Notes

Contents

Part I

Basics

Chapter 1

Cross Ratios

1.1 Projective Geometry and Cross ratios

Definition 1.1.1. The *projective plane* $\mathbb{P}^2$ is the set of lines through an observation point O in three dimensional space. A *projective line* l is a plane passing through O, and a *projective point* P is a line passing through O. If the line defining P is contained in the plane defining l, we say that $P \in l$.

If $\mathbb{A}^2$ is an ordinary plane which does not pass through O, then we can identify most projective points of $\mathbb{P}^2$ with ordinary points on $\mathbb{A}^2$ by taking the intersection of the line defining the projective point with $\mathbb{A}^2$. The projective line which is defined by a plane passing through O and parallel to $\mathbb{A}^2$ is called the *line at infinity*, or the *horizon line*. Projective points contained in the line at infinity are called *infinite points*.

If we take $O = (0, 0, 0)$, then we can put coordinates on the projective plane as follows. Every projective point P is a line through O and some other point (p, q, r). Then every point on the line defining P is of the form $(\lambda p, \lambda q, \lambda r)$ for some λ. We write $P = [p : q : r]$, where the colons indicate that we only care about the ratios of the coordinates. If $\mathbb{A}^2$ is the plane $z = 1$, then the ordinary point on $\mathbb{A}^2$ corresponding to P is $(\frac{p}{r}, \frac{q}{r}, 1)$, or if we ignore the z-coordinate it is just $(\frac{p}{r}, \frac{q}{r})$. If $r = 0$, then P is an infinite point with *slope* $\frac{q}{p}$.

We can define projective coordinates for projective lines as well. A projective line l is defined by a single linear equation

$$dx + ey + fz = 0,$$

with not all of d, e, f equal to 0. Furthermore, this equation defines the same line if all of d, e, f are rescaled by the same nonzero λ. Thus we say that $l = (d : e : f)$. If $P = [p : q : r]$, then we have $P \in l$ if and only if

$$dp + eq + fr = 0.$$

The intersection of l with the ordinary plane $\mathbb{A}^2$ defined by $z = 1$ is just the line $dx + ey + f = 0$. The line at infinity has coordinates $(0 : 0 : 1)$.

The coordinate system described above can be called *cartesian projective coordinates*. There are other projective coordinate systems, one of the most useful of which is the *barycentric coordinate* system. In the barycentric coordinate system, a triangle ABC in $\mathbb{A}^2$ is fixed and the coordinates of three dimensional space are chosen such that $A = (1, 0, 0), B = (0, 1, 0), C = (0, 0, 1)$ - so the plane $\mathbb{A}^2$ is now defined by the equation $x + y + z = 1$. If P is an ordinary point in $\mathbb{A}^2$, then the projective coordinates $[p : q : r]$ of P are defined to be any three numbers p, q, r, not all zero, proportional to the three directed areas $[PBC], [APC], [ABP]$. In the barycentric coordinate system, a line $l = (d : e : f)$ is the set of points P such that

$$d[PBC] + e[APC] + f[ABP] = 0.$$

The line at infinity has barycentric coordinates $(1 : 1 : 1)$.

1.1.1 Cross Ratios

First we recall the definition of the ratio.

Definition 1.1.2. If A, B, C are three points on a line, not all equal, then we define their *ratio* to be

$$(A, B; C) = \frac{AC}{BC},$$

where the ratio is taken to be positive if the rays AC and BC point in the same direction, and negative otherwise. If l_1, l_2, l_3 are three directed lines passing through a point, not all equal, then their ratio is defined by

$$(l_1, l_2; l_3) = \frac{\sin \angle l_1 l_3}{\sin \angle l_2 l_3},$$

where the angles are oriented in the counterclockwise sense.

Exercise 1.1.1. (a) Show that if $A \neq B$ then there is a bijection between points C on the line AB and ratios $(A, B; C)$. Thus we can use the ratio as a coordinate on the line AB.

(b) Show that the ratio $(l_1, l_2; l_3)$ does not depend on the orientation of line l_3. Show that if $l_1 \neq l_2$ we can use the ratio $(l_1, l_2; l_3)$ as a coordinate on the set of lines through the point $l_1 \cap l_2$.

Exercise 1.1.2. Suppose that points A, B, C, not all equal, are on a line, and that point P is not on that line. Show that

$$\frac{(A, B; C)}{(PA, PB; PC)} = \frac{|PA|}{|PB|}.$$

Definition 1.1.3. If A, B, C, D are four points on a line, no three of them equal, then we define their *cross ratio* to be

$$(A, B; C, D) = \frac{(A, B; C)}{(A, B; D)} = \frac{AC}{CB} \Big/ \frac{AD}{DB}.$$

If l_1, l_2, l_3, l_4 are four lines passing through a point, no three of them equal, then their cross ratio is defined by picking an orientation for each line, and then setting

$$(l_1, l_2; l_3, l_4) = \frac{(l_1, l_2; l_3)}{(l_1, l_2; l_4)} = \frac{\sin \angle l_1 l_3}{\sin \angle l_3 l_2} \Big/ \frac{\sin \angle l_1 l_4}{\sin \angle l_4 l_2}.$$

Theorem 1.1.4 (The fundamental theorem of cross ratios)**.** *If A, B, C, D are on a line, no three of them equal, and if E is a point not on that line, then*

$$(EA, EB; EC, ED) = (A, B; C, D).$$

We would like to extend the above definitions to any four points or lines in the projective plane. One way to do this is to make special definitions if one of A, B, C, D is an infinite point: for instance, if ∞ is the infinite point on line AB, then we have

$$(A, B; C, \infty) = (A, B; C) = -\frac{AC}{CB}.$$

Similarly, if all of A, B, C, D are infinite points with slopes a, b, c, d, then their cross ratio is

$$(a, b; c, d) = \frac{c-a}{b-c}\bigg/\frac{d-a}{b-d}.$$

However, the best way to do this is to simply change perspectives to get a coordinate system where none of A, B, C, D is an infinite point. In other words, we find a new plane $\mathbb{A}'^2$ not passing through the observation point O, which intersects the four lines corresponding to the projective points OA, OB, OC, OD at some new points A', B', C', D'. Then for finite points A, B, C, D we have

$$(A, B; C, D) = (OA, OB; OC, OD) = (A', B'; C', D'),$$

so the cross ratio in the new coordinate system will be the same as the original cross ratio. If one of A, B, C, D is an infinite point we use this formula as the *definition* of the cross ratio.

To check your understanding, calculate the cross ratio of four parallel lines in terms of the distances between them (parallel lines intersect at the infinite point corresponding to their common slope).

Exercise 1.1.3. Let ABC be a triangle, let M be the midpoint of AC, and let N be a point on line BM such that AN is parallel to BC. Let P be any point on line AC, and let Q be the intersection of line BP with line AN. Use cross ratios to prove that

$$\frac{AQ}{QN} = \frac{1}{2}\frac{AP}{PM}.$$

Exercise 1.1.4. Suppose a painter is painting a square-tiled floor which he is looking at from an angle. Given that the painter draws the four corners of one of the squares at the four points $ABCD$, construct the rest of the points that the painter draws using only a straightedge. If the next two points that the painter draws along the line AB are X and Y, compute the value of the cross ratio $(A, B; X, Y)$.

Exercise 1.1.5. (a) Check that for any number λ we have $(\lambda, 1; 0, \infty) = \lambda$.

(b) Show that $(A, B; D, C) = \frac{1}{(A,B;C,D)}$.

(c) Show that $(A, C; D, B) = \frac{1}{1-(A,B;C,D)}$.

Exercise 1.1.6. (a) Show that if $A \neq B$ and $(A, B; C, X) = (A, B; C, Y)$ then $X = Y$.

(b) Show that if $(A, B; C, D) = 1$ then either $A = B$ or $C = D$.

(c) Show that if $A \neq B$, $C \neq D$, and $(A, B; C, D) = (A, B; D, C)$ then $(A, B; C, D) = -1$.

Definition 1.1.5. If $(A, B; C, D) = -1$, then the four points A, B, C, D are called *harmonic.* We also say that D is the *harmonic conjugate* of C with respect to A, B. Sometimes we say that A, B, C, D are harmonic when three of them are equal.

Example 1.1.1. (i) If M is the midpoint of AB and if ∞ is the infinite point along line AB, then $(A, B; M, \infty) = -1$.

(ii) If ABC is a triangle, and if X, Y are the feet of the internal and external angle bisectors through C, then $(A, B; X, Y) = -1$ by the angle bisector theorem.

(iii) We have $(1, -1; x, \frac{1}{x}) = -1$ and $(0, \infty; x, -x) = -1$ for any x.

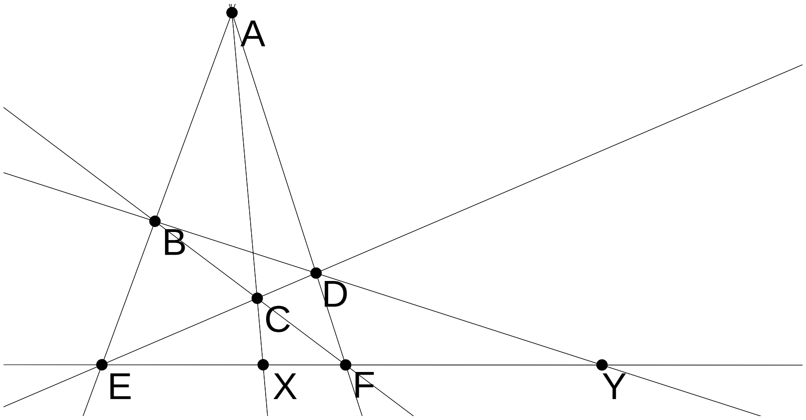

Figure 1.1: Quadrilateral Theorem

Theorem 1.1.6 (Quadrilateral Theorem)**.** *Let $ABCD$ be any quadrilateral. Let E be the intersection of sides AB and CD, and let F be the intersection of sides BC and DA. Let X be the intersection of diagonal AC with the line EF, and let Y be the intersection of diagonal BD with line EF. Then*

$$(E, F; X, Y) = -1.$$

Proof 1, using Ceva and Menelaus. By Ceva applied to triangle AEF and point C, we have

$$\frac{AB}{BE}\frac{EX}{XF}\frac{FD}{DA} = 1.$$

By Menelaus applied to triangle AEF and line BD, we have

$$\frac{AB}{BE}\frac{EY}{YF}\frac{FD}{DA} = -1.$$

Dividing these two equations, we get $(E, F; X, Y) = -1$. □

Proof 2, using cross ratios. Let P be the intersection of the diagonals AC and BD. We have

$$(E, F; X, Y) = (AE, AF; AX, AY) = (B, D; P, Y) = (CB, CD; CP, CY) = (F, E; X, Y).$$

Since $E \neq F$ and $X \neq Y$, we conclude that $(E, F; X, Y) = -1$. □

If EA, EB, EC, ED intersect a line l at points A', B', C', D', it often saves space to abbreviate the inference

$$(A, B; C, D) = (EA, EB; EC, ED) = (A', B'; C', D')$$

by just writing

$$(A, B; C, D) \overset{E}{=} (A', B'; C', D').$$

Now let's use this notation to give a compact proof of Desargues' Theorem:

Theorem 1.1.7 (Desargues' Theorem)**.** *Suppose that triangles ABC and XYZ are* perspective from a point*, that is, suppose that the lines AX, BY, CZ all meet at a point P. Then the triangles ABC and XYZ are* perspective from a line*, that is, the intersections $AB \cap XY$, $BC \cap YZ$, $CA \cap ZX$ all lie on a line.*

Proof. Let $U = BC \cap YZ$, $V = CA \cap ZX$, $W = AB \cap XY$. We want to show that U, V, W lie on a line, so we may as well suppose that $V \neq W$. Let Q, M, N be the intersections of line BY with the lines WV, AC, XZ, respectively. Then we have

$$(W, V; Q, BC \cap VW) \overset{B}{=} (A, V; M, C) \overset{P}{=} (X, V; N, Z) \overset{Y}{=} (W, V; Q, YZ \cap VW).$$

Thus $BC \cap VW = YZ \cap VW$, so the three lines BC, YZ, VW meet at the point U. □

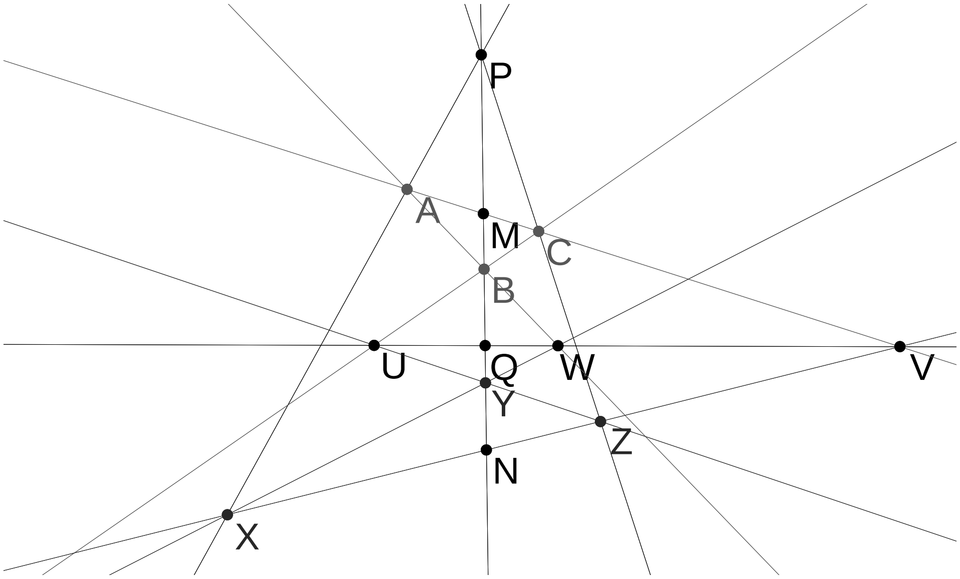

Figure 1.2: Desargues' Theorem

Exercise 1.1.7 (Pappus's Hexagon Theorem). Let A, B, C be on a line, and let D, E, F be on another line. Let $X = AE \cap BD, Y = BF \cap CE, Z = CD \cap AF$. Use cross ratios to show that X, Y, Z are on a line. (Hint: let $P = CD \cap BF$, and show that $(C, D; P, Z) = (C, D; P, CD \cap XY)$.)

Theorem 1.1.8 (Cross Ratio Equality)**.** *Let A, B, C, D be on a line, and let E, F, G, H be on another line. Let $X = AF \cap BE, Y = BG \cap CF, Z = CH \cap DG$. Then X, Y, Z are on a line if and only if $(A, B; C, D) = (E, F; G, H)$.*

Proof. Let $P = AG \cap CE, Q = CG \cap XY$. By Pappus's Theorem, P is on line XY. Projecting through G, we have $(A, B; C, D) \overset{G}{=} (P, Y; Q, DG \cap XY)$, and projecting through C, we have $(E, F; G, H) \overset{C}{=} (P, Y; Q, CH \cap XY)$. Thus $(A, B; C, D) = (E, F; G, H)$ if and only if CH, DG, and XY meet at a point. □

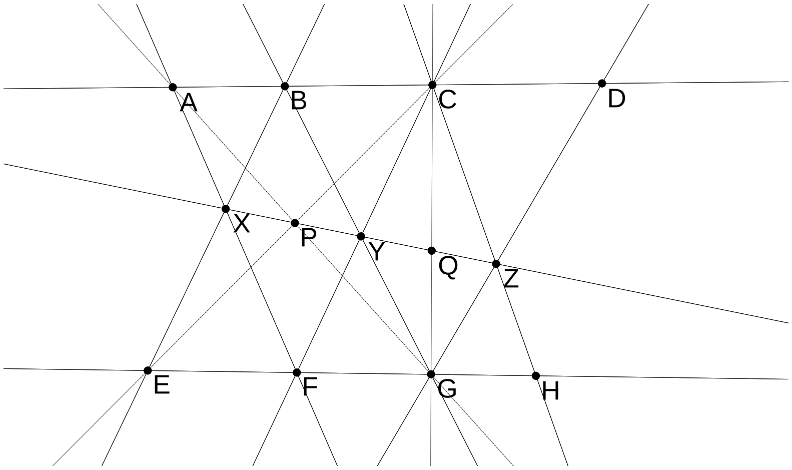

Figure 1.3: Equal cross ratios

1.1.2 Cross Ratios on a Conic Section

Proposition 1.1.9. *Suppose that A, C, B, D are on circle ω, and that the (directed) arcs AC, CB, BD, DA of ω have central angles $2\alpha, 2\beta, 2\gamma, 2\delta$. Let E be any other point on ω. Then*

$$(EA, EB; EC, ED) = -\frac{\sin\alpha}{\sin\beta}\Big/\frac{\sin\delta}{\sin\gamma}.$$

In particular, we have

$$(EA, EB; EC, ED) = \pm\frac{|AC||BD|}{|AD||BC|},$$

where the sign is negative if and only if the points A, B separate the points C, D.

Corollary 1.1.10. *Let ω be any* conic section*, that is, any intersection of a cone $\mathcal{C}$ through the observation point O with the plane $\mathbb{A}^2$. If A, B, C, D, E, F are any six points on ω, then we have*

$$(EA, EB; EC, ED) = (FA, FB; FC, FD).$$

Proof. First we prove it when ω is a circle. By Proposition 2, we have

$$(EA, EB; EC, ED) = -\frac{\sin\alpha}{\sin\beta}\Big/\frac{\sin\delta}{\sin\gamma} = (FA, FB; FC, FD).$$

For the general case, we choose another plane $\mathbb{A}'^2$ such that $\mathcal{C} \cap \mathbb{A}'^2$ is a circle. Let $A', B', \ldots$ be the intersections of lines $OA, OB, \ldots$ with the plane $\mathbb{A}'^2$. Then we have

$$(EA, EB; EC, ED) \overset{O}{=} (E'A', E'B'; E'C', E'D') = (F'A', F'B'; F'C', F'D') \overset{O}{=} (FA, FB; FC, FD).$$

□

Definition 1.1.11. If A, B, C, D are four points on a conic section ω, then we define the cross ratio of A, B, C, D with respect to ω by choosing any fifth point E on ω and setting

$$(A, B; C, D)_\omega = (EA, EB; EC, ED).$$

By Corollary 2, this doesn't depend on the choice of E.

Exercise 1.1.8. Check that the cross-ratio formula $(A, B; C, D)_\omega = 1 - (A, C; B, D)_\omega$ is equivalent to Ptolemy's theorem when ω is a circle.

Remark 1.1.1. The concept of *separation* can be defined on lines and conics as follows: we say that the points A, B separate the points C, D on ω if deleting the points A, B from ω cuts ω into two disconnected components, one of which contains C and the other of which contains D. To make sense of this definition on a hyperbola, parabola, or line, it is necessary to include the points at infinity in the conic ω. Then we have

$$(A, B; C, D)_\omega < 0$$

exactly when the points A, B separate the points C, D on ω.

Separation is the fundamental ordering-like concept which is appropriate when we study real projective geometry. For Euclidean geometry, the analogous concept is *betweenness*: if A, B, C lie on a line ℓ, then we say that C is between A and B when deleting C from ℓ cuts ℓ into two disconnected components (ignoring the point at infinity), one of which contains A and the other of which contains B. Betweenness is a special case of separation: C is between A, B on the line ℓ exactly when the points A, B separate the points C, ∞_ℓ along ℓ. There turn out to be exactly four fundamental ordering-like concepts on lines and circles:

- order, for two points on a directed line,
- betweenness, for three points on an undirected line,
- cyclic order, for three points on an oriented circle, and
- separation, for four points on an unoriented circle.

Each of these concepts has an elegant axiomatic system which goes along with it. Facts about betweenness are often used without explicit mention in Euclidean geometry, and were left out of Euclid's five axioms for geometry but included in Hilbert's more careful list of axioms for geometry. In two-dimensional Euclidean geometry, the main nontrivial fact about betweenness is called *Pasch's axiom*, which states that if a line meets one side of a triangle internally, then it meets one of the other two sides of the triangle internally.

Our first application of the cross ratio on a conic is to give a short proof of Pascal's theorem.

Theorem 1.1.12 (Pascal's Theorem)**.** *If $ABCDEF$ is any hexagon with vertices lying on a conic ω, then the three intersections of opposite sides $AB \cap DE$, $BC \cap EF$, $CD \cap FA$ lie on a line.*

Proof. Let $L = BC \cap EF$, $M = CD \cap FA$, $N = AB \cap DE$ be the intersections of opposite sides of the hexagon. Let $P = AF \cap BC$ and $Q = AB \cap CD$. Then

$$(C, L; P, B) \overset{F}{=} (C, E; A, B)_\omega \overset{D}{=} (Q, N; A, B) \overset{M}{=} (C, MN \cap BC; P, B).$$

Thus $L = MN \cap BC$, so L is on the line MN. □

Exercise 1.1.9.

(a) Given points A, B, C, D, E and a line l through A construct, using only a straightedge, the second point of intersection F between the line l and the conic through the points A, B, C, D, E.

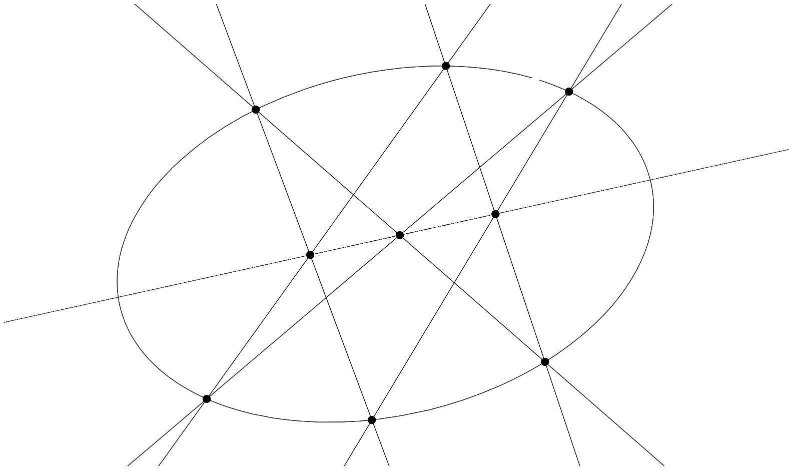

Figure 1.4: Pascal's Theorem

(b) Given points A, B, C, D, E construct, using only a straightedge, the line l which is tangent to the conic through the points A, B, C, D, E at A.

Exercise 1.1.10. Suppose points A, B, C, D, E, F, G, H lie on a conic ω. Let $X = AF \cap BE, Y = BG \cap CF, Z = CH \cap DG$. Show that $(A, B; C, D)_\omega = (E, F; G, H)_\omega$ if and only if X, Y, Z are on a line.

Another easy application is a short proof of the butterfly theorem.

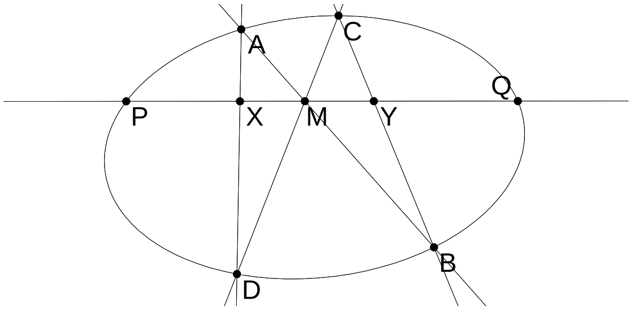

Figure 1.5: The Projective Butterfly Theorem

Theorem 1.1.13 (Projective Butterfly Theorem)**.** *Let ω be a conic, and let PQ be a chord on ω through the point M. Let AB and CD be two more chords of ω passing through M, and set $X = AD \cap PQ, Y = BC \cap PQ$. Then $(P, Q; M, X) = (Q, P; M, Y)$. In particular, if M is the midpoint of PQ then $|MX| = |MY|$.*

Proof.

$$(P, Q; M, X) \overset{A}{=} (P, Q; B, D)_\omega \overset{C}{=} (P, Q; Y, M) = (Q, P; M, Y).$$

We leave the proof of the last claim as an easy exercise to the reader. □

Definition 1.1.14. A cyclic quadrilateral $ACBD$ is called *harmonic* if $A \neq B, C \neq D$, and $|AC||BD| = |AD||BC|$.

Exercise 1.1.11. (a) Suppose P is a point outside circle ω. Let the two tangents from P to ω meet ω at A and B. Let l be a line through P meeting ω at two points C and D. Show that $ACBD$ is a harmonic quadrilateral.

(b) Let P, ω, A, B, C, D be as in (a), and let Q be the intersection of AB and CD. Show that $(C, D; P, Q) = -1$.

(c) Let P, ω, A, B be as in (a). Show that P', the inverse P with respect to ω, is on the line AB.

Exercise 1.1.12. Let ω be the unit circle, given in affine coordinates by the equation $x^2 + y^2 = 1$. Let $A = (1, 0), B = (0, 1), C = (-1, 0)$ in affine coordinates. Find the affine coordinates of the point D on ω such that $ACBD$ is a harmonic quadrilateral.

Exercise 1.1.13. Let $ABCDE$ be a regular pentagon inscribed in a circle ω. Compute $(A, B; C, D)_\omega$.

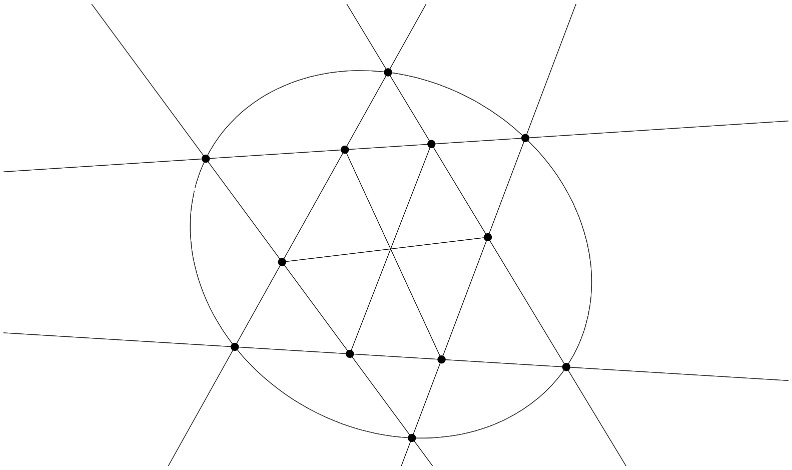

Figure 1.6: Exercise 1.1.14

Exercise 1.1.14. Let A, B, C, D, E, F be six distinct points in the plane. Let $U = BC \cap DE, V = CA \cap EF, W = AB \cap FD, X = AB \cap EF, Y = BC \cap FD, Z = CA \cap DE$, so that hexagon $UZVXWY$ is the intersection of triangles ABC and DEF if it is convex. Show that the lines UX, VY, WZ meet in a point if and only if the points A, B, C, D, E, F lie on a conic.

1.1.3 Cross Ratios on the Inversive Plane

Just as we used three projective coordinates for the projective plane, we use two projective coordinates to describe a projective line. Specifically, the projective point $[s : t]$ will correspond to the ordinary point with coordinate $z = \frac{s}{t}$ if $t \neq 0$, and to the point ∞ if $t = 0$. When we allow s, t to be *complex* numbers, we get what is sometimes called the *complex projective line* $\mathbb{CP}^1$, the *inversive plane*, or the *Riemann sphere*.

We define cross ratios on the inversive plane the same way we define cross ratios on a line:

$$(a, b; c, d) = \frac{c-a}{b-c} \bigg/ \frac{d-a}{b-d},$$

where now a, b, c, d are complex numbers corresponding to ordinary points A, B, C, D in the inversive plane.

Proposition 1.1.15. *The points A, B, C, D corresponding to the complex numbers a, b, c, d are on a circle or a line if and only if $(a, b; c, d)$ is a real number. If they are on a line, we have $(a, b; c, d) = (A, B; C, D)$, and if they are on a circle ω, we have $(a, b; c, d) = (A, B; C, D)_\omega$.*

Proof. Left as an exercise. □

Inversion around the unit circle is given by the simple formula $z \mapsto \frac{1}{\bar{z}}$ in the inversive plane. We have

$$\left(\frac{1}{\bar{a}}, \frac{1}{\bar{b}}; \frac{1}{\bar{c}}, \frac{1}{\bar{d}}\right) = \overline{\frac{\frac{1}{c} - \frac{1}{a}}{\frac{1}{b} - \frac{1}{c}} \Big/ \frac{\frac{1}{d} - \frac{1}{a}}{\frac{1}{b} - \frac{1}{d}}} = \overline{\frac{a-c}{c-b} \Big/ \frac{a-d}{d-b}} = \overline{(a, b; c, d)},$$

so inversion takes cross ratios to their complex conjugates. As a consequence, we see that inversion takes circles and lines to circles and lines, and furthermore it takes harmonic quadrilaterals to harmonic quadrilaterals.

Definition 1.1.16. To every two by two matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant $ad - bc$ not equal to zero, we associate a transformation f_M of the inversive plane as follows. In projective coordinates $[s : t]$, we write

$$f_M([s : t]) = [as + bt : cs + dt].$$

In ordinary coordinates $z = \frac{s}{t}$, we write

$$f_M(z) = \frac{az + b}{cz + d}.$$

The maps f_M are called *Möbius Transformations.*

Exercise 1.1.15. Show that for any two by two matrix M with nonzero determinant, and for any four points a, b, c, d on the inversive plane, we have

$$(f_M(a), f_M(b); f_M(c), f_M(d)) = (a, b; c, d).$$

Exercise 1.1.16. Check that composition of Möbius transformations corresponds to matrix multiplication, i.e. that for any two matrices M, N and any point $[s : t]$ we have

$$f_M(f_N([s : t])) = f_{MN}([s : t]).$$

Exercise 1.1.17. Let A, B, C, X, Y, Z be six points on the projective line, no two of A, B, C equal and no two of X, Y, Z equal. Prove that there is a Möbius transformation f such that $f(A) = X, f(B) = Y, f(C) = Z$.

1.1.4 Invertible functions on the line

Suppose a projective line $\mathbb{P}^1$ has coordinate z, and we have defined an invertible map $f : \mathbb{P}^1 \to \mathbb{P}^1$ via some geometric procedure that has no "configuration issues" (so for instance, taking the *leftmost* intersection of a circle with a line would not count). Since any geometrically defined map can be described algebraically by writing every point out in coordinates, our function f may be written as an algebraic function of z, and if there are no "configuration issues", then f must be a rational function, i.e. a ratio of two polynomials:

$$f(z) = \frac{p(z)}{q(z)}.$$

Since f is invertible, the equation $f(z) = w$ should have exactly one solution, so the polynomial

$$p(z) - wq(z)$$

should have degree 1 for every constant w. Thus p and q are both linear polynomials, and we can write

$$f(z) = \frac{az + b}{cz + d}.$$

Thus, f is in fact a Möbius transformation, and so f preserves the cross ratio. We record this as an informal theorem.

Theorem 1.1.17. *If f is an invertible function from a line to a line that is defined by a geometric procedure that has no "configuration issues", then f preserves the cross ratio. Furthermore, in this case f is a Möbius transformation.*

Exercise 1.1.18. Prove the converse: if $f : \mathbb{P}^1 \to \mathbb{P}^1$ is any function that preserves the cross ratio, prove that f is a Möbius transformation, and find a geometric construction of the function f.

As an application, we consider the harmonic conjugation map. For any points A, B on $\mathbb{P}^1$, we define

$$h_{A,B}(C) = D \text{ if } (A, B; C, D) = -1.$$

We can construct D geometrically using the Quadrilateral Theorem, and $h_{A,B}$ is clearly invertible, so by the above discussion $h_{A,B}$ is a Möbius transformation. In coordinates, if A has coordinate a and B has coordinate b, we have

$$h_{a,b}(z) = \frac{(a + b)z - 2ab}{2z - a - b}.$$

Harmonic conjugation has the property that $h_{A,B}(h_{A,B}(C)) = C$ - in other words, harmonic conjugation is always an *involution*. In fact, this property characterizes harmonic conjugation.

Theorem 1.1.18. *If f is a Möbius transformation with the further property that f is an involution, i.e. $f(f(C)) = C$ for all points C, then f is either the identity map or there is a pair of (possibly imaginary) points A, B such that $f = h_{A,B}$.*

Proof. In coordinates, the equation $f(z) = z$ becomes a quadratic after clearing the denominator. If f is not the identity map, this quadratic will have two solutions, corresponding to two distinct points A, B. For any point C, write $D = f(C)$. Since f preserves the cross ratio, we have

$$(A, B; C, D) = (f(A), f(B); f(C), f(D)) = (A, B; D, C),$$

so the points A, B, C, D are harmonic. □

1.1.5 Angles and the circle points

Two special points in the projective plane allow us to talk about angles using cross ratios. These points are both infinite and imaginary, but we can treat them the same way we treat any other points in projective geometry. This allows us to solve many problems that are traditionally thought to be out of the scope of projective geometry.

Definition 1.1.19. The *circle points* are the points $\wp = [i : 1 : 0]$ and $\bar{\wp} = [1 : i : 0]$. These are the points at infinity of slope $-i$ and i.

Theorem 1.1.20 (Angle Theorem)**.** *If lines l, m intersect the line at infinity in points L, M, then*

$$(L, M; \wp, \bar{\wp}) = e^{2i\angle lm}.$$

In particular, lines l and m are orthogonal if and only if points $L, M, \wp, \bar{\wp}$ are harmonic.

Proof. Let s be the slope of line l and let t be the slope of line m. By the tangent subtraction formula, we have

$$\tan(\angle lm) = \frac{t-s}{1+st}.$$

We have

$$\begin{aligned}
(L, M; \wp, \bar{\wp}) &= (s, t; -i, i) \\
&= \frac{s+i}{-i-t}\Big/\frac{s-i}{i-t} \\
&= \frac{(s+i)^2(t-i)^2}{(s^2+1)(t^2+1)} \\
&= \frac{(st+1)^2 - (t-s)^2 + 2i(t-s)(st+1)}{(st+1)^2 + (t-s)^2} \\
&= \frac{1-\tan^2(\angle lm)}{1+\tan^2(\angle lm)} + i\frac{2\tan(\angle lm)}{1+\tan^2(\angle lm)} \\
&= \cos(2\angle lm) + i\sin(2\angle lm) \\
&= e^{2i\angle lm}.
\end{aligned}$$

□

Theorem 1.1.21. *A conic ω is a circle if and only if it passes through the two circle points.*

Proof. First, suppose ω is a circle with center (a, b) and radius r. In projective coordinates, ω is the set of points $[x : y : z]$ such that

$$(x - az)^2 + (y - bz)^2 = (rz)^2.$$

Plugging in, we can check that $[x : y : z] = [i : 1 : 0]$ and $[x : y : z] = [1 : i : 0]$ satisfy the equation defining ω.

Now suppose ω is any conic passing through $\wp, \bar{\wp}$. Let A, B, C, D be any four points on ω. Then we have

$$e^{2i\angle CAD} = (AC, AD; A\wp, A\bar{\wp}) = (C, D; \wp, \bar{\wp})_\omega = (BC, BD; B\wp, B\bar{\wp}) = e^{2i\angle CBD},$$

so the directed angles $\angle CAD$ and $\angle CBD$ are congruent modulo π. Thus A, B, C, D are concyclic.

□

Corollary 1.1.22. *Let A, B be two points on a circle ω with center O. Then*

$$(A, B; \wp, \bar{\wp})_\omega = e^{i\angle AOB}.$$

In particular, if A, B are diametrically opposite then $A, B, \wp, \bar{\wp}$ are harmonic.

Exercise 1.1.19. Say that four distinct points A, B, C, D on a line are *melodic* if we have

$$(A, B; C, D) = (A, D; B, C),$$

and make a similar definition for four points on a conic. Let $ABCDEF$ be a regular hexagon inscribed in a circle ω. Prove that the four points $A, B, \wp, \bar{\wp}$ are melodic with respect to ω.

1.1.6 Polar maps

Definition 1.1.23. We say that two points P, Q are *harmonic conjugates* with respect to a conic ω if P, Q, X, Y are harmonic, where X, Y are the (possibly imaginary) points of intersection of ω and PQ.

Theorem 1.1.24. *Let P be a point and ω a conic. Then the locus p of harmonic conjugates of P with respect to ω is a line.*

Proof 1, using tangents. Let U, V be the feet of the two tangents from P to ω. We will show that every point Q on the line UV is a harmonic conjugate of P with respect to ω. Let the line PQ meet ω at X, Y.

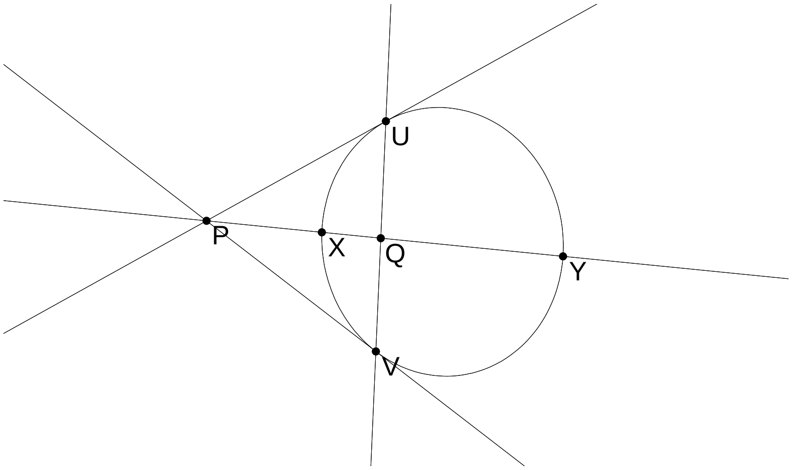

Figure 1.7: Proving P, Q are conjugate

Chasing cross ratios, we have

$$(P, Q; X, Y) \overset{U}{=} (U, V; X, Y)_\omega \overset{V}{=} (Q, P; X, Y),$$

so P, Q, X, Y are harmonic. □

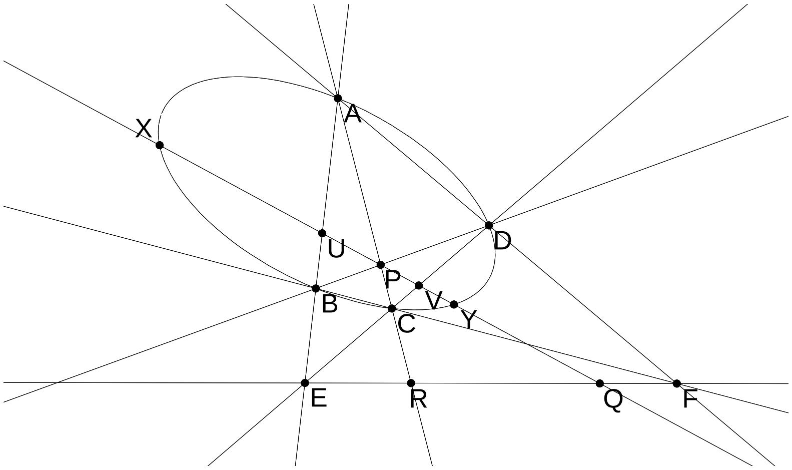

Figure 1.8: Proving P, Q are conjugate

Proof 2, using chords. Let AC and BD be any two chords of ω passing through P. Let $E = AB \cap CD, F = AD \cap BC$. We will show that every point Q on the line EF is a harmonic conjugate of P with respect to ω. Let $AP \cap EF = R$, let ω meet PQ at X, Y, and let $U = AB \cap PQ, V = CD \cap PQ$.

By the quadrilateral theorem applied to the quadrilateral $BEDF$, the points A, C, P, R are harmonic. Projecting through E, we see that the four points U, V, P, Q are harmonic. Furthermore, by the projective butterfly theorem we have

$$(X, Y; P, U) = (Y, X; P, V).$$

Now suppose that Q' is the harmonic conjugate of P with respect to X, Y. Then if $h_{PQ'}$ denotes harmonic conjugation with respect to P, Q' we have

$$(X, Y; P, U) = (h_{PQ'}(X), h_{PQ'}(Y); h_{PQ'}(P), h_{PQ'}(U)) = (Y, X; P, h_{PQ'}(U)),$$

so $h_{PQ'}(U) = V$. Thus U, V, P, Q' are harmonic, so in fact we have $Q = Q'$. □

Definition 1.1.25. If P is a point, ω a conic, and p is the locus of harmonic conjugates of P with respect to ω then we say that P is the *pole* of the line p, and p is the *polar* of the point P. When several conics are around, we will usually write ρ_ω for the *polar map* taking a point P to its polar p with respect to ω and taking a line p to its pole P with respect to ω.

Proposition 1.1.26. *Every line p has a unique pole P with respect to ω.*

Proof. Let Q, R be any two distinct points on p, and let their polars be q, r. Then q, r intersect in at least one point P. By definition, P is conjugate to Q and R with respect to ω, so the polar of P must be the line $QR = p$. Uniqueness is left as an exercise (consider the line joining two distinct poles of p). □

Proposition 1.1.27. *Let ω be a conic, let P, Q be points, and let p, q be their polars with respect to ω.*

(a) P is on q if and only if Q is on p.

(b) P is on p if and only if P is on ω, in which case p is tangent to ω.

(c) If X, Y are the feet of the tangent lines from P to ω, then $p = XY$.

Proof. The claims (a) and (b) are obvious from the definitions, while (c) follows easily from (a) and (b). □

Proposition 1.1.28. *Let ω be a conic. The either ω is a parabola or ω is centrally symmetric around a point O. If ω is a hyperbola, then O is the intersection of the asymptotes of ω.*

Proof. Let O be the pole of the line at infinity. If O is infinite, then ω must be tangent to the line at infinity at O, in which case ω is a parabola.

Now assume O is finite. Then for any chord X, Y through O, the points X, Y, O, and the point at infinity along XY are harmonic conjugates, so O is the midpoint of XY, i.e. ω is centrally symmetric around O. If ω is a hyperbola, then the asymptotes intersect at the pole of the line at infinity, which is O (this is still true if ω is an ellipse or a circle, but in that case the asymptotes have imaginary slopes). □

Theorem 1.1.29. *Let ω be a circle with center O. Let $P \neq O$ be a finite point, and let P' be its inverse with respect to the circle ω. Then the polar of P passes through P' and is perpendicular to the line OP.*

Proof. Let ω meet OP in the points X, Y. When we restrict inversion to the line OP, we see that it is a nontrivial involution fixing X and Y, so it must be harmonic conjugation with respect to X, Y. Thus X, Y, P, P' are harmonic conjugates (this can also be checked using coordinates, or alternatively by drawing tangents and using facts we have already proven about harmonic quadrilaterals).

Now let OP meet the line at infinity in the point L, and let M be the harmonic conjugate of L with respect to the circle points $\mathfrak{o}, \bar{\mathfrak{o}}$. Let l, m, o, p denote the polars of L, M, O, P, respectively. Since the circle points are the intersection of ω with the line at infinity, L and M are conjugate with respect to ω, so $M = o \cap l$ and thus $m = OL$. Since P is on m, M must be on p, so $p = MP'$. By the angle theorem, MP' is perpendicular to OP, so we are done. □

Theorem 1.1.30. *Let ω be a conic, and let points A, B, C, D on a line l have polars a, b, c, d. Then we have*

$$(A, B; C, D) = (a, b; c, d).$$

Proof. Note that all four lines a, b, c, d pass through L, the pole of l. First suppose that l is not tangent to ω. Let ω intersect l in points X, Y, and let a, b, c, d intersect l at the points A', B', C', D'. Then by the definition of the polar, the points A', B', C', D' are the harmonic conjugates of A, B, C, D with respect to X, Y. Thus if h_{XY} denotes harmonic conjugation with respect to X, Y, we have

$$(A, B; C, D) = (h_{XY}(A), h_{XY}(B); h_{XY}(C), h_{XY}(D)) = (A', B'; C', D') \overset{L}{=} (a, b; c, d).$$

Now suppose the line l is tangent to ω. Let M be any point not on l or ω, and let m be its polar with respect to ω. Then by the previous case applied to the line m,

$$(A, B; C, D) = (MA, MB; MC, MD) = (m \cap a, m \cap b; m \cap c; , m \cap d) = (a, b; c, d). \qquad \square$$

Exercise 1.1.20. Give a direct proof of Theorem 1.1.30 in the case that the line l is tangent to ω. (Hint: consider the map from l to ω taking a point P on l to the foot of the second tangent from P to ω. Prove that this map preserves the cross ratio.)

Exercise 1.1.21. Suppose ω_1, ω_2 are two conics which intersect at points A, B, C, D, and let $P = AB \cap CD$. Show that the the polar of P with respect to ω_1 and the polar of P with respect to ω_2 are the same.

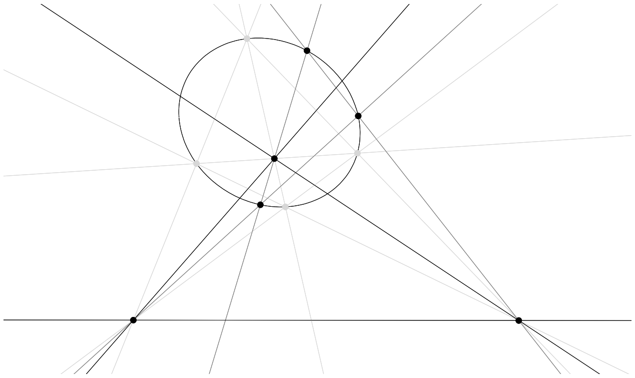

Figure 1.9: A self-polar triangle (Exercise 1.1.22)

Exercise 1.1.22.

(a) Let $ABCD$ be a quadrilateral inscribed in a conic ω. Let $E = AB \cap CD, F = AD \cap BC$ be the intersections of the opposite sides, and let $G = AC \cap BD$ be the intersection of the diagonals. Prove that the triangle EFG is *self-polar* with respect to ω, that is, that the polars of E, F, G are FG, GE, EF, respectively.

(b) Let ABC be a self-polar triangle with respect to a conic ω, and let X, Y, Z be points on ω such that Z, A, Y are collinear and X, B, Z are collinear. Prove that Y, C, X are collinear.

Exercise 1.1.23. If a, b, c, d, e are lines tangent to a conic ω, define the cross ratio of a, b, c, d with respect to ω by

$$(a, b; c, d)_\omega = (a \cap e, b \cap e; c \cap e, d \cap e).$$

(a) Show that $(a, b; c, d)_\omega$ is independent of the choice of e.

(b) If a, b, c, d meet ω at A, B, C, D, show that

$$(a, b; c, d)_\omega = (A, B; C, D)_\omega.$$

Exercise 1.1.24 (Anders Kaseorg). Let ω, Ω be distinct circles, and let ρ_ω, ρ_Ω be the polar maps with respect to ω, Ω. Show that the composite map $\rho_\omega \circ \rho_\Omega \circ \rho_\omega \circ \rho_\Omega \circ \rho_\omega \circ \rho_\Omega$ is the identity if and only if the circles ω, Ω have equal radii and intersect in 60° arcs.

1.1.7 Coharmonic points

For any two pairs of distinct points $\{A, X\}$ and $\{B, Y\}$ on a line, we can find a Möbius transformation f satisfying $f(A) = X$, $f(X) = A$, $f(B) = Y$ (since Möbius transformations have three independent parameters). Since f preserves the cross ratio, for any other point C we must have

$$(X, A; f(C), C) = (A, X; C, f(C)) = (f(A), f(X); f(C), f(f(C))) = (X, A; f(C), f(f(C))),$$

so $C = f(f(C))$ and f is a harmonic conjugation in a pair of points $\{M, N\}$. Motivated by this fact, we make the following definition.

Definition 1.1.31. Three pairs of points $\{A, X\}, \{B, Y\}, \{C, Z\}$ on the same line are called *coharmonic* if there is another pair of (possibly imaginary) points $\{M, N\}$ such that

$$(M, N; A, X) = (M, N; B, Y) = (M, N; C, Z) = -1.$$

Remark 1.1.2. Most geometers use the phrase "quadrangular hexad" to describe a collection of six coharmonic points.

Theorem 1.1.32 (Main theorem of coharmonic points)**.** *Let A, B, C, X, Y, Z be on a line, no three the same, and suppose $A \neq X$. The following are equivalent:*

(a) *The three pairs of points $\{A, X\}, \{B, Y\}, \{C, Z\}$ are coharmonic.*

(b) *There is a Möbius transformation f satisfying $f(A) = X, f(B) = Y, f(C) = Z$ which is an involution.*

(c) $(A, X; B, C) = (X, A; Y, Z)$.

(d) $\frac{AY}{YC}\frac{CX}{XB}\frac{BZ}{ZA} = -1$.

Proof. By the above discussion, (a) and (b) are clearly equivalent. To see the equivalence of (b) and (c), let f be the Möbius function satisfying $f(A) = X, f(X) = A, f(B) = Y$. Then since f preserves the cross ratio, we have

$$(A, X; B, C) = (f(A), f(X); f(B), f(C)) = (X, A; Y, f(C)),$$

so $f(C) = Z$ if and only if $(A, X; B, C) = (X, A; Y, Z)$.

Now we show that (b) implies (d). We start by making the definition

$$(A, B, C; X, Y, Z) = \frac{AY}{YC}\frac{CX}{XB}\frac{BZ}{ZA}.$$

This can also be written as

$$(A, B, C; X, Y, Z) = -(A, C; Y, B)(B, A; Z, C)(C, B; X, A),$$

so it is preserved by any Möbius transformation. Thus

$$(A, B, C; X, Y, Z) = (f(A), f(B), f(C); f(X), f(Y), f(Z)) = (X, Y, Z; A, B, C) = 1/(A, B, C; X, Y, Z),$$

so $(A, B, C; X, Y, Z) = \pm 1$. To determine whether it is 1 or -1, we need to work with coordinates. Since $(A, B, C; X, Y, Z)$ is a projective invariant, we can choose coordinates so that the fixed points of f are 0 and ∞. Then $f(z) = -z$ for any z. Let the coordinates of A, B, C be a, b, c so the coordinates of X, Y, Z are $-a, -b, -c$. Then

$$(A, B, C; X, Y, Z) = \frac{a+b}{-b-c} \cdot \frac{c+a}{-a-b} \cdot \frac{b+c}{-c-a} = -1.$$

Finally, to see that (d) implies (b), note that for any A, B, C, X, Y there is a unique Z such that $(A, B, C; X, Y, Z) = -1$, and if f is a Möbius involution taking A to X and B to Y, then $(A, B, C; X, Y, f(C)) = -1$ by the above. □

Theorem 1.1.33 (Three Conic Law). *Let A, B, C, D be any four points, no three on a line. Let l be a line passing through at most one of A, B, C, D. Let $\omega_1, \omega_2, \omega_3$ be three (possibly degenerate) conics passing through A, B, C, D. For each $i = 1, 2, 3$, let X_i, Y_i be the two points of intersection of conic ω_i with line l. Then the three pairs $\{X_1, Y_1\}, \{X_2, Y_2\}, \{X_3, Y_3\}$ are coharmonic.*

Proof. Consider the following map f from the line l to itself. For any point P on l, let ω_P be the conic passing through the points A, B, C, D, P, and define $f(P)$ to be the second point of intersection of ω_P with the line l. By Theorem 1.1.17, or more concretely by the solution to Exercise 1.1.9, f is a Möbius transformation. Since f is clearly also an involution satisfying $f(X_i) = Y_i$ for $i = 1, 2, 3$, the main theorem of coharmonic points shows that $\{X_1, Y_1\}, \{X_2, Y_2\}, \{X_3, Y_3\}$ are coharmonic. □

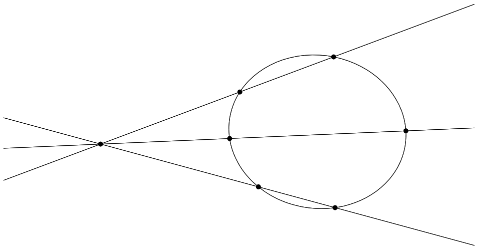

Figure 1.10: Coharmonic points on a conic

Exercise 1.1.25. (a) Let ω be a conic, and let P be a point not on ω, and let A, B, C be three points on ω. Let X, Y, Z be the second intersections of the lines PA, PB, PC with ω. Show that the three pairs $\{A, X\}, \{B, Y\}, \{C, Z\}$ are coharmonic with respect to the conic ω. (Hint: see Exercise 5.)

(b) Suppose that $ABCDEF$ is a convex hexagon inscribed in a circle ω. Show, using part (a), that the lines AD, BE, CF meet in a point if and only if

$$|AB||CD||EF| = |BC||DE||FA|.$$

(Hint: define $(A, E, C; D, B, F)_\omega$ for any conic ω, and calculate it in the special case that ω is a circle.) How is this related to the trigonometric form of Ceva's Theorem?

Exercise 1.1.26. Suppose that A, B, C, X, Y, Z are six points on a conic ω. Let U be the intersection between the line BC and the tangent to ω at X, and similarly let V be the intersection between AC and the tangent to ω at Y, and W the intersection between AB and the tangent to ω at Z. Show that if $\{A, X\}, \{B, Y\}, \{C, Z\}$ are coharmonic with respect to ω, then U, V, W are collinear. Is the converse true?

Exercise 1.1.27. Apply the Three Conic Law to give a second proof of the projective Butterfly Theorem: if ω is a conic, PQ is a chord on ω, M is a point on PQ, AB and CD are two more chords of ω passing through M, and $X = AD \cap PQ, Y = BC \cap PQ$, then $(P, Q; M, X) = (Q, P; M, Y)$. (Hint: show that $\{P, Q\}, \{M, M\}, \{X, Y\}$ are coharmonic.)

Exercise 1.1.28. Apply a degenerate case of the Three Conic Law to give a second proof of the Quadrilateral Theorem. (Hint: what does it mean for $\{X, Y\}, \{E, E\}, \{F, F\}$ to be coharmonic?)

Exercise 1.1.29. Apply the Three Conic Law to give a second proof of Desargues' Theorem. (Hint: In the notation of Theorem 1.1.7, show that $\{PA \cap VW, BC \cap VW\}, \{PB \cap VW, V\}, \{PC \cap VW, W\}$ are coharmonic, and compare the corresponding statement with A, B, C replaced by X, Y, Z.)

Exercise 1.1.30. Let ω, Ω be a pair of circles intersecting at points A, B, and let P be a point on the line AB. Let l be a line through P, let X, Y be the points of intersection between l and ω, and let U, V be the points of intersection between l and Ω. Show that

$$PX \cdot PY = PU \cdot PV.$$

Exercise 1.1.31. Let A, B, C, D, E lie on a conic ω, and let l be a line which is tangent to ω at E. Construct, using only a straightedge, the point $F \neq E$ on l such that the conic ω' passing through A, B, C, D, F is tangent to the line l at F.

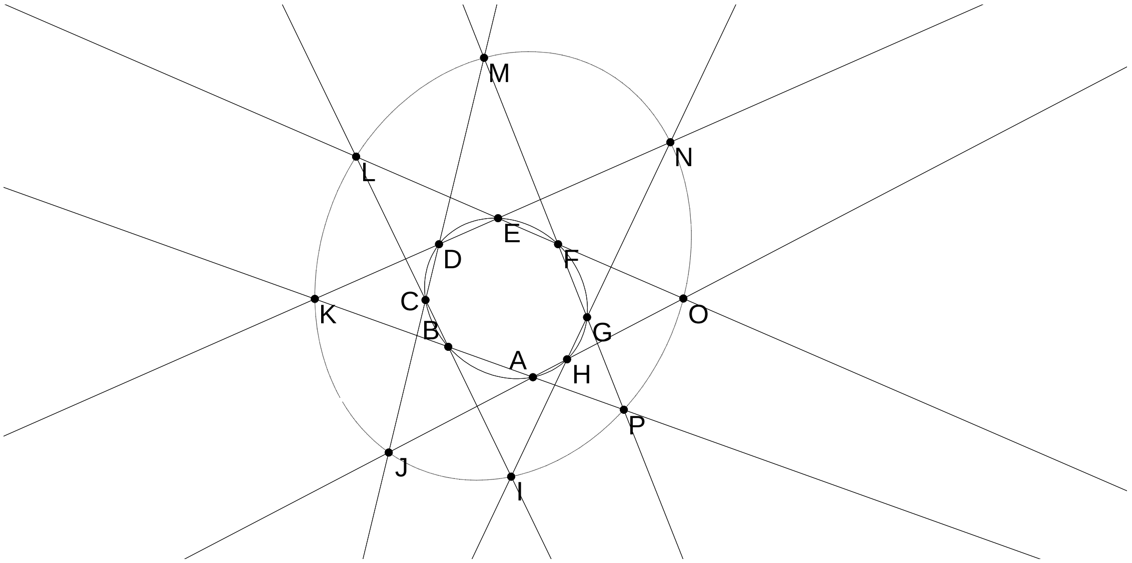

Figure 1.11: Octagrammum Mysticum

Theorem 1.1.34 (Octagrammum Mysticum)**.** *Let A, B, C, D, E, F, G, H be eight points, no three on a line. Let $I = GH \cap BC, J = HA \cap CD, K = AB \cap DE$, etc., as in Figure 1.11. Then A, B, C, D, E, F, G, H lie on a conic if and only if I, J, K, L, M, N, O, P lie on a conic.*

Proof 1 (using coharmonicity). Suppose that I, J, K, L, M, N, O, P lie on a conic ω. It's enough to show that $(AF, AD; AH, AB) = (J, P; L, N)_\omega$, since then by symmetry we will have

$$(J, P; L, N)_\omega = (CF, CD; CH, CB) = (EF, ED; EH, EB) = (GF, GD; GH, GB),$$

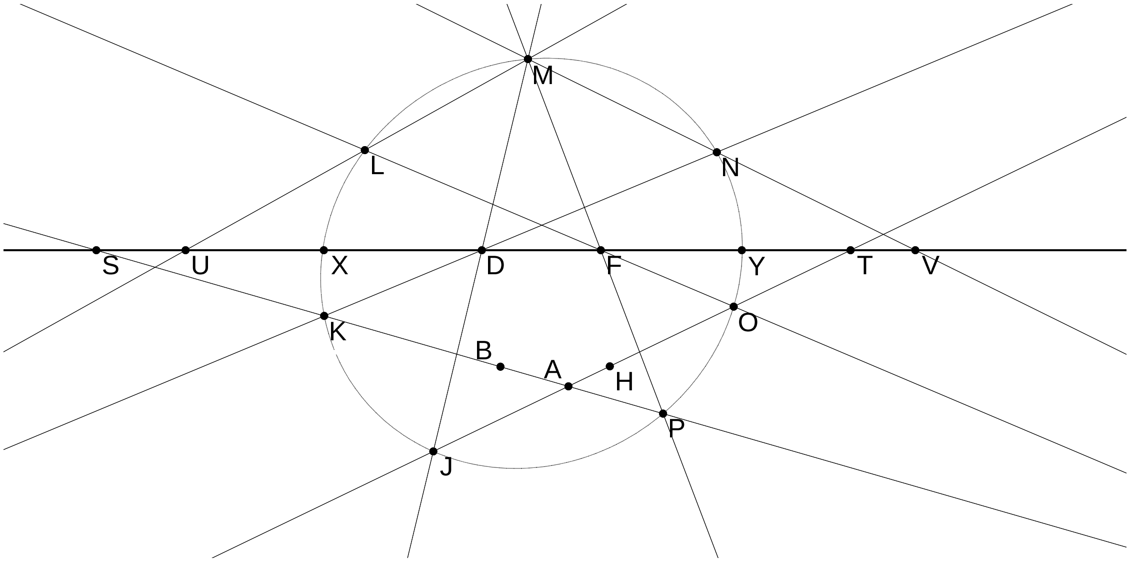

Figure 1.12: Proving $(AF, AD; AH, AB) = (J, P; L, N)_\omega$

from which we can conclude that C, E, G are on the conic through A, F, D, H, B. To this end, we project everything onto the line DF. Let $S = AB \cap DF, T = AH \cap DF, U = ML \cap DF, V = MN \cap DF$, and let X, Y be the (possibly imaginary) points of intersection between ω and DF. We have

$$(AF, AD; AH, AB) = (F, D; T, S)$$

and

$$(J, P; L, N)_\omega \overset{M}{=} (D, F; U, V),$$

so by Theorem 1.1.32 it's enough to show that $\{D, F\}, \{U, T\}, \{S, V\}$ are coharmonic.

Applying Three Conic Law to the points M, L, J, O, the line DF, the conic ω and the degenerate conics $ML \cup JO, MJ \cup LO$, we see that $\{D, F\}, \{X, Y\}, \{U, T\}$ are coharmonic. Similarly, applying the Three Conic Law to the points M, N, K, P, the line DF, the conic ω and the conics $MN \cup KP, MP \cup NK$, we see that $\{D, F\}, \{X, Y\}, \{S, V\}$ are coharmonic.

Thus the harmonic conjugation map that exchanges D with F and exchanges X with Y also exchanges U with T and S with V, so $\{D, F\}, \{U, T\}, \{S, V\}$ are coharmonic and we are done. □

Proof 2 (from [74], using Pascal's Theorem). Again, we assume that I, J, K, L, M, N, O, P lie on a conic ω. It's enough to show that G, H, A, B, C, D lie on a conic, since then by symmetry we have H, A, B, C, D, E on a conic, etc.

Let X be the intersection of lines KP and IJ. Applying Pascal's Theorem to the hexagon $MPKNIJ$ inscribed in the conic ω, we see that D, G, X lie on a line. From this we see that I, J, X are the intersections of the opposite sides of the hexagon $GHABCD$, so by the converse to Pascal's Theorem $GHABCD$ is also inscribed in a conic. □

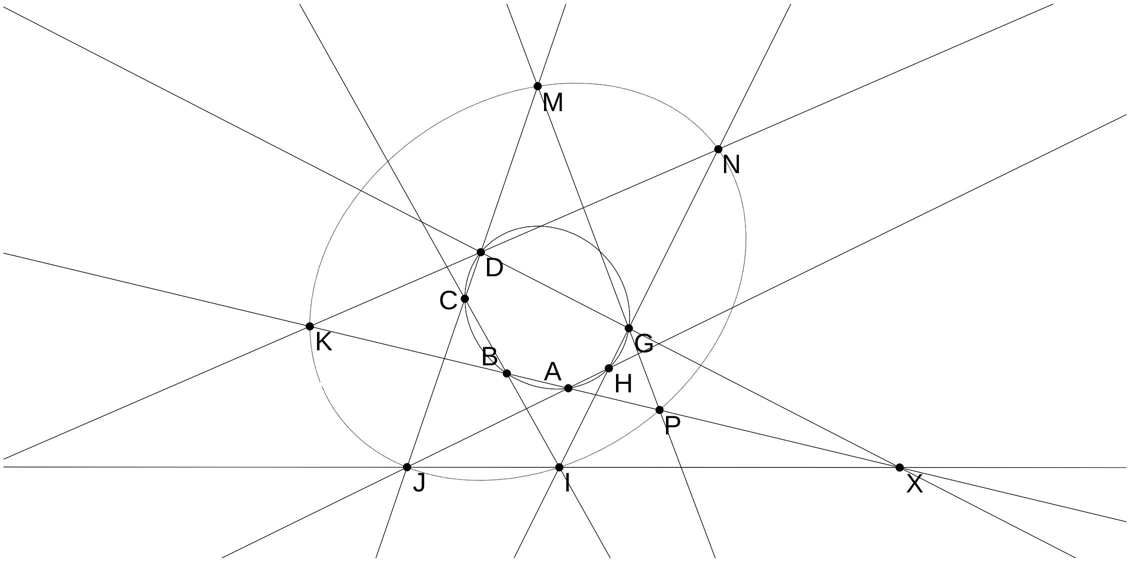

Figure 1.13: Applying Pascal

1.1.8 Symmetries of the plane

Definition 1.1.35. Let $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ be a three by three matrix with nonzero determinant. The map $f_M : \mathbb{P}^2 \to \mathbb{P}^2$ defined by $f([x : y : z]) = [ax + by + cz : dx + ey + fz : gx + hy + iz]$ is called a *projective transformation* of the plane.

