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Chapter 1

Algebra

1.1 Noncommutative rings

Definition 1. If R is a ring, then the *Jacobson radical* $J(R)$ (sometimes written $\mathrm{rad}(R)$) is the intersection of the annihilators of all simple left R-modules.

Definition 2. A submodule N of M is *superfluous*, written $N \subseteq_s M$ or $N \ll M$, if for all H we have $N + H = M \implies H = M$.

Theorem 1. *We can replace "left" by "right" in the definition of the Jacobson radical of a ring. Furthermore, we have the following equivalent definitions:*

- $J(R)$ *is the intersection of all maximal left ideals of* R*,*
- $J(R)$ *is the sum of all superfluous left ideals of* R*,*
- $J(R)$ *is the maximal left ideal of* R *such that for all* $x \in J(R)$*,* $1 - x$ *has a left inverse,*
- $J(R) = \{x \in R \mid 1 + RxR \subseteq R^\times\}$.

Lemma 1 (Nakayama's Lemma)**.** *If* M *is a finitely generated left* R*-module with* $M = J(R)M$*, then* $M = 0$.

Proof. Consider a minimal generating set $x_1, ..., x_n$ of M, and use $\sum x_i \in J(R)M$ to write x_n as a linear combination of $x_1, ..., x_{n-1}$. □

Proposition 1. $J(R/J(R)) = 0$.

1.1.1 Artinian Rings

Proposition 2. *If* R*, considered as a left* R*-module over itself, has a composition series of length* k*, then* $J(R)^k = 0$.

Theorem 2 (Hopkins' Theorem)**.** *If* M *is a left module over a left Artinian ring, then the following are equivalent:*

- M *is finitely generated,*

- *M has finite length,*
- *M is Noetherian,*
- *M is Artinian.*

Theorem 3 (Hopkins-Levitzki)**.** *If R is* semiprimary *- that is, if $R/J(R)$ is semisimple and $J(R)$ is nilpotent - then for left R-modules, being Noetherian, being Artinian, and having a composition series are equivalent.*

Proposition 3. *If $J(R) = 0$, then every minimal left ideal of R is a direct summand of R.*

Theorem 4. *R is semisimple if and only if it is left Artinian and has $J(R) = 0$.*

1.2 Commutative Algebra

Definition 3. If R is a commutative ring, then $I \lhd R$ means that I is an ideal of R.

Definition 4. If $I, J \lhd R$, set $(I : J) = \{r \in R \mid rJ \subseteq I\}$. If $a \in R$, we abbreviate $(I : (a))$ to $(I : a)$.

1.2.1 Primary Ideals

Definition 5. $Q \lhd R$ is *primary* if $\forall a, b \in R$ with $ab \in Q$, either $b \in Q$ or $\exists n$ such that $a^n \in Q$.

Definition 6. If $I \lhd R$, then $\mathrm{rad}(I) = \{r \in R \mid \exists n \; r^n \in I\}$.

Proposition 4. *Q is primary if and only if $\mathrm{rad}(Q)$ is prime. If Q_1, Q_2 are primary and $\mathrm{rad}(Q_1) = \mathrm{rad}(Q_2)$, then $Q_1 \cap Q_2$ is primary. If R is Noetherian and $Q \lhd R$, then $\exists n$ such that $\mathrm{rad}(Q)^n \subseteq Q$.*

Theorem 5 (Primary Decomposition)**.** *If R is Noetherian and $I \lhd R$, then $\exists k$ and $Q_1, ..., Q_k \lhd R$ primary such that $I = Q_1 \cap \cdots \cap Q_k$.*

Proof. By R Noetherian, $\forall a \in R \, \exists n$ with $(I : a^n) = (I : a^{n+1})$, and for this n we have $(I + (a^n)) \cap (I : a) = I$, so either I is already primary or we can write I as an intersection of bigger ideals, and apply Noetherian induction. □

Lemma 2. *If R is Noetherian, then for any $I \lhd R$ and $r \in R \setminus I$, there exists $s \in R$ such that $(I : rs)$ is prime.*

Theorem 6 (Uniqueness of radicals)**.** *If R is Noetherian, $I = Q_1 \cap \cdots \cap Q_k$ with $Q_i \lhd R$ primary and no Q_i containing $\cap_{j \neq i} Q_j$, and if $\mathfrak{p} \lhd R$ is prime, then $\exists r \in R$ with $(I : r) = \mathfrak{p}$ if and only if there is an i with $\mathrm{rad}(Q_i) = \mathfrak{p}$. In particular, the set $\{\mathrm{rad}(Q_i)\}_{i \le k}$ is uniquely determined by I.*

Theorem 7 (Uniqueness of primaries with minimal radical)**.** *If R is Noetherian, $I = Q_1 \cap \cdots \cap Q_k$ with $Q_i \lhd R$ primary and $\mathrm{rad}(Q_i) \not\subseteq \mathrm{rad}(Q_1)$ for $i > 1$, then for n sufficiently large we have $(I : \mathrm{rad}(Q_2)^n \cdots \mathrm{rad}(Q_k)^n) = Q_1$, so Q_1 is uniquely determined by I and $\mathrm{rad}(Q_1)$.*

Chapter 2

Analysis

2.1 Basic Facts

2.1.1 Point Set Stuff

Definition 7. A topological space is *normal*, or T_4, if any two disjoint closed sets have disjoint open neighborhoods.

Proposition 5. *Compact Hausdorff spaces are normal.*

Lemma 3 (Urysohn's Lemma). *A topological space X is normal iff for any disjoint closed subsets $A, B \subseteq X$ there exists a continuous $f : X \to [0, 1]$ such that $f(A) \subseteq \{0\}$ and $f(B) \subseteq \{1\}$.*

Proof. Let $U(1) = X \setminus B, V(0) = X \setminus A$. For each dyadic rational $r = \frac{2a+1}{2^{n+1}} \in (0, 1)$ we construct disjoint open subsets $U(r), V(r) \subseteq X$ such that $X \setminus V(\frac{a}{2^n}) \subseteq U(\frac{2a+1}{2^{n+1}})$ and $X \setminus U(\frac{a+1}{2^n}) \subseteq V(\frac{2a+1}{2^{n+1}})$. Then for every r we have $A \subseteq U(r)$, $B \subseteq V(r)$, for $r \leq s$ we have $U(r) \cap V(s) = \emptyset$, and for $r < s$ we have $V(r) \cup U(s) = X$. Thus for $r < s$, the closure of $U(r)$ is contained in $U(s)$. Finally, define f by $f(x) = \min(1, \inf\{r \mid x \in U(r)\})$. $\square$

Lemma 4 (Locally Compact Urysohn's Lemma). *If X is locally compact Hausdorff and $K \subseteq U \subseteq X$ with K compact and U open, then there exists a continuous $f : X \to [0, 1]$ such that $f(K) \subseteq \{1\}$ and f is supported on a compact subset of U.*

Proof. Find a precompact open set V with $K \subseteq V \subseteq \overline{V} \subseteq U$, then $\overline{V}$ is normal (since it is compact and Hausdorff), so by Urysohn's Lemma there is a continuous $f : \overline{V} \to [0, 1]$ with $f(K) \subseteq \{1\}$ and $f(\partial V) \subseteq \{0\}$. $\square$

Theorem 8 (Tietze Extension Theorem). *If X is a normal space, $A \subseteq X$ is closed, and $f : A \to \mathbb{R}$ is continuous, then there exists a continuous $F : X \to \mathbb{R}$ with $F|_A = f$.*

Proof. Assume without loss of generality that $f(A) \subseteq [0, 1]$. We'll find a sequence of functions $g_i : X \to [0, \frac{2^{i-1}}{3^i}]$ with $0 \leq f - \sum_{i=1}^n g_i \leq \frac{2^n}{3^n}$ for all n, and finish by taking $F = \sum_i g_i$. It's enough to show how to find g_1: we apply Urysohn's Lemma to find $g_1 : X \to [0, \frac{1}{3}]$ with $g_1(x) = 0$ for $x \in f^{-1}([0, \frac{1}{3}])$ and $g_1(x) = \frac{1}{3}$ for $x \in f^{-1}([\frac{2}{3}, 1])$. $\square$

Corollary 1 (Locally Compact Tietze)**.** *If X is locally compact Hausdorff and $K \subseteq U \subseteq X$ with K compact and U open, then for every continuous $f : K \to [0,1]$ there exists a continuous $F : X \to [0,1]$ such that $F|_K = f$ and F is supported on a compact subset of U.*

Proposition 6. *If X is locally compact Hausdorff, then $C_0(X)$ is the closure of $C_c(X)$ in the uniform metric.*

Theorem 9 (Stone-Weierstrauss, lattice version)**.** *If X compact, $B \subseteq C(X, \mathbb{R})$ such that for any $x, y \in X$ and $a, b \in \mathbb{R}$ there exists $g \in B$ with $g(x) = a, g(y) = b$, and such that B contains $\max(f, g), \min(f, g)$ whenever it contains f, g, then B is dense in $C(X, \mathbb{R})$.*

Proof. Let $f \in C(X, \mathbb{R})$, and for all $x, y \in X$ pick $g_{xy} \in B$ with $g_{xy}(x) = f(x), g_{xy}(y) = f(y)$. Fix $\epsilon > 0$. Take $U_{xy} = \{z \mid f(z) < g_{xy}(z) + \epsilon\}, V_{xy} = \{z \mid f(z) > g_{xy} - \epsilon\}$. For any y, some finite subcollection of the U_{xy}s cover X, corresponding to $x_1, ..., x_n$, take $g_y = \max(g_{x_1y}, ..., g_{x_ny})$ and $V_y = \cap V_{x_jy}$, then $f < g_y + \epsilon$ and for $x \in V_y$ we have $f(x) > g_y(x) - \epsilon$. Now take a finite subcollection of the V_ys which covers X, and let g be the minimum of the corresponding g_ys, then $g \in B$ and $|f - g| \leq \epsilon$. □

Definition 8. The *Bernstein polynomials* are defined by

$$b_{\nu,n}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}.$$

Theorem 10 (Weierstrauss approximation)**.** *If $f : [a, b] \to \mathbb{C}$ is continuous, then $\forall \epsilon > 0$ there exists a polynomial $p \in \mathbb{C}[x]$ such that $\forall x \in [a, b]$, we have $|f(x) - p(x)| < \epsilon$.*

Proof. Suppose $[a, b] = [0, 1]$, and define $B_n(f)$ by

$$B_n(f) = \sum_{\nu=0}^{n} f(\tfrac{\nu}{n}) b_{\nu,n}.$$

If k is the number of times we flip heads in n independent random coinflips with bias x, then

$$\mathbb{E}[f(\tfrac{k}{n})] = B_n(f)(x),$$

so the law of large numbers shows that $B_n(f)$ approximates f. □

Theorem 11 (Stone-Weierstrauss for $\mathbb{R}$)**.** *X compact Hausdorff, A a subalgebra of $C(X, \mathbb{R})$ which contains a non-zero constant. Then A is dense in $C(X, \mathbb{R})$ iff it separates points.*

Proof. It's enough to show that if $f \in A$, then $|f|$ is in the closure of A, since then the closure of A will be closed under max and min. To do this, we find $p \in \mathbb{R}[x]$ such that $\forall x \in f(X)$ we have $||x| - p(x)| < \epsilon$, then $p \circ f \in A$ and $||f| - p \circ f| < \epsilon$. □

Theorem 12 (Stone-Weierstrauss for $\mathbb{C}$)**.** *X compact Hausdorff, $S \subseteq C(X, \mathbb{C})$ separates points. Then the complex unital $*$-algebra generated by S is dense in $C(X, \mathbb{C})$.*

Theorem 13 (Tychonoff's Theorem)**.** *If $\{X_a\}_{a \in A}$ is a family of compact sets, then $X = \prod_{a \in A} X_a$ is compact in the product topology.*

Proof. With nets and Zorn: Suppose $\{U_i\}_{i\in I}$ is an open cover of X with no finite subcover. For each finite subset $J \subseteq I$, let x_J be a point of X not contained in $\cup_{j\in J} U_j$. We show that for every $B \subseteq A$, the net $\{\pi_B(x_J)\}_{J\subseteq I}$ has a cluster point, by transfinite induction on B, and taking $B = A$ gives a contradiction.

With Alexander Subbase Theorem 14: Suppose there is an open cover by cylinder sets with no finite subcover. Then for each $a \in A$, there is some $x_a \in X_a$ not covered by the cylinder sets corresponding to coordinate a, and the corresponding point $(x_a)_{a\in A} \in X$ is not covered by any of the cylinders. □

Theorem 14 (Alexander Subbase Theorem)**.** *If X is a topological space with subbase B, and every open cover of X by elements of B has a finite subcover, then X is compact.*

Proof. Suppose not, let C be a maximal open cover of X which has no finite subcover (alternatively, take C to be a maximal proper ideal of $\mathcal{P}(X)$ containing an open cover of X). Take $x \in X$ not contained in any element of $C \cap B$, then there is $U \in C$ with $x \in U$, and $S_1, ..., S_n \in B$ with $x \in S_1 \cap \cdots \cap S_n \subseteq U$. For each S_i, since $S_i \notin C$ there must be a finite subset $C_i \subseteq C$ such that $\{S_i\} \cup C_i$ covers X, but then $\{U\} \cup C_1 \cup \cdots \cup C_n$ is a finite subset of C which covers X. □

The next bit is from `https://math.stackexchange.com/a/6338`.

Definition 9. A topological space is called a *continuum* if it is a compact connected Hausdorff space.

Lemma 5. *Let X be a continuum. If F is a non-trivial closed subset of X, then for every component C of F we have that $\partial F \cap C$ is non-empty.*

Proof. Since X is Hausdorff compact, quasicomponents coincide with components, so C is the intersection of all clopen sets in F which contain C. Suppose that C is disjoint from ∂F. Then, by compactness of ∂F, there is a single clopen set A in F containing C and disjoint from ∂F. Take an open set U such that $A = U \cap F$. $A \cap \partial F = \emptyset$ implies that $A = U \cap \operatorname{int}(F)$, so A is open in X. But A is also closed in X, and contains C, so $A = X$. But then $\partial F = \emptyset$, which is not possible since F would be non-trivial clopen in X. □

Theorem 15 (Sierpiński [6])**.** *If a continuum X has a countable cover $\{X_i\}_{i=1}^{\infty}$ by pairwise disjoint closed subsets, then at most one of the sets X_i is non-empty.*

Proof. Assume that at least two of the sets X_i are non-empty. First we show that for every i there exists a continuum $C \subseteq X$ such that $C \cap X_i = \emptyset$ and at least two sets in the sequence $C \cap X_1, C \cap X_2, \ldots$ are non-empty. If X_i is empty then we can take $C = X$; thus we can assume that X_i is non-empty. Take $j \neq i$ such that $X_j \neq \emptyset$. Since X is Hausdorff compact, there are disjoint open sets $U, V \subseteq X$ satisfying $X_i \subseteq U$ and $X_j \subseteq V$. Let C be a component of $\overline{V}$ which meets X_j. Clearly, C is a continuum, $C \cap X_i = \emptyset$ and $C \cap X_j \neq \emptyset$. By the previous lemma, $C \cap \partial(\overline{V}) \neq \emptyset$ and since $X_j \subseteq \operatorname{int}(\overline{V})$, there exist a $k \neq j$ such that $C \cap X_k \neq \emptyset$.

It follows that there exists a decreasing sequence $C_1 \supseteq C_2 \supseteq \ldots$ of continua contained in X such that $C_i \cap X_i = \emptyset$ and $C_i \neq \emptyset$ for $i = 1, 2, \ldots$ The first part implies that $\bigcap_{i=1}^{\infty} C_i = \emptyset$ and from the second part and compactness of X it follows that $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$. □

Definition 10. A subset S of a topological space is *perfect* if it is closed and every point of S is a limit point.

Definition 11. A *Polish space* is a separable completely metrizable topological space.

