z \rightarrow Z \times Z, f(x, y) = (x + 1, 2y)$

> **One-to-one**, but <u>not</u> onto.
>
> For example, there is no integer pair (x, y) such that $(x + 1, 2y) = (1, 5)$, moreover, there is no y such that $2y = 5$.

(k) $f: Z^+ \times Z^+ \rightarrow Z^+, f(x, y) = 2^x + y$

> **One-to-one**, but <u>not</u> onto.
>
> For example, there is no positive integer pair (x, y), such that $2^x + y = 1$

Question 4, Part 1, B: Exercise 4.2.4, sections b, c, d, g

For each of the functions below, indicate whether the function is onto, one-to-one, neither or both. If the function is not onto or not one-to-one, give an example showing why.

(b) $f: \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1.
For example, $f(001) = 101$ and $f(110) = 110$.

> **Neither.**
>
> Not one-to-one because $f(001) = f(101) = 101$.
> Not onto because there is no x such that $f(x) = 001$.

(c) $f: \{0,1\}^3 \to \{0,1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example $f(011) = 110$.

> **Both** one-to-one and onto**.**

(d) $f: \{0,1\}^3 \to \{0,1\}^4$. The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string. For example, $f(100) = 1001$.

> **One-to-one,** but <u>not</u> onto.
>
> Not onto because there is no x such that $f(x) = 1000$.

(g) Let A be defined to be the set $\{1,2,3,4,5,6,7,8\}$ and let $B = \{1\}$. $f: P(A) \to P(A)$.
For $X \subseteq A$, $f(X) = X - B$. Recall that for a finite set A, $P(A)$ denotes the power set of A which is the set of all subsets of A.

> **Neither.**
>
> Not one-to-one because $f(\{1,2,3\}) = f(\{2,3\}) = \{2,3\}$.
> Not onto because there is no x such that $f(x) = \{1,2,3\}$.

Question 4, Part 2:

Give an example of a function from the set of integers to the set of positive integers that is:

(a) one-to-one, but not onto.

$$f\colon Z \to \mathbb{Z}^{+}, f(x) = x^2$$

(b) onto, but not one-to-one.

$$f\colon Z \to \mathbb{Z}^{+}, f(x) = |\,x\,| + 1$$

(c) one-to-one and onto.

$$f : Z \to Z^{+}, f(x) = \begin{cases} 2x & \text{if } x > 0 \\ 2|x| + 1 & \text{if } x \leq 0 \end{cases}$$

(d) neither one-to-one nor onto.

$$f\colon Z \to \mathbb{Z}^{+}, f(x) = 2$$

Question 5, A: Exercise 4.3.2, sections c, d, g, i

For each of the following functions, indicate whether the function has a well-defined inverse. If the inverse is well-defined, give the input/output relationship of f^{-1}.

(c) $f: R \to R.\ f(x) = 2x + 3$

$$y = 2x + 3$$
$$y - 2x = 3$$
$$-2x = 3 - y$$

$x = -\dfrac{3-y}{2}$

$x = \dfrac{y-3}{2}$

$f^{-1}(x) = \dfrac{x-3}{2}$

(d) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ $f: P(A) \to \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.
For $X \subseteq A, f(X) = |\, X \,|$.

The inverse is <u>not</u> well-defined because multiple subsets within A can have the same cardinality. For example: $|\{2, 3\}| = |\{1, 2\}|$

(g) $f: \{0, 1\}^3 \to \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits. For example, $f(011) = 110$.

f^{-1} is obtained by taking the input string and reversing the bits, just like f, so $f^{-1} = f$.

(i) $f: Z \times Z \to Z \times Z, f(x, y) = (x + 5, y - 2)$

$$x = x + 5$$
$$y - 5 = x$$
$$x = y - 2$$
$$x + 2 = y$$

$$f^{-1}(x, y) = (x - 5, y + 2)$$

Question 5, B: Exercise 4.4.8, sections c, d

The domain and target set of functions f, g, and h are $\mathbb{Z}$. The functions are defined as:

$$f(x) = 2x + 3$$
$$g(x) = 5x + 7$$
$$h(x) = x^2 + 1$$

Give an explicit formula for each function given below.