Exercise 1.1.32.

(a) Show that every projective transformation sends straight lines to straight lines, sends conics to conics, and preserves cross ratios.

(b) Show that if M, N are three by three matrices with nonzero determinants, then $f_M \circ f_N = f_{MN}$.

(c) Show that if A, B, C, D are any four points with no three on a line, and E, F, G, H are any four points with no three on a line, then there is a projective transformation f with $f(A) = E, f(B) = F, f(C) = G, f(D) = H$.

Definition 1.1.36. A bijection $f : \mathbb{P}^2 \to \mathbb{P}^2$ is a *collineation* if it takes straight lines to straight lines.

Exercise 1.1.33.

(a) Let A, B, C, D, E, F be six distinct points on a line. Show that $\{A, B\}, \{C, D\}, \{E, F\}$ are coharmonic if and only if

$$(A, B; C, D) = (A, B; C, E)(A, B; C, F).$$

(b) Given distinct points A, B, C, D, E on a line, construct points F and G on the same line such that

$$(A, B; C, F) = (A, B; C, D)(A, B; C, E)$$

and

$$(A, B; C, G) = (A, B; C, D) + (A, B; C, E)$$

using only a straightedge.

Exercise 1.1.34. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(1) = 1$ and such that for any $x, y \in \mathbb{R}$ we have $f(xy) = f(x)f(y)$ and $f(x + y) = f(x) + f(y)$. Show that $f(x) = x$ for all $x \in \mathbb{R}$.

Theorem 1.1.37 (Fundamental theorem of projective geometry)**.** *A bijection $f : \mathbb{RP}^2 \to \mathbb{RP}^2$ is a collineation if and only if it is a projective transformation (here we write $\mathbb{RP}^2$ for the* real *points of the projective plane).*

Proof. We start by showing that if f is a collineation then it must preserve cross ratios. If A, B, C, D are distinct points on a line and E, F, G, H are distinct points on another line, then by Theorem 1.1.8 we can check whether $(A, B; C, D) = (E, F; G, H)$ by checking whether the points $X = AF \cap BE, Y = BG \cap CF, Z = CH \cap DG$ lie on a line. Since f is a collineation, we have $f(X) = f(AF) \cap f(BE) = f(A)f(F) \cap f(B)f(E)$ and so on, and $f(X), f(Y), f(Z)$ lie on a line if and only if X, Y, Z lie on a line, so

$$(A, B; C, D) = (E, F; G, H) \iff (f(A), f(B); f(C), f(D)) = (f(E), f(F); f(G), f(H)).$$

Thus we get a well-defined bijection $\tilde{f} : \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$ by taking

$$\tilde{f}((A, B; C, D)) = (f(A), f(B); f(C), f(D)).$$

This bijection automatically satisfies $\tilde{f}(0) = 0, \tilde{f}(1) = 1, \tilde{f}(\infty) = \infty$. By Exercise 1.1.33 we have $\tilde{f}(xy) = \tilde{f}(x)\tilde{f}(y)$ and $\tilde{f}(x+y) = \tilde{f}(x)+\tilde{f}(y)$ for any real x, y, and thus by Exercise 1.1.34 we must have $\tilde{f}(x) = x$ for all real x. Thus f preserves cross ratios.

To finish, note that by Exercise 1.1.32 we may assume without loss of generality that f fixes some collection of four points A, B, C, D such that no three are on a line. Letting $P = AB \cap CD$, we see that $f(P) = P$, and thus for any point X on AB we have

$$(A, B; P, X) = (f(A), f(B); f(P), f(X)) = (A, B; P, f(X)),$$

so $f(X) = X$. Thus if l is any line through C, and $X = l \cap AB$, then $f(l) = f(C)f(X) = CX = l$, so every line through C is sent to itself. Similarly, every line through A or B is sent to itself. Since any point E is determined by the three lines AE, BE, CE, every point E must be sent to itself, and we are done. □

Remark 1.1.3. A collineation of $\mathbb{CP}^2$ might not preserve cross ratios: for instance, the map $[x : y : z] \mapsto [\bar{x} : \bar{y} : \bar{z}]$ taking every point to its complex conjugate sends every cross ratio to its complex conjugate. More generally, if $\tilde{f} : \mathbb{C} \to \mathbb{C}$ satisfies $\tilde{f}(1) = 1, \tilde{f}(xy) = \tilde{f}(x)\tilde{f}(y), \tilde{f}(x+y) = \tilde{f}(x)+\tilde{f}(y)$, then the map $[x : y : z] \mapsto [\tilde{f}(x) : \tilde{f}(y) : \tilde{f}(z)]$ is called an *automorphic collineation*, and sends a set of four points on a line with cross ratio c to a set of four points with cross ratio $\tilde{f}(c)$.

The same argument as above can be used to show that every collineation of $\mathbb{CP}^2$ can be written as the composition of an automorphic collineation and a projective transformation.

Definition 1.1.38. Let P be a point and l be a line not passing through P. Define the *projective reflection* $r_{P,l}$ by sending a point $Q \neq P$ to the harmonic conjugate of Q with respect to $P, PQ \cap l$ along the line PQ, and sending P to P.

Example 1.1.2. (a) Let l intersect the line at infinity at L. If P is on the line at infinity with $L, P, \wp, \bar{\wp}$ harmonic, then $r_{P,l}$ is (ordinary) reflection across the line l. As a consequence, (ordinary) reflections always interchange the two circle points.

(b) If l is the line at infinity, then $r_{P,l}$ is a 180° rotation around P (sometimes called a reflection through the point P).

Theorem 1.1.39. *For any point P and any line l not passing through P, the projective reflection $r_{P,l}$ is a projective transformation.*

Proof. We just need to show that $r_{P,l}$ sends lines to lines and preserves cross ratios. We leave this as an easy exercise to the reader. □

Definition 1.1.40. If A, B, C, D are four points with no three on a line and $\sigma : \{A, B, C, D\} \to \{A, B, C, D\}$ is a permutation, define r_σ to be the projective transformation taking A to $\sigma(A)$, B to $\sigma(B)$, etc. We will often write σ using its cycle decomposition, including the cycles of length 1, so that for instance $r_{(A)(B)(C\ D)}$ is the projective transformation taking A and B to themselves, and swapping C and D.

Exercise 1.1.35. Suppose A, B, C, D are four points with no three on a line.

(a) If $P = AB \cap CD$ and l is the line connecting $AC \cap BD$ to $AD \cap BC$, show that $r_{(A\ B)(C\ D)}$ is the projective reflection $r_{P,l}$.

(b) Show that if ω is a conic passing through A, B, C, D then $r_{(A\ B)(C\ D)}(\omega) = \omega$.

(c) Show that if ω is as in (b) and P, l are as in (a), then l is the polar of P with respect to ω.

Exercise 1.1.36. (a) Show that for every permutation $\sigma : \{A, B, C, D\} \to \{A, B, C, D\}$ we can write r_σ as a composition of two projective reflections.

(b) Show that a projective transformation defined by a three by three matrix M can be written as a composition of two projective reflections if and only if the eigenvalues of M are in a geometric progression.

Exercise 1.1.37. Let $f(p, q, r), g(p, q, r), h(p, q, r)$ be homogeneous polynomials of the same degree having no common factor. The map $[p : q : r] \mapsto [f(p, q, r) : g(p, q, r) : h(p, q, r)]$ is called *biregular* if it is defined everywhere (i.e. f, g, h are never simultaneously 0 unless p, q, r are all 0) and is a bijection of the complex points of the projective plane. Prove that every biregular map is a projective transformation.

One rather boring way to use symmetries of the plane is to choose a coordinate system in which four points A, B, C, D in general position are assigned the coordinates $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1]$. If a geometric configuration is completely determined by the locations of five points A, B, C, D, E, then every other point has coordinates given by homogenous algebraic functions of the coordinates $[x : y : z]$ of the point E. Problems involving such configurations can then be straightforwardly transformed into simple algebra problems, which typically will state that if one homogenous polynomial of the coordinates $[x : y : z]$ of E vanishes, then so does another (often these polynomials will be linear or quadratic). Many problems in triangle geometry have this form: the five relevant points are the vertices A, B, C of the triangle, and the two circle points $\wp$ and $\bar{\wp}$.

Theorem 1.1.41. *If A, B, C, D, E are five points in general position, and if we choose a coordinate system where A, B, C, D, E are assigned the coordinates $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1], [x : y : z]$, respectively, then we have*

$$\frac{y}{z} = (AB, AC; AD, AE)$$

and

$$\frac{x}{z} = (BA, BC; BD, BE).$$

In particular, the pair of values of these two cross ratios completely determines which statements of projective geometry are true of the configuration $ABCDE$.

Proof. We will only prove the first equality, the second one is similar. We have

$$\begin{aligned}(AB, AC; AD, AE) &= (B, C; AD \cap BC, AE \cap BC)\\ &= ([0 : 1 : 0], [0 : 0 : 1]; [0 : 1 : 1], [0 : y : z]) = (\infty, 0; 1, y/z) = y/z. \qquad \square\end{aligned}$$

In the special case where A, B, C are the vertices of a triangle and D, E are the circle points $\wp, \bar{\wp}$, the previous theorem becomes the statement that every triangle ABC is determined up to direct similarity by the ordered pair of directed angles $\angle BAC$ and $\angle ABC$ modulo π. So for instance, the correct projective generalization of the concept of an isosceles triangle is a configuration of five points $ABCDE$, no three on a line, which satisfies the symmetry

$$r_{(A\ B)(D\ E)}(C) = C,$$

and the projective analogue of an equilateral triangle will additionally satisfy the symmetry

$$r_{(A\ C)(D\ E)}(B) = B.$$

Exercise 1.1.38. Show that if no three of A, B, C, D, E are on a line, and if the configuration $ABCDE$ satisfies the symmetries $r_{(A\ B)(D\ E)}(C) = C$ and $r_{(A\ C)(D\ E)}(B) = B$, then it also satisfies the symmetry $r_{(B\ C)(D\ E)}(A) = A$. Show that in this case, the cross ratio $(EA, EB; EC, ED)$ is *melodic* in the sense of Exercise 1.1.19.

Exercise 1.1.39. Show that if no three of A, B, C, D, E are on a line, and if the configuration $ABCDE$ satisfies the symmetries $r_{(B\ E)(C\ D)}(A) = A$ and $r_{(A\ C)(D\ E)}(B) = B$, then it also satisfies the symmetry $r_{(A\ E)(B\ D)}(C) = C$. Show that in this case, the cross ratio $(EA, EB; EC, ED)$ is either the golden ratio ϕ or its algebraic conjugate $-1/\phi$.

1.1.9 The Cross Cross Ratio

Since any four points (no three on a line) can be sent to any other four points (no three on a line) by a projective transformation, there are no interesting invariants of four general points in the plane. If we have five general points A, B, C, D, E, then we can form the cross ratio $(EA, EB; EC, ED)$. Going one step further, we have the following natural definition.

Definition 1.1.42. Let A, B, C, D, E, F be six points in the plane, such that either none of ACE, ADF, BCF, BDE are lines or none of ACF, ADE, BCE, BDF are lines. Define their *cross cross ratio* to be

$$(A, B; C, D; E, F) = \frac{(EA, EB; EC, ED)}{(FA, FB; FC, FD)}.$$

First we will prove that this definition is more symmetric than it seems.

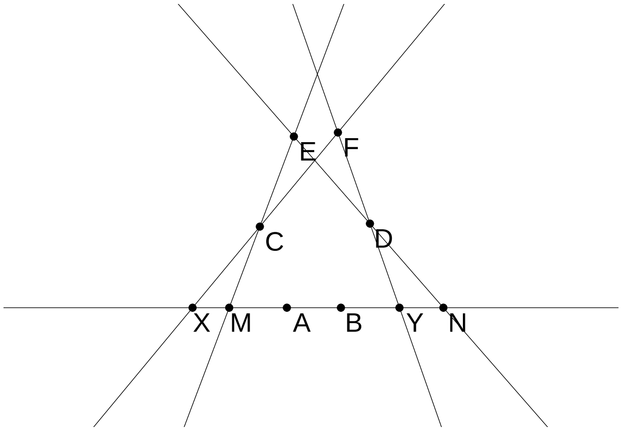

Figure 1.14: Symmetry of the cross cross ratio

Theorem 1.1.43. *Let A, B, C, D, E, F be as above. Then we have*

$$(A, B; C, D; E, F) = (A, B; E, F; C, D).$$

Proof. We start by projecting everything onto the line AB. Let $M = EC \cap AB, N = ED \cap AB, X = FC \cap AB, Y = FD \cap AB$. Then we have

$$\begin{aligned}(A, B; C, D; E, F) &= \frac{(EA, EB; EC, ED)}{(FA, FB; FC, FD)} = \frac{(A, B; M, N)}{(A, B; X, Y)} \\ &= \frac{(A, B; M)}{(A, B; N)} \Big/ \frac{(A, B; X)}{(A, B; Y)} = \frac{(A, B; M)}{(A, B; X)} \Big/ \frac{(A, B; N)}{(A, B; Y)} \\ &= \frac{(A, B; M, X)}{(A, B; N, Y)} = \frac{(CA, CB; CE, CF)}{(DA, DB; DE, DF)} = (A, B; E, F; C, D).\end{aligned}$$ □

Proposition 1.1.44. *Let two circles ω, ω' intersect at points A, B, and let C be a point on ω, D a point on ω'. Let θ be the (directed) angle of intersection between the circles ω, ω' at A. Then we have*

$$(A, B; \wp, \bar{\wp}; C, D) = e^{2i\theta}.$$

In particular, ω and ω' are orthogonal if and only if $(A, B; \wp, \bar{\wp}; C, D) = -1$.

Proof.

$$(A, B; \wp, \bar{\wp}; C, D) = \frac{(A, B; \wp, \bar{\wp})_\omega}{(A, B; \wp, \bar{\wp})_{\omega'}} = e^{2i(\angle ACB - \angle ADB)} = e^{2i\theta}.$$ □

Definition 1.1.45. If conics ω, ω' meet in points A, B, C, D, set

$$(A, B; C, D; \omega, \omega') = \frac{(A, B; C, D)_\omega}{(A, B; C, D)_{\omega'}}.$$

If $(A, B; C, D; \omega, \omega') = -1$, we say that the conics ω, ω' are *projectively orthogonal with respect to the partition* $\{A, B\}, \{C, D\}$ of their intersection points.

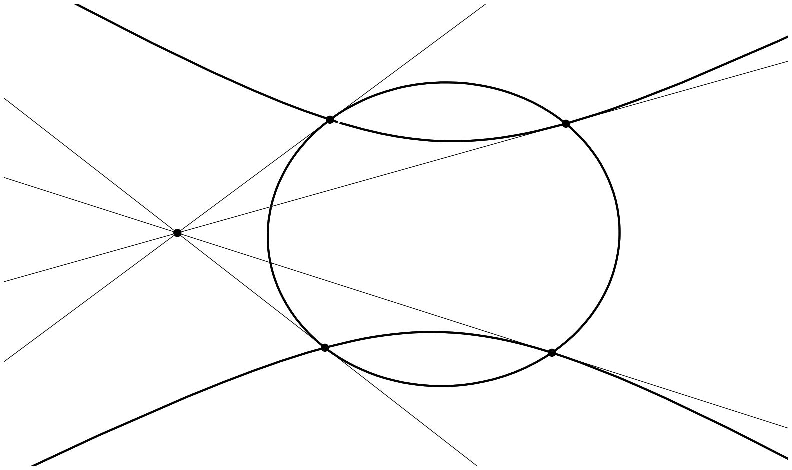

Figure 1.15: Projectively orthogonal conics

Theorem 1.1.46. *Two conics ω, Ω meeting in points A, B, C, D are projectively orthogonal with respect to the partition $\{A, B\}, \{C, D\}$ if and only if the two tangents to ω at A and B meet the two tangents to Ω at C and D.*

Proof. Let $E = AC \cap BD, F = AD \cap BC$. We will project everything onto the line EF: let $X = AB \cap EF$, let $Y = CD \cap EF$, let P be the intersection of the tangent to ω at A with EF, and let Q be the intersection of the tangent to Ω at C with EF.

Projecting through A or B, we have

$$(A, B; C, D)_\omega \overset{A}{=} (P, X; E, F) \overset{B}{=} (BP \cap \omega, A; D, C)_\omega,$$

so BP is also tangent to ω, and similarly we have

$$(A, B; C, D)_\Omega \overset{C}{=} (E, F; Q, Y) \overset{D}{=} (B, A; DQ \cap \Omega, C)_\Omega$$

and DQ is tangent to Ω. By the quadrilateral theorem, we have

$$(E, F; X, Y) = -1,$$

so

$$\frac{(A, B; C, D)_\omega}{(A, B; C, D)_\Omega} = \frac{(E, F; P, X)}{(E, F; Q, Y)} = \frac{(E, F; P, Q)}{(E, F; X, Y)} = -(E, F; P, Q).$$

Thus $(A, B; C, D; \omega, \Omega) = -1$ if and only if $P = Q$. □

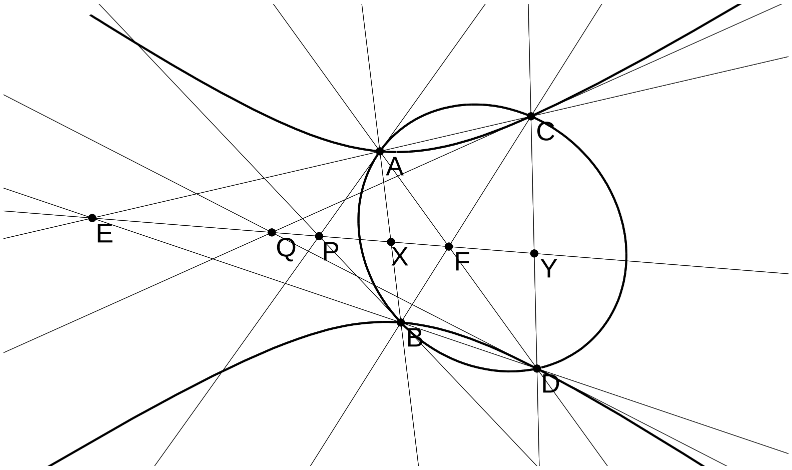

Figure 1.16: Checking orthogonality

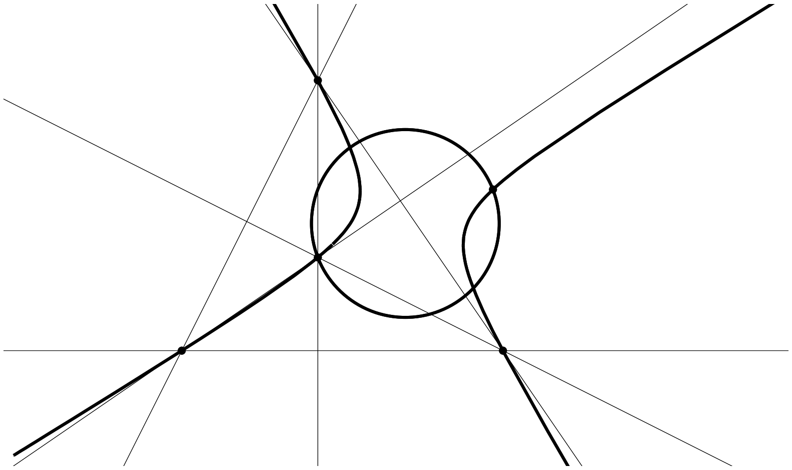

Figure 1.17: Exercise 1.1.40

Exercise 1.1.40. Let H be the orthocenter of triangle ABC, and let P be any point other than H. Let ω be the circle with diameter HP, and let Ω be the conic through A, B, C, H, P.

(a) Show that the asymptotes to Ω meet at a right angle.

(b) Show that if ω, Ω also meet at points X, Y, then ω is projectively orthogonal to Ω with respect to the partition $\{H, P\}, \{X, Y\}$.

Exercise 1.1.41. Suppose conics ω, Ω meet at A, B, C, D and are projectively orthogonal with respect to the partition $\{A, B\}, \{C, D\}$ of their intersection points. Let l be a line meeting ω at P, Q and meeting Ω at R, S.

(a) Show that $(P, Q; A, B)_\omega = -1$ if and only if $(R, S; C, D)_\Omega = -1$.

(b) Show that if $(P, Q; A, B)_\omega = -1$ then $(P, Q; R, S) = -1$.

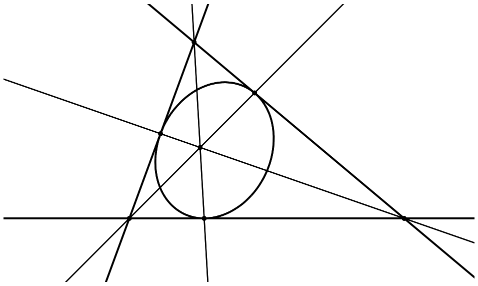

Figure 1.18: Exercise 1.1.42(a)

1.1.10 A few miscellaneous exercises

Exercise 1.1.42.

(a) Let ABC be a triangle, let D be a point on BC, let E be a point on CA, and let F be a point of AB. Show that the lines AD, BE, CF meet in a point if and only if there is a conic ω which is tangent to BC at D, tangent to CA at E, and tangent to AB at F.

(b) Let ABC be a triangle and let P be a point not lying on any edge of ABC. Let U, X be points on BC with $X = r_{(A)(P)(B\,C)}(U)$, let V, Y be points on CA with $Y = r_{(B)(P)(A\,C)}(V)$, and let W, Z be points on AB with $Z = r_{(C)(P)(A\,B)}(W)$. Show that U, V, W, X, Y, Z lie on a conic.

Exercise 1.1.43 (Holden Mui). Suppose $\Omega, \omega_1, \omega_2$ are conics such that Ω is tangent to ω_1 at A and B and Ω is tangent to ω_2 at C and D. Let $P = AB \cap CD$, and let X, Y, Z, W be the four points of intersection between ω_1 and ω_2.

(a) Show that there is a way to order X, Y, Z, W such that $XZ \cap YW = P$.

(b) Show that if X, Y, Z, W are ordered as in (a), then the four lines $AB, CD; XZ, YW$ are harmonic.

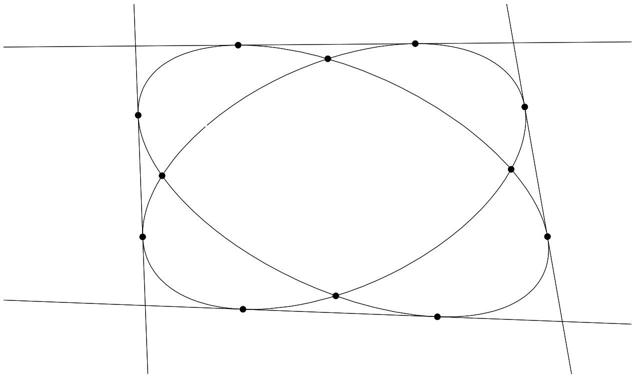

Figure 1.19: Exercise 1.1.44

Exercise 1.1.44.

(a) Given points A, B, C, D and a line e, there are two conics ω, Ω passing through A, B, C, D and tangent to e. Construct the other three common tangent lines f, g, h to the conics ω, Ω using only the points A, B, C, D, the line e, and a straightedge.

(b) Show that if you order e, f, g, h correctly, you have

$$(A, B; C, D)_\omega = (e, f; g, h)_\Omega.$$

(c) Show that polar maps send projectively orthogonal pairs of conics to projective orthogonal pairs of conics.

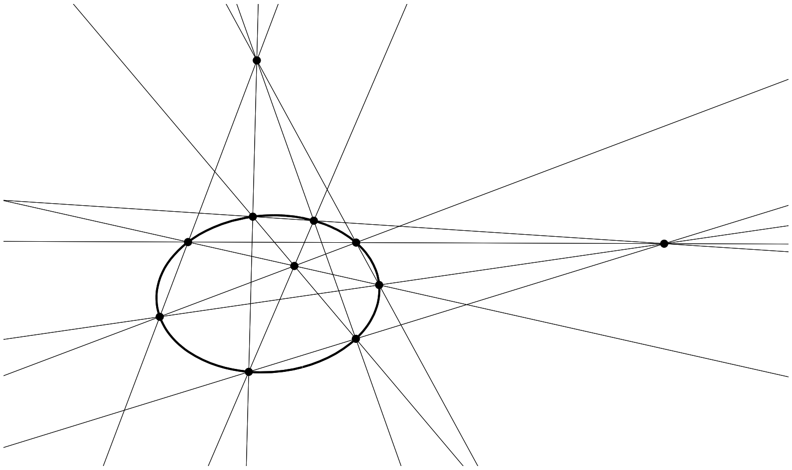

Figure 1.20: Exercise 1.1.45

Exercise 1.1.45. Suppose that A, B, C, D, E, F, G, H are eight distinct points in the plane such that the four lines AB, CD, EF, GH meet in a point, the four lines AC, BD, EG, FH meet in a point, and the four lines AD, BC, EH, FG meet in a point. Show that A, B, C, D, E, F, G, H all lie on a single conic.

Exercise 1.1.46 (Triangular grid lemma). Let $a_1, a_2, a_3, a_4, b_1, b_2$ be six distinct lines. Let c_1 be the line through $a_3 \cap b_1$ and $a_2 \cap b_2$. Let c_2 be the line through $a_4 \cap b_1$ and $a_3 \cap b_2$. Let b_3 be the line through $a_1 \cap c_1$ and $a_2 \cap c_2$. Let c_3 be the line through $a_4 \cap b_2$ and $a_3 \cap b_3$. Let b_4 be the line through $a_1 \cap c_2$ and $a_2 \cap c_3$. Let c_4 be the line through $a_4 \cap b_3$ and $a_3 \cap b_4$. Let b_5 be the line through $a_1 \cap c_3$ and $a_2 \cap c_4$. Let c_5 be the line through $a_4 \cap b_4$ and $a_3 \cap b_5$. Show that the three points $b_1 \cap c_3, b_2 \cap c_4, b_3 \cap c_5$ are on a line. (Hint: use Theorem 1.1.8.)

1.2 Cross ratios in other geometries

1.2.1 Cremona involutions and blow ups

Let A, B, C, D be four points in the projective plane, no three on a line. Choose projective coordinates such that $A = [1 : 0 : 0], B = [0 : 1 : 0], C = [0 : 0 : 1], D = [1 : 1 : 1]$ (one way to do this is

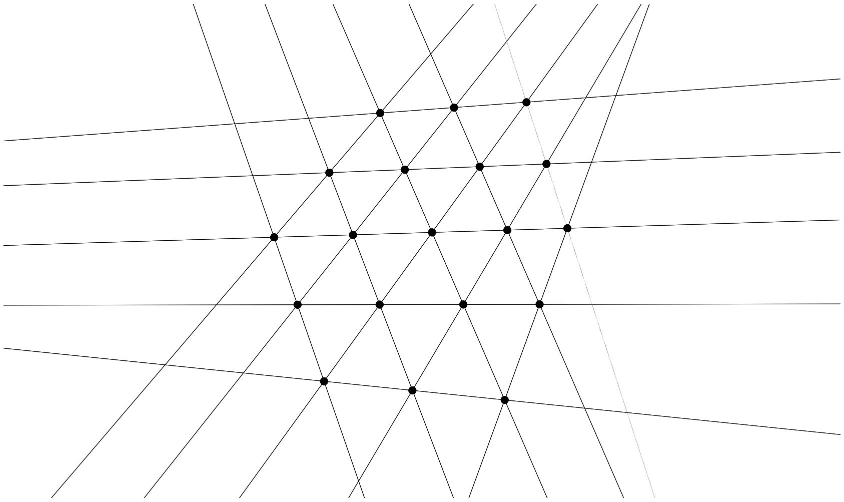

Figure 1.21: Triangular grid lemma

to start with barycentric coordinates on the triangle A, B, C, and then rescale the coordinates to make $D = [1 : 1 : 1]$). For future reference, let $E = [-1 : 1 : 1], F = [1 : -1 : 1], G = [1 : 1 : -1]$ in this coordinate system.

Exercise 1.2.1. Show that E, F, G satisfy $DE \cap FG = A, DF \cap EG = B, DG \cap EF = C$, and that they are uniquely determined by these conditions. Show that E is the harmonic conjugate of D with respect to $A, AD \cap BC$.

One of the simplest nonlinear functions we can write down is the Cremona involution: if p, q, r are all nonzero, it takes the point $P = [p : q : r]$ in the above coordinate system to the point $f_{ABCD}(P) = [\frac{1}{p} : \frac{1}{q} : \frac{1}{r}]$. We would like to extend this to an involution of the plane. Clearing denominators, we get $f_{ABCD}(P) = [qr : pr : pq]$, and this lets us define $f_{ABCD}(P)$ as long as no two of p, q, r are 0, i.e. as long as P is not equal to one of A, B, C. If P is on line BC, then $p = 0$, so $f_{ABCD}(P) = [qr : 0 : 0] = A$, and f_{ABCD} is not injective. We can fix these problems by "blowing up" the points A, B, C.

Definition 1.2.1. If A, B, C are three distinct points in the projective plane, we set

$$\mathrm{Bl}_{ABC}\,\mathbb{P}^2 = \{(P, l_A, l_B, l_C) \mid P \in \mathbb{P}^2, \{A, P\} \subset l_A, \{B, P\} \subset l_B, \{C, P\} \subset l_C\}.$$

If $(P, l_A, l_B, l_C) \in \mathrm{Bl}_{ABC}\,\mathbb{P}^2$ has $P \neq A, B, C$, then $l_A = AP, l_B = BP, l_C = CP$, and we write P as shorthand for (P, l_A, l_B, l_C). Let e_A be the set of points (P, l_A, l_B, l_C) in $\mathrm{Bl}_{ABC}\,\mathbb{P}^2$ with $P = A$, that is,

$$e_A = \{(A, l, AB, AC) \mid A \in l\},$$

and define e_B, e_C similarly. If $(A, l, AB, AC) \in e_A$, we write (A, l) as shorthand for it. If $P = (A, l)$, we write AP as shorthand for l. The three lines e_A, e_B, e_C are called the *exceptional lines* above A, B, C. We say that a curve ω passing through A intersects the exceptional line e_A in the point (A, l_A) if line l_A is tangent to ω at A.

In coordinates, we have

$$\mathrm{Bl}_{ABC}\,\mathbb{P}^2 = \{([p : q : r], (0 : a : b), (c : 0 : d), (e : f : 0)) \mid aq + br = cp + dr = ep + fq = 0\}.$$

Proposition 1.2.2. *The map $[p : q : r] \mapsto [\frac{1}{p} : \frac{1}{q} : \frac{1}{r}]$, defined for $p, q, r \neq 0$, extends to an involution $f_{ABCD} : \mathrm{Bl}_{ABC}\,\mathbb{P}^2 \to \mathrm{Bl}_{ABC}\,\mathbb{P}^2$. The extended involution f_{ABCD} takes e_A (resp. e_B, e_C) bijectively to BC (resp. AC, AB). The fixed points of f_{ABCD} are D, E, F, G, and we have $f_{ABCD} = f_{ABCE} = f_{ABCF} = f_{ABCG}$.*

Proof. In coordinates, if $P = ([p : q : r], (0 : a : b), (c : 0 : d), (e : f : 0))$ we set

$$f_{ABCD}(P) = \begin{cases} ([aq : ap : -bp], (0 : b : a), (d : 0 : c), (f : e : 0)) & \text{if } p \neq 0, \\ ([cq : cp : -dq], (0 : b : a), (d : 0 : c), (f : e : 0)) & \text{if } q \neq 0, \\ ([er : -fr : ep], (0 : b : a), (d : 0 : c), (f : e : 0)) & \text{if } r \neq 0. \end{cases}$$

Checking that this is well-defined, along with checking the other claims of the proposition, is left as an easy exercise to the reader. □

Remark 1.2.1. More generally, for any three homogeneous polynomials $f(p, q, r), g(p, q, r), h(p, q, r)$ of the same degree having no common factor we can define a map $[p : q : r] \to [f(p, q, r) : g(p, q, r) : h(p, q, r)]$, which is well-defined whenever f, g, h are not simultaneously zero. Such a map is called a *rational map.* It is called *birational* if it is usually one-to-one - in this case you can write down a rational function which inverts it whenever both are defined. Noether and Castelnuovo have proved that every birational map $\mathbb{P}^2 \to \mathbb{P}^2$ can be built out of projective transformations and Cremona involutions.

Proposition 1.2.3. *Let A, B, C, D, E, F, G be such that $A = DE \cap FG, B = DF \cap EG, C = DG \cap EF$, and suppose that $f_{ABCD}(P) = Q$. Then we have*

$$(AP, AQ; DE, FG) = (BP, BQ; DF, EG) = (CP, CQ; DG, EF) = -1.$$

In other words, AQ is the harmonic conjugate of AP with respect to AD, AF, and similarly for BQ, CQ.

Proof. By symmetry, it's enough to show that $(AP, AQ; DE, FG) = -1$. In the coordinate system described above, suppose that $AP = (0 : a : b)$. We then have $DE = (0 : 1 : -1), FG = (0 : 1 : 1), AQ = (0 : b : a)$, so

$$(AP, AQ; DE, FG) = (a/b, b/a; -1, 1) = -1.$$ □

Example 1.2.1. Let G be the centroid of triangle ABC, and suppose $f_{ABCG}(P) = Q$. Let FED have parallel sides to ABC, such that A is the midpoint of DE, B is the midpoint of DF, and C is the midpoint of EF. Let $M = AG \cap BC, X = AP \cap BC, Y = AQ \cap BC, \infty = AD \cap BC$. Then

$$(X, Y; \infty, M) = (AP, AQ; DE, FG) = -1,$$

so X is the reflection of Y across M, the midpoint of BC. Similarly, $BP \cap AC$ is the reflection of $BQ \cap AC$ across the midpoint of AC, etc. The point Q is called the *isotomic conjugate* of P.

Corollary 1.2.4. *Let $f = f_{ABCD}$. For any four points $P, Q, R, S \in \mathrm{Bl}_{ABC}\,\mathbb{P}^2$ we have*

$$(AP, AQ; AR, AS) = (Af(P), Af(Q); Af(R), Af(S)).$$

Proof. Harmonic conjugation preserves the cross ratio. □

Theorem 1.2.5. *Let l be a line which does not pass through any of A, B, C. Then $f_{ABCD}(l)$ is a* circumconic*, that is, a conic passing through all three of A, B, C. Conversely, if ω is a circumconic then $f_{ABCD}(\omega)$ is a line which does not pass through any of A, B, C.*

Proof. Write $f = f_{ABCD}$, and let P, Q, R be any three points on l. Let $S = l \cap BC$, so that $f(S) \in e_A$. By the Corollary, we have

$$(Bf(P), Bf(Q); Bf(R), BA) = (P, Q; R, S) = (Cf(P), Cf(Q); Cf(R), CA),$$

so $f(R)$ lies on the conic ω through $A, B, C, f(P), f(Q)$. The converse is left as an exercise. □

Exercise 1.2.2. Let I be the incenter of triangle ABC. The map f_{ABCI} is called *isogonal conjugation.*

(a) Show that $f_{ABCI}(\wp) = \bar{\wp}$.

(b) Let Ω be the circumcircle of triangle ABC. Show that $f_{ABCI}(\Omega)$ is the line at infinity.

(c) Let m be the median through A. Show that $f_{ABCI}(m)$ passes through the pole of BC with respect to Ω. (Hint: show that the intersections of m, BC, AB, AC with the line at infinity are harmonic, then apply f_{ABCI}.)

(d) Let ω be the circumcircle of triangle BCI. Show that $f_{ABCI}(\omega) = \omega$.

Exercise 1.2.3. Write $f = f_{ABCD}$, let l be a line which doesn't pass through any of A, B, C, let $\omega = f(l)$, and let P, Q, R, S be any four points on l. Show that

$$(P, Q; R, S) = (f(P), f(Q); f(R), f(S))_\omega.$$

Exercise 1.2.4. Let A, B, C, D be in general position, and let ω be a conic passing through A, B, and C. Let X be the second intersection of the line AD with the conic ω, and let U be the intersection between the line BC and the tangent to ω at X. Show that $U \in f_{ABCD}(\omega)$. In particular, if we define points $V \in AC, W \in AB$ similarly, then U, V, W are collinear.

Theorem 1.2.6. *If ω is a conic which passes through B and C but not A, then $f_{ABCD}(\omega)$ is also a conic passing through B and C but not A. We have $f_{ABCD}(\omega) = \omega$ if and only if ω either passes through D and E or passes through F and G.*

Proof. Write $f = f_{ABCD}$, and let P, Q, R, S be any four points on ω. By Corollary 1.2.4, we have

$$(Bf(P), Bf(Q); Bf(R), Bf(S)) \overset{B}{=} (P, Q; R, S)_\omega \overset{C}{=} (Cf(P), Cf(Q); Cf(R), Cf(S)),$$

so $B, C, f(P), f(Q), f(R), f(S)$ are on a conic. If $f(\omega)$ passed through A, then ω would need to be tangent to BC at either B or C, which is impossible.

Note that if $f(\omega) = \omega$ then f defines an involution from ω to itself, and so f must fix exactly two points of ω, which can't both be contained in the same line through B or C. Conversely, suppose for instance that D, E are on ω, and let X be any other point on ω. The conic through $B, C, D, X, f(X)$ is sent to itself, so it must contain E. Thus $f(X)$ must be on ω. □

Aside: some basic intersection theory

We recall (without proof) a famous theorem of Bézout.

Theorem 1.2.7 (Bézout)**.** *If Ω, ω are distinct curves in $\mathbb{P}^2$ defined by irreducible polynomial equations of degrees m, n, respectively, then the number of intersection points between Ω and ω is exactly mn, if you count points "with multiplicity" and remember to include imaginary points and points at infinity.*

In particular, any two curves in $\mathbb{P}^2$ meet in at least one point. $\mathrm{Bl}_{ABC}\,\mathbb{P}^2$ doesn't have this property: for instance, the line AB doesn't intersect either of the lines e_C, BC in $\mathrm{Bl}_{ABC}\,\mathbb{P}^2$. Luckily, it's easy to modify Bézout's theorem to make it work for $\mathrm{Bl}_{ABC}\,\mathbb{P}^2$.

Definition 1.2.8. If ω is a curve in $\mathrm{Bl}_{ABC}\,\mathbb{P}^2$ defined by an irreducible polynomial equation of degree m, which passes through A, B, C with multiplicities a, b, c, respectively, we say that ω is a *curve of type* $(m, -a, -b, -c)$. If $\omega = e_A$, we say that ω is a curve of type $(0, 1, 0, 0)$, and similarly e_B has type $(0, 0, 1, 0)$, e_C has type $(0, 0, 0, 1)$.

Theorem 1.2.9. *If Ω, ω are distinct irreducible algebraic curves in $\mathrm{Bl}_{ABC}\,\mathbb{P}^2$ of types $(m, p, q, r), (n, x, y, z)$, then the number of intersection points between Ω and ω in $\mathrm{Bl}_{ABC}\,\mathbb{P}^2$ is exactly $mn - px - qy - rz$, if you count points "with multiplicity" and remember to include imaginary points and points at infinity.*

Definition 1.2.10. If ω has type (m, p, q, r), then the *self-intersection number* of ω is defined to be $m^2 - p^2 - q^2 - r^2$.

Proposition 1.2.11. *If ω has type (m, p, q, r) then $f_{ABCD}(\omega)$ has type $(2m + p + q + r, -m - q - r, -m - p - r, -m - p - q)$.*

Exercise 1.2.5.

(a) Prove Proposition 1.2.11.

(b) Using Proposition 1.2.11 and Theorem 1.2.9, check that the number of intersection points between ω and Ω is the same as the number of intersection points between $f_{ABCD}(\omega)$ and $f_{ABCD}(\Omega)$. In particular, the self-intersection number of ω is the same as the self-intersection number of $f_{ABCD}(\omega)$.

(c) Use Proposition 1.2.11 to give another proof of Theorem 1.2.5.

(d) Find all curves in $\mathrm{Bl}_{ABC}\,\mathbb{P}^2$ which have self-intersection number at most 0.

1.2.2 Hyperbolic geometry

There are many models of hyperbolic geometry. The easiest ones to understand are the models which live inside disks in the inversive plane $\mathbb{CP}^1$.

Definition 1.2.12. A *disk* in $\mathbb{CP}^1$ is a circle or line $\Omega \subseteq \mathbb{CP}^1$, together with a choice of one of the two connected components of $\mathbb{CP}^1 \setminus \Omega$, which we call the *interior* of the disk (the other connected component of $\mathbb{CP}^1$ is called the *exterior* of the disk). The circle Ω is the *boundary* of the disk.

Note that the choice of which region of $\mathbb{CP}^1 \setminus \Omega$ should be the interior and which should be the exterior is arbitrary, since an inversion around the center of Ω (or a reflection across Ω, if Ω is a line) interchanges these two regions. So we really need to explicitly specify which region should be considered the interior of the disk in order to be unambiguous. When Ω is a circle, generally people take the interior of Ω to be the region of $\mathbb{CP}^1$ which does not contain the point at infinity - this way, the disk can be drawn using a finite amount of paper.

Definition 1.2.13. Let D be a disk in $\mathbb{CP}^1$ with boundary Ω. The associated *disk model* of hyperbolic geometry works as follows:

- the *points* of the disk model consist of the points in the interior of D,
- for every circle ω which intersects Ω at a 90-degree angle, the set $\omega \cap D$ is a *hyperbolic line* of the disk model, and
- every point on the boundary Ω is a *point at infinity* (aka a *rimpoint*) of the disk model.

The *angle* between hyperbolic lines $\omega_1 \cap D, \omega_2 \cap D$ are computed in the disk model by computing the ordinary angle between ω_1 and ω_2 at their point of intersection inside D; distances are more complicated and will be defined later. The *symmetries* of the disk model are defined to be the set of Möbius transformations and complex conjugates of Möbius transformations of $\mathbb{CP}^1$ which send D bijectively to itself (note that these are all angle-preserving, so our definition of angles is compatible with our definition of symmetries).

In order to be a legitimate geometry, our model should satisfy some basic properties.

Proposition 1.2.14. *Suppose D is a disk in $\mathbb{CP}^1$, and let $P \neq Q$ be points in D. Then there is a unique hyperbolic line $\ell = \omega \cap D$ which goes through P and Q.*

Proof. We can assume without loss of generality that the boundary Ω of D is a straight line, by inverting around a point on Ω if necessary. Let p be the perpendicular bisector of PQ: if p intersects Ω at a finite point O then ω must be the circle with center O and radius OP. If p is parallel to Ω, then ω must be the line PQ. □

Proposition 1.2.15. *If D is a disk in $\mathbb{CP}^1$ and ℓ, m are hyperbolic lines of D, then ℓ and m intersect in at most one point of D.*

If ℓ meets the boundary of D at X and Y, and m meets the boundary of D at U and V, then ℓ and m intersect in the interior of D if and only if $(X,Y;U,V) < 0$.

Proof. Suppose that $P \in \ell \cap m$, and that $\ell = \alpha \cap D$ and $m = \beta \cap D$, where α, β are circles or lines in $\mathbb{CP}^1$. Then inversion around the center of the disk D (or reflecting across its boundary, if the boundary is a line) sends α and β to themselves by Proposition 1.2.14, so it sends P to the second intersection point between α and β. As a consequence, the second intersection point of α and β is either on the exterior of D, or is P itself (if P is on the boundary of D).

There are two very different ways to prove the second statement. The straightforward way is to apply a Möbius transformation which takes X to 0, Y to ∞, and U to 1, at which point the statement becomes obvious. The more visual way is to note that $(X,Y;U,V) < 0$ exactly when X and Y *separate* the points U and V along the boundary of the disk D. Therefore, if $(X,Y;U,V) < 0$, then the number of intersection points (counted with multiplicity) between *any* smooth path connecting X to Y within D and any smooth path connecting U to V within D must be odd, for purely topological reasons, while if $(X, Y; U, V) > 0$ then the number of intersection points within D must be even. Since the number of intersection points within D is at most one, this proves the second claim. □

Proposition 1.2.16. *Suppose D is a disk in $\mathbb{CP}^1$, P, Q are points in the interior of D, and ℓ, m are hyperbolic lines with $P \in \ell$ and $Q \in m$. Then there are exactly four symmetries of the disk model which take P to Q and ℓ to m.*

Proof. Let Ω be the boundary of D, let A and B be the intersections of ℓ with Ω, and let C and D be the intersections of m with Ω. Then there is a unique Möbius transformation f which takes A to C, B to D, and P to Q. Thus we have $f(\ell) = m$ and $f(P) = Q$, and we need to check that $f(\Omega) = \Omega$. By the Proposition 1.2.14, Ω is the unique circle or line which is perpendicular to ℓ at A and B. Therefore $f(\Omega)$ is the unique circle or line which is perpendicular to m at C and D, which is also Ω. The other symmetries which take P to Q and ℓ to m are the Möbius transformation which takes A to D, B to C, and P to Q, and the variants of these which involve complex conjugation. □

A consequence of the last proposition is that there are no symmetries of the hyperbolic plane which fix a point and a line through it, but rescale *distances* by a positive amount. So unlike Euclidean geometry and projective geometry, in hyperbolic geometry all symmetries will end up being distance-preserving, once we get around to defining what hyperbolic distance *is*.

The main advantages of the disk model of hyperbolic geometry are that angles are not distorted, and that the symmetries are easy to describe. A disadvantage is that if a painter was living in a three-dimensional hyperbolic space (defined as the interior of a three-dimensional ball in a similar way to the disk model), and if they were to paint what they saw as they looked at a geometric configuration in some two-dimensional hyperbolic plane (which would be a portion of a sphere which is perpendicular to the ball they lived within), then the hyperbolic lines in the picture they would paint would be perfectly straight, not curved. Of course, when a painter paints a picture of a plane, angles will generally *not* be preserved in their painting. So the true *projective* model of hyperbolic space will consist of the interior of a conic section, where the hyperbolic lines are perfectly straight - this is called the *Klein model* of hyperbolic space, and we will go over it later.

We will start investigating the geometry of hyperbolic space by looking at the least elegant model: the upper halfplane model. The reason for starting with this model is that the calculations involving distances and areas are easiest to describe in the upper halfplane.

Upper halfplane model

The upper halfplane is a disk in $\mathbb{CP}^1$, with boundary equal to the real line and interior corresponding to the points with positive imaginary parts. The hyperbolic lines of the upper halfplane model are just the upright semicircles which have their centers on the real line, together with the upright half-lines which are perpendicular to the real line. Points P of the upper halfplane model are often written in the form $x + iy$, where $x \in \mathbb{R}$ and $y > 0$.

What are the symmetries of the upper half-plane? Any Möbius transformation that takes the real line to itself must have the form $f : z \mapsto \frac{az+b}{cz+d}$, where a, b, c, d are all real numbers, with $ad \neq bc$. To see whether such a Möbius transformation takes the upper halfplane to itself, we just need to check whether it takes the point i to a point with positive imaginary part:

$$f(i) = \frac{ai+b}{ci+d} = \frac{(ai+b)(-ci+d)}{c^2+d^2} = \frac{ac+bd+(ad-bc)i}{c^2+d^2}.$$

Since $c^2 + d^2 > 0$, we see that $f(i)$ has positive imaginary part if and only if $ad - bc > 0$. Since multiplying all of a, b, c, d by the same thing doesn't change the Möbius transformation but does scale the value of $ad-bc$ by a square, people usually normalize the symmetries of the upper halfplane by assuming that $ad - bc = 1$ (this is still slightly redundant: if we negate all of a, b, c, d, we get the same Möbius transformation, and $ad - bc$ is still 1). This gives us a three-dimensional family of symmetries - just enough for the symmetries to be able to take any point and line through it to any other point and line through it.

Let's dig a little deeper into the symmetries of the upper halfplane. Suppose that $f : z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. What are the fixed points of f? Solving the equation

$$z = \frac{az+b}{cz+d},$$

we get

$$cz^2 + (d-a)z - b = 0,$$

so

$$z = \frac{a-d \pm \sqrt{(a-d)^2+4bc}}{2c} = \frac{a-d \pm \sqrt{(a+d)^2-4}}{2c}.$$

We get three different cases, depending on whether or not $|a + d|$ is greater than 2, less than 2, or equal to 2.

If $|a + d| > 2$, then the fixed points of f are both *real*, that is, they are *points at infinity* in the upper halfplane model. The hyperbolic line connecting these fixed points is then preserved by f. To understand this case better, we may as well assume that the fixed points of f are at 0 and ∞ (by applying a different Möbius transformation, if necessary). In this case we must have $b = c = 0$, and $d = 1/a$, so our Möbius transformation is just the map

$$z \mapsto a^2 z.$$

The fact that this map is supposed to preserve hyperbolic distances gives us a hint that along the hyperbolic line from 0 to ∞ (i.e., the positive part of the imaginary axis), distances will be related to the *logarithm* of the imaginary part.

If $|a + d| = 2$, then the Möbius transformation f has exactly one real fixed point, at $\frac{a-d}{2c}$. Again, we may as well assume that this fixed point is at ∞, in which case we must have $c = 0$ and $a = d = \pm 1$. If we take $a = d = +1$ (by negating b if necessary), then our Möbius transformation f is just the map

$$z \mapsto z + b.$$

Finally, if $|a + d| < 2$, then the Möbius transformation f has a pair of complex fixed points, which are conjugates of each other. Exactly one of these fixed points will be in the upper halfplane. We may as well assume that this fixed point is i, in which case the formula for $f(i)$ we had earlier implies that $c^2 + d^2 = 1$ and $ac + bd = 0$. A little algebra shows that we must have $a = d$ and $b = -c$, so we can write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

for some angle θ. This can be thought of as the hyperbolic geometry analogue of a counterclockwise rotation around i - but it will be a rotation of angle 2θ, not θ, since flipping the signs of all of the entries a, b, c, d does not change the Möbius transformation. Note that if we take $\theta = \pi/2$, so that $2\theta = \pi$, then we see that the analogue of a 180 degree rotation around i is the Möbius transformation

$$z \mapsto -1/z.$$

Since rotations around i should certainly preserve the distance to i, this gives us another hint that hyperbolic distances along the positive part of the imaginary axis will be related to logarithms. In fact, we can now justify the claim that the scaling map $z \mapsto az$ should preserve hyperbolic distances: we can build this map by composing the 180 degree rotation $z \mapsto -1/z$ around i with the 180 degree rotation $z \mapsto -a/z$ around ai.

Now let's think seriously about how distances should be defined in hyperbolic geometry. Let's start by thinking about infinitesimal distances: we start from a point $x+iy$, and change x by dx and y by dy. Let ds be the corresponding infinitesimal amount of hyperbolic distance that we travel. In ordinary Euclidean geometry, we would have $ds^2 = dx^2 + dy^2$ by the Pythagorean theorem. In a general "smooth" geometry, we might instead have

$$ds^2 = \alpha(x, y)\ dx^2 + 2\beta(x, y)\ dx\ dy + \gamma(x, y)\ dy^2,$$

where α, β, γ could be any (smooth) functions we like of x and y, subject to the conditions $\alpha > 0, \gamma > 0$, and $\alpha\gamma > \beta^2$ (to guarantee that the right hand side is always positive). Since $z \mapsto z + b$ is a symmetry of our geometry, we immediately see that the functions α, β, γ can't depend on x, and are only functions of y:

$$ds^2 = \alpha(y)\ dx^2 + 2\beta(y)\ dx\ dy + \gamma(y)\ dy^2.$$

Since the map $z \mapsto az$ is a symmetry of our geometry which should preserve distances, we see that in fact $\alpha(y), \beta(y), \gamma(y)$ should all be proportional to $1/y^2$, so we can write

$$ds^2 = \frac{\alpha\ dx^2 + 2\beta\ dx\ dy + \gamma\ dy^2}{y^2}$$

for some constants α, β, γ. To compute α, β, γ, we may as well assume that $x+iy = i$, that is, $x = 0$ and $y = 1$. Since the negated complex conjugation $z \mapsto -\bar{z}$ is a symmetry of the upper halfplane, we see that when $x + iy = i$ the map $(dx, dy) \mapsto (dx, -dy)$ has to preserve distances, so $\beta = 0$. To figure out the relationship between α and γ, we consider the hyperbolic 90 degree rotation around i, which is given by

$$z \mapsto \frac{1+z}{1-z}.$$

Plugging in $z = dx + i$ (and $dy = 0$) and expanding to first order in dx (i.e. ignoring larger powers of dx), we get

$$\frac{1 + dx + i}{1 - dx - i} = \frac{(1 + dx + i)(1 - dx + i))}{1 + (1 - dx)^2} = \frac{2i}{2 - 2dx} = i(1 + dx),$$

so this 90 degree rotation turns an infinitesimal step in the real direction into an infinitesimal step in the imaginary direction of the same length. This shows that we must have $\alpha = \gamma$, and we may as well take $\alpha = 1$, in which case our formula for infinitesimal distances becomes

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y}.$$

Let's check that this formula really is compatible with our symmetries.

Proposition 1.2.17. *Suppose that $x + iy$ is in the upper halfplane, and let $f : z \mapsto \frac{az+b}{cz+d}$ be a Möbius transformation with $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$. If*

$$f(x + dx + i(y + dy)) = u + du + i(v + dv)$$

to first order, then we have

$$\frac{\sqrt{dx^2 + dy^2}}{y} = \frac{\sqrt{du^2 + dv^2}}{v}.$$

Proof. First we find u and v:

$$\frac{a(x + iy) + b}{c(x + iy) + d} = \frac{(ax + b)(cx + d) + acy^2 + i(ad - bc)y}{(cx + d)^2 + c^2y^2} = \frac{(ax + b)(cx + d) + acy^2}{(cx + d)^2 + c^2y^2} + \frac{iy}{(cx + d)^2 + c^2y^2}.$$

In particular, we have

$$v = \frac{y}{(cx + d)^2 + c^2y^2}.$$

Now let $z = x + iy$, so to first order we have

$$\frac{az + b + a\ dz}{cz + d + c\ dz} = \frac{az + b}{cz + d} + \frac{a(cz + d) - (az + b)c}{(cz + d)^2}\ dz = u + iv + \frac{dz}{(cz + d)^2}.$$

Expanding out the last term, we get

$$du + i\ dv = \frac{dx + i\ dy}{(cx + d + icy)^2},$$

so

$$|du + i\ dv| = \frac{|dx + i\ dy|}{(cx + d)^2 + c^2y^2}.$$

Thus we have

$$\frac{\sqrt{du^2 + dv^2}}{v} = \frac{\sqrt{dx^2 + dy^2}}{(cx + d)^2 + c^2y^2} \Big/ \frac{y}{(cx + d)^2 + c^2y^2} = \frac{\sqrt{dx^2 + dy^2}}{y}.$$ □

Intuitively, the formula for infinitesimal distances can be thought of as saying the following:

As you get closer to the real line, you become smaller, in proportion to the imaginary part of your current position.

In particular, if we were to try to walk directly towards the real line, we would find ourselves shrinking as we did so, and as a result we would never be able to actually reach the real line. This is why we can think of the real line as the collection of "points at infinity" of the hyperbolic plane. Additionally, the shortest path between two points with the same imaginary parts could contain a detour through points with larger imaginary part: when we walk away from the real line, we get larger, so we can travel more quickly.

Now that we've figured out what infinitesimal distances should look like, we can figure out what the shortest path between more distant points looks like. We start by considering the easiest case: distances along the positive part of the imaginary axis.

Proposition 1.2.18. *If infinitesimal distances are given by* $ds = \frac{\sqrt{dx^2+dy^2}}{y}$, *then the shortest path from* ai *to* bi *travels directly along the positive imaginary axis and has length* $|\log(b/a)|$.

Proof. Moving the real part back and forth on a path from ai to bi obviously can only increase the total distance traveled, so it's always best to travel directly along the positive imaginary axis. If $b > a$, then the length of this path is given by

$$\int_a^b \frac{dy}{y} = \log(b) - \log(a). \qquad \square$$

Using the fact that infinitesimal distances are preserved by symmetries, we can now understand the general case of shortest paths in hyperbolic space.

Theorem 1.2.19. *If infinitesimal distances are given by* $ds = \frac{\sqrt{dx^2+dy^2}}{y}$, *then the shortest path from* P *to* Q *travels directly along the hyperbolic line* ℓ *connecting* P *to* Q*. If* ℓ *meets the real line at* X *and* Y*, then the length of this path is equal to*

$$\left|\log\Big((P,Q;X,Y)\Big)\right|.$$

Proof. Let f be a Möbius transformation which takes the upper halfplane to itself, maps P to i, and maps ℓ to the positive imaginary axis. Since $Q \in \ell$, Q is mapped to ai for some real number a. Since f preserves infinitesimal distances, f turns a shortest path from P to Q into a shortest path from i to ai, and the length of these shortest paths are equal. Since the shortest path from i to ai travels directly along $f(\ell)$, the shortest path from P to Q must travel directly along ℓ. Since f preserves cross ratios, the length of this path is given by

$$|\log(a)| = \left|\log\Big((i,ai;0,\infty)\Big)\right| = \left|\log\Big((f(P),f(Q);f(X),f(Y))\Big)\right| = \left|\log\Big((P,Q;X,Y)\Big)\right|. \qquad \square$$

Now we can make the result of the previous theorem into a definition.

Definition 1.2.20. If D is a disk in $\mathbb{CP}^1$, points P, Q are in the interior of D, and the hyperbolic line ℓ through P and Q meets the boundary of D at points X and Y, then the *hyperbolic distance* from P to Q is defined to be

$$\delta_D(P,Q) = \left|\log\Big((P,Q;X,Y)\Big)\right|.$$

Corollary 1.2.21. *If* D *is a disk in* $\mathbb{CP}^1$ *and points* P, Q, R *are in the interior of* D*, then the hyperbolic distances between* $P, Q,$ *and* R *satisfy the triangle inequality:*

$$\delta_D(P,R) \leq \delta_D(P,Q) + \delta_D(Q,R).$$

Exercise 1.2.6. Prove the triangle inequality for hyperbolic distance directly from its definition.

We can also relate *angles* in hyperbolic geometry to cross ratios between points at infinity.

Theorem 1.2.22. *If* ℓ, m *are intersecting hyperbolic lines in the upper halfplane model such that* ℓ *meets the real line at* X *and* Y *and* m *meets the real line at* U *and* V*, and if* ℓ *is directed from* X *to* Y *and* m *is directed from* U *to* V*, then the angle* θ *between* ℓ *and* m *satisfies*

$$\cos(\theta) = \frac{1 + (X,Y;U,V)}{1 - (X,Y;U,V)}.$$

In particular, the hyperbolic lines ℓ *and* m *meet at a right angle if and only if* X, Y, U, V *are harmonic.*

Proof. We may assume without loss of generality that X and Y are 0 and ∞, i.e. that ℓ is the positive part of the imaginary axis. Let u, v be the real numbers corresponding to the points U and V, and note that ℓ and m intersect if and only if u and v have opposite signs. Computing the power of the point 0 with respect to the circle with diameter UV in two different ways, we see that ℓ and m intersect in the point $i\sqrt{-uv}$. Some angle chasing reveals that the angle θ between ℓ and the tangent to m at $i\sqrt{-uv}$ is equal to twice the angle of the right triangle formed by U, V, and $i\sqrt{-uv}$ at V, which is also equal to the angle of the right triangle formed by 0, V, and $i\sqrt{-uv}$ at V. Thus we have

$$|\tan(\theta/2)| = \frac{\sqrt{-uv}}{|v|} = \sqrt{-\frac{u}{v}} = \sqrt{-(0,\infty;u,v)} = \sqrt{-(X,Y;U,V)}.$$

To finish, we apply the formula

$$\cos(\theta) = \frac{1-\tan^2(\theta/2)}{1+\tan^2(\theta/2)}. \qquad \square$$

Chapter 2

Inequalities

2.1 Mechanical procedures

2.1.1 Quadratic inequalities: Keep completing the square!

Suppose someone hands you a quadratic polynomial in several variables, such as

$$x^2 + 2xy - 2xz + 2y^2 + 2yz + 6z^2 - z + 1,$$

and asks you to check whether it is always ≥ 0. How do you do it?

The trick to this is a slight generalization of the high school procedure known as "completing the square", which I like to call "keep completing the square" (I stumbled on this method after meditating on what the Cholesky decomposition really *meant* in terms of quadratic polynomials). We start by trying to write down a square that agrees with our polynomial at least as far as x is concerned, that is, we try to solve the equation

$$(x + Ay + Bz + C)^2 = x^2 + 2xy - 2xz + ...,$$

for A, B, C (and ignoring the ..., since it doesn't involve x). In this case, we can take $A = 1, B = -1, C = 0$, and we get

$$(x + y - z)^2 = x + 2xy - 2xz + y^2 - 2yz + z^2.$$

Since that doesn't completely match our polynomial, we look at the difference:

$$(x^2 + 2xy - 2xz + 2y^2 + 2yz + 6z^2 - z + 1) - (x + y - z)^2 = y^2 + 4yz + 5z^2 - z + 1.$$

Now we complete the square again, this time with y, and so on. Writing the whole process in one string of equalities, we get

$$\begin{aligned} x^2 + 2xy - 2xz + 2y^2 + 2yz + 6z^2 - 2z + 1 &= (x + y - z)^2 + y^2 + 4yz + 5z^2 - z + 1 \\ &= (x + y - z)^2 + (y + 2z)^2 + z^2 - z + 1 \\ &= (x + y - z)^2 + (y + 2z)^2 + (z - \tfrac{1}{2})^2 + \tfrac{3}{4}, \end{aligned}$$

and this is clearly positive, since it is a sum of squares.

Let's do a more complicated example (the previous example was clearly chosen to let you avoid taking any square roots). What if we are faced with something like

$$6x^2 - 4xy + 2xz + 3y^2 - 4yz + 2z^2?$$

At the very first step, it seems like we'll have to take the square root of 6. What a mess! Here's how to avoid the mess: instead of starting with a square like

$$(\sqrt{6}x + Ay + Bz)^2,$$

instead we start by looking for something like

$$6(x + Ay + Bz)^2.$$

Now we can find A, B by simple division, and we get $A = -\frac{1}{3}, B = \frac{1}{6}$. Continuing, we get

$$\begin{aligned} 6x^2 - 4xy + 2xz + 3y^2 - 4yz + 2z^2 &= 6(x - \tfrac{1}{3}y + \tfrac{1}{6}z)^2 + \tfrac{7}{3}y^2 - \tfrac{10}{3}yz + \tfrac{11}{6}z^2 \\ &= 6(x - \tfrac{1}{3}y + \tfrac{1}{6}z)^2 + \tfrac{7}{3}(y - \tfrac{5}{7}z)^2 + \tfrac{9}{14}z^2, \end{aligned}$$

which is again obviously positive since it has been written as a sum of squares with positive coefficients. (By the way, I came up this polynomial by expanding out $(x - y)^2 + (x + y - z)^2 + (2x - y + z)^2$ - so we see that there can be multiple ways to write the same polynomial as a sum of squares. If we had processed the variables in a different order, we could come up with yet another way to write it as a sum of squares!)