Theorem 16 (Cantor)**.** *Every nonempty perfect subset of a Polish space has cardinality at least* $2^{\aleph_0}$.

Definition 12. A *condensation point* of a subset S of a topological space is a point x such that every neighborhood of x intersects S in uncountably many points.

Theorem 17 (Cantor-Bendixson)**.** *Every closed subset S of a Polish space X can be written uniquely as a disjoint union of a perfect set and a countable set.*

Proof. Ordinal proof: For any set S, let S' be the set of limit points of S. Define a sequence S_α indexed by ordinals by $S_0 = S$, $S_{\alpha+1} = S'_\alpha$, and $S_\beta = \cap_{\alpha<\beta} S_\alpha$ for β a limit ordinal. Since each closed set S_α is determined by the collection of open subsets of a basis of X which do not intersect it, and since every well-ordered chain contained in $\mathcal{P}(\mathbb{N})$ is countable, there is some countable ordinal β such that $S_\beta = S_{\beta+1}$. Since the number of isolated points of any S_α is countable, we see that $S \setminus S_\beta$ must be countable.

Condensation point proof: Let P be the set of condensation points of S. Then $S \setminus P$ is contained in a countable union of open sets of a basis of X which each intersect S in countably many points, so $S \setminus P$ is countable and P is perfect.

For uniqueness: note that every point in a perfect subset of S must be a condensation point of S. ☐

2.1.2 Metric Spaces

Definition 13. A metric space is *complete* if every Cauchy sequence has a limit. It is *totally bounded* if it can be covered by finitely many subsets of size ϵ, for every $\epsilon > 0$.

Theorem 18. *A metric space is compact iff it is complete and totally bounded.*

Definition 14. A metric space is *sequentially compact* if every sequence has a bounded subsequence.

Theorem 19 (Bolzano-Weierstrauss)**.** *A subset of $\mathbb{R}^n$ is sequentially compact iff it is closed and bounded.*

Proposition 7. *A closed subset of a complete space is complete, and a complete subset of a metric space is closed.*

Theorem 20 (Baire Category Theorem)**.** *If M is either a complete metric space or a locally compact Hausdorff space, then a union of countably many nowhere dense subsets of M has empty interior.*

Definition 15. A space is called a *Baire space* if the intersection of any countable collection of open dense sets is dense.

Theorem 21 (Banach Fixed Point)**.** *Contraction mappings on complete metric spaces have unique fixed points.*

Corollary 2 (Picard-Lindelöf)**.** *The initial value problem $y'(t) = f(t, y(t)), y(t_0) = y_0$ for $t \in [t_0 - \epsilon, t_0 + \epsilon]$ has a unique solution for some $\epsilon > 0$ if f is Lipschitz continuous in y and continuous in t.*

Definition 16. If X, Y are Banach spaces, $U \subseteq X$ open, then $f : U \to Y$ is called *Frechét differentiable* at x if there exists a bounded linear operator $A : X \to Y$ such that $\|f(x+h) - f(x) - Ah\|_Y = o(\|h\|_X)$ as $h \to 0$. In this case we write $Df_x = A$.

Corollary 3 (Inverse Function Theorem). *If X, Y are Banach spaces, U an open neighborhood of 0 in X, $F : U \to Y$ continuously (Fréchet) differentiable and $DF_0 : X \to Y$ a bounded isomorphism from X to Y (with bounded inverse), then there exists an open neighborhood $V \subseteq Y$ of $F(0)$ and a continuously differentiable map $G : V \to X$ such that $F(G(y)) = y$ for all $y \in V$.*

Definition 17. A topological space is called *separable* if it contains a countable dense set. It is called *second countable* if its topology has a countable base.

Proposition 8. *Every second countable space is separable, and every separable metric space is second countable.*

Definition 18. If X, Y are metric spaces, then $f : X \to Y$ is called *uniformly continuous* if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x, y \in X$ such that $d_X(x, y) < \delta$, we have $d_Y(f(x), f(y)) < \epsilon$.

Definition 19. A family of functions F is called *equicontinuous* at $x_0 \in X$ if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall f \in F, x \in X$ such that $d(x_0, x) < \delta$ we have $d(f(x_0), f(x)) < \epsilon$. F is *uniformly equicontinuous* if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall f \in F, x, y$ such that $d(x, y) < \delta$ we have $d(f(x), f(y)) < \epsilon$.

Theorem 22 (Arzelà-Ascoli). *If $(f_n)_{n \in \mathbb{N}}$ defined on $[a, b]$ is uniformly bounded and equicontinuous, then there is a subsequence which converges uniformly.*

Theorem 23 (Ascoli Version 2). *If X is compact Hausdorff, then a subset of $C(X)$ (with the uniform norm) is compact iff it is closed, pointwise bounded, and equicontinuous.*

Lemma 6 (Finite Vitali Covering Lemma). *If $B_1, ..., B_n$ are balls in a metric space, then there is a subcollection $B_{j_1}, ..., B_{j_k}$ which are disjoint, and which satisfy*

$$B_1 \cup \cdots \cup B_n \subseteq 3B_{j_1} \cup \cdots \cup 3B_{j_k},$$

where $3B_j$ is the ball with the same center as B_j and three times the radius.

Proof. Keep adding the biggest ball which is disjoint from the ones you have chosen so far to your collection. Then every ball you haven't chosen will intersect a larger ball that you have chosen. $\square$

Lemma 7 (Infinite Vitali Covering Lemma). *If $(B_i)_{i \in I}$ is a collection of balls in a metric space such that $\sup_{i \in I} \operatorname{rad}(B_i) < \infty$, then for any $c > 1$ there is a subcollection $J \subseteq I$ such that the B_j with $j \in J$ are disjoint, and $\cup_{i \in I} B_i \subseteq \cup_{j \in J} (1 + 2c) B_j$.*

Proof. Let $R = \sup \operatorname{rad}(B_i)$, and for each n choose a maximal disjoint subcollection of the balls with radius between R/c^n and R/c^{n+1} which are disjoint from the balls you have already chosen so far. Then every ball you haven't chosen will intersect a ball you have chosen, whose radius is at most a factor of c smaller. $\square$

Lemma 8 (Besicovitch Covering Lemma). *For every n there exists a constant c_n such that for $E \in \mathbb{R}^n$ bounded and for a collection of balls $\mathcal{B}$ such that every point of E is the center of some ball B in $\mathcal{B}$, there is a collection of c_n families $\mathcal{B}_i \subseteq \mathcal{B}$ of pairwise disjoint balls, such that $E \subseteq \cup_{i \leq c_n} \cup_{B \in \mathcal{B}_i} B$.*

Proof. WLOG assume all balls in $\mathcal{B}$ are contained in a big ball B_0. Make a countable sequence of balls $B_i \in \mathcal{B}$ such that each B_i has its center not contained in $B_1, ..., B_{i-1}$, and its radius within a factor of $1-\epsilon$ of the sup of the radii of such balls. If we shrink each B_i by a factor of $1+\frac{1}{1-\epsilon}$ to make a ball B_i', the B_i's are pairwise disjoint. Since the volume of B_0 is at least the sum of the volumes of the B_i's, the radii of the B_is goes to zero, so $E \subseteq \cup_i B_i$.

To finish, we just need to show that each B_i intersects less than c_n of the balls $B_1, ..., B_{i-1}$. To do this, we divide the balls $B_1, ..., B_{i-1}$ which intersect B_i into two groups based on whether their radii are at most M times the radius of B_i. The group of smaller balls is bounded because the B_j's are disjoint and contained in a ball of radius $2M+1$ times the radius of B_i (and the radii of the B_j with $j < i$ are at least $1-\epsilon$ times the radius of B_i). The group of larger balls is bounded by showing that the angles between the rays connecting the center of B_i with the centers of the B_js must be large (approaching $\frac{\pi}{3}$) if M is big enough and ϵ small enough (using the law of cosines). $\square$

2.1.3 Topologies on $C(X, Y)$

Definition 20. The *compact-open* topology on $C(X, Y)$ has a subbase given by

$$V(K, U) = \{f : X \to Y \mid f(K) \subseteq U\}$$

for K compact and U open.

Proposition 9. *If Y is a metric space then $f_n \to f$ in the compact-open topology iff $\forall K \subseteq X$ compact we have $f_n \to f$ uniformly on K, so in this case the compact-open topology is the "topology of compact convergence". If X is compact as well, this becomes the uniform convergence topology.*

Proposition 10. *If Y is locally compact Hausdorff, composition $\circ : C(Y, Z) \times C(X, Y) \to C(X, Z)$ is continuous in the compact-open topology.*

Definition 21. If X, Y Banach spaces, $U \subseteq X$ open, $\mathcal{C}^m(U, Y)$ the m-times continuously Fréchet-differentiable functions $U \to Y$, then the "compact-open" topology on $\mathcal{C}^m(U, Y)$ is induced by the seminorms

$$\rho_K(f) = \sup\{\|D^j f_x\| \mid x \in K,\ 0 \le j \le m\}$$

for $K \subseteq U$ compact.

Definition 22. The topology of *compact convergence* is defined by $f_n \to f$ iff for all K compact, $f_n|_K \to f|_K$ converges uniformly.

Proposition 11. *A set F of functions is called* normal *if every sequence of functions from F contains a subsequence that converges compactly to a continuous function.*

Theorem 24 (Montel)**.** *Any uniformly bounded family of holomorphic functions defined on an open subset of $\mathbb{C}$ is normal.*

Definition 23. The topology of *pointwise convergence* is the product topology on Y^X - this has $f_n \to f$ iff $f_n(x) \to f(x)$ for all x.

2.1.4 Measure

Definition 24. Two subsets A, B of $\mathbb{R}^n$ are called *equidecomposable* if A can be cut into finitely many disjoint pieces which can be translated and rotated to give a disjoint decomposition of B. More generally, if G is a group acting on a set X, then two subsets A, B of X are *G-equidecomposable* if we can write $A = A_1 \dot\cup \cdots \dot\cup A_n$ and if there are $g_i \in G$ with $B = g_1 A_1 \dot\cup \cdots \dot\cup g_n A_n$.

Proposition 12 (Banach-Cantor-Schröder-Bernstein)**.** *If A is equidecomposable with a subset of B and B is equidecomposable with a subset of A, then A, B are equidecomposable.*

Definition 25. If G acts on X and $Y \subseteq X$, we say that Y is *G-paradoxical* if there are disjoint $A, B \subseteq Y$ which are both G-equidecomposable with Y.

Proposition 13. *If F_2 is the free group on two generators a, b, then we can write $F = A_0 \dot\cup A_1 \dot\cup A_2 \dot\cup B_1 \dot\cup B_2$, with $F = A_0 \dot\cup aA_1 \dot\cup A_2 = bB_1 \dot\cup B_2$. In particular, F_2 is F_2-paradoxical.*

Proof. For any word w, let $W(w)$ be the set of elements of F_2 that begin with w. Take $A_0 = \{a^{-n} \mid n \geq 0\}, A_1 = W(a^{-1}) \setminus A_0, A_2 = W(a), B_1 = W(b^{-1}), B_2 = W(b)$. □

Proposition 14. *If G is G-paradoxical and acts on X without fixed points, then X is G-paradoxical.*

Lemma 9. *$SO(3)$ contains a free group of rank 2.*

Proof. Let σ, τ be the matrices

$$\sigma = \frac{1}{3}\begin{pmatrix} 1 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \tau = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -2\sqrt{2} \\ 0 & 2\sqrt{2} & 1 \end{pmatrix}.$$

It's easy to check by induction on the length of w that if w is a word of length k ending with σ, then $w \cdot (1\ 0\ 0)^T = \frac{1}{3^k}(a\ b\sqrt{2}\ c)^T$ with $3 \nmid b$, and that if w starts with $\sigma^{\pm}$ then $a \equiv \pm b \pmod 3$ and $c \equiv 0 \pmod 3$, while if w begins with $\tau^{\pm}$ then $c \equiv \pm b \pmod 3$ and $a \equiv 0 \pmod 3$. Thus σ, τ generate a free group of rank 2. □

Proposition 15. *If E is a subset of S^2 with $|E| < 2^{\aleph_0}$, then S^2 is equidecomposable with $S^2 \setminus E$.*

Proof. We just need to find a rotation ρ with $\rho^n(E) \cap E = \emptyset$ for all $n > 0$. We take the axis to be any line through the origin which doesn't pass through any point of E, and then choose the angle of the rotation avoiding $|E| \times |E| \times |\mathbb{N}|$ bad angles. □

Corollary 4 (Banach-Tarski Paradox)**.** *Any ball in $\mathbb{R}^3$ is paradoxical.*

Corollary 5 (Strong Banach-Tarski Paradox)**.** *If $A, B \subset \mathbb{R}^3$ have nonempty interior and are bounded, then they are equidecomposable.*

Definition 26. A set of subsets Σ of X is a *σ-algebra* over X if Σ satisfies: $\emptyset \in \Sigma$, $\forall A \in \Sigma$ we have $X \setminus A \in \Sigma$, and for any sequence $(A_n)_{n\in\mathbb{N}}$ of elements of Σ we have $\cup_n A_n \in \Sigma$.

Proposition 16. *If Σ is a σ-algebra and Σ is infinite, then $|\Sigma| \geq 2^{\aleph_0}$. If Σ is generated by at most $2^{\aleph_0}$ sets, then $|\Sigma| \leq 2^{\aleph_0}$ (more generally, if Σ is generated by κ sets, then $|\Sigma| \leq \kappa^{\aleph_0}$).*

Definition 27. If X is a topological space, the *Borel* σ-algebra is the smallest σ-algebra containing the open subsets of X (some authors replace "open" by "compact" in this definition).

Proposition 17. *If X is metric, then the Borel σ-algebra can be generated from the open sets by iteratively taking closure under countable unions and intersections at most ω_1 times.*

Proof. Every open subset of X is a countable union of closed subsets of X, and ω_1 has uncountable cofinality. $\square$

Corollary 6. *The Borel σ-algebra on $\mathbb{R}$ has cardinality $2^{\aleph_0}$.*

Proposition 18. *If E is in the σ-algebra generated by $\mathcal{A} \subseteq \mathcal{P}(X)$, then there is a countable subset $\{A_1, A_2, ...\}$ of $\mathcal{A}$ such that E is in the σ-algebra generated by $A_1, A_2, ...$ In particular, E can be written as a disjoint union of at most $2^{\aleph_0}$ countable intersections of elements of $\mathcal{A}$.*

Corollary 7 (Nedoma's Pathology)**.** *If $|X| > 2^{\aleph_0}$, then the set $\Delta_X = \{(x, x) \mid x \in X\}$ is not in the σ-algebra generated by the collection of all rectangles $E \times F$, where E, F are arbitrary subsets of X.*

Definition 28. $\mu : \Sigma \to [0, \infty]$ is a *measure* if $\mu(\emptyset) = 0$ and $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ whenever $E_i \in \Sigma$ and $E_i \cap E_j = \emptyset$ for all $i \neq j$. (X, Σ, μ) is called a *measure space* if Σ is a σ-algebra over X and $\mu : \Sigma \to [0, \infty]$ is a measure.

Proposition 19. *There is no translation invariant measure μ on the collection of all subsets of $\mathbb{R}$ which satisfies $\mu([0, 1]) = 1$.*

Proof. Let G be any additive subgroup of $\mathbb{R}$ which contains $\mathbb{Z}$ and has $[G : \mathbb{Z}] = \aleph_0$ (we could take $G = \mathbb{Q}$, $G = \mathbb{Z}[\sqrt{2}]$, etc.). Let A be a set of representatives of $\mathbb{R}/G$ which are all in $[0, 1]$. Then there is a set $X \subset G$ with $|X| = \aleph_0$ such that $[0, 1] \subseteq A + X \subseteq [-1, 2]$. Thus $\mu(A) \leq \frac{\mu([-1,2])}{n} = \frac{3}{n}$ for all $n > 0$, so $\mu(A) = 0$, so $\mu([0, 1]) = 0$ by countable additivity. $\square$

Proposition 20. *If μ is a measure and $E_1 \subseteq E_2 \subseteq \cdots$ are measurable, then $\mu(\cup_{i=1}^{\infty} E_i) = \sup_i \mu(E_i)$. If $F_1 \supseteq F_2 \supseteq \cdots$ are measurable and $\mu(F_1) < \infty$, then $\mu(\cap_{i=1}^{\infty} F_i) = \inf_i \mu(F_i)$.*

Definition 29. A set E is *σ-finite* with respect to a measure μ if E can be written as a countable union of sets with finite μ-measure. We say that μ is σ-finite if the whole space X is σ-finite with respect to μ. We say that μ is *decomposable* if X can be written as a disjoint union of σ-finite subsets X_i such that for any $A \subseteq X$, A is measurable iff $A \cap X_i$ is measurable for all i, and $\mu(A) = \sum_{i \in I} \mu(A \cap X_i)$.