(c) $f \circ h$

$$f \circ h(x) = 2(x^2 + 1) + 3$$
$$f \circ h(x) = 2x^2 + 2 + 3$$
$$f \circ h(x) = 2x^2 + 5$$

(d) $h \circ f$

$$h \circ f(x) = (2x + 3)^2 + 1$$
$$h \circ f(x) = 4x^2 + 12x + 9 + 1$$
$$h \circ f(x) = 4x^2 + 12x + 10$$

Question 5, C: Exercise 4.4.2, sections b - d

Consider three functions f, g, and h, whose domain and target are $\mathbb{Z}$. Let

$$f(x) = x^2 \qquad g(x) = 2^x \qquad h(x) = \left\lceil \frac{x}{5} \right\rceil$$

(b) Evaluate $f \circ h(52)$

$$h(52) = 11$$
$$f(11) = 121$$

$$f \circ h(52) = 121$$

(c) Evaluate $g \circ h \circ f(4)$

$$f(4) = 16$$
$$h(16) = 4$$
$$g(4) = 16$$

$$g \circ h \circ f(4) = 16$$

(d) Give a mathematical expression for $h \circ f$.

$$h \circ f(x) = \left\lceil \frac{x^2}{5} \right\rceil$$

Question 5, D: Exercise 4.4.6, sections c - e

Define the following functions f, g, and h:

- $f: \{0,1\}^3 \rightarrow \{0,1\}^3$.
  - The output of f is obtained by taking the input string and replacing the first bit by 1, regardless of whether the first bit is a 0 or 1. For example, $f(001) = 101$and $f(110) = 110$.
- $g: \{0,1\}^3 \rightarrow \{0,1\}^3$.
  - The output of g is obtained by taking the input string and reversing the bits. For example, $g(011) = 110$.
- $h: \{0,1\}^3 \rightarrow \{0,1\}^3$.
  - The output of h is obtained by taking the input string x, and replacing the last bit with a copy of the first bit. For example, $h(011) = 010$.

(c) What is $h \circ f(010)$?

$$f(010) = 110$$
$$h(110) = 111$$

$$h \circ f(010) = 111$$

(d) What is the range of $h \circ f$?

$$\{0,1\}^3 = \{000, 001, 010, 100, 011, 110, 111, 101\}$$
$$Range\ of\ f = \{100, 101, 110, 111\}$$

$$Range\ of\ h \circ f = \{101, 111\}$$

(e) What is the range of $g \circ f$?

$$\{0,1\}^3 = \{000, 001, 010, 011, 100, 110, 111, 101\}$$
$$Range\ of\ f = \{100, 101, 110, 111\}$$

$$Range\ of\ g \circ f = \{001, 010, 001, 000\}$$

Question 5, E: Exercise 4.4.4, sections c, d **(Extra Credit)**

Let $f: X \to Y$ and $g: Y \to Z$ be two functions.

(c) Is it possible that f is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

No, it is not possible because given that f is the first function that needs to be solved and it is <u>not</u> one-to-one, then g cannot be one-to-one.

(d) Is it possible that g is not one-to-one and $g \circ f$ is one-to-one? Justify your answer. If the answer is "yes", give a specific example for f and g.

Yes, it is possible for g to not be a one-to-one function and $g \circ f$ to be a one-to-one function.

Given $g = x^2$, g returns the same value for both x and $-x$, making it <u>not</u> one-to-one. However, given $f = |\, x + 1 \,|$, g's input will always be positive, and the $x + 1$ guarantees it will return different values for the positive and negative version of the same number, making $g \circ f$ one-to-one.

Example:
Let $g = x^2$ and $f = |\, x + 1 \,|$

$$g \circ f(2) = 9$$
$$g \circ f(-2) = 1$$