What happens if we try to do this to a quadratic polynomial which *isn't* always ≥ 0? Obviously, something has to go wrong. Let's try the polynomial

$$x^2 - 4xy + 2xz + y^2 - 2yz + 2z^2.$$

The first step goes just fine: we get

$$x^2 - 4xy + 2xz + y^2 - 2yz + 2z^2 = (x - 2y + z)^2 - 3y^2 + 2yz + z^2.$$

But now we have a problem: the coefficient of y^2 is negative. Could our polynomial still be ≥ 0? Maybe the z^2 and the $(x - 2y + z)^2$ somehow always conspire to be larger than $3y^2$? Nope! To see why, just set z to 0, and choose x to make $x - 2y + z$ equal to 0, for instance, take $z = 0, y = 1, x = 2$.

In the previous example, we had a problem because the coefficient of y^2 was negative. What if the coefficient of y^2 comes out to exactly 0? For an example, let's consider the polynomial

$$x^2 - 2xy - 2xz + y^2 - 2yz + 10z^2.$$

After the first step, we get

$$x^2 - 2xy - 2xz + y^2 - 2yz + 2z^2 = (x - y - z)^2 - 4yz + 9z^2.$$

To show that this sometimes goes negative, we will take z to be whatever nonzero value we like - say, take $z = 1$ - and then pick y to make $-4yz + 9z^2$ come out negative (we can do this since, for any fixed nonzero z, $-4yz + 9z^2$ is a linear function of y with a nonzero y-coefficient), and finally pick x to make $x - y - z$ equal to 0. For instance, we can take $z = 1, y = 3, x = 4$.

At the end of the day, we have a procedure that starts with a quadratic polynomial in any number of variables, and either writes it as a sum of squares with positive coefficients, or spits out a point where it is negative! We summarize in the following theorem.

Theorem 2.1.1. *Suppose that* $Q(x_1, ..., x_n) = \sum_{i,j} a_{ij}x_ix_j + \sum_i a_ix_i + a$*, where* a_{ij}, a_i, a *are some coefficients. Then either we can write* Q *in the form*

$$Q(x_1, ..., x_n) = \sum_{i=1}^{n} c_i(x_i + b_{i(i+1)}x_{i+1} + \cdots + b_{in}x_n + b_i)^2 + c$$

with $c_i \geq 0$ *for all* i *and* $c \geq 0$*, or else we can find a point* $(x_1, ..., x_n)$ *such that* $Q(x_1, ..., x_n) < 0$*.*

In the case of homogeneous quadratic polynomials, people often like to represent their coefficients in a symmetric matrix. In the three variable case, the matrix

$$\begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}$$

corresponds to the polynomial

$$ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2.$$

Why the random factors of 2? This is because we have the nice formula

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2.$$

When we follow the "keep completing the square" procedure for this general three variable homogeneous quadratic, we get

$$\begin{aligned} ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2 &= a(x + \tfrac{b}{a}y + \tfrac{d}{a}z)^2 + \tfrac{ac-b^2}{a}y^2 + 2\tfrac{ae-bd}{a}yz + \tfrac{af-d^2}{a}z^2 \\ &= a(x + \tfrac{b}{a}y + \tfrac{d}{a}z)^2 + \tfrac{ac-b^2}{a}(y + \tfrac{ae-bd}{ac-b^2}z)^2 + \tfrac{(af-d^2)(ac-b^2)-(ae-bd)^2}{a(ac-b^2)}z^2 \\ &= a(x + \tfrac{b}{a}y + \tfrac{d}{a}z)^2 + \tfrac{ac-b^2}{a}(y + \tfrac{ae-bd}{ac-b^2}z)^2 + \tfrac{acf+2bde-ae^2-b^2f-cd^2}{ac-b^2}z^2. \end{aligned}$$

Curiously, the coefficients in that last formula happen to be ratios of determinants:

$$\begin{aligned} \det \begin{bmatrix} a \end{bmatrix} &= a, \\ \det \begin{bmatrix} a & b \\ b & c \end{bmatrix} &= ac - b^2, \\ \det \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} &= acf + 2bde - ae^2 - b^2f - cd^2. \end{aligned}$$

So we've proved that a three variable homogeneous quadratic is ≥ 0 if those three determinants are all positive!

Exercise 2.1.1. Generalize this determinant formula to any number of variables.

2.1.2 Systems of linear inequalities: Fourier-Motzkin Elimination

Suppose that someone hands you a system of linear inequalities in several variables, such as

$$\begin{aligned} 2x + y &\le 2z, \\ x + z &\le 2y + 1, \\ x + 3 &\le 2y, \\ 3x &\le y + z, \\ y + z &\le 2x + 3, \end{aligned}$$

and asks you whether or not this system of inequalities has a solution. In fact, to make it more interesting, suppose they ask you to find all possible values of x which can occur in a solution (x, y, z) to this system of inequalities. How do you do it?

There are several different ways to solve this sort of problem, but the simplest and most direct method is known as *Fourier-Motzkin elimination.* The idea is to pick one of the variables and to *eliminate* it, getting a system of linear inequalities in the remaining variables which captures every last bit of information that doesn't involve the eliminated variable.

Let's start by trying to eliminate the variable z from our system of linear inequalities. To start, we rearrange all of our inequalities into three different categories based on how they involve z.

- Some of the inequalities give us *lower bounds* on z. In our example, the first and fourth inequalities give us lower bounds on z, and we rewrite them to make this more obvious:

$$\begin{aligned} x + y/2 &\le z, \\ 3x - y &\le z. \end{aligned}$$

- Some of the inequalities give us *upper bounds* on z. In our example, the second and fifth inequalities give us lower bounds on z, and we rewrite them to make this more obvious:

$$\begin{aligned} z &\le -x + 2y + 1, \\ z &\le 2x - y + 3. \end{aligned}$$

- Some of the inequalities don't involve z at all. In our example, the third inequality didn't involve z at all:

$$x + 3 \le 2y.$$

To eliminate z, we need to figure out which pairs of values (x, y) allow us to pick a z which satisfies all three types of inequalities above. If (x, y) satisfy all of the inequalities that don't involve z, then obviously the only possible thing that can go wrong when we try to find a value for z is that one of the lower bounds for z might be bigger than one of the upper bounds for z. As long as each lower bound for z is at most as large as each upper bound for z, we will be fine. So the new system of inequalities, after we eliminate z, is just

$$\begin{aligned} x + 3 &\le 2y, \\ x + y/2 &\le -x + 2y + 1, \\ x + y/2 &\le 2x - y + 3, \\ 3x - y &\le -x + 2y + 1, \\ 3x - y &\le 2x - y + 3. \end{aligned}$$

As long as x and y satisfy this new system of inequalities, we can pick any z which satisfies

$$\max(x + y/2, 3x - y) \leq z \leq \min(-x + 2y + 1, 2x - y + 3)$$

to solve the original system of linear inequalities. We have successfully eliminated the variable z!

Now we slightly rearrange each one of our new linear inequalities to clear denominators and so on:

$$\begin{aligned} x + 3 &\leq 2y, \\ 4x &\leq 3y + 2, \\ 3y &\leq 2x + 6, \\ 4x &\leq 3y + 1, \\ x &\leq 3. \end{aligned}$$

The second of these is clearly redundant, so we can forget about it. Now we want to eliminate y from our new system of linear inequalities. Once again, we divide our inequalities into three different categories based on how they involve y.

- Some of the inequalities give us *lower bounds* on y. In our new system of linear inequalities, the first and fourth inequalities give us lower bounds on y, and we rewrite them to make this more obvious:

$$\begin{aligned} x/2 + 3/2 &\leq y, \\ 4x/3 - 1/3 &\leq y. \end{aligned}$$

- Some of the inequalities give us *upper bounds* on y. In our new system of linear inequalities, the third inequality gives us a lower bound on y, and we rewrite it to make this more obvious:

$$y \leq 2x/3 + 2.$$

- Some of the inequalities don't involve y at all. In our new system of linear inequalities, the fifth inequality didn't involve y at all:

$$x \leq 3.$$

Once again, we keep every inequality that doesn't involve y at all, and we compare every lower bound on y to every upper bound on y. This gives us the following system of linear inequalities involving only x:

$$\begin{aligned} x &\leq 3, \\ x/2 + 3/2 &\leq 2x/3 + 2, \\ 4x/3 - 1/3 &\leq 2x/3 + 2. \end{aligned}$$

As long as x satisfies this system of linear inequalities, we can pick any y which satisfies

$$\max(x/2 + 3/2, 4x/3 - 1/3) \leq y \leq 2x/3 + 2.$$

We have now successfully eliminated both y and z!

Now we slightly rearrange our inequalities on x:

$$\begin{aligned} x &\leq 3, \\ 0 &\leq x + 3, \\ 2x &\leq 7. \end{aligned}$$

The third inequality on x is clearly redundant, and we see that the possible values for x are exactly the values of x between -3 and 3!

Now suppose that we want to convince a skeptical friend that the possible values for x are exactly the values between -3 and 3. First, we verify that we really can fill in values for y and z when x is -3 or 3. If x is -3, then y can be any value which satisfies

$$0 = \max(x/2 + 3/2, 4x/3 - 1/3) \leq y \leq 2x/3 + 2 = 0,$$

so we have to choose $y = 0$. Then z can be any value which satisfies

$$-3 = \max(x + y/2, 3x - y) \leq z \leq \min(-x + 2y + 1, 2x - y + 3) = -3,$$

so we have to choose $z = -3$. So we show our friend that $(x, y, z) = (-3, 0, -3)$ solves our system of linear inequalities. A similar calculation leads us to the fact that $(x, y, z) = (3, 4, 5)$ also solves our system of linear inequalities. By taking convex combinations of these two solutions, we see that for any $x \in [-3, 3]$, the point

$$(x, y, z) = (x, 2x/3 + 2, 4x/3 + 1)$$

will solve our system of linear inequalities.

How do we convince our skeptical friend that x can't be bigger than 3 or smaller than -3? We just work backwards through our elimination procedure to see how we derived the inequalities $x \leq 3$ and $0 \leq x + 3$. For the upper bound $x \leq 3$, we see that this was one of the bounds which didn't involve y at all, after we had eliminated z, and that we had obtained it by simplifying the inequality

$$3x - y \leq 2x - y + 3.$$

This inequality was obtained by comparing the lower bound $3x - y \leq z$, which corresponded to the fourth inequality in our original system of inequalities, to the upper bound $z \leq 2x - y + 3$, which corresponded to the fifth inequality of our original system of inequalities. So we can simply tell our friend that we added together the inequalities

$$\begin{aligned} 3x &\leq y + z, \\ y + z &\leq 2x + 3 \end{aligned}$$

and simplified, to deduce the upper bound $x \leq 3$. Note that each of these two inequalities has equality when $(x, y, z) = (3, 4, 5)$.

As for the lower bound $0 \leq x + 3$, this was a simplification of the inequality

$$x/2 + 3/2 \leq 2x/3 + 2$$

which we found by rearranging and multiplying both sides by 6. That inequality, in turn, was obtained by comparing the lower bound $x/2 + 3/2 \leq y$ to the upper bound $y \leq 2x/3 + 2$, and these two inequalities were rescaled versions of the inequalities $x + 3 \leq 2y$ and $3y \leq 2x + 6$. So the inequality $0 \leq x + 3$ follows from adding together the two inequalities

$$\begin{aligned} 3 \times \Big(x + 3 \leq 2y \qquad\Big),\\ 2 \times \Big(\quad 3y \leq 2x + 6\Big).\end{aligned}$$

The inequality $x + 3 \leq 2y$ was one of the original inequalities which didn't involve z, while the inequality $3y \leq 2x + 6$ was a rescaled version of the inequality $x + y/2 \leq 2x - y + 3$. The inequality $x + y/2 \leq 2x - y + 3$ was built out of the inequalities $2x + y \leq 2z$ and $y + z \leq 2x + 3$ by eliminating z - that is, we derived it by adding the inequalities

$$\begin{aligned} 2x + y \leq 2z,\\ 2 \times \Big(\ y + z \leq 2x + 3\Big).\end{aligned}$$

All together, we see that the inequality $0 \leq x + 3$ was derived from the original system of linear inequalities by adding together the following three multiples of the first, third, and fifth inequalities from our original system of linear inequalities:

$$\begin{aligned} 2 \times \Big(2x + y \leq 2z \qquad\Big),\\ 3 \times \Big(\ x + 3 \leq 2y \qquad\Big),\\ 4 \times \Big(\ y + z \leq 2x + 3\Big).\end{aligned}$$

Note that each of these three inequalities has equality when $(x, y, z) = (-3, 0, -3)$.

It's also possible to deal with mixtures of linear inequalities and linear equations - in fact, as long as there is at least one nontrivial linear equation around, we can use it to eliminate a variable directly, as in Gaussian elimination. We can summarize the main idea behind this procedure in the following result.

Theorem 2.1.2 (Fourier-Motzkin Elimination)**.** *Suppose that we have a system S_n of m linear inequalities and linear equations in n unknowns $x_1, ..., x_n$. Then we can find a new system S_{n-1} of at most* $\max(m, m^2/4)$ *linear inequalities and linear equations in the $n - 1$ unknowns $x_1, ..., x_{n-1}$, with the following properties:*

- *the values $(x_1, ..., x_{n-1})$ satisfy the new system S_{n-1} if and only if there is some x_n such that the values $(x_1, ..., x_{n-1}, x_n)$ satisfy the original system S_n, and*
- *every linear inequality or linear equation in the new system S_{n-1} is either one of the inequalities/equations from S_n which did not involve the variable x_n, or can be written as a weighted combination of two inequalities/equations from S_n with weights chosen to cancel out the coefficient of the variable x_n.*

Corollary 2.1.3. *If a system of linear inequalities and linear equations has no solutions, then by summing multiples of these inequalities and equations, we can derive a false inequality*

$$a \leq b,$$

where a, b are constants with $a > b$. Equivalently, in this case we can derive the false inequality

$$1 \leq 0$$

by summing multiples of our inequalities and equations.

Corollary 2.1.4. *If a system S of linear inequalities and linear equations in the variables $x_1, ..., x_n$ implies the inequality $x_1 \leq c$ for some constant c, then this inequality can be derived by summing multiples of the inequalities and equations from S.*

If there is also a solution $x^* = (x_1^*, ..., x_n^*)$ to the system S which has $x_1^* = c$, then every single inequality which occurs in the sum which we used to derive the inequality $x_1 \leq c$ must have equality at the point x^*.

By changing variables, we can prove the following stronger-looking result.

Corollary 2.1.5. *If a system S of linear inequalities and linear equations in the variables $x_1, ..., x_n$ implies a linear inequality $\sum_i a_i x_i \leq c$, then this inequality can be derived by summing multiples of the inequalities and equations from S.*

If the number m of inequalities and the number n of variables in our original system of linear inequalities are both very large, then the systems of inequalities produced by Fourier-Motzkin elimination can grow out of control. The *simplex method* is a better method for solving larger systems - the idea is to examine points where n of the inequalities have equality, and to compare them to "neighboring" points which have equality at a slightly different collection of n of the inequalities, where $n - 1$ of the inequalities are the same as before and one is new. More advanced procedures, such as Khachiyan's *ellipsoid method* and Karmarkar's *interior point algorithm* are based on finding approximate solutions to high accuracy and then rounding them - the ellipsoid method is the basis of an important theoretical result about the computational complexity of convex optimization, and interior point algorithms are fast even for enormous problems.

When discussing large systems of linear inequalities, it's convenient to use the notation of linear algebra. We package our collection of variables $x_1, ..., x_n$ into a column vector $x \in \mathbb{R}^n$, and we package the system of m linear inequalities into the inequality

$$Ax \leq b,$$

where $A \in \mathbb{R}^{m \times n}$ is an $m \times n$ matrix whose m rows correspond to the individual inequalities, and $b \in \mathbb{R}^m$ is a column vector whose entries correspond to the constants which show up in the linear inequalities. If $y \in \mathbb{R}^m$ is a column vector of *weights* which satisfies $y \geq 0$, then the weighted combination of the inequalities which corresponds to y is given by

$$y^T A x \leq y^T b,$$

where y^T is the *transpose* of y (which is a row vector - note that $y^T b$ is just another way of writing the dot product $y \cdot b$). We can rephrase what we have proved so far in this language, as follows.

Theorem 2.1.6 (Theorem of the Alternatives)**.** *The system of inequalities $Ax \leq b$ has no solution $x \in \mathbb{R}^n$ if and only if there is some vector $y \in \mathbb{R}^m$ such that*

- $y \geq 0,$

- $A^T y = 0$ *(or equivalently* $y^T A = 0^T$*), and*
- $b^T y < 0$ *(or equivalently* $y^T b < 0$*).*

An alternative standard way of expressing systems of linear equations and inequalities, which is more useful in some applications, is to introduce new variables which correspond to the amount of "slack" that we have in each inequality - so each new variable corresponds to the difference between the right hand side and the left hand side of one of our original inequalities - and to work out a system of equations that has to be satisfied by these "slack" variables. This gives us a possibly complicated system of equations, together with a very simple system of inequalities:

$$\begin{aligned} Ax &= b, \\ x &\geq 0. \end{aligned}$$

The same argument that gave us the Theorem of the Alternatives gives us an analogous result for systems of this form, known as Farkas' Lemma.

Lemma 2.1.7 (Farkas Lemma)**.** *The system* $Ax = b, x \geq 0$ *has no solution if and only if there is some vector* y *such that* $A^T y \geq 0$ *and* $b^T y < 0$*.*

2.1.3 Single-variable polynomials: Sturm chains

Suppose you are handed a single-variable polynomial, such as

$$p(x) = x^3 - 6x^2 + 4x + 12$$

and you are asked to determine whether this polynomial is positive for all x between -1 and 2. How can you accomplish this?

As it turns out, there is a general procedure which allows you to work out the exact number of real roots of any real polynomial in any half-open interval $(a, b]$, using only pencil and paper. The only special fact we will need about the real numbers (as opposed to, say, the rational numbers) is the following weak version of the intermediate value theorem.

Theorem 2.1.8 (Intermediate Value Theorem for Polynomials)**.** *If* $p(x)$ *is a real polynomial, and if* $p(a)$ *and* $p(b)$ *have opposite signs for some real values* $a < b$*, then there is some real number* $c \in (a, b)$ *such that* $p(c) = 0$*.*

Proof. This is a special case of the intermediate value theorem, which applies to all continuous functions (the fact that polynomials are continuous follows from the binomial theorem and the triangle inequality). The standard proof is as follows: supposing without loss of generality that $p(a) < 0$ and $p(b) > 0$, we define the set $S \subseteq [a, b]$ by

$$S = \{x \in [a, b] \mid p(x) \leq 0\}.$$

Since S is a nonempty, bounded subset of $\mathbb{R}$, it has a supremum (by the defining property of the real numbers - in some developments this is an axiom, in others it is proved in terms of a particular construction of the real number system), so we take $c = \sup S$. Since p is continuous, we must have $p(c) \leq 0$, since there are values x which are arbitrarily close to c satisfying $p(x) \leq 0$. From $p(c) \leq 0$ we conclude $c \neq b$, so for every positive $\epsilon < b - c$ we have $p(c + \epsilon) > 0$ (otherwise c would be less than $\sup S$), and applying the continuity of p once more we see that $p(c) \geq 0$ as well, so we must have $p(c) = 0$. □

Versions of Theorem 2.1.8 are true for other number systems as well, such as the collection of *algebraic* real numbers (recall that a number is called algebraic if it is a root of a polynomial which has integer coefficients), or certain number systems involving infinitesimals or power series. An ordered number system which satisfies a version of Theorem 2.1.8 is called a *real closed field*, and for the purpose of studying algebraic inequalities, all real closed fields are essentially indistinguishable from one another.

We will also need a basic algebraic fact about derivatives, which follows from the binomial theorem and is true in any ordered number system.

Proposition 2.1.9. *If the polynomial $p(x)$ is given by*

$$p(x) = \sum_{i=0}^{d} a_i x^i$$

and its derivative *$p'(x)$ is defined by*

$$p'(x) = \sum_{i=1}^{d} i a_i x^{i-1},$$

then there is a two-variable polynomial $r(x, y)$ such that

$$p(x + y) = p(x) + y \cdot p'(x) + y^2 \cdot r(x, y).$$

In particular, if $p'(x) > 0$, then there is some $\epsilon > 0$ such that p is strictly increasing in the interval $(x - \epsilon, x + \epsilon)$.

Ok, back to the original problem: we want to count the number of roots of $p(x)$ between -1 and 2. A good starting point is to compute $p(-1)$ and $p(2)$ (which come out to $p(-1) = 1$ and $p(2) = 4$, in this example problem), and to check whether they have the same sign or not. Since $p(-1)$ and $p(2)$ are both positive, it seems like we can conclude that $p(x)$ has an even number of roots between -1 and 2. Or can we?

In order to definitively conclude that p has an even number of roots between -1 and 2 from the fact that $p(-1)$ and $p(2)$ are both positive, we need to either be certain that p doesn't have any double (or triple, etc.) roots between -1 and 2, or to be careful to count any multiple roots "with multiplicity". For instance, the polynomial

$$q(x) = (x - 1)^2(x + 2) = x^3 - 3x + 2$$

has $q(-1) = 4$ and $q(2) = 4$, and it has a double root at $x = 1$. In order the detect this multiple root, we use the fact that any multiple root of a polynomial $q(x)$ will also be a root of the derivative $q'(x)$, e.g.:

$$\begin{aligned} q'(x) &= 2(x - 1)(x + 2) + (x - 1)^2 \\ &= (x - 1)(2(x + 2) + (x - 1)) \\ &= 3(x^2 - 1). \end{aligned}$$

If we didn't already know the factorization of $q(x)$, then we could use the Euclidean algorithm to compute the gcd of $q(x)$ and $q'(x)$:

$$\begin{aligned} \gcd(q(x), q'(x)) &= \gcd(x^3 - 3x + 2,\ 3(x^2 - 1)) \\ &= \gcd(-2(x-1),\ 3(x^2-1)) \\ &= \gcd(-2(x-1),\ 0) \\ &= x - 1. \end{aligned}$$

Using the Euclidean algorithm several times, it is always possible to split any polynomial into pieces which correspond to the roots of various multiplicities. We state this result semi-formally below.

Proposition 2.1.10. *If $p(x)$ is a polynomial with coefficients from a real closed field, then we can use the Euclidean algorithm several times to write p in the form*

$$p(x) = c \cdot p_1(x)^{m_1} p_2(x)^{m_2} \cdots p_k(x)^{m_k},$$

where c is a constant, each $p_i(x)$ has no common factor with its derivative $p_i'(x)$, and no $p_i(x)$ has a common factor with $p_j(x)$ for any $i \neq j$.

So we only need to worry about the case where $p(x)$ has no common factor with its derivative $p'(x)$ (such a polynomial is called *squarefree*). Let's check that our running example $p(x) = x^3 - 6x^2 + 4x + 12$ has this property:

$$\begin{aligned} \gcd(p(x), p'(x)) &= \gcd(x^3 - 6x^2 + 4x + 12,\ 3x^2 - 12x + 4) \\ &= \gcd(-(4/3)(4x - 11),\ 3x^2 - 12x + 4) \\ &= \gcd(-(4/3)(4x - 11),\ -101/16) \\ &= 1. \end{aligned}$$

So now we can conclude that $p(x)$ has an even number of roots between -1 and 2, and that the issue of multiplicity is not relevant to in this case. It seems that looking at the derivative $p'(x)$ is generally helpful for thinking about the roots, so we take a peek at the values of $p'(-1)$ and $p'(2)$: in this example, we have $p'(-1) = 19$ and $p'(2) = -8$, so $p(x)$ is increasing around $x = -1$ and $p(x)$ is decreasing around $x = 2$. Additionally, the Polynomial Intermediate Value Theorem 2.1.8 tells us that since $p'(-1)$ and $p'(2)$ have opposite signs, the derivative $p'(x)$ must have at least one root between -1 and 2. This feels like a hint of an inductive procedure: perhaps we can count the number of real roots of $p'(x)$ between -1 and 2, use that to count the number of local minima and maxima of $p(x)$, etc.?

As it turns out, the Sturm chain trick doesn't require us to count the number of real roots of the derivative $p'(x)$. It is entirely based on rewriting the computations we made during the Euclidean algorithm, when we checked that $p(x)$ and $p'(x)$ have no common factor, in the following equivalent form:

$$\begin{aligned} 3 \cdot (x^3 - 6x^2 + 4x + 12) + 4 \cdot (4x - 11) &= (x - 2) \cdot (3x^2 - 12x + 4), \\ 16 \cdot (3x^2 - 12x + 4) + 101 \cdot 1 &= (12x - 15) \cdot (4x - 11). \end{aligned}$$

Introducing the notation

$$\begin{aligned}
p_0(x) &= p(x) = x^3 - 6x^2 + 4x + 12, \\
p_1(x) &= p'(x) = 3x^2 - 12x + 4, \\
p_2(x) &= 4x - 11, \\
p_3(x) &= 1,
\end{aligned}$$

we see that for each $i = 1, 2$ we have an equation of the form

$$\text{positive} \cdot p_{i-1}(x) + \text{positive} \cdot p_{i+1}(x) = \text{something} \cdot p_i(x).$$

In particular, we have

$$p_i(x) = 0 \quad \Longrightarrow \quad p_{i-1}(x) \text{ and } p_{i+1}(x) \text{ have opposite signs.}$$

Since the gcd of any consecutive pair $\gcd(p_i(x), p_{i+1}(x))$ is 1, when $p_i(x) = 0$ we don't have to worry about the possibility of $p_{i-1}(x)$ or $p_{i+1}(x)$ also being 0, so the statement above is always meaningful.

Now let's see what happens to the signs of our four polynomials $p_0(x), p_1(x), p_2(x), p_3(x)$ as x goes from $-\infty$ to $+\infty$. For now, we will cheat by using a computer to find all of the roots of these polynomials: the roots of $p_0(x)$ occur at

$$-1.05..., 2.51..., 4.53...,$$

the roots of $p_1(x)$ occur at

$$0.36..., 3.63...,$$

and the root of $p_2(x)$ occurs at

$$2.75.$$

Armed with these numerical computations, together with the fact that the polynomials $p_i(x)$ can't change sign without passing through a 0 (by the Polynomial Intermediate Value Theorem 2.1.8), we can make the following table:

	$\cdots$	-1.05	$\cdots$	0.36	$\cdots$	2.51	$\cdots$	2.75	$\cdots$	3.63	$\cdots$	4.53	$\cdots$
p_0	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	$-$	$-$	0	$+$
p_1	$+$	$+$	$+$	0	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$
p_2	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
p_3	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$

What do we notice when we stare at this table? Visually, the $-$ and $+$ signs seem to "flow around" the 0s in the table, and the whole pattern of $+$ and $-$ signs seems to simplify as we move from left to right. More precisely, the pattern seems to simplify *exactly when we pass through a root of* $p_0(x) = p(x)$!

Before concluding too much from this example, let's try another example, with a cubic polynomial that only has one real root. Suppose we take

$$p(x) = x^3 - x + 1,$$

and we want to know how many roots $p(x)$ has and where they are. Once again, we compute the gcd of $p(x)$ and $p'(x) = 3x^2 - 1$, but we arrange our computation in the slightly weird way we did before:

$$\begin{aligned} 3 \cdot (x^3 - x + 1) + 1 \cdot (2x - 3) &= x \cdot (3x^2 - 1) \\ 4 \cdot (3x^2 - 1) + 23 \cdot (-1) &= (6x + 9) \cdot (2x - 3). \end{aligned}$$

Defining p_0, p_1, p_2, p_3 by

$$\begin{aligned} p_0(x) &= p(x) = x^3 - x + 1, \\ p_1(x) &= p'(x) = 3x^2 - 1, \\ p_2(x) &= 2x - 3, \\ p_3(x) &= -1, \end{aligned}$$

we see once again that for each $i = 1, 2$ we have an equation of the form

$$\text{positive} \cdot p_{i-1}(x) + \text{positive} \cdot p_{i+1}(x) = \text{something} \cdot p_i(x),$$

so

$$p_i(x) = 0 \quad \Longrightarrow \quad p_{i-1}(x) \text{ and } p_{i+1}(x) \text{ have opposite signs.}$$

The single real root of $p_0(x)$ is located at $-1.32..$, the roots of $p_1(x)$ are located at $\pm 0.57...$, and the root of $p_2(x)$ is located at 1.5. Making a table as before, we get:

	$\cdots$	-1.32	$\cdots$	-0.57	$\cdots$	0.57	$\cdots$	1.5	$\cdots$
p_0	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$
p_1	$+$	$+$	$+$	0	$-$	0	$+$	$+$	$+$
p_2	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$
p_3	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$

Once again, the $+$ and $-$ signs appear to flow around the 0s in the table, and the pattern of $+$ and $-$ signs simplifies as we move past the single root of $p_0(x) = p(x)$.

Why does this happen? Well, as long as

$$p_i(x) = 0 \quad \Longrightarrow \quad p_{i-1}(x) \text{ and } p_{i+1}(x) \text{ have opposite signs,}$$

if we zoom in around a root r of the polynomial $p_i(x)$, we will always see either the pattern

	$\cdots$	r	$\cdots$
p_{i-1}	$+$	$+$	$+$
p_i	?	0	?
p_{i+1}	$-$	$-$	$-$

or the pattern

	$\cdots$	r	$\cdots$
p_{i-1}	$-$	$-$	$-$
p_i	?	0	?
p_{i+1}	$+$	$+$	$+$

and regardless of how the ?s are filled in with + or − signs, the sequence of signs of $p_{i-1}(x), p_i(x), p_{i+1}(x)$ will flip from + to − (or vice versa) exactly once for any $x \approx r$.

That explains what is going on around the internal 0s of the table - how about the 0s at the top? The claim is that every time $p_0(x) = p(x)$ passes through the value 0, the pattern of + and − signs always simplifies as we go from left to right, and never becomes more complex. To see why this is true, we need to use the fact that $p_1(x) = p'(x)$ is the *derivative* of $p(x)$:

- if $p(r) = 0$ and $p'(r) > 0$, then $p(x)$ is increasing for $x \approx r$, so for x just below r we have $p(x) < 0$, and for x just above r we have $p(x) > 0$, while
- if $p(r) = 0$ and $p'(r) < 0$, then $p(x)$ is decreasing for $x \approx r$, so for x just below r we have $p(x) > 0$, and for x just above r we have $p(x) < 0$.

If we zoom in around the root r of $p(x)$, in the first case we see the pattern

	$\cdots$	r	$\cdots$
p_0	$-$	0	$+$
p_1	$+$	$+$	$+$

while in the second case we see the pattern

	$\cdots$	r	$\cdots$
p_0	$+$	0	$-$
p_1	$-$	$-$	$-$

and in either case, the signs of $p_0(x), p_1(x)$ are different for x just below r and are the same for x just above r.

Now we formalize what we have discovered.

Definition 2.1.11. If $a_1, ..., a_k$ is a sequence of numbers in a real closed field, then we define the number of *sign changes* in the sequence, written $\mathrm{sc}(a_1, ..., a_k)$, to be one less than the maximum length of a sequence of indices $1 \leq i_0 < i_2 < \cdots < i_s \leq k$ such that

$$a_{i_j} \cdot a_{i_{j+1}} < 0$$

for all $j < s$.

Theorem 2.1.12 (Sturm). *Suppose a polynomial $p(x)$ with coefficients from a real closed field has no common factor with its derivative $p'(x)$, and we have a sequence of polynomials $p_0(x), ..., p_k(x)$ such that*

- *the polynomial $p_0(x)$ always has the same sign as $p(x)$,*
- *the polynomial $p_1(x)$ always has the same sign as $p'(x)$,*
- *for each $1 \leq i < k$, if $p_i(x) = 0$ then $p_{i-1}(x)$ and $p_{i+1}(x)$ have opposite signs (and are nonzero), and*
- *the polynomial $p_k(x)$ has a constant (nonzero) sign.*

Then for any $a \leq b$ (including $\pm\infty$), the number of roots of $p(x)$ in the half-open interval $(a, b]$ is exactly

$$\operatorname{sc}\big(p_0(a), ..., p_k(a)\big) - \operatorname{sc}\big(p_0(b), ..., p_k(b)\big).$$

A sequence of polynomials $p_0, p_1, ..., p_k$ as in Theorem 2.1.12 is called a *Sturm chain* for the polynomial $p(x)$. A slight variation of Theorem 2.1.12 lets us check whether a different polynomial $q(x)$ is positive at the roots of $p(x)$.

Theorem 2.1.13. *Suppose that polynomials $p(x), q(x)$ with coefficients from a real closed field have no common factor, that $p(x)$ has no common factor with $p'(x)$, and that we have a sequence of polynomials $p_0(x), p_1(x), ..., p_k(x)$ as in Theorem 2.1.12 but with the condition on $p_1(x)$ modified to:*

- *the polynomial $p_1(x)$ always has the same sign as $p'(x) \cdot q(x)$.*

Then for any $a \leq b$ (including $\pm\infty$), the difference between the number of roots of $p(x)$ in the half-open interval $(a, b]$ where $q(x)$ is positive and the number of roots of $p(x)$ in the half-open interval $(a, b]$ where $q(x)$ is negative is exactly

$$\operatorname{sc}\big(p_0(a), ..., p_k(a)\big) - \operatorname{sc}\big(p_0(b), ..., p_k(b)\big).$$

Let's try computing a Sturm chain for the generic cubic polynomial

$$p(x) = x^3 - ax + b.$$

We have $p'(x) = 3x^2 - a$, and our Euclidean algorithm computation goes as follows:

$$\begin{aligned} &3 \cdot (x^3 + ax + b) + 1 \cdot (2ax - 3b) = x \cdot (3x^2 - a), \\ &4a^2 \cdot (3x^2 - a) + 1 \cdot (4a^3 - 27b^2) = (6ax + 9b) \cdot (2ax - 3b), \end{aligned}$$

so we get the Sturm chain

$$\begin{aligned} p_0(x) &= p(x) = x^3 - ax + b, \\ p_1(x) &= p'(x) = 3x^2 - a, \\ p_2(x) &= 2ax - 3b, \\ p_3(x) &= 4a^3 - 27b^2. \end{aligned}$$

In particular, the number of real roots of $p(x)$ is given by

$$\operatorname{sc}(-1, 3, -2a, 4a^3 - 27b^2) - \operatorname{sc}(1, 3, 2a, 4a^3 - 27b^2) = \begin{cases} 3 & 4a^3 > 27b^2, \\ 1 & 4a^3 < 27b^2. \end{cases}$$

On top of being useful for proving one-variable polynomial inequalities, Sturm chains can be useful for numerically approximating the real roots of a given polynomial $p(x)$:

- first, use Proposition 2.1.10 to split $p(x)$ into a product of powers of polynomials which have no repeated roots, and have no roots in common with each other,

- second, supposing for simplicity that $p(x)$ has no repeated roots, use the Euclidean algorithm to compute a Sturm chain $p_0, p_1, ..., p_k$ for $p(x)$,
- third, use Theorem 2.1.12 to count the number of real roots in $(-\infty, +\infty]$, and repeatedly chop this half-open interval into pieces using something like binary search until we get a sequence of half-open intervals $(-\infty, a_1], (a_1, a_2], ..., (a_m, +\infty]$ which each contain at most one root of $p(x)$,
- fourth, for each interval $(a_i, a_{i+1}]$ containing one of the roots r of $p(x)$, start with a decent approximation to r and repeatedly refine it using Newton's method until you have computed r to as many digits of precision as desired.

In practice, there are faster ways to numerically compute the real roots of a one-variable polynomial, based on variations of Descartes' rule of signs (which only bounds the number of roots in an interval - but this is sometimes good enough). One convenient way to rephrase Descartes' rule of signs is to use the fact that the coefficients of a polynomial $p(x)$ are equal to the (higher) derivatives

$$p(0), p'(0), p''(0)/2, ..., p^{(\deg p)}(0)/(\deg p)!,$$

where $p^{(k)}$ is an abbreviation for the kth derivative of p.

Theorem 2.1.14 (Fourier-Budan Theorem)**.** *If $p(x)$ is a polynomial of degree d with coefficients from a real closed field, then for any $a \le b$, there is a whole number $k \ge 0$ such that the number of roots of $p(x)$ in the interval $(a, b]$, counted "with multiplicity", is given by*

$$\operatorname{sc}\left(p(a), p'(a), ..., p^{(d)}(a)\right) - \operatorname{sc}\left(p(b), p'(b), ..., p^{(d)}(b)\right) - 2k.$$

The proof of the Fourier-Budan Theorem 2.1.14 is similar to the proof of Sturm's Theorem 2.1.12 - we just analyze how the number of sign changes in the sequence $p(x), ..., p^{(d)}(x)$ changes as x passes through a root of $p(x)$ or a root of some $p^{(i)}(x)$.

There is a clever trick we can use to deal with the fact that the Fourier-Budan Theorem 2.1.14 sometimes gives an overestimate of the number of roots in an interval $(a, b]$. Consider once again the cubic polynomial

$$p(x) = x^3 - x + 1$$

which we studied before. If we apply the Fourier-Budan Theorem 2.1.14 to try to bound the number of roots in the interval $(0, 1]$, we get

$$\begin{aligned}\operatorname{sc}\left(p(0), p'(0), p''(0), p^{(3)}(0)\right) - \operatorname{sc}\left(p(1), p'(1), p''(1), p^{(3)}(1)\right) &= \operatorname{sc}\left(1, -1, 0, 6\right) - \operatorname{sc}\left(1, 2, 6, 6\right)\\ &= 2 - 0,\end{aligned}$$

even though $p(x)$ has no positive roots. To probe the interval $(0, 1]$, the trick is to make the change of variables

$$x = \frac{1}{1+y},$$

so that $x \in (0, 1]$ exactly when $y \ge 0$. We then define a new polynomial $q(y)$ by

$$\begin{aligned}q(y) &= (1+y)^3 p\Big(\frac{1}{1+y}\Big)\\ &= 1 - (1+y)^2 + (1+y)^3\\ &= y^3 + 2y^2 + y + 1.\end{aligned}$$

Descartes' rule of signs then shows that $q(y)$ has no positive roots, so we see that $p(x)$ has no roots in the interval $(0, 1]$. In some cases, it may be necessary to make a sequence of several changes of variables

$$x_i = a_i + \frac{1}{1 + x_{i+1}}$$

in order to rule out the spurious "roots" suggested by the Fourier-Budan Theorem 2.1.14 - and this approach naturally leads to a procedure that finds continued fraction approximations of the actual roots which remain. In order to understand when these changes of variables will finally get rid of all of the extra sign changes, we use the following result.

Theorem 2.1.15 (Vincent's Theorem)**.** *Suppose that $p(x)$ has real coefficients, let $a < b$ be real numbers, and define a polynomial $q(y)$ by*

$$q(y) = (y+1)^{\deg p} p\Big(\frac{ay+b}{y+1}\Big).$$

Then:

- *if $p(x)$ has no real roots in the interval (a, b), and has no complex roots in the disk*

$$\Big|x - \frac{a+b}{2}\Big| < \frac{b-a}{2},$$

  then $q(y)$ has nonnegative coefficients, and

- *if $p(x)$ has exactly one real root (counted with multiplicity) in the interval (a, b), and has no other complex roots in the disk*

$$\Big|x - \frac{a+b}{2} - i\frac{b-a}{2\sqrt{3}}\Big| < \frac{b-a}{\sqrt{3}},$$

  or in its complex conjugate, then the sequence of coefficients of $q(y)$ has exactly one sign change.

This is a consequence of the following two results, applied to $q(y)$. The first follows directly from the Fundamental Theorem of Algebra, while the second is a bit harder.

Proposition 2.1.16. *If $p(x)$ has real coefficients, and if none of the (possibly complex) roots of $p(x)$ have a positive real part, then all of the coefficients of $p(x)$ are nonnegative.*

Theorem 2.1.17 (Obreschkoff's Cone Theorem)**.** *If $p(x)$ has real coefficients and has exactly one positive real root (counted with multiplicity), and if every other root $a + bi$ of $p(x)$ satisfies*

$$a\sqrt{3} + |b| \le 0,$$

then the sequence of coefficients of $p(x)$ has exactly one sign change.

Proof sketch. Write $p(x) = (x - r)q(x)$. Then by the Fundamental Theorem of Algebra, $q(x)$ can be written as a product of linear and quadratic factors corresponding to roots of $p(x)$. If $a + bi$ is a complex root of $p(x)$, then the corresponding quadratic factor of $q(x)$ is

$$(x-a)^2 + b^2 = x - 2ax + a^2 + b^2.$$

If $a \leq 0$, then the inequality $a\sqrt{3} + |b| \leq 0$ is equivalent to

$$1 \cdot (a^2 + b^2) \leq (2a)^2,$$

so the sequence of coefficients of each quadratic factor of $q(x)$ are *log-concave*, where we say that a sequence of positive numbers $a_0, ..., a_n$ is log-concave when we have

$$a_{i-1}a_{i+1} \leq a_i^2$$

for all $i = 1, ..., n-1$. It's a standard (and slightly tricky to prove) fact that a product of two polynomials with positive, log-concave sequences of coefficients will itself have a positive, log-concave sequence of coefficients, so we can conclude that the sequence of coefficients of $q(x)$ is also positive and log-concave.

To finish, we just check that if r is positive and if the coefficients of $q(x)$ form a positive and log-concave sequence, then the sequence of coefficients of $p(x) = (x - r)q(x)$ has exactly one sign change. □

If we just want to study polynomials which are positive on some fixed half-open interval $(a, b]$, then there is a way to convert this to the simpler problem of studying polynomials which are positive everywhere, based on a similar change of variables. We use the fact that the rational function

$$\phi_{a,b} : y \mapsto \frac{ay^2 + b}{y^2 + 1}$$

has

$$\phi_{a,b}(\mathbb{R}) = (a, b],$$

so $p(x)$ is positive on the interval $(a, b]$ if and only if the polynomial

$$q(y) = (y^2 + 1)^{\deg p} p\Big(\frac{ay^2 + b}{y^2 + 1}\Big)$$

is positive for all $y \in \mathbb{R}$. The collection of polynomials $q(y)$ which are positive everywhere has the following nice characterization, based on the Fundamental Theorem of Algebra (which has a generalization that applies to every real closed field).

Theorem 2.1.18. *A single variable polynomial $p(x)$ with coefficients in a real closed field satisfies the inequality*

$$p(x) \geq 0$$

for all x if and only if there are polynomials $f(x)$ and $g(x)$ with coefficients in the same real closed field which satisfy

$$p(x) = f(x)^2 + g(x)^2.$$

If p has degree $2k$, then this occurs if and only if there is a multivariable quadratic polynomial $Q(x_0, x_1, ..., x_k)$ such that

$$Q(x_0, x_1, ..., x_k) \geq 0$$

for all $x_0, x_1, ..., x_k$ and

$$p(x) = Q(1, x, ..., x^k).$$

Proof. By the Fundamental Theorem of Algebra, there are numbers r_i and multiplicities m_i, pairs (a_j, b_j) and multiplicities n_j, and a constant c such that

$$p(x) = c\prod_i (x - r_i)^{m_i} \prod_j \big((x - a_j)^2 + b_j^2\big)^{n_j}.$$

If $p(x) \geq 0$ for all x, then each multiplicity m_i must be even and c must be positive, so $p(x)$ can be written as a product of expressions which can each be written as a sum of two squares. The formula

$$(x^2 + y^2)(z^2 + w^2) = (xz - yw)^2 + (xw + yz)^2$$

can then be used to show that $p(x)$ itself can be written as a sum of two squares.

Now for the second statement. Clearly if $p(x) = Q(1, x, ..., x^k)$ and $Q(x_0, ..., x_k) \geq 0$, then $p(x) \geq 0$. For the other direction, if

$$p(x) = f(x)^2 + g(x)^2$$

with

$$f(x) = \sum_{i=0}^{k} a_i x^i$$

and

$$g(x) = \sum_{i=0}^{k} b_i x^i,$$

then we can take

$$Q(x_0, ..., x_k) = \Big(\sum_{i=0}^{k} a_i x_i\Big)^2 + \Big(\sum_{i=0}^{k} b_i x_i\Big)^2. \qquad \square$$

For instance, the fourth degree polynomial

$$p(x) = x^4 + ax^2 + bx + c$$

is always ≥ 0 if and only if there is a constant $d \geq 0$ such that the three-variable quadratic polynomial

$$Q(x, y, z) = x^2 + (a - d)xz + dy^2 + byz + cz^2$$

is always ≥ 0. If $d > 0$, then this quadratic form is always ≥ 0 as long as the determinant

$$\det \begin{bmatrix} 1 & 0 & (a-d)/2 \\ 0 & d & b/2 \\ (a-d)/2 & b/2 & c \end{bmatrix} = cd - b^2/4 - d(a - d)^2/4$$

is positive, that is, as long as

$$\big(4c - (a - d)^2\big)d \geq b^2.$$

Incidentally, if we find a $d > 0$ which gives equality in the inequality above, then we can use it to factor $p(x)$ into a product of two positive quadratic polynomials.

2.2 Some notes on Olympiad inequalities

Good online resources for Olympiad level inequalities and techniques include Kiran Kedlaya's $A < B$ and Mildorf's notes.

2.2.1 Algebraic inequalities

How hard are inequalities?

Theorem 2.2.1 (Artin). *If $P \in \mathbb{R}[x_1, ..., x_n]$ is a polynomial satisfying $P(x_1, ..., x_n) \geq 0$ for all $x_1, ..., x_n \in \mathbb{R}$, then there is an integer k and a collection of polynomials $Q_i, R_i \in \mathbb{R}[x_1, ..., x_n]$, $i = 1, ..., k$, satisfying*

$$P = \sum_{i=1}^{k} \frac{Q_i^2}{R_i^2}.$$

Remark 2.2.1. It is not always possible to write a nonnegative polynomial as a sum of squares of polynomials. One famous example, due to Motzkin, is the polynomial

$$x^4y^2 + x^2y^4 - 3x^2y^2 + 1,$$

which is easily seen to be positive by AM-GM, but is not a sum of squares of polynomials. Artin's theorem tells us that it is possible to write it as a sum of squares of rational functions, but unfortunately gives us no bounds on how large the denominators of those rational functions may need to be.

Although Artin's Theorem does not tell us what type of denominators to look for, in most cases the simplest possible denominators will work.

Theorem 2.2.2 (Polya). *If $F(x_1, ..., x_n)$ is a homogeneous polynomial such that $F(x_1, ..., x_n) > 0$ whenever $x_i \geq 0$ for $1 \leq i \leq n$ and not all x_i are equal to 0, then there is a number p such that every coefficient of $(x_1 + \cdots + x_n)^p F(x_1, ..., x_n)$ is positive.*

Theorem 2.2.3 (Tarski). *There is an algorithm that can decide in a bounded amount of time the truth or falsity of any statement about real numbers built up from arithmetic operations, logical connectives, and logical quantifiers.*

Remark 2.2.2. Unfortunately, Tarski's algorithm is very slow - his original algorithm's running time satisfied a recurrence similar to that of the Ackermann function. The modern form of this algorithm is called Cylindrical Algebraic Decomposition, and in the worst case the running time is doubly exponential in the size of the input. (This algorithm is implemented in many computer algebra programs, such as Mathematica.)

Theorem 2.2.4 (Stengle's Positivstellansatz). *Let $h_1, ..., h_k$ and $g_1, ..., g_l$ be polynomials in $\mathbb{R}[x_1, ..., x_n]$. Then the set*

$$\{(x_1, ..., x_n) \in \mathbb{R}^n \mid h_i(x_1, ..., x_n) = 0, g_j(x_1, ..., x_n) \geq 0 \text{ for } 1 \leq i \leq k, 1 \leq j \leq l\}$$

is empty if and only if there are polynomials $t_1, ..., t_k$ and finitely many tuples $(i_1, ..., i_m)$ and sums of squares $s_{i_1,...,i_m}$ such that

$$-1 = \sum_{i=1}^{k} t_i h_i + \sum_{(i_1,...,i_m)} s_{i_1,...,i_m} g_{i_1} g_{i_2} \cdots g_{i_m}.$$

Examples

Proposition 2.2.5 (Lagrange's identity)**.** *For any $x_1, ..., x_n$, $y_1, ..., y_n$ we have*

$$\Big(\sum_i x_i^2\Big)\Big(\sum_i y_i^2\Big) - \left(\sum_i x_i y_i\right)^2 = \sum_{i<j}(x_i y_j - x_j y_i)^2.$$

Proposition 2.2.6. *If $a \geq b$ and $x, y \in \mathbb{R}$, then*

$$x^{a+1}y^{b-1} + x^{b-1}y^{a+1} - x^a y^b - x^b y^a = x^{b-1}y^{b-1}(x-y)^2(x^a + x^{a-1}y + \cdots + y^a).$$

Proposition 2.2.7 (AM-GM)**.** *For any $x_1, ..., x_n$, we have*

$$\frac{x_1^n + \cdots + x_n^n}{n} - x_1 \cdots x_n = \frac{1}{2n!}\sum_{sym}(x_{n-1} - x_n)^2\Big(\sum_{j=0}^{n-2} x_1 \cdots x_j(x_{n-1}^{n-2-j} + x_{n-1}^{n-2-j-1}x_n + \cdots + x_n^{n-2-j})\Big).$$

Problems

1. Write $x^6 + y^6 + z^6 - 3x^2y^2z^2$ as a sum of squares of polynomials.
2. Prove that $\frac{1}{2}x^2 + y^2 + 1 \geq xy + x$ by writing the difference of both sides as a sum of squares.
3. Prove that $x^4 + y^4 + z^2 \geq \sqrt{8}xyz$ by writing the difference of both sides as a sum of squares.
4. (Mitrinović) Show that if $0 < b \leq a$ then we have
$$\frac{1}{8}\frac{(a-b)^2}{a} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{1}{8}\frac{(a-b)^2}{b}.$$
5. Show that
$$\sqrt{a^2+1}^3 \geq a^3 + \frac{6}{5}a + \frac{3}{5}.$$
6. Show that if $n \geq 2$ and $x_1, ..., x_n$ are positive real numbers satisfying
$$\Big(\sum_i x_i\Big)\Big(\sum_i \frac{1}{x_i}\Big) \leq n^2 + 1,$$
then $\frac{x_1}{x_2} \leq \frac{3+\sqrt{5}}{2}$.
7. For $a, b, c > 0$ show that
$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$
8. Show that
$$\sum_{sym} x^4 + 3x^2y^2 \geq \sum_{sym} 4x^3y.$$
9. Show that
$$\sum_{sym} 3x^4 + 2x^2yz \geq \sum_{sym} 4x^3y + x^2y^2.$$

10. Find a way to write the polynomial

$$(x^2+y^2+1)(x^4y^2+x^2y^4-3x^2y^2+1)$$

as a sum of squares.

11. (Nesbitt) Show that for $a,b,c>0$, we have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

12. Prove that for $a,b,c>0$ and $abc=1$, we have

$$\sum_{cyc}\frac{ab}{a^5+ab+b^5}\leq 1.$$

13. (Schur) Show that for $x,y,z\geq 0$, we have

$$\sum_{sym}x^3+xyz\geq\sum_{sym}2x^2y.$$

14. (Crux) Prove that if $a,b,c,d>0$ and $c^2+d^2=(a^2+b^2)^3$, then

$$\frac{a^3}{c}+\frac{b^3}{d}\geq 1.$$

What is the equality case?

2.2.2 Functional Inequalities

Useful facts about convex functions

Definition 2.2.8. A function f is called *convex* on the interval I if for all $x,y\in I$ and $\lambda\in[0,1]$, we have

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y).$$

If $-f$ is convex, then f is called *concave.*

Theorem 2.2.9. *If f has a derivative f', then f is convex if and only if f' is increasing.*

Theorem 2.2.10 (Weighted Jensen)**.** *If f is a convex function, $w_1,...,w_n$ are positive real numbers, and $x_1,...,x_n$ are any real numbers, then*

$$f\left(\frac{\sum_i w_ix_i}{\sum_i w_i}\right)\leq\frac{\sum_i w_if(x_i)}{\sum_i w_i}.$$

Proposition 2.2.11. *If f is convex on the interval $[a,b]$ and $x\in[a,b]$, then*

$$f(x)\leq\max(f(a),f(b)).$$

Definition 2.2.12. If $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots b_n$, then we say that $(a_1, ..., a_n) \succ (b_1, ..., b_n)$, or $(a_1, ..., a_n)$ *majorizes* $(b_1, ..., b_n)$, if

$$\begin{gathered} a_1 \geq b_1, \\ a_1 + a_2 \geq b_1 + b_2, \\ \cdots \\ a_1 + \cdots + a_{n-1} \geq b_1 + \cdots + b_{n-1}, \text{ and} \\ a_1 + \cdots + a_n = b_1 + \cdots + b_n. \end{gathered}$$

Theorem 2.2.13 (Karamata)**.** *If f is convex and $(a_1, ...a_n) \succ (b_1, ..., b_n)$, then*

$$\sum_{i=1}^{n} f(a_i) \geq \sum_{i=1}^{n} f(b_i).$$

Theorem 2.2.14. *If f is convex on the interval $[\alpha, \infty)$ and concave on $(-\infty, \alpha]$, then for any whole number n and any real numbers $x_1, ..., x_n$ we can find a, b with $a + (n-1)b = x_1 + \cdots + x_n$ and*

$$f(x_1) + \cdots + f(x_n) \leq f(a) + (n-1)f(b).$$

Problems

1. Show that if f is convex, $a \leq c$, and $b \leq d$, then

$$\frac{f(b) - f(a)}{b - a} \leq \frac{f(d) - f(c)}{d - c}.$$

2. Prove that each of the functions $|x|, x^2, e^x$ is convex, and that each of $\frac{1}{x}, -\sqrt{x}, -\ln(x)$ is convex when restricted to $x > 0$, without using differential calculus.

3. Prove that if f is convex, then for any x, y, z we have

$$f(4x) + f(4y) + f(4z) \geq f(2x + y + z) + f(x + 2y + z) + f(x + y + 2z).$$

4. (Popoviciu) Prove that if f is a convex function, then for any x, y, z we have

$$f(x) + f(y) + f(z) + 3f\left(\frac{x + y + z}{3}\right) \geq 2f\left(\frac{x + y}{2}\right) + 2f\left(\frac{y + z}{2}\right) + 2f\left(\frac{z + x}{2}\right).$$

5. Prove Karamata's inequality.

6. Prove Theorem 2.2.14.

7. Prove that if $a, b, c \in [0, 1]$, we have

$$\frac{a}{b + c - 1} + \frac{b}{c + a - 1} + \frac{c}{a + b - 1} + (1 - a)(1 - b)(1 - c) \leq 1.$$

8. Prove that if one tetrahedron is contained in another tetrahedron, then the sums of the lengths of the edges of the inner tetrahedron is at most $\frac{4}{3}$ as large as the sum of the lengths of the edges of the outer tetrahedron. (Hint: the distance between two points is a convex function of either point.)

9. Prove that if n is a whole number and $x_1, ..., x_n$ satisfy $x_1 \cdots x_n = 1$, then
$$\frac{1}{n-1+x_1} + \cdots + \frac{1}{n-1+x_n} \leq 1.$$

2.2.3 The Equally Moving Variables technique

For more about this technique, see Pham Kim Hung's posts on the Art of Problem Solving forums. (His user name is hungkhtn.)

Theorem 2.2.15. *Suppose $F : \mathbb{R}^n \to \mathbb{R}$ is an n-variable function satisfying the following:*

- $F(x_1, x_2, ..., x_{n-1}, x_n) = F(x_2, x_3, ..., x_n, x_1)$ *for all* $x_1, ..., x_n$.
- $F(x_1, ..., x_{n-1}, 0) \geq 0$ *for all* $x_1, ..., x_{n-1} \geq 0$.
- $F(x_1 + t, ..., x_n + t) \geq F(x_1, ..., x_n)$ *for all* $x_1, ..., x_n, t \geq 0$.

Then $F(x_1, ..., x_n) \geq 0$ *for all* $x_1, ..., x_n \geq 0$.

It is often convenient to check the third condition with differential calculus. We define an operator D by
$$DF(x_1, ..., x_n) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F(x_1, ..., x_n) = \frac{\partial}{\partial t} F(x_1 + t, ..., x_n + t) \mid_{t=0}.$$
To check the third condition, it is enough to check that $DF(x_1, ..., x_n) \geq 0$ for all $x_1, ..., x_n \geq 0$.

D satisfies the Liebniz rule: if F, G are two n-variable functions and FG is their product, then
$$D(FG)(x_1, ..., x_n) = F(x_1, ..., x_n)DG(x_1, ..., x_n) + G(x_1, ..., x_n)DF(x_1, ..., x_n).$$
If $F(x_1, ..., x_n) = (x_i - x_j)^k$ for a fixed i, j, k, then we have $DF(x_1, ..., x_n) = 0$.

Problems

1. (Schur's inequality) Prove, by induction on r, that for any $x, y, z \geq 0$ we have
$$\sum_{sym} x^{r+2} + x^r yz \geq \sum_{sym} 2x^{r+1}y.$$

2. (Pham Kim Hung) Prove that for $a, b, c \geq 0$ we have
$$a^3 + b^3 + c^3 - 3abc \geq 4(a-b)(b-c)(c-a).$$

3. Prove that for $x, y, z \geq 0$ we have
$$\sum_{sym} 4x^4y^2 + 2x^3y^2z \geq \sum_{sym} 3x^4yz + 3x^3y^3.$$

4. (Suranji) Prove, by induction on n, that for any $a_1, ..., a_n \geq 0$ we have

$$(n-1)(a_1^n + \cdots + a_n^n) + na_1 \cdots a_n \geq (a_1 + \cdots + a_n)(a_1^{n-1} + \cdots + a_n^{n-1}).$$

5. Prove that for $x, y, z \geq 0$ we have

$$3(x^3 + y^3 + z^3) + 6xyz \geq 5(x^2y + y^2z + z^2x).$$

6. Prove that for $x, y, z \geq 0$ we have

$$\sum_{sym} x^9y^4 + x^7y^3z^3 \geq \sum_{sym} x^9y^3z + x^6y^6z.$$

2.3 A few fewnomial exercises

1. Prove that for $x \geq 0$, we have

$$x^7 + x^4 + x^3 + 1 \geq 2x^6 + 2x.$$

2. Prove that for $x \geq 0$, we have

$$x^{\sqrt{2}} + 2\sqrt{2} \geq 2^{\frac{3-\sqrt{2}}{2}}x + 2.$$

3. Prove that for all x we have

$$3x^2e^x + 4e^2 \geq 8xe^x.$$

4. Prove that for $x \geq 0$, we have

$$x^{22} + x^{11} + x^9 + 1 \geq x^{21} + x^{15} + x^4 + x^2.$$

5. Prove that for $x > 0$, we have

$$x^{\sqrt{2}} + 2 + \frac{1}{x^{\sqrt{2}}} \geq 2x + \frac{2}{x}.$$

6. Prove that for $x \geq 0$, we have

$$x^9 + 281x^3 + 100 \geq 22x^6 + 360x^2.$$

7. For $x, y > 0$, define their *logarithmic mean* to be

$$\mathrm{LM}(x, y) = \frac{x - y}{\ln(x) - \ln(y)},$$

where $\ln(x)$ is the natural logarithm of x. Prove that

$$\frac{x+y}{2} \geq \mathrm{LM}(x, y) \geq \sqrt{xy}.$$

8. Prove that for any $x > 0$ we have

$$x^{66} + x^{29} + x^{26} + \frac{1}{x^{26}} + \frac{1}{x^{29}} + \frac{1}{x^{66}} \geq x^{62} + x^{45} + x^2 + \frac{1}{x^2} + \frac{1}{x^{45}} + \frac{1}{x^{62}}.$$

9. Prove that if f is differentiable and f' is convex, then we have

$$f(3) + 3f(1) \geq 3f(2) + f(0), \text{ and}$$
$$f(6) + f(2) + f(1) \geq f(5) + f(4) + f(0).$$

10. (Vasc) Prove that if f is differentiable and f' is convex, then for any $x \geq y \geq z$ we have

$$f(2x + y) + f(2y + z) + f(2z + x) \geq f(2x + z) + f(2z + y) + f(2y + x).$$

11. Popoviciu defines the divided differences of a polynomial inductively, as follows:

$$\begin{aligned} [a; f] &= f(a), \\ [a, b; f] &= \frac{f(b) - f(a)}{b - a}, \\ [a_0, ..., a_n; f] &= \frac{[a_1, ..., a_n; f] - [a_0, ..., a_{n-1}; f]}{a_n - a_0}. \end{aligned}$$

Prove, by induction on n, that if the nth derivative of f exists and is nonnegative, that for any $a_0, ..., a_n$ we have

$$[a_0, ..., a_n; f] \geq 0.$$

12. Show that $[a_0, ..., a_n; f]$ is a symmetric function of $a_0, ..., a_n$.

13. Let n be an integer which is at least 3. Suppose f is a function such that for every $a_0, ..., a_n$ we have $[a_0, ..., a_n; f] \geq 0$. Show that f is differentiable and that for any $b_0, ..., b_{n-1}$ we have

$$[b_0, ..., b_{n-1}; f'] \geq 0.$$

14. Suppose that f is a function such that for every integer n and every $a_0, ..., a_n$ we have $[a_0, ..., a_n; f] \geq 0$. Prove that

$$f(2) + 4f(0) \geq 4f(1).$$

15. Prove that if $a_1, ..., a_9$ are real numbers satisfying $\sum_{i=1}^{9} a_i = \sum_{i=1}^{9} a_i^3 = 0$ and $\sum_{i=1}^{9} a_i^2 = 8$, then for any $x > 0$ we have

$$x^2 + 7 + \frac{1}{x^2} \geq \sum_{i=1}^{9} x^{a_i} \geq 4x + 1 + \frac{4}{x}.$$

16. Prove that for any $x, y, z > 0$ we have

$$\sum_{sym} \frac{x^2}{y^2} + \sum_{sym} \frac{x^{\sqrt{2}}}{y^{\sqrt{2}}} \geq \sum_{sym} \frac{x^2}{yz} + \sum_{sym} \frac{xy}{z^2}.$$

Part II

Foundational Material

Chapter 1

Analysis

1.1 Basic Facts

1.1.1 Point Set Stuff

Definition 1.1.1. A topological space is *normal*, or T_4, if any two disjoint closed sets have disjoint open neighborhoods.