Definition 30. A *signed measure* is a map $\mu : \Sigma \to [-\infty, \infty]$ which is countably additive (and doesn't take both $\infty, -\infty$ as values).

Theorem 25 (Hahn decomposition Theorem)**.** *If μ is a signed measure, then there exist measurable sets P, N such that $P \cup N = X, P \cap N = \emptyset$, and for all $E \subseteq P$ measurable we have $\mu(E) \geq 0$, while for all $E \subseteq N$ measurable we have $\mu(E) \leq 0$. This decomposition is unique up to null sets.*

Proof. Assume WLOG that μ doesn't take the value $-\infty$. Say a measurable set is *negative* if every measurable subset has measure ≤ 0. First we show that for any measurable D with $\mu(D) \leq 0$ there is a negative set $A \subseteq D$ with $\mu(A) \leq \mu(D)$: define a sequence of sets A_n, $A_0 = D$, each A_{n+1} given by removing a set of positive measure from A_n whose measure is at least half as large as the sup of measures of subsets (if finite), or at least 1 otherwise, and take $A = \cap_n A_n$. Next, we define N by making a sequence N_n with $N_0 = \emptyset$, and N_{n+1} given by adding a negative set to N_n whose measure is at least half as negative as the inf of measure of subsets (if finite), or at most -1 otherwise, and take $N = \cup_n N_n$. $\square$

Theorem 26 (Jordan decomposition Theorem)**.** *If μ is a signed measure, there is a unique decomposition $\mu = \mu^+ - \mu^-$ where μ^+, μ^- are positive measures (at least one of which is finite), such that $\mu^+(E)$ is 0 for any negative set E and μ^- is 0 for any positive set E.*

Definition 31. If μ is a signed measure and $\mu = \mu^+ - \mu^-$ is its Jordan decomposition, then we set $|\mu| = \mu^+ + \mu^-$.

Definition 32. A *complex measure* is a countably additive function $\mu : \Sigma \to \mathbb{C}$. Equivalently, it is a complex combination of finite measures.

Definition 33. If μ is a complex measure, we define the *total variation* of μ to be the positive measure $|\mu|$ given by $|\mu|(E) = \sup\{\sum_{i=1}^n |\mu(E_i)| \mid E = E_1 \cup \cdots \cup E_n\}$.

Definition 34. If μ, ν are (possibly signed) measures, then μ is *absolutely continuous* with respect to ν, written $\mu \ll \nu$, if $|\nu|(A) = 0 \implies |\mu|(A) = 0$.

Proposition 21. *If μ is finite, then $\mu \ll \nu$ iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|\nu|(A) < \delta \implies |\mu|(A) < \epsilon$.*

Proof. Assume μ, ν are positive. For every $n \geq 1$, let $n\nu - \mu$ have Hahn decomposition (P_n, N_n), and let $N = \bigcap_n N_n$. Since $n\nu(N) \leq \mu(N)$ for all n and $\mu(N) < \infty$, we have $\nu(N) = 0$, so we must have $\mu(N) = 0$ by $\mu \ll \nu$. Thus there is some n such that $\mu(N_n) < \frac{\epsilon}{2}$, and we can take $\delta = \frac{\epsilon}{2n}$: if $\nu(A) < \delta$, then $\mu(A) = \mu(A \cap P_n) + \mu(A \cap N_n) \leq n\nu(A) + \mu(N_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. $\square$

Definition 35. We say that two (possibly signed or complex) measures μ, ν on X are *singular*, written $\mu \perp \nu$, if there are measurable sets A, B with $A \cup B = X$ such that B is μ-null and A is ν-null.

Theorem 27 (Lebesgue decomposition Theorem)**.** *If μ, ν are (possibly signed) σ-finite measures over X, then there is a unique pair of σ-finite measure μ_{ac}, μ_s such that $\mu = \mu_{ac} + \mu_s$, $\mu_{ac} \ll \nu$, and $\mu_s \perp \nu$.*

Proof. We just need to prove this in the finite, unsigned case. Let $\mathcal{N}$ be the collection of ν-null sets. Define μ_{ac} by

$$\mu_{ac}(A) = \inf_{N \in \mathcal{N}} \mu(A \setminus N).$$

μ_{ac} is clearly nonnegative and countably additive, and we clearly have $\mu_{ac} \ll \nu$. Set $\mu_s = \mu - \mu_{ac}$, taking $A = X$ and noting that the infimum must actually be attained, we see that there is a ν-null set N such that $\mu_s(X \setminus N) = 0$, so $\mu_s \perp \nu$.

For uniqueness, suppose that $\mu = \mu_1 + \mu_2$ with $\mu_1 \ll \nu, \mu_2 \perp \nu$. Since $\mu_1 \leq \mu$ and $\mu_1 \ll \nu$, we have

$$\mu_1(A) = \inf_{N \in \mathcal{N}} \mu_1(A \setminus N) \leq \inf_{N \in \mathcal{N}} \mu(A \setminus N) = \mu_{ac}(A),$$

so $\mu_1 \leq \mu_{ac}$. Thus $\mu_{ac} - \mu_1 = \mu_2 - \mu_s$ is both ν-absolutely continuous and ν-singular, so $\mu_1 = \mu_{ac}$. $\square$

Constructing measures

Definition 36. On any set, the *counting measure* takes every finite set to its size and every infinite set to ∞. If $S = \{s_1, ...\}$ is a countable subset of X and $a_1, ... \in [0, \infty]$, then the *discrete measure* $\sum_i a_i \delta_{s_i}$ is given by $E \mapsto \sum_{s_i \in E} a_i$. More generally, if $f : X \to [0, \infty]$, we can define a measure $A \mapsto \sum_{a \in A} f(a)$, where the sum over A is defined to be the supremum of all the sums over finite subsets of A.

Definition 37. A measure space (X, Σ, μ) is *complete* if every subset of a null set (that is, a set with measure 0) is in Σ. If Z is the collection of all subsets of null sets, then define Σ_0 to be the σ-algebra generated by Σ and Z, and $\mu_0(C) = \inf\{\mu(D) \mid C \subseteq D \in \Sigma\}$, and define the *completion* of (X, Σ, μ) to be (X, Σ_0, μ_0).

Proposition 22. *The completion of a measure space is always a complete measure space, and in fact* $\Sigma_0 = \{A \cup B \mid A \in \Sigma, B \in Z\}$.

Definition 38. $\varphi : \mathcal{P}(X) \to [0, \infty]$ is an *outer measure* if $\varphi(\emptyset) = 0$, $A \subseteq B \implies \varphi(A) \leq \varphi(B)$, and for any sequence $(A_n)_{n \in \mathbb{N}}$ we have have $\varphi(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \varphi(A_i)$.

Definition 39. If φ is an outer measure over X, we say that E is *φ-measurable* if $\forall A \subseteq X$, we have $\varphi(A) = \varphi(A \cap E) + \varphi(A \cap E^c)$. We write Σ_φ for the collection of all φ-measurable sets.

Theorem 28. *If φ is an outer measure, then Σ_φ is a σ-algebra, and the restriction of φ to Σ_φ is a complete measure.*

Proof. If $E_i \in \Sigma_\varphi$ are pairwise disjoint and $E = \cup_{i=1}^{\infty} E_i$, then for any A we have

$$\varphi(A) \leq \varphi(A \cap E^c) + \varphi(A \cap E) \leq \varphi(A \cap E^c) + \sum_{i=1}^{\infty} \varphi(A \cap E_i) = \sup_n \Big(\varphi(A \cap E^c) + \sum_{i=1}^{n} \varphi(A \cap E_i) \Big) \leq \varphi(A).$$

Taking $A = E$ shows that $\varphi(E) = \sum_{i=1}^{\infty} \varphi(E_i)$. □

Definition 40. An outer measure φ is *regular* if for every set E there exists a φ-measurable set $A \supseteq E$ with $\varphi(E) = \varphi(A)$. (Note: don't confuse regular outer measures with "outer regular" Borel measures in the topological setting!)

Proposition 23. *If φ is an outer measure and $\varphi(A) = 0$, then A is φ-measurable. More generally, if $B \subseteq A$ with B φ-measurable, $\varphi(A) < \infty$, and $\varphi(A) = \varphi(B)$, then A is φ-measurable.*

Proposition 24. *If φ is a regular outer measure, $A \subseteq B$ with B φ-measurable, $\varphi(B) < \infty$, $\varphi(A)$, and $\varphi(B) = \varphi(A) + \varphi(B \setminus A)$, then A is φ-measurable.*

Definition 41. If X is a metric space and φ is an outer measure over X, we say that φ is a *metric outer measure* if $d(E, F) > 0 \implies \varphi(E \cup F) = \varphi(E) + \varphi(F)$.

Theorem 29. *If φ is a metric outer measure, then all Borel sets are φ-measurable.*

Proof. If U is open, let $U_n = \{x \in U \mid B(x, \frac{1}{n}) \subseteq U\}$, and note that for any n, $d(U_n, U_{n+1}^c) \geq \frac{1}{n(n+1)} > 0$. For any A with $\varphi(A) < \infty$ we then have

$$\sum_{n \text{ odd}} \varphi(A \cap (U_{n+1} \setminus U_n)) \leq \varphi(A) < \infty,$$

and similarly for n even, so the tails of the sum go to zero. Then for any A we have

$$\varphi(A) \leq \varphi(A\cap U^c)+\varphi(A\cap U) \leq \inf_n \Big(\varphi(A\cap U^c)+\varphi(A\cap U_n)+\sum_{m\geq n}\varphi(A\cap(U_{m+1}\setminus U_m))\Big) \leq \varphi(A). \quad \square$$

Definition 42. A G_δ set is any countable intersection of open sets, and an F_σ set is any countable union of closed sets.

Proposition 25. *In a metric space, every closed set is a G_δ set and every open set is an F_σ set.*

Proposition 26. *If X is a topological space, Y a metric space, and $f : X \to Y$ is any function, then the set of points of continuity of f is a G_δ set.*

Proof. Let C be the set of points of continuity of f, and for each $c \in C$ and each $n \in \mathbb{N}^+$, pick an open set $c \in U_n^c \subseteq X$ such that $x \in U_n^c \implies d_Y(f(x), f(c)) < \frac{1}{n}$. Then $C = \cap_n \cup_{c\in C} U_n^c$. $\square$

Definition 43. A collection of sets S is a *semi-ring* if $\emptyset \in S$, for any $A, B \in S$ we have $A\cap B \in S$, and for any $A, B \in S$ there exists n and pairwise disjoint $C_1, ..., C_n \in S$ such that $A \setminus B = \cup_{i=1}^n C_i$.

Definition 44. If S is a collection of sets, then a map $\mu : S \to [0,\infty]$ is a *pre-measure* if $\mu(\emptyset) = 0$ and for any sequence A_n of pairwise disjoint sets in S such that $\cup_{i=1}^\infty A_i \in S$, we have $\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$.

Theorem 30 (Carathéodory Extension Theorem)**.** *If S is a semi-ring of subsets of X and $\mu_0 : S \to [0,\infty]$ is a pre-measure, then if we define μ^* by*

$$\mu^*(E) = \inf\Big\{\sum_{i=1}^\infty \mu_0(A_i) \mid A_i \in S,\ E \subseteq \bigcup_{i=1}^\infty A_i\Big\},$$

then μ^* is an outer measure over X with $\mu^*(A) = \mu_0(A)$ for all $A \in S$, and $S \subseteq \Sigma_{\mu^*}$.

Definition 45. A pre-measure $\mu : S \to [0,\infty]$ with S a collection of subsets of X is *σ-finite* if there exists a sequence $A_n \in S$ with $\mu(A_i) < \infty$ and $X = \cup_{i=1}^\infty A_i$.

Theorem 31 (Hahn-Kolmogorov)**.** *If μ_0 is a pre-measure on a semi-ring S, then it extends to a measure μ on the σ-algebra Σ generated by S. If μ_0 is σ-finite, then this extension is unique.*

Proof. Let μ^* be the associated outer measure from the Carathéodory extension theorem, and suppose μ' is a different measure extending μ on $\Sigma' \supseteq S$. Then for any $E \in \Sigma' \cap \Sigma_{\mu^*}$, we clearly have $\mu'(E) \leq \mu^*(E)$. By σ-finiteness and the fact that μ' is countably additive, we can assume WLOG that $\mu^*(X) = \mu'(X) < \infty$, but then $\mu'(E^c) \leq \mu^*(E^c)$ implies $\mu'(E) = \mu^*(E)$ since E is μ^*-measurable. $\square$

Proposition 27. *Let $\mu_0, \mu^*, \mu, S, \Sigma, \Sigma_{\mu^*}$ be as above. If μ_0 is σ-finite, then Σ_{μ^*} is the completion of Σ - in fact, for any $E \in \Sigma_{\mu^*}$, there is a countable intersection of countable unions of elements of S which contains E and differs from it in a null set.*

Theorem 32 (Lebesgue outer measure)**.** *Let S be the collection of half-open intervals $[a, b)$ for $a \leq b \in \mathbb{R}$, and define $\lambda_0 : S \to [0,\infty)$ by $\lambda_0([a, b)) = b - a$. Then S is a semi-ring, λ_0 is a pre-measure, and the associated outer measure λ^* is a translation-invariant metric outer measure over $\mathbb{R}$ with $\lambda^*([0, 1]) = 1$.*

Proof. Suppose that $[a, b) = \cup_{i=1}^{\infty} A_i$, where the A_i are pairwise disjoint half-open intervals. Then the set of left endpoints of the A_i is well-ordered (any descending sequence must have a limit in $[a, b)$, and this limit must be contained in some A_i), so we can show by well-founded induction that if $A_i = [c, d)$, then $\sum_{A_j < A_i} \lambda_0(A_j) = c - a$.

Alternate proof: Let $A' = [a, b - \epsilon]$, and if $A_i = [c_i, d_i)$ let $A_i' = (c_i - \epsilon/2^i, d_i)$. Then by compactness, some finite subset of the A_i's cover A'. $\square$

Definition 46. If λ^* is constructed as above, then a set is called *Lebesgue-measurable* if it is in Σ_{λ^*}, and $\lambda^* \mid_{\Sigma_{\lambda^*}}$ is called the *Lebesgue measure*, and written as λ.

Theorem 33 (Lebesgue-Stieltjes measure)**.** *If I is an interval and $g : I \to \mathbb{R}$ is monotone increasing, set $g_-(x) = \sup_{y<x} g(y)$, then there is a unique Borel measure μ_g such that $\mu_g([a, b)) = g_-(b) - g_-(a)$. If g is continuous, then for any E we have $\mu_g(E) = \lambda(g(E))$.*

Definition 47. If g has bounded variation, then we define the *signed Lebesgue-Stieltjes measure* μ_g by writing $g = g_1 - g_2$ with g_1, g_2 monotone increasing, and $\mu_g = \mu_{g_1} - \mu_{g_2}$.

Definition 48. A Borel measure μ is *locally finite* if every point has an open neighborhood of finite measure. It is *inner regular* on B if $\mu(B)$ is the supremum of $\mu(K)$ over all compact $K \subseteq B$. It is *outer regular* if for all Borel sets B, $\mu(B)$ is the infimum of $\mu(U)$ over all open U containing B. A measure is *Radon* if it is inner regular on open sets, outer regular, and locally finite.