Proposition 1.1.2. *Compact Hausdorff spaces are normal.*

Lemma 1.1.3 (Urysohn's Lemma)**.** *A topological space X is normal iff for any disjoint closed subsets $A, B \subseteq X$ there exists a continuous $f : X \to [0,1]$ such that $f(A) \subseteq \{0\}$ and $f(B) \subseteq \{1\}$.*

Proof. Let $U(1) = X \setminus B, V(0) = X \setminus A$. For each dyadic rational $r = \frac{2a+1}{2^{n+1}} \in (0,1)$ we construct disjoint open subsets $U(r), V(r) \subseteq X$ such that $X \setminus V(\frac{a}{2^n}) \subseteq U(\frac{2a+1}{2^{n+1}})$ and $X \setminus U(\frac{a+1}{2^n}) \subseteq V(\frac{2a+1}{2^{n+1}})$. Then for every r we have $A \subseteq U(r)$, $B \subseteq V(r)$, for $r \le s$ we have $U(r) \cap V(s) = \emptyset$, and for $r < s$ we have $V(r) \cup U(s) = X$. Thus for $r < s$, the closure of $U(r)$ is contained in $U(s)$. Finally, define f by $f(x) = \min(1, \inf\{r \mid x \in U(r)\})$. □

Lemma 1.1.4 (Locally Compact Urysohn's Lemma)**.** *If X is locally compact Hausdorff and $K \subseteq U \subseteq X$ with K compact and U open, then there exists a continuous $f : X \to [0,1]$ such that $f(K) \subseteq \{1\}$ and f is supported on a compact subset of U.*

Proof. Find a precompact open set V with $K \subseteq V \subseteq \overline{V} \subseteq U$, then $\overline{V}$ is normal (since it is compact and Hausdorff), so by Urysohn's Lemma there is a continuous $f : \overline{V} \to [0,1]$ with $f(K) \subseteq \{1\}$ and $f(\partial V) \subseteq \{0\}$. □

Theorem 1.1.5 (Tietze Extension Theorem)**.** *If X is a normal space, $A \subseteq X$ is closed, and $f : A \to \mathbb{R}$ is continuous, then there exists a continuous $F : X \to \mathbb{R}$ with $F|_A = f$.*

Proof. Assume without loss of generality that $f(A) \subseteq [0,1]$. We'll find a sequence of functions $g_i : X \to [0, \frac{2^{i-1}}{3^i}]$ with $0 \le f - \sum_{i=1}^n g_i \le \frac{2^n}{3^n}$ for all n, and finish by taking $F = \sum_i g_i$. It's enough to show how to find g_1: we apply Urysohn's Lemma to find $g_1 : X \to [0, \frac{1}{3}]$ with $g_1(x) = 0$ for $x \in f^{-1}([0, \frac{1}{3}])$ and $g_1(x) = \frac{1}{3}$ for $x \in f^{-1}([\frac{2}{3}, 1])$. □

Corollary 1.1.6 (Locally Compact Tietze). *If X is locally compact Hausdorff and $K \subseteq U \subseteq X$ with K compact and U open, then for every continuous $f : K \to [0,1]$ there exists a continuous $F : X \to [0,1]$ such that $F|_K = f$ and F is supported on a compact subset of U.*

Proposition 1.1.7. *If X is locally compact Hausdorff, then $C_0(X)$ is the closure of $C_c(X)$ in the uniform metric.*

Theorem 1.1.8 (Stone-Weierstrauss, lattice version). *If X compact, $B \subseteq C(X,\mathbb{R})$ such that for any $x, y \in X$ and $a, b \in \mathbb{R}$ there exists $g \in B$ with $g(x) = a, g(y) = b$, and such that B contains $\max(f,g), \min(f,g)$ whenever it contains f, g, then B is dense in $C(X,\mathbb{R})$.*

Proof. Let $f \in C(X,\mathbb{R})$, and for all $x, y \in X$ pick $g_{xy} \in B$ with $g_{xy}(x) = f(x), g_{xy}(y) = f(y)$. Fix $\epsilon > 0$. Take $U_{xy} = \{z \mid f(z) < g_{xy}(z) + \epsilon\}, V_{xy} = \{z \mid f(z) > g_{xy} - \epsilon\}$. For any y, some finite subcollection of the U_{xy}s cover X, corresponding to $x_1, ..., x_n$, take $g_y = \max(g_{x_1y}, ..., g_{x_ny})$ and $V_y = \cap V_{x_jy}$, then $f < g_y + \epsilon$ and for $x \in V_y$ we have $f(x) > g_y(x) - \epsilon$. Now take a finite subcollection of the V_ys which covers X, and let g be the minimum of the corresponding g_ys, then $g \in B$ and $|f - g| \leq \epsilon$. □

Definition 1.1.9. The *Bernstein polynomials* are defined by

$$b_{\nu,n}(x) = \binom{n}{\nu} x^\nu (1-x)^{n-\nu}.$$

Theorem 1.1.10 (Weierstrauss approximation). *If $f : [a,b] \to \mathbb{C}$ is continuous, then $\forall \epsilon > 0$ there exists a polynomial $p \in \mathbb{C}[x]$ such that $\forall x \in [a,b]$, we have $|f(x) - p(x)| < \epsilon$.*

Proof. Suppose $[a,b] = [0,1]$, and define $B_n(f)$ by

$$B_n(f) = \sum_{\nu=0}^{n} f(\tfrac{\nu}{n}) b_{\nu,n}.$$

If k is the number of times we flip heads in n independent random coinflips with bias x, then

$$\mathbb{E}[f(\tfrac{k}{n})] = B_n(f)(x),$$

so the law of large numbers shows that $B_n(f)$ approximates f. □

Theorem 1.1.11 (Stone-Weierstrauss for $\mathbb{R}$). *X compact Hausdorff, A a subalgebra of $C(X,\mathbb{R})$ which contains a non-zero constant. Then A is dense in $C(X,\mathbb{R})$ iff it separates points.*

Proof. It's enough to show that if $f \in A$, then $|f|$ is in the closure of A, since then the closure of A will be closed under max and min. To do this, we find $p \in \mathbb{R}[x]$ such that $\forall x \in f(X)$ we have $||x| - p(x)| < \epsilon$, then $p \circ f \in A$ and $||f| - p \circ f| < \epsilon$. □

Theorem 1.1.12 (Stone-Weierstrauss for $\mathbb{C}$). *X compact Hausdorff, $S \subseteq C(X,\mathbb{C})$ separates points. Then the complex unital $*$-algebra generated by S is dense in $C(X,\mathbb{C})$.*

Theorem 1.1.13 (Tychonoff's Theorem). *If $\{X_a\}_{a \in A}$ is a family of compact sets, then $X = \prod_{a \in A} X_a$ is compact in the product topology.*

Proof. With nets and Zorn: Suppose $\{U_i\}_{i\in I}$ is an open cover of X with no finite subcover. For each finite subset $J \subseteq I$, let x_J be a point of X not contained in $\cup_{j\in J}U_j$. We show that for every $B \subseteq A$, the net $\{\pi_B(x_J)\}_{J\subseteq I}$ has a cluster point, by transfinite induction on B, and taking $B = A$ gives a contradiction.

With Alexander Subbase Theorem 1.1.14: Suppose there is an open cover by cylinder sets with no finite subcover. Then for each $a \in A$, there is some $x_a \in X_a$ not covered by the cylinder sets corresponding to coordinate a, and the corresponding point $(x_a)_{a\in A} \in X$ is not covered by any of the cylinders. □

Theorem 1.1.14 (Alexander Subbase Theorem)**.** *If X is a topological space with subbase B, and every open cover of X by elements of B has a finite subcover, then X is compact.*

Proof. Suppose not, let C be a maximal open cover of X which has no finite subcover (alternatively, take C to be a maximal proper ideal of $\mathcal{P}(X)$ containing an open cover of X). Take $x \in X$ not contained in any element of $C \cap B$, then there is $U \in C$ with $x \in U$, and $S_1, ..., S_n \in B$ with $x \in S_1 \cap \cdots \cap S_n \subseteq U$. For each S_i, since $S_i \notin C$ there must be a finite subset $C_i \subseteq C$ such that $\{S_i\} \cup C_i$ covers X, but then $\{U\} \cup C_1 \cup \cdots \cup C_n$ is a finite subset of C which covers X. □

The next bit is from `https://math.stackexchange.com/a/6338`.

Definition 1.1.15. A topological space is called a *continuum* if it is a compact connected Hausdorff space.

Lemma 1.1.16. *Let X be a continuum. If F is a non-trivial closed subset of X, then for every component C of F we have that $\partial F \cap C$ is non-empty.*

Proof. Since X is Hausdorff compact, quasicomponents coincide with components, so C is the intersection of all clopen sets in F which contain C. Suppose that C is disjoint from ∂F. Then, by compactness of ∂F, there is a single clopen set A in F containing C and disjoint from ∂F. Take an open set U such that $A = U \cap F$. $A \cap \partial F = \emptyset$ implies that $A = U \cap \operatorname{int}(F)$, so A is open in X. But A is also closed in X, and contains C, so $A = X$. But then $\partial F = \emptyset$, which is not possible since F would be non-trivial clopen in X. □

Theorem 1.1.17 (Sierpiński [173])**.** *If a continuum X has a countable cover $\{X_i\}_{i=1}^{\infty}$ by pairwise disjoint closed subsets, then at most one of the sets X_i is non-empty.*

Proof. Assume that at least two of the sets X_i are non-empty. First we show that for every i there exists a continuum $C \subsetneq X$ such that $C \cap X_i = \emptyset$ and at least two sets in the sequence $C \cap X_1, C \cap X_2, \ldots$ are non-empty. If X_i is empty then we can take $C = X$; thus we can assume that X_i is non-empty. Take $j \neq i$ such that $X_j \neq \emptyset$. Since X is Hausdorff compact, there are disjoint open sets $U, V \subseteq X$ satisfying $X_i \subseteq U$ and $X_j \subseteq V$. Let C be a component of $\overline{V}$ which meets X_j. Clearly, C is a continuum, $C \cap X_i = \emptyset$ and $C \cap X_j \neq \emptyset$. By the previous lemma, $C \cap \partial(\overline{V}) \neq \emptyset$ and since $X_j \subseteq \operatorname{int}(\overline{V})$, there exist a $k \neq j$ such that $C \cap X_k \neq \emptyset$.

It follows that there exists a decreasing sequence $C_1 \supseteq C_2 \supseteq \ldots$ of continua contained in X such that $C_i \cap X_i = \emptyset$ and $C_i \neq \emptyset$ for $i = 1, 2, \ldots$ The first part implies that $\bigcap_{i=1}^{\infty} C_i = \emptyset$ and from the second part and compactness of X it follows that $\bigcap_{i=1}^{\infty} C_i \neq \emptyset$. □

Corollary 1.1.18. *If $f, p_i : \mathbb{R}^n \to X$ are continuous functions from $\mathbb{R}^n$ to a topological space X, such that*

- *$n \geq 2$,*
- *for every $x \in \mathbb{R}^n$, there is some i such that $f(x) = p_i(x)$,*
- *the collection of functions p_i is countable, and*
- *for all $i \neq j$, the set of points $x \in \mathbb{R}^n$ such that $p_i(x) = p_j(x)$ is countable,*

then there is some i such that $f(x) = p_i(x)$ for all $x \in \mathbb{R}^n$. The same is true if we replace $\mathbb{R}^n$ by any convex subset of $\mathbb{R}^n$ which has dimension ≥ 2.

Proof. For each i, let C_i be the set of points x such that $f(x) = p_i(x)$, and note that each C_i is closed by the continuity of f, p_i. The assumptions imply that the C_i are a countable closed cover of $\mathbb{R}^n$ such that $C_i \cap C_j$ is countable for all $i \neq j$. Let B be the set of points x such that $p_i(x) = p_j(x)$ for some $i \neq j$, and note that B is countable.

Since there are only countably many C_is, there must be some i such that C_i is uncountable. Suppose for contradiction that $C_i \neq \mathbb{R}^n$. Pick any point x_0 in C_i which is not in B. Since B is countable, there are at most countably many lines connecting x_0 to points in B. Since the closure of the complement of any countable set of lines through x_0 is $\mathbb{R}^n$, for $n \geq 2$, there must be some point $x_1 \notin C_i$ such that the line segment ℓ connecting x_0 to x_1 doesn't meet any point of B. But then the sets $\ell \cap C_j$ form a countable cover of ℓ by pairwise disjoint closed subsets, at least two of which are nonempty, contradicting the previous result. □

Definition 1.1.19. A subset S of a topological space is *perfect* if it is closed and every point of S is a limit point.

Definition 1.1.20. A *Polish space* is a separable completely metrizable topological space.

Theorem 1.1.21 (Cantor)**.** *Every nonempty perfect subset of a Polish space has cardinality at least $2^{\aleph_0}$.*

Definition 1.1.22. A *condensation point* of a subset S of a topological space is a point x such that every neighborhood of x intersects S in uncountably many points.

Theorem 1.1.23 (Cantor-Bendixson)**.** *Every closed subset S of a Polish space X can be written uniquely as a disjoint union of a perfect set and a countable set.*

Proof. Ordinal proof: For any set S, let S' be the set of limit points of S. Define a sequence S_α indexed by ordinals by $S_0 = S$, $S_{\alpha+1} = S'_\alpha$, and $S_\beta = \cap_{\alpha<\beta} S_\alpha$ for β a limit ordinal. Since each closed set S_α is determined by the collection of open subsets of a basis of X which do not intersect it, and since every well-ordered chain contained in $\mathcal{P}(\mathbb{N})$ is countable, there is some countable ordinal β such that $S_\beta = S_{\beta+1}$. Since the number of isolated points of any S_α is countable, we see that $S \setminus S_\beta$ must be countable.

Condensation point proof: Let P be the set of condensation points of S. Then $S \setminus P$ is contained in a countable union of open sets of a basis of X which each intersect S in countably many points, so $S \setminus P$ is countable and P is perfect.

For uniqueness: note that every point in a perfect subset of S must be a condensation point of S. □

1.1.2 Metric Spaces

Definition 1.1.24. A metric space is *complete* if every Cauchy sequence has a limit. It is *totally bounded* if it can be covered by finitely many subsets of size ϵ, for every $\epsilon > 0$.

Theorem 1.1.25. *A metric space is compact iff it is complete and totally bounded.*

Definition 1.1.26. A metric space is *sequentially compact* if every sequence has a bounded subsequence.

Theorem 1.1.27 (Bolzano-Weierstrauss)**.** *A subset of $\mathbb{R}^n$ is sequentially compact iff it is closed and bounded.*

Proposition 1.1.28. *A closed subset of a complete space is complete, and a complete subset of a metric space is closed.*

Theorem 1.1.29 (Baire Category Theorem)**.** *If M is either a complete metric space or a locally compact Hausdorff space, then a union of countably many nowhere dense subsets of M has empty interior.*

Definition 1.1.30. A space is called a *Baire space* if the intersection of any countable collection of open dense sets is dense.

Theorem 1.1.31 (Banach Fixed Point)**.** *Contraction mappings on complete metric spaces have unique fixed points.*

Corollary 1.1.32 (Picard-Lindelöf)**.** *The initial value problem $y'(t) = f(t, y(t)), y(t_0) = y_0$ for $t \in [t_0 - \epsilon, t_0 + \epsilon]$ has a unique solution for some $\epsilon > 0$ if f is Lipschitz continuous in y and continuous in t.*

Definition 1.1.33. If X, Y are Banach spaces, $U \subseteq X$ open, then $f : U \to Y$ is called *Frechét differentiable* at x if there exists a bounded linear operator $A : X \to Y$ such that $\|f(x+h) - f(x) - Ah\|_Y = o(\|h\|_X)$ as $h \to 0$. In this case we write $Df_x = A$.

Corollary 1.1.34 (Inverse Function Theorem)**.** *If X, Y are Banach spaces, U an open neighborhood of 0 in X, $F : U \to Y$ continuously (Fréchet) differentiable and $DF_0 : X \to Y$ a bounded isomorphism from X to Y (with bounded inverse), then there exists an open neighborhood $V \subseteq Y$ of $F(0)$ and a continuously differentiable map $G : V \to X$ such that $F(G(y)) = y$ for all $y \in V$.*

Definition 1.1.35. A topological space is called *separable* if it contains a countable dense set. It is called *second countable* if its topology has a countable base.

Proposition 1.1.36. *Every second countable space is separable, and every separable metric space is second countable.*

Definition 1.1.37. If X, Y are metric spaces, then $f : X \to Y$ is called *uniformly continuous* if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x, y \in X$ such that $d_X(x, y) < \delta$, we have $d_Y(f(x), f(y)) < \epsilon$.

Definition 1.1.38. A family of functions F is called *equicontinuous* at $x_0 \in X$ if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall f \in F, x \in X$ such that $d(x_0, x) < \delta$ we have $d(f(x_0), f(x)) < \epsilon$. F is *uniformly equicontinuous* if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall f \in F, x, y$ such that $d(x, y) < \delta$ we have $d(f(x), f(y)) < \epsilon$.

Theorem 1.1.39 (Arzelà-Ascoli)**.** *If $(f_n)_{n\in\mathbb{N}}$ defined on $[a,b]$ is uniformly bounded and equicontinuous, then there is a subsequence which converges uniformly.*

Theorem 1.1.40 (Ascoli Version 2)**.** *If X is compact Hausdorff, then a subset of $C(X)$ (with the uniform norm) is compact iff it is closed, pointwise bounded, and equicontinuous.*

Lemma 1.1.41 (Finite Vitali Covering Lemma)**.** *If $B_1, ..., B_n$ are balls in a metric space, then there is a subcollection $B_{j_1}, ..., B_{j_k}$ which are disjoint, and which satisfy*

$$B_1 \cup \cdots \cup B_n \subseteq 3B_{j_1} \cup \cdots \cup 3B_{j_k},$$

where $3B_j$ is the ball with the same center as B_j and three times the radius.

Proof. Keep adding the biggest ball which is disjoint from the ones you have chosen so far to your collection. Then every ball you haven't chosen will intersect a larger ball that you have chosen. □

Lemma 1.1.42 (Infinite Vitali Covering Lemma)**.** *If $(B_i)_{i\in I}$ is a collection of balls in a metric space such that $\sup_{i\in I} \operatorname{rad}(B_i) < \infty$, then for any $c > 1$ there is a subcollection $J \subseteq I$ such that the B_j with $j \in J$ are disjoint, and $\cup_{i\in I} B_i \subseteq \cup_{j\in J}(1+2c)B_j$.*

Proof. Let $R = \sup \operatorname{rad}(B_i)$, and for each n choose a maximal disjoint subcollection of the balls with radius between R/c^n and R/c^{n+1} which are disjoint from the balls you have already chosen so far. Then every ball you haven't chosen will intersect a ball you have chosen, whose radius is at most a factor of c smaller. □

Lemma 1.1.43 (Besicovitch Covering Lemma)**.** *For every n there exists a constant c_n such that for $E \in \mathbb{R}^n$ bounded and for a collection of balls $\mathcal{B}$ such that every point of E is the center of some ball B in $\mathcal{B}$, there is a collection of c_n families $\mathcal{B}_i \subseteq \mathcal{B}$ of pairwise disjoint balls, such that $E \subseteq \cup_{i\le c_n} \cup_{B\in\mathcal{B}_i} B$.*

Proof. WLOG assume all balls in $\mathcal{B}$ are contained in a big ball B_0. Make a countable sequence of balls $B_i \in \mathcal{B}$ such that each B_i has its center not contained in $B_1, ..., B_{i-1}$, and its radius within a factor of $1-\epsilon$ of the sup of the radii of such balls. If we shrink each B_i by a factor of $1 + \frac{1}{1-\epsilon}$ to make a ball B_i', the B_i's are pairwise disjoint. Since the volume of B_0 is at least the sum of the volumes of the B_i's, the radii of the B_is goes to zero, so $E \subseteq \cup_i B_i$.

To finish, we just need to show that each B_i intersects less than c_n of the balls $B_1, ..., B_{i-1}$. To do this, we divide the balls $B_1, ..., B_{i-1}$ which intersect B_i into two groups based on whether their radii are at most M times the radius of B_i. The group of smaller balls is bounded because the B_j's are disjoint and contained in a ball of radius $2M+1$ times the radius of B_i (and the radii of the B_j with $j < i$ are at least $1-\epsilon$ times the radius of B_i). The group of larger balls is bounded by showing that the angles between the rays connecting the center of B_i with the centers of the B_js must be large (approaching $\frac{\pi}{3}$) if M is big enough and ϵ small enough (using the law of cosines). □

1.1.3 Topologies on $C(X, Y)$

Definition 1.1.44. The *compact-open* topology on $C(X, Y)$ has a subbase given by

$$V(K, U) = \{f : X \to Y \mid f(K) \subseteq U\}$$

for K compact and U open.

Proposition 1.1.45. *If Y is a metric space then $f_n \to f$ in the compact-open topology iff $\forall K \subseteq X$ compact we have $f_n \to f$ uniformly on K, so in this case the compact-open topology is the "topology of compact convergence". If X is compact as well, this becomes the uniform convergence topology.*

Proposition 1.1.46. *If Y is locally compact Hausdorff, composition $\circ : C(Y,Z) \times C(X,Y) \to C(X,Z)$ is continuous in the compact-open topology.*

Definition 1.1.47. If X, Y Banach spaces, $U \subseteq X$ open, $\mathcal{C}^m(U,Y)$ the m-times continuously Frechét-differentiable functions $U \to Y$, then the "compact-open" topology on $\mathcal{C}^m(U,Y)$ is induced by the seminorms

$$\rho_K(f) = \sup\{\|D^j f_x\| \mid x \in K,\ 0 \leq j \leq m\}$$

for $K \subseteq U$ compact.

Definition 1.1.48. The topology of *compact convergence* is defined by $f_n \to f$ iff for all K compact, $f_n|_K \to f|_K$ converges uniformly.

Proposition 1.1.49. *A set F of functions is called* normal *if every sequence of functions from F contains a subsequence that converges compactly to a continuous function.*

Theorem 1.1.50 (Montel)**.** *Any uniformly bounded family of holomorphic functions defined on an open subset of $\mathbb{C}$ is normal.*

Definition 1.1.51. The topology of *pointwise convergence* is the product topology on Y^X - this has $f_n \to f$ iff $f_n(x) \to f(x)$ for all x.

1.1.4 Measure

Definition 1.1.52. Two subsets A, B of $\mathbb{R}^n$ are called *equidecomposable* if A can be cut into finitely many disjoint pieces which can be translated and rotated to give a disjoint decomposition of B. More generally, if G is a group acting on a set X, then two subsets A, B of X are *G-equidecomposable* if we can write $A = A_1 \uplus \cdots \uplus A_n$ and if there are $g_i \in G$ with $B = g_1 A_1 \uplus \cdots \uplus g_n A_n$.

Proposition 1.1.53 (Banach-Cantor-Schröder-Bernstein)**.** *If A is equidecomposable with a subset of B and B is equidecomposable with a subset of A, then A, B are equidecomposable.*

Definition 1.1.54. If G acts on X and $Y \subseteq X$, we say that Y is *G-paradoxical* if there are disjoint $A, B \subseteq Y$ which are both G-equidecomposable with Y.

Proposition 1.1.55. *If F_2 is the free group on two generators a, b, then we can write $F = A_0 \uplus A_1 \uplus A_2 \uplus B_1 \uplus B_2$, with $F = A_0 \uplus aA_1 \uplus A_2 = bB_1 \uplus B_2$. In particular, F_2 is F_2-paradoxical.*

Proof. For any word w, let $W(w)$ be the set of elements of F_2 that begin with w. Take $A_0 = \{a^{-n} \mid n \geq 0\}, A_1 = W(a^{-1}) \setminus A_0, A_2 = W(a), B_1 = W(b^{-1}), B_2 = W(b)$. □

Proposition 1.1.56. *If G is G-paradoxical and acts on X without fixed points, then X is G-paradoxical.*

Lemma 1.1.57. *$SO(3)$ contains a free group of rank 2.*

Proof. Let σ, τ be the matrices

$$\sigma = \frac{1}{3}\begin{pmatrix} 1 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \tau = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & -2\sqrt{2} \\ 0 & 2\sqrt{2} & 1 \end{pmatrix}.$$

It's easy to check by induction on the length of w that if w is a word of length k ending with σ, then $w \cdot (1\ 0\ 0)^T = \frac{1}{3^k}(a\ b\sqrt{2}\ c)^T$ with $3 \nmid b$, and that if w starts with $\sigma^{\pm}$ then $a \equiv \pm b \pmod 3$ and $c \equiv 0 \pmod 3$, while if w begins with $\tau^{\pm}$ then $c \equiv \pm b \pmod 3$ and $a \equiv 0 \pmod 3$. Thus σ, τ generate a free group of rank 2. □

Proposition 1.1.58. *If E is a subset of S^2 with $|E| < 2^{\aleph_0}$, then S^2 is equidecomposable with $S^2 \setminus E$.*

Proof. We just need to find a rotation ρ with $\rho^n(E) \cap E = \emptyset$ for all $n > 0$. We take the axis to be any line through the origin which doesn't pass through any point of E, and then choose the angle of the rotation avoiding $|E| \times |E| \times |\mathbb{N}|$ bad angles. □

Corollary 1.1.59 (Banach-Tarski Paradox)**.** *Any ball in $\mathbb{R}^3$ is paradoxical.*

Corollary 1.1.60 (Strong Banach-Tarski Paradox)**.** *If $A, B \subset \mathbb{R}^3$ have nonempty interior and are bounded, then they are equidecomposable.*

Definition 1.1.61. A set of subsets Σ of X is a *σ-algebra* over X if Σ satisfies: $\emptyset \in \Sigma$, $\forall A \in \Sigma$ we have $X \setminus A \in \Sigma$, and for any sequence $(A_n)_{n\in\mathbb{N}}$ of elements of Σ we have $\cup_n A_n \in \Sigma$.

Proposition 1.1.62. *If Σ is a σ-algebra and Σ is infinite, then $|\Sigma| \ge 2^{\aleph_0}$. If Σ is generated by at most $2^{\aleph_0}$ sets, then $|\Sigma| \le 2^{\aleph_0}$ (more generally, if Σ is generated by κ sets, then $|\Sigma| \le \kappa^{\aleph_0}$).*

Definition 1.1.63. If X is a topological space, the *Borel* σ-algebra is the smallest σ-algebra containing the open subsets of X (some authors replace "open" by "compact" in this definition).

Proposition 1.1.64. *If X is metric, then the Borel σ-algebra can be generated from the open sets by iteratively taking closure under countable unions and intersections at most ω_1 times.*

Proof. Every open subset of X is a countable union of closed subsets of X, and ω_1 has uncountable cofinality. □

Corollary 1.1.65. *The Borel σ-algebra on $\mathbb{R}$ has cardinality $2^{\aleph_0}$.*

Proposition 1.1.66. *If E is in the σ-algebra generated by $\mathcal{A} \subseteq \mathcal{P}(X)$, then there is a countable subset $\{A_1, A_2, ...\}$ of $\mathcal{A}$ such that E is in the σ-algebra generated by $A_1, A_2, ...$ In particular, E can be written as a disjoint union of at most $2^{\aleph_0}$ countable intersections of elements of $\mathcal{A}$.*

Corollary 1.1.67 (Nedoma's Pathology)**.** *If $|X| > 2^{\aleph_0}$, then the set $\Delta_X = \{(x,x) \mid x \in X\}$ is not in the σ-algebra generated by the collection of all rectangles $E \times F$, where E, F are arbitrary subsets of X.*

Definition 1.1.68. $\mu : \Sigma \to [0,\infty]$ is a *measure* if $\mu(\emptyset) = 0$ and $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ whenever $E_i \in \Sigma$ and $E_i \cap E_j = \emptyset$ for all $i \ne j$. (X, Σ, μ) is called a *measure space* if Σ is a σ-algebra over X and $\mu : \Sigma \to [0,\infty]$ is a measure.

Proposition 1.1.69. *There is no translation invariant measure μ on the collection of all subsets of $\mathbb{R}$ which satisfies $\mu([0,1]) = 1$.*

Proof. Let G be any additive subgroup of $\mathbb{R}$ which contains $\mathbb{Z}$ and has $[G : \mathbb{Z}] = \aleph_0$ (we could take $G = \mathbb{Q}$, $G = \mathbb{Z}[\sqrt{2}]$, etc.). Let A be a set of representatives of $\mathbb{R}/G$ which are all in $[0,1]$. Then there is a set $X \subset G$ with $|X| = \aleph_0$ such that $[0,1] \subseteq A + X \subseteq [-1,2]$. Thus $\mu(A) \le \frac{\mu([-1,2])}{n} = \frac{3}{n}$ for all $n > 0$, so $\mu(A) = 0$, so $\mu([0,1]) = 0$ by countable additivity. □

Proposition 1.1.70. *If μ is a measure and $E_1 \subseteq E_2 \subseteq \cdots$ are measurable, then $\mu(\cup_{i=1}^{\infty} E_i) = \sup_i \mu(E_i)$. If $F_1 \supseteq F_2 \supseteq \cdots$ are measurable and $\mu(F_1) < \infty$, then $\mu(\cap_{i=1}^{\infty} F_i) = \inf_i \mu(F_i)$.*

Example 1.1.1. The assumption that $\mu(F_1) < \infty$ is necessary: for instance, consider the case $F_i = [i, \infty)$, where each F_i has infinite measure, but $\cap_{i=1}^{\infty} F_i = \emptyset$ has measure 0.

Definition 1.1.71. A set E is *σ-finite* with respect to a measure μ if E can be written as a countable union of sets with finite μ-measure. We say that μ is σ-finite if the whole space X is σ-finite with respect to μ. We say that μ is *decomposable* if X can be written as a disjoint union of σ-finite subsets X_i such that for any $A \subseteq X$, A is measurable iff $A \cap X_i$ is measurable for all i, and $\mu(A) = \sum_{i \in I} \mu(A \cap X_i)$.

Definition 1.1.72. A *signed measure* is a map $\mu : \Sigma \to [-\infty, \infty]$ which is countably additive (and doesn't take both $\infty, -\infty$ as values).

Theorem 1.1.73 (Hahn decomposition Theorem)**.** *If μ is a signed measure, then there exist measurable sets P, N such that $P \cup N = X, P \cap N = \emptyset$, and for all $E \subseteq P$ measurable we have $\mu(E) \ge 0$, while for all $E \subseteq N$ measurable we have $\mu(E) \le 0$. This decomposition is unique up to null sets.*

Proof. Assume WLOG that μ doesn't take the value $-\infty$. Say a measurable set is *negative* if every measurable subset has measure ≤ 0. First we show that for any measurable D with $\mu(D) \le 0$ there is a negative set $A \subseteq D$ with $\mu(A) \le \mu(D)$: define a sequence of sets A_n, $A_0 = D$, each A_{n+1} given by removing a set of positive measure from A_n whose measure is at least half as large as the sup of measures of subsets (if finite), or at least 1 otherwise, and take $A = \cap_n A_n$. Next, we define N by making a sequence N_n with $N_0 = \emptyset$, and N_{n+1} given by adding a negative set to N_n whose measure is at least half as negative as the inf of measure of subsets (if finite), or at most -1 otherwise, and take $N = \cup_n N_n$. □

Theorem 1.1.74 (Jordan decomposition Theorem)**.** *If μ is a signed measure, there is a unique decomposition $\mu = \mu^+ - \mu^-$ where μ^+, μ^- are positive measures (at least one of which is finite), such that $\mu^+(E)$ is 0 for any negative set E and μ^- is 0 for any positive set E.*

Definition 1.1.75. If μ is a signed measure and $\mu = \mu^+ - \mu^-$ is its Jordan decomposition, then we set $|\mu| = \mu^+ + \mu^-$.

Definition 1.1.76. A *complex measure* is a countably additive function $\mu : \Sigma \to \mathbb{C}$. Equivalently, it is a complex combination of finite measures.

Definition 1.1.77. If μ is a complex measure, we define the *total variation* of μ to be the positive measure $|\mu|$ given by $|\mu|(E) = \sup\{\sum_{i=1}^{n} |\mu(E_i)| \mid E = E_1 \sqcup \cdots \sqcup E_n\}$.

Definition 1.1.78. If μ, ν are (possibly signed) measures, then μ is *absolutely continuous* with respect to ν, written $\mu \ll \nu$, if $|\nu|(A) = 0 \implies |\mu|(A) = 0$.

Proposition 1.1.79. *If μ is finite, then $\mu \ll \nu$ iff for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|\nu|(A) < \delta \implies |\mu|(A) < \epsilon$.*

Proof. Assume μ, ν are positive. For every $n \geq 1$, let $n\nu - \mu$ have Hahn decomposition (P_n, N_n), and let $N = \bigcap_n N_n$. Since $n\nu(N) \leq \mu(N)$ for all n and $\mu(N) < \infty$, we have $\nu(N) = 0$, so we must have $\mu(N) = 0$ by $\mu \ll \nu$. Thus there is some n such that $\mu(N_n) < \frac{\epsilon}{2}$, and we can take $\delta = \frac{\epsilon}{2n}$: if $\nu(A) < \delta$, then $\mu(A) = \mu(A \cap P_n) + \mu(A \cap N_n) \leq n\nu(A) + \mu(N_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. □

Definition 1.1.80. We say that two (possibly signed or complex) measures μ, ν on X are *singular*, written $\mu \perp \nu$, if there are measurable sets A, B with $A \cup B = X$ such that B is μ-null and A is ν-null.

Theorem 1.1.81 (Lebesgue decomposition Theorem)**.** *If μ, ν are (possibly signed) σ-finite measures over X, then there is a unique pair of σ-finite measure μ_{ac}, μ_s such that $\mu = \mu_{ac} + \mu_s$, $\mu_{ac} \ll \nu$, and $\mu_s \perp \nu$.*

Proof. We just need to prove this in the finite, unsigned case. Let $\mathcal{N}$ be the collection of ν-null sets. Define μ_{ac} by

$$\mu_{ac}(A) = \inf_{N \in \mathcal{N}} \mu(A \setminus N).$$

μ_{ac} is clearly nonnegative and countably additive, and we clearly have $\mu_{ac} \ll \nu$. Set $\mu_s = \mu - \mu_{ac}$, taking $A = X$ and noting that the infimum must actually be attained, we see that there is a ν-null set N such that $\mu_s(X \setminus N) = 0$, so $\mu_s \perp \nu$.

For uniqueness, suppose that $\mu = \mu_1 + \mu_2$ with $\mu_1 \ll \nu, \mu_2 \perp \nu$. Since $\mu_1 \leq \mu$ and $\mu_1 \ll \nu$, we have

$$\mu_1(A) = \inf_{N \in \mathcal{N}} \mu_1(A \setminus N) \leq \inf_{N \in \mathcal{N}} \mu(A \setminus N) = \mu_{ac}(A),$$

so $\mu_1 \leq \mu_{ac}$. Thus $\mu_{ac} - \mu_1 = \mu_2 - \mu_s$ is both ν-absolutely continuous and ν-singular, so $\mu_1 = \mu_{ac}$. □

Constructing measures

Definition 1.1.82. On any set, the *counting measure* takes every finite set to its size and every infinite set to ∞. If $S = \{s_1, ...\}$ is a countable subset of X and $a_1, ... \in [0, \infty]$, then the *discrete measure* $\sum_i a_i \delta_{s_i}$ is given by $E \mapsto \sum_{s_i \in E} a_i$. More generally, if $f : X \to [0, \infty]$, we can define a measure $A \mapsto \sum_{a \in A} f(a)$, where the sum over A is defined to be the supremum of all the sums over finite subsets of A.

Definition 1.1.83. A measure space (X, Σ, μ) is *complete* if every subset of a null set (that is, a set with measure 0) is in Σ. If Z is the collection of all subsets of null sets, then define Σ_0 to be the σ-algebra generated by Σ and Z, and $\mu_0(C) = \inf\{\mu(D) \mid C \subseteq D \in \Sigma\}$, and define the *completion* of (X, Σ, μ) to be (X, Σ_0, μ_0).

Proposition 1.1.84. *The completion of a measure space is always a complete measure space, and in fact $\Sigma_0 = \{A \cup B \mid A \in \Sigma, B \in Z\}$.*

Definition 1.1.85. $\varphi : \mathcal{P}(X) \to [0, \infty]$ is an *outer measure* if $\varphi(\emptyset) = 0$, $A \subseteq B \implies \varphi(A) \leq \varphi(B)$, and for any sequence $(A_n)_{n \in \mathbb{N}}$ we have have $\varphi(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \varphi(A_i)$.

Definition 1.1.86. If φ is an outer measure over X, we say that E is *φ-measurable* if $\forall A \subseteq X$, we have $\varphi(A) = \varphi(A \cap E) + \varphi(A \cap E^c)$. We write Σ_φ for the collection of all φ-measurable sets.

Theorem 1.1.87. *If φ is an outer measure, then Σ_φ is a σ-algebra, and the restriction of φ to Σ_φ is a complete measure.*

Proof. If $E_i \in \Sigma_\varphi$ are pairwise disjoint and $E = \cup_{i=1}^\infty E_i$, then for any A we have

$$\varphi(A) \leq \varphi(A \cap E^c) + \varphi(A \cap E) \leq \varphi(A \cap E^c) + \sum_{i=1}^\infty \varphi(A \cap E_i) = \sup_n \Big(\varphi(A \cap E^c) + \sum_{i=1}^n \varphi(A \cap E_i)\Big) \leq \varphi(A).$$

Taking $A = E$ shows that $\varphi(E) = \sum_{i=1}^\infty \varphi(E_i)$. □

Definition 1.1.88. An outer measure φ is *regular* if for every set E there exists a φ-measurable set $A \supseteq E$ with $\varphi(E) = \varphi(A)$. (Note: don't confuse regular outer measures with "outer regular" Borel measures in the topological setting!)

Proposition 1.1.89. *If φ is an outer measure and $\varphi(A) = 0$, then A is φ-measurable. More generally, if $B \subseteq A$ with B φ-measurable, $\varphi(A) < \infty$, and $\varphi(A) = \varphi(B)$, then A is φ-measurable.*

Proposition 1.1.90. *If φ is a regular outer measure, $A \subseteq B$ with B φ-measurable, $\varphi(B) < \infty$, $\varphi(A)$, and $\varphi(B) = \varphi(A) + \varphi(B \setminus A)$, then A is φ-measurable.*

Definition 1.1.91. If X is a metric space and φ is an outer measure over X, we say that φ is a *metric outer measure* if $d(E, F) > 0 \implies \varphi(E \cup F) = \varphi(E) + \varphi(F)$.

Theorem 1.1.92. *If φ is a metric outer measure, then all Borel sets are φ-measurable.*

Proof. If U is open, let $U_n = \{x \in U \mid B(x, \frac{1}{n}) \subseteq U\}$, and note that for any n, $d(U_n, U_{n+1}^c) \geq \frac{1}{n(n+1)} > 0$. For any A with $\varphi(A) < \infty$ we then have

$$\sum_{n \text{ odd}} \varphi(A \cap (U_{n+1} \setminus U_n)) \leq \varphi(A) < \infty,$$

and similarly for n even, so the tails of the sum go to zero. Then for any A we have

$$\varphi(A) \leq \varphi(A \cap U^c) + \varphi(A \cap U) \leq \inf_n \Big(\varphi(A \cap U^c) + \varphi(A \cap U_n) + \sum_{m \geq n} \varphi(A \cap (U_{m+1} \setminus U_m))\Big) \leq \varphi(A).$$ □

Definition 1.1.93. A G_δ set is any countable intersection of open sets, and an F_σ set is any countable union of closed sets.

Proposition 1.1.94. *In a metric space, every closed set is a G_δ set and every open set is an F_σ set.*

Proposition 1.1.95. *If X is a topological space, Y a metric space, and $f : X \to Y$ is any function, then the set of points of continuity of f is a G_δ set.*

Proof. Let C be the set of points of continuity of f, and for each $c \in C$ and each $n \in \mathbb{N}^+$, pick an open set $c \in U_n^c \subseteq X$ such that $x \in U_n^c \implies d_Y(f(x), f(c)) < \frac{1}{n}$. Then $C = \cap_n \cup_{c \in C} U_n^c$. □

Definition 1.1.96. A collection of sets S is a *semi-ring* if $\emptyset \in S$, for any $A, B \in S$ we have $A \cap B \in S$, and for any $A, B \in S$ there exists n and pairwise disjoint $C_1, ..., C_n \in S$ such that $A \setminus B = \cup_{i=1}^n C_i$.

Definition 1.1.97. If S is a collection of sets, then a map $\mu : S \to [0, \infty]$ is a *pre-measure* if $\mu(\emptyset) = 0$ and for any sequence A_n of pairwise disjoint sets in S such that $\cup_{i=1}^{\infty} A_i \in S$, we have $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

Theorem 1.1.98 (Carathéodory Extension Theorem)**.** *If S is a semi-ring of subsets of X and $\mu_0 : S \to [0, \infty]$ is a pre-measure, then if we define μ^* by*

$$\mu^*(E) = \inf\Big\{ \sum_{i=1}^{\infty} \mu_0(A_i) \mid A_i \in S,\ E \subseteq \bigcup_{i=1}^{\infty} A_i \Big\},$$

then μ^* is an outer measure over X with $\mu^*(A) = \mu_0(A)$ for all $A \in S$, and $S \subseteq \Sigma_{\mu^*}$.

Definition 1.1.99. A pre-measure $\mu : S \to [0, \infty]$ with S a collection of subsets of X is *σ-finite* if there exists a sequence $A_n \in S$ with $\mu(A_i) < \infty$ and $X = \cup_{i=1}^{\infty} A_i$.

Theorem 1.1.100 (Hahn-Kolmogorov)**.** *If μ_0 is a pre-measure on a semi-ring S, then it extends to a measure μ on the σ-algebra Σ generated by S. If μ_0 is σ-finite, then this extension is unique.*

Proof. Let μ^* be the associated outer measure from the Carathéodory extension theorem, and suppose μ' is a different measure extending μ on $\Sigma' \supseteq S$. Then for any $E \in \Sigma' \cap \Sigma_{\mu^*}$, we clearly have $\mu'(E) \le \mu^*(E)$. By σ-finiteness and the fact that μ' is countably additive, we can assume WLOG that $\mu^*(X) = \mu'(X) < \infty$, but then $\mu'(E^c) \le \mu^*(E^c)$ implies $\mu'(E) = \mu^*(E)$ since E is μ^*-measurable. □

Proposition 1.1.101. *Let $\mu_0, \mu^*, \mu, S, \Sigma, \Sigma_{\mu^*}$ be as above. If μ_0 is σ-finite, then Σ_{μ^*} is the completion of Σ - in fact, for any $E \in \Sigma_{\mu^*}$, there is a countable intersection of countable unions of elements of S which contains E and differs from it in a null set.*

Theorem 1.1.102 (Lebesgue outer measure)**.** *Let S be the collection of half-open intervals $[a, b)$ for $a \le b \in \mathbb{R}$, and define $\lambda_0 : S \to [0, \infty)$ by $\lambda_0([a, b)) = b - a$. Then S is a semi-ring, λ_0 is a pre-measure, and the associated outer measure λ^* is a translation-invariant metric outer measure over $\mathbb{R}$ with $\lambda^*([0, 1]) = 1$.*

Proof. Suppose that $[a, b) = \cup_{i=1}^{\infty} A_i$, where the A_i are pairwise disjoint half-open intervals. Then the set of left endpoints of the A_i is well-ordered (any descending sequence must have a limit in $[a, b)$, and this limit must be contained in some A_i), so we can show by well-founded induction that if $A_i = [c, d)$, then $\sum_{A_j < A_i} \lambda_0(A_j) = c - a$.

Alternate proof: Let $A' = [a, b - \epsilon]$, and if $A_i = [c_i, d_i)$ let $A_i' = (c_i - \epsilon/2^i, d_i)$. Then by compactness, some finite subset of the A_i's cover A'. □

Definition 1.1.103. If λ^* is constructed as above, then a set is called *Lebesgue-measurable* if it is in Σ_{λ^*}, and $\lambda^* |_{\Sigma_{\lambda^*}}$ is called the *Lebesgue measure*, and written as λ.

Theorem 1.1.104 (Lebesgue-Stieltjes measure)**.** *If I is an interval and $g : I \to \mathbb{R}$ is monotone increasing, set $g_-(x) = \sup_{y<x} g(y)$, then there is a unique Borel measure μ_g such that $\mu_g([a, b)) = g_-(b) - g_-(a)$. If g is continuous, then for any E we have $\mu_g(E) = \lambda(g(E))$.*

Definition 1.1.105. If g has bounded variation, then we define the *signed Lebesgue-Stieltjes measure* μ_g by writing $g = g_1 - g_2$ with g_1, g_2 monotone increasing, and $\mu_g = \mu_{g_1} - \mu_{g_2}$.

Definition 1.1.106. A Borel measure μ is *locally finite* if every point has an open neighborhood of finite measure. It is *inner regular* on B if $\mu(B)$ is the supremum of $\mu(K)$ over all compact $K \subseteq B$. It is *outer regular* if for all Borel sets B, $\mu(B)$ is the infimum of $\mu(U)$ over all open U containing B. A measure is *Radon* if it is inner regular on open sets, outer regular, and locally finite.

Proposition 1.1.107. *Every locally finite Borel measure over $\mathbb{R}$ is a Lebesgue-Stieltjes measure, and every Lebesgue-Stieltjes measure is a Radon measure. More generally, every locally finite Borel measure on $\mathbb{R}^n$ is Radon.*

Theorem 1.1.108 (Besicovitch Covering Theorem for Radon Measures)**.** *If E is a bounded subset of $\mathbb{R}^n$ and μ is a Radon measure on $\mathbb{R}^n$ with associated outer measure μ^*, and if $\mathcal{B}$ is a collection of closed balls such that every point in E is the center of an arbitrarily small ball of $\mathcal{B}$, then there exists a countable collection of disjoint balls $\{B_i\} \subseteq \mathcal{B}$ such that $\mu^*(E \setminus \cup_i B_i) = 0$.*

Proof. By the Besicovitch Covering Lemma 1.1.43, we can find a finite number c_n of families $\mathcal{B}_i \subseteq \mathcal{B}$ of disjoint balls such that $E \subseteq \cup_i \cup_{B \in \mathcal{B}_i} B$. Then for some i we must have $\mu^*(E \cap \cup_{B \in \mathcal{B}_i} B) \geq \mu^*(E)/c_n$. Pick some finite subset $B_1, ..., B_k$ of $\mathcal{B}_i$ such that $\mu^*(E \cap \cup_{j \leq k} B_j) \geq \mu^*(E)/2c_n$. Now replace E by $E \setminus \cup_{j \leq k} B_j$ and replace $\mathcal{B}$ by the set of balls of $\mathcal{B}$ which do not intersect the closed set $\cup_{j \leq k} B_j$, and iterate. □

Theorem 1.1.109 (Product measures)**.** *If μ, ν are pre-measures on semi-rings S, T, respectively, then the collection of rectangles $S \times T$ is a semi-ring, and $\mu \times \nu$ is a pre-measure on $S \times T$.*

Proof. Suppose $E \times F \in S \times T$ is a countable union of disjoint rectangles $E_i \times F_i$. We'll show that for any $M < \mu(E)$ and $N < \nu(F)$, we have $MN \leq \sum_i \mu(E_i)\nu(F_i)$. Let $A_n = \{x \in E \mid \sum_{i=1}^n 1_{x \in E_i} \cdot \nu(F_i) \geq N\}$. Each A_n is a finite union of elements of S, and $\cup_n A_n = E$ since for each $x \in E$, the collection of F_is with $x \in E_i$ is disjoint and covers F, so some finite subset of them must have measure at least N. Thus there is some n such that $\mu(A_n) \geq M$, and for this n we have $MN \leq \sum_{i=1}^n \mu(E_i)\nu(F_i)$. □

Theorem 1.1.110 (Infinite products)**.** *Let I be any index set. If μ_i are pre-measures on semi-rings S_i, such that each S_i has an element X_i with $\mu_i(X_i) = 1$, and if we let $S = \prod'_{i \in I} S_i$ be the set of rectangles $\prod_{i \in I} A_i$ such that $A_i = X_i$ for all but finitely many i and define $\mu = \prod_i \mu_i$, then S is a semi-ring and μ is a pre-measure on S.*

Proof. Suppose that $A = \cup_{n=1}^\infty A_n$ with $A, A_n \in S$ and the A_ns disjoint, but that $\mu(A) > \sum_n \mu(A_n)$. Each A_n only has finitely many coordinates i which are not equal to X_i, so at most countably many coordinates in I are relevant - rename these relevant coordinates as $1, 2, ...$ Write $A = E \times F$, $A_n = E_n \times F_n$, with $E, E_n \in S_1$ and $F, F_n \in \prod'_{i \neq 1} S_i$, and write $\mu^1 = \prod_{i \neq 1} \mu_i$. By the argument for the finite case, there is some $x_1 \in E$ such that $\mu^1(F) > \sum_n 1_{x_1 \in E_n} \cdot \mu^1(F_n)$. Continuing inductively, we find a sequence of coordinates $x_1, x_2, ...$ such that for each k, when we restrict the first k coordinates to be $x_1, ..., x_k$, the two sides don't add up. But then no point with $(x_1, x_2, ...)$ as the relevant countably many coordinates can be an element of any A_n (take k to be larger than the finitely many coordinates i of A, A_n which are not equal to X_i), contradicting the assumption $A = \cup_n A_n$. □

Corollary 1.1.111 (Lebesgue measure on $\mathbb{R}^n$)**.** *For every n, there is a translation-invariant metric outer measure λ^* on $\mathbb{R}^n$ with $\lambda^*([0,1]^n) = 1$. If T is a linear transformation and $A \subseteq \mathbb{R}^n$, then $\lambda^*(T(A)) = |\det(T)|\lambda^*(A)$. The associated measure λ is a Radon measure.*

Proof. For the statement about linear transformations, it's enough to check this for shear and stretch transformations in the case A is a box, and this can done done using a standard dissection argument (the pieces are Borel sets). □

Definition 1.1.112. If X, Y are measure spaces with measures μ, ν, then $X \times Y$ has a measure $\mu \times \nu$ given by applying the Carathéodory extension Theorem 1.1.98 to the product pre-measure contructed in Theorem 1.1.109 - this measure is called the *maximal product measure* on $X \times Y$.

Proposition 1.1.113. *If $A \subseteq X \times Y$ is $\mu \times \nu$-null, then the set of $y \in Y$ such that $A_y = \{x \in X \mid (x, y) \in A\}$ is not μ-null is ν-null.*

Proof. Pick $\epsilon > 0$, and let E be the set of $y \in Y$ such that $\mu(A_y) > \epsilon$. If $A \subseteq \cup_{n=1}^{\infty} R_n$ such that the R_n are measurable rectangles, and E_k is the set of y such that $\mu((\cup_{n=1}^{k} R_k)_y) > \epsilon$, then $\cup_k E_k = E$, so if $\nu(E) > \delta$ then some $\nu(E_k) > \delta/2$, so $\mu \times \nu(\cup_n R_n) > \epsilon\delta/2$. □

I haven't been able to find a nice proof of the following result which doesn't use integration. Once the integral is defined, it will be a quick consequence of Tonelli's Theorem (1.1.173).

Theorem 1.1.114 (Cavalieri Principle)**.** *If X, Y are σ-finite measure spaces and $A, B \subseteq X \times Y$ are measurable with $\mu(A_y) = \mu(B_y)$ for ν-almost every $y \in Y$, then $\mu \times \nu(A) = \mu \times \nu(B)$.*

Example 1.1.2. To see σ-finiteness is necessary, take X to be $[0, 1]$ with counting measure, Y to be $[0, 1]$ with Lebesgue measure, A to be $\{0\} \times Y$, and B to be the diagonal.

Theorem 1.1.115 (Lebesgue Density Theorem)**.** *If $E \subseteq \mathbb{R}^n$, then for Lebesgue-a.e. x in E we have*

$$\lim_{r \to 0} \frac{\lambda^*(E \cap B_r(x))}{\lambda(B_r(x))} = 1.$$

Proof. Let A_t be the set of points such that the left hand side (with a lim inf instead) is less than $1-t$, and let U_ϵ be an open set containing A_t with $\lambda^*(U_\epsilon \setminus A_t) \leq \epsilon$. Then for each point x in A_t, we can find an r such that the left hand side of the above is at most $1 - t$ and such that $B_r(x) \subseteq U_\epsilon$. Now apply the Vitali Covering Lemma to get a collection $(B_i)_{i \in I}$ of disjoint balls contained in U_ϵ such that $A_t \subseteq \cup_i 5B_i$. Then since $\cup_i B_i \subseteq U_\epsilon$, we have

$$\lambda(\cup_i B_i) - \epsilon \leq \lambda^*(A \cap (\cup_i B_i)) \leq \lambda^*(E \cap (\cup_i B_i)) \leq \sum_i (1-t)\lambda(B_i) = (1-t)\lambda(\cup_i B_i),$$

so $\lambda(\cup_i B_i) \leq \epsilon/t$, and since $A_t \subseteq \cup_i 5B_i$ we get $\lambda^*(A_t) \leq 5^n \epsilon/t$. Since $\epsilon > 0$ was arbitrary, $\lambda^*(A_t) = 0$. □

Definition 1.1.116. If X is a metric space and $S \subseteq X$, we set

$$H_\delta^d(S) = \inf\Big\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^d \mid S \subseteq \bigcup_{i=1}^{\infty} U_i,\ \operatorname{diam}(U_i) < \delta\Big\}$$

and

$$H^d(S) = \sup_{\delta > 0} H_\delta^d(S).$$

This is a metric outer measure, called the *Hausdorff measure*.

Theorem 1.1.117. *In $\mathbb{R}^n$, we have $H^n(B) = 2^n$, where B is the unit ball.*

Proof. This follows from the isodiametric inequality: the volume of a set of diameter 2 is at most the volume of the unit ball. Suppose that K has diameter 2, then $K - K \subseteq 2B$, so by Brunn-Minkowski we have $\lambda(K) \le \lambda(\frac{1}{2}(K - K)) \le \lambda(B)$. □

Definition 1.1.118. If X is a metric space and $S \subseteq X$, we define the *Hausdorff content* of S to be

$$C^d(S) = \inf\Big\{\sum_{i=1}^{\infty} r_i^d \mid S \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i)\Big\}.$$

Proposition 1.1.119. *If X is a metric space and $S \subseteq X$, then $C^d(S) = 0$ iff $H^d(S) = 0$.*

Definition 1.1.120. If X is a metric space and $S \subseteq X$, the *Hausdorff dimension* of S is defined to be the infimum of the set of d such that $C^d(S) = 0$.

Proposition 1.1.121. *If X is a metric space and $S \subseteq X$, then there is a G_δ set which contains S and has the same Hausdorff dimension as S.*

Proposition 1.1.122. *If X is a metric space and $S \subseteq X$, then there is a G_δ set which contains S and has the same Hausdorff measure H^d as S (so H^d is a regular outer measure). If additionally S is H^d-measurable and $H^d(S) < \infty$, then there is an F_σ set contained in S with the same Hausdorff measure as S.*

Proof. For the first part, note that in the definition of $H_\delta^d(S)$ we may restrict the covers to be covers by open sets without changing the inf, and take an intersection of open covers over a sequence of δs going to 0. For the second part, let $\cap_i O_i \supseteq S$ be the G_δ set from the first part, and write each O_i as an F_σ set by $O_i = \cup_j C_{ij}$. For each i, find j_i such that $H^d(S \setminus C_{ij_i}) < \epsilon/2^i$, and let $C_\epsilon = \cap_i C_{ij_i}$. Then $H^d(C_\epsilon) > H^d(S) - \epsilon$, and $H^d(C_\epsilon \setminus S) \le H^d(\cap_i O_i \setminus S) = 0$, so we can find an H^d-null G_δ set containing $C_\epsilon \setminus S$, and removing it from C_ϵ we get an F_σ set $C'_\epsilon \subseteq S$. Now take a union over a sequence of ϵs going to 0. □

Theorem 1.1.123 (Vitali Covering Theorem for Hausdorff Measure)**.** *If E is a subset of a metric space, and $\mathcal{V}$ is a collection of sets such that every point of E is contained in an element of $\mathcal{V}$ of arbitrarily small nonzero diameter, then there is a countable disjoint subcollection $\{U_i\} \subseteq \mathcal{V}$ such that either $H^d(E \setminus \cup_i U_i) = 0$ or $\sum_i \operatorname{diam}(U_i)^d = \infty$.*

Furthermore, if $H^d(E) < \infty$, then for any $\epsilon > 0$ we may choose this subcollection such that $H^d(E) \le \sum_i \operatorname{diam}(U_i)^d + \epsilon$.

Proof. Let ρ be small, and assume WLOG that all diameters of sets in $\mathcal{V}$ are at most ρ. At each step, choose U_i to be disjoint from $U_1, ..., U_{i-1}$ with diameter at least $\frac{1}{2}$ the sup of the diameters of such disjoint sets. For each i, let B_i be a ball with center in U_i and radius $3\operatorname{diam}(U_i)$, then $E \setminus \cup_{i\le k} U_i \subseteq \cup_{i>k} B_i$. If $\sum_i \operatorname{diam}(U_i)^d < \infty$, then $\operatorname{diam}(B_i) \to 0$ and the tails of $\sum_i \operatorname{diam}(B_i)^d$ go to 0, so each $H_\delta^d(E \setminus \cup_i U_i) = 0$, etc. □

Corollary 1.1.124. *An arbitrary union of full-dimensional closed convex subsets of $\mathbb{R}^n$ is Lebesgue measurable.*

Proof. Reduce to the case where everything is contained in a big ball and all the convex sets C satisfy $\mathrm{diam}(C)^n \le k\lambda(C)$ for some integer k, then apply the Vitali Covering Theorem 1.1.123 to the collection of homothetic images of these convex sets which are contained within the union, to see that the union can be decomposed into a countable disjoint union of closed convex sets together with a set of measure 0. □

For exceedingly large (in terms of cardinality) spaces, issues like Nedoma's pathology 1.1.67 require us to use a different approach to constructing measures, by making use of topological structure.

Definition 1.1.125. If X is a locally compact Hausdorff space, then a Borel measure μ is called a *Radon measure* if it is locally finite, outer regular, and inner regular on open sets.

Definition 1.1.126. A Borel measure μ is called an *inner Radon measure* if it is locally finite and inner regular on all Borel sets.

Proposition 1.1.127. *If X is Hausdorff, μ is a Radon measure, and E is σ-finite, then* $\mu(E) = \sup\{\mu(K) \mid K \subseteq E,\ K \text{ compact}\}$.

Proof. It's enough to prove this when $\mu(E) < \infty$. Take an open set $U \supseteq E$ with $\mu(U) < \infty$, take a compact $K \subseteq U$ with $\mu(U \setminus K) < \epsilon$, and take an open V with $K \setminus E \subseteq V$, $\mu(V \setminus (K \setminus E)) < \epsilon$, then $K \setminus V$ is compact, contained in E, and $\mu(K \setminus V) > \mu(E) - 2\epsilon$. □

Proposition 1.1.128. *In a locally compact Hausdorff space, if $K \subseteq U$ with K compact and U open, then there is a compact L with $K \subseteq \mathrm{int}(L)$ and $L \subseteq U$.*

Definition 1.1.129. Call a collection $\mathcal{K}$ of compact subsets of a locally compact Hausdorff space X *splittable* if $\mathcal{K}$ is a local base of neighborhoods of X which is closed under finite unions.

Proposition 1.1.130. *Suppose that $\mathcal{K}$ is a splittable collection of compact subsets. Then for any K compact and U_1, U_2 open with $K \subseteq U_1 \cup U_2$ there are $K_1, K_2 \in \mathcal{K}$ with $K_i \subseteq U_i$ and $K \subseteq K_1 \cup K_2$.*

Definition 1.1.131. A *content* on a splittable collection of compact sets $\mathcal{K}$ is a function $\lambda : \mathcal{K} \to [0, \infty)$, such that $\lambda(K)$ is increasing in K, $\lambda(K_1 \cup K_2) \le \lambda(K_1) + \lambda(K_2)$, and such that for any $K_1, K_2 \in \mathcal{K}$ disjoint, we have $\lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$. A content λ is *regular* if for any $K \in \mathcal{K}$, we have $\lambda(K) = \inf\{\lambda(L) \mid K \subseteq \mathrm{int}(L)\}$.

Theorem 1.1.132. *For every content λ on a splittable collection of compact subsets of a locally compact Hausdorff space X, there is a unique Radon measure μ on X such that for all open sets U we have $\mu(U) = \sup\{\lambda(K) \mid K \subseteq U\}$. If λ is a regular content, then μ extends λ.*

Proof. Define μ on open sets as in the theorem statement, and define $\mu^* : \mathcal{P}(X) \to [0, \infty]$ by $\mu^*(A) = \inf\{\mu(U) \mid A \subseteq U\}$. μ is finite on the interior of any compact set, so μ^* is locally finite.

First we show that μ^* is an outer measure: If $A = \cup_{n=1}^{\infty} A_n$, then pick U_n open with $A_n \subseteq U_n$ and $\mu^*(U_n) \le \mu^*(A_n) + \epsilon/2^n$, and let $U = \cup_n U_n$. Pick $K \subseteq U$ compact with $\mu(U) \le \lambda(K) + \epsilon$, then some finite subset of the U_n cover K, say $U_1, ..., U_k$. We just need to show that $\lambda(K) \le \sum_{i=1}^{k} \mu(U_i)$, and this follows if we can construct compact $K_i \subseteq U_i$ with $K \subseteq \cup_i K_i$, which follows from splittability.

Now we show that open sets are μ^*-measurable. Let U be open and $A \subseteq X$ be arbitrary. We want to show that for any open $V \supseteq A$, we have $\mu(V) \ge \mu^*(A \cap U) + \mu^*(A \cap U^c)$, so we just need to show that $\mu(V) \ge \mu(V \cap U) + \mu^*(V \setminus U)$. For any compact $K \subseteq V \cap U$, let $W = V \setminus K$, then for any compact $L \subseteq W$ we have $\mu(V) \ge \lambda(K \cup L) = \lambda(K) + \lambda(L)$, so $\mu(V) \ge \lambda(K) + \mu(W) \ge \lambda(K) + \mu^*(V \setminus U)$, so $\mu(V) \ge \mu(V \cap U) + \mu^*(V \setminus U)$. □

Corollary 1.1.133. *Every regular content on a locally compact Hausdorff space extends to a unique inner Radon measure. There is a bijection between Radon measures and inner Radon measures on such spaces.*

Proof. Suppose λ is a regular content, and let μ be the associated Radon measure. Define μ_{in} on Borel sets E by $\mu_{in}(E) = \sup\{\mu(K) \mid K \subseteq E,\ K \text{ compact}\}$. To show μ_{in} is a Radon inner measure, we just need to check it is countably additive. Let $E = \bigcup_i E_i$ with $E_i \cap E_j = \emptyset$ for $i \neq j$. We clearly have $\mu_{in}(E) \geq \sum_i \mu_{in}(E_i)$. For the other direction, if $K \subseteq E$ is compact, then $\mu(K) < \infty$ implies that $\mu(K \cap E_i) = \mu_{in}(K \cap E_i)$ for all i by Proposition 1.1.127, so $\mu_{in}(K) = \mu(K) = \sum_i \mu(K \cap E_i) = \sum_i \mu_{in}(K \cap E_i)$.