Proposition 28. *Every locally finite Borel measure over $\mathbb{R}$ is a Lebesgue-Stieltjes measure, and every Lebesgue-Stieltjes measure is a Radon measure. More generally, every locally finite Borel measure on $\mathbb{R}^n$ is Radon.*

Theorem 34 (Besicovitch Covering Theorem for Radon Measures)**.** *If E is a bounded subset of $\mathbb{R}^n$ and μ is a Radon measure on $\mathbb{R}^n$ with associated outer measure μ^*, and if $\mathcal{B}$ is a collection of closed balls such that every point in E is the center of an arbitrarily small ball of $\mathcal{B}$, then there exists a countable collection of disjoint balls $\{B_i\} \subseteq \mathcal{B}$ such that $\mu^*(E \setminus \cup_i B_i) = 0$.*

Proof. By the Besicovitch Covering Lemma 8, we can find a finite number c_n of families $\mathcal{B}_i \subseteq \mathcal{B}$ of disjoint balls such that $E \subseteq \cup_i \cup_{B \in \mathcal{B}_i} B$. Then for some i we must have $\mu^*(E \cap \cup_{B \in \mathcal{B}_i} B) \geq \mu^*(E)/c_n$. Pick some finite subset $B_1, ..., B_k$ of $\mathcal{B}_i$ such that $\mu^*(E \cap \cup_{j \leq k} B_j) \geq \mu^*(E)/2c_n$. Now replace E by $E \setminus \cup_{j \leq k} B_j$ and replace $\mathcal{B}$ by the set of balls of $\mathcal{B}$ which do not intersect the closed set $\cup_{j \leq k} B_j$, and iterate. $\square$

Theorem 35 (Product measures)**.** *If μ, ν are pre-measures on semi-rings S, T, respectively, then the collection of rectangles $S \times T$ is a semi-ring, and $\mu \times \nu$ is a pre-measure on $S \times T$.*

Proof. Suppose $E \times F \in S \times T$ is a countable union of disjoint rectangles $E_i \times F_i$. We'll show that for any $M < \mu(E)$ and $N < \nu(F)$, we have $MN \leq \sum_i \mu(E_i)\nu(F_i)$. Let $A_n = \{x \in E \mid \sum_{i=1}^n 1_{x \in E_i} \cdot \nu(F_i) \geq N\}$. Each A_n is a finite union of elements of S, and $\cup_n A_n = E$ since for each $x \in E$, the collection of F_is with $x \in E_i$ is disjoint and covers F, so some finite subset of them must have measure at least N. Thus there is some n such that $\mu(A_n) \geq M$, and for this n we have $MN \leq \sum_{i=1}^n \mu(E_i)\nu(F_i)$. $\square$

Theorem 36 (Infinite products)**.** *Let I be any index set. If μ_i are pre-measures on semi-rings S_i, such that each S_i has an element X_i with $\mu_i(X_i) = 1$, and if we let $S = \prod_{i \in I}' S_i$ be the set of rectangles $\prod_{i \in I} A_i$ such that $A_i = X_i$ for all but finitely many i and define $\mu = \prod_i \mu_i$, then S is a semi-ring and μ is a pre-measure on S.*

Proof. Suppose that $A = \cup_{n=1}^{\infty} A_n$ with $A, A_n \in S$ and the A_ns disjoint, but that $\mu(A) > \sum_n \mu(A_n)$. Each A_n only has finitely many coordinates i which are not equal to X_i, so at most countably many coordinates in I are relevant - rename these relevant coordinates as $1, 2, \ldots$. Write $A = E \times F$, $A_n = E_n \times F_n$, with $E, E_n \in S_1$ and $F, F_n \in \prod'_{i\neq 1} S_i$, and write $\mu^1 = \prod_{i\neq 1} \mu_i$. By the argument for the finite case, there is some $x_1 \in E$ such that $\mu^1(F) > \sum_n 1_{x_1 \in E_n} \cdot \mu^1(F_n)$. Continuing inductively, we find a sequence of coordinates $x_1, x_2, \ldots$ such that for each k, when we restrict the first k coordinates to be $x_1, \ldots, x_k$, the two sides don't add up. But then no point with $(x_1, x_2, \ldots)$ as the relevant countably many coordinates can be an element of any A_n (take k to be larger than the finitely many coordinates i of A, A_n which are not equal to X_i), contradicting the assumption $A = \cup_n A_n$. □

Corollary 8 (Lebesgue measure on $\mathbb{R}^n$)**.** *For every n, there is a translation-invariant metric outer measure λ^* on $\mathbb{R}^n$ with $\lambda^*([0,1]^n) = 1$. If T is a linear transformation and $A \subseteq \mathbb{R}^n$, then $\lambda^*(T(A)) = |\det(T)|\lambda^*(A)$. The associated measure λ is a Radon measure.*

Proof. For the statement about linear transformations, it's enough to check this for shear and stretch transformations in the case A is a box, and this can done done using a standard dissection argument (the pieces are Borel sets). □

Definition 49. If X, Y are measure spaces with measures μ, ν, then $X \times Y$ has a measure $\mu \times \nu$ given by applying the Carathéodory extension Theorem 30 to the product pre-measure contructed in Theorem 35 - this measure is called the *maximal product measure* on $X \times Y$.

Proposition 29. *If $A \subseteq X \times Y$ is $\mu \times \nu$-null, then the set of $y \in Y$ such that $A_y = \{x \in X \mid (x, y) \in A\}$ is not μ-null is ν-null.*

Proof. Pick $\epsilon > 0$, and let E be the set of $y \in Y$ such that $\mu(A_y) > \epsilon$. If $A \subseteq \cup_{n=1}^{\infty} R_n$ such that the R_n are measurable rectangles, and E_k is the set of y such that $\mu((\cup_{n=1}^{k} R_k)_y) > \epsilon$, then $\cup_k E_k = E$, so if $\nu(E) > \delta$ then some $\nu(E_k) > \delta/2$, so $\mu \times \nu(\cup_n R_n) > \epsilon\delta/2$. □

Theorem 37 (Cavalieri Principle)**.** *If X, Y are σ-finite measure spaces and $A, B \subseteq X \times Y$ are measurable with $\mu(A_y) = \mu(B_y)$ for ν-almost every $y \in Y$, then $\mu \times \nu(A) = \mu \times \nu(B)$.*

Example 1. To see σ-finiteness is necessary, take X to be $[0, 1]$ with counting measure, Y to be $[0, 1]$ with Lebesgue measure, A to be $\{0\} \times Y$, and B to be the diagonal.

Theorem 38 (Lebesgue Density Theorem)**.** *If $E \subseteq \mathbb{R}^n$, then for Lebesgue-a.e. x in E we have*

$$\lim_{r\to 0} \frac{\lambda^*(E \cap B_r(x))}{\lambda(B_r(x))} = 1.$$

Proof. Let A_t be the set of points such that the left hand side (with a lim inf instead) is less than $1-t$, and let U_ϵ be an open set containing A_t with $\lambda^*(U_\epsilon \setminus A_t) \le \epsilon$. Then for each point x in A_t, we can find an r such that the left hand side of the above is at most $1 - t$ and such that $B_r(x) \subseteq U_\epsilon$. Now apply the Vitali Covering Lemma to get a collection $(B_i)_{i\in I}$ of disjoint balls contained in U_ϵ such that $A_t \subseteq \cup_i 5B_i$. Then since $\cup_i B_i \subseteq U_\epsilon$, we have

$$\lambda(\cup_i B_i) - \epsilon \le \lambda^*(A \cap (\cup_i B_i)) \le \lambda^*(E \cap (\cup_i B_i)) \le \sum_i (1-t)\lambda(B_i) = (1-t)\lambda(\cup_i B_i),$$

so $\lambda(\cup_i B_i) \le \epsilon/t$, and since $A_t \subseteq \cup_i 5B_i$ we get $\lambda^*(A_t) \le 5^n\epsilon/t$. Since $\epsilon > 0$ was arbitrary, $\lambda^*(A_t) = 0$. □

Definition 50. If X is a metric space and $S \subseteq X$, we set

$$H^d_\delta(S) = \inf\Big\{\sum_{i=1}^\infty \operatorname{diam}(U_i)^d \mid S \subseteq \bigcup_{i=1}^\infty U_i,\ \operatorname{diam}(U_i) < \delta\Big\}$$

and

$$H^d(S) = \sup_{\delta>0} H^d_\delta(S).$$

This is a metric outer measure, called the *Hausdorff measure*.

Theorem 39. *In $\mathbb{R}^n$, we have $H^n(B) = 2^n$, where B is the unit ball.*

Proof. This follows from the isodiametric inequality: the volume of a set of diameter 2 is at most the volume of the unit ball. Suppose that K has diameter 2, then $K - K \subseteq 2B$, so by Brunn-Minkowski we have $\lambda(K) \le \lambda(\frac{1}{2}(K-K)) \le \lambda(B)$. $\square$

Definition 51. If X is a metric space and $S \subseteq X$, we define the *Hausdorff content* of S to be

$$C^d(S) = \inf\Big\{\sum_{i=1}^\infty r_i^d \mid S \subseteq \bigcup_{i=1}^\infty B(x_i, r_i)\Big\}.$$

Proposition 30. *If X is a metric space and $S \subseteq X$, then $C^d(S) = 0$ iff $H^d(S) = 0$.*

Definition 52. If X is a metric space and $S \subseteq X$, the *Hausdorff dimension* of S is defined to be the infimum of the set of d such that $C^d(S) = 0$.

Proposition 31. *If X is a metric space and $S \subseteq X$, then there is a G_δ set which contains S and has the same Hausdorff dimension as S.*

Proposition 32. *If X is a metric space and $S \subseteq X$, then there is a G_δ set which contains S and has the same Hausdorff measure H^d as S (so H^d is a regular outer measure). If additionally S is H^d-measurable and $H^d(S) < \infty$, then there is an F_σ set contained in S with the same Hausdorff measure as S.*

Proof. For the first part, note that in the definition of $H^d_\delta(S)$ we may restrict the covers to be covers by open sets without changing the inf, and take an intersection of open covers over a sequence of δs going to 0. For the second part, let $\cap_i O_i \supseteq S$ be the G_δ set from the first part, and write each O_i as an F_σ set by $O_i = \cup_j C_{ij}$. For each i, find j_i such that $H^d(S \setminus C_{ij_i}) < \epsilon/2^i$, and let $C_\epsilon = \cap_i C_{ij_i}$. Then $H^d(C_\epsilon) > H^d(S) - \epsilon$, and $H^d(C_\epsilon \setminus S) \le H^d(\cap_i O_i \setminus S) = 0$, so we can find an H^d-null G_δ set containing $C_\epsilon \setminus S$, and removing it from C_ϵ we get an F_σ set $C'_\epsilon \subseteq S$. Now take a union over a sequence of ϵs going to 0. $\square$

Theorem 40 (Vitali Covering Theorem for Hausdorff Measure)**.** *If E is a subset of a metric space, and $\mathcal{V}$ is a collection of sets such that every point of E is contained in an element of $\mathcal{V}$ of arbitrarily small nonzero diameter, then there is a countable disjoint subcollection $\{U_i\} \subseteq \mathcal{V}$ such that either $H^d(E \setminus \cup_i U_i) = 0$ or $\sum_i \operatorname{diam}(U_i)^d = \infty$.*

Furthermore, if $H^d(E) < \infty$, then for any $\epsilon > 0$ we may choose this subcollection such that $H^d(E) \le \sum_i \operatorname{diam}(U_i)^d + \epsilon$.

Proof. Let ρ be small, and assume WLOG that all diameters of sets in $\mathcal{V}$ are at most ρ. At each step, choose U_i to be disjoint from $U_1, ..., U_{i-1}$ with diameter at least $\frac{1}{2}$ the sup of the diameters of such disjoint sets. For each i, let B_i be a ball with center in U_i and radius $3\operatorname{diam}(U_i)$, then $E \setminus \cup_{i\le k} U_i \subseteq \cup_{i>k} B_i$. If $\sum_i \operatorname{diam}(U_i)^d < \infty$, then $\operatorname{diam}(B_i) \to 0$ and the tails of $\sum_i \operatorname{diam}(B_i)^d$ go to 0, so each $H^d_\delta(E \setminus \cup_i U_i) = 0$, etc. □

Corollary 9. *An arbitrary union of full-dimensional closed convex subsets of $\mathbb{R}^n$ is Lebesgue measurable.*

Proof. Reduce to the case where everything is contained in a big ball and all the convex sets C satisfy $\operatorname{diam}(C)^n \le k\lambda(C)$ for some integer k, then apply the Vitali Covering Theorem 40 to the collection of homothetic images of these convex sets which are contained within the union, to see that the union can be decomposed into a countable disjoint union of closed convex sets together with a set of measure 0. □

For exceedingly large (in terms of cardinality) spaces, issues like Nedoma's pathology 7 require us to use a different approach to constructing measures, by making use of topological structure.

Definition 53. If X is a locally compact Hausdorff space, then a Borel measure μ is called a *Radon measure* if it is locally finite, outer regular, and inner regular on open sets.

Definition 54. A Borel measure μ is called an *inner Radon measure* if it is locally finite and inner regular on all Borel sets.

Proposition 33. *If X is Hausdorff, μ is a Radon measure, and E is σ-finite, then* $\mu(E) = \sup\{\mu(K) \mid K \subseteq E,\ K \text{ compact}\}$.

Proof. It's enough to prove this when $\mu(E) < \infty$. Take an open set $U \supseteq E$ with $\mu(U) < \infty$, take a compact $K \subseteq U$ with $\mu(U \setminus K) < \epsilon$, and take an open V with $K \setminus E \subseteq V$, $\mu(V \setminus (K \setminus E)) < \epsilon$, then $K \setminus V$ is compact, contained in E, and $\mu(K \setminus V) > \mu(E) - 2\epsilon$. □

Proposition 34. *In a locally compact Hausdorff space, if $K \subseteq U$ with K compact and U open, then there is a compact L with $K \subseteq \operatorname{int}(L)$ and $L \subseteq U$.*

Definition 55. Call a collection $\mathcal{K}$ of compact subsets of a locally compact Hausdorff space X *splittable* if $\mathcal{K}$ is a local base of neighborhoods of X which is closed under finite unions.

Proposition 35. *Suppose that $\mathcal{K}$ is a splittable collection of compact subsets. Then for any K compact and U_1, U_2 open with $K \subseteq U_1 \cup U_2$ there are $K_1, K_2 \in \mathcal{K}$ with $K_i \subseteq U_i$ and $K \subseteq K_1 \cup K_2$.*

Definition 56. A *content* on a splittable collection of compact sets $\mathcal{K}$ is a function $\lambda : \mathcal{K} \to [0, \infty)$, such that $\lambda(K)$ is increasing in K, $\lambda(K_1 \cup K_2) \le \lambda(K_1) + \lambda(K_2)$, and such that for any $K_1, K_2 \in \mathcal{K}$ disjoint, we have $\lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$. A content λ is *regular* if for any $K \in \mathcal{K}$, we have $\lambda(K) = \inf\{\lambda(L) \mid K \subseteq \operatorname{int}(L)\}$.

Theorem 41. *For every content λ on a splittable collection of compact subsets of a locally compact Hausdorff space X, there is a unique Radon measure μ on X such that for all open sets U we have $\mu(U) = \sup\{\lambda(K) \mid K \subseteq U\}$. If λ is a regular content, then μ extends λ.*

Proof. Define μ on open sets as in the theorem statement, and define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by $\mu^*(A) = \inf\{\mu(U) \mid A \subseteq U\}$. μ is finite on the interior of any compact set, so μ^* is locally finite.

First we show that μ^* is an outer measure: If $A = \cup_{n=1}^{\infty} A_n$, then pick U_n open with $A_n \subseteq U_n$ and $\mu^*(U_n) \leq \mu^*(A_n) + \epsilon/2^n$, and let $U = \cup_n U_n$. Pick $K \subseteq U$ compact with $\mu(U) \leq \lambda(K) + \epsilon$, then some finite subset of the U_n cover K, say $U_1, ..., U_k$. We just need to show that $\lambda(K) \leq \sum_{i=1}^{k} \mu(U_i)$, and this follows if we can construct compact $K_i \subseteq U_i$ with $K \subseteq \cup_i K_i$, which follows from splittability.

Now we show that open sets are μ^*-measurable. Let U be open and $A \subseteq X$ be arbitrary. We want to show that for any open $V \supseteq A$, we have $\mu(V) \geq \mu^*(A \cap U) + \mu^*(A \cap U^c)$, so we just need to show that $\mu(V) \geq \mu(V \cap U) + \mu^*(V \setminus U)$. For any compact $K \subseteq V \cap U$, let $W = V \setminus K$, then for any compact $L \subseteq W$ we have $\mu(V) \geq \lambda(K \cup L) = \lambda(K) + \lambda(L)$, so $\mu(V) \geq \lambda(K) + \mu(W) \geq \lambda(K) + \mu^*(V \setminus U)$, so $\mu(V) \geq \mu(V \cap U) + \mu^*(V \setminus U)$. $\square$

Corollary 10. *Every regular content on a locally compact Hausdorff space extends to a unique inner Radon measure. There is a bijection between Radon measures and inner Radon measures on such spaces.*

Proof. Suppose λ is a regular content, and let μ be the associated Radon measure. Define μ_{in} on Borel sets E by $\mu_{in}(E) = \sup\{\mu(K) \mid K \subseteq E,\ K \text{ compact}\}$. To show μ_{in} is a Radon inner measure, we just need to check it is countably additive. Let $E = \bigcup_i E_i$ with $E_i \cap E_j = \emptyset$ for $i \neq j$. We clearly have $\mu_{in}(E) \geq \sum_i \mu_{in}(E_i)$. For the other direction, if $K \subseteq E$ is compact, then $\mu(K) < \infty$ implies that $\mu(K \cap E_i) = \mu_{in}(K \cap E_i)$ for all i by Proposition 33, so $\mu_{in}(K) = \mu(K) = \sum_i \mu(K \cap E_i) = \sum_i \mu_{in}(K \cap E_i)$.

The uniqueness of the extension of λ and the correspondence between Radon measures and inner Radon measures will both follow if we show that any Radon inner measure ν is outer regular on compact sets. So suppose K is compact, and let U be any open set which contains K and is contained in a compact set. Then $\nu(U) < \infty$, so for any ϵ there is a compact set $L \subseteq U \setminus K$ such that $\nu(U \setminus K) \leq \nu(L) + \epsilon$. Then $U \setminus L$ is an open set which contains K and has $\nu(U \setminus L) < \nu(K) + \epsilon$. $\square$

Example 2. Consider the product topology on $\mathbb{R} \times X$, where X is an uncountable set with the discrete topology. Let λ be the natural content on finite unions of closed intervals in copies of $\mathbb{R}$, and let μ, μ_{in} be the associated Radon measure and Radon inner measure. Then the set $\{0\} \times X$ is not σ-finite with respect to μ, but has μ_{in}-measure 0.

Definition 57. If μ is a Borel measure, define its *support* to be the set of points p such that $p \in U$, U open imply $\mu(U) \neq 0$.

Proposition 36. *The support of a Borel measure on a topological space X is always closed. If μ is inner regular on open sets, then $\mu(X \setminus \operatorname{supp} \mu) = 0$.*

Definition 58. If μ is a Borel measure, then a family $\mathcal{D}$ of disjoint nonempty compact sets is called a *concassage* for μ if

- for any $K \in \mathcal{D}$ and any U open with $K \cap U \neq \emptyset$ we have $\mu(K \cap D) > 0$, and
- for any Borel set E, we have $\mu(E) = \sum_{K \in \mathcal{D}} \mu(E \cap K)$.

Proposition 37. *Every inner Radon measure μ on a locally compact Hausdorff space has a concassage.*

Proof. Let $\mathcal{D}$ be any maximal disjoint collection of nonempty disjoint compact sets K such that the restriction of μ to K has full support (such $\mathcal{D}$ exists by Zorn's Lemma). First, suppose for contradiction that there is a Borel set E such that $E \cap \bigcup_{K\in\mathcal{D}} K = \emptyset$ but $\mu(E) > 0$. Then by inner regularity we may assume that E is compact, and then by considering the support of the restriction of μ to E we see that $\mathcal{D}$ is not maximal.

Now let C be any compact set, and let U be an open set containing C which is contained in a compact set. Then from $\mu(U) < \infty$, we see that U can intersect at most countably many sets K in $\mathcal{D}$, so the same is true for C, and we have $\mu(C) = \mu(C \cap \bigcup_{K\in\mathcal{D}} K) = \sum_{K\in\mathcal{D}} \mu(C \cap K)$. By inner regularity, this implies that for any Borel set E, we have $\mu(E) \le \sum_{K\in\mathcal{D}} \mu(E \cap K)$, and the other inequality follows from disjointness of the sets in $\mathcal{D}$. □

Definition 59. If X, Y are locally compact Hausdorff spaces with Radon measures μ, ν, then we define the *Radon product measure* $\mu\widehat{\times}\nu$ on $X \times Y$ to be the unique Radon measure such that for every open subset U of $X \times Y$, we have $\mu\widehat{\times}\nu(U) = \sup_{K\in\mathcal{K}} \mu \times \nu(K)$, where $\mathcal{K}$ is the collection of finite unions of products of compact subsets of X and Y. (Note that if μ, ν are σ-finite, then the restriction of $\mu\widehat{\times}\nu$ to the product σ-algebra on $X \times Y$ is $\mu \times \nu$.)

Proposition 38. *Suppose that (X, μ) is a σ-finite Radon measure space, Y is a topological space, and that U is an open subset of $X \times Y$. Then for any $r \in \mathbb{R}$, the set of $y \in Y$ such that $\mu(U_y) > r$ is an open subset of Y. In fact, for any y with $\mu(U_y) > r$, there are open sets $V \subseteq X$ and $W \subseteq Y$ such that $\mu(V) > r$, $y \in W$, and $V \times W \subseteq U$.*

Proof. Suppose that $\mu(U_y) > r$. By inner regularity, there is some compact set $K \subseteq U_y$ with $\mu(K) > r$. Now cover the compact set $K \times \{y\}$ with finitely many open rectangles contained in U. □

Lemma 10 (Weak Fubini for Radon Products)**.** *Let (X, μ) and (Y, ν) be σ-finite Radon spaces, and suppose that $E \subseteq X \times Y$ is a countable intersection of open subsets of $X \times Y$. If $\nu(E_x) = 0$ for μ-a.e. $x \in X$, then for ν-a.e. $y \in Y$ we have $\mu(E_y) = 0$.*

Proof. We may assume WLOG that $\mu(X) = \nu(Y) = 1$. Suppose for contradiction that there is some $\epsilon > 0$ such that the set of $y \in Y$ with $\mu(E_y) > \epsilon$ has ν-measure greater than ϵ. Write $E = \cap_n U_n$ with $U_1 \supseteq U_2 \supseteq \cdots$ open subsets of $X \times Y$. Then by the previous Proposition for each n there is a compact set $K_n \subseteq Y$ with $\nu(K_n) > \epsilon$, and such that for all $y \in K_n$ we have $\mu((U_n)_y) > \epsilon$, and there is a set $V_n \subseteq U_n$ which is a finite union of open rectangles, such that for all $y \in K_n$ we have $\mu((V_n)_y) > \epsilon$. Since $\mu \times \nu(V_n) > \epsilon^2$, we see that the set of x such that $\nu((V_n)_x) > \epsilon^2/2$ has μ-measure at least $\epsilon^2/2$, and thus the same is true if we replace V_n by U_n. Taking a decreasing limit of measurable subsets of X, we see that the μ-measure of the set of x such that $\nu(E_x) > \epsilon^2/2$ is at least $\epsilon^2/2$, a contradiction. □

Definition 60. If G is a locally compact Hausdorff group and μ is a Borel measure on G, then μ is a *left Haar measure* on G if $\mu(gE) = \mu(E)$ for $g \in G$ and E Borel, and μ is Radon.

Theorem 42 (Haar measure)**.** *If G is a locally compact Hausdorff group, then there is a unique (up to scale) left Haar measure on G.*

Proof. For K compact and V with nonempty interior, let $(K : V)$ be the minimum number of left translates of V that are needed to cover K. Pick K_0 compact with nonempty interior. For every

U, define μ_U on compact sets by
$$\mu_U(K) = \frac{(K : U)}{(K_0 : U)}.$$
Then for all K, U we have $0 \le \mu_U(K) \le (K : K_0)$. We consider each μ_U as a point in $\prod_K [0, (K : K_0)]$. For each open V, let $C(V)$ be the closure of the set of μ_Us with $U \subseteq V$. By compactness, there exists $\mu \in \cap_V C(V)$. For K_1, K_2 disjoint, find V open such that $K_1V^{-1} \cap K_2V^{-1} = \emptyset$, then from $\mu \in C(V)$ we see that $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$. Thus μ defines a left-invariant content on the compact sets of G, so there is a left-invariant Radon measure on G by Theorem 41.

To prove uniqueness, suppose μ, ν are left Haar measures and K, L are compact, L with nonempty interior (so $\mu(L), \nu(L) > 0$). Let G_0 be the subgroup generated by K, L, so that G_0 is σ-compact (as well as clopen and locally compact Hausdorff), and restrict everything to G_0. Suppose for contradiction that $\nu(K)/\nu(L) \neq \mu(K)/\mu(L)$, and rescale μ, ν so that $\mu(K) - \nu(K)$ and $\mu(L) - \nu(L)$ have opposite signs. For every $\epsilon > 0$, let (P_ϵ, N_ϵ) be a Hahn decomposition (Theorem 25) for $\mu - (1+\epsilon)\nu$ restricted to G_0, and let $N = \cap_{n>0} N_{1/n}$. Then for any $g \in G_0$, $gN \setminus N = \cup_{n>0}\ gN \cap P_{1/n}$ is a null set with respect to $\mu + \nu$. Now consider the set $\{(g, x) \in G_0 \times G_0 \mid x \in N \iff x \notin gN\}$ - this is a Borel subset of $G_0 \times G_0$ such that every column is null, and to finish we just need to show that some row is null, which can be shown using Lemma 10: we choose $N_0 \subseteq N \subseteq N_1$ such that N_1 and $G_0 \setminus N_0$ are countable intersections of open subsets of G_0 with $N_1 \setminus N_0$ null, so $\{(g, x) \mid g^{-1}x \in N_1, x \notin N_0\}$ is a countable intersection of open subsets of $G_0 \times G_0$, etc. □

Proposition 39. *Let G be a locally compact Hausdorff group with left Haar measure μ, let K be a compact subset of G with nonempty interior, and let G_0 be the clopen, σ-finite subgroup of G generated by K. Then a Borel subset E of G is σ-finite iff it intersects at most countably many left cosets of G_0, and if so we have $\mu(E) = \sum_{h \in G/G_0} \mu(E \cap hG_0)$.*

The corresponding inner Radon measure μ_{in} decomposes: for any Borel set E, we have $\mu_{in}(E) = \sum_{h \in G/G_0} \mu(E \cap hG_0)$.

Definition 61. If G is a locally compact Hausdorff group, then the *modular function* $\Delta : G \to \mathbb{R}^+$ is defined by $\mu(Kg) = \Delta(g)\mu(K)$, where μ is a left Haar-measure on G. If $\Delta(G) = \{1\}$, then G is called *unimodular*.

Proposition 40. *The modular function is continuous.*

Proof. Fix $\epsilon > 0$. Let K be a compact set with nonempty interior, and let $U \supseteq K$ be open such that $\mu(K) \le \mu(U) < (1+\epsilon)\mu(K)$. By compactness, there is a neighborhood of the identity V with $KV \subseteq U$. For $g \in V$, we have $\Delta(g) = \frac{\mu(Kg)}{\mu(K)} \le \frac{\mu(U)}{\mu(K)} < 1 + \epsilon$, and for $g \in V^{-1}$ we have $\Delta(g) = \frac{\mu(Ug)}{\mu(U)} \ge \frac{\mu(K)}{\mu(U)} > 1 - \epsilon$. □

Proposition 41. *If G is a locally compact Hausdorff group with left Haar measure μ and $\mu_{in}(A) > 0$, then AA^{-1} is a neighborhood of the identity.*

Proof. Choose $K \subseteq A$ compact with $\mu(K) > 0$, choose $U \supseteq K$ open such that $\mu(U) < 2\mu(K)$, and find V a neighborhood of the identity such that $VK \subseteq U$. Then for any $g \in V$, we have $\mu(gK \cup K) \le \mu(VK) \le \mu(U) < 2\mu(K)$, so $gK \cap K \neq \emptyset$. □

Next we'll use this to prove that measurable homomorphisms of locally compact Hausdorff groups are continuous, following [5]. (An even stronger statement is proved there.)

Lemma 11. *For N a subgroup of G, TFAE:*

1. *for all $x \in G$, $[N : xNx^{-1} \cap N] \leq \aleph_0$,*
2. *each double coset NxN is a union of countably many left N cosets,*
3. *for each x there is a countable set D such that $Nx \subseteq DN$,*
4. *if $C \subseteq G$ is countable, and M is the subgroup generated by $N \cup C$, then $[M : N] \leq \aleph_0$.*

Proof. For (1) $\iff$ (2), the double coset NxN is the orbit of $xN \in G/N$ under left translation by N, and the stabilizer is $xNx^{-1} \cap N$. (2) $\iff$ (3) is obvious, and (3) $\implies$ (4), (4) $\implies$ (2) are easy. □

Definition 62. A subgroup N of G is called *asoo* if it satisfies the equivalent conditions of Lemma 11.

Proposition 42. *Countable subgroups and normal subgroups are asoo. Any open σ-compact subgroup of a topological group is asoo. If $\phi : G \to H$ is a homomorphism and L is asoo in H then $\phi^{-1}(L)$ is asoo in G. If ϕ is onto and N is asoo in G, then $\phi(N)$ is asoo in H. If N is asoo and C is countable, then the subgroup generated by $N \cup C$ is asoo.*

Proposition 43. *If G is σ-compact and U_n is a countable family of neighborhoods of the identity, then there is a compact normal subgroup K of G such that $K \subseteq \bigcap_n U_n$ and G/K is separable.*

Proof. Let $G = \bigcup_n F_n$, with F_n an increasing sequence of compact subsets of G. Let V_0 be a compact neighborhood of the identity. For each n, we can find a symmetric neighborhood of the identity V_{n+1} such that $V_{n+1}^2 \subseteq V_n \cap U_n$ and for all $x \in F_n$ we have $xV_{n+1}x^{-1} \subseteq V_n$. Take $K = \bigcap_n V_n$. To finish, we need to show that for any open W containing the identity, there is some n such that $V_n \subseteq WK$. Otherwise, each $V_n \setminus WK$ is a compact nonempty subset of V_0, so by the finite intersection property $K \setminus WK$ is nonempty, contradiction. □

Lemma 12. *Let G be a locally compact Hausdorff group which is either separable or σ-compact and N a null asoo subgroup of G. Then there is a nonmeasurable set $S \subseteq G$ with $S = NS$.*

Proof. Let μ be left Haar measure. First, assume G is separable, and let $U_1 \supseteq U_2 \supseteq \cdots$ be a basis of neighborhoods of the identity. For each n take $x_n \in U_n \setminus N$ (which is nonempty since $\mu(U_n) > 0$). Let M be generated by N and the x_ns, and let Y be a set of right coset representatives of M in G. Take $S = NY$. Suppose S measurable, and let X be a set of left coset representatives of N in M. Since X is countable and $G = MY = XNY = XS$, we have $\mu_{in}(S) > 0$, so by Proposition 41 there is an n with $U_n \subseteq SS^{-1}$. But then $x_n \in SS^{-1}$, so $S \cap x_nS \neq \emptyset$, so $N \cap x_nN \neq \emptyset$, contradiction.