The uniqueness of the extension of λ and the correspondence between Radon measures and inner Radon measures will both follow if we show that any Radon inner measure ν is outer regular on compact sets. So suppose K is compact, and let U be any open set which contains K and is contained in a compact set. Then $\nu(U) < \infty$, so for any ϵ there is a compact set $L \subseteq U \setminus K$ such that $\nu(U \setminus K) \leq \nu(L) + \epsilon$. Then $U \setminus L$ is an open set which contains K and has $\nu(U \setminus L) < \nu(K) + \epsilon$. □

Example 1.1.3. Consider the product topology on $\mathbb{R} \times X$, where X is an uncountable set with the discrete topology. Let λ be the natural content on finite unions of closed intervals in copies of $\mathbb{R}$, and let μ, μ_{in} be the associated Radon measure and Radon inner measure. Then the set $\{0\} \times X$ is not σ-finite with respect to μ, but has μ_{in}-measure 0.

Definition 1.1.134. If μ is a Borel measure, define its *support* to be the set of points p such that $p \in U$, U open imply $\mu(U) \neq 0$.

Proposition 1.1.135. *The support of a Borel measure on a topological space X is always closed. If μ is inner regular on open sets, then $\mu(X \setminus \operatorname{supp} \mu) = 0$.*

Definition 1.1.136. If μ is a Borel measure, then a family $\mathcal{D}$ of disjoint nonempty compact sets is called a *concassage* for μ if

- for any $K \in \mathcal{D}$ and any U open with $K \cap U \neq \emptyset$ we have $\mu(K \cap U) > 0$, and
- for any Borel set E, we have $\mu(E) = \sum_{K \in \mathcal{D}} \mu(E \cap K)$.

Proposition 1.1.137. *Every inner Radon measure μ on a locally compact Hausdorff space has a concassage.*

Proof. Let $\mathcal{D}$ be any maximal disjoint collection of nonempty disjoint compact sets K such that the restriction of μ to K has full support (such $\mathcal{D}$ exists by Zorn's Lemma). First, suppose for contradiction that there is a Borel set E such that $E \cap \bigcup_{K \in \mathcal{D}} K = \emptyset$ but $\mu(E) > 0$. Then by inner regularity we may assume that E is compact, and then by considering the support of the restriction of μ to E we see that $\mathcal{D}$ is not maximal.

Now let C be any compact set, and let U be an open set containing C which is contained in a compact set. Then from $\mu(U) < \infty$, we see that U can intersect at most countably many sets K in $\mathcal{D}$, so the same is true for C, and we have $\mu(C) = \mu(C \cap \bigcup_{K \in \mathcal{D}} K) = \sum_{K \in \mathcal{D}} \mu(C \cap K)$. By inner regularity, this implies that for any Borel set E, we have $\mu(E) \leq \sum_{K \in \mathcal{D}} \mu(E \cap K)$, and the other inequality follows from disjointness of the sets in $\mathcal{D}$. □

Definition 1.1.138. If X, Y are locally compact Hausdorff spaces with Radon measures μ, ν, then we define the *Radon product measure* $\mu \widehat{\times} \nu$ on $X \times Y$ to be the unique Radon measure such that for every open subset U of $X \times Y$, we have $\mu \widehat{\times} \nu(U) = \sup_{K \in \mathcal{K}} \mu \times \nu(K)$, where $\mathcal{K}$ is the collection of finite unions of products of compact subsets of X and Y. (Note that if μ, ν are σ-finite, then the restriction of $\mu \widehat{\times} \nu$ to the product σ-algebra on $X \times Y$ is $\mu \times \nu$.)

Proposition 1.1.139. *Suppose that (X, μ) is a σ-finite Radon measure space, Y is a topological space, and that U is an open subset of $X \times Y$. Then for any $r \in \mathbb{R}$, the set of $y \in Y$ such that $\mu(U_y) > r$ is an open subset of Y. In fact, for any y with $\mu(U_y) > r$, there are open sets $V \subseteq X$ and $W \subseteq Y$ such that $\mu(V) > r$, $y \in W$, and $V \times W \subseteq U$.*

Proof. Suppose that $\mu(U_y) > r$. By inner regularity, there is some compact set $K \subseteq U_y$ with $\mu(K) > r$. Now cover the compact set $K \times \{y\}$ with finitely many open rectangles contained in U. □

Lemma 1.1.140 (Weak Fubini for Radon Products)**.** *Let (X, μ) and (Y, ν) be σ-finite Radon spaces, and suppose that $E \subseteq X \times Y$ is a countable intersection of open subsets of $X \times Y$. If $\nu(E_x) = 0$ for μ-a.e. $x \in X$, then for ν-a.e. $y \in Y$ we have $\mu(E_y) = 0$.*

Proof. We may assume WLOG that $\mu(X) = \nu(Y) = 1$. Suppose for contradiction that there is some $\epsilon > 0$ such that the set of $y \in Y$ with $\mu(E_y) > \epsilon$ has ν-measure greater than ϵ. Write $E = \cap_n U_n$ with $U_1 \supseteq U_2 \supseteq \cdots$ open subsets of $X \times Y$. Then by the previous Proposition for each n there is a compact set $K_n \subseteq Y$ with $\nu(K_n) > \epsilon$, and such that for all $y \in K_n$ we have $\mu((U_n)_y) > \epsilon$, and there is a set $V_n \subseteq U_n$ which is a finite union of open rectangles, such that for all $y \in K_n$ we have $\mu((V_n)_y) > \epsilon$. Since $\mu \times \nu(V_n) > \epsilon^2$, we see that the set of x such that $\nu((V_n)_x) > \epsilon^2/2$ has μ-measure at least $\epsilon^2/2$, and thus the same is true if we replace V_n by U_n. Taking a decreasing limit of measurable subsets of X, we see that the μ-measure of the set of x such that $\nu(E_x) > \epsilon^2/2$ is at least $\epsilon^2/2$, a contradiction. □

Definition 1.1.141. If G is a locally compact Hausdorff group and μ is a Borel measure on G, then μ is a *left Haar measure* on G if $\mu(gE) = \mu(E)$ for $g \in G$ and E Borel, and μ is Radon.

Theorem 1.1.142 (Haar measure)**.** *If G is a locally compact Hausdorff group, then there is a unique (up to scale) left Haar measure on G.*

Proof. For K compact and V with nonempty interior, let $(K : V)$ be the minimum number of left translates of V that are needed to cover K. Pick K_0 compact with nonempty interior. For every U, define μ_U on compact sets by

$$\mu_U(K) = \frac{(K : U)}{(K_0 : U)}.$$

Then for all K, U we have $0 \le \mu_U(K) \le (K : K_0)$. We consider each μ_U as a point in $\prod_K [0, (K : K_0)]$. For each open V, let $C(V)$ be the closure of the set of μ_Us with $U \subseteq V$. By compactness, there exists $\mu \in \cap_V C(V)$. For K_1, K_2 disjoint, find V open such that $K_1 V^{-1} \cap K_2 V^{-1} = \emptyset$, then from $\mu \in C(V)$ we see that $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$. Thus μ defines a left-invariant content on the compact sets of G, so there is a left-invariant Radon measure on G by Theorem 1.1.132.

To prove uniqueness, suppose μ, ν are left Haar measures and K, L are compact, L with nonempty interior (so $\mu(L), \nu(L) > 0$). Let G_0 be the subgroup generated by K, L, so that G_0 is σ-compact (as well as clopen and locally compact Hausdorff), and restrict everything to G_0.

Suppose for contradiction that $\nu(K)/\nu(L) \neq \mu(K)/\mu(L)$, and rescale μ, ν so that $\mu(K) - \nu(K)$ and $\mu(L) - \nu(L)$ have opposite signs. For every $\epsilon > 0$, let (P_ϵ, N_ϵ) be a Hahn decomposition (Theorem 1.1.73) for $\mu - (1+\epsilon)\nu$ restricted to G_0, and let $N = \cap_{n>0} N_{1/n}$. Then for any $g \in G_0$, $gN \setminus N = \cup_{n>0}\, gN \cap P_{1/n}$ is a null set with respect to $\mu + \nu$. Now consider the set $\{(g,x) \in G_0 \times G_0 \mid x \in N \iff x \notin gN\}$ - this is a Borel subset of $G_0 \times G_0$ such that every column is null, and to finish we just need to show that some row is null, which can be shown using Lemma 1.1.140: we choose $N_0 \subseteq N \subseteq N_1$ such that N_1 and $G_0 \setminus N_0$ are countable intersections of open subsets of G_0 with $N_1 \setminus N_0$ null, so $\{(g,x) \mid g^{-1}x \in N_1, x \notin N_0\}$ is a countable intersection of open subsets of $G_0 \times G_0$, etc. □

Proposition 1.1.143. *Let G be a locally compact Hausdorff group with left Haar measure μ, let K be a compact subset of G with nonempty interior, and let G_0 be the clopen, σ-finite subgroup of G generated by K. Then a Borel subset E of G is σ-finite iff it intersects at most countably many left cosets of G_0, and if so we have $\mu(E) = \sum_{h \in G/G_0} \mu(E \cap hG_0)$.*

The corresponding inner Radon measure μ_{in} decomposes: for any Borel set E, we have $\mu_{in}(E) = \sum_{h \in G/G_0} \mu(E \cap hG_0)$.

Definition 1.1.144. If G is a locally compact Hausdorff group, then the *modular function* $\Delta : G \to \mathbb{R}^+$ is defined by $\mu(Kg) = \Delta(g)\mu(K)$, where μ is a left Haar-measure on G. If $\Delta(G) = \{1\}$, then G is called *unimodular*.

Proposition 1.1.145. *The modular function is continuous.*

Proof. Fix $\epsilon > 0$. Let K be a compact set with nonempty interior, and let $U \supseteq K$ be open such that $\mu(K) \le \mu(U) < (1+\epsilon)\mu(K)$. By compactness, there is a neighborhood of the identity V with $KV \subseteq U$. For $g \in V$, we have $\Delta(g) = \frac{\mu(Kg)}{\mu(K)} \le \frac{\mu(U)}{\mu(K)} < 1 + \epsilon$, and for $g \in V^{-1}$ we have $\Delta(g) = \frac{\mu(Ug)}{\mu(U)} \ge \frac{\mu(K)}{\mu(U)} > 1 - \epsilon$. □

Proposition 1.1.146. *If G is a locally compact Hausdorff group with left Haar measure μ and $\mu_{in}(A) > 0$, then AA^{-1} is a neighborhood of the identity.*

Proof. Choose $K \subseteq A$ compact with $\mu(K) > 0$, choose $U \supseteq K$ open such that $\mu(U) < 2\mu(K)$, and find V a neighborhood of the identity such that $VK \subseteq U$. Then for any $g \in V$, we have $\mu(gK \cup K) \le \mu(VK) \le \mu(U) < 2\mu(K)$, so $gK \cap K \neq \emptyset$. □

Next we'll use this to prove that measurable homomorphisms of locally compact Hausdorff groups are continuous, following [120]. (An even stronger statement is proved there.)

Lemma 1.1.147. *For N a subgroup of G, TFAE:*

1. *for all $x \in G$, $[N : xNx^{-1} \cap N] \le \aleph_0$,*

2. *each double coset NxN is a union of countably many left N cosets,*

3. *for each x there is a countable set D such that $Nx \subseteq DN$,*

4. *if $C \subseteq G$ is countable, and M is the subgroup generated by $N \cup C$, then $[M : N] \le \aleph_0$.*

Proof. For (1) $\iff$ (2), the double coset NxN is the orbit of $xN \in G/N$ under left translation by N, and the stabilizer is $xNx^{-1} \cap N$. (2) $\iff$ (3) is obvious, and (3) $\implies$ (4), (4) $\implies$ (2) are easy. □

Definition 1.1.148. A subgroup N of G is called *asoo* if it satisfies the equivalent conditions of Lemma 1.1.147.

Proposition 1.1.149. *Countable subgroups and normal subgroups are asoo. Any open σ-compact subgroup of a topological group is asoo. If $\phi : G \to H$ is a homomorphism and L is asoo in H then $\phi^{-1}(L)$ is asoo in G. If ϕ is onto and N is asoo in G, then $\phi(N)$ is asoo in H. If N is asoo and C is countable, then the subgroup generated by $N \cup C$ is asoo.*

Proposition 1.1.150. *If G is σ-compact and U_n is a countable family of neighborhoods of the identity, then there is a compact normal subgroup K of G such that $K \subseteq \bigcap_n U_n$ and G/K is separable.*

Proof. Let $G = \bigcup_n F_n$, with F_n an increasing sequence of compact subsets of G. Let V_0 be a compact neighborhood of the identity. For each n, we can find a symmetric neighborhood of the identity V_{n+1} such that $V_{n+1}^2 \subseteq V_n \cap U_n$ and for all $x \in F_n$ we have $xV_{n+1}x^{-1} \subseteq V_n$. Take $K = \bigcap_n V_n$. To finish, we need to show that for any open W containing the identity, there is some n such that $V_n \subseteq WK$. Otherwise, each $V_n \setminus WK$ is a compact nonempty subset of V_0, so by the finite intersection property $K \setminus WK$ is nonempty, contradiction. □

Lemma 1.1.151. *Let G be a locally compact Hausdorff group which is either separable or σ-compact and N a null asoo subgroup of G. Then there is a nonmeasurable set $S \subseteq G$ with $S = NS$.*

Proof. Let μ be left Haar measure. First, assume G is separable, and let $U_1 \supseteq U_2 \supseteq \cdots$ be a basis of neighborhoods of the identity. For each n take $x_n \in U_n \setminus N$ (which is nonempty since $\mu(U_n) > 0$). Let M be generated by N and the x_ns, and let Y be a set of right coset representatives of M in G. Take $S = NY$. Suppose S measurable, and let X be a set of left coset representatives of N in M. Since X is countable and $G = MY = XNY = XS$, we have $\mu_{in}(S) > 0$, so by Proposition 1.1.146 there is an n with $U_n \subseteq SS^{-1}$. But then $x_n \in SS^{-1}$, so $S \cap x_n S \neq \emptyset$, so $N \cap x_n N \neq \emptyset$, contradiction.

Next, assume that N is not closed, and take $x \in \overline{N} \setminus N$. Let M be generated by N and x, and define Y, S, X as before. If S is measurable, then as above we see there is an open neighborhood of the identity $U \subseteq SS^{-1}$. Since $x \in \overline{N}$, we have $xN \cap U \neq \emptyset$, so $S \cap xNS \neq \emptyset$, so $N \cap xN \neq \emptyset$, contradiction.

Finally, suppose that N is closed and G is σ-compact. Then there is a compact normal subgroup K of G such that G/K is separable. Let ϕ be the quotient map. If $\phi(N)$ is null in G/K, then the first case lets us finish. Otherwise, since $\phi(N)$ is a subgroup of G/K, Proposition 1.1.146 shows $\phi(N)$ is open, so NK is open in G, so $[NK : N] > \aleph_0$. Let C be a countable subset of NK with infinite image in NK/N, and let M be the subgroup generated by $N \cup C$. If M is not closed, the second part lets us finish. If M is closed, then it is locally compact Hausdorff, and from $[M : N] = \aleph_0$ we see that N has positive inner M-Haar measure, so by Proposition 1.1.146 N is open in M. Thus M/N is a discrete closed subset of the compact set NK/N, so it is finite, contradiction. □

Theorem 1.1.152. *Suppose $\phi : G \to H$ is a homomorphism of locally compact Hausdorff groups such that for every open set $U \subseteq H$, $\phi^{-1}(U)$ is measurable (with respect to the completion of the Haar measure). Then ϕ is continuous.*

Proof. We may assume WLOG that G is compactly generated. By Proposition 1.1.146 and the fact that every neighborhood U of the identity contains a neighborhood V with $VV^{-1} \subseteq U$, it's enough to show that for every open neighborhood V of the identity, $\phi^{-1}(V)$ is not null. Suppose $\phi^{-1}(V)$ is null, and let L be an open σ-compact subgroup of H. Then L is asoo and is contained in a union of countably many left translates of V, so $\phi^{-1}(L)$ is a null asoo subgroup of G. By the Lemma, there is a nonmeasurable $S \subseteq G$ with $S = \phi^{-1}(L)S$. We have $L\phi(S)$ open, so by hypothesis $S = \phi^{-1}(L)S = \phi^{-1}(L\phi(S))$ is measurable, contradiction. □

1.1.5 Integration

Definition 1.1.153. If $f : X \to Y$ and $\mathcal{B}$ is a σ-algebra on Y, then $\sigma(f)$ is the σ-algebra on X generated by $f^{-1}(S)$ for $S \in \mathcal{B}$. We say that $f : (X, \Sigma) \to (Y, \mathcal{B})$ is Σ-*measurable*, or just *measurable* if Σ is clear, if $\sigma(f) \subseteq \Sigma$ (if unspecified, $\mathcal{B}$ is usually taken to be the Borel sets of Y).

Proposition 1.1.154. *$f : (X, \Sigma) \to [-\infty, \infty]$ is measurable iff $f^{-1}([-\infty, a]) \in \Sigma$ for all $a \in \mathbb{R}$. If $f_1, ..., f_n$ are measurable and $g : \mathbb{R}^n \to [-\infty, \infty]$ is Borel measurable, then $g(f_1, ..., f_n)$ is measurable. If f_k is a sequence of measurable functions, then $\sup f_k$ is measurable.*

Proposition 1.1.155. *If $f_k : X \to Y$ is a sequence of measurable functions to a metric space and $f_k \to f$ pointwise, then f is measurable.*

Proof. For any open set U the collection of $x \in X$ such that $f_k(x)$ are eventually all in U is measurable, and this set contains $f^{-1}(U)$ and is contained $f^{-1}(\overline{U})$. Since every open set in a metric space is a countable union of open subsets whose closures are contained in it, the preimage of every open set is measurable. □

Definition 1.1.156. A *simple function* is a function which can be written as a finite linear combination of indicator functions of measurable sets. Equivalently, a function is simple if it is measurable and has a finite range.

Definition 1.1.157. For $f \geq 0$ measurable (up to a null set), we define the *integral* of f with respect to a measure $\mu : \Sigma \to [0, \infty]$ to be

$$\int f \ d\mu = \sup\Big\{\sum_{i=1}^{k} c_i\mu(A_i) \mid c_1, ..., c_k \geq 0,\ A_1, ..., A_k \in \Sigma,\ \sum_{i=1}^{k} c_i \cdot 1_{x\in A_i} \leq f(x)\Big\}.$$

A measurable (up to a null set) complex-valued function f is *integrable* if $\int |f| \ d\mu < \infty$. We extend the integral to all integrable functions by linearity.

Theorem 1.1.158 (Markov's Inequality)**.** *For $t > 0$, $\mu(\{|f| \geq t\}) \leq \frac{1}{t}\int |f| \ d\mu$.*

Proposition 1.1.159. *For $f, g \geq 0$ measurable, we have $\int f + g \ d\mu = \int f \ d\mu + \int g \ d\mu$.*

Proof. For any finite $S \subset [0, \infty]$, define f_S by

$$f_S(x) = \max\{s \in S \mid s \leq f(x)\}.$$

Note f_S is a simple function and $\int f \ d\mu = \sup_S \int f_S \ d\mu$. For any S and any n, if we let $S_n = \{\frac{k}{n}s \mid k \leq n,\ s \in S\}$, then $(f + g)_S \leq \frac{n-1}{n}(f_{S_n} + g_{S_n})$. □

Proposition 1.1.160. *Any Riemann integrable function $f : [0,1] \to \mathbb{C}$ is Lebesgue integrable, with the same integral.*

Proposition 1.1.161. *If $f : X \to [0,\infty]$ is measurable, then $\{(x,t) \mid 0 \leq t \leq f(x)\}$ is measurable in $X \times [0,\infty]$, with $\mu \times \lambda$-measure $\int_X f \; d\mu = \int_0^\infty \mu(\{x \mid f(x) \geq t\}) \; dt$.*

Proof. For any $c > 1$, if we round positive values of f up or down to the nearest c^n, $n \in \mathbb{Z}$, we see that the product outer measure of $\{(x,t) \mid 0 \leq t \leq f(x)\}$ is at most c times $\int_X f \; d\mu$. □

Theorem 1.1.162 (Monotone Convergence Theorem)**.** *If f_k is a sequence of measurable functions with $0 \leq f_k \leq f_{k+1}$ for all k and f is the pointwise limit of the f_k, then f is measurable and $\int f \; d\mu = \lim_k \int f_k \; d\mu$.*

Proof. It's enough to prove this when f is the characteristic function of a measurable set A. Fix $\epsilon > 0$, and for each k set $A_k = \{x \mid f_k(x) \geq 1-\epsilon\}$, then from $\cup_k A_k = A$, we have $\lim_k \mu(A_k) = \mu(A)$, so $\lim_k \int f_k \; d\mu \geq (1-\epsilon)\mu(A)$. □

Lemma 1.1.163 (Fatou's Lemma)**.** *If $f_k \geq 0$ are measurable, then $\int \liminf_k f_k \; d\mu \leq \liminf_k \int f_k \; d\mu$.*

Proof. $\int \liminf_k f_k \; d\mu = \lim_k \int \inf_{l \geq k} f_l \; d\mu \leq \liminf_k \int f_k \; d\mu$. □

Corollary 1.1.164. *If f_k measurable, $|f_k| \leq g$, g integrable, then*

$$\int \liminf f_k \; d\mu \leq \liminf \int f_k \; d\mu \leq \limsup \int f_k \; d\mu \leq \int \limsup f_k \; d\mu.$$

Theorem 1.1.165 (Dominated Convergence Theorem)**.** *If f_k measurable, $|f_k| \leq g$, g integrable, $f_k \to f$ pointwise, then $\lim_k \int f_k \; d\mu = \int f \; d\mu$, and $\lim_k \int |f_k - f| \; d\mu = 0$.*

Theorem 1.1.166 (Jensen)**.** *If $\mu(X) = 1$, g real μ-integrable, φ convex, then $\varphi(\int_X g \; d\mu) \leq \int_X \varphi \circ g \; d\mu$.*

Proof. Let $x_0 = \int_X g \; d\mu$. Since φ is convex, there are $a, b \in \mathbb{R}$ such that $ax + b \leq \varphi(x)$ and $ax_0 + b = \varphi(x_0)$. Integrating both sides of $ag(t) + b \leq \varphi(g(t))$ gives the inequality. □

Theorem 1.1.167 (Radon-Nikodym Theorem)**.** *If μ, ν are σ-finite measures on X (ν possibly signed or complex) and $\nu \ll \mu$, then there exists a measurable function f (unique up to a μ-null set) such that for any measurable set A, $\nu(A) = \int_A f \; d\mu$.*

Proof. We just need to prove this in the positive, finite case. Let $\mathcal{F}$ be the family of measurable functions f such that for all measurable A, $\nu(A) \geq \int_A f \; d\mu$. Note that $\mathcal{F}$ is closed under maximum, and by the Monotone Convergence Theorem 1.1.162 $\mathcal{F}$ is closed under countable monotone limits, so there is some $f \in \mathcal{F}$ with $\int_X f \; d\mu = \sup_{g \in \mathcal{F}} \int_X g \; d\mu$. Let $\nu_0 = \nu - \int f \; d\mu$. If $\nu_0(X) > 0$, take $\epsilon > 0$ such that $\nu_0(X) > \epsilon\mu(X)$, and let (N, P) be a Hahn decomposition 1.1.73 of $\nu_0 - \epsilon\mu$. But then $f + \epsilon \cdot 1_P \in \mathcal{F}$ and $\mu(P) > 0$, contradicting our choice of f. □

Definition 1.1.168. If μ, ν have $\nu = \int f \; d\mu$, then the *Radon-Nikodym derivative* $\frac{d\nu}{d\mu}$ is defined to be the equivalence class of f when we quotient by μ-null functions.

Proposition 1.1.169. *If μ is a complex measure, then $\frac{d\mu}{d|\mu|}$ has absolute value 1 $|\mu|$-almost everywhere.*

Proof. Let f be a representative of $\frac{d\mu}{d|\mu|}$. For any $\epsilon > 0$, the set where $|f| < 1-\epsilon$ has measure 0, since otherwise its $|\mu|$ measure would be smaller than itself by a factor of $1-\epsilon$. By dividing up the set where $|f| > 1+\epsilon$ into $O(\frac{1}{\sqrt{\epsilon}})$ many subsets based on the argument of f, we see that it must also have measure 0, since otherwise its $|\mu|$ measure would be larger than itself by a factor of $1+\frac{\epsilon}{2}$. □

Proposition 1.1.170. *Where the relevant Radon-Nikodym derivatives make sense, we have* $\frac{d(\nu+\mu)}{d\lambda} = \frac{d\mu}{d\lambda} + \frac{d\nu}{d\lambda}$, $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$, $\frac{d|\nu|}{d\mu} = |\frac{d\nu}{d\mu}|$, *and* $\int g \ d\mu = \int g \frac{d\mu}{d\lambda} \ d\lambda$.

Proposition 1.1.171. *If $E \subseteq X \times Y$ is measurable and $\mu \times \nu(E) < \infty$, then for μ-almost every $x \in X$ E_x is measurable up to a ν-null set, the function $g(x) = \mu(E_x)$ is measurable up to a μ-null set, and $\int g \ d\mu = \mu \times \nu(E)$.*

Proof. By definition of $\mu \times \nu$, there is an $F \supseteq E$ which is a countable decreasing intersection of countable unions of measurable rectangles, such that $\mu \times \nu(E) = \mu \times \nu(F)$. Since $\mu \times \nu(E) < \infty$, $F \setminus E$ is $\mu \times \nu$-null, so we may replace E by F without changing g (aside from on a μ-null set) by Proposition 1.1.113 and then apply monotone 1.1.162 and dominated 1.1.165 convergence to reduce to the case of a finite union of measurable rectangles. □

Theorem 1.1.172 (Fubini's Theorem)**.** *If $\int_{X\times Y} |f(x,y)| \ d(x,y) < \infty$, where $d(x,y)$ is the maximal product measure on $X \times Y$, then for a.e. $x \in X$ $f(x,y)$ is integrable in y, and we have $\int_{X\times Y} f(x,y) \ d(x,y) = \int_X \int_Y f(x,y) \ dy \ dx$.*

Theorem 1.1.173 (Tonelli's Theorem)**.** *If X, Y are σ-finite, then $\int_{X\times Y} |f(x,y)| \ d(x,y) = \int_X \int_Y |f(x,y)| \ dy \ dx$.*

Proof. Assume $f \geq 0$. The assumptions of either Fubini or Tonelli imply that f can be written as the pointwise limit of an increasing sequence ϕ_n of nonnegative simple functions that each vanish outside a set of finite measure. Thus, using Proposition 1.1.171, for almost every fixed x the function $y \mapsto f(x,y) = \lim_n \phi_n(x,y)$ is measurable up to a null set, and by monotone convergence 1.1.162 the function $x \mapsto \int_Y f(x,y) \ dy = \lim_n \int_Y \phi_n(x,y) \ dy$ is measurable up to a null set. Applying monotone convergence and Proposition 1.1.171 again, we get

$$\begin{aligned}\int_X \int_Y f(x,y) \ dy \ dx &= \lim_n \int_X \int_Y \phi_n(x,y) \ dy \ dx \\ &= \lim_n \int_{X\times Y} \phi_n(x,y) \ d(x,y) = \int_{X\times Y} f(x,y) \ d(x,y).\end{aligned}$$ □

A lot of the next bits are from [79].

Proposition 1.1.174. *If X, Y are locally compact Hausdorff with Radon measures μ, ν and U is open in $X \times Y$, then $x \mapsto \nu(U_x)$ is lower semicontinuous and $\mu \widehat{\times} \nu(U) = \int \nu(U_x) \ d\mu(x)$.*

Proof. This follows directly from Proposition 1.1.139, the definition of the integral, and the fact that Radon measures are inner regular on open sets. □

Theorem 1.1.175 (Fubini-Tonelli for Radon Products)**.** *If μ, ν are σ-finite Radon measures on locally compact Hausdorff spaces X, Y, f is Borel measurable on $X \times Y$, and either $f \geq 0$ or $|f|$ is integrable, then $\int_{X\times Y} f \ d\mu\widehat{\times}\nu = \int_X \int_Y f \ d\nu \ d\mu$.*

Proposition 1.1.176. *If X, Y are locally compact Hausdorff, then every $f \in C_c(X \times Y)$ is measurable with respect to the product of the Borel σ-algebras.*

Proof. Follows from Stone-Weierstrauss, Urysohn's Lemma, and the fact that pointwise limits of measurable functions are measurable. □

Definition 1.1.177. Write $f \prec U$ if $0 \le f \le \chi_U$ and $\mathrm{supp}(f) \subseteq U$.

Theorem 1.1.178 (Riesz Representation Theorem)**.** *If X is a locally compact Hausdorff space and I is a positive linear functional on $C_c(X)$, then there is a unique Radon measure μ such that $I(f) = \int f\ d\mu$ for all $f \in C_c(X)$. This μ satisfies $\mu(K) = \inf\{I(f) \mid f \ge \chi_K\}$ for all compact K and $\mu(U) = \sup\{I(f) \mid f \prec U\}$ for all open U.*

Proof. Uniqueness and the formulas for $\mu(U), \mu(K)$ follow from Urysohn's Lemma. For existence, we need to check that the formula for $\mu(K)$ defines a regular content and that the formula $I(f) = \int f\ d\mu$ holds. For the regularity, note that if K is compact and $f \ge \chi_K$, then if we let $U_\epsilon = \{x \mid f(x) > 1 - \epsilon\}$, then for any $g \prec U_\epsilon$ we have $I(g) \le I(f)/(1-\epsilon)$, so $\mu(U_\epsilon) \le I(f)/(1-\epsilon)$.

For the integral formula, if $f \le 1$, for any N we define $K_j = \{x \mid f(x) \ge j/N\}$ and $K_0 = \mathrm{supp}(f)$, and $f_j = \min((f - \frac{j-1}{N})_+, \frac{1}{N})$, so $f = \sum f_j$ and $\chi_{K_j} \le N f_j \le \chi_{K_{j-1}}$. Then we have $\mu(K_j) \le N \int f_j\ d\mu \le \mu(K_{j-1})$ and $\mu(K_j) \le N I(f_j) \le \mu(K_{j-1})$ (last inequality using outer regularity), so $|I(f) - \int f\ d\mu| \le \mu(K_0)/N$. □

Lemma 1.1.179. *If X is locally compact Hausdorff, then every bounded real linear functional I on $C_0(X)$ can be written as the difference between two positive linear functionals.*

Proof. For $f \ge 0$ in $C_0(X)$, set $I^+(f) = \sup\{I(g) \mid 0 \le g \le f\}$ (this is finite since I is bounded) and $I^- = I^+ - I$. The only tricky bit is to check that $I^+(f_1 + f_2) \le I^+(f_1) + I^+(f_2)$: if $0 \le g \le f_1 + f_2$, then set $g_1 = \min(g, f_1)$ and $g_2 = g - g_1 = \max(0, g - f_1)$, so $0 \le g_i \le f_i$, so $I(g) \le I^+(f_1) + I^+(f_2)$. □

Theorem 1.1.180 (Riesz Representation Theorem for $C_0(X)$)**.** *If X is a locally compact Hausdorff space, then $\mu \mapsto I_\mu = (f \mapsto \int f\ d\mu)$ is an isometric isomorphism between complex Radon measures on X under the total variation norm $\|\mu\| = |\mu|(X)$ and bounded linear functionals on $C_0(X)$ under the operator norm $\|I\| = \sup\{|I(f)| \mid \sup_x |f(x)| = 1\}$.*

Proof. Apply Lusin's Theorem (below) to $\frac{d\mu}{d|\mu|}$ to show that $\|\mu\| \le \|I_\mu\|$; the other inequality is easy. □

Convergence in Measure

Definition 1.1.181. A sequence of measurable functions f_n converges to f *globally in measure* if $\forall \epsilon > 0$, we have $\lim_n \mu(\{x \mid |f(x) - f_n(x)| \ge \epsilon\}) = 0$, and $f_n \to f$ *locally in measure* if $\forall \epsilon > 0$ and for all $F \in \Sigma$ with $\mu(F) < \infty$ we have $\lim_n \mu(\{x \in F \mid |f(x) - f_n(x)| \ge \epsilon\}) = 0$.

Theorem 1.1.182 (Riesz)**.** *If $f_n \to f$ globally in measure (or locally in measure on a σ-finite space) then some subsequence converges to f pointwise almost everywhere.*

Proof. Choose a subsequence n_k such that $\mu(\{x \mid |f(x) - f_{n_k}(x)| \ge \frac{1}{k}\}) < 2^{-k}$. □

Proposition 1.1.183. *If all subsequences of f_n have a subsequence which converges to f almost everywhere (and f is finite almost everywhere), then $f_n \to f$ locally in measure.*

Proof. Suppose there is some $F \in \Sigma$ with $\mu(F) < \infty$ and $\epsilon > 0$ such that $\mu(\{x \in F \mid |f(x) - f_n(x)| \geq \epsilon\})$ doesn't converge to 0. Then there is a $\delta > 0$ and a subsequence n_k such that $\mu(\{x \in F \mid |f(x) - f_{n_k}(x)| \geq \epsilon\}) > \delta$ for all k. No such subsequence f_{n_k} can converge almost everywhere to f: otherwise, there would be some K such that the set of $x \in F$ with $|f(x) - f_{n_k}(x)| < \epsilon$ for all $k > K$ has measure at least $\mu(F) - \delta$. □

Theorem 1.1.184 (Egoroff's Theorem)**.** *If M is a separable metric space and f_n is a sequence of measurable functions from A to M, with $\mu(A) < \infty$, such that $f_n \to f$ pointwise almost everywhere, then for every $\epsilon > 0$ there is $B \subseteq A$ such that $\mu(B) < \epsilon$ and $f_n \to f$ uniformly on $A \setminus B$.*

Proof. For every k, choose n_k such that $\mu(\{x \in A \mid \exists m > n_k \; d(f(x), f_m(x)) \geq \frac{1}{k}\}) < \frac{\epsilon}{2^k}$ (to see that $x \mapsto d(f(x), f_m(x))$ is measurable, we use separability of M). □

Theorem 1.1.185 (Lusin's Theorem)**.** *If $f : [a, b] \to \mathbb{C}$ is measurable, then $\forall \epsilon > 0$ there exists a compact $E \subseteq [a, b]$ such that $f|_E$ is continuous and $\mu(E) > b - a - \epsilon$. More generally, if (X, μ) is a Radon measure space and Y is second-countable, and $f : A \to Y$ is measurable with $\mu(A) < \infty$, then $\forall \epsilon > 0$ there is a compact set $E \subseteq A$ with $\mu(A \setminus E) < \epsilon$ such that $f|_E$ is continuous.*

Proof. (From [78]) Let U_j be an enumeration of a base of open sets for Y, and for each j choose V_j open in X such that $f^{-1}(U_j) \subseteq V_j$ and $\mu(V_j \setminus f^{-1}(U_j)) < \frac{\epsilon}{2^j}$. Take $E_1 = A \setminus \bigcup_j (V_j \setminus f^{-1}(U_j))$, so $f^{-1}(U_j) \cap E_1 = V_j \cap E_1$, then let E be a compact set contained in E_1 with sufficiently close measure. □

Lebesgue Integral and Derivatives

Definition 1.1.186. The *Hardy-Littlewood maximal operator* M takes a locally integrable $f : \mathbb{R}^n \to \mathbb{C}$ to the function Mf given by

$$Mf(x) = \sup_{r>0} \frac{\int_{B_r(x)} |f(y)| \, dy}{\lambda(B_r)}.$$

Theorem 1.1.187 (Weak type Hardy-Littlewood maximal inequality)**.** *For any integrable function $f : \mathbb{R}^n \to \mathbb{C}$, we have $\lambda(\{Mf > t\}) \leq \frac{3^n}{t} \int |f| \, d\lambda$.*

Proof. Let $A_t = \{Mf > t\}$, and let K be any compact set contained in A_t. For each $x \in K$, we can find an $r > 0$ such that $\int_{B_r(x)} |f(y)| \, dy > t\lambda(B_r(x))$, and finitely many of these balls $B_r(x)$ cover K. Apply the Finite Vitali Covering Lemma 1.1.41 to get a collection B_i of disjoint balls among these such that $K \subseteq \bigcup_i 3B_i$, then $\lambda(K) \leq 3^n \sum_i \lambda(B_i) \leq \frac{3^n}{t} \int |f(y)| \, dy$. □

Theorem 1.1.188 (Lebesgue Differentiation Theorem)**.** *If $f : \mathbb{R}^n \to \mathbb{C}$ is locally integrable, then for Lebesgue-a.e. x we have*

$$\lim_{r \to 0} \frac{\int_{B_r(x)} |f(y) - f(x)| \, dy}{\lambda(B_r)} = 0.$$

Proof. First proof: approximate f by a simple function, and apply the Lebesgue Density Theorem 1.1.115.

Second proof: Assume f is supported on a finite ball. By Lusin's Theorem 1.1.185 and the Tietze Extension Theorem (Corollary 1.1.6), we can find $g \in C_c(\mathbb{R}^n)$ with $\int |f - g| \, d\lambda < \epsilon$. Then

$$\frac{1}{\lambda(B)} \int_B |f(y) - f(x)| \, dy \leq \frac{1}{\lambda(B)} \int_B |f(y) - g(y)| \, d\lambda + \frac{1}{\lambda(B)} \int_B |g(y) - g(x)| \, d\lambda + |f(x) - g(x)|.$$

By Theorem 1.1.187 the first summand is at most t away from a set of measure at most $\frac{3^n}{t}\int|f-g|\,d\lambda < \frac{3^n\epsilon}{t}$, and by Markov's inequality the third summand is at most t away from a set of measure at most $\frac{\epsilon}{t}$, while the second summand goes to 0 as r goes to 0 since $g \in C_c(\mathbb{R}^n)$. □

Proposition 1.1.189. *If f is nondecreasing, then f has only jump discontinuities, and only countably many of them.*

Lemma 1.1.190 (Riesz's Rising Sun Lemma)**.** *If $U \subseteq \mathbb{R}$ is open and $g : U \to \mathbb{R}$ is continuous, then the set $U_g = \{x \in U \mid \exists y > x \text{ s.t. } (x, y) \subseteq U \text{ and } g(x) < g(y)\}$ is also open, and if (a, b) is a component of U_g then $g(a) \le g(b)$.*

Theorem 1.1.191 (Lebesgue)**.** *If f is nondecreasing, then f is differentiable almost everywhere, and $\int_a^b f'(x)\,dx \le f(b) - f(a)$. If E, Z, I are the sets where f is not differentiable, has derivative 0, and has derivative ∞, respectively, then $f(E), f(Z), I$ have measure 0.*

Proof. (Following [75]) Set $D^+f(x) = \limsup_{h\downarrow 0} \frac{f(x+h)-f(x)}{h}$, $D_+f(x) = \liminf_{h\downarrow 0} \frac{f(x+h)-f(x)}{h}$, and similarly define D^-, D_- with h approaching 0 from below.

First we show that if f is continuous and E is any set where $D^+f > u$, then $\lambda^*(f(E)) \ge u\lambda^*(E)$: if U is any open set containing $f(E)$, then $f^{-1}(U)$ is an open set containing E, and the rising sun lemma 1.1.190 applied to $g(x) = f(x) - ux$ and $f^{-1}(U)$ shows that $\lambda(U) \ge u\lambda(f^{-1}(U)_g) \ge u\lambda^*(E)$.

Next we show that if f is strictly increasing and E is any set where $D_+ < v$, then $\lambda^*(f(E)) \le v\lambda^*(E)$: let $g(y) = \inf\{z \mid f(z) \ge y\}$ be inverse to f, suppose WLOG that no point of discontinuity of f is in E, then for any $x \in E$ we have $D^+g(f(x)) > \frac{1}{v}$, and we can reduce to the previous case. We extend this from strictly increasing f to all f by replacing f by $h(x) = f(x) + x$ and noting that $\lambda^*(h(E)) \ge \lambda^*(f(E)) + \lambda^*(E)$ (take any open set containing $h(E)$ and break it into connected components). From this we see that we can drop the continuity assumption in the first case by considering the function g inverse to f again and ignoring the countably many points where either f or g is discontinuous.

Applying the above to $-f(-x)$, we get similar statements for D^-, D_-. We'll show that $D^+ \le D_-$ almost everywhere, and similarly $D^- \le D_+$ a.e., so we will have $D^+ \le D_- \le D^- \le D_+ \le D^+$ almost everywhere. Let E_{uv} be the set of x with $D^+f(x) > u > v > D_-f(x)$. Then $u\lambda^*(E_{uv}) \le \lambda^*(f(E_{uv})) \le v\lambda^*(E_{uv})$, so $\lambda^*(E_{uv}), \lambda^*(f(E_{uv}))$ must be 0.

For the statement about $\int_a^b f'(x)\,dx$, apply Fatou's Lemma to the sequence of functions $f_n(x) = n(f(x + \frac{1}{n}) - f(x))$ (assuming WLOG that $f(x) = f(b)$ for $x > b$). □

Corollary 1.1.192. *If f has bounded variation, then f is differentiable almost everywhere and f' is Lebesgue integrable.*

Corollary 1.1.193. *If f is increasing, μ_f is the Lebsgue-Stieltjes measure 1.1.104, and $(\mu_f)_{ac}$ is the absolutely continuous part of the Lebesgue decomposition 1.1.81 of μ_f with respect to λ, then $\frac{d(\mu_f)_{ac}}{d\lambda} = f'$. The singular part $(\mu_f)_s$ can be written as the sum of a discrete measure and some μ_c with c continuous and $c' = 0$ almost everywhere.*

Proof. Let $g(x) = \int_0^x f'(t)\,dt$, and note that $\mu_g \le \mu_f$ and $\mu_g \ll \lambda$ since $\mu_g(E) = \int_E f'(t)\,dt$ for any Borel set E. By the Lebesgue differentiation theorem 1.1.188 and the fact that g' exists almost everywhere, we have $g' = f'$ almost everywhere. To finish, we need to check that if c is continuous and $c' = 0$ almost everywhere then $\mu_c \perp \lambda$: if Z is the set where $c' = 0$, then $\mu_c(Z) = \lambda(c(Z)) = 0$. □

Definition 1.1.194. A function $f : I \to \mathbb{R}$ is *absolutely continuous* on I if $\forall \epsilon > 0\ \exists \delta > 0$ such that if $(x_k, y_k) \subseteq I$ are disjoint subintervals with $\sum_k |y_k - x_k| < \delta$ then $\sum_k |f(y_k) - f(x_k)| < \epsilon$.

Proposition 1.1.195. *If f is absolutely continuous then f has bounded variation, and the variation of f is also absolutely continuous.*

Theorem 1.1.196 (Fundamental Theorem of the Lebesgue Integral)**.** *A function f is absolutely continuous on $[a, b]$ iff there exists g integrable with $f(x) = f(a) + \int_a^x g(t)\ dt$ for all $x \in [a, b]$. In this case we have $g = f'$ almost everywhere.*

Corollary 1.1.197. *If f is absolutely continuous and has $f' = 0$ almost everywhere, then f is constant.*

Proof. Direct proof based on Vitali Covering Theorem 1.1.123: Let $\mathcal{V}$ be the family of intervals $[x, y] \subseteq [a, b]$ such that $f'(x) = 0$ and $|\frac{f(y)-f(x)}{y-x}| < \epsilon$, then we can find a finite disjoint subset of intervals of $\mathcal{V}$ that cover all but δ of $[a, b]$ for δ sufficiently small, so $|f(b) - f(a)| \le \epsilon|b - a| + \epsilon$. □

Example 1.1.4. The Cantor function (aka the Devil's Staircase, defined by writing a number in ternary, ignoring every digit after the first 1, replacing every 2 with a 1, and interpreting the result in binary) is uniformly continuous, but not absolutely continuous, and has derivative 0 almost everywhere.

This example leads to other pathologies: let $f : [0, 1] \to [0, 1]$ be the Cantor function, let $h(x) = f(x) + x : [0, 1] \to [0, 2]$, let $g = h^{-1} : [0, 2] \to [0, 1]$, let $C \subseteq [0, 1]$ be the Cantor set, and let $D = [0, 1] \setminus C$. Since D is a union of open intervals whose lengths sum to 1 and f is constant on these intervals, $h(D)$ is measurable with measure 1, so $h(C)$ is also measurable with measure 1. Any measurable subset with positive measure contains a set which isn't measurable (usual argument with equivalence classes based on rationals works), so let $A \subseteq h(C)$ be such a nonmeasurable set and let $B = g(A)$. Then $B \subseteq C$, so B is Lebesgue measurable (but not Borel measurable), so χ_B, g are both measurable but $\chi_B \circ g$ is not measurable. Additionally, although g, h are continuous and strictly increasing, we have $A = g^{-1}(B) = h(B)$ is not measurable even though B is.

A lot of the following is from [86].

Proposition 1.1.198. *Absolutely continuous functions map null sets to null sets and map measurable sets to measurable sets.*

Proposition 1.1.199. *If f_n is a sequence of equi-absolutely continuous functions (i.e. for each ϵ, a single δ works for all of them), and $\lim_n f_n = f$ pointwise, then f is absolutely continuous (and similarly for a sequence with uniformly bounded variation). In particular, if a sequence g_n of absolutely continous functions has $\sum_n g_n$ convergent and the sum of the variations of the g_ns is finite, then $\sum_n g_n$ is absolutely continuous.*

Proposition 1.1.200. *If f has bounded variation on $[a, b]$, $V(x)$ is the variation of f on $[a, x]$, and f is continuous at c, then V is also continuous at c. In particular, if f is continuous and has bounded variation on $[a, b]$, and is absolutely continuous on $[a, c]$ for all $a < c < b$, then f is absolutely continuous on $[a, b]$.*

Proposition 1.1.201. *Suppose $f : [a, b] \to \mathbb{R}$ is bounded, and let $m_f(x) = \limsup_{t\to x} f(t)$. Then m_f can be written as a pointwise limit of step functions which each exceed f, and the upper Riemann integral of f is equal to the Lebesgue integral $\int m_f$. In particular, a bounded function on $[a, b]$ is Riemann integrable iff it is continuous a.e.*

Proposition 1.1.202. *There is a perfect, nowhere dense subset of* $[0, 1]$ *with positive Lebesgue measure.*

Proof. The construction is the same as the Cantor set, but shrink each removed interval by a constant factor. Alternatively, consider the set of numbers whose base 5 expansions contain no 2s. □

Proposition 1.1.203. *There is a function* $f : [0, 1] \to \mathbb{R}$ *such that* f' *exists and is bounded everywhere on* $[0, 1]$*, but* f' *is discontinuous on a set of positive measure and is therefore not Riemann integrable.*

Proof. Let E be a perfect, nowhere dense subset of $[0, 1]$ with positive measure. The plan is to make f equal to 0 on E, and on each open interval (a, b) in $[0, 1] \setminus E$ to choose f such that $|f(x)| \le |x - a|^2, |b - x|^2$, but such that $|f'(x)| = 1$ for points $x \in (a, b)$ arbitrarily close to a and b. To construct such functions, start with the function $x \mapsto (x - a)^2 \sin(1/(x - a))$ around a, and connect it to a similar function around b. □

Proposition 1.1.204. *If* f_i *are nondecreasing functions and* $f = \sum_i f_i$ *converges, then* $f' = \sum_i f_i'$ *a.e.*

Proof. Let $g_k = \sum_{i>k} f_i$, then it's enough to prove that $\lim_k g_k' = 0$ a.e. To see this, pick a subsequence k_i such that $\sum_i g_{k_i}$ converges at points a, b, and note that $\int_a^b \sum_i g_{k_i}' = \sum_i \int_a^b g_{k_i}' = \sum_i g_{k_i}(b) - g_{k_i}(a) < \infty$, so $\sum_i g_{k_i}'$ must converge a.e. on $[a, b]$, so $\lim_k g_k' = \lim_i g_{k_i}' = 0$ a.e. on $[a, b]$. □

Proposition 1.1.205. *If* $f : \mathbb{R} \to \mathbb{R}$ *is measurable and* $0 < \alpha < \beta$*, then the function* $x \mapsto \sup\{\frac{f(y)-f(x)}{y-x} \mid x+\alpha < y < x+\beta\}$ *is measurable. In particular, all four derivates* D^+f, D^-f, D_+f, D_-f *are measurable.*

Proof. If $\sup\{\frac{f(y)-f(x)}{y-x} \mid x + \alpha < y < x + \beta\} > r$, then x is contained in one of countably many sets which are preimages of open intervals under f, intersected with open sets that guarantee the sup is large so long as $f(x)$ is in the given range. □

Proposition 1.1.206. *If* $f : \mathbb{R} \to \mathbb{R}$ *is any function, then* $\overline{D}f : x \mapsto \limsup_{y\to x} \frac{f(y)-f(x)}{y-x}$ *is measurable, as is the similarly defined* $\underline{D}f$*.*

Proof. Let $r \in \mathbb{R}$, and for any k, n let E_n^k be the union of all intervals $[a, b]$ such that $b - a < \frac{1}{k}$ and $\frac{f(b)-f(a)}{b-a} > r + \frac{1}{n}$. Then E_n^k is measurable since it is a union of closed intervals, and the set where $\overline{D}f > r$ is equal to $\cup_n \cap_k E_n^k$. □

Example 1.1.5. If $E \subseteq [0, 1]$ is null, then there is a nondecreasing absolutely continuous function f with $D_-f = D_+f = +\infty$ on E. To see this, for each n find an open set O_n containing E with measure at most $\frac{1}{2^n}$, let f_n be the integral of χ_{O_n}, and let $f = \sum_n f_n$.

Example 1.1.6. Let E be a perfect nowhere dense set of positive measure in $[0, 1]$, and let f be the integral of $\chi_{[0,1]\setminus E}$. Then f is strictly increasing and absolutely continuous, but $f' = 0$ on a set of positive measure. Additionally, by summing countably many dilated copies of the Cantor function, we can make a strictly increasing continuous function which has derivative 0 almost everywhere.

Gauge Integral

A lot of this material is from [8].

Definition 1.1.207. If I is a closed interval, then a *partition* of I is a finite collection $\mathcal{P}$ of closed subintervals $I_i \subseteq I$ such that $\cup_i I_i = I$ and such that any pair of distinct intervals $I_i, I_j \in \mathcal{P}$ intersect in at most one point. A *subpartition* of I is a subset of a partition of I, that is, a collection of closed subintervals which pairwise don't overlap in more than a single point.

A *tagged partition* of I is a finite collection $\dot{\mathcal{P}}$ of pairs (t_i, I_i) such that $\{I_i\}$ is a partition of I, and $t_i \in I_i$ for each i. An *improperly tagged partition* is the same as a tagged partition, but where we drop the condition $t_i \in I_i$, replacing it with $t_i \in I$ instead.

Definition 1.1.208. A *gauge* on I is any function $\delta : I \to \mathbb{R}^+$, such that $\delta(x) > 0$ for all $x \in I$.

A (possibly improperly) tagged partition $\dot{\mathcal{P}} = \{(t_i, I_i)\}$ is *δ-fine*, written $\dot{\mathcal{P}} \ll \delta$, if

$$I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$$

for each i.

Lemma 1.1.209 (Cousin's lemma)**.** *If an interval I is compact, then for any gauge $\delta : I \to \mathbb{R}^+$ there exists a δ-fine tagged partition $\dot{\mathcal{P}}$.*

Proposition 1.1.210. *For any $c \in I$, there is a gauge $\delta : I \to \mathbb{R}^+$ such that for every δ-fine tagged partition $\dot{\mathcal{P}}$, one of the tags is forced to be c.*

Proof. Take $\delta(c) = 1$ and $\delta(x) = \frac{1}{2}|x - c|$ for all $x \neq c$. □

Definition 1.1.211. If $\dot{\mathcal{P}} = \{(t_i, [a_i, b_i])\}$ is a tagged partition of I and $f : I \to \mathbb{C}$, then the *Riemann sum* of f with respect to $\dot{\mathcal{P}}$ is defined by

$$S(f, \dot{\mathcal{P}}) := \sum_i f(t_i)(b_i - a_i).$$

More generally, if $g : I \to \mathbb{C}$ is another function, then the *Riemann-Stieltjes sum* of $f\ dg$ with respect to $\dot{\mathcal{P}}$ is defined by

$$S(f, dg, \dot{\mathcal{P}}) := \sum_i f(t_i)(g(b_i) - g(a_i)).$$

Definition 1.1.212. A function $f : I \to \mathbb{C}$ is said to be *gauge-integrable* with integral C if for all $\epsilon > 0$ there exists a gauge $\delta_\epsilon : I \to \mathbb{R}^+$ such that for every tagged partition $\dot{\mathcal{P}}$, we have

$$\dot{\mathcal{P}} \ll \delta_\epsilon \implies |S(f, \dot{\mathcal{P}}) - C| \leq \epsilon.$$

More generally, we write

$$\int_I f(x)\ dg(x) = C$$

if for all $\epsilon > 0$ there exists a gauge $\delta_\epsilon : I \to \mathbb{R}^+$ such that for every δ_ϵ-fine tagged partition $\dot{\mathcal{P}}$, we have $|S(f, dg, \dot{\mathcal{P}}) - C| \leq \epsilon$.

Theorem 1.1.213 (McShane integral)**.** *A function $f : I \to \mathbb{C}$ is Lebesgue-integrable with integral C iff for all $\epsilon > 0$ there exists a gauge $\delta_\epsilon : I \to \mathbb{R}^+$ such that for every δ_ϵ-fine* improperly *tagged partition $\dot{\mathcal{P}}$, we have $|S(f, \dot{\mathcal{P}}) - C| \leq \epsilon$.*

Proposition 1.1.214. *If $f : I \to \mathbb{C}$ is 0 a.e., then the gauge integral of f is 0.*

Proof. For each positive integer m, let $Z_m = \{x \in I : |f(x)| \in (m-1, m]\}$. For any ϵ, pick countable collections $U_{m,i}$ of open intervals such that $Z_m \subseteq \cup_i U_{m,i}$ and such that $\sum_i \lambda(U_{m,i}) \leq \frac{\epsilon}{m2^m}$ for each m. Then define the gauge δ_ϵ by $\delta_\epsilon(x) = 1$ if $f(x) = 0$, and otherwise pick some m, i such that $x \in U_{m,i}$ and let $\delta_\epsilon(x)$ be anything small enough to guarantee that $[x - \delta_\epsilon(x), x + \delta_\epsilon(x)] \subseteq U_{m,i}$. □

Proposition 1.1.215. *The gauge integral has the following properties:*

- *the gauge-integrable functions form a vector space, and the gauge integral is linear,*
- *if $f \leq g$, then $\int_I f \leq \int_I g$,*
- *if $a < c < b$, then f is gauge-integrable on $[a, b]$ iff f is gaugeintegrable on $[a, c]$ and $[c, b]$, and $\int_a^b f = \int_a^c f + \int_c^b f$,*
- *(squeeze) if for every $\epsilon > 0$ there are gauge-integrable g, h with $g \leq f \leq h$ and $\int_I h - g \leq \epsilon$, then f is gauge-integrable on I.*

Proposition 1.1.216. *A function $f : I \to \mathbb{C}$ is a uniform limit of step functions iff f has left and right limits at every point of I. In particular, any such function is Riemann-integrable and has at most countably many points of discontinuity.*

Proof. It's easy to see that any uniform limit of step functions has left and right limits everywhere. For the converse, let $\epsilon > 0$ and pick a gauge $\delta_\epsilon : I \to \mathbb{R}^+$ such that for any $y, z \in (x, x + \delta_\epsilon(x)]$ we have $|f(y) - f(z)| \leq \epsilon$, and similarly for $y, z \in [x - \delta_\epsilon(x), x)$. By Cousin's Lemma there is a δ_ϵ-fine tagged partition $\dot{\mathcal{P}} = \{(t_i, [a_i, b_i])\}$, so we can define a step function s by $s(x) = f(a_i)$ for $x \in (a_i, t_i)$, $s(x) = f(b_i)$ for $x \in (t_i, b_i)$, and $s(x) = f(x)$ for $x \in \{t_i, a_i, b_i\}$. □

Definition 1.1.217. Call a function $f : I \to \mathbb{C}$ *regulated* if f is a uniform limit of step functions.

Proposition 1.1.218. *If f is bounded below and gauge-integrable, and if g is regulated, then fg is gauge-integrable.*

Definition 1.1.219. A function F is an *a.e.-primitive* of f if F is continuous and if there is a null set E such that F' exists and equals f away from E. If we can take E countable, then we say that F is a *c.e.-primitive* of f.

Theorem 1.1.220. *If F is a c.e.-primitive of f, then f is gauge-integrable and $\int_a^b f = F(b) - F(a)$.*

Proof. Let $E = \{e_n\}$ be the exceptional set, and assume WLOG that $f(e_n) = 0$ for all n. For any $\epsilon > 0$, choose the gauge $\delta_\epsilon : [a, b] \to \mathbb{R}^+$ as follows. For $x \notin E$, pick $\delta_\epsilon(x)$ such that for all y with $|y - x| \leq \delta_\epsilon(x)$, we have

$$|F(x) - F(y) - f(x)(x - y)| < \epsilon|x - y|.$$

For $x = e_n$, use continuity of F to pick $\delta_\epsilon(x)$ such that for all y with $|y - x| \leq \delta_\epsilon(x)$, we have

$$|F(x) - F(y)| < \frac{\epsilon}{2^n}.$$ □

Lemma 1.1.221 (Saks-Henstock). *If $f : I \to \mathbb{C}$ is gauge-integrable, and if δ_ϵ is a gauge on I such that $\dot{\mathcal{P}} \ll \delta_\epsilon$ implies $|S(f, \dot{\mathcal{P}}) - \int_I f| \leq \epsilon$ for all tagged partitions $\dot{\mathcal{P}}$, then for all δ_ϵ-fine tagged* subpartitions $\dot{\mathcal{P}}_0$ *we also have $|S(f, \dot{\mathcal{P}}_0) - \int_{\cup \mathcal{P}_0} f| \leq \epsilon$.*

Proof. Complete $\mathcal{P}_0$ to a δ_ϵ-fine partition $\mathcal{P}$ which is $\delta_{\epsilon'}$-fine outside of $\cup\mathcal{P}_0$ for arbitrarily small ϵ'. □

Corollary 1.1.222. *If $f : I \to \mathbb{R}$ is gauge integrable, and if δ_ϵ is a gauge on I such that $\dot{\mathcal{P}} \ll \delta_\epsilon$ implies $|S(f,\dot{\mathcal{P}}) - \int_I f| \le \epsilon$ for all tagged partitions $\dot{\mathcal{P}}$, then for all δ_ϵ-fine tagged partitions $\dot{\mathcal{P}} = \{(t_i,[a_i,b_i])\}$ we have*

$$\sum_i \max\left(0, f(t_i)(b_i - a_i) - \int_{[a_i,b_i]} f\right) \le \epsilon$$

and

$$\sum_i \left| f(t_i)(b_i - a_i) - \int_{[a_i,b_i]} f \right| \le 2\epsilon.$$

Corollary 1.1.223. *If $f : I \to \mathbb{C}$ is gauge-integrable, and if δ_ϵ is a gauge on I such that $\dot{\mathcal{P}} \ll \delta_\epsilon$ implies $|S(f,\dot{\mathcal{P}}) - \int_I f| \le \epsilon$ for all tagged partitions $\dot{\mathcal{P}}$, then for all δ_ϵ-fine tagged partitions $\dot{\mathcal{P}} = \{(t_i,[a_i,b_i])\}$ we have*

$$\sum_i \left| f(t_i)(b_i - a_i) - \int_{[a_i,b_i]} f \right| \le \pi\epsilon.$$

Proof. Pick a uniformly random angle θ, and apply the previous corollary to $\Re(e^{i\theta} f)$. □

Corollary 1.1.224. *If $f : [a,b] \to \mathbb{C}$ is gauge-integrable then the indefinite integral $x \mapsto \int_{[a,x]} f$ is continuous on $[a,b]$.*

Proof. To show continuity at $c \in [a,b]$, pick a gauge δ_ϵ and apply the Saks-Henstock Lemma 1.1.221 to the subpartition

$$\dot{\mathcal{P}}_0 = \{(c,[c-\delta,c]),(c,[c,c+\delta])\}$$

for any δ such that $\delta \le \delta_\epsilon(c)$ and such that $\delta|f(c)| \le \epsilon$. □

Theorem 1.1.225. *If $f : [a,b] \to \mathbb{C}$ is gauge-integrable and $F : x \mapsto \int_{[a,x]} f$ is the indefinite integral, then F is an a.e.-primitive of f.*

Proof. We've already proved that F is continuous, so we just need to prove that $F'(x) = f(x)$ almost everywhere. Let E_n be the set of points $x \in [a,b]$ such that for all $\delta > 0$ there exists some y,z with $x \in [y,z] \subseteq [x-\delta,x+\delta]$ such that

$$\left| F(z) - F(y) - f(x)(z-y) \right| > \frac{|z-y|}{n}.$$

Let $\delta_\epsilon : [a,b] \to \mathbb{R}^+$ be any gauge such that $\dot{\mathcal{P}} \ll \delta_\epsilon \implies |S(f,\dot{\mathcal{P}}) - \int_{[a,b]} f| \le \epsilon$. By the Vitali Covering Lemma 1.1.42, we can find a disjoint collection of tagged intervals $(x_i,[y_i,z_i])$ such that

$$\left| F(z_i) - F(y_i) - f(x_i)(z_i-y_i) \right| > \frac{|z_i-y_i|}{n}$$

and

$$x_i \in [y_i,z_i] \subseteq [x_i - \delta_\epsilon(x_i), x_i + \delta_\epsilon(x_i)]$$

for each i, and such that $E_n \setminus \bigcup_i [y_i,z_i]$ is a null set. By the Saks-Henstock Lemma 1.1.221 and its corollaries, we also have

$$\sum_i \left| F(z_i) - F(y_i) - f(x_i)(z_i-y_i) \right| \le \pi\epsilon,$$

so $\sum_i |z_i - y_i| \le \pi n\epsilon$. Since ϵ was arbitrary, we see that E_n is null. □

$L^p(X, \mu)$

Definition 1.1.226. We say that a function is *null* if it vanishes outside of a set of measure 0.

Definition 1.1.227. For $p > 0$, the *p-norm* of a measurable function $f : X \to \mathbb{C}$ (possibly undefined or infinite on a set of measure zero) with respect to the measure μ is defined by $\|f\|_p = (\int_X |f|^p \, d\mu)^{1/p}$ for $p < \infty$, and $\|f\|_\infty = \inf\{C \geq 0 \mid |f(x)| \leq C \text{ a.e. } x \in X\}$. We let $\mathcal{L}^p(X, \mu)$ be the vector space of functions on X with $\|f\|_p < \infty$, and we let L^p be the quotient of $\mathcal{L}^p$ by the set of null functions.