Next, assume that N is not closed, and take $x \in \overline{N} \setminus N$. Let M be generated by N and x, and define Y, S, X as before. If S is measurable, then as above we see there is an open neighborhood of the identity $U \subseteq SS^{-1}$. Since $x \in \overline{N}$, we have $xN \cap U \neq \emptyset$, so $S \cap xNS \neq \emptyset$, so $N \cap xN \neq \emptyset$, contradiction.

Finally, suppose that N is closed and G is σ-compact. Then there is a compact normal subgroup K of G such that G/K is separable. Let ϕ be the quotient map. If $\phi(N)$ is null in G/K, then the first case lets us finish. Otherwise, since $\phi(N)$ is a subgroup of G/K, Proposition 41 shows $\phi(N)$ is open, so NK is open in G, so $[NK : N] > \aleph_0$. Let C be a countable subset of NK with infinite image in NK/N, and let M be the subgroup generated by $N \cup C$. If M is not closed, the second part lets us finish. If M is closed, then it is locally compact Hausdorff, and from $[M : N] = \aleph_0$ we see that N has positive inner M-Haar measure, so by Proposition 41 N is open in M. Thus M/N is a discrete closed subset of the compact set NK/N, so it is finite, contradiction. □

Theorem 43. *Suppose $\phi : G \to H$ is a homomorphism of locally compact Hausdorff groups such that for every open set $U \subseteq H$, $\phi^{-1}(U)$ is measurable (with respect to the completion of the Haar measure). Then ϕ is continuous.*

Proof. We may assume WLOG that G is compactly generated. By Proposition 41 and the fact that every neighborhood U of the identity contains a neighborhood V with $VV^{-1} \subseteq U$, it's enough to show that for every open neighborhood V of the identity, $\phi^{-1}(V)$ is not null. Suppose $\phi^{-1}(V)$ is null, and let L be an open σ-compact subgroup of H. Then L is asoo and is contained in a union of countably many left translates of V, so $\phi^{-1}(L)$ is a null asoo subgroup of G. By the Lemma, there is a nonmeasurable $S \subseteq G$ with $S = \phi^{-1}(L)S$. We have $L\phi(S)$ open, so by hypothesis $S = \phi^{-1}(L)S = \phi^{-1}(L\phi(S))$ is measurable, contradiction. □

2.1.5 Integration

Definition 63. If $f : X \to Y$ and $\mathcal{B}$ is a σ-algebra on Y, then $\sigma(f)$ is the σ-algebra on X generated by $f^{-1}(S)$ for $S \in \mathcal{B}$. We say that $f : (X, \Sigma) \to (Y, \mathcal{B})$ is Σ-*measurable*, or just *measurable* if Σ is clear, if $\sigma(f) \subseteq \Sigma$ (if unspecified, $\mathcal{B}$ is usually taken to be the Borel sets of Y).

Proposition 44. *$f : (X, \Sigma) \to [-\infty, \infty]$ is measurable iff $f^{-1}([-\infty, a]) \in \Sigma$ for all $a \in \mathbb{R}$. If $f_1, ..., f_n$ are measurable and $g : \mathbb{R}^n \to [-\infty, \infty]$ is Borel measurable, then $g(f_1, ..., f_n)$ is measurable. If f_k is a sequence of measurable functions, then $\sup f_k$ is measurable.*

Proposition 45. *If $f_k : X \to Y$ is a sequence of measurable functions to a metric space and $f_k \to f$ pointwise, then f is measurable.*

Proof. For any open set U the collection of $x \in X$ such that $f_k(x)$ are eventually all in U is measurable, and this set contains $f^{-1}(U)$ and is contained $f^{-1}(\overline{U})$. Since every open set in a metric space is a countable union of open subsets whose closures are contained in it, the preimage of every open set is measurable. □

Definition 64. A *simple function* is a function which can be written as a finite linear combination of measurable sets. Equivalently, a function is simple if it is measurable and has a finite range.

Definition 65. For $f \geq 0$ measurable (up to a null set), we define the *integral* of f with respect to a measure $\mu : \Sigma \to [0, \infty]$ to be

$$\int f \; d\mu = \sup \Big\{ \sum_{i=1}^{k} c_i \mu(A_i) \mid c_1, ..., c_k \geq 0,\ A_1, ..., A_k \in \Sigma,\ \sum_{i=1}^{k} c_i \cdot 1_{x \in A_i} \leq f(x) \Big\}.$$

A measurable (up to a null set) complex-valued function f is *integrable* if $\int |f| d\mu < \infty$. We extend the integral to all integrable functions by linearity.

Theorem 44 (Markov's Inequality)**.** *For $t > 0$, $\mu(\{|f| \geq t\}) \leq \frac{1}{t} \int |f| \; d\mu$.*

Proposition 46. *For $f, g \geq 0$ measurable, we have $\int f + g \; d\mu = \int f \; d\mu + \int g \; d\mu$.*

Proof. For any finite $S \subset [0, \infty]$, define f_S by

$$f_S(x) = \max\{s \in S \mid s \leq f(x)\}.$$

Note f_S is a simple function and $\int f \, d\mu = \sup_S \int f_S \, d\mu$. For any S and any n, if we let $S_n = \{\frac{k}{n} s \mid k \leq n, \ s \in S\}$, then $(f+g)_S \leq \frac{n-1}{n}(f_{S_n} + g_{S_n})$. □

Proposition 47. *Any Riemann integrable function $f : [0,1] \to \mathbb{C}$ is Lebesgue integrable, with the same integral.*

Proposition 48. *If $f : X \to [0, \infty]$ is measurable, then $\{(x,t) \mid 0 \leq t \leq f(x)\}$ is measurable in $X \times [0, \infty]$, with $\mu \times \lambda$-measure $\int_X f \, d\mu = \int_0^\infty \mu(\{x \mid f(x) \geq t\}) \, dt$.*

Proof. For any $c > 1$, if we round positive values of f up or down to the nearest c^n, $n \in \mathbb{Z}$, we see that the product outer measure of $\{(x,t) \mid 0 \leq t \leq f(x)\}$ is at most c times $\int_X f \, d\mu$. □

Theorem 45 (Monotone Convergence Theorem)**.** *If f_k is a sequence of measurable functions with $0 \leq f_k \leq f_{k+1}$ for all k and f is the pointwise limit of the f_k, then f is measurable and $\int f \, d\mu = \lim_k \int f_k \, d\mu$.*

Proof. It's enough to prove this when f is the characteristic function of a measurable set A. Fix $\epsilon > 0$, and for each k set $A_k = \{x \mid f_k(x) \geq 1 - \epsilon\}$, then from $\cup_k A_k = A$, we have $\lim_k \mu(A_k) = \mu(A)$, so $\lim_k \int f_k \, d\mu \geq (1-\epsilon)\mu(A)$. □

Lemma 13 (Fatou's Lemma)**.** *If $f_k \geq 0$ are measurable, then $\int \liminf_k f_k \, d\mu \leq \liminf_k \int f_k \, d\mu$.*

Proof. $\int \liminf_k f_k \, d\mu = \lim_k \int \inf_{l \geq k} f_l \, d\mu \leq \liminf_k \int f_k \, d\mu$. □

Corollary 11. *If f_k measurable, $|f_k| \leq g$, g integrable, then*

$$\int \liminf f_k \, d\mu \leq \liminf \int f_k \, d\mu \leq \limsup \int f_k \, d\mu \leq \int \limsup f_k \, d\mu.$$

Theorem 46 (Dominated Convergence Theorem)**.** *If f_k measurable, $|f_k| \leq g$, g integrable, $f_k \to f$ pointwise, then $\lim_k \int f_k \, d\mu = \int f \, d\mu$, and $\lim_k \int |f_k - f| \, d\mu = 0$.*

Theorem 47 (Jensen)**.** *If $\mu(X) = 1$, g real μ-integrable, φ convex, then $\varphi(\int_X g \, d\mu) \leq \int_X \varphi \circ g \, d\mu$.*

Proof. Let $x_0 = \int_X g \, d\mu$. Since φ is convex, there are $a, b \in \mathbb{R}$ such that $ax + b \leq \varphi(x)$ and $ax_0 + b = \varphi(x_0)$. Integrating both sides of $ag(t) + b \leq \varphi(g(t))$ gives the inequality. □

Theorem 48 (Radon-Nikodym Theorem)**.** *If μ, ν are σ-finite measures on X (ν possibly signed or complex) and $\nu \ll \mu$, then there exists a measurable function f (unique up to a μ-null set) such that for any measurable set A, $\nu(A) = \int_A f \, d\mu$.*

Proof. We just need to prove this in the positive, finite case. Let $\mathcal{F}$ be the family of measurable functions f such that for all measurable A, $\nu(A) \geq \int_A f \, d\mu$. Note that $\mathcal{F}$ is closed under maximum, and by the Monotone Convergence Theorem 45 $\mathcal{F}$ is closed under countable monotone limits, so there is some $f \in \mathcal{F}$ with $\int_X f \, d\mu = \sup_{g \in \mathcal{F}} \int_X g \, d\mu$. Let $\nu_0 = \nu - \int f \, d\mu$. If $\nu_0(X) > 0$, take $\epsilon > 0$ such that $\nu_0(X) > \epsilon\mu(X)$, and let (N, P) be a Hahn decomposition 25 of $\nu_0 - \epsilon\mu$. But then $f + \epsilon \cdot 1_P \in \mathcal{F}$ and $\mu(P) > 0$, contradicting our choice of f. □

Definition 66. If μ, ν have $\nu = \int f \, d\mu$, then the *Radon-Nikodym derivative* $\frac{d\nu}{d\mu}$ is defined to be the equivalence class of f when we quotient by μ-null functions.

Proposition 49. *If μ is a complex measure, then $\frac{d\mu}{d|\mu|}$ has absolute value 1 $|\mu|$-almost everywhere.*

Proof. Let f be a representative of $\frac{d\mu}{d|\mu|}$. For any $\epsilon > 0$, the set where $|f| < 1 - \epsilon$ has measure 0, since otherwise its $|\mu|$ measure would be smaller than itself by a factor of $1 - \epsilon$. By dividing up the set where $|f| > 1 + \epsilon$ into $O(\frac{1}{\sqrt{\epsilon}})$ many subsets based on the argument of f, we see that it must also have measure 0, since otherwise its $|\mu|$ measure would be larger than itself by a factor of $1 + \frac{\epsilon}{2}$. $\square$

Proposition 50. *Where the relevant Radon-Nikodym derivatives make sense, we have $\frac{d(\nu+\mu)}{d\lambda} = \frac{d\mu}{d\lambda} + \frac{d\nu}{d\lambda}$, $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$, $\frac{d|\nu|}{d\mu} = |\frac{d\nu}{d\mu}|$, and $\int g \, d\mu = \int g \frac{d\mu}{d\lambda} \, d\lambda$.*

Proposition 51. *If $E \subseteq X \times Y$ is measurable and $\mu \times \nu(E) < \infty$, then for μ-almost every $x \in X$ E_x is measurable up to a ν-null set, the function $g(x) = \mu(E_x)$ is measurable up to a μ-null set, and $\int g \, d\mu = \mu \times \nu(E)$.*

Proof. By definition of $\mu \times \nu$, there is an $F \supseteq E$ which is a countable decreasing intersection of countable unions of measurable rectangles, such that $\mu \times \nu(E) = \mu \times \nu(F)$. Since $\mu \times \nu(E) < \infty$, $F \setminus E$ is $\mu \times \nu$-null, so we may replace E by F without changing g (aside from on a μ-null set) by Proposition 29 and then apply monotone 45 and dominated 46 convergence to reduce to the case of a finite union of measurable rectangles. $\square$

Theorem 49 (Fubini's Theorem). *If $\int_{X\times Y} |f(x,y)| \, d(x,y) < \infty$, where $d(x,y)$ is the maximal product measure on $X \times Y$, then for a.e. $x \in X$ $f(x,y)$ is integrable in y, and we have $\int_{X\times Y} f(x,y) \, d(x,y) = \int_X \int_Y f(x,y) \, dy \, dx$.*

Theorem 50 (Tonelli's Theorem). *If X, Y are σ-finite, then $\int_{X\times Y} |f(x,y)| \, d(x,y) = \int_X \int_Y |f(x,y)| \, dy \, dx$.*

Proof. Assume $f \geq 0$. The assumptions of either Fubini or Tonelli imply that f can be written as the pointwise limit of an increasing sequence ϕ_n of nonnegative simple functions that each vanish outside a set of finite measure. Thus, using Proposition 51, for almost every fixed x the function $y \mapsto f(x,y) = \lim_n \phi_n(x,y)$ is measurable up to a null set, and by monotone convergence 45 the function $x \mapsto \int_Y f(x,y) \, dy = \lim_n \int_Y \phi_n(x,y) \, dy$ is measurable up to a null set. Applying monotone convergence and Proposition 51 again, we get

$$\begin{aligned} \int_X \int_Y f(x,y) \, dy \, dx &= \lim_n \int_X \int_Y \phi_n(x,y) \, dy \, dx \\ &= \lim_n \int_{X\times Y} \phi_n(x,y) \, d(x,y) = \int_{X\times Y} f(x,y) \, d(x,y). \end{aligned}$$ $\square$

A lot of the next bits are from [3].

Proposition 52. *If X, Y are locally compact Hausdorff with Radon measures μ, ν and U is open in $X \times Y$, then $x \mapsto \nu(U_x)$ is lower semicontinuous and $\mu \widehat{\times} \nu(U) = \int \nu(U_x) \, d\mu(x)$.*

Proof. This follows directly from Proposition 38, the definition of the integral, and the fact that Radon measures are inner regular on open sets. $\square$

Theorem 51 (Fubini-Tonelli for Radon Products)**.** *If μ, ν are σ-finite Radon measures on locally compact Hausdorff spaces X, Y, f is Borel measurable on $X \times Y$, and either $f \geq 0$ or $|f|$ is integrable, then $\int_{X\times Y} f \; d\mu \widehat{\times} \nu = \int_X \int_Y f \; d\nu \; d\mu$.*

Proposition 53. *If X, Y are locally compact Hausdorff, then every $f \in C_c(X \times Y)$ is measurable with respect to the product of the Borel σ-algebras.*

Proof. Follows from Stone-Weierstrauss, Urysohn's Lemma, and the fact that pointwise limits of measurable functions are measurable. □

Definition 67. Write $f \prec U$ if $0 \leq f \leq \chi_U$ and $\mathrm{supp}(f) \subseteq U$.

Theorem 52 (Riesz Representation Theorem)**.** *If X is a locally compact Hausdorff space and I is a positive linear functional on $C_c(X)$, then there is a unique Radon measure μ such that $I(f) = \int f \; d\mu$ for all $f \in C_c(X)$. This μ satisfies $\mu(K) = \inf\{I(f) \mid f \geq \chi_K\}$ for all compact K and $\mu(U) = \sup\{I(f) \mid f \prec U\}$ for all open U.*

Proof. Uniqueness and the formulas for $\mu(U), \mu(K)$ follow from Urysohn's Lemma. For existence, we need to check that the formula for $\mu(K)$ defines a regular content and that the formula $I(f) = \int f \; d\mu$ holds. For the regularity, note that if K is compact and $f \geq \chi_K$, then if we let $U_\epsilon = \{x \mid f(x) > 1 - \epsilon\}$, then for any $g \prec U_\epsilon$ we have $I(g) \leq I(f)/(1-\epsilon)$, so $\mu(U_\epsilon) \leq I(f)/(1-\epsilon)$.