Proposition 1.1.228. *If $f, g \in L^p$ then $f + g \in L^p$. If $0 < p \leq 1$ then $d_p(f, g) = \|f - g\|_p^p$ defines a metric on L^p.*

Proof. For $1 \leq p < \infty$, we have $|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p)$ by convexity of $|\cdot|^p$, while for $0 < p \leq 1$ we have $|f + g|^p \leq 0^p + (|f| + |g|)^p \leq |f|^p + |g|^p$ by concavity of $(\cdot)^p$ on $\mathbb{R}^+$. □

Proposition 1.1.229. *If $f \in L^\infty \cap L^q$ for some $q < \infty$, then $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.*

Lemma 1.1.230 (Young's Inequality)**.** *If $a, b, p, q \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, with equality when $a^p = b^q$.*

Theorem 1.1.231 (Hölder)**.** *If $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p, g \in L^q$, then $\|fg\|_1 \leq \|f\|_p \|g\|_q$.*

Conversely, if $p < \infty$ and $f \in L^p$, then $\|f\|_p = \max\{|\int_X fg \, d\mu| \text{ s.t. } \|g\|_q \leq 1\}$, and the same holds for $p = \infty$ with the max *replaced by a* sup *if every set of infinite measure contains a subset of finite nonzero measure.*

Proof. We may assume without loss of generality that $\|f\|_p = \|g\|_q = 1$. Then $\int |fg| \, d\mu \leq \int \frac{|f|^p}{p} + \frac{|g|^q}{q} \, d\mu = 1$. Without the assumption that $\|f\|_p = \|g\|_q = 1$, the argument goes as follows:

$$\int |fg| = \|f\|_p\|g\|_q \int \frac{|f|}{\|f\|_p} \frac{|g|}{\|g\|_q} \leq \|f\|_p\|g\|_q \int \frac{|f|^p}{p\|f\|_p^p} + \frac{|g|^q}{q\|g\|_q^q} = \|f\|_p\|g\|_q.$$

For the converse, take $g = \frac{|f|^p}{f\|f\|_p^{p-1}}$. □

Corollary 1.1.232. *If μ is σ-finite, then for $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have $f \in L^p$ iff there exists some M such that $|\int fg| \leq M\|g\|_q$ for all simple functions g.*

Proof. Approximate $|f|$ from below by simple functions in L^p, and apply the converse to Hölder 1.1.231. □

Theorem 1.1.233 (Minkowski)**.** *If $p \geq 1$, then $\|f + g\|_p \leq \|f\|_p + \|g\|_p$, and if $1 < p < \infty$ we have equality iff $f = \lambda g$ with $\lambda \geq 0$ or $g = 0$ (a.e.).*

Proof. By Hölder 1.1.231, for any $h \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have $\int |f+g|h \leq \|f\|_p\|h\|_q + \|g\|_p\|h\|_q$, and taking $h = |f + g|^{p-1}$ gives the result (this h has $\int |f + g|h = \|f + g\|_p\|h\|_q$).

For the equality case, note that by the equality case of Young's inequality in the proof of Hölder, we must have $\frac{|f|^p}{\|f\|_p^p} = \frac{|f+g|^{(p-1)q}}{\|f+g\|_p^{(p-1)q}} = \frac{|g|^p}{\|g\|_p^p}$ (a.e.). □

Theorem 1.1.234 (Riesz-Fischer for L^p)**.** *L^p is complete with respect to the p-norm for $0 < p \leq \infty$.*

Proof. It's enough to show that if $\sum_i \|u_i\|_p < \infty$ (or $\sum_i \|u_i\|_p^p < \infty$ in the case $0 < p < 1$) then $\sum_i u_i$ is the L^p-limit of its partial sums. This follows from the monotone convergence theorem (to show that $\sum_i |u_i|$ is in L^p), followed by the dominated convergence theorem to show that the tail sums converge to 0 in the p-norm. □

Corollary 1.1.235. *If f_k converge in L^p to f, then there is a subsequence f_{k_i} that converge pointwise a.e. to f.*

Proof. Choose any subsequence such that $\sum_i \|f_{k_{i+1}} - f_{k_i}\|_p < \infty$. □

Proposition 1.1.236. *The integrable simple functions are dense in L^p for every $0 < p < \infty$, and the simple functions are dense in L^∞.*

Theorem 1.1.237 (Riesz Representation for L^p)**.** *The natural map $L^p \to L^{q*}$ is an isometric isomorphism if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If μ is σ-finite, then so is the map $L^\infty \to L^{1*}$.*

Proof. By Hölder 1.1.231, we just need to check that $L^p \to L^{q*}$ is surjective. Let I be a bounded linear functional on L^{q*}, we just need to construct an $f \in L^p$ such that $I(\chi_E) = \int f\chi_E$ for all sets E with $\mu(E) < \infty$.

If μ is σ-finite, with the full space written as a disjoint union $\bigcup_i X_i$ with $\mu(X_i) < \infty$, then $\nu : E \mapsto \sum_i I(\chi_{E\cap X_i})$ defines a measure, and since $\mu(E) = 0 \implies \|\chi_E\|_q = 0 \implies I(\chi_E) = 0$, we have $\nu \ll \mu$. Then taking f to be the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$, we get $I(g) = \int fg$ for all integrable simple functions g, and since I is a bounded functional on L^q we see from the corollary to Hölder 1.1.231 that $f \in L^p$.

For the general case, use the previous case to define functions $f_E \in L^p$ supported on E for every σ-finite set E, such that $I(g) = \int f_E g$ for $g \in L^q$ supported on E. For any $E \subseteq E'$ σ-finite, by uniqueness we have $f_E = f_{E'}|_E$ a.e., and $\|f_E\|_p \le \|f_{E'}\|_p \le \|I\|$. Choose a sequence E_i of σ-finite sets with $\lim_i \|f_{E_i}\|_p = \sup_E \|f_E\|_p$, and let $X = \bigcup_i E_i$. Then X is σ-finite and $\|f_X\|_p = \sup_E \|f_E\|_p$, so for any σ-finite E we have f_E supported on $E \cap X$ up to a set of measure 0. For any $g \in L^q$, the support of g is a σ-finite set E, so $I(g) = \int f_E g = \int f_{E\cap X} g = \int f_X g$. Thus we may take $f = f_X$. □

Proposition 1.1.238. *If $0 < p \le q \le \infty$ and $\mu(X) < \infty$ then $\|f\|_p \le \mu(X)^{\frac{1}{p}-\frac{1}{q}}\|f\|_q$.*

Proof. By raising both sides to the pth power and replacing f with $|f|^p$ and q with $\frac{q}{p}$, we see that it's enough to prove $\|f\|_1 \le \mu(X)^{1-\frac{1}{q}}\|f\|_q$ for $1 \le q \le \infty$. This follows from Hölder 1.1.231 applied to the functions 1 and f. □

Proposition 1.1.239. *If $f \in L^p, g \in L^q$, $\alpha \in [0,1]$, and $\frac{1}{r} = \alpha\frac{1}{p} + (1-\alpha)\frac{1}{q}$, then $\||f|^\alpha|g|^{1-\alpha}\|_r \le \|f\|_p^\alpha\|g\|_q^{1-\alpha}$. In particular, if $f \in L^p \cap L^q$, then $f \in L^r$ for all $r \in [p,q]$.*

Proof. Apply Hölder 1.1.231 to $|f|^{\alpha r} \in L^{p/\alpha r}$ and $|g|^{(1-\alpha)r} \in L^{q/(1-\alpha)r}$. □

Proposition 1.1.240. *If X is metrizable and σ-finite and Σ is the Borel σ-algebra, then $C(X)\cap L^p$ is dense in L^p.*

Proposition 1.1.241. *If μ is a Radon measure on a locally compact Hausdorff space, then continuous functions with compact support are dense in L^p for $0 < p < \infty$.*

Proof. It's enough to approximate χ_E for every Borel set E with $\mu(E) < \infty$. For such E and for any $\epsilon > 0$, there is an open set U and a compact set K with $K \subseteq E \subseteq U$ with $\mu(U \setminus K) < \epsilon$. By locally compact Urysohn 1.1.4, there is a continuous function f taking values in $[0, 1]$ which is supported on a compact subset of U, with $f|_K = 1$. □

Corollary 1.1.242. *Integrable step functions are dense in $L^p(\mathbb{R}^n)$ for $0 < p < \infty$.*

1.1.6 Banach spaces and Banach algebras

Definition 1.1.243. A *functional* of a vector space V is a linear map from V to the scalars. If V is a topological vector space over a topological field, then the (continuous) *dual space* V^* is the vector space of continuous functionals of V.

The *weak-$*$ topology* on V^* is the weakest topology such that for each $v \in V$, the map $\varphi \mapsto \varphi(v)$ defines a continuous functional on V^*.

Theorem 1.1.244 (Alaoglu)**.** *If V is a topological vector space over $\mathbb{R}$ or $\mathbb{C}$, and if $U \subseteq V$ is any neighborhood of 0, then the polar*

$$U^\circ = \{\varphi \in V^* \textit{ s.t. } \sup_{u \in U} |\varphi(u)| \le 1\}$$

is compact with respect to the weak-$*$ topology on V^*.

Proof. Let D be the set of scalars with absolute value at most 1. The set D^U of functions $f : U \to D$ is compact under the product topology by Tychonoff's Theorem 1.1.13, and U° is a closed subset of D^U since linearity of $\varphi : U \to D$ can be expressed as a collection of linear equations each relating at most three coordinates of D^U and since continuity is automatic for linear functionals φ which map U into D. □

Definition 1.1.245. If V is a vector space (over $\mathbb{R}$ or $\mathbb{C}$), then $p : V \to [0, \infty)$ is a *seminorm* if $p(0) = 0$, $p(cv) = |c|p(v)$ for c a scalar and $v \in V$, and $p(v + w) \le p(v) + p(w)$ for $v, w \in V$. The seminorm p is a *norm* if additionally $p(v) = 0 \iff v = 0$.

Proposition 1.1.246. *Any seminorm p is a convex function.*

Proof. For any $\alpha \in [0, 1]$, we have

$$p(\alpha x + (1 - \alpha)y) \le p(\alpha x) + p((1 - \alpha)y) = \alpha p(x) + (1 - \alpha)p(y).$$ □

Theorem 1.1.247 (Hahn-Banach Dominated Extension Theorem)**.** *If V is a vector space and $p : V \to [0, \infty)$ is a seminorm, and if φ is a partial functional defined on a subspace $U \subseteq V$ which satisfies $|\varphi(x)| \le p(x)$ for all $x \in U$, then there is some functional ψ defined on V such that $\psi(x) = \varphi(x)$ for $x \in U$ and $|\psi(x)| \le p(x)$ for all $x \in V$.*

Proof. First we prove the real case. By Zorn's Lemma we just need to show that if $v \notin U$ then we can extend φ to a functional ϕ on $U + \langle v \rangle$ which satisfies $|\phi(x)| \le p(x)$. The extension is completely determined by the choice of $\phi(v)$. We just need to ensure that for every $u \in U$ and every $r \in [0, \infty)$ we have

$$\phi(rv + u) = r\phi(v) + \varphi(u) \le p(rv + u)$$

and

$$\phi(-rv+u) = -r\phi(v) + \varphi(u) \leq p(-rv+u).$$

These give us a collection of upper and lower bounds on $\phi(v)$, which are satisfiable as long as

$$\inf_{r \geq 0, u \in U} \frac{p(rv+u) - \varphi(u)}{r} \geq \sup_{r' \geq 0, u' \in U} \frac{\varphi(u') - p(-r'v+u')}{r'}.$$

This follows directly from the convexity of p and our assumption on φ:

$$\begin{aligned}\frac{p(rv+u)}{r} + \frac{p(-r'v+u')}{r'} &\geq \frac{r+r'}{rr'} p\Big(\frac{r'u + ru'}{r+r'}\Big) \\ &\geq \frac{r+r'}{rr'} \varphi\Big(\frac{r'u + ru'}{r+r'}\Big) \\ &= \frac{\varphi(u)}{r} + \frac{\varphi(u')}{r'}.\end{aligned}$$

For the complex case, we extend the real part of φ to a real functional $f : V \to \mathbb{R}$ satisfying $f(x) = \Re(\varphi(x))$ for $x \in U$ and $|f(x)| \leq p(x)$ for all $x \in V$. Then we define ψ by

$$\psi(x) = f(x) - if(ix).$$

□

Definition 1.1.248. A *Banach space* X is a vector space over $\mathbb{R}$ or $\mathbb{C}$ with a norm $x \mapsto \|x\|$ such that X is complete with respect to $\|\cdot\|$, considered as a topological vector space with respect to the topology generated by the open balls $B_r(x) = \{y \mid \|x-y\| < r\}$.

The next bit is stolen from this blogoverflow post.

Lemma 1.1.249 (Zabreiko's Lemma)**.** *If X is a Banach space and $p : X \to [0,\infty)$ is a seminorm such that for all absolutely convergent series $\sum_{n=1}^{\infty} x_n$ in X we have $p(\sum_n x_n) \leq \sum_n p(x_n)$, then p is continuous, that is, $p(x) \ll \|x\|$.*

Proof. Let $A_n = p^{-1}([0,n])$, then since $X = \cup_n \overline{A_n}$, there is some n such that $\overline{A_n}$ has nonempty interior by the Baire category theorem. Since $\overline{A_n}$ is convex and symmetric, some open ball $B_R(0)$ around 0 is contained in $\overline{A_n}$. We claim that $B_R(0) \subseteq A_n$ as well: if $\|x\| < R$, pick $0 < q < 1$ such that $\frac{\|x\|}{1-q} < R$, set $y = \frac{R}{\|x\|}x$, then since $y \in \overline{A_n}$ there exists $y_0 \in A_n$ with $\|y - y_0\| < qR$, and then inductively we find $y_0, y_1, ... \in A_n$ such that for each k, we have $\|y - \sum_{i<k} y_i\| < q^k R$: y_k is taken to be a point in A_n with $\|q^{-k}(y - \sum_{i<k} y_i) - y_k\| < qR$. Since $\|y_k\| < R + qR$ for each k, the sum $\sum_k q^k y_k = y$ is absolutely convergent, so by hypothesis $p(y) \leq \sum_k q^k p(y_k) \leq \frac{n}{1-q}$, so $p(x) \leq \frac{\|x\|}{R}\frac{n}{1-q} < n$, so $x \in A_n$. □

Theorem 1.1.250 (Open Mapping Theorem)**.** *If X, Y Banach spaces, $A : X \to Y$ surjective and continuous, then A takes open sets to open sets.*

Proof. For $y \in Y$, set $p(y) = \inf\{\|x\| \mid Ax = y\}$ in Zabreiko's Lemma. □

Theorem 1.1.251 (Bounded Inverse Theorem)**.** *If X, Y Banach spaces, $A : X \to Y$ bijective and continuous, then A^{-1} is also bounded.*

Theorem 1.1.252 (Closed Graph Theorem)**.** *If X, Y Banach spaces, then $A : X \to Y$ is bounded iff the graph is closed in $X \times Y$.*

Proof. For $x \in X$, set $p(x) = \|Ax\|$ in Zabreiko's Lemma. □

Theorem 1.1.253 (Uniform Boundedness Theorem/Banach Steinhaus)**.** *Suppose X is Banach, Y is a normed vector space, and F is a set of continuous linear functions $T : X \to Y$. If for all $x \in X$ we have $\sup_{T\in F} \|T(x)\| < \infty$, then $\sup_{T\in F} \|T\| < \infty$.*

Proof. Set $p(x) \in \sup_{T\in F} \|T(x)\|$ in Zabreiko's Lemma. □

Corollary 1.1.254. *If a sequence of bounded operators from a Banach space to a normed space converges pointwise, then the pointwise limit is a bounded operator.*

There is a cute representation-theoretic application of the Uniform Boundedness Theorem.

Definition 1.1.255. Suppose that V is a topological vector space and G is a topological group. A representation $\rho : G \to \mathrm{Aut}(V)$ is a *continuous representation* if the map $(g, v) \mapsto \rho_g(v)$ is a continuous map from $G \times V$ to V.

Theorem 1.1.256. *If V is a Banach space, G is a locally compact topological group, and $\rho : G \to \mathrm{Aut}(V)$ is a representation such that for all $v \in V$ the map $g \mapsto \rho_g(v)$ is a continuous map from G to V, then ρ is a continuous representation of G.*

Proof. Let K be a compact neighborhood of the identity in G. Then for any $v \in V$, the image of K under $g \mapsto \rho_g(v)$ is compact, so $\sup_{g\in K} \|\rho_g(v)\| < \infty$. Then by the Uniform Boundedness Theorem, we see that $\sup_{g\in K} \|\rho_g\| < \infty$. □

Definition 1.1.257. A *Banach algebra* $\mathcal{A}$ is a normed algebra over $\mathbb{R}$ or $\mathbb{C}$ which is a Banach space such that for all $a, b \in \mathcal{A}$ we have $\|ab\| \le \|a\|\|b\|$. The Banach algebra $\mathcal{A}$ is *unital* if a multiplicative identity $1 \in \mathcal{A}$ exists and satisfies $\|1\| = 1$.

Proposition 1.1.258. *Every Banach algebra $\mathcal{A}$ embeds isometrically into a unital Banach algebra $\mathcal{A}_1$.*

Proof. Let $\mathcal{A}_1$ be the set of formal linear combinations $c + a$ where c is a scalar and $a \in \mathcal{A}$, with the norm $\|c + a\| = |c| + \|a\|$. We just need to check the norm inequality for multiplication:

$$\begin{aligned}\|(c + a)(d + b)\| &= |cd| + \|cb + da + ab\| \\ &\le |cd| + \|cb\| + \|da\| + \|ab\| \\ &\le |cd| + |c|\|b\| + |d|\|a\| + \|a\|\|b\| \\ &= \|c + a\|\|d + b\|.\end{aligned}$$

□

Proposition 1.1.259. *Every real Banach algebra $\mathcal{A}$ has an isometric embedding into a complex Banach algebra $\mathcal{A}_{\mathbb{C}}$.*

Proof. As a real vector space, we will take $\mathcal{A}_{\mathbb{C}}$ to be $\mathcal{A} \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{A} \oplus i\mathcal{A}$. The only challenge is defining the norm, which needs to be invariant under multiplication by complex numbers with absolute value 1. We'll define it by the rule

$$\|a + ib\| = \inf \Big\{ \sum_{k\le n} |c_k|\|d_k\| \text{ s.t. } n \in \mathbb{N}, c_k \in \mathbb{C}, d_k \in \mathcal{A}, \sum_{k\le n} c_k d_k = a + ib \Big\}.$$

This satisfies $\|c(a+ib)\| = |c|\|a+ib\|$ for all $c \in \mathbb{C}$, as well as the inequality $\|(a+ib)(c+id)\| \leq \|a+ib\|\|c+id\|$. By taking real parts of the c_ks, we have $\|a+ib\| \geq \|a\|$, and similarly $\|a+ib\| \geq \|b\|$, so nonzero elements of $\mathcal{A}_{\mathbb{C}}$ have positive norm. □

Proposition 1.1.260. *If V is a Banach space, then the algebra* $\mathrm{End}(V)$ *of continuous linear operators $V \to V$ is a Banach algebra, with norm given by the operator norm.*

Proposition 1.1.261. *If $\mathcal{A}$ is a Banach algebra, then there is a continuous algebra homomorphism $\mathcal{A} \to \mathrm{End}(\mathcal{A})$ given by left multiplication, i.e. $a \mapsto \rho_a$ where $\rho_a(b) = ab$. If $\mathcal{A}$ is unital then $\|a\| = \|\rho_a\|$.*

Proposition 1.1.262. *If $\mathcal{A}$ is a unital Banach algebra, then the set of invertible elements $\mathcal{A}^\times$ is open, and the map $x \mapsto x^{-1}$ is a homeomorphism from $\mathcal{A}^\times$ to itself.*

Proof. If a is invertible and $\|b-a\| < 1/\|a^{-1}\|$, then $\|(a-b)a^{-1}\| < 1$, so we can use the geometric series:

$$b^{-1} = a^{-1}(1-(a-b)a^{-1})^{-1} = a^{-1}\sum_{i\geq 0}((a-b)a^{-1})^i.$$

□

Definition 1.1.263. For $\mathcal{A}$ a complex unital Banach algebra and $a \in \mathcal{A}$, we define the *spectrum* of a, written $\sigma(a)$, to be the set

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda \notin \mathcal{A}^\times\}.$$

We define the *spectral radius* of a, written $r(a)$, to be

$$r(a) = \sup_{\lambda\in\sigma(a)} |\lambda|.$$

Proposition 1.1.264. *For any element $a \in \mathcal{A}$, $\mathcal{A}$ a complex unital Banach algebra, the spectrum $\sigma(a)$ is a compact subset of $\mathbb{C}$ and $r(a) \leq \|a\|$.*

Proof. That $\sigma(a)$ is closed follows from the fact that $\mathcal{A}^\times$ is open, and the bound on $r(a)$ follows from the fact that $(1-a/\lambda)^{-1} = \sum_{i\geq 0}(a/\lambda)^i$ as long as $\|a\| < \lambda$. □

Proposition 1.1.265. *For any element $a \in \mathcal{A}$, $\mathcal{A}$ a complex unital Banach algebra, and for any polynomial $p(x) \in \mathbb{C}[x]$, we have $p(\sigma(a)) \subseteq \sigma(p(a))$.*

Proof. For any $\lambda \in \sigma(a)$, the difference $a - \lambda$ left-divides $p(a) - p(\lambda)$, so $p(\lambda) \in \sigma(p(a))$. □

Theorem 1.1.266. *For any element $a \in \mathcal{A}$, $\mathcal{A}$ a complex unital Banach algebra, the spectrum $\sigma(a)$ is nonempty and $r(a) = \lim_{n\to\infty}\|a^n\|^{1/n}$.*

Proof. By the previous results, we have $r(a) \leq r(a^n)^{1/n} \leq \|a^n\|^{1/n}$ for all n. Next let φ be any element of the dual Banach space $\mathcal{A}^*$, i.e. φ is any bounded linear map $\varphi : \mathcal{A} \to \mathbb{C}$. Define $f : \mathbb{C} \setminus \sigma(a) \to \mathbb{C}$ by

$$f(\lambda) = \varphi\big((\lambda - a)^{-1}\big),$$

then f is holomorphic on its domain (by the geometric series trick), and for $|\lambda| > \|a\|$ we have

$$|f(\lambda)| \leq \sum_{n\geq 0}|\lambda|^{-n-1}\|\varphi\|\|a\|^n = \frac{\|\varphi\|}{|\lambda| - \|a\|}.$$

In particular, if the spectrum $\sigma(a)$ is empty then f is a bounded holomorphic function which goes to zero at infinity, so f is identically 0, and since this would have to be true for all $\varphi \in \mathcal{A}^*$, we see that $a^{-1} = 0$, which is impossible.

For the statement about the spectral radius, note that $f(1/x)$ extends to a holomorphic function around $x = 0$ with series expansion

$$f(\lambda) = \sum_{n\geq 0} \lambda^{-n-1}\varphi(a^n)$$

for $|\lambda| > r(a)$. By the Cauchy integral formula applied to $f(1/x)$, we have

$$|\varphi(a^n)| \leq r^{n+1}\|\varphi\| \sup_{\theta\in[0,2\pi)} \|re^{i\theta} - a\|$$

for any $r > r(a)$, and by the Hahn-Banach Theorem we can choose φ such that $|\varphi(a^n)| = \|a^n\|$ and $\|\varphi\| = 1$, so for any $r > r(a)$ we have

$$\|a^n\|^{1/n} \leq r^{1+1/n} \sup_{\theta\in[0,2\pi)} \|re^{i\theta} - a\|^{1/n}.$$

Taking the lim sup of both sides, we get $\limsup_n \|a^n\|^{1/n} \leq r(a)$. □

Corollary 1.1.267. *If a complex unital Banach algebra $\mathcal{A}$ is a division ring, then $\mathcal{A} \cong \mathbb{C}$.*

Corollary 1.1.268. *If a real unital Banach algebra $\mathcal{A}$ is a division ring, then $\mathcal{A}$ is isomorphic (as an $\mathbb{R}$-algebra) to either $\mathbb{R}$, $\mathbb{C}$, or the quaternions $\mathbb{H}$.*

Proof. Embed $\mathcal{A}$ into a complex Banach algebra $\mathcal{A}_{\mathbb{C}}$. For $a \in \mathcal{A}$, let $\sigma_{\mathbb{C}}(a)$ be the spectrum of a considered as an element of $\mathcal{A}_{\mathbb{C}}$. Since $\sigma_{\mathbb{C}}(a)$ is nonempty, there is some $\lambda \in \mathbb{C}$ such that $a - \lambda$ is not invertible in $\mathcal{A}_{\mathbb{C}}$. If λ is real, then since $a - \lambda \in \mathcal{A}$ and $a - \lambda$ is not invertible, we must have $a = \lambda$. If λ has a nonzero imaginary part, then we have

$$a^2 - 2\Re(\lambda)a + |\lambda|^2 = (a - \lambda)(a - \bar{\lambda}) \in \mathcal{A} \setminus \mathcal{A}^\times = \{0\},$$

so we see that every element of $\mathcal{A}$ is either real or satisfies a quadratic polynomial which has real coefficients and no real roots. By subtracting $\Re(\lambda)$ from a and rescaling, we see that any $a \in \mathcal{A}$ can be written in the form

$$a = x + yu$$

where $x, y \in \mathbb{R}$ are given by $x = \Re(\lambda), y = \Im(\lambda)$, and $u \in \mathcal{A}$ satisfies $u^2 = -1$ and $au = ua$.

Now suppose that $u, v \in \mathcal{A}$ satisfy $u^2 = v^2 = -1$, but $u \neq \pm v$. Writing

$$uv = x + yw$$

where $x, y \in \mathbb{R}$ and $w \in \mathcal{A}, w^2 = -1$, we see that

$$vu = -u(uv)u = x - yuwu$$

and $(uwu)^2 = -1$. We also have $(uv)(vu) = 1$, so

$$vu = (uv)^{-1} = \frac{x - yw}{x^2 + y^2}.$$

By matching real parts, we see that $x^2+y^2=1$, so $uwu = w$ (since $y \neq 0$ if $u \neq \pm v$). In particular, we have $wu = -uw$, and

$$v = -u(uv) = -xu - yuw,$$

so v is a linear combination of u and uw. By adding xu to v and rescaling, we may as well assume that $x = 0$ and $y = 1$, so $uv = w, vu = -w$.

To finish the argument, we just need to show that every $z \in \mathcal{A}$ which satisfies $z^2 = -1$ and $uz = -zu$ must be a linear combination of v and w. Suppose not, for the sake of contradiction. By subtracting a multiple of v from z and rescaling, we may assume without loss of generality that z also satisfies $vz = -zv$, and similarly that $wz = -zw$. But then we have

$$-z = uvwz = -zuvw = z,$$

so $z = 0$, contradicting $z^2 = -1$. □

Now we will focus on commutative Banach algebras over $\mathbb{C}$. The obvious examples of these are the spaces of continuous functions on compact Hausdorff spaces, where the norm of f is given by $\sup_x |f(x)|$.

Definition 1.1.269. A *character* of a commutative unital Banach algebra $\mathcal{A}$ over $\mathbb{C}$ is a $\mathbb{C}$-algebra map $\gamma : \mathcal{A} \to \mathbb{C}$. We write $\widehat{\mathcal{A}}$ for the set of characters of $\mathcal{A}$.

Proposition 1.1.270. *If $\mathcal{A}$ is a commutative complex unital Banach algebra, then every maximal ideal $\mathfrak{m}$ of $\mathcal{A}$ is closed.*

Proof. Maximality of $\mathfrak{m}$ implies that $x \notin \mathfrak{m}$ is equivalent the existence of some $y \in \mathfrak{m}$ such that $x + y$ is invertible, so $\mathfrak{m}$ is the complement of $\mathfrak{m} + \mathcal{A}^\times$, which is open since $\mathcal{A}^\times$ is open. □

Proposition 1.1.271. *If $\mathcal{A}$ is a commutative complex unital Banach algebra, then there is a bijection between the maximal ideals $\mathfrak{m}$ of $\mathcal{A}$ and the characters $\gamma \in \widehat{\mathcal{A}}$.*

Proof. If $\gamma \in \widehat{\mathcal{A}}$, then $\ker \gamma$ is a maximal ideal of $\mathcal{A}$. Conversely, if $\mathfrak{m}$ is a maximal ideal then $\mathfrak{m}$ is closed, so $\mathcal{A}/\mathfrak{m}$ is a complex unital Banach algebra (under the norm $\|x + \mathfrak{m}\| = \inf_{y \in \mathfrak{m}} \|x + y\|$, which satisfies $\|1 + \mathfrak{m}\| = 1$ since any y with $\|1 - y\| < 1$ must be invertible) which is also a division ring, and therefore we have an isomorphism $\mathcal{A}/\mathfrak{m} \cong \mathbb{C}$, corresponding to a character $\gamma : \mathcal{A} \to \mathbb{C}$ with $\ker \gamma = \mathfrak{m}$. □

Proposition 1.1.272. *If $\mathcal{A}$ is a commutative complex unital Banach algebra, then every character $\gamma \in \widehat{\mathcal{A}}$ is continuous.*

Proof. There is only one norm on $\mathbb{C} \cong \mathcal{A}/\ker \gamma$ which satisfies $\|1\| = 1$. □

Proposition 1.1.273. *If $\mathcal{A}$ is a commutative complex unital Banach algebra, then for every $a \in \mathcal{A}$ the spectrum $\sigma(a)$ is given by*

$$\sigma(a) = \{\gamma(a) \mid \gamma \in \widehat{\mathcal{A}}\}.$$

Proof. We have $\lambda \in \sigma(a)$ iff $\lambda - a \notin \mathcal{A}^\times$, which occurs iff some maximal ideal $\mathfrak{m}$ contains $\lambda - a$, and $\gamma \in \widehat{\mathcal{A}}$ sends a to λ iff $a - \lambda \in \ker \gamma$. □

Definition 1.1.274. If $\mathcal{A}$ is a commutative complex unital Banach algebra, then the *Gelfand topology* on $\widehat{\mathcal{A}}$ is the restriction of the weak-$*$ topology to $\widehat{\mathcal{A}} \subseteq \mathcal{A}^*$ - so the open sets are generated by the sets $\{\gamma \in \widehat{\mathcal{A}} \mid \gamma(a) \in U\}$ for $a \in \mathcal{A}$ and $U \subseteq \mathbb{C}$ open.

Chapter 2

Algebra

2.1 Noncommutative rings

Definition 2.1.1. If R is a ring, then the *Jacobson radical* $J(R)$ (sometimes written $\operatorname{rad}(R)$) is the intersection of the annihilators of all simple left R-modules.

Definition 2.1.2. A submodule N of M is *superfluous*, written $N \subseteq_s M$ or $N \ll M$, if for all H we have $N + H = M \implies H = M$.

Theorem 2.1.3. *We can replace "left" by "right" in the definition of the Jacobson radical of a ring. Furthermore, we have the following equivalent definitions:*

- *$J(R)$ is the intersection of all maximal left ideals of R,*
- *$J(R)$ is the sum of all superfluous left ideals of R,*
- *$J(R)$ is the maximal left ideal of R such that for all $x \in J(R)$, $1 - x$ has a left inverse,*
- *$J(R) = \{x \in R \mid 1 + RxR \subseteq R^\times\}$.*

Lemma 2.1.4 (Nakayama's Lemma)**.** *If M is a finitely generated left R-module with $M = J(R)M$, then $M = 0$.*

Proof. Consider a minimal generating set $x_1, ..., x_n$ of M, and use $\sum x_i \in J(R)M$ to write x_n as a linear combination of $x_1, ..., x_{n-1}$. □

Proposition 2.1.5. *$J(R/J(R)) = 0$.*

2.1.1 Artinian Rings

Proposition 2.1.6. *If R, considered as a left R-module over itself, has a composition series of length k, then $J(R)^k = 0$.*

Theorem 2.1.7 (Hopkins' Theorem)**.** *If M is a left module over a left Artinian ring, then the following are equivalent:*

- *M is finitely generated,*

- *M has finite length,*
- *M is Noetherian,*
- *M is Artinian.*

Theorem 2.1.8 (Hopkins-Levitzki)**.** *If R is* semiprimary *- that is, if $R/J(R)$ is semisimple and $J(R)$ is nilpotent - then for left R-modules, being Noetherian, being Artinian, and having a composition series are equivalent.*

Proposition 2.1.9. *If $J(R) = 0$, then every minimal left ideal of R is a direct summand of R.*

Theorem 2.1.10. *R is semisimple if and only if it is left Artinian and has $J(R) = 0$.*

2.2 Commutative Algebra

Definition 2.2.1. If R is a commutative ring, then $I \lhd R$ means that I is an ideal of R.

Definition 2.2.2. If $I, J \lhd R$, set $(I : J) = \{r \in R \mid rJ \subseteq I\}$. If $a \in R$, we abbreviate $(I : (a))$ to $(I : a)$.

2.2.1 Primary Ideals

Definition 2.2.3. $Q \lhd R$ is *primary* if $\forall a, b \in R$ with $ab \in Q$, either $b \in Q$ or $\exists n$ such that $a^n \in Q$.

Definition 2.2.4. If $I \lhd R$, then $\mathrm{rad}(I) = \{r \in R \mid \exists n\ r^n \in I\}$.

Proposition 2.2.5. *Q is primary if and only if $\mathrm{rad}(Q)$ is prime. If Q_1, Q_2 are primary and $\mathrm{rad}(Q_1) = \mathrm{rad}(Q_2)$, then $Q_1 \cap Q_2$ is primary. If R is Noetherian and $Q \lhd R$, then $\exists n$ such that $\mathrm{rad}(Q)^n \subseteq Q$.*

Theorem 2.2.6 (Primary Decomposition)**.** *If R is Noetherian and $I \lhd R$, then $\exists k$ and $Q_1, ..., Q_k \lhd R$ primary such that $I = Q_1 \cap \cdots \cap Q_k$.*

Proof. By R Noetherian, $\forall a \in R\ \exists n$ with $(I : a^n) = (I : a^{n+1})$, and for this n we have $(I + (a^n)) \cap (I : a) = I$, so either I is already primary or we can write I as an intersection of bigger ideals, and apply Noetherian induction. □

Lemma 2.2.7. *If R is Noetherian, then for any $I \lhd R$ and $r \in R \setminus I$, there exists $s \in R$ such that $(I : rs)$ is prime.*

Theorem 2.2.8 (Uniqueness of radicals)**.** *If R is Noetherian, $I = Q_1 \cap \cdots \cap Q_k$ with $Q_i \lhd R$ primary and no Q_i containing $\cap_{j \neq i} Q_j$, and if $\mathfrak{p} \lhd R$ is prime, then $\exists r \in R$ with $(I : r) = \mathfrak{p}$ if and only if there is an i with $\mathrm{rad}(Q_i) = \mathfrak{p}$. In particular, the set $\{\mathrm{rad}(Q_i)\}_{i \leq k}$ is uniquely determined by I.*

Theorem 2.2.9 (Uniqueness of primaries with minimal radical)**.** *If R is Noetherian, $I = Q_1 \cap \cdots \cap Q_k$ with $Q_i \lhd R$ primary and $\mathrm{rad}(Q_i) \not\subseteq \mathrm{rad}(Q_1)$ for $i > 1$, then for n sufficiently large we have $(I : \mathrm{rad}(Q_2)^n \cdots \mathrm{rad}(Q_k)^n) = Q_1$, so Q_1 is uniquely determined by I and $\mathrm{rad}(Q_1)$.*

Chapter 3

Sheaf Cohomology

3.1 Grothendieck Abelian Categories

The material in this section is mostly from the stacks project, specifically [176, Tag 05NM], [176, Tag 079A], and [176, Tag 05AB].

A note: most references are not up front about what type of categories they consider. In this paper all categories $\mathcal{C}$ under consideration will be locally small: for any two objects $A, B \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Mor}_{\mathcal{C}}(A, B)$ is a set. In an additive category, I will write Hom instead of Mor.

Definition 3.1.1. An additive locally small category $\mathcal{C}$ is a *Grothendieck Abelian Category* if it has the following four properties:

(AB) $\mathcal{C}$ is an abelian category. In other words $\mathcal{C}$ has kernels and cokernels, and the canonical map from the coimage to the image is always an isomorphism.

(AB3) AB holds and $\mathcal{C}$ has direct sums indexed by arbitrary sets. Note this implies that colimits over small categories exist (since colimits over small categories can be written as cokernels of direct sums over sets).

(AB5) AB3 holds and filtered colimits over small categories are exact. (A colimit over a small category $\mathcal{D}$ is *filtered* if any two objects $i, j \in \mathrm{Ob}(\mathcal{D})$ have maps to a common object k, if any two maps $i \to j$, $i \to j'$ can be extended to a commutative diagram with everything mapping to another object k, and if for any two maps from i to j we can find a map from j to k coequalizing them. This is meant to be a generalization of a directed set.)

(GEN) $\mathcal{C}$ has a generator. A generator is an object U such that for any proper subobject $N \subsetneq M$ of any object M, we can find a map $U \to M$ that does not factor through N.

Remark 3.1.1. Tamme [179] claims that the following is an equivalent reformulation of the AB5 condition:

(AB5') AB3 holds, and for each directed set of subobjects A_i of an object A of $\mathcal{C}$, and each system of morphisms $u_i : A_i \to B$ such that u_i is induced from u_j if $A_i \subseteq A_j$, there is a morphism $u : \Sigma_i A_i \to B$ inducing the u_i. Here $\Sigma_i A_i$ is the internal sum of the A_is in A, i.e. $\Sigma_i A_i = \mathrm{im}(\bigoplus_i A_i \to A)$.

I haven't worked through the proof of the equivalence, but it probably isn't too hard.

Example 3.1.1. If R is a ring, then the category of R-modules forms a Grothendieck abelian category. AB5' is easy to verify, so if we believe Tamme then we only need to find a generator. One such generator is R, considered as an R-module in the obvious way.

3.1.1 The size of an object

Definition 3.1.2. If M is an object of $\mathcal{C}$, we define $|M|$ to be the cardinality of the smallest set of subobjects of M containing one subobject from each equivalence class of subobjects, or ∞ if there is no such set.

Proposition 3.1.3. *Let $\mathcal{C}$ be a Grothendieck abelian category with a generator U. Then for any object M of $\mathcal{C}$, we have*

- $|M| \le 2^{\mathrm{Hom}_{\mathcal{C}}(U,M)}$
- *If $|M| \le \kappa$, then there is an epimorphism $\bigoplus_\kappa U \twoheadrightarrow M$.*

Proof. For the second claim, find for every proper subobject N of M a map $U \to M$ not factoring through N. The direct sum of this collection of maps can't factor through any proper subobject of M, so it must be an epimorphism.

For the first claim, we just have to check that since U is a generator every subobject N of M is determined up to equivalence by the set of maps $U \to M$ which factor through N. This follows from the proof of the second claim, applied to N. □

We will need the following technical lemma later. Recall that the cofinality of a poset is the smallest cardinality of a cofinal subset of the poset.

Lemma 3.1.4. *Let $\mathcal{C}$ be a Grothendieck abelian category, and let M be an object of $\mathcal{C}$. Suppose α is an ordinal with cofinality greater than $|M|$, and let $\{B_\beta\}_{\beta\in\alpha}$ be a directed system such that each map $B_\beta \to B_\gamma$ is an injection for $\beta \subseteq \gamma$. Then any map $f : M \to \varinjlim B_\beta$ factors through some B_β.*

Proof. By applying AB5 to the exact sequences

$$0 \to B_\beta \to \varinjlim B_\gamma \to (\varinjlim B_\gamma)/B_\beta \to 0,$$

$$0 \to f^{-1}(B_\beta) \to M \to (\varinjlim B_\gamma)/B_\beta$$

we have $\varinjlim f^{-1}(B_\beta) = M$. Since each $f^{-1}(B_\beta)$ is a subobject of M, we can choose a collection of at most $|M|$ β_is such that each $f^{-1}(B_\beta)$ is equivalent to some $f^{-1}(B_{\beta_i})$. Since the cofinality of α is greater than $|M|$, we can find an upper bound $\gamma \in \alpha$ of all of the β_is. Then $f^{-1}(B_\gamma) = M$, so f factors through B_γ. □

3.1.2 Injectives

The next lemma generalizes the fact an abelian group is injective if and only if it is divisible.

Lemma 3.1.5. *Let $\mathcal{C}$ be a Grothendieck abelian category with generator U. Then an object I of $\mathcal{C}$ is injective if and only if we can fill in the dashed arrow in any diagram of the form*

$$\begin{tikzcd}
M \arrow[r] \arrow[d, hook] & I \\
U \arrow[ur, dashed] &
\end{tikzcd}$$

Proof. We need to show that we can fill in the dashed arrow in any diagram of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & I \\
B \arrow[ur, dashed] &
\end{tikzcd}$$

By Zorn's lemma and AB5', we can assume without loss of generality that there is no larger subobject A' of B such that we can find a map $A' \to I$ extending $A \to I$. Suppose for a contradiction that $A \neq B$.

Choose a map $\varphi : U \to B$ that does not factor through A, and set $M = \varphi^{-1}(\varphi(U) \cap A)$. By assumption we can extend the obvious map $M \to I$ to a map $U \to I$. By construction the map $U \to I$ vanishes on $\ker(U \to B)$, and the induced map $\varphi(U) \to I$ agrees with $A \to I$ on $\varphi(U) \cap A$. Thus $A \to I$ extends to a map $A + \varphi(U) \to I$, contradicting the choice of A. □

Theorem 3.1.6 (Grothendieck abelian categories have enough injectives)**.** *Let $\mathcal{C}$ be a Grothendieck abelian category. Then there is a functor taking an object M of $\mathcal{C}$ to a monomorphism $M \hookrightarrow I$ from M to an injective object I.*

Proof. Define the functor J by taking $J(M)$ to be the pushout

$$\begin{tikzcd}
\bigoplus_{N \subseteq U} \bigoplus_{\mathrm{Hom}(N,M)} N \arrow[r] \arrow[d, hook] & M \arrow[d, hook, dashed] \\
\bigoplus_{N \subseteq U} \bigoplus_{\mathrm{Hom}(N,M)} U \arrow[r, dashed] & J(M)
\end{tikzcd}$$

where here N runs over a set of representatives for the subobjects of U, of cardinality $|U|$.

Now we inductively define a sequence of functors J_α indexed by ordinals. Set $J_0 = J$, set $J_{\alpha+1} = J \circ J_\alpha$, and for α a limit ordinal set $J_\alpha = \varinjlim_{\beta \in \alpha} J_\beta$.

Pick, once and for all, an α with cofinality greater than $|U|$ (for instance, we can pick α to be the smallest infinite ordinal with cardinality greater than $|U|$). Then for any M the map $M \to J_\alpha(M)$ is injective (by Zorn's lemma and AB5), so we just need to check that $J_\alpha(M)$ is injective to finish.

By Lemma 3.1.5, we just need to check that for each subobject N of U we can extend any map $N \to J_\alpha(M)$ to a map $U \to J_\alpha(M)$. By Lemma 3.1.4, such a map factors through some $J_\beta(M)$ for some $\beta \in \alpha$, and by the definition of J the map $N \to J_\beta(M)$ extends to a map $U \to J_{\beta+1}(M)$. Since α is a limit ordinal, we have $\beta + 1 \in \alpha$ as well, so $U \to J_{\beta+1}(M) \to J_\alpha(M)$ is the desired extension of $N \to J_\alpha(M)$. □

3.2 Grothendieck Spectral Sequence

For this section we will need a few facts about Cartan-Eilenberg resolutions of complexes.

Exercise 3.2.1. Let $C^\bullet$ be a complex in an abelian category $\mathcal{C}$ with enough injectives. Show that we can find a resolution

$$0 \to C^\bullet \to I^{\bullet,0} \to I^{\bullet,1} \to \cdots$$

such that

- each $I^{i,j}$ is injective,
- if $C^i = 0$, then $I^{i,j} = 0$ for all j,
- each of the sequences

$$\begin{aligned} &0 \to C^i \to I^{i,0} \to I^{i,1} \to \cdots \\ &0 \to B^i(C^\bullet) \to B^i(I^{\bullet,0}) \to B^i(I^{\bullet,1}) \to \cdots \\ &0 \to Z^i(C^\bullet) \to Z^i(I^{\bullet,0}) \to Z^i(I^{\bullet,1}) \to \cdots \\ &0 \to H^i(C^\bullet) \to H^i(I^{\bullet,0}) \to H^i(I^{\bullet,1}) \to \cdots \end{aligned}$$

  is an injective resolution (B^i is the ith coboundary group, and Z^i is the ith cocycle group).

Such a resolution is called a Cartan-Eilenberg resolution. Hint: apply the horseshoe lemma to the exact sequences

$$0 \to B^i(C^\bullet) \to Z^i(C^\bullet) \to H^i(C^\bullet) \to 0$$

and

$$0 \to Z^i(C^\bullet) \to C^i \to B^{i+1}(C^\bullet) \to 0.$$

Exercise 3.2.2. For extra credit, show that for any exact sequence of complexes

$$0 \to C'^\bullet \to C^\bullet \to C''^\bullet \to 0$$

we can find an exact sequence of Cartan-Eilenberg resolutions

$$0 \to I'^{\bullet,\bullet} \to I^{\bullet,\bullet} \to I''^{\bullet,\bullet} \to 0.$$

Theorem 3.2.1 (Grothendieck spectral sequence)**.** *Let $F : \mathcal{C} \to \mathcal{C}'$, $G : \mathcal{C}' \to \mathcal{C}''$ be left exact additive functors of abelian categories, and let $\mathcal{C}, \mathcal{C}'$ have enough injectives. If F maps injective objects of $\mathcal{C}$ to G-acyclic (M is G-acyclic means $R^pG(M) = 0$ for all $p > 0$) objects of $\mathcal{C}'$, then we have a functorial spectral sequence taking $A \in \mathrm{Ob}(\mathcal{C})$ to*

$$E_2^{p,q} = R^pG(R^qF(A)) \ \Rightarrow \ E^n = R^n(G \circ F)(A).$$

Proof. Choose an injective resolution $0 \to A \to I^\bullet$, and then choose a Cartan-Eilenberg resolution $0 \to F(I^\bullet) \to J^{\bullet,\bullet}$. Let $K^\bullet$ be the total complex of $G(J^{\bullet,\bullet})$.

We compute the cohomology of $K^\bullet$ in two ways by means of the two spectral sequences E, E' coming from the double complex $G(J^{\bullet,\bullet})$. Here E is the spectral sequnce we get by first taking cohomology in the first index, and E' is the spectral sequence we get by first taking cohomology in the second index. E' is the easier spectral sequence: we have

$$E_1'^{p,q} = H^q(G(J^{p,\bullet})) = R^qG(F(I^p)) = \begin{cases} (G\circ F)(I^p) \text{ if } q = 0 \\ 0 \text{ if } q > 0 \end{cases},$$

since $F(I^p)$ was assumed to be G-acyclic. Thus $E_2'^{p,q} = \begin{cases} R^p(G\circ F)(A) \text{ if } q = 0 \\ 0 \text{ if } q > 0 \end{cases}$, and the spectral sequence abuts to $E'^n = R^n(G \circ F)(A)$.

As for E, we have (after switching the roles of p and q)

$$E_1^{p,q} = H^q(G(J^{\bullet,p})) = G(H^q(J^{\bullet,p})),$$

since each of the exact sequences

$$0 \to B^q(J^{\bullet,p}) \to Z^q(J^{\bullet,p}) \to H^q(J^{\bullet,p}) \to 0,$$

$$0 \to Z^q(J^{\bullet,p}) \to J^{q,p} \to B^{q+1}(J^{\bullet,p}) \to 0,$$

$$0 \to Z^q(J^{\bullet,p}) \to J^{q,p} \to J^{q+1,p}$$

has all terms injective and thus remains exact when we apply the functor G. Since $0 \to H^q(F(I^\bullet)) \to H^q(J^{\bullet,\bullet})$ is an injective resolution and $H^q(F(I^\bullet)) = R^qF(A)$, we have

$$E_2^{p,q} = H^p(G(H^q(J^{\bullet,\bullet}))) = R^pG(H^q(F(I^\bullet))) = R^pG(R^qF(A)).$$

This abuts to $E^n = H^n(K^\bullet) = E'^n = R^n(G \circ F)(A)$. □

Corollary 3.2.2 (Exact sequence of low degree)**.** *If F, G are as above, then for any A we have an exact sequence*

$$0 \to R^1G(F(A)) \to R^1(G\circ F)(A) \to G(R^1F(A)) \to R^2G(F(A)) \to R^2(G\circ F)(A).$$

We also have the following strengthening of the exact sequence of low degree, from [179].

Corollary 3.2.3. *If F, G are as above, and if $R^pG(R^qF(A)) = 0$ for $0 < q < n$, then*

$$R^mG(F(A)) \cong R^m(G\circ F)(A) \text{ for } m < n,$$

and we have an exact sequence

$$0 \to R^nG(F(A)) \to R^n(G\circ F)(A) \to G(R^nF(A)) \to R^{n+1}G(F(A)) \to R^{n+1}(G\circ F)(A).$$

Example 3.2.1. Let N be a normal subgroup of a group G. Then the functor $A \mapsto A^N$ takes G-modules to G/N-modules, and the functor $B \mapsto B^{G/N}$ takes G/N-modules to abelian groups. The category of G-modules satisfies AB5 and has the generator $\mathbb{Z}[G]$, so it has enough injectives by Theorem 3.1.6, and similarly for the category of G/N-modules. It's easy to check that the functor $A \mapsto A^N$ takes injective G-modules to injective G/N-modules (essentially, since every G/N-module can be regarded as a G-module invariant under N), so we can apply Corollary 3.2.2 to obtain the inflation-restriction exact sequence of group cohomology:

$$0 \to H^1(G/N, A^N) \xrightarrow{inf} H^1(G,A) \xrightarrow{res} H^1(N,A)^{G/N} \xrightarrow{tr} H^2(G/N, A^N) \xrightarrow{inf} H^2(G,A).$$

3.3 Sheaf Cohomology

First we recall the definition of a topology. I'm going to follow Tamme's presentation from [179].

Definition 3.3.1. A *topology* (or *site*) T is a small category $\mathrm{cat}(T)$ (objects of $\mathrm{cat}(T)$ will be called *opens*) together with a set $\mathrm{cov}(T)$ of families $\{U_i \overset{\varphi_i}{\to} U\}_{i\in I}$, called *coverings* of T, satisfying the following axioms.

T1 For $\{U_i \to U\}$ any covering and any morphism $V \to U$, the fiber products $U_i \times_U V$ exist and $\{U_i \times_U V \to V\}$ is also a covering.

T2 For $\{U_i \to U\}$ any covering and for any family of coverings $\{V_{ij} \to U_i\}$, $\{V_{ij} \to U\}$ is also a covering.

T3 If $U' \to U$ is an isomorphism, then $\{U' \to U\}$ is a covering.

A *morphism of topologies* is a functor taking coverings to coverings and commuting with all fiber products that show up in T1.

Example 3.3.1. Let X be a topological space. The site T_X with underlying category the category of open sets of X and coverings given by open coverings satisfies the axioms of a topology. If U, V are open sets contained in the open set W, then we have $U \times_W V = U \cap V$.

If $f : X \to Y$ is a continuous map, then $f^{-1} : T_Y \to T_X$ is a morphism of topologies.

Our main concern is the case of a topology T_X, where X is a scheme.

3.3.1 Sheaves and Presheaves

Let $\mathcal{P}$ be the category of presheaves on T - that is, the category of contravariant functors from $\mathrm{cat}(T)$ to the category of abelian groups. For any open U we define the section functor by $\Gamma(U, F) = F(U)$, for F a presheaf.

Proposition 3.3.2. *$\mathcal{P}$ is a Grothendieck abelian category. A sequence of presheaves is exact if and only if it is exact on each open U.*

Proof. The only nontrivial part of this theorem is that $\mathcal{P}$ has a generator. Rather than constructing a single generator, it is convenient to construct a set of generators, that is a set of presheaves $\{Z_i\}$ such that for any $N \subsetneq M$ we can find an i and a map $Z_i \to M$ which does not factor through N. Then we may take $Z = \bigoplus_i Z_i$ as a generator for $\mathcal{P}$.

Our family of generators is defined as follows. For any open U, we define the presheaf Z_U by

$$Z_U(V) = \bigoplus_{\mathrm{Mor}(V,U)} \mathbb{Z}.$$

For any presheaf F, we have $F(U) = \mathrm{Hom}(Z_U, F)$. Now it's easy to see that $\{Z_U\}_{U\in\mathrm{Ob}(\mathrm{cat}(T))}$ is a family of generators for $\mathcal{P}$. Note that Z_U represents the section functor $\Gamma(U, \cdot)$. □

Now we let $\mathcal{S}$ be the category of sheaves on T. The objects of $\mathcal{S}$ are presheaves F which satisfy the sheaf axiom, which states that for all coverings $\{U_i \to U\}$, the sequence

$$0 \to F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$$

is exact. A morphism of sheaves is then defined to be a morphism of presheaves, making $\mathcal{S}$ a full subcategory of $\mathcal{P}$. Let $\iota : \mathcal{S} \to \mathcal{P}$ be the natural inclusion.

Define a functor $\dagger : \mathcal{P} \to \mathcal{P}$ by

$$F^{\dagger}(U) = \varinjlim_{\{U_i \to U\}} \ker(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)).$$

The index category of the limit is the category of coverings of U with morphisms given by refinements of coverings. (A *refinement* $\{U'_j \to U\}_{j \in J} \to \{U_i \to U\}_{i \in I}$ is a map $\epsilon : J \to I$ together with a map $U'_j \to U_{\epsilon(j)}$ for each $j \in J$.) In the case that our site comes from a topological space, this index category is filtered, so we can conclude that $\dagger$ is a left exact functor (in general we can do some shenanigans to replace the index category with another category which is filtered - see [179] for details).

Definition 3.3.3. A presheaf is called *separated* if the map $F(U) \to \prod_i F(U_i)$ is an injection for every covering $\{U_i \to U\}$.

Exercise 3.3.1. Show that if F is a presheaf then $F^{\dagger}$ is separated, and if F is a separated presheaf then $F^{\dagger}$ is a sheaf. Show that for any sheaf G, any map $F \to G$ factors through $F^{\dagger}$.

If we now define $\# = \dagger \circ \dagger : \mathcal{P} \to \mathcal{S}$, we see that $\#$ is left adjoint to ι. $\#$ is called *sheafification.*

Proposition 3.3.4. *$\mathcal{S}$ is a Grothendieck abelian category. ι is left exact and $\#$ is exact.*

Proof. The presheaf kernel of a morphism of sheaves is easily seen to be a sheaf (since limits commute with limits). Using the adjointness of ι and $\#$, we see that the cokernel of a morphism of sheaves is just the sheafification of the presheaf cokernel.

Since $\dagger: \mathcal{P} \to \mathcal{P}$ is left exact, and since the presheaf kernel agrees with the sheaf kernel, we see that $\#$ is left exact. The left exactness of $\#$ implies that the coimage and the image of a morphism agree (easy exercise). Thus $\mathcal{S}$ satisfies AB.

That ι is left exact also follows from the fact that the presheaf kernel and the sheaf kernel agree. From the adjointness of ι and $\#$ we see that $\#$ is right exact. Combining this with the above, we see that $\#$ is exact.

For AB3, note that to calculate a colimit in $\mathcal{S}$, we just calculate the colimit in $\mathcal{P}$ and then sheafify (using the adjointness of ι and $\#$). For AB5, note that if filtered colimits are exact in $\mathcal{P}$ then they remain exact in $\mathcal{S}$ (since $\#$ is exact).

Finally, we must construct a family of generators for $\mathcal{S}$. We take as generators the sheaves $Z_U^{\#}$: for any sheaf F, we have

$$F(U) = \mathrm{Hom}_{\mathcal{P}}(Z_U, \iota(F)) = \mathrm{Hom}_{\mathcal{S}}(Z_U^{\#}, F).$$

Note that this shows that the sheaf $Z_U^{\#}$ represents the functor $\Gamma(U, \cdot)$. □

3.3.2 Čech Cohomology

Čech Cohomology is most naturally defined on the category of presheaves.

Definition 3.3.5. Let $\{U_i \to U\}$ be a covering. The *derived Čech Cohomology groups* of a presheaf F with respect to the covering $\{U_i \to U\}$ are

$$H^0(\{U_i \to U\}, F) = \ker(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)),$$

and

$$H^p(\{U_i \to U\}, F) = R^p H^0(\{U_i \to U\}, F).$$

These groups can be computed by means of the Čech complex. For the sake of my sanity, we make the abbreviation $U_{i_0,\ldots,i_p} = U_{i_0} \times_U \cdots \times_U U_{i_p}$.

Definition 3.3.6. For F a presheaf and $\{U_i \to U\}$ a covering, the *Čech Complex* is given by

$$C^p(\{U_i \to U\}, F) = \prod_{(i_0,\ldots,i_p)} F(U_{i_0,\ldots,i_p}),$$

with differentials $d^p : C^p(\{U_i \to U\}, F) \to C^{p+1}(\{U_i \to U\}, F)$ given by

$$(d^p s)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k F(U_{i_0,\ldots,i_{p+1}} \to U_{i_0,\ldots,\widehat{i_k},\ldots,i_{p+1}})(s_{i_0,\ldots,\widehat{i_k},\ldots,i_{p+1}}),$$

where the hat over a term means that that term is omitted. In the case of a topological space, this reduces to the usual definition.

Theorem 3.3.7 (Čech Cohomology is a derived functor)**.** *For any presheaf F and any covering $\{U_i \to U\}$, we have*

$$H^p(\{U_i \to U\}, F) = H^p(C^\bullet(\{U_i \to U\}, F)).$$

Proof. Set $Z_{\{U_i \to U\}} = \operatorname{coker}(\bigoplus_{i,j} Z_{U_{i,j}} \rightrightarrows \bigoplus_i Z_{U_i})$. Then we have

$$H^0(\{U_i \to U\}, F) = \ker(\operatorname{Hom}(\bigoplus_i Z_{U_i}, F) \rightrightarrows \operatorname{Hom}(\bigoplus_{i,j} Z_{U_{i,j}}, F)) = \operatorname{Hom}(Z_{\{U_i \to U\}}, F),$$

so in fact

$$H^p(\{U_i \to U\}, F) = \operatorname{Ext}^p(Z_{\{U_i \to U\}}, F).$$

Furthermore, we have

$$C^p(\{U_i \to U\}, F) = \operatorname{Hom}(\bigoplus_{(i_0,\ldots,i_p)} Z_{U_{i_0,\ldots,i_p}}, F),$$

and the maps d^p are induced by maps $d_{p+1} : \bigoplus_{(i_0,\ldots,i_{p+1})} Z_{U_{i_0,\ldots,i_{p+1}}} \to \bigoplus_{(i_0,\ldots,i_p)} Z_{U_{i_0,\ldots,i_p}}$.

For any open V the functor $\operatorname{Hom}(Z_V, \cdot) = \Gamma(V, \cdot) : \mathcal{P} \to \mathrm{Ab}$ is right exact, so in fact all of the presheaves Z_V are projective. Thus, to show that

$$\operatorname{Ext}^p(Z_{\{U_i \to U\}}, F) = H^p(C^\bullet(\{U_i \to U\}, F)),$$

it's enough to show that the projective resolution

$$0 \leftarrow Z_{\{U_i \to U\}} \leftarrow \bigoplus_i Z_{U_i} \xleftarrow{d_1} \bigoplus_{i,j} Z_{U_{i,j}} \xleftarrow{d_2} \cdots$$

is exact. By construction, we already know that it is exact at $Z_{\{U_i \to U\}}$ and at $\bigoplus_i Z_{U_i}$.

To check the exactness everywhere else, it is enough to check it is exact when we plug in any open V. Using $Z_U(V) = \bigoplus_{\mathrm{Mor}(V,U)} \mathbb{Z}$, we see that we just need to prove the exactness of

$$\bigoplus_{i} \bigoplus_{\mathrm{Mor}(V,U_i)} \mathbb{Z} \xleftarrow{d_1} \bigoplus_{i,j} \bigoplus_{\mathrm{Mor}(V,U_{i,j})} \mathbb{Z} \xleftarrow{d_2} \bigoplus_{i,j,k} \bigoplus_{\mathrm{Mor}(V,U_{i,j,k})} \mathbb{Z} \xleftarrow{d_3} \cdots$$

Now we split this up into non-interacting sequences based on the overall map $\varphi : V \to U$ (note that this step is incredibly silly in the case of topological spaces). Let S_φ be the set of commuting diagrams of the form

$$\begin{tikzcd} V \arrow[r] \arrow[dr, "\varphi"'] & U_i \arrow[d] \\ & U \end{tikzcd}$$

Then the subset of $\coprod_{i_0,\dots,i_p} \mathrm{Mor}(V, U_{i_0,\dots,i_p})$ that maps to φ in $\mathrm{Mor}(V,U)$ is identified with S_φ^{p+1}. Thus we just have to prove the exactness of the sequence

$$\bigoplus_{S_\varphi} \mathbb{Z} \xleftarrow{d_1} \bigoplus_{S_\varphi \times S_\varphi} \mathbb{Z} \xleftarrow{d_2} \bigoplus_{S_\varphi \times S_\varphi \times S_\varphi} \mathbb{Z} \xleftarrow{d_3} \cdots$$

If we label the generators of the different copies of $\mathbb{Z}$ by $e_{s_0,\dots,s_p}$, $s_i \in S_\varphi$, then we have

$$d_p(e_{s_0,\dots,s_p}) = \sum_{k=0}^{p} (-1)^k e_{i_0,\dots,\widehat{i_k},\dots,i_p}.$$

Most likely, you already know a proof that this sequence is exact (probably involving an explicit chain homotopy). □

Definition 3.3.8. For any presheaf F and any open U, the *Čech cohomology groups* of F on the open U are

$$\check{H}^p(U,F) = \varinjlim_{\{U_i \to U\}} H^p(\{U_i \to U\}, F).$$

Remark 3.3.1. We have $\check{H}^0(U,F) = F^\dagger(U)$, and $\check{H}^p(U,F) = R^p\check{H}^0(U,F)$.

Remark 3.3.2. For a general site there is a subtle technical problem with the previous definition: it is possible that a cover $\{U_i \to U\}_i$ is refined by another cover $\{V_{ij} \to U\}_{ij}$ in multiple ways, since a refinement of a cover comes with a collection of maps $\varphi_{ij} : V_{ij} \to U_i$ over U. In order to fix this, one shows that for any two such collections of maps $\varphi_{ij}, \varphi'_{ij}$, the two induced maps from $H^p(\{U_i \to U\}_i, F)$ to $H^p(\{V_{ij} \to U\}_{ij}, F)$ agree. For details see Tamme's book [179].

3.3.3 Sheaf Cohomology

Definition 3.3.9. If $F \in \mathcal{S}$ is a sheaf, we define the *sheaf cohomology groups* of F on the open U by

$$H^p(U,F) = R^p\Gamma(U,F),$$

and the *sheaf cohomology presheaves* of F by

$$\mathcal{H}^p(F) = R^p\iota(F).$$

Remark 3.3.3. Since ι is right adjoint to a left exact functor, ι takes injective objects to injective objects. Thus we may apply the Grothendieck spectral sequence to composite functors $G \circ \iota$, where G is a left exact additive functor with domain $\mathcal{P}$.

Since the functor $\Gamma(U, \cdot) : \mathcal{P} \to \mathrm{Ab}$ is exact, and since $\Gamma(U, F) = \Gamma(U, \iota(F))$, a trivial spectral sequence shows that for every open U we have $\mathcal{H}^p(F)(U) = H^p(U, F)$. The next proposition shows that the sheaf cohomology presheaves are not very sheafy for $p > 0$.

Proposition 3.3.10. *For any $F \in \mathcal{S}$ we have $\check{H}^0(U, \mathcal{H}^p(F)) = 0$ for all $p > 0$.*

Proof. The map $G \to G^\dagger$ is a monomorphism for any separated presheaf G, so it's enough to show that $\mathcal{H}^p(F)^\# = 0$ for all $p > 0$. Since $\mathrm{id}_\mathcal{S} = \# \circ \iota$ and $\#$ is exact, a trivial spectral sequence shows that $\mathcal{H}^p(F)^\# = R^p\mathrm{id}_\mathcal{S}(F)$, and this is 0 for $p > 0$ since $\mathrm{id}_\mathcal{S}$ is exact. □

Theorem 3.3.11 (Čech to derived)**.** *For any sheaf F we have the following spectral sequences:*

- $H^p(\{U_i \to U\}, \mathcal{H}^q(F)) \;\Rightarrow\; H^{p+q}(U, F),$
- $\check{H}^p(U, \mathcal{H}^q(F)) \;\Rightarrow\; H^{p+q}(U, F).$

Proof. These follow from the Grothendieck spectral sequence applied to the identities

$$\Gamma(U, \cdot) = H^0(\{U_i \to U\}, \cdot) \circ \iota = \check{H}^0(U, \cdot) \circ \iota. \qquad \square$$

Corollary 3.3.12. *If $\{U_i \to U\}$ is a covering of U satisfying $H^q(U_{i_0,\dots,i_r}, F) = 0$ for all $q > 0$ and all $(i_0, ..., i_r)$, then the canonical map*

$$H^p(\{U_i \to U\}, F) \to H^p(U, F)$$

is an isomorphism.

Corollary 3.3.13. *The map*

$$\check{H}^1(U, F) \to H^1(U, F)$$

is always an isomorphism, and the map

$$\check{H}^2(U, F) \to H^2(U, F)$$

is always a monomorphism.

Proof. Since $\check{H}^0(U, \mathcal{H}^1(F)) = 0$, the exact sequence of low degree from the spectral sequence $\check{H}^p(U, \mathcal{H}^q(F)) \;\Rightarrow\; H^{p+q}(U, F)$ is just

$$0 \to \check{H}^1(U, F) \to H^1(U, F) \to 0 \to \check{H}^2(U, F) \to H^2(U, F). \qquad \square$$

Proposition 3.3.14. *Suppose that for every presheaf P with $P^\# = 0$ we have $\check{H}^p(X, P) = 0$ for all $p \geq 0$. Then for every presheaf P and every $p \geq 0$ the natural map $\check{H}^p(X, P) \to H^p(X, P^\#)$ is an isomorphism.*

Proof. Consider the exact sequence of presheaves

$$0 \to P \to P^\# \to P^\#/P \to 0.$$

Since sheafification is an exact functor, we see that $(P^\#/P)^\# = 0$, so by assumption we have $\check{H}^p(X, P^\#/P) = 0$ for all p. By the long exact sequence of Čech cohomology associated to any short exact sequence of presheaves, we see that the natural map $\check{H}^p(X, P) \to \check{H}^p(X, P^\#)$ is an isomorphism for every presheaf P and every p.

Now consider the spectral sequence $\check{H}^p(X, \mathcal{H}^q(P^\#)) \to H^{p+q}(X, P^\#)$ of Theorem 3.3.11. By Proposition 3.3.10 we have $(\mathcal{H}^q(P^\#))^\# = 0$ for $q > 0$, so by assumption we have $\check{H}^p(X, \mathcal{H}^q(P^\#)) = 0$ for $q > 0$, and then by Corollary 3.2.3 the natural maps $\check{H}^p(X, P^\#) \to H^p(X, P^\#)$ are isomorphisms. □

Exercise 3.3.2. Use the previous Proposition to show that for every presheaf P on a paracompact Hausdorff topological space X the natural maps $\check{H}^p(X, P) \to H^p(X, P^\#)$ are isomorphisms. (Hint: given a Čech cocycle of a presheaf P with $P^\# = 0$ which is defined on some cover, try to construct a refinement of the cover on which every component of the cocycle vanishes.)

3.3.4 Torsors and H^1

Since H^1 always agrees with $\check{H}^1$ for abelian sheaves, we will extend the definition of H^1 to noncommutative sheaves G as follows.

Definition 3.3.15. Let G be a sheaf of (possibly noncommutative) groups on X. For any open cover $\{U_i \to U\}_i$, we define a *cocycle* to be an element $\varphi \in \prod_{i,j} G(U_{i,j})$ satisfying

$$G(U_{i,j,k} \to U_{i,j})(\varphi_{i,j}) \cdot G(U_{i,j,k} \to U_{j,k})(\varphi_{j,k}) = G(U_{i,j,k} \to U_{i,k})(\varphi_{i,k})$$

for all i, j, k. Two cocycles φ, ϕ are *equivalent* if there exists an element $g \in \prod_i G(U_i)$ satisfying

$$G(U_{i,j} \to U_i)(g_i) \cdot \varphi_{i,j} = \phi_{i,j} \cdot G(U_{i,j} \to U_j)(g_j)$$

for all i, j. The *trivial* cocycle is the cocycle all of whose components are the identity of G. The set of cocycles up to equivalence forms a pointed set, which we call $H^1(\{U_i \to U\}_i, G)$. Finally, we set

$$H^1(U, G) = \varinjlim_{\{U_i \to U\}} H^1(\{U_i \to U\}, G).$$

Definition 3.3.16. Let G be a sheaf of (possibly noncommutative) groups on X. A *left G-torsor* on X is a sheaf of sets P with a left action $G \times P \to P$ such that there is an open cover $\{U_i \to X\}_i$ such that P restricted to each U_i is isomorphic, as a sheaf of sets with left G action, to G with the action defined by left multiplication.

Exercise 3.3.3. Check that there is a natural bijection between $H^1(X, G)$ and the set of left G-torsors on X up to isomorphism.

Exercise 3.3.4. Let

$$1 \to A \to B \to C \to 1$$

be a short exact sequence of sheaves of groups, and suppose that A is contained in the center of B. Show that for every open U we have an exact sequence (of pointed sets)

$$1 \to H^0(U, A) \to H^0(U, B) \to H^0(U, C) \to H^1(U, A) \to H^1(U, B) \to H^1(U, C) \to H^2(U, A).$$

(Hint: start by constructing a map $H^1(U, C) \to \check{H}^2(U, A)$, then use the injectivity of the natural map $\check{H}^2(U, A) \to H^2(U, A)$.)

3.4 Flask Sheaves

The following lemma from Milne [145] explains the properties that we want from the family of flask sheaves.

Lemma 3.4.1 (Acyclic Cohomology)**.** *Let $F : \mathcal{C} \to \mathcal{C}'$ be a left exact functor of abelian categories, and assume that $\mathcal{C}$ has enough injectives. Let T be a class of objects in $\mathcal{C}$ such that*

(a) *for every object $A \in \mathcal{C}$ there is a monomorphism from A to an object of T (i.e. $\mathcal{C}$ has enough T-objects),*

(b) *if $A \oplus A' \in T$ then $A \in T$,*

(c) *if $0 \to A' \to A \to A'' \to 0$ is exact and $A', A \in T$, then we have $A'' \in T$ and the sequence $0 \to F(A') \to F(A) \to F(A'') \to 0$ is exact.*

Then all elements of T are F-acyclic, and so T-resolutions can be used to calculate R^pF. Furthermore, all injective objects of $\mathcal{C}$ are in T.

Proof. Since every monomorphism from an injective object to an object of T splits, (a) and (b) imply that every injective object of $\mathcal{C}$ is in T. Now let A be any object in T, and choose an injective resolution

$$0 \to A \to I^0 \to I^1 \to \cdots$$

of A. Split this resolution up into short exact sequences

$$\begin{aligned} 0 \to Z^0 \to & I^0 \to Z^1 \to 0 \\ 0 \to Z^1 \to & I^1 \to Z^2 \to 0 \\ & \cdots \end{aligned}$$

where $Z^0 = A$. Then by (c) and induction on i, each Z^i is in T, and so each sequence

$$0 \to F(Z^p) \to F(I^p) \to F(Z^{p+1}) \to 0$$

is exact in $\mathcal{C}'$. Thus $0 \to F(A) \to F(I^\bullet)$ is exact, and so $R^pF(A) = 0$ for all $p > 0$. □

Tamme [179] gives the following definition of flask sheaves.

Definition 3.4.2. A sheaf F is *flask* if for every covering $\{U_i \to U\}$ and for every $p > 0$, we have

$$H^p(\{U_i \to U\}, F) = 0.$$

Proposition 3.4.3. *The class of flask sheaves satisfies conditions* (a), (b), (c) *of Lemma 3.4.1 for the functor $\iota : \mathcal{S} \to \mathcal{P}$. Furthermore, for any sheaf $F \in \mathcal{S}$ the following are equivalent:*

(i) *F is flask.*

(ii) *$\mathcal{H}^p(F) = 0$ for all $p > 0$, or equivalently $H^p(U, F) = 0$ for all opens U and all $p > 0$.*

Proof. Recall that ι takes injectives to injectives. Thus for any injective object I of $\mathcal{S}$, $H^p(\{U_i \to U\}, I) = R^pH^0(\{U_i \to U\}, \iota(I)) = 0$ for $p > 0$, and so I is flask. Since $\mathcal{S}$ has enough injectives, the class of flask sheaves satisfies condition (a).

Since the functor $H^p(\{U_i \to U\}, \cdot)$ commutes with finite direct sums, the class of flask sheaves also satisfies condition (b).

Finally, the long exact sequence of Čech cohomology and the fact that $\check{H}^1(U, \cdot) = H^1(U, \cdot)$ (Corollary 3.3.13) show that the class of flask sheaves satisfies condition (c).

Now Lemma 3.4.1 shows that (i) implies (ii). The reverse implication follows from the first spectral sequence of Theorem 3.3.11. □

If we suppose that our site has the form T_X for some topological space X, then we can make the following simpler definition.

Definition 3.4.4. A sheaf F on a topological space X is called *flabby* if for every inclusion of opens $V \subseteq U$ the restriction map $F(V \to U)$ is surjective.

Proposition 3.4.5. *The class of flabby sheaves on a topological space satisfies conditions* (a), (b), (c) *of Lemma 3.4.1 for the functor $\iota : \mathcal{S} \to \mathcal{P}$. If a sheaf F on a topological space is flabby, then it is also flask.*

Proof. For (a), we note that any sheaf injects into the product of the skyscraper sheaves corresponding to its stalks, and that such a product is a flabby sheaf. The condition (b) is trivial. Now suppose that

$$0 \to F' \to F \to F'' \to 0$$

is an exact sequence of sheaves with F, F' flabby. Let P be the presheaf $\iota(F)/\iota(F')$, so we have $F'' = P^{\#}$. An easy application of the snake lemma shows that every restriction map $P(V \to U)$ is surjective, so to check (c) we just have to check that P is a sheaf, or equivalently that $P = P^{\dagger}$. By the long exact sequence of Čech cohomology, it suffices to check that $H^1(\{U_i \to U\}, F') = 0$ for every cover $\{U_i \to U\}_{i \in I}$.

So suppose that $s = (s_{i,j}) \in C^1(\{U_i \to U\}_{i\in I}, F')$ is a coboundary. Since all three maps $U_{i,i,i} \to U_{i,i}$ defined by omitting one of the three factors are the identity in the case of a topological space, we see that

$$0 = (d^1s)_{i,i,i} = s_{i,i} - s_{i,i} + s_{i,i} = s_{i,i}$$

for every $i \in I$. Similarly, since the two maps $U_{i,j,i} \to U_{j,i}$ and $U_{i,j,i} \to U_{i,j}$ defined by omitting either the first or the last factor are the identity on a topological space, we have

$$0 = (d^1s)_{i,j,i} = s_{j,i} - F(U_{i,j,i} \to U_{i,i})s_{i,i} + s_{i,j} = s_{j,i} + s_{i,j}$$

for all $i, j \in I$. Now well-order the index set I. We will inductively define sections s_i such that $F'(U_{j,i} \to U_i)s_i - F'(U_{j,i} \to U_j)s_j = s_{j,i}$ for all $j < i$. Let $V = \bigcup_{j<i} U_{j,i}$. Let $j, k < i$. Then we have

$$F'(U_{k,j,i} \to U_{j,i})(s_{j,i} + F'(U_{j,i} \to U_j)s_j) - F'(U_{k,j,i} \to U_{k,i})(s_{k,i} + F'(U_{k,i} \to U_k)s_k) =$$
$$F'(U_{k,j,i} \to U_{j,i})(s_{j,i}) - F'(U_{k,j,i} \to U_{k,i})(s_{k,i}) + F'(U_{k,j,i} \to U_{k,j})(s_{k,j}) = (d^1 s)_{k,j,i} = 0,$$

so by the sheaf condition for F' applied to the cover $\{U_{j,i} \to V\}_{j<i}$ the sections $\tilde{s}_{j,i} = s_{j,i} + F'(U_{j,i} \to U_j)s_j$ on $U_{j,i}$ glue to a section $\tilde{s}$ of $F'(V)$. Now we take s_i to be any section of $F'(U_i)$ such that $F'(V \to U_i)(s_i) = \tilde{s}$.

Thus we have constructed $(s_i) \in C^0(\{U_i \to U\}_{i\in I}, F')$ such that $(s_{i,j}) = d^0(s_i)$. This calculation shows that $H^1(\{U_i \to U\}, F') = 0$, and so we have verified condition (c) for the class of flabby sheaves.

Now by Lemma 3.4.1, a flabby sheaf F is ι-acyclic, and so $\mathcal{H}^p(F) = R^p\iota(F) = 0$ for every $p > 0$. Thus by Proposition 3.4.3 F is flask. □

Remark 3.4.1. Even in the case of a topological space, flask does not necessarily imply flabby. For instance, if X is the Sierpinski space, then all sheaves on X are flask, but not all sheaves on X are flabby.

Remark 3.4.2. Milne [145] mentions a third class of sheaves, which I will call *flasque* sheaves, that satisfies the conditions of Lemma 3.4.1. A sheaf F is *flasque* if for every sheaf of sets S, F is acyclic for the functor $\mathrm{Mor}(S, \cdot)$. Flasque sheaves are easily seen to be flask.

3.5 $\mathcal{O}_X$-module cohomology

Proposition 3.5.1. *Let X be a scheme. The category of $\mathcal{O}_X$-modules is a Grothendieck abelian category. Injective $\mathcal{O}_X$-modules are flabby.*

Proof. It's easy to check that AB5 is satisfied. Let U be any open set of X, and let $j : U \to X$ be the inclusion. Then we can form the $\mathcal{O}_X$-module $j_!\mathcal{O}_U$, which is the sheafification of the presheaf which sends an open V to $\mathcal{O}_V$ if $V \subseteq U$ and sends V to 0 otherwise. If F is an $\mathcal{O}_X$-module, then we have

$$\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{O}_U, F) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, F \mid_U) = F(U),$$

so the collection $j_!\mathcal{O}_U$ forms a family of generators as U varies over the open sets of X.

To see that an injective $\mathcal{O}_X$-module I is flabby, let $V \subseteq U$ be any inclusion of opens. Then the natural map $j_!\mathcal{O}_V \to j_!\mathcal{O}_U$ is a monomorphism, and so the induced map $\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{O}_U, I) \to \mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{O}_V, I)$ must be surjective. But this map is just the restriction map $I(U) \to I(V)$. □

By the proposition, $\mathcal{O}_X$-module cohomology and sheaf cohomology are the same thing, since any injective resolution in the category of $\mathcal{O}_X$-modules will automatically be a flabby, hence flask resolution in the category of sheaves.

Lemma 3.5.2 (Zariski Poincaré Lemma)**.** *Let F be a quasi-coherent sheaf on an affine scheme X. Then $\check{H}^p(X, F) = 0$ for all $p > 0$.*

Proof. Let $X = \mathrm{Spec}(A)$, and let $M = \Gamma(X, F)$, so $F = \widetilde{M}$. Since the collection of finite covers by principal open sets is cofinal in the collection of all covers, it suffices to show that if $(f_1, ..., f_n) = 1$ then $H^p(\{\mathrm{Spec}(A_{f_i}) \to \mathrm{Spec}(A)\}_{i\in\{1,...,n\}}, \widetilde{M}) = 0$ for $p > 0$.

Let $s = (s_{i_0,...,i_p}) \in Z^p(\{\mathrm{Spec}(A_{f_i}) \to \mathrm{Spec}(A)\}_{i\in\{1,...,n\}}, \widetilde{M})$. Then we can write $s_{i_0,...,i_p} = \frac{m_{i_0,...,i_p}}{(f_{i_0}\cdots f_{i_p})^k}$ with $m_{i_0,...,i_p} \in M$ for each $i_0, ..., i_p$. We may assume without loss of generality that each k is 1 by replacing the f_is with large enough powers of themselves. For each $i_0, ..., i_{p+1}$ we have an identity

$$0 = (d^p s)_{i_0,...,i_{p+1}} = \sum_{k=0}^{p+1}(-1)^k s_{i_0,...,\widehat{i_k},...,i_{p+1}} |_{\mathrm{Spec}(A_{f_{i_0}\cdots f_{i_{p+1}}})} = \sum_{k=0}^{p+1}(-1)^k \frac{f_{i_k} m_{i_0,...,\widehat{i_k},...,i_{p+1}}}{f_{i_0}\cdots f_{i_{p+1}}},$$

so the numerator of the sum is killed by some power of $f_{i_0}\cdots f_{i_{p+1}}$. If we replace each f_i by a sufficiently large power of itself then the numerator of the sum will actually vanish, and we obtain

$$\sum_{k=0}^{p+1}(-1)^k f_{i_k} m_{i_0,...,\widehat{i_k},...,i_{p+1}} = 0.$$

Finally, replacing each f_i with a multiple of itself we can assume that $\sum_{i=1}^n f_i = 1$, so that the f_is form a partition of unity.

Now for each $i_1, ..., i_p$, set $s'_{i_1,...,i_p} = \sum_{j=1}^n \frac{m_{j,i_1,...,i_p}}{f_{i_1}\cdots f_{i_p}}$. Morally speaking, we have

$$\text{“}s'_{i_1,...,i_p} = \sum_{j=1}^n f_j s_{j,i_1,...,i_p}\text{”},$$

so $s'_{i_1,...,i_p}$ is acting like a weighted average of the $s_{j,i_1,...,i_p}$s. Then we have

$$\begin{aligned}(d^{p-1}s')_{i_0,...,i_p} &= \sum_{k=0}^{p}(-1)^k \sum_{j=1}^n \frac{f_{i_k} m_{j,i_0,...,\widehat{i_k}...,i_p}}{f_{i_0}\cdots f_{i_p}} \\ &= \sum_{j=1}^n \frac{\sum_{k=0}^p (-1)^k f_{i_k} m_{j,i_0,...,\widehat{i_k}...,i_p}}{f_{i_0}\cdots f_{i_p}} \\ &= \sum_{j=1}^n \frac{f_j m_{i_0,...,i_p}}{f_{i_0}\cdots f_{i_p}} = s_{i_0,...,i_p}.\end{aligned}$$

□

Finally, we have arrived at the main course.

Theorem 3.5.3. *Let X be a separated scheme and let F be a quasicoherent sheaf on X. Then $H^p(X, F) = \check{H}^p(X, F)$ for all p.*

Proof. By Corollary 3.3.12 and the fact that the intersection of two affine opens is affine on a separated scheme, it is enough to check that when X is affine we have $H^p(X, F) = 0$ for $p > 0$. We will prove this by strong induction on p.

By Theorem 3.3.11 we have a spectral sequence $\check{H}^p(X, \mathcal{H}^q(F)) \Rightarrow H^{p+q}(X, F)$. By Lemma 3.5.2, we have $\check{H}^p(X, F) = 0$ for $p > 0$, and by Proposition 3.3.10 we have $\check{H}^0(X, \mathcal{H}^p(F)) = 0$ for $p > 0$. By the induction hypothesis, the presheaf $\mathcal{H}^a(F)$ vanishes on every affine open U for every $0 < a < p$. Since affine covers are cofinal in the collection of all covers, we have $\check{H}^{p-a}(X, \mathcal{H}^a(F)) = 0$ for $0 < a < p$. Putting everything together we see that $\check{H}^{p-a}(X, \mathcal{H}^a(F)) = 0$ for all a, so by the spectral sequence we must have $H^p(X, F) = 0$. □

In fact, the proof gives the following (more useful for computations) result.

Corollary 3.5.4. *Let X be a separated scheme, let F be a quasicoherent sheaf on X, and let $\{U_i \to X\}$ be any affine cover of X. Then $H^p(X, F) = H^p(\{U_i \to X\}, F)$ for all $p > 0$.*

3.6 Higher pushforwards

Let $\pi : X \to Y$ be a map of schemes. Let $\mathcal{P}_X$ denote the category of presheaves on X, and similarly for $\mathcal{P}_Y, \mathcal{S}_X, \mathcal{S}_Y$. Then we can define two functors $\pi_p : \mathcal{P}_X \to \mathcal{P}_Y$ and $\pi_* : \mathcal{S}_X \to \mathcal{S}_Y$ by

$$\pi_p(F)(U) = F(\pi^{-1}(U))$$

and $\pi_* = \# \circ \pi_p \circ \iota$. Since $\# \circ \pi_p$ is a composite of two exact functors it is exact, and so a trivial spectral sequence gives

$$R^p\pi_*F = (\pi_p\mathcal{H}^p(F))^{\#}.$$

From this we see that flask sheaves are acyclic for π_*, so we may calculate $R^p\pi_*$ by taking flask resolutions (so $R^p\pi_*$ is the same as the higher direct image on the category of $\mathcal{O}_X$-modules, for instance).

Theorem 3.6.1. *Let $\pi : X \to Y$ be a separated map of schemes, and let F be a quasicoherent sheaf on X. Then for every affine open U of Y we have $R^p\pi_*F(U) = \check{H}^p(\pi^{-1}(U), F)$. Furthermore, $R^p\pi_*F$ is a quasicoherent sheaf on Y.*

Proof. By Theorem 3.5.3 we have $\pi_p\mathcal{H}^p(F)(U) = H^p(\pi^{-1}(U), F) = \check{H}^p(\pi^{-1}(U), F)$ for every affine open U on Y. Let T_Y^{aff} denote the topology of affine opens of Y. Since affine opens form a base of open sets on Y, it's enough to show that the presheaf $U \mapsto \check{H}^p(\pi^{-1}(U), F)$ is a quasicoherent sheaf on T_Y^{aff}. This follows from the easy fact that Čech cohomology commutes with localization for quasicoherent sheaves. □

3.7 Hypercohomology

Let $\mathcal{C}$ be an abelian category with enough injectives. Let Ch^+ denote the category of cochain complexes $C^\bullet$ of objects in $\mathcal{C}$ with $C^i = 0$ for $i < 0$.

Definition 3.7.1. A cochain map $C^\bullet \to D^\bullet$ is a *quasiisomorphism* if the induced maps on cohomology are isomorphisms.

Definition 3.7.2. An *injective resolution* of $C^\bullet$ is a quasiisomorphism $C^\bullet \hookrightarrow I^\bullet$ from $C^\bullet$ to a complex of injectives $I^\bullet$ such that each map $C^i \to I^i$ is a monomorphism.

Exercise 3.7.1. Show that the total complex of a Cartan-Eilenberg resolution of $C^\bullet$ is an injective resolution of $C^\bullet$.

Theorem 3.7.3. *Let $C^\bullet \hookrightarrow I^\bullet$ be a quasiisomorphism with each $C^i \to I^i$ a monomorphism, and let $\varphi^\bullet : C^\bullet \to J^\bullet$ be any cochain map from $C^\bullet$ to a complex of injectives $J^\bullet$. Then $\varphi^\bullet$ extends to a map $\psi^\bullet : I^\bullet \to J^\bullet$, and $\psi^\bullet$ is unique up to cochain homotopy.*

Proof. We will construct the maps $\psi^i : I^i \to J^i$ by induction on i. Suppose we have already constructed $\psi^0, ..., \psi^{i-1}$. Since φ^{i-1} induces a well-defined map $H^{i-1}(C^\bullet) \to H^{i-1}(J^\bullet)$ and since the natural map $H^{i-1}(C^\bullet) \to H^{i-1}(I^\bullet)$ is an isomorphism, we have $\psi^{i-1}(Z^{i-1}(I^\bullet)) \subseteq Z^{i-1}(J^\bullet)$. Thus there is a well-defined map $\bar{\psi} : B^i(I^\bullet) \to J^i$ induced by $d^{i-1} \circ \psi^{i-1}$.

If we now write $B^i(I^\bullet) \cap C^i = \ker(B^i(I^\bullet) \oplus C^i \to I^i)$, then since the map $B^i(I^\bullet) \cap C^i \to H^i(I^\bullet) \cong H^i(C^\bullet)$ is trivial, and since $B^i(I^\bullet) \cap C^i \subseteq Z^i(C^\bullet)$ (by the fact that $C^{i+1} \to I^{i+1}$ is a monomorphism), we have $B^i(I^\bullet) \cap C^i = B^i(C^\bullet)$. Thus the maps $\bar{\psi}$ and φ^i agree on $B^i(I^\bullet) \cap C^i$, and we can define a map $\widetilde{\psi} : B^i(I^\bullet) + C^i \to J^i$ that agrees with $\bar{\psi}$ on $B^i(I^\bullet)$ and φ^i on C^i. Since J^i is injective, we can extend $\widetilde{\psi}$ to a map $\psi^i : I^i \to J^i$.

We have constructed a cochain map $\psi^\bullet$ extending $\varphi^\bullet$. To check that any two such extensions are homotopic, it's enough to check that if $\varphi^\bullet = 0$ then $\psi^\bullet$ is homotopic to 0.

We will construct a homotopy $h^\bullet : I^\bullet \to J^{\bullet-1}$ that vanishes on $C^\bullet$ inductively. Assume we've already constructed $h^0, ..., h^{i-1}$ such that $h^{i-1}(C^{i-1}) = 0$. Then

$$(\psi^{i-1} - d^{i-2} \circ h^{i-1}) \circ d^{i-2} = d^{i-2} \circ (\psi^{i-2} - h^{i-1} \circ d^{i-2}) = d^{i-2} \circ d^{i-3} \circ h^{i-2} = 0,$$

so $\psi^{i-1} - d^{i-2} \circ h^{i-1}$ vanishes on $B^{i-1}(I^\bullet)$. Since both ψ^{i-1} and $d^{i-2} \circ h^{i-1}$ vanish on C^{i-1}, and since $H^{i-1}(I^\bullet) \cong H^{i-1}(C^\bullet)$, we see that $\psi^{i-1} - d^{i-2} \circ h^{i-1}$ vanishes on $Z^{i-1}(I^\bullet)$. Thus the map $\psi^{i-1} - d^{i-2} \circ h^{i-1}$ descends to a well-defined map $\bar{h} : B^i(I^\bullet) \to J^{i-1}$, which vanishes on $B^i(I^\bullet) \cap C^i = B^i(C^\bullet)$ by construction. From this we construct $\widetilde{h} : B^i(I^\bullet) + C^i \to J^{i-1}$ agreeing with $\bar{h}$ on $B^i(I^\bullet)$ and with 0 on C^i, and since J^{i-1} is injective we can extend this to $h^i : I^i \to J^{i-1}$. □

Definition 3.7.4. If $F : \mathcal{C} \to \mathcal{C}'$ is a left exact additive functor, then the *hypercohomology* of a cochain complex $C^\bullet$ with respect to F is given by

$$\mathbb{H}^p(C^\bullet) = H^p(F(I^\bullet)),$$

where $C^\bullet \hookrightarrow I^\bullet$ is any injective resolution. By Theorem 3.7.3, $\mathbb{H}^p$ is a well-defined functor from Ch^+ to $\mathcal{C}'$, and any quasiisomorphism $C^\bullet \to D^\bullet$ induces isomorphisms on hypercohomology.

Remark 3.7.1. If $C^i = 0$ for all $i > 0$, then $\mathbb{H}^p(C^\bullet) = R^pF(C^0)$ for all p.

Theorem 3.7.5. (a) *A short exact sequence* $0 \to C'^\bullet \to C^\bullet \to C''^\bullet \to 0$ *induces a long exact sequence*

$$0 \to \mathbb{H}^0(C'^\bullet) \to \mathbb{H}^0(C^\bullet) \to \mathbb{H}^0(C''^\bullet) \to \mathbb{H}^1(C'^\bullet) \to \mathbb{H}^1(C^\bullet) \to \mathbb{H}^1(C''^\bullet) \to \cdots$$

(b) *We have a spectral sequence* $E_2^{p,q} = R^pF(H^q(C^\bullet)) \Rightarrow E^n = \mathbb{H}^n(C^\bullet)$.

(c) *We have a spectral sequence* $E_1^{p,q} = R^qF(C^p) \Rightarrow E^n = \mathbb{H}^n(C^\bullet)$.

Proof. Exercise. □

Definition 3.7.6. If $C^\bullet$ is a complex of presheaves we write $\check{H}^p(U, C^\bullet)$ for the pth Čech hypercohomology of $C^\bullet$ on U, and similarly if $C^\bullet$ is a complex of sheaves we write $H^p(U, C^\bullet)$ for the pth sheaf hypercohomology of $C^\bullet$.

Exercise 3.7.2. If $C^\bullet$ is a complex of sheaves, show there is a natural map $\check{H}^p(U, C^\bullet) \to H^p(U, C^\bullet)$.

3.8 Soft and fine sheaves

For this section, we only consider paracompact topological spaces.

Definition 3.8.1. A sheaf F on a paracompact topological space X is *soft* if for every closed set K, the map $\Gamma(X, F) \to \Gamma(K, F|_K)$ is surjective.

Proposition 3.8.2. *If F is a flabby sheaf on a paracompact topological space X then F is soft.*

Proof. Let K be a closed subset of X, and let s be a section of $F|_K$. Write s_p for the germ of s at a point p of K. Then by the definition of $F|_K$, for each point $p \in K$ we can find an open neighborhood U_p and a section s^p of F on U_p such that $s^p_q = s_q$ for all $q \in U_p \cap K$. Since X is paracompact, we can find a locally finite refinement $\{X \setminus K \to X, V_i \to X\}$ of the cover $\{X \setminus K \to X, U_p \to X\}$. If $V_i \subseteq U_p$, let $s^i = s^p|_{V_i}$.

Now for each point $p \in K$, if we let $i_1, ..., i_n$ be the finite set of indices i such that $p \in V_i$, then each of the stalks $s^{i_j}_p$ agrees with s_p. Thus we can find an open neighborhood W_p of p such that $s^{i_1}|_{W_p} = \cdots = s^{i_n}|_{W_p}$. Thus the section s extends to a section of F on $\bigcup_p W_p$. Since F is flabby and $\bigcup_p W_p$ is open, we can extend this to a global section of F. □

Proposition 3.8.3. *Suppose F is a soft sheaf on a paracompact topological space X. For any closed set $K \subseteq X$, section s of $F|_K$, and locally finite cover $\{U_i \to X\}_i$ we can find sections $s^i \in F(X)$ with $\operatorname{supp}(s^i) \subseteq U_i$ and $s = \sum_i s^i|_K$.*

Proof. Assume the index set of the U_is is well-ordered. We will construct the s^is inductively, such that for every i, if we write $K_i = K \setminus (\cup_{j>i} U_j)$, then we have $s|_{K_i} = \sum_{j\le i} s^j|_{K_i}$. Suppose that we have already constructed s^j for all $j < i$. Then at any point p of $K_i \setminus U_i$ we have $s_p = \sum_{j<i} s^j_p$ by the inductive hypothesis, since there is a maximal $j < i$ with $p \in U_j$ by the local finiteness of the cover. Now we just take $s^i \in F(X)$ to be any extension of the section of $F|_{K_i \cup (X \setminus U_i)}$ which is equal to 0 on $X \setminus U_i$ and is equal to $s|_{K_i} - \sum_{j<i} s^j|_{K_i}$ on K_i. □

Proposition 3.8.4. *The class of soft sheaves on a paracompact Hausdorff topological space X satisfies conditions* (a), (b), (c) *of Lemma 3.4.1 for $\Gamma(X, \cdot)$, so soft sheaves are acyclic for $\Gamma(X, \cdot)$.*

Proof. For condition (a) we use the fact that there are enough flabby sheaves and Proposition 3.8.2. Condition (b) is trivial.

Now we show that for any soft sheaf F we have $H^1(X, F) = 0$. Let $\{U_i \to X\}_{i\in I}$ be any locally finite open cover. Let $\{V_i \to X\}_i$ be a *shrinking* of this cover, i.e. an open cover of X such that for each i we have $\overline{V}_i \subseteq U_i$ (this exists since X is paracompact Hausdorff). It's enough to show that $\operatorname{Im}(H^1(\{U_i \to X\}_i, F) \to H^1(\{V_i \to X\}_i, F)) = 0$. The proof of this closely mimics the proof of Proposition 3.4.5, once we note that for any $J \subseteq I$ the set $\cup_{j\in J}\overline{V}_j$ is closed by local finiteness.

Now let

$$0 \to F' \to F \to F'' \to 0$$

be an exact sequence of sheaves with F', F soft. Let $K \subseteq X$ be any closed set. Then $F'|_K$ is soft, so $H^1(K, F'|_K) = 0$, and thus the sequence

$$0 \to \Gamma(K, F'|_K) \to \Gamma(K, F|_K) \to \Gamma(K, F''|_K) \to 0$$

is exact. Now since F is soft, we see that any section of $F''|_K$ can be lifted to a section of $F|_K$ and then to a global section of F, so F'' is soft as well. □

Definition 3.8.5. A sheaf F is *fine* if $\mathcal{H}om(F, F)$ is soft.

Proposition 3.8.6. *Let X be a paracompact topological space, and let F be a sheaf on X. The following are equivalent:*

(a) *F is fine,*

(b) *for any closed disjoint sets $A, B \subseteq X$ there is an endomorphism of F which restricts to the identity on A and restricts to 0 on B,*

(c) *there is a sheaf of rings A acting on F such that for any locally finite open cover $\{U_i \to X\}_i$ there is a collection of elements $a_i \in A(X)$ with $\operatorname{supp}(a_i) \subseteq U_i$ and $1 = \sum_i a_i$.*

Furthermore, every fine sheaf is soft.

Proposition 3.8.7. *If F is a fine sheaf on a paracompact topological space X, then $H^p(X, F) = 0$ for every $p > 0$.*

Proof. Let A be a sheaf of rings as in (c) of Proposition 3.8.6. Then we can find an acyclic resolution

$$0 \to F \to I^0 \to I^1 \to \cdots$$

of F such that $I^\bullet$ is a complex of A-modules and each map is an A-module map (one way to do this is to use the functoriality of the injective embeddings constructed in Theorem 3.1.6). Let $s \in \Gamma(X, I^p)$ with $ds = 0$, then by exactness X is covered by open sets U_i such that for each i there is an element $t_i \in \Gamma(U_i, I^{p-1})$ with $s|_{U_i} = dt_i$. By passing to a refinement we may assume that the cover $\{U_i \to X\}_i$ is locally finite. Let $a_i \in A(X)$ be as in (c) of Proposition 3.8.6. Then for each i we have $a_i t_i \in \Gamma(X, I^{p-1})$ and $a_i s = d(a_i t_i)$, so

$$s = \sum_i a_i s = d\Big(\sum_i a_i t_i\Big). \qquad \square$$

3.8.1 Sheaves on manifolds

First we show that singular cohomology and sheaf cohomology agree on a locally contractible space X. For any ring R we associate a sheaf R_X, the sheaf of locally constant R-valued functions on X (this is the sheafification of the constant presheaf which takes every open set to R).

Theorem 3.8.8. *Let X be a locally contractible topological space, and let R be any ring. Then for each $p \geq 0$ there is a natural isomorphism*

$$H^p_{\text{sing}}(X, R) \simeq H^p(X, R_X).$$

Proof. For each open $U \subseteq X$, let $C^\bullet(U)$ be the singular cochain complex with values in R associated to U. Let $C^\bullet$ be the associated complex of presheaves. Let $V^\bullet$ be the complex of locally vanishing cochains, where we say a cochain vanishes near p if there is an open set containing p such that any simplex mapping into this neighborhood is assigned the value 0 by the cochain. The sheafification $(C^\bullet)^\#$ is then equal to $(C/V)^\bullet$. Since the complex $C^\bullet(U)$ is exact for every contractible U (using the usual chain homotopy induced by taking any simplex to its image under a fixed contraction of U), the complex

$$0 \to R_X \to (C/V)^0 \to (C/V)^1 \to \cdots$$

is a flabby resolution of R_X. Thus we have $H^p(X, R_X) = H^p((C/V)^\bullet(X))$ for each p, and by the definition of singular cohomology we have $H^p_{\text{sing}}(X, R) = H^p(C^\bullet(X))$.

To finish, we just need to show that $C^\bullet(X) \to (C/V)^\bullet(X)$ is a quasiisomorphism, or equivalently that $V^\bullet(X)$ is exact. To see this, let φ be a locally vanishing i-cocycle, and let σ be an $i-1$-simplex. Using barycentric subdivision, construct an i-chain c_σ with boundary equal to σ plus a collection of $i-1$-simplices contained in open sets on which φ vanishes. Note that $\varphi(c_\sigma)$ independent of c_σ: for any c'_σ satisfying the same conditions, $c_\sigma - c'_\sigma$ is homologous to a sum of i-simplices contained in sets on which φ vanishes. Thus we can use the map $\sigma \mapsto \varphi(c_\sigma)$ to define an $i-1$-cochain, the boundary of which is easily seen to be φ. □

Now we specialize to the case X is a paracompact smooth manifold of dimension n. Let $\Omega^\bullet$ be the complex of sheaves of smooth differential forms. Then by the Poincaré Lemma we have an exact sequence

$$0 \to \mathbb{R}_X \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \to 0,$$

and each Ω^p is fine since it is a C^∞-module. Setting $H^p_{\text{dR}}(X, \mathbb{R}) = H^p(\Omega^\bullet(X))$, this gives the following theorem.

Theorem 3.8.9 (de Rham)**.** *For a paracompact smooth manifold X, we have $H^p(X, \mathbb{R}_X) = H^p_{\text{dR}}(X, \mathbb{R})$.*

Now we consider the case X is a paracompact complex manifold of dimension n. For any p, q we let $\Omega^{p,q}$ be the sheaf of complex C^∞ differential forms of type (p, q), and let $\Omega^r_{\mathbb{C}} = \Omega^r \otimes_{\mathbb{R}} \mathbb{C} = \oplus_{p+q=r} \Omega^{p,q}$. We let $\Omega^p_{\text{hol}} \subseteq \Omega^{p,0}$ be the sheaf of holomorphic differential p-forms.

Lemma 3.8.10 ($\bar{\partial}$-Poincaré Lemma)**.** *For a complex manifold X of dimension n and for any p the sequence*

$$0 \to \Omega^p_{\text{hol}} \to \Omega^{p,0} \xrightarrow{\bar{\partial}} \Omega^{p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{p,n} \to 0$$

is exact.

Proof. It's enough to prove this for $p = 0$, since we can get the general result by tensoring with the locally free $\mathcal{O}_X$-module Ω^p_{hol} (here $\mathcal{O}_X$ is the sheaf of holomorphic functions on X). Since exactness is a local property, we may assume that X is a polydisc.

First we show this for $n = 1$. Recall the general Cauchy integral formula: if D is a disk, $f \in C^\infty(\overline{D})$, $z \in D$, then

$$2\pi i f(z) = \int_{\partial D} \frac{f(w)}{w - z} dw + \int_D \frac{\partial f}{\partial \bar{w}}(w) \frac{dw \wedge d\bar{w}}{w - z},$$

which follows from Stokes' Theorem applied to the form $\frac{f(w)}{w-z} dw$ and some bounds for the contribution from w near z. Now if we set

$$g(z) = \frac{1}{2\pi i} \int_D \frac{f(w)}{w - z} dw \wedge d\bar{w},$$

then by writing f as the sum of a function which vanishes near z and a function which vanishes near ∂D we can show that $g \in C^\infty(D)$, with $\bar{\partial} g = f d\bar{z}$ on D.

For general n, we show that if a form ω which only involves $d\bar{z}^1, ..., d\bar{z}^k$ has $\bar{\partial}\omega = 0$, then we can find a form φ such that $\omega - \bar{\partial}\varphi$ only involves $d\bar{z}^1, ..., d\bar{z}^{k-1}$. Write

$$\omega = \omega_1 \wedge d\bar{z}^k + \omega_2,$$

with ω_1, ω_2 only involving $d\bar{z}^1, ..., d\bar{z}^{k-1}$. Then for each $l > k$ we have $\frac{\partial}{\partial \bar{z}^l}\omega_1 = 0$ since $\bar{\partial}\omega_2$ doesn't have any terms involving $d\bar{z}^k \wedge d\bar{z}^l$. Thus we can apply the construction for the case $n = 1$ to each coefficient of ω_1 to get φ. □

Corollary 3.8.11. *For any paracompact complex manifold X of dimension n the sequence*

$$0 \to \mathbb{C}_X \to \Omega^0_{\text{hol}} \xrightarrow{d} \Omega^1_{\text{hol}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\text{hol}} \to 0$$

is exact. Thus $H^p(X, \mathbb{C}_X) = H^p(X, \Omega^\bullet_{\text{hol}})$.

Proof. This is an immediate application of the spectral sequence associated to the double complex $\Omega^{p,q}$, since by the $\bar{\partial}$-Poincaré Lemma the columns are exact and by the usual Poincaré Lemma the total complex is exact. □

For any p, q, we define the Dolbeault cohomology group $H^{p,q}_{\bar{\partial}}(X)$ of X to be the qth cohomology group of the complex

$$0 \to \Omega^{p,0}(X) \xrightarrow{\bar{\partial}} \Omega^{p,1}(X) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \Omega^{p,n}(X) \to 0.$$

The spectral sequence of the double complex $\Omega^{p,q}$ gives us a spectral sequence

$$E^{p,q}_1 = H^{p,q}_{\bar{\partial}}(X) \;\Rightarrow\; E^n = H^n_{dR}(X, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Since each $\Omega^{p,q}$ is fine, the $\bar{\partial}$-Poincaré Lemma gives the following theorem.

Theorem 3.8.12 (Dolbeault)**.** *Let X be a paracompact complex manifold. For every p, q we have $H^q(X, \Omega^p_{\text{hol}}) = H^{p,q}_{\bar{\partial}}(X)$.*

3.9 Descent

3.9.1 Galois descent

Let L/K be a Galois extension of fields with Galois group Γ. If V is a vector space over L, we say that a group action $\sigma : \Gamma \times V \to V$ is a *semilinear action* of Γ on V if, setting $\sigma_g(v) = \sigma(g, v)$ for $g \in \Gamma, v \in V$, we have $\sigma_g : V \to V$ additive for every $g \in \Gamma$ and

$$\sigma_g(lv) = g(l)\sigma_g(v)$$

for all $g \in \Gamma, l \in L, v \in V$.

Theorem 3.9.1. *There is an equivalence of categories*

$$\{\mathit{Vect}/K\} \leftrightarrow \{(V, \sigma) \mid V \in \mathit{Vect}/L,\ \sigma : \Gamma \times V \to V \ \textit{semilinear}\}$$

defined by

$$\begin{aligned} &W \mapsto (W \otimes_K L, \sigma_g : w \otimes l \mapsto w \otimes g(l)), \\ &V^\Gamma \leftarrow\!\shortmid (V, \sigma). \end{aligned}$$

Proof. We just need to show that for any (V, σ) the natural map $V^\Gamma \otimes_K L \to V$ is an isomorphism.

Suppose first that this map is not injective, and consider the minimal relation $\sum_i l_i w_i = 0$, $w_i \in V^\Gamma$ linearly independent over K, $l_i \in L$. Without loss of generality we may take $l_n = 1$. Then for every $g \in \Gamma$ we have

$$\sum_i g(l_i) w_i = \sum_i \sigma_g(l_i w_i) = \sigma_g \left(\sum_i l_i w_i \right) = 0,$$

so $\sum_{i<n}(g(l_i) - l_i) w_i = 0$, and by minimality we must have $l_i = g(l_i)$ for all $g \in \Gamma$, so each l_i is in K, contradicting the independence of the w_i over K.

Now suppose that the map is not surjective, and set $V' = V/V^\Gamma \otimes_K L$. Set $\operatorname{Tr}(v') = \sum_{g \in \Gamma} \sigma_g(v')$. If $v' \in V' \setminus \{0\}$, then the map

$$l \mapsto \operatorname{Tr}(lv') = \sum_{g \in \Gamma} g(l) \sigma_g(v')$$

is not identically 0 by Artin's theorem on the linear independence of characters applied to the characters (of $L^\times$) $g : L^\times \to L^\times$, $g \in \Gamma$. Choose l such that $\operatorname{Tr}(lv') \neq 0$, and choose $v \in V$ mapping to lv' in V'. Then we have $\operatorname{Tr}(v) \notin V^\Gamma \otimes_K L$, but clearly $\operatorname{Tr}(v)$ is invariant under the action of Γ, a contradiction. □

Corollary 3.9.2. *For every $n \in \mathbb{N}$ we have $H^1(\Gamma, \mathrm{GL}_n(L)) = 1$.*

3.9.2 Faithfully flat descent

For a ring map $A \to B$ and an A-module M, define the *Amitsur complex* to be

$$0 \to M \otimes_A B \to M \otimes_A B \otimes_A B \to \cdots,$$

where the pth differential is given by

$$d^p(m \otimes b_0 \otimes \cdots \otimes b_p) = \sum_{i=0}^{p+1} (-1)^i m \otimes b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_p.$$

Note this is the same as the Čech complex $C^\bullet(\{\text{Spec } B \to \text{Spec } A\}, \widetilde{M})$.

Lemma 3.9.3 (Fpqc Poincaré Lemma)**.** *If the map $A \to B$ is such that either*

a) *there is a section $s : B \to A$, or*

b) *the map $A \to B$ is faithfully flat,*

then the Amitsur complex $C^\bullet(\{\text{Spec } B \to \text{Spec } A\}, \widetilde{M})$ is quasiisomorphic to the complex

$$0 \to M \to 0 \to 0 \to \cdots.$$

Proof. We just need to show that

$$0 \to M \to M \otimes_A B \to M \otimes_A B \otimes_A B \to \cdots$$

is exact.

In case a), we have the chain homotopy

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] & M \otimes_A B \arrow[r] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] \arrow[dl, "h"'] & M \otimes_A B \otimes_A B \arrow[r] \arrow[d, shift right, "0"'] \arrow[d, shift left, "1"] \arrow[dl, "h"'] & \cdots \\
0 \arrow[r] & M \arrow[r] & M \otimes_A B \arrow[r] & M \otimes_A B \otimes_A B \arrow[r] & \cdots
\end{tikzcd}$$

given by

$$h(m \otimes b_0 \otimes b_1 \otimes \cdots \otimes b_p) = s(b_0)m \otimes b_1 \otimes \cdots \otimes b_p.$$

In case b), by faithful flatness it is enough to check exactness after applying the functor $B \otimes_A \cdot$. We have

$$\begin{tikzcd}
0 \arrow[r] & B \otimes_A M \arrow[r] \arrow[d, equal] & B \otimes_A M \otimes_A B \arrow[r] \arrow[d, equal] & B \otimes_A M \otimes_A B \otimes_A B \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & B \otimes_A M \arrow[r] & (B \otimes_A M) \otimes_B (B \otimes_A B) \arrow[r] & (B \otimes_A M) \otimes_B (B \otimes_A B) \otimes_B (B \otimes_A B) \arrow[r] & \cdots
\end{tikzcd}$$

i.e. $B \otimes_A C^\bullet(\{\text{Spec } B \to \text{Spec } A\}, \widetilde{M}) = C^\bullet(\{\text{Spec } B \otimes_A B \to \text{Spec } B\}, \widetilde{B \otimes_A M})$, where the map $B \to B \otimes_A B$ is given by $b \mapsto 1 \otimes b$. This map has the section $s : B \otimes_A B \to B$ given by $s(b \otimes b') = bb'$, so we are done by case a). □

Example 3.9.1. Suppose that $f_1, ..., f_n \in A$ are such that $(f_1, ..., f_n) = 1$. Then $\{\text{Spec } A_{f_i} \to \text{Spec } A\}_i$ is an open cover of Spec A by principal open sets. Setting $B = \prod_{i=1}^n A_{f_i}$, we see that $A \to B$ is faithfully flat, and we can apply the fpqc Poincaré lemma to give another proof of the Zariski Poincaré lemma.

Definition 3.9.4. A *descent datum* (for a ring map $A \to B$) is a pair (N, φ), where N is a B module and $\varphi : N \otimes_A B \simeq B \otimes_A N$ is an isomorphism of $B \otimes_A B$ modules such that the diagram

$$\begin{tikzcd}
N \otimes_A B \otimes_A B \arrow[rr, "\varphi_{13}"] \arrow[dr, "\varphi_{12}"'] & & B \otimes_A B \otimes_A N \\
& B \otimes_A N \otimes_A B \arrow[ur, "\varphi_{23}"'] &
\end{tikzcd}$$

commutes (this is the cocycle condition).

Theorem 3.9.5. *If $A \to B$ is faithfully flat, we have an equivalence of categories*

$$\{M \in A\text{-mod}\} \leftrightarrow \{(N, \varphi) \text{ descent datum}\}$$

given by

$$M \mapsto (B \otimes_A M,\ \varphi : (b \otimes m) \otimes b' \mapsto b \otimes (b' \otimes m)),$$
$$\ker(n \mapsto \varphi(n \otimes 1) - 1 \otimes n) \leftarrow\!\shortmid (N, \varphi).$$

Proof. First we need to check that if we start from M, then go to (N, φ), then go back we get something naturally isomorphic to M. This follows immediately from the exactness of

$$0 \to M \to M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B.$$

Now we check that if we start from (N, φ), go to M, and go back we get something naturally isomorphic to (N, φ). By the cocycle condition, if $\varphi(n \otimes 1) = \sum_i b_i \otimes n_i$ then $\sum_i b_i \otimes 1 \otimes n_i = \sum_i b_i \otimes \varphi(n_i \otimes 1)$, so

$$\varphi(n \otimes 1) \in \ker(b \otimes n \mapsto b \otimes (\varphi(n \otimes 1) - 1 \otimes n)),$$

and the right hand side is $B \otimes_A M$ by the flatness of $A \to B$. This defines a natural map $N \overset{\varphi}{\to} B \otimes_A M$. For $b \in B, m \in M$ we have

$$\varphi(bm \otimes 1) = (b \otimes 1)\varphi(m \otimes 1) = (b \otimes 1)(1 \otimes m) = b \otimes m,$$

so the composite map $B \otimes_A M \to N \overset{\varphi}{\to} B \otimes_A M$ is the identity, hence $N \overset{\varphi}{\to} B \otimes_A M$ is surjective. Since $A \to B$ is faithfully flat the natural map $N \to N \otimes_A B$ is injective, and φ is injective by assumption, so the composite map $N \overset{\varphi}{\to} B \otimes_A M$ is also injective, hence an isomorphism. Finally, we have to check that the original φ matches the new φ: for any $b, b' \in B, m \in M$, we have

$$\varphi((bm) \otimes b') = (b \otimes b')\varphi(m \otimes 1) = (b \otimes b')(1 \otimes m) = b \otimes (b'm). \qquad \square$$

Definition 3.9.6. A family of maps $\{Y_i \to X\}_i$ of schemes is called an *fpqc cover* (fpqc stands for "faithfully flat quasi-compact" in French) if each $Y_i \to X$ is flat, and if for every affine open subset U of X there is a finite collection of affine open subsets of the Y_is which map surjectively onto U.

Remark 3.9.1. It's easy to see that a family $\{Y_i \to X\}_i$ is an fpqc cover if and only if the map $\coprod_i Y_i \to X$ is an fpqc cover.

Corollary 3.9.7. *Let $Y \to X$ be an fpqc cover. Let p_1, p_2 be the projections from $Y \times_X Y$ to Y, and let π_1, π_2, π_3 be the three projections from $Y \times_X Y \times_X Y$ to Y. Then we have an equivalence of categories*

$$\{\mathcal{F} \text{ qcoh}/X\} \leftrightarrow \{(\mathcal{G}, \varphi), \mathcal{G} \text{ qcoh}/Y, \varphi : p_1^*\mathcal{G} \simeq p_2^*\mathcal{G} \textit{ s.t. } \varphi_{23} \circ \varphi_{12} = \varphi_{13} : \pi_1^*\mathcal{G} \to \pi_3^*\mathcal{G}\}.$$

Proof. Left as an exercise. □

Example 3.9.2. We say that a cover $Y \to X$ is *Galois* if there exists a finite group Γ of automorphisms of Y over X such that $\Gamma \times Y \simeq Y \times_X Y, (\sigma, y) \mapsto (\sigma y, y)$. Then we have $\Gamma \times \Gamma \times Y \simeq Y \times_X Y \times_X Y$, $(\sigma, \tau, y) \mapsto (\sigma\tau y, \tau y, y)$.

In particular we can consider the case $Y = \text{Spec } L, X = \text{Spec } K$, L/K a Galois field extension. In this case we have $L \otimes_K L \simeq \prod_{g \in \Gamma} L$ by $\prod_{g\in\Gamma} l_g \mapsto \sum_{g \in \Gamma} g(l_g) \otimes l_g$ (that this is an isomorphism follows from Artin's linear independence of characters). A descent datum (V, φ) over L is then easily seen to be the same thing as a Galois semilinear action $\sigma : \Gamma \times V \to V$ via

$$\varphi(lv \otimes g(l)) = l \otimes g(l)\sigma_g(v).$$

Theorem 3.9.8. *Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_X$-module on a scheme X, and define a presheaf on the category of schemes over X taking $\pi : Y \to X$ to $\Gamma(Y, \pi^*\mathcal{F})$. Then this presheaf is a sheaf in the fpqc topology.*

Proof. Note that for any $\pi : Y \to X$ we have $\Gamma(Y, \pi^*\mathcal{F}) = \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \pi^*\mathcal{F})$. If $\pi : Y \to X$ is any fpqc cover, the natural bijection between maps $\mathcal{O}_X \to \mathcal{F}$ and descent data for maps $\mathcal{O}_Y \to \pi^*\mathcal{F}$ shows that our presheaf satisfies the sheaf condition for this cover. □

Theorem 3.9.9. *Any representable functor is a sheaf of sets in the fpqc topology. In particular, every abelian group scheme represents an abelian sheaf in the fpqc topology.*

Proof. We'll just prove this in the affine case. Let $A \to B$ be a faithfully flat map of rings, and let C be our representing ring. We need to show that every map Spec $B \to$ Spec C such that the two induced maps Spec $B \otimes_A B \rightrightarrows$ Spec C agree is induced by a unique map Spec $A \to$ Spec C. This follows from the exactness of the sequence

$$0 \to A \to B \rightrightarrows B \otimes_A B,$$

which follows from the special case $M = A$ of Lemma 3.9.3. □

Remark 3.9.2. Since the category of schemes is not a small category, we technically shouldn't call the fpqc topology a "topology", and it doesn't necessarily make sense to define cohomology groups with respect to the fpqc topology. Instead we usually focus on small subcategories with topologies whose open covers are a subset of the fpqc covers (such as the Zariski, étale, or fppf topologies). The above theorems clearly continue to apply to such topologies.

Theorem 3.9.10. *Let X be a separated scheme and let $\mathcal{F}$ be a quasicoherent sheaf on X. Let T be a topology containing the Zariski topology on X, whose opens are a small subcategory of the category of schemes over X, such that every cover of an affine scheme over X can be refined to a faithfully flat cover by a finite collection of affine schemes. Extend $\mathcal{F}$ to a sheaf on T as in Theorem 3.9.9. Then for any $p \geq 0$ we have $H^p(T, X, \mathcal{F}) = \check{H}^p(X, \mathcal{F})$ (i.e. the usual Zariski Čech cohomology).*

Proof. The proof is almost identical to the proof of Theorem 3.5.3, with the fpqc Poincaré lemma taking the place of the Zariski Poincaré lemma. □

Theorem 3.9.11. *Let X, T be as in the previous theorem. Then*

$$H^1(T, X, \mathrm{GL}_n) = \{\text{rank } n \text{ vector bundles}/X\}/\simeq$$

for every $n \in \mathbb{N}$. In particular, $H^1(T, X, \mathbb{G}_m) = \mathrm{Pic}(X)$.

Part III

Number Theory

Chapter 1

Weil bounds

1.1 Introduction

These notes are a work in progress, based on a student seminar series at Stanford. The eventual goal is to understand the proof of Deligne's Weil II, as well as the theory of trace functions, without learning French.

1.2 Hasse bound for elliptic curves

1.2.1 Manin's elementary proof for characteristic not equal to 2 or 3

The exposition here follows Chahal's paper [53] extremely closely.

Theorem 1.2.1 (Hasse bound). *Let $q = p^m$, p a prime other than 2, and let $a, b \in \mathbb{F}_q$ be such that $\Delta = 4a^3 + 27b^2 \neq 0$. Let*

$$N_q = \#\{(x, y) \in \mathbb{F}_q^2 \mid y^2 = x^3 + ax + b\}$$

(note that this does not *count the point at infinity). Then*

$$|N_q - q| \leq 2\sqrt{q}.$$

Proof. We work in the function field $\mathbb{F}_q(t)$. Set $\lambda = \lambda(t) = t^3 + at + b$ throughout, and define the *twisted* curve E_λ by

$$\lambda y^2 = x^3 + ax + b.$$

The addition formulae for two points P_1, P_2 on E_λ are given by

$$x(P_1 + P_2) = \lambda\left(\frac{y_1 - y_2}{x_1 - x_2}\right)^2 - (x_1 + x_2)$$

if $P_1 \neq P_2$, or

$$x(2P) = \frac{(3x^2 + a)^2}{4(x^3 + ax + b)} - 2x$$

if $P_1 = P_2 = P$.

We now define a sequence of points P_n on E_λ for $n \in \mathbb{Z}$ by

$$P_n = (t^q, (t^3 + at + b)^{\frac{q-1}{2}}) + n \cdot (t, 1)$$

(that P_0 is on E_λ follows from well-known properties of Frobenius). Setting $(x_n, y_n) = P_n$ and writing $x_n = \frac{f_n}{g_n}$ with f_n, g_n relatively prime elements of $\mathbb{F}_q[t]$ whenever P_n is not the zero element of the curve E_λ (which we will denote O), we define a sequence d_n by

$$d_n = \begin{cases} \deg f_n & P_n \neq O, \\ 0 & P_n = O. \end{cases}$$

By the definition of P_0, we clearly have $d_0 = q$.

Claim: $d_{-1} = N_q + 1$. To see this, note that by the addition formula we have

$$x_{-1} = (t^3 + at + b)\left(\frac{(t^3 + at + b)^{\frac{q-1}{2}} + 1}{t^q - t}\right)^2 - (t^q + t) = \frac{t^{2q+1} + O(t^{2q})}{(t^q - t)^2}.$$

Thus, to compute the degree of f_{-1} we just need to know how many factors of the denominator cancel with factors of the numerator in the fraction

$$\frac{(t^3 + at + b)((t^3 + at + b)^{\frac{q-1}{2}} + 1)^2}{(t^q - t)^2}.$$

The denominator factors as $\prod_{\alpha \in \mathbb{F}_q}(t - \alpha)^2$, and $t - \alpha$ divides the numerator once if $\alpha^3 + a\alpha + b = 0$, and twice if $(\frac{\alpha^3 + a\alpha + b}{q}) = -1$ (here $(\frac{\alpha}{q})$ is the quadratic residue symbol for $\mathbb{F}_q$). Thus,

$$d_{-1} = 2q + 1 - \sum_{\alpha \in \mathbb{F}_q} \left(1 - (\tfrac{\alpha^3 + a\alpha + b}{q})\right) = N_q + 1.$$

Lemma: If $P_n \neq O$, then $x_n \neq 0$ and $\deg f_n > \deg g_n$.

Basic Identity: $d_{n-1} + d_{n+1} = 2d_n + 2$.

Proof of the Hasse bound given these: From the Basic Identity, we easily see that

$$d_n = n^2 - (d_{-1} - d_0 - 1)n + d_0 = n^2 - (N_q - q)n + q.$$

Let r_0, r_1 be the roots of the quadratic $n \mapsto n^2 - (N_q - q)n + q$. We have $(r_0 - r_1)^2 = (N_q - q)^2 - 4q \in \mathbb{Z}$, so if r_0, r_1 were real and distinct then their difference would be at least 1, so there would then necessarily be some n such that either $d_n < 0$ or $d_n = 0 = d_{n+1}$. Either one of these possibilities contradicts the Lemma, so we must have $(r_0 - r_1)^2 \leq 0$, or equivalently,

$$|N_q - q| \leq 2\sqrt{q}.$$

Proof of Lemma: The plan is to formally evaluate x_n, y_n at $t = \infty$ (equivalently, we are looking at the ratio of the leading term of the numerator and the leading term of the denominator), and to induct on $|n|$. Note that from $y_n^2 = \frac{x_n^3 + ax_n + b}{t^3 + at + b}$, we see that if $x_n|_\infty \neq \infty$ then $y_n|_\infty = 0$.

Assume that the Lemma holds for n but fails for $n + 1$ (the reverse case, for $n < 0$, is handled similarly). Since

$$(x_{n+1}, -y_{n+1}) + (x_n, y_n) + (t, 1) = O,$$

the three summands on the left hand side are collinear, so

$$1-(-y_{n+1})=\frac{1-y_n}{t-x_n}(t-x_{n+1}),$$

so from the assumption $\frac{x_{n+1}}{t}|_\infty=0$, we get

$$0=y_{n+1}|_\infty=\left(\frac{1-y_n}{1-\frac{x_n}{t}}\Big(1-\frac{x_{n+1}}{t}\Big)-1\right)\Bigg|_\infty,$$

and thus

$$\frac{1-y_n}{1-\frac{x_n}{t}}\Bigg|_\infty=1.$$

From

$$x_{n+1}=\lambda\Big(\frac{1-y_n}{t-x_n}\Big)^2-t-x_n,$$

we get

$$0=\frac{x_{n+1}}{t}\bigg|_\infty=\left(\Big(\frac{1-y_n}{1-\frac{x_n}{t}}\Big)^2\Big(1+\frac{a}{t^2}+\frac{b}{t^3}\Big)-1-\frac{x_n}{t}\right)\Bigg|_\infty=-\frac{x_n}{t}\bigg|_\infty\neq 0,$$

a contradiction.

Proof of the Basic Identity: If $P_n=O$, then this is trivial. If $P_{n-1}=O$, then

$$x_{n+1}=x(2\cdot(t,1))=\frac{(3t^2+a)^2}{4(t^3+at+b)}-2t=\frac{t^4+O(t^3)}{4(t^3+at+b)},$$

and since $\Delta\neq 0$ we know that $3t^2+a$ has no common factor with t^3+at+b, so $d_{n+1}=4$, and the identity holds in this case (as we trivially have $d_{n-1}=0, d_n=1$). The case $P_{n+1}=O$ is identical, so we may assume from here on that none of P_{n-1},P_n,P_{n+1} are O.