For the integral formula, if $f \leq 1$, for any N we define $K_j = \{x \mid f(x) \geq j/N\}$ and $K_0 = \mathrm{supp}(f)$, and $f_j = \min((f - \frac{j-1}{N})_+, \frac{1}{N})$, so $f = \sum f_j$ and $\chi_{K_j} \leq N f_j \leq \chi_{K_{j-1}}$. Then we have $\mu(K_j) \leq N \int f_j \; d\mu \leq \mu(K_{j-1})$ and $\mu(K_j) \leq N I(f_j) \leq \mu(K_{j-1})$ (last inequality using outer regularity), so $|I(f) - \int f \; d\mu| \leq \mu(K_0)/N$. □

Lemma 14. *If X is locally compact Hausdorff, then every bounded real linear functional I on $C_0(X)$ can be written as the difference between two positive linear functionals.*

Proof. For $f \geq 0$ in $C_0(X)$, set $I^+(f) = \sup\{I(g) \mid 0 \leq g \leq f\}$ (this is finite since I is bounded) and $I^- = I^+ - I$. The only tricky bit is to check that $I^+(f_1 + f_2) \leq I^+(f_1) + I^+(f_2)$: if $0 \leq g \leq f_1 + f_2$, then set $g_1 = \min(g, f_1)$ and $g_2 = g - g_1 = \max(0, g - f_1)$, so $0 \leq g_i \leq f_i$, so $I(g) \leq I^+(f_1) + I^+(f_2)$. □

Theorem 53 (Riesz Representation Theorem for $C_0(X)$)**.** *If X is a locally compact Hausdorff space, then $\mu \mapsto I_\mu = (f \mapsto \int f \; d\mu)$ is an isometric isomorphism between complex Radon measures on X under the total variation norm $\|\mu\| = |\mu|(X)$ and bounded linear functionals on $C_0(X)$ under the operator norm $\|I\| = \sup\{|I(f)| \mid \sup_x |f(x)| = 1\}$.*

Proof. Apply Lusin's Theorem (below) to $\frac{d\mu}{d|\mu|}$ to show that $\|\mu\| \leq \|I_\mu\|$; the other inequality is easy. □

Convergence in Measure

Definition 68. A sequence of measurable functions f_n converges to f *globally in measure* if $\forall \epsilon > 0$, we have $\lim_n \mu(\{x \mid |f(x) - f_n(x)| \geq \epsilon\}) = 0$, and $f_n \to f$ *locally in measure* if $\forall \epsilon > 0$ and for all $F \in \Sigma$ with $\mu(F) < \infty$ we have $\lim_n \mu(\{x \in F \mid |f(x) - f_n(x)| \geq \epsilon\}) = 0$.

Theorem 54 (Riesz)**.** *If $f_n \to f$ globally in measure (or locally in measure on a σ-finite space) then some subsequence converges to f pointwise almost everywhere.*

Proof. Choose a subsequence n_k such that $\mu(\{x \mid |f(x) - f_{n_k}(x)| \geq \frac{1}{k}\}) < 2^{-k}$. $\square$

Proposition 54. *If all subsequences of f_n have a subsequence which converges to f almost everywhere (and f is finite almost everywhere), then $f_n \to f$ locally in measure.*

Proof. Suppose there is some $F \in \Sigma$ with $\mu(F) < \infty$ and $\epsilon > 0$ such that $\mu(\{x \in F \mid |f(x) - f_n(x)| \geq \epsilon\})$ doesn't converge to 0. Then there is a $\delta > 0$ and a subsequence n_k such that $\mu(\{x \in F \mid |f(x) - f_{n_k}(x)| \geq \epsilon\}) > \delta$ for all k. No such subsequence f_{n_k} can converge almost everywhere to f: otherwise, there would be some K such that the set of $x \in F$ with $|f(x) - f_{n_k}(x)| < \epsilon$ for all $k > K$ has measure at least $\mu(F) - \delta$. $\square$

Theorem 55 (Egoroff's Theorem)**.** *If M is a separable metric space and f_n is a sequence of measurable functions from A to M, with $\mu(A) < \infty$, such that $f_n \to f$ pointwise almost everywhere, then for every $\epsilon > 0$ there is $B \subseteq A$ such that $\mu(B) < \epsilon$ and $f_n \to f$ uniformly on $A \setminus B$.*

Proof. For every k, choose n_k such that $\mu(\{x \in A \mid \exists m > n_k\ d(f(x), f_m(x)) \geq \frac{1}{k}\}) < \frac{\epsilon}{2^k}$ (to see that $x \mapsto d(f(x), f_m(x))$ is measurable, we use separability of M). $\square$

Theorem 56 (Lusin's Theorem)**.** *If $f : [a, b] \to \mathbb{C}$ is measurable, then $\forall \epsilon > 0$ there exists a compact $E \subseteq [a, b]$ such that $f|_E$ is continuous and $\mu(E) > b - a - \epsilon$. More generally, if (X, μ) is a Radon measure space and Y is second-countable, and $f : A \to Y$ is measurable with $\mu(A) < \infty$, then $\forall \epsilon > 0$ there is a compact set $E \subseteq A$ with $\mu(A \setminus E) < \epsilon$ such that $f|_E$ is continuous.*

Proof. (From [2]) Let U_j be an enumeration of a base of open sets for Y, and for each j choose V_j open in X such that $f^{-1}(U_j) \subseteq V_j$ and $\mu(V_j \setminus f^{-1}(U_j)) < \frac{\epsilon}{2^j}$. Take $E_1 = A \setminus \bigcup_j (V_j \setminus f^{-1}(U_j))$, so $f^{-1}(U_j) \cap E_1 = V_j \cap E_1$, then let E be a compact set contained in E_1 with sufficiently close measure. $\square$

Lebesgue Integral and Derivatives

Definition 69. The *Hardy-Littlewood maximal operator* M takes a locally integrable $f : \mathbb{R}^n \to \mathbb{C}$ to the function Mf given by

$$Mf(x) = \sup_{r>0} \frac{\int_{B_r(x)} |f(y)|\ dy}{\lambda(B_r)}.$$

Theorem 57 (Weak type Hardy-Littlewood maximal inequality)**.** *For any integrable function $f : \mathbb{R}^n \to \mathbb{C}$, we have $\lambda(\{Mf > t\}) \leq \frac{3^n}{t} \int |f|\ d\lambda$.*

Proof. Let $A_t = \{Mf > t\}$, and let K be any compact set contained in A_t. For each $x \in K$, we can find an $r > 0$ such that $\int_{B_r(x)} |f(y)|\ dy > t\lambda(B_r(x))$, and finitely many of these balls $B_r(x)$ cover K. Apply the Finite Vitali Covering Lemma 6 to get a collection B_i of disjoint balls among these such that $K \subseteq \bigcup_i 3B_i$, then $\lambda(K) \leq 3^n \sum_i \lambda(B_i) \leq \frac{3^n}{t} \int |f(y)|\ dy$. $\square$

Theorem 58 (Lebesgue Differentiation Theorem)**.** *If $f : \mathbb{R}^n \to \mathbb{C}$ is locally integrable, then for Lebesgue-a.e. x we have*

$$\lim_{r \to 0} \frac{\int_{B_r(x)} |f(y) - f(x)|\ dy}{\lambda(B_r)} = 0.$$

Proof. First proof: approximate f by a simple function, and apply the Lebesgue Density Theorem 38.

Second proof: Assume f is supported on a finite ball. By Lusin's Theorem 56 and the Tietze Extension Theorem (Corollary 1), we can find $g \in C_c(\mathbb{R}^n)$ with $\int |f - g| \, d\lambda < \epsilon$. Then

$$\frac{1}{\lambda(B)} \int_B |f(y) - f(x)| \, dy \leq \frac{1}{\lambda(B)} \int_B |f(y) - g(y)| \, d\lambda + \frac{1}{\lambda(B)} \int_B |g(y) - g(x)| \, d\lambda + |f(x) - g(x)|.$$

By Theorem 57 the first summand is at most t away from a set of measure at most $\frac{3^n}{t} \int |f - g| \, d\lambda < \frac{3^n \epsilon}{t}$, and by Markov's inequality the third summand is at most t away from a set of measure at most $\frac{\epsilon}{t}$, while the second summand goes to 0 as r goes to 0 since $g \in C_c(\mathbb{R}^n)$. $\square$

Proposition 55. *If f is nondecreasing, then f has only jump discontinuities, and only countably many of them.*

Lemma 15 (Riesz's Rising Sun Lemma)**.** *If $U \subseteq \mathbb{R}$ is open and $g : U \to \mathbb{R}$ is continuous, then the set $U_g = \{x \in U \mid \exists y > x \text{ s.t. } (x, y) \subseteq U \text{ and } g(x) < g(y)\}$ is also open, and if (a, b) is a component of U_g then $g(a) \leq g(b)$.*

Theorem 59 (Lebesgue)**.** *If f is nondecreasing, then f is differentiable almost everywhere, and $\int_a^b f'(x) \, dx \leq f(b) - f(a)$. If E, Z, I are the sets where f is not differentiable, has derivative 0, and has derivative ∞, respectively, then $f(E), f(Z), I$ have measure 0.*

Proof. (Following [1]) Set $D^+ f(x) = \limsup_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}$, $D_+ f(x) = \liminf_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}$, and similarly define D^-, D_- with h approaching 0 from below.

First we show that if f is continuous and E is any set where $D^+ f > u$, then $\lambda^*(f(E)) \geq u\lambda^*(E)$: if U is any open set containing $f(E)$, then $f^{-1}(U)$ is an open set containing E, and the rising sun lemma 15 applied to $g(x) = f(x) - ux$ and $f^{-1}(U)$ shows that $\lambda(U) \geq u\lambda(f^{-1}(U)_g) \geq u\lambda^*(E)$.

Next we show that if f is strictly increasing and E is any set where $D_+ < v$, then $\lambda^*(f(E)) \leq v\lambda^*(E)$: let $g(y) = \inf\{z \mid f(z) \geq y\}$ be inverse to f, suppose WLOG that no point of discontinuity of f is in E, then for any $x \in E$ we have $D^+ g(f(x)) > \frac{1}{v}$, and we can reduce to the previous case. We extend this from strictly increasing f to all f by replacing f by $h(x) = f(x) + x$ and noting that $\lambda^*(h(E)) \geq \lambda^*(f(E)) + \lambda^*(E)$ (take any open set containing $h(E)$ and break it into connected components). From this we see that we can drop the continuity assumption in the first case by considering the function g inverse to f again and ignoring the countably many points where either f or g is discontinuous.

Applying the above to $-f(-x)$, we get similar statements for D^-, D_-. We'll show that $D^+ \leq D_-$ almost everywhere, and similarly $D^- \leq D_+$ a.e., so we will have $D^+ \leq D_- \leq D^- \leq D_+ \leq D^+$ almost everywhere. Let E_{uv} be the set of x with $D^+ f(x) > u > v > D_- f(x)$. Then $u\lambda^*(E_{uv}) \leq \lambda^*(f(E_{uv})) \leq v\lambda^*(E_{uv})$, so $\lambda^*(E_{uv}), \lambda^*(f(E_{uv}))$ must be 0.

For the statement about $\int_a^b f'(x) \, dx$, apply Fatou's Lemma to the sequence of functions $f_n(x) = n(f(x + \frac{1}{n}) - f(x))$ (assuming WLOG that $f(x) = f(b)$ for $x > b$). $\square$

Corollary 12. *If f has bounded variation, then f is differentiable almost everywhere and f' is Lebesgue integrable.*

Corollary 13. *If f is increasing, μ_f is the Lebsgue-Stieltjes measure 33, and $(\mu_f)_{ac}$ is the absolutely continuous part of the Lebesgue decomposition 27 of μ_f with respect to λ, then $\frac{d(\mu_f)_{ac}}{d\lambda} = f'$. The* *singular part $(\mu_f)_s$ can be written as the sum of a discrete measure and some μ_c with c continuous and $c' = 0$ almost everywhere.*

Proof. Let $g(x) = \int_0^x f'(t)\,dt$, and note that $\mu_g \le \mu_f$ and $\mu_g \ll \lambda$ since $\mu_g(E) = \int_E f'(t)\,dt$ for any Borel set E. By the Lebesgue differentiation theorem 58 and the fact that g' exists almost everywhere, we have $g' = f'$ almost everywhere. To finish, we need to check that if c is continuous and $c' = 0$ almost everywhere then $\mu_c \perp \lambda$: if Z is the set where $c' = 0$, then $\mu_c(Z) = \lambda(c(Z)) = 0$. □

Definition 70. A function $f : I \to \mathbb{R}$ is *absolutely continuous* on I if $\forall \epsilon > 0\ \exists \delta > 0$ such that if $(x_k, y_k) \subseteq I$ are disjoint subintervals with $\sum_k |y_k - x_k| < \delta$ then $\sum_k |f(y_k) - f(x_k)| < \epsilon$.

Proposition 56. *If f is absolutely continuous then f has bounded variation, and the variation of f is also absolutely continuous.*

Theorem 60 (Fundamental Theorem of the Lebesgue Integral)**.** *A function f is absolutely continuous on $[a,b]$ iff there exists g integrable with $f(x) = f(a) + \int_a^x g(t)\,dt$ for all $x \in [a,b]$. In this case we have $g = f'$ almost everywhere.*

Corollary 14. *If f is absolutely continuous and has $f' = 0$ almost everywhere, then f is constant.*

Proof. Direct proof based on Vitali Covering Theorem 40: Let $\mathcal{V}$ be the family of intervals $[x,y] \subseteq [a,b]$ such that $f'(x) = 0$ and $|\frac{f(y)-f(x)}{y-x}| < \epsilon$, then we can find a finite disjoint subset of intervals of $\mathcal{V}$ that cover all but δ of $[a,b]$ for δ sufficiently small, so $|f(b) - f(a)| \le \epsilon|b-a| + \epsilon$. □

Example 3. The Cantor function (aka the Devil's Staircase, defined by writing a number in ternary, ignoring every digit after the first 1, replacing every 2 with a 1, and interpreting the result in binary) is uniformly continuous, but not absolutely continuous, and has derivative 0 almost everywhere.

This example leads to other pathologies: let $f : [0,1] \to [0,1]$ be the Cantor function, let $h(x) = f(x) + x : [0,1] \to [0,2]$, let $g = h^{-1} : [0,2] \to [0,1]$, let $C \subseteq [0,1]$ be the Cantor set, and let $D = [0,1] \setminus C$. Since D is a union of open intervals whose lengths sum to 1 and f is constant on these intervals, $h(D)$ is measurable with measure 1, so $h(C)$ is also measurable with measure 1. Any measurable subset with positive measure contains a set which isn't measurable (usual argument with equivalence classes based on rationals works), so let $A \subseteq h(C)$ be such a nonmeasurable set and let $B = g(A)$. Then $B \subseteq C$, so B is Lebesgue measurable (but not Borel measurable), so χ_B, g are both measurable but $\chi_B \circ g$ is not measurable. Additionally, although g, h are continuous and strictly increasing, we have $A = g^{-1}(B) = h(B)$ is not measurable even though B is.

A lot of the following is from [4].

Proposition 57. *Absolutely continuous functions map null sets to null sets and map measurable sets to measurable sets.*

Proposition 58. *If f_n is a sequence of equi-absolutely continuous functions (i.e. for each ϵ, a single δ works for all of them), and $\lim_n f_n = f$ pointwise, then f is absolutely continuous (and similarly for a sequence with uniformly bounded variation). In particular, if a sequence g_n of absolutely continous functions has $\sum_n g_n$ convergent and the sum of the variations of the g_ns is finite, then $\sum_n g_n$ is absolutely continuous.*

Proposition 59. *If f has bounded variation on $[a,b]$, $V(x)$ is the variation of f on $[a,x]$, and f is continuous at c, then V is also continuous at c. In particular, if f is continuous and has bounded variation on $[a,b]$, and is absolutely continuous on $[a,c]$ for all $a < c < b$, then f is absolutely continuous on $[a,b]$.*

Proposition 60. *Suppose $f : [a,b] \to \mathbb{R}$ is bounded, and let $m_f(x) = \limsup_{t\to x} f(t)$. Then m_f can be written as a pointwise limit of step functions which each exceed f, and the upper Riemann integral of f is equal to the Lebesgue integral $\int m_f$. In particular, a bounded function on $[a,b]$ is Riemann integrable iff it is continuous a.e.*

Proposition 61. *There is a perfect, nowhere dense subset of $[0,1]$ with positive Lebesgue measure.*

Proof. The construction is the same as the Cantor set, but shrink each removed interval by a constant factor. Alternatively, consider the set of numbers whose base 5 expansions contain no 2s. $\square$

Proposition 62. *There is a function $f : [0,1] \to \mathbb{R}$ such that f' exists and is bounded everywhere on $[0,1]$, but f' is discontinuous on a set of positive measure and is therefore not Riemann integrable.*

Proof. Let E be a perfect, nowhere dense subset of $[0,1]$ with positive measure. The plan is to make f equal to 0 on E, and on each open interval (a,b) in $[0,1] \setminus E$ to choose f such that $|f(x)| \le |x-a|^2, |b-x|^2$, but such that $|f'(x)| = 1$ for points $x \in (a,b)$ arbitrarily close to a and b. To construct such functions, start with the function $x \mapsto (x-a)^2\sin(1/(x-a))$ around a, and connect it to a similar function around b. $\square$

Proposition 63. *If f_i are nondecreasing functions and $f = \sum_i f_i$ converges, then $f' = \sum_i f_i'$ a.e.*

Proof. Let $g_k = \sum_{i>k} f_i$, then it's enough to prove that $\lim_k g_k' = 0$ a.e. To see this, pick a subsequence k_i such that $\sum_i g_{k_i}$ converges at points a, b, and note that $\int_a^b \sum_i g_{k_i}' = \sum_i \int_a^b g_{k_i}' = \sum_i g_{k_i}(b) - g_{k_i}(a) < \infty$, so $\sum_i g_{k_i}'$ must converge a.e. on $[a,b]$, so $\lim_k g_k' = \lim_i g_{k_i}' = 0$ a.e. on $[a,b]$. $\square$

Proposition 64. *If $f : \mathbb{R} \to \mathbb{R}$ is measurable and $0 < \alpha < \beta$, then the function $x \mapsto \sup\{\frac{f(y)-f(x)}{y-x} \mid x+\alpha < y < x+\beta\}$ is measurable. In particular, all four derivates D^+f, D^-f, D_+f, D_-f are measurable.*

Proof. If $\sup\{\frac{f(y)-f(x)}{y-x} \mid x+\alpha < y < x+\beta\} > r$, then x is contained in one of countably many sets which are preimages of open intervals under f, intersected with open sets that guarantee the sup is large so long as $f(x)$ is in the given range. $\square$

Proposition 65. *If $f : \mathbb{R} \to \mathbb{R}$ is any function, then $\overline{D}f : x \mapsto \limsup_{y\to x} \frac{f(y)-f(x)}{y-x}$ is measurable, as is the similarly defined $\underline{D}f$.*

Proof. Let $r \in \mathbb{R}$, and for any k, n let E_n^k be the union of all intervals $[a,b]$ such that $b - a < \frac{1}{k}$ and $\frac{f(b)-f(a)}{b-a} > r + \frac{1}{n}$. Then E_n^k is measurable since it is a union of closed intervals, and the set where $\overline{D}f > r$ is equal to $\cup_n \cap_k E_n^k$. $\square$

Example 4. If $E \subseteq [0,1]$ is null, then there is a nondecreasing absolutely continuous function f with $D_-f = D_+f = +\infty$ on E. To see this, for each n find an open set O_n containing E with measure at most $\frac{1}{2^n}$, let f_n be the integral of χ_{O_n}, and let $f = \sum_n f_n$.

Example 5. Let E be a perfect nowhere dense set of positive measure in $[0,1]$, and let f be the integral of $\chi_{[0,1]\setminus E}$. Then f is strictly increasing and absolutely continuous, but $f' = 0$ on a set of positive measure. Additionally, by summing countably many dilated copies of the Cantor function, we can make a strictly increasing continuous function which has derivative 0 almost everywhere.

$L^p(X,\mu)$

Definition 71. We say that a function is *null* if it vanishes outside of a set of measure 0.

Definition 72. For $p > 0$, the *p-norm* of a measurable function $f : X \to \mathbb{C}$ (possibly undefined or infinite on a set of measure zero) with respect to the measure μ is defined by $\|f\|_p = (\int_X |f|^p \, d\mu)^{1/p}$ for $p < \infty$, and $\|f\|_\infty = \inf\{C \geq 0 \mid |f(x)| \leq C \text{ a.e. } x \in X\}$. We let $\mathcal{L}^p(X,\mu)$ be the vector space of functions on X with $\|f\|_p < \infty$, and we let L^p be the quotient of $\mathcal{L}^p$ by the set of null functions.

Proposition 66. *If $f, g \in L^p$ then $f + g \in L^p$. If $0 < p \leq 1$ then $d_p(f,g) = \|f-g\|_p^p$ defines a metric on L^p.*

Proof. For $1 \leq p < \infty$, we have $|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p)$ by convexity of $|\cdot|^p$, while for $0 < p \leq 1$ we have $|f+g|^p \leq 0^p + (|f|+|g|)^p \leq |f|^p + |g|^p$ by concavity of $(\cdot)^p$ on $\mathbb{R}^+$. □

Proposition 67. *If $f \in L^\infty \cap L^q$ for some $q < \infty$, then $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.*

Lemma 16 (Young's Inequality)**.** *If $a, b, p, q \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, with equality when $a^p = b^q$.*

Theorem 61 (Hölder)**.** *If $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p, g \in L^q$, then $\|fg\|_1 \leq \|f\|_p \|g\|_q$.*

Conversely, if $p < \infty$ and $f \in L^p$, then $\|f\|_p = \max\{|\int_X fg \, d\mu| \text{ s.t. } \|g\|_q \leq 1\}$, and the same holds for $p = \infty$ with the max *replaced by a* sup *if every set of infinite measure contains a subset of finite nonzero measure.*

Proof. We may assume without loss of generality that $\|f\|_p = \|g\|_q = 1$. Then $\int |fg| \, d\mu \leq \int \frac{|f|^p}{p} + \frac{|g|^q}{q} \, d\mu = 1$. Without the assumption that $\|f\|_p = \|g\|_q = 1$, the argument goes as follows:

$$\int |fg| = \|f\|_p \|g\|_q \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \leq \|f\|_p \|g\|_q \int \frac{|f|^p}{p\|f\|_p^p} + \frac{|g|^q}{q\|g\|_q^q} = \|f\|_p \|g\|_q.$$

For the converse, take $g = \frac{|f|^p}{f\|f\|_p^{p-1}}$. □

Corollary 15. *If μ is σ-finite, then for $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have $f \in L^p$ iff there exists some M such that $|\int fg| \leq M\|g\|_q$ for all simple functions g.*

Proof. Approximate $|f|$ from below by simple functions in L^p, and apply the converse to Hölder 61. □

Theorem 62 (Minkowski)**.** *If $p \geq 1$, then $\|f+g\|_p \leq \|f\|_p + \|g\|_p$, and if $1 < p < \infty$ we have equality iff $f = \lambda g$ with $\lambda \geq 0$ or $g = 0$ (a.e.).*

Proof. By Hölder 61, for any $h \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have $\int |f+g| h \leq \|f\|_p \|h\|_q + \|g\|_p \|h\|_q$, and taking $h = |f+g|^{p-1}$ gives the result (this h has $\int |f+g| h = \|f+g\|_p \|h\|_q$).

For the equality case, note that by the equality case of Young's inequality in the proof of Hölder, we must have $\frac{|f|^p}{\|f\|_p^p} = \frac{|f+g|^{(p-1)q}}{\|f+g\|_p^{(p-1)q}} = \frac{|g|^p}{\|g\|_p^p}$ (a.e.). □

Theorem 63 (Riesz-Fischer for L^p)**.** *L^p is complete with respect to the p-norm for $0 < p \le \infty$.*

Proof. It's enough to show that if $\sum_i \|u_i\|_p < \infty$ (or $\sum_i \|u_i\|_p^p < \infty$ in the case $0 < p < 1$) then $\sum_i u_i$ is the L^p-limit of its partial sums. This follows from the monotone convergence theorem (to show that $\sum_i |u_i|$ is in L^p), followed by the dominated convergence theorem to show that the tail sums converge to 0 in the p-norm. □

Corollary 16. *If f_k converge in L^p to f, then there is a subsequence f_{k_i} that converge pointwise a.e. to f.*

Proof. Choose any subsequence such that $\sum_i \|f_{k_{i+1}} - f_{k_i}\|_p < \infty$. □

Proposition 68. *The integrable simple functions are dense in L^p for every $0 < p < \infty$, and the simple functions are dense in L^∞.*

Theorem 64 (Riesz Representation for L^p)**.** *The natural map $L^p \to L^{q*}$ is an isometric isomorphism if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If μ is σ-finite, then so is the map $L^\infty \to L^{1*}$.*

Proof. By Hölder 61, we just need to check that $L^p \to L^{q*}$ is surjective. Let I be a bounded linear functional on L^{q*}, we just need to construct an $f \in L^p$ such that $I(\chi_E) = \int f\chi_E$ for all sets E with $\mu(E) < \infty$.

If μ is σ-finite, with the full space written as a disjoint union $\bigcup_i X_i$ with $\mu(X_i) < \infty$, then $\nu : E \mapsto \sum_i I(\chi_{E\cap X_i})$ defines a measure, and since $\mu(E) = 0 \implies \|\chi_E\|_q = 0 \implies I(\chi_E) = 0$, we have $\nu \ll \mu$. Then taking f to be the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$, we get $I(g) = \int fg$ for all integrable simple functions g, and since I is a bounded functional on L^q we see from the corollary to Hölder 61 that $f \in L^p$.

For the general case, use the previous case to define functions $f_E \in L^p$ supported on E for every σ-finite set E, such that $I(g) = \int f_E g$ for $g \in L^q$ supported on E. For any $E \subseteq E'$ σ-finite, by uniqueness we have $f_E = f_{E'}|_E$ a.e., and $\|f_E\|_p \le \|f_{E'}\|_p \le \|I\|$. Choose a sequence E_i of σ-finite sets with $\lim_i \|f_{E_i}\|_p = \sup_E \|f_E\|_p$, and let $X = \bigcup_i E_i$. Then X is σ-finite and $\|f_X\|_p = \sup_E \|f_E\|_p$, so for any σ-finite E we have f_E supported on $E \cap X$ up to a set of measure 0. For any $g \in L^q$, the support of g is a σ-finite set E, so $I(g) = \int f_E g = \int f_{E\cap X} g = \int f_X g$. Thus we may take $f = f_X$. □

Proposition 69. *If $0 < p \le q \le \infty$ and $\mu(X) < \infty$ then $\|f\|_p \le \mu(X)^{\frac{1}{p}-\frac{1}{q}}\|f\|_q$.*

Proof. By raising both sides to the pth power and replacing f with $|f|^p$ and q with $\frac{q}{p}$, we see that it's enough to prove $\|f\|_1 \le \mu(X)^{1-\frac{1}{q}}\|f\|_q$ for $1 \le q \le \infty$. This follows from Hölder 61 applied to the functions 1 and f. □

Proposition 70. *If $f \in L^p, g \in L^q$, $\alpha \in [0,1]$, and $\frac{1}{r} = \alpha\frac{1}{p} + (1-\alpha)\frac{1}{q}$, then $\||f|^\alpha |g|^{1-\alpha}\|_r \le \|f\|_p^\alpha \|g\|_q^{1-\alpha}$. In particular, if $f \in L^p \cap L^q$, then $f \in L^r$ for all $r \in [p,q]$.*

Proof. Apply Hölder 61 to $|f|^{\alpha r} \in L^{p/\alpha r}$ and $|g|^{(1-\alpha)r} \in L^{q/(1-\alpha)r}$. □

Proposition 71. *If X is metrizable and σ-finite and Σ is the Borel σ-algebra, then $C(X) \cap L^p$ is dense in L^p.*

Proposition 72. *If μ is a Radon measure on a locally compact Hausdorff space, then continuous functions with compact support are dense in L^p for $0 < p < \infty$.*

Proof. It's enough to approximate χ_E for every Borel set E with $\mu(E) < \infty$. For such E and for any $\epsilon > 0$, there is an open set U and a compact set K with $K \subseteq E \subseteq U$ with $\mu(U \setminus K) < \epsilon$. By locally compact Urysohn 4, there is a continuous function f taking values in $[0,1]$ which is supported on a compact subset of U, with $f|_K = 1$. □

Corollary 17. *Integrable step functions are dense in $L^p(\mathbb{R}^n)$ for $0 < p < \infty$.*

2.1.6 Banach spaces

Definition 73. If V is a vector space (over $\mathbb{R}$ or $\mathbb{C}$), then $p : V \to [0,\infty)$ is a *seminorm* if $p(0) = 0$, $p(cv) = |c|p(v)$ for c a scalar and $v \in V$, and $p(v+w) \le p(v) + p(w)$ for $v, w \in V$. p is a *norm* if additionally $p(v) = 0 \iff v = 0$.

The next bit is stolen from this blogoverflow post.

Lemma 17 (Zabreiko's Lemma). *If X is a Banach space and $p : X \to [0,\infty)$ is a seminorm such that for all absolutely convergent series $\sum_{n=1}^{\infty} x_n$ in X we have $p(\sum_n x_n) \le \sum_n p(x_n)$, then p is continuous, that is, $p(x) \ll \|x\|$.*

Proof. Let $A_n = p^{-1}([0,n])$, then since $X = \cup_n \overline{A_n}$, there is some n such that $\overline{A_n}$ has nonempty interior by the Baire category theorem. Since $\overline{A_n}$ is convex and symmetric, some open ball $B_R(0)$ around 0 is contained in $\overline{A_n}$. We claim that $B_R(0) \subseteq A_n$ as well: if $\|x\| < R$, pick $0 < q < 1$ such that $\frac{\|x\|}{1-q} < R$, set $y = \frac{R}{\|x\|}x$, then since $y \in \overline{A_n}$ there exists $y_0 \in A_n$ with $\|y - y_0\| < qR$, and then inductively we find $y_0, y_1, ... \in A_n$ such that for each k, we have $\|y - \sum_{i<k} y_i\| < q^k R$: y_k is taken to be a point in A_n with $\|q^{-k}(y - \sum_{i<k} y_i) - y_k\| < qR$. Since $\|y_k\| < R + qR$ for each k, the sum $\sum_k q^k y_k = y$ is absolutely convergent, so by hypothesis $p(y) \le \sum_k q^k p(y_k) \le \frac{n}{1-q}$, so $p(x) \le \frac{\|x\|}{R}\frac{n}{1-q} < n$, so $x \in A_n$. □

Theorem 65 (Open Mapping Theorem). *If X, Y Banach spaces, $A : X \to Y$ surjective and continuous, then A takes open sets to open sets.*

Proof. For $y \in Y$, set $p(y) = \inf\{\|x\| \mid Ax = y\}$ in Zabreiko's Lemma. □

Theorem 66 (Bounded Inverse Theorem). *If X, Y Banach spaces, $A : X \to Y$ bijective and continuous, then A^{-1} is also bounded.*

Theorem 67 (Closed Graph Theorem). *If X, Y Banach spaces, then $A : X \to Y$ is bounded iff the graph is closed in $X \times Y$.*

Proof. For $x \in X$, set $p(x) = \|Ax\|$ in Zabreiko's Lemma. □

Theorem 68 (Uniform Boundedness Theorem/Banach Steinhaus). *If X is Banach, Y a normed vector space, F a set of continuous linear functions $T : X \to Y$. If $\forall x \in X$ $\sup_{T\in F} \|T(x)\| < \infty$, then $\sup_{T\in F} \|T\| < \infty$.*

Proof. Set $p(x) \in \sup_{T\in F} \|T(x)\|$ in Zabreiko's Lemma. □

Corollary 18. *If a sequence of bounded operators from a Banach space to a normed space converges pointwise, then the pointwise limit is a bounded operator.*

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