Computing x_{n-1}, we have

$$\begin{aligned}x_{n-1}&=\lambda\Big(\frac{y_n+1}{x_n-t}\Big)^2-(x_n+t)=\frac{\lambda(y_n+1)^2-(x_n+t)(x_n-t)^2}{(x_n-t)^2}\\&=\frac{x_n^3+ax_n+b+t^3+at+b-(x_n+t)(x_n-t)^2+2\lambda y_n}{(x_n-t)^2}\\&=\frac{(x_n+t)(tx_n+a)+2b+2\lambda y_n}{(x_n-t)^2}\\&=\frac{(f_n+tg_n)(tf_n+ag_n)+2bg_n^2+2\lambda y_ng_n^2}{(f_n-tg_n)^2}=:\frac{R}{(f_n-tg_n)^2},\end{aligned}$$

say. From

$$(\lambda y_ng_n^2)^2=\lambda g_n^4(x_n^3+ax_n+b)=\lambda g_n(f_n^3+af_ng_n^2+bg_n^3)\in\mathbb{F}_q[t],$$

we get $\lambda y_ng_n^2\in\mathbb{F}_q[t]$ (by the rational root theorem), so $R\in\mathbb{F}_q[t]$. A similar calculation gives $x_{n+1}=\frac{S}{(f_n-tg_n)^2}$, where

$$S=(f_n+tg_n)(tf_n+ag_n)+2bg_n^2-2\lambda y_ng_n^2.$$

Since x_{n-1} and x_{n+1} differ only in the sign of $2\lambda y_n$, we can apply the difference of squares formula to see

$$\begin{aligned}
x_{n-1}x_{n+1} &= \frac{((x_n+t)(tx_n+a)+2b)^2 - 4(t^3+at+b)(x^3+ax+b)}{(x_n-t)^4} \\
&= \frac{((x^3+ax+t^3+at-(x_n+t)(x_n-t)^2)+2b)^2 - 4(t^3+at+b)(x^3+ax+b)}{(x_n-t)^4} \\
&= \frac{((x_n+t)(tx_n+a))^2 - (2t^2x_n+2ax_n)(2tx_n^2+2at) - 4b(x_n+t)(x_n-t)^2}{(x_n-t)^4} \\
&= \frac{(tx_n-a)^2 - 4b(x_n+t)}{(x_n-t)^2} \\
&= \frac{(tf_n-ag_n)^2 - 4bg_n(f_n+tg_n)}{(f_n-tg_n)^2} =: \frac{T}{(f_n-tg_n)^2} \\
&= \frac{t^2f_n^2 + O(tf_n^2)}{(f_n-tg_n)^2}.
\end{aligned}$$

Noting that T has degree $2d_n+2$ (by the last line, which implicitly used the Lemma), we just have to show that no extra cancellation occurs. Since $RS = (f_n - tg_n)^2T$, there must exist $r, s \in \mathbb{F}_q[t]$ such that $r \mid R, s \mid S$, and $rs = (f_n - tg_n)^2$. Thus we have

$$\frac{f_{n-1}}{g_{n-1}} = \frac{R}{(f_n-tg_n)^2} = \frac{R/r}{s},$$

and similarly $\frac{f_{n+1}}{g_{n+1}} = \frac{S/s}{r}$, and of course $(R/r)(S/s) = T$. Thus we just have to show that R/r and s have no common factor, and that S/s and r have no common factor. If not, then (in either case) any irreducible common factor must divide R, S, T, and $f_n - tg_n$, so it divides

$$\begin{aligned}
\gcd(f_n-tg_n,R,S,T) &= \gcd(f_n-tg_n, 2\lambda(1+y_n)g_n^2, 2\lambda(1-y_n)g_n^2, ((t^2-a)^2-8bt)g_n) \\
&\mid \gcd(2\lambda(1+y_n), 2\lambda(1-y_n), (t^2-a)^2-8bt)\gcd(f_n-tg_n,g_n)^2 \\
&= \gcd(t^3+at+b, t^4-2at^2-8bt+a^2) \\
&= \gcd(t^3+at+b, 9t^4+6at^2+a^2) \\
&= \gcd(t^3+at+b, (3t^2+a)^2) = 1,
\end{aligned}$$

(since $\Delta \neq 0$), which is a contradiction. □

What's really going on in this proof: In a follow up paper by Chahal [54], the following high level explanation of Manin's argument is given, and used to derive a similar elementary proof in characteristic 2.

Suppose we are given an elliptic curve E over a field k. By taking a quotient of the first projection map $\pi_1 : E \times E \to E$, we get a map $E \times E/(-1,-1) \to E/(-1) \cong \mathbb{P}^1$. Taking the generic fiber of this map gives $E^{\text{tw}} \to \operatorname{Spec} k(t)$, where E^{tw} is a twist of E which trivializes over the degree 2 extension $k(E)/k(t)$. Thus

$$E^{\text{tw}}(k(E)) \cong E(k(E)) \cong \operatorname{Mor}_k(E,E),$$

and

$$E^{\text{tw}}(k(t)) \cong \{\phi \in \text{Mor}_k(E,E) \mid \phi \circ (-1) = (-1) \circ \phi\}.$$

Thus the maps 1 and Frob (as well as all the other isogenies of E) correspond to $k(t)$ points on E^{tw}, and the degree of an isogeny should correspond to a naïve height of the corresponding point on E^{tw}.

1.2.2 Aside: binomial coefficients, Jacobi sums, and trinomial plane curves

Chevalley-Warning trick

How many mod-p points are there on the curve $x^m + y^n = k$? We can compute the number of points on this curve mod p by following the proof of Chevalley-Warning and seeing how badly it fails:

$$\begin{aligned}
\#\{(x,y) \in \mathbb{F}_p^2 \mid x^m + y^n = k\} &\equiv \sum_{x,y\in\mathbb{F}_p} 1 - (x^m + y^n - k)^{p-1}\\
&\equiv -\sum_{x,y\in\mathbb{F}_p}\sum_{a+b+c=p-1}\binom{p-1}{a,b,c}x^{am}y^{bn}(-k)^c\\
&\equiv -\sum_{a+b+c=p-1} k^c\binom{p-1-c}{a}\sum_{x,y\in\mathbb{F}_p} x^{am}y^{bn}\\
&\equiv -\sum_{\substack{a+b+c=p-1\\ a,b>0\\ p-1\mid am,bn}} k^c\binom{p-1-c}{a} \pmod{p}.
\end{aligned}$$

Note that the number of summands depends only on m and n, and the number of summands which are not trivially congruent to ±1 is

$$\frac{(m-1)(n-1) - (\gcd(m,n)-1)}{2},$$

which is precisely the geometric genus of the plane curve $x^m + y^n = k$ (as one can easily check with the Riemann-Hurwitz formula). A similar calculation applies to any plane curve defined by an equation involving three monomials whose exponent vectors are affinely independent.

Example 1.2.1. Applying this to the genus 1 curve $y^2 = x^3 + k$, we see that when $p \equiv 1 \pmod{6}$ we have

$$\begin{aligned}
\#\{(x,y) \in \mathbb{F}_p^2 \mid y^2 = x^3 + k\} &\equiv -k^{\frac{p-1}{6}}\binom{\frac{5(p-1)}{6}}{\frac{p-1}{3}}\\
&\equiv p - k^{\frac{p-1}{6}}\binom{\frac{p-1}{2}}{\frac{p-1}{3}} \pmod{p}.
\end{aligned}$$

Suppose $p > 16$, so that $p > 4\sqrt{p}$. Letting w be a solution to $w^2 + w + 1 \equiv 0 \pmod{p}$, and letting a be the least (in absolute value) remainder of $\binom{\frac{p-1}{2}}{\frac{p-1}{3}}$ mod p and b be the least (in absolute value) remainder of $w\binom{\frac{p-1}{2}}{\frac{p-1}{3}}$ mod p, we see from the Hasse bound that $|a|, |b|, |a+b| < 2\sqrt{p}$. From this we easily conclude that

$$p \mid a^2 + ab + b^2 = \frac{a^2 + b^2 + (a+b)^2}{2} < 4p,$$

so $a^2 + ab + b^2$ is either p, $2p$, or $3p$. Since the number of points on the curve $y^2 = x^3 + k$ is congruent to 2 modulo 3 whenever k is a quadratic residue mod p, we see that both a and b are congruent to 2 modulo 3, so we must have $a^2 + ab + b^2 = 3p$. Similarly, since the number of points on $y^2 = x^3 + k$ is odd exactly when k is a cubic residue mod p, we see that a is even and b is odd. Thus, setting $A = \frac{a}{2}$ and $B = \frac{a+2b}{6}$, we have $A, B \in \mathbb{Z}$,

$$A^2 + 3B^2 = p, \quad A \equiv 1 \pmod 3,$$

and

$$\binom{\frac{p-1}{2}}{\frac{p-1}{3}} \equiv 2A \pmod p.$$

Example 1.2.2. Similar reasoning applied to the curve $y^2 = x^4 + k$ (or alternatively to the curve $y^2 = x^3 - kx$) shows that if $p \equiv 1 \pmod 4$ then there are integers a, b such that

$$a^2 + b^2 = p, \quad a \equiv 1 \pmod 4,$$

and

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv 2a \pmod p.$$

Remark 1.2.1. Trying the same approach with the elliptic curve $y^2 = x^3 + x + k$, we get

$$\#\{(x,y) \in \mathbb{F}_p \mid y^2 = x^3 + x + k\} \equiv -\sum_{\frac{p-1}{4} \le n \le \frac{p-1}{3}} k^{2n - \frac{p-1}{2}} \binom{\frac{p-1}{2}}{n, p-1-3n, 2n-\frac{p-1}{2}} \pmod p.$$

Considering the right hand side as a polynomial in k, we see that it has degree at most $\frac{p-1}{6}$ and always takes values in $(-2\sqrt{p}, 2\sqrt{p}) + p\mathbb{Z}$. Out of curiosity, I tried factoring these polynomials (in $\mathbb{F}_p[k]$) for p up to 3000, and found that they always seem to split into a product of factors of degrees $1, 2$, and 4 - can anyone explain this?

Jacobi sums and binomial coefficients

Definition 1.2.2. Let p be a prime and let χ be any Dirichlet character modulo p. Define the *Gauss sum* $g(\chi)$ to be

$$g(\chi) = \sum_{j=0}^{p-1} \chi(j) e^{2\pi i j/p}.$$

Proposition 1.2.3. *If χ is a nontrivial Dirichlet character mod p, then $|g(\chi)| = \sqrt{p}$ and $g(\chi)g(\overline{\chi}) = \chi(-1)p$. If χ is the real quadratic character, then $g(\chi)$ is either $\sqrt{p}$ or $i\sqrt{p}$ depending on whether p is 1 or -1 modulo 4.*

Definition 1.2.4. Let p be a prime and let χ, ψ be any two Dirichlet characters modulo p. Define the *Jacobi sum* $J(\chi, \psi)$ to be

$$J(\chi, \psi) = \sum_{j=0}^{p-1} \chi(j)\psi(1-j).$$

Proposition 1.2.5. *If χ, ψ are Dirichlet characters mod p such that $\chi\psi$ is nontrivial, then*

$$J(\chi, \psi) = \frac{g(\chi)g(\psi)}{g(\chi\psi)}.$$

Let n be a positive integer, write $\zeta_n = e^{2\pi i/n}$, and let p be a prime with $p \equiv 1 \pmod{n}$. From the existence of primitive roots modulo p, we see that there are $\varphi(n)$ congruence classes $z \pmod{p}$ with $\mathrm{ord}_p(z) = n$. Picking one of these congruence classes, we can define the prime ideal $P = (p, \zeta_n - z)$ of $\mathbb{Z}[\zeta_n]$. Note that every element of $\mathbb{Z}[\zeta_n]$ is congruent to an element of $\mathbb{Z}$ modulo P, i.e. $\mathbb{Z}[\zeta_n]/P = \mathbb{Z}/p$, and that $\mathrm{Nm}(P) = p$.

Definition 1.2.6. Let P be a prime ideal of $\mathbb{Z}[\zeta_n]$ which does not divide n, and let a be any element of $\mathbb{Z}[\zeta_n]$. Define the nth *power residue symbol* of a on P by

$$\left(\frac{a}{P}\right)_n \equiv a^{\frac{\mathrm{Nm}(P)-1}{n}} \pmod{P}$$

and

$$\left(\frac{a}{P}\right)_n \in \{0, 1, \zeta_n, \zeta_n^2, ..., \zeta_n^{n-1}\}.$$

Theorem 1.2.7 (Theorem 5.1 of [100])**.** *Let n, p, P be as above, so $p \equiv 1 \pmod{n}$ and P is a prime ideal of $\mathbb{Z}[\zeta_n]$ lying over p. Let χ_n be the Dirichlet character mod p defined by $\chi_n(a) = (\frac{a}{P})_n$. Then for any $0 < k, l < n$ we have*

$$\binom{\frac{k(p-1)}{n}}{\frac{l(p-1)}{n}} \equiv (-1)^{\frac{l(p-1)}{n}+1} J(\chi_n^k, \chi_n^{n-l}) \pmod{P}.$$

Proof. From $\chi_n(a) \equiv a^{\frac{p-1}{n}} \pmod{P}$, we have

$$\begin{aligned} J(\chi_n^{n-l}, \chi_n^k) &= \sum_{j=0}^{p-1} \chi_n(j)^{n-l}\chi_n(1-j)^k \\ &\equiv \sum_{j=0}^{p-1} j^{p-1-\frac{l(p-1)}{n}}(1-j)^{\frac{k(p-1)}{n}} \\ &= \sum_{j=0}^{p-1} j^{p-1-\frac{l(p-1)}{n}} \sum_m \binom{\frac{k(p-1)}{n}}{m}(-j)^m \\ &= \sum_m (-1)^m \binom{\frac{k(p-1)}{n}}{m} \sum_{j=0}^{p-1} j^{p-1+m-\frac{l(p-1)}{n}} \\ &\equiv -(-1)^{\frac{l(p-1)}{n}} \binom{\frac{k(p-1)}{n}}{\frac{l(p-1)}{n}} \pmod{P}. \end{aligned}$$

□

Example 1.2.3. Take $n = 4$, and let p be a prime which is $1 \pmod 4$. Writing $p = a^2 + b^2$ with $a \equiv 1 \pmod 4$, let $P = (a + bi)$. Define the Dirichlet character χ_4 by $\chi_4(k) = (\frac{k}{a+bi})_4$, and let $\chi_2 = \chi_4^2$ be the quadratic character mod p. Then

$$J(\chi_2, \chi_4) \equiv -\binom{\frac{p-1}{4}}{\frac{p-1}{2}} = 0 \pmod P$$

and

$$\overline{J(\chi_2, \chi_4)} = J(\chi_2, \chi_4^3) \equiv (-1)^{\frac{p-1}{4}+1}\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \pmod P,$$

so

$$\mathrm{Tr}(J(\chi_2, \chi_4)) \equiv (-1)^{\frac{p-1}{4}+1}\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \pmod p.$$

From $|J(\chi_2, \chi_4)| = \sqrt{p}$ and $(a + bi) \mid J(\chi_2, \chi_4)$, we see that $J(\chi_2, \chi_4) = i^k(a + bi)$ for some k. By computing $J(\chi_2, \chi_4)$ modulo 4, one can show that in fact we have

$$J(\chi_2, \chi_4) = (-1)^{\frac{p-1}{4}+1}(a + bi),$$

giving us a second proof of the congruence

$$\binom{\frac{p-1}{2}}{\frac{p-1}{4}} \equiv 2a \pmod p.$$

1.3 Weil's argument for diagonal hypersurfaces

This section follows Weil's paper [185]. Let q be a power of a prime p. Let $a_0, ..., a_r \in \mathbb{F}_q^\times$ and let $n_0, ..., n_r \in \mathbb{N}^+$. We want to count

$$N = \#\{(x_0, ..., x_r) \in \mathbb{F}_q^{r+1} \mid a_0x_0^{n_0} + \cdots + a_rx_r^{n_r} = 0\}.$$

Set $d_i = \gcd(n_i, q - 1)$.

The plan is to use Fourier analysis, so the first step is to pick additive and multiplicative characters.

Definition 1.3.1. Define $\psi_q : \mathbb{F}_q \to \mathbb{C}^\times$ by

$$\psi_q(a) = e^{\frac{2\pi i \,\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)}{p}}.$$

Proposition 1.3.2. *The character ψ_q is not identically equal to 1, and every additive character of $\mathbb{F}_q$ can be written as $a \mapsto \psi_q(ca)$ for some $c \in \mathbb{F}_q$.*

Proof. This follows immediately from Artin's theorem on the linear independence of characters. □

Definition 1.3.3. Fix once and for all an injective multiplicative map $\phi : \overline{\mathbb{F}}_q^\times \to \mathbb{C}^\times$. For $\alpha \in \mathbb{Q}/\mathbb{Z}$ and $n \in \mathbb{N}$ such that $(q^n - 1)\alpha \equiv 0 \pmod 1$, define $\chi_{\alpha,n} : \mathbb{F}_{q^n}^\times \to \mathbb{C}^\times$ by

$$\chi_{\alpha,n}(x) = \phi(x)^{(q^n-1)\alpha}.$$

Extend this to $\mathbb{F}_{q^n}$ by

$$\chi_{\alpha,n}(0) = \begin{cases} 0 & \alpha \not\equiv 0 \pmod 1, \\ 1 & \alpha \equiv 0 \pmod 1, \end{cases}$$

and set $\chi_\alpha = \chi_{\alpha,1}$.

Proposition 1.3.4. *If $(q-1)\alpha \equiv 0 \pmod 1$, then $\chi_{\alpha,n}(x) = \chi_\alpha(\mathrm{Nm}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(x))$.*

Proof.

$$\chi_{\alpha,n}(x) = \phi(x)^{(q^n-1)\alpha} = (\phi(x)^{(q-1)\alpha})^{q^{n-1}+\cdots+1} = \chi_\alpha(x^{q^{n-1}+\cdots+1}) = \chi_\alpha(\mathrm{Nm}_{\mathbb{F}_{q^n}/\mathbb{F}_q}(x)). \qquad \square$$

Proposition 1.3.5. *If $d = \gcd(n, q-1)$ and $u \in \mathbb{F}_q$ then number of $x \in \mathbb{F}_q$ such that $x^n = u$ is $\sum_{d\alpha \equiv 0 \pmod 1} \chi_\alpha(u)$.*

From this we see that

$$\begin{aligned} N &= \sum_{\substack{\alpha=(\alpha_0,\dots,\alpha_r) \\ d_i\alpha_i \equiv 0 \pmod 1}} \sum_{\substack{u=(u_0,\dots,u_r) \\ \sum a_i u_i = 0}} \chi_{\alpha_0}(u_0)\cdots\chi_{\alpha_r}(u_r) \\ &= q^r + \sum_{\substack{\alpha=(\alpha_0,\dots,\alpha_r) \\ 0<\alpha_i<1 \\ d_i\alpha_i \equiv 0 \pmod 1}} \chi_{\alpha_0}(a_0^{-1})\cdots\chi_{\alpha_r}(a_r^{-1}) \sum_{u_0+\cdots+u_r=0} \chi_{\alpha_0}(u_0)\cdots\chi_{\alpha_r}(u_r), \end{aligned}$$

where the second equality follows from the fact that the inner sum is 0 if some but not all of the α_i are $0 \pmod 1$. For $0 < \alpha_0 < 1$, we can simplify the inner sum further by restricting to $u_0 \neq 0$ and setting $u_i = u_0 v_i$:

$$\begin{aligned} \sum_{u_0+\cdots+u_r} \chi_{\alpha_0}(u_0)\cdots\chi_{\alpha_r}(u_r) &= \sum_{u_0 \neq 0} \chi_{\alpha_0+\cdots+\alpha_r}(u_0) \sum_{1+v_1+\cdots+v_r=0} \chi_{\alpha_1}(v_1)\cdots\chi_{\alpha_r}(v_r) \\ &= \begin{cases} 0 & \alpha_0+\cdots+\alpha_r \not\equiv 0 \pmod 1, \\ (q-1)\sum_{1+v_1+\cdots+v_r=0} \chi_{\alpha_1}(v_1)\cdots\chi_{\alpha_r}(v_r) & \alpha_0+\cdots+\alpha_r \equiv 0 \pmod 1. \end{cases} \end{aligned}$$

Definition 1.3.6. For $\alpha = (\alpha_0, ..., \alpha_r)$ with $\alpha_0 + \cdots + \alpha_r \equiv 0 \pmod 1$, define the *Jacobi sum* $j(\alpha)$ by

$$\begin{aligned} j(\alpha) &= \frac{1}{q-1} \sum_{u_0+\cdots+u_r} \chi_{\alpha_0}(u_0)\cdots\chi_{\alpha_r}(u_r) \\ &= \sum_{1+v_1+\cdots+v_r=0} \chi_{\alpha_1}(v_1)\cdots\chi_{\alpha_r}(v_r). \end{aligned}$$

In terms of the Jacobi sums, we have

$$N = q^r + (q-1) \sum_{\substack{\alpha_0+\cdots+\alpha_i \equiv 0 \pmod 1 \\ d_i\alpha_i \equiv 0 \pmod 1 \\ 0<\alpha_i<1}} \chi_{\alpha_0}(a_0^{-1})\cdots\chi_{\alpha_r}(a_r^{-1}) j(\alpha).$$

Note that the number of summands is bounded by a constant which depends only on r and $d_0, ..., d_r$. In order to evaluate the Jacobi sums, we will use Gauss sums.

Definition 1.3.7. If $\chi : \mathbb{F}_q \to \mathbb{C}$ is a multiplicative character, then the *Gauss sum* $g(\chi)$ is

$$g(\chi) = \sum_{x \in \mathbb{F}_q} \chi(x)\psi_q(x).$$

Proposition 1.3.8. *If $\chi \neq \chi_0$ then $|g(\chi)| = \sqrt{q}$, $g(\chi)g(\overline{\chi}) = \chi(-1)q$, and $g(\chi_0) = 0$. For $\chi \neq \chi_0$, we have*

$$\chi(t) = \frac{g(\chi)}{q} \sum_{x \in \mathbb{F}_q} \overline{\chi}(x)\overline{\psi}_q(tx).$$

Proof. The first statement is easy. For the second, note that for any $t \neq 0$ we have

$$\frac{q}{g(\chi)} = \overline{g}(\chi) = \overline{\chi}(t) \sum_{x \in \mathbb{F}_q} \overline{\chi}(x)\overline{\psi}_q(tx). \qquad \square$$

Proposition 1.3.9. *If $\alpha = (\alpha_0, ..., \alpha_r)$ with $\alpha_0 + \cdots + \alpha_r \equiv 0 \pmod 1$, then*

$$j(\alpha) = \frac{g(\chi_{\alpha_0}) \cdots g(\chi_{\alpha_r})}{q}$$

and $|j(\alpha)| = q^{\frac{r-1}{2}}$.

Proof. Expanding out each $\chi_{\alpha_i}(u_i)$ in the definition of $j(\alpha)$, we get

$$(q-1)j(\alpha) = \frac{g(\chi_{\alpha_0}) \cdots g(\chi_{\alpha_r})}{q^{r+1}} \sum_{x_0,\ldots,x_r} \overline{\chi}_{\alpha_0}(x_0) \cdots \overline{\chi}_{\alpha_r}(x_r) \sum_{u_0+\cdots+u_r=0} \overline{\psi}_q(x_0u_0 + \cdots + x_ru_r),$$

and the inner sum is 0 unless $x_0 = \cdots = x_r$, in which case it is q^r. $\square$

Next we want to understand how N changes when we replace $\mathbb{F}_q$ with $\mathbb{F}_{q^\nu}$. The main difficulty is understanding what happens to Gauss sums.

Theorem 1.3.10 (Davenport, Hasse)**.** *If $(q-1)\alpha \equiv 0 \pmod 1$, then $-g(\chi_{\alpha,\nu}) = (-g(\chi_\alpha))^\nu$.*

Proof. For $F(x) = x^n + c_1x^{n-1} + \cdots + c_n \in \mathbb{F}_q[x]$ monic, set

$$\lambda_\alpha(F) = \chi_\alpha(c_n)\psi_q(c_1).$$

Note that $\lambda_\alpha(F_1F_2) = \lambda_\alpha(F_1)\lambda_\alpha(F_2)$, so by unique factorization for polynomials in $\mathbb{F}_q[x]$ we have

$$\sum_{F \in \mathbb{F}_q[x] \text{ monic}} \lambda_\alpha(F)T^{\deg F} = \prod_{P \in \mathbb{F}_q[x] \text{ irred.}} (1 - \lambda_\alpha(P)T^{\deg P})^{-1},$$

and the left hand side is easily seen to be equal to $1 + g(\chi_\alpha)T$. Defining $\lambda_{\alpha,\nu}$ for functions in $\mathbb{F}_{q^\nu}[x]$ similarly, we have

$$1 + g(\chi_{\alpha,\nu})T = \prod_{P' \in \mathbb{F}_{q^\nu}[x] \text{ irred.}} (1 - \lambda_{\alpha,\nu}(P')T^{\deg P'})^{-1}.$$

Suppose that $P(x) = x^n + bx^{n-1} + \cdots + a$ is irreducible in $\mathbb{F}_q[x]$ and $P'(x) = x^{n'} + b'x^{n'-1} + \cdots + a'$ is an irreducible factor of $P(x)$ in $\mathbb{F}_{q^\nu}[x]$. Then by Galois theory we have $n' = \frac{n}{(n,\nu)}$ and

$$\begin{aligned}\lambda_{\alpha,\nu}(P') &= \chi_{\alpha,\nu}(a')\psi_{q^\nu}(b') \\ &= \chi_\alpha(\mathrm{Nm}_{\mathbb{F}_{q^\nu}/\mathbb{F}_q}(a'))\psi_q(\mathrm{Tr}_{\mathbb{F}_{q^\nu}/\mathbb{F}_q}(b')) \\ &= \chi_\alpha\big(a^{\frac{\nu}{(n,\nu)}}\big)\psi_q\big(\tfrac{\nu}{(n,\nu)}b\big) \\ &= \lambda_\alpha(P)^{\frac{\nu}{(n,\nu)}}.\end{aligned}$$

Thus we have

$$\begin{aligned}\prod_{P'|P}(1-\lambda_{\alpha,\nu}(P')T^{\nu\deg P'})^{-1} &= (1-\lambda_\alpha(P)T^{\frac{n\nu}{(n,\nu)}})^{-(n,\nu)} \\ &= \prod_{a=0}^{\nu-1}(1-\lambda_\alpha(P)(e^{\frac{2\pi i a}{\nu}}T)^n)^{-1},\end{aligned}$$

so

$$\begin{aligned}1 + g(\chi_{\alpha,\nu})T^\nu &= \prod_{a=0}^{\nu-1}\prod_{P\in\mathbb{F}_q[x]\text{ irred.}}(1-\lambda_\alpha(P)(e^{\frac{2\pi i a}{\nu}}T)^{\deg P})^{-1} \\ &= \prod_{a=0}^{\nu-1}(1+g(\chi_\alpha)e^{\frac{2\pi i a}{\nu}}T) \\ &= 1-(-g(\chi_\alpha))^\nu T^\nu. \qquad \square\end{aligned}$$

Now we restrict to the special case $n_0 = \cdots = n_r = n$, and set

$$\overline{N}_\nu = \#\{[x_0 : \cdots : x_r] \in \mathbb{P}^r_{\mathbb{F}_{q^\nu}} \mid a_0x_0^n + \cdots + a_rx_r^n = 0\}.$$

From the formula we derived for N, we have

$$\overline{N}_\nu = \frac{N_\nu}{q^\nu - 1} = q^{(r-1)\nu} + \cdots + q^\nu + 1 + \sum_{\substack{\alpha_0+\cdots+\alpha_r\equiv 0 \pmod 1 \\ (n,q^\nu-1)\alpha_i\equiv 0 \pmod 1 \\ 0<\alpha_i<1}} \overline{\chi}_{\alpha_0,\nu}(a_0)\cdots\overline{\chi}_{\alpha_r,\nu}(a_r)j_\nu(\alpha).$$

We want to compute the generating function $\exp(\sum_{\nu\geq 1}\overline{N}_\nu\frac{T^\nu}{\nu})$ (this is the zeta function of the diagonal hypersurface in $\mathbb{P}^r$ given by $a_0x_0^n + \cdots + a_rx_r^n = 0$). Setting

$$\mu(\alpha) = \min\{\mu \mid (q^\mu - 1)\alpha \equiv \vec{0} \pmod 1\},$$

we have

$$\exp\left(\sum_{\nu\geq 1}\overline{N}_\nu\frac{T^\nu}{\nu}\right) = \frac{1}{(1-T)(1-qT)\cdots(1-q^{r-1}T)}\prod_{\substack{\alpha_0+\cdots+\alpha_r\equiv 0 \pmod 1 \\ (n,q^\nu-1)\alpha_i\equiv 0 \pmod 1 \\ 0<\alpha_i<1}}(1-C(\alpha)T^{\mu(\alpha)})^{\frac{(-1)^r}{\mu(\alpha)}},$$

where

$$C(\alpha) = (-1)^{r+1}\overline{\chi}_{\alpha_0,\mu(\alpha)}(a_0)\cdots\overline{\chi}_{\alpha_r,\mu(\alpha)}(a_r)j_{\mu(\alpha)}(\alpha),$$

and $|C(\alpha)| = q^{\frac{(r-1)\mu(\alpha)}{2}}$. Furthermore, we have $C(q\alpha) = C(\alpha)$ since $a_i^q = a_i$, $\mu(q\alpha) = \mu(\alpha)$, and $j_{\mu(\alpha)}(q\alpha) = j_{\mu(\alpha)}(\alpha)$, so by grouping the terms in the product corresponding to $\alpha, q\alpha, ..., q^{\mu(\alpha)-1}\alpha$ we see that in fact the zeta function of our diagonal hypersurface is a rational function of T.

Furthermore, either the last product or its inverse is a polynomial with integer coefficients (since $j(\alpha)$, being a sum of roots of unity, is always an algebraic integer), and the degree of this polynomial is the number of tuples $(\alpha_0, ..., \alpha_r)$ such that $0 < \alpha_i < 1$ for all i, each α_i has denominator dividing n and coprime to p, and $\alpha_0 + \cdots + \alpha_r \equiv 0 \pmod 1$. Since α_0 is determined by $\alpha_1, ..., \alpha_r$, we see that the number of such tuples is

$$(n-1)^r - ((n-1)^{r-1} - \cdots) = \frac{(n-1)((n-1)^r - (-1)^r)}{n}$$

if n is relatively prime to p.

1.4 Ho Chung's notes on rationality of the zeta function for curves

1.4.1 Introduction

So one goal of the seminar is to perhaps give bounds of sum of trace functions on a variety.

Some examples of what we care about

Example 1.4.1 (Gauss sum for Dirichlet characters mod p). We want to understand the size of

$$\tau(\chi) = \sum_{a\in\mathbb{A}^1(\mathbb{F}_p)} \chi(a)e\left(\frac{a}{p}\right)$$

for a non-trivial multiplicative character $\chi : \mathbb{F}_p^* \to \mathbb{C}$, extending to domain $\mathbb{F}_p$ by zero.

This is a classical Gauss sum, where it is known that $|g(\chi)| = \sqrt{p}$.

Example 1.4.2 (Kloosterman sums). We want to understand the size of

$$S(a, b; p) = \sum_{x\in(\mathbb{A}^1-0)(\mathbb{F}_p)} e\left(\frac{ax + b\overline{x}}{p}\right)$$

for $a, b \in \mathbb{F}_p^*$. Here $\overline{x}$ means multiplicative inverse of $x \mod p$.

Here Weil bound says that

$$|S(a, b; p)| \leq 2\sqrt{p}$$

so we do attain square-root cancellation.

Example 1.4.3 (Hasse-Weil). We want to understand the size of

$$|\#E(\mathbb{F}_p) - p - 1|$$

for an elliptic curve $E : y^2 = f(x)$ with $f(x) = x^3 + ax + b$ over $\mathbb{F}_p$.

Note that the number of solutions of $x^2 \equiv a \mod p$ equals $1+\left(\frac{a}{p}\right)$. Thus, after first subtracting off the point at infinity,

$$|\#E(\mathbb{F}_p) - p - 1| = \left| \sum_{x\in\mathbb{A}^1(\mathbb{F}_p)} \left(1 + \left(\frac{x^3+ax+b}{p}\right) - p\right) \right| = \left| \sum_{x\in\mathbb{A}^1(\mathbb{F}_p)} \left(\frac{x^3+ax+b}{p}\right) \right|$$

Here Hasse-Weil bound says that

$$|\#E(\mathbb{F}_p) - p - 1| \leq 2\sqrt{p}$$

It can be considered a square-root cancellation type result for the functions $\chi(f(x))$ where χ is the Legendre symbol/non-trivial quadratic character for $\mathbb{F}_p^*$.

The general setup

What is the most general setup here; how to unify the multiplicative characters and the additive characters?

The most general set up here would be

- Let $k = \mathbb{F}_q$ be a finite field Consider any separated scheme X/k of finite type, any constructible $\mathbb{Q}_l$-sheaf $\mathcal{F}$ on X, any finite extension E/k. For any $x \in X(E)$, denote
$$\mathrm{Frob}_{E,x}(\mathcal{F})$$
the action of geometric Frobenius $\mathrm{Frob}_E \in \mathrm{Gal}(\overline{E}/E)$ on the pullback $\mathcal{F}$ to $\mathrm{Spec}(E)$ by the point $x \in X(E)$ viewed as a map $\mathrm{Spec}(E) \to X$. Write
$$t_{\mathcal{F}}(E,x) = \mathrm{Tr}(\mathrm{Frob}_{E,x} | \mathcal{F})$$
In other words, $t_{\mathcal{F}}(E,x)$ is the trace of Frob_E action on the stalk $\mathcal{F}_x$. A simplified but good enough case would be X/k is quasi-projective, $\mathcal{F}$ is locally constant (synonym: lisse) sheaf on X.

Extremely sketchy

What is constructible sheaf, local system,...

- We have a version of Lefschetz trace formula here:
$$\sum_{x\in X(E)} t_{\mathcal{F}}(E,x) = \sum_i (-1)^i \,\mathrm{Tr}\left(\mathrm{Frob}_E \,|H^i_c(X \otimes_k \overline{k}, \mathcal{F}\right)$$
- Deligne's work (seems to be mainly Weil II, Theorem 3.3.1) buys us something of the sort
  - There are generally hard Lefschetz type result on cohomology; and in special cases concentration of cohomology results, that roughly says most of the cohomology groups (say, all but the middle one) vanish.
  - The dimension of the nonvanishing cohomology group can be written down.
  - Purity result - The cohomology groups are mixed with some weight in general; pure with some weight in nice cases. A cohomology group being pure of weight n means that all eigenvalues of Frob_k acting on this cohomology has complex absolute value $|k|^{n/2}$, once you fixed the isomorphism between $\overline{\mathbb{Q}_l} \cong \mathbb{C}$.

  Triangle inequality then gives us square cancellation we look for.
- Fouvry, Kowalski, Michel et al's work seems to focus on the case X being a dense open subset of $\mathbb{P}^1$, and $E = k$ so far.

Execution of this plan for the three examples

Extremely sketchy END

1.4.2 Zeta function for varieties over $\mathbb{F}_q$

It does not hurt to replace all the "scheme of finite type" below with "quasi-projective variety"

Two definitions of zeta function

Lemma 1.4.1. *Let X be a scheme of finite type over $\mathbb{F}_q$. Then*

$$\# X(\mathbb{F}_{q^n}) \leq O\left(q^{n \cdot \dim X}\right) \quad \textit{as } n \to \infty$$

Proof. Without loss of generality, assume that X is affine and integral. Then the result follows from Noether normalization. □

Definition 1.4.2 (Local zeta function)**.** Let X be a scheme of finite type over $\mathbb{F}_q$. Define the local zeta function to be

$$Z\left(X/\mathbb{F}_q, T\right) = \exp\left(\sum_{n=1}^{\infty} \# X(\mathbb{F}_{q^n}) \frac{T^n}{n}\right) \in \mathbb{Q}[\![T]\!]$$

Say a word on what $X(\mathbb{F}_{q^n})$ is, closed point etc for analytic number theorist in audience

Remark 1.4.1. The previous lemma implies that $Z(X, q^{-s})$ converges to a holomorphic function on $\Re s > \dim X$.

Example 1.4.4 ($\mathbb{A}^0 = Spec(\mathbb{F}_q)$). For each n there is only one point for $\mathbb{A}^0(\mathbb{F}_{q^n})$, which is already rational over $\overline{\mathbb{F}_q}$. Thus the zeta function is

$$Z(\mathbb{A}^0/\mathbb{F}_q, T) = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n}\right) = exp\left(-\log(1-T)\right) = \frac{1}{1-T} \in \mathbb{Z}[\![T]\!]$$

which corresponds to the Euler factor of $\zeta(s)$ once we substitute $T = p^{-s}$. In general, the Euler factor of Dedekind zeta function can be obtained in the same way.

Example 1.4.5 ($\mathbb{A}^k$). Clearly $\#\mathbb{A}^k(\mathbb{F}_{q^n}) = (q^n)^k = q^{kn}$. Thus the zeta function is

$$Z(\mathbb{A}^k/\mathbb{F}_q, T) = \exp\left(\sum_{n=1}^{\infty} q^{kn} \frac{T^n}{n}\right) = exp\left(-\log(1-q^kT)\right) = \frac{1}{1-q^kT} \in \mathbb{Z}[\![T]\!]$$

Example 1.4.6 ($\mathbb{P}^k$). Clearly

$$\#\mathbb{P}^k(\mathbb{F}_{q^n}) = (q^n)^k + (q^n)^{k-1} + \cdots + 1$$

Thus the zeta function is

$$Z(\mathbb{P}^k/\mathbb{F}_q, T) = \exp\left(\sum_{n=1}^{\infty} (q^{kn} + \cdots + 1) \frac{T^n}{n}\right) = exp\left(-\sum_{i=0}^{k} \log(1-q^iT)\right) = \prod_{i=1}^{k} \frac{1}{1-q^iT} \in \mathbb{Z}[\![T]\!]$$

Here is another way of writing down the local zeta function.

Proposition 1.4.3. *Let X be a quasi-projective variety $\mathbb{F}_q$. For each closed point x we define $deg(x)$ to be the degree of the extension $k_{X,x}/\mathbb{F}_q$. Then*

$$Z(X/\mathbb{F}_q, T) = \prod_{x \in |X|} \left(1 - T^{deg(x)}\right)^{-1}$$

We ignore convergence issues as we are merely considering formal power series.

Proof. When we count $\#X(\mathbb{F}_{q^n})$, separate the counting for each $x \in |X|$ and use the fact that a closed point $x \in |X|$ will show up in $X(\mathbb{F}_{q^n})$ if and only if $deg(x)|n$. □

Corollary 1.4.4. *We actually have $Z(X/\mathbb{F}_q, T) \in \mathbb{Z}[\![T]\!]$.*

Remark 1.4.2. Note also that

$$Z(X/\mathbb{F}_q, q^{-s}) = \prod_{x \in |X|} \left(1 - |k_{X,x}|^{-s}\right)^{-1}$$

This can be used as a definition of (global) zeta function for scheme of finite type over $\mathbb{Z}$. In this set up the above proposition may be regarded as the analogue (in the local case) of Euler product factorization for Riemann zeta function.

Proposition 1.4.5 (Properties of local zeta functions)**.**

- *If $C \hookrightarrow X$ is a closed subscheme, $U = X - C$ an open subscheme of X, then*

$$Z(X, T) = Z(C, T)Z(U, T)$$

- *If X is reduced, $X = X_1 \cup X_2$ is a union of two closed subschemes, and $X_1 \cap X_2$ is equipped with reduced induced subscheme structure, then*

$$Z(X, T) = \frac{Z(X_1, T)Z(X_2, T)}{Z(X_1 \cap X_2, T)}$$

These two properties are useful in doing reduction arguments. For example, to prove rationality of zeta function, these properties reduce it to the case where X is affine and integral, which is birational to an irreducible hypersurface in $\mathbb{A}_{\mathbb{F}_{q^n}}$.

- *If X is defined over $\mathbb{F}_q$, then*

$$Z(X \times_{\mathbb{F}_q} \mathbb{F}_{q^r}, T^r) = \prod_{i=1}^{r} Z(X, \xi_r^i T)$$

where ξ_r is a primitive r-th root of unity.

Statement of Weil conjectures

The properties of this local zeta function was conjectured by Weil and proved by Deligne.

Theorem 1.4.6 (Deligne)**.** *For a smooth, projective, geometrically irreducible variety $X/\mathbb{F}_q$ we have,*

(Rationality) *$Z(X/\mathbb{F}_q, T)$ is a rational function in T. If $\dim X = n$ we can write it as*

$$Z(X/\mathbb{F}_q, T) = \frac{P_1(T)P_3(T)\cdots P_{2n-1}(T)}{P_0(T)P_2(T)\cdots P_{2n}(T)}$$

where each $P_i(T)$ has integral coefficients with leading coefficient 1.

Functional equation) *Define $\chi = \chi(X) = \sum_i (-1)^i deg(P_i)$. We have*

$$Z\left(X, \frac{1}{q^n T}\right) = \epsilon q^{n\chi/2} T^{\chi} Z(X, T)$$

Here the root number ϵ is defined as follows.

$$\epsilon = \begin{cases} (-1)^{\chi} & \text{if } n \text{ is odd} \\ (-1)^{\chi} & \text{if } n \text{ is even and ground field } \mathbb{F}_q \text{ is large enough} \end{cases}$$

FIX: What is the exact root number

iemann Hypothesis) *We can pin down $P_0(T) = 1 - T$ and $P_{2n}(T) = 1 - q^{2n}T$. For $1 \le i \le 2n-1$,*

$$P_i(T) = \prod_j (1 - \alpha_i(j)T)$$

with $|\alpha_i(j)| = q^{i/2}$ for every archimedean place of $\mathbb{Q}(\alpha_i(j)) \hookrightarrow \mathbb{C}$.

1.4.3 The case of curves

In this section, we will show rationality/functional equation of the zeta function via Riemann-Roch. For the Riemann hypothesis, there is also an elementary approach due to Bombieri-Stepanov.

Divisors on curves

Let $k = \mathbb{F}_q$ and X/k be a smooth, projective, geometrically irreducible curve. We use $\overline{X} = X_{\overline{k}}$ to denote its base change to $\overline{k}$.

- A divisor
$$D = \sum_{x \in |X|} n_x \cdot x$$
is a formal finite linear combination of closed points of X, with integer coefficients n_x. An effective divisor is one where each $n_x \ge 0$ - we use the notation $D \ge 0$ to denote effectiveness.
- $\mathrm{Div}(X)$ is the set of divisors.

- The degree of a divisor $D = \sum_{x \in |X|} n_x \cdot x$ is

$$\deg(D) := \sum_{x \in |X|} n_x \cdot \deg(x)$$

- Let $k(X)$ be the field of rational functions of X over k. For $f \in k(X)$, we can define the order of zeros/poles of f at each closed point. (Smoothness gives you a local uniformizer at each closed point). Denote the order of f at closed point x by $ord_x(f)$. We can then define the principal divisors

$$\operatorname{div}(f) = \sum_{x \in |X|} ord_x(f)x$$

  Since X is projective, $\deg(\operatorname{div}(f)) = 0$ for all $f \in k(X)$.

Picard group

- We define an equivalence relation on divisors: $D \sim D'$ iff $D = D' + \operatorname{div}(f)$ for some $f \in k(X)$. The Picard group is then defined as

$$\operatorname{Pic}(X) = \operatorname{Div}(X)/\sim$$

- Degree map descends so we can define

$$\operatorname{Pic}(X) = \operatorname{Div}(X)/\sim \xrightarrow{deg} \mathbb{Z}$$

  Define $\operatorname{Pic}^0(X)$ to be the kernel of this map.

Section of line bundles

- For any divisor D on X, define

$$L(D) = \{f \in k(X) : \operatorname{div}(f) + D \geq 0\}$$

  which is a k-vector space. We also use $l(D)$ to denote the dimension of $l(D)$ as a k-vector space. Clearly $l(D) \geq 0$. It is also finite - this can be considered part of Riemann-Roch.

- Note that

$$\deg(\operatorname{div}(f) + D) = \deg(\operatorname{div}(f)) + \deg(D) = \deg(D)$$

  So if $\deg(D) < 0$, so that some coefficients of D is negative, $L(D) = \emptyset$ and $l(D) = 0$.

Riemann-Roch

Theorem 1.4.7 (Riemann-Roch + Serre Duality)**.** *There is a canonical divisor K on X such that for any divisor D on X, we have*

$$l(D) - l(K - D) = \deg(D) + 1 - g$$

where $g := l(K)$.

Corollary 1.4.8.

- $\deg(K) = 2g - 2$.
- *$l(D) \leq \deg(D) + 1 - g$ for all divisor D (Riemann's inequality), and thus is finite.*
- *If $n > 2g - 2$, then $l(D) = \deg(D) + 1 - g$.*

Corollary 1.4.9. *Implications on Picard group:*

- *For $n > 2g - 2$, each equivalence class in $\operatorname{Div}(n)/\sim$ has a representative by effective divisor. This follows from Riemann Roch.*
- *$\operatorname{Pic}^0(X)$ is finite. Note that from lemma 2.1, for fixed n there are finitely many effective divisors of degree $\leq n$. Last bullet point then implies that $\operatorname{Div}(n)/\sim$ is finite for all n large. But these are all cosets for $\operatorname{Pic}^0(X)$, hence $\operatorname{Pic}^0(X)$ is also finite.*

Rationality of zeta function of curves

Let X_0/k be a smooth, projective, geometrically irreducible curve over k. We saw that the zeta function is

$$\begin{aligned} Z(X,T) &= \prod_{x \in |X|} \left(1 - T^{\deg(x)}\right)^{-1} \\ &= \prod_{x \in |X|} (1 + T^{\deg(x)} + T^{2\deg(x)} + \cdots) \\ &= \sum_{D \geq 0} T^{\deg(D)} \\ &= \sum_{n=0}^{\infty} T^n \ \#\{\text{effective divisors of degree } n\} \qquad (\star) \end{aligned}$$

The constant term is the number of effective divisors of degree 0, which is 1.

Suppose that the degree map of Picard group maps onto $d\mathbb{Z}$, and let $\mathfrak{a}$ be a divisor of degree d. Then,

- If $d|n$, we see that $\operatorname{Div}(n)/\sim$ are cosets of $\operatorname{Pic}^0(X)$. In particular,
$$|\operatorname{Div}(n)/\sim| = |\operatorname{Pic}^0(X)|$$
- If $d \nmid n$, $\operatorname{Div}(n)/\sim$ is empty.

We will eventually show that $d = 1$, but for now, $(\star)$is

Any direct proof of this?

$$\sum_{\substack{n=0 \\ d|n}}^{\infty} T^n \ \#\{\text{effective divisors of degree } n\}$$

Note that for $n > 2g-2$ (and $d|n$), the degree n effective divisors surjects onto $\mathrm{Div}(n)/\sim$ (by our second bullet point in last section.) This means that

$$\begin{aligned}
\#\{\text{effective divisors of degree } n\} &= \sum_{D\in \mathrm{Div}(n)/\sim} \#\{\text{effective divisors of degree } n \text{ equivalent to } D\} \\
&= \sum_{D\in \mathrm{Div}(n)/\sim} |\mathbb{P}(L(D))| \\
&= \sum_{D\in \mathrm{Div}(n)/\sim} \frac{q^{l(D)}-1}{q-1}
\end{aligned}$$

Note also that $l(D) = n+1-g$ since $n > 2g-2$, and that $|\mathrm{Div}(n)/\sim| = |\operatorname{Pic}^0(X)|$

$$= |\operatorname{Pic}^0(X)|\frac{q^{n+1-g}-1}{q-1}$$

Therefore the $n > 2g-2$ part in ($\star$) is

$$\begin{aligned}
\sum_{\substack{n=2g-2+d \\ d|n}}^{\infty} T^n|\operatorname{Pic}^0(X)|\frac{q^{n+1-g}-1}{q-1} &= \frac{|\operatorname{Pic}^0(X)|}{q-1}\left(\sum_{\substack{n=2g-2+d \\ d|n}}^{\infty} q^{n+1-g}T^n - \sum_{\substack{n=2g-2+d \\ d|n}}^{\infty} T^n\right) \\
&= \frac{|\operatorname{Pic}^0(X)|}{q-1}\left(q^{1-g}\frac{(qT)^{2g-2+d}}{1-(qT)^d} - \frac{T^{2g-2+d}}{1-T^d}\right)
\end{aligned}$$

and is of the shape

$$\frac{\text{Polynomial in } T^d}{(1-T^d)(1-(qT)^d)}$$

For the $0 \le n \le 2g-2$ part of ($\star$), it is clearly a polynomial in T^d of degree at most $\frac{2g-2}{d}$. In particular, we get that

$$Z(X,T) = \frac{\text{Polynomial in } T^d}{(1-T^d)(1-(qT)^d)}$$

where the polynomial in numerator has degree at most $\frac{2g-2}{d}+2$. Notice that $Z(X,T)$ is a rational function in T^d.

We now seek more refined information about d and the numerator of $Z(X,T)$.

Claim 1. $d = 1$.

Proof. If ξ_d is a primitive d-th root of unity, recall that

$$Z(X \times_{\mathbb{F}_q} \mathbb{F}_{q^d}, T^d) = \prod_{i=1}^{d} Z(X, \xi_d^i T) = Z(X,T)^d$$

where the last equality is because $Z(X,T)$ is rational function in T^d as we have shown.

Now same proof (of rationality of zeta) shows that $Z(X \times_{\mathbb{F}_q} \mathbb{F}_{q^d}, T)$ has a pole of order 1 at $T = 1$, so same is true for $Z(X \times_{\mathbb{F}_q} \mathbb{F}_{q^d}, T^d) = Z(X,T)^d$. But this is impossible unless $d = 1$. □

So far we saw that

$$Z(X,T) = \frac{\text{Polynomial in } T}{(1-T)(1-qT)}$$

with degree of numerator at most $2g$. We now show that it is exactly $2g$.

- Contribution from $n \geq 2g-1$ term to $Z(X,T)$ is of the shape:

$$\frac{|\operatorname{Pic}^0(X)|}{q-1}\left(q^g \frac{T^{2g-1}}{1-qT} - \frac{T^{2g-1}}{1-T}\right) = \frac{|\operatorname{Pic}^0(X_0)|}{q-1} \cdot \frac{(q-q^g)T^{2g} + (q^g-1)T^{2g-1}}{(1-qT)(1-T)}$$

- Contribution from $n \leq 2g-2$ term to $Z(X,T)$ is of the shape

$$T^{2g-2}\#\{\text{effective divisors of degree } 2g-2\} = T^{2g-2} \sum_{\substack{D \in \operatorname{Div}(2g-2)/\sim \\ D \text{ effective}}} \frac{q^{l(D)}-1}{q-1}$$

- For $D \in \operatorname{Div}(2g-2)/\sim$,
  - if $D \sim K$, then $l(D) = l(K) = g$.
  - If $D \not\sim K$, then $l(D) = g-1$. This is by Riemann-Roch, and note that as $K - D$ is a divisor of degree 0 that is not equivalent to 0, we must have $l(K-D) = 0$.
- This would imply that after we clear the fraction, the leading term in the numerator of $Z(X,T)$ will not be cancelled.

Thus we conclude that

Theorem 1.4.10 (Rationality of zeta function)**.** *For a smooth, projective, geometrically irreducible curve X over $\mathbb{F}_q$, we have*

$$Z(X,T) = \frac{P_1(T)}{(1-T)(1-qT)}$$

where $P_1(T) \in \mathbb{Z}[T]$ is of degree $2g$.

We mention that functional equation can be argued in a bare-hand way along this line, while Riemann Hypothesis would be more involved.

1.5 Weil bound for curves

1.5.1 Bombieri-Stepanov

Given a smooth proper curve X over $\mathbb{F}_q$, our strategy is to count the $\mathbb{F}_q$ points of X by finding the points of $\overline{X} = X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ whose coordinates are unchanged by raising them to the qth power. Algebraically, we are looking for fixed points of frobenius. Since there are several versions of frobenius, we'll give a concrete description of the two versions of frobenius we will be using and how they are different.

Relative frobenius is defined as $F_q =$ (frobenius on X) $\times$ (identity on $\overline{\mathbb{F}}_q$), while absolute frobenius just raises everything to the pth power. What this really means, with an example:

"If $(x, y) \in \overline{X}$ satisfies $x^q = \sqrt{2}$, then $(x', y') = F_q((x, y))$ satisfies $x' = \sqrt{2}$" vs "If $(x, y) \in \overline{X}$ satisfies $x^p = \sqrt{2}^p$, then $(x', y') = (x^p, y^p)$ satisfies $x' = \sqrt{2}$."

So absolute frobenius doesn't do anything interesting other than change multiplicites of roots by multiples of p, while relative frobenius changes the coordinates of $\overline{\mathbb{F}}_q$-points of $\overline{X}$. Thus, we want to count fixed points of *relative* frobenius.

One of the main tricks in the proof of the Riemann hypothesis is based on the following Lemma, which, when combined with the rationality of the zeta function, turns asymptotic bounds with poor implicit constants into more precise bounds.

Lemma 1.5.1. *If $\alpha_1, ..., \alpha_n \in \mathbb{C}$ and $c \in \mathbb{R}^+$ are such that $\Re(\sum_{i=1}^n \alpha_i^k) = O(c^k)$, then for all i we have $|\alpha_i| \leq c$.*

Proof. Either one can apply the Pigeonhole Principle several times to show that there exist arbitrarily large integers k such that for all i, $\arg(\alpha_i) \cdot k$ is very close to an element of $2\pi\mathbb{Z}$, or alternatively one can look at the radius of convergence of the power series $\sum_{k \geq 0}(\sum_{i=1}^n \alpha_i^k) z^k = \sum_{i=1}^n \frac{1}{1-\alpha_i z}$. □

Main Idea: Suppose $Y/\mathbb{P}^1$ is Galois, that is, that $\mathbb{F}_q(Y)/\mathbb{F}_q(\mathbb{P}^1)$ is a Galois extension of fields, of degree d (so $d = |\operatorname{Gal}(Y/\mathbb{P}^1)|$). Since Y is proper and of dimension 1, every element of $g \in \operatorname{Gal}(Y/\mathbb{P}^1)$ gives a well defined regular function $g : Y \to Y$ which is defined over $\mathbb{F}_q$ (a priori, we only knew that g was a rational function). All but finitely many points $x \in \mathbb{P}^1(\overline{\mathbb{F}}_q)$ have exactly d preimages in $Y(\overline{\mathbb{F}}_q)$, and these d preimages will be permuted by $\operatorname{Gal}(Y/\mathbb{P}^1)$ (the points with less than d preimages are the "ramification points", and the number of ramification points is bounded by $2d + 2g - 2$, where g is the genus of Y). If $x \in \mathbb{P}^1(\mathbb{F}_q)$ and $y \mapsto x$ is unramified, then there exists a unique $g \in \operatorname{Gal}(Y/\mathbb{P}^1)$ such that $g(y) = F_q(y)$ (since y and $F_q(y)$ are both preimages of $x = F_q(x)$). Thus, we have

$$1 + q = |\mathbb{P}^1(\mathbb{F}_q)| = \frac{1}{|\operatorname{Gal}(Y/\mathbb{P}^1)|} \sum_{g \in \operatorname{Gal}} |\operatorname{Fix}(g^{-1} \circ F_q \text{ on } \overline{Y})| + O(2d + 2g - 2).$$

Although this is good enough for our purposes, we can actually get rid of the error term. We use the following group theoretic fact: if a group acts on a set, then the expected number of fixed points of a random element of the group is equal to the number of orbits of the action. We know that away from a finite collection of points $x \in \mathbb{P}^1(\overline{\mathbb{F}}_q)$, the group $\operatorname{Gal}(Y/\mathbb{P}^1)$ acts transitively on the preimages of x, and the set of points x such that the action on the preimages of x is *not* transitive is easily seen to be open, so it must be empty. Thus, the number of orbits of the action of $\operatorname{Gal}(Y/\mathbb{P}^1)$ on the set of preimages of any $x \in \mathbb{P}^1(\overline{\mathbb{F}}_q)$ is always 1, so

$$1 + q = \frac{1}{|\operatorname{Gal}(Y/\mathbb{P}^1)|} \sum_{g \in \operatorname{Gal}} |\operatorname{Fix}(g^{-1} \circ F_q \text{ on } \overline{Y})|.$$

The upshot of all this is that if we can get good upper bounds on $|\operatorname{Fix}(g^{-1} \circ F_q \text{ on } \overline{Y})|$ for any $g \in \operatorname{Gal}(Y/\mathbb{P}^1)$, then we can get decent lower bounds on the same quantity by applying the upper bounds to $|\operatorname{Fix}(h^{-1} \circ F_q \text{ on } \overline{Y})|$ for $h \neq g$ (the error terms will get multiplied by $|\operatorname{Gal}| - 1$ in the process).

For general $X/\mathbb{P}^1$, we let Y be the Galois closure of X over $\mathbb{P}^1$. Let $G = \operatorname{Gal}(Y/\mathbb{P}^1)$, and let $H = \operatorname{Gal}(Y/X)$. Then a similar argument to the above gives us the formula

$$|X(\mathbb{F}_q)| = \frac{1}{|H|} \sum_{h \in H} |\operatorname{Fix}(h^{-1} \circ F_q \text{ on } \overline{Y})|.$$

Then good upper and lower bounds for $|\operatorname{Fix}(h^{-1} \circ F_q \text{ on } \overline{Y})|$ give us good upper and lower bounds for $|X(\mathbb{F}_q)|$.

Theorem 1.5.2 (Bombieri-Stepanov)**.** *Suppose X is a proper smooth curve over $\mathbb{F}_q$ of genus g, and let $g \in \operatorname{Aut}(X/\mathbb{F}_q)$. Set $\varphi = g^{-1} \circ F_q$. Assume that $q = p^\alpha$, with α even, and that $q > (g+1)^4$. Then*

$$|\operatorname{Fix}(\varphi \text{ on } \overline{X})| \leq 1 + q + (2g+1)\sqrt{q}.$$

Proof. The general strategy is to show that there is a nonzero function of low degree which vanishes at every fixed point of φ, and we will produce such a function by doing a dimension count, using the fact that the collection of pth powers of functions forms a vector space. Suppose there is some $x_0 \in \operatorname{Fix}(\varphi)$ (if there is no such x_0 then we are done). We will treat x_0 as the "point at infinity" on X, study functions on X which only have poles at x_0, and measure the degree of such a function by the order of its pole at x_0. Formally, we set

$$L_m = \Gamma(\overline{X}, \mathcal{O}_{\overline{X}}(mx_0)) \subseteq \overline{\mathbb{F}}_q(X),$$

so L_m is the collection of functions of degree at most m which only have poles at x_0, and we let $l_m = \dim L_m$. Recall that Riemann-Roch implies that

$$m + 1 - g \leq l_m \leq m + 1,$$

and that $l_m = m + 1 - g$ if $m > 2g - 2$. We'll also set

$$L_m^\varphi = \{f \circ \varphi \mid f \in L_m\}, L_l^{p^\mu} = \{f^{p^\mu} \mid f \in L_l\},$$

the images of L_m and L_l under composition with φ and powers of absolute frobenius, respectively. Since g is an automorphism and F_q has order q, we have

$$L_m \xrightarrow[\approx]{\varphi} L_m^\varphi \hookrightarrow \Gamma(\overline{X}, \mathcal{O}_{\overline{X}}(mqx_0)) = L_{mq}, \;\; L_l \xrightarrow[\approx]{} L_l^{p^\mu} \subseteq L_{lp^\mu}.$$

Lemma 1.5.3. *If $lp^\mu < q$, then $L_l^{p^\mu} \otimes_{\overline{\mathbb{F}}_q} L_m^\varphi \to L_{lp^\mu + mq}$ is injective.*

Proof. Look at the Laurent expansion at x_0. □

Corollary 1.5.4. *If $lp^\mu < q$, there exists a well-defined map $\delta : L_l^{p^\mu} \cdot L_m^\varphi \to L_l^{p^\mu} \cdot L_m \subseteq L_{m+lp^\mu}$, given by*

$$\delta : \sum_i g_i^{p^\mu} \cdot (f_i \circ \varphi) \mapsto \sum_i g_i^{p^\mu} f_i.$$

If $l_m l_l > l_{m+lp^\mu}$, then $\ker \delta \neq 0$.

Suppose that $lp^\mu < q, l_m l_l > l_{m+lp^\mu}$, and let $f = \sum_i g_i^{p^\mu} \cdot (f_i \circ \varphi) \neq 0$ be in the kernel of δ. From $lp^\mu < q$, we see that f is a p^μth power, and for $x \in \operatorname{Fix}(\varphi), x \neq x_0$, we have

$$f(x) = \sum_i g_i(x)^{p^\mu} f_i(\varphi(x)) = \sum_i g_i(x)^{p^\mu} f_i(x) = 0,$$

so

$$p^\mu(|\operatorname{Fix}(\varphi)| - 1) \leq \#\text{zeroes of } f \leq lp^\mu + mq,$$

since every root of f occurs with multiplicity at least p^μ. Dividing by p^μ, this becomes

$$|\operatorname{Fix}(\varphi)| \le l + m\frac{q}{p^\mu} + 1.$$

Now we just need to choose values of l, m, μ in order to get a good bound. We take $p^\mu = \sqrt{q}, m = \sqrt{q} + 2g, l = g + 1 + \lfloor \frac{g}{g+1}\sqrt{q}\rfloor$:

- $lp^\mu < q$ is the same as $l < \sqrt{q}$, which follows from $g + 1 < \frac{\sqrt{q}}{g+1}$.
- To check that $l_l l_m > l_{m+lp^\mu}$, note that $l_l \ge l+1-g, l_m \ge m+1-g$, and $l_{m+lp^\mu} = m+lp^\mu+1-g$, so we just need to check that $(l-g)(m+1-g) > lp^\mu = l\sqrt{q}$, or equivalently $l(m+1-q-\sqrt{q}) > g(m+1-g)$. Simplifying, this becomes $l(g+1) > g(\sqrt{q}+g+1)$, or $l > \frac{g}{g+1}\sqrt{q} + g$.
- Finally, we get

$$\begin{aligned}|\operatorname{Fix}(\varphi)| &\le g + 1 + \lfloor \frac{g}{g+1}\sqrt{q}\rfloor + \sqrt{q}(\sqrt{q}+2g) + 1\\ &\le q + (2g+1)\sqrt{q} + 1 - (\tfrac{\sqrt{q}}{g+1} - (g+1)).\end{aligned}$$ □

1.5.2 Improvements to the Weil bound

This section follows Schoof's exposition [171]. Recall that for a proper smooth curve $X/\mathbb{F}_q$ of genus g, the zeta function attached to X is rational, of the form

$$Z(X,T) = \frac{P_X(T)}{(1-T)(1-qT)},$$

where $P_X(T)$ has integral coefficients and constant term 1, and $Z(X,T)$ satisfies the functional equation

$$Z\Big(X, \frac{1}{qT}\Big) = q^{1-g}T^{2-2g}Z(X,T).$$

Thus the leading coefficient of $P_X(T)$ is q^g, and

$$P_X(T) = \prod_{i=1}^{2g}(1-\alpha_i T)$$

for some algebraic integers α_i. From the functional equation, we see that for factor $1-\alpha_i T$ of $P_X(T)$ there must be a corresponding factor $1 - \frac{q}{\alpha_i}T$ with the same multiplicity. Together with the fact that there are an even number of α_is and that their product is q^g, we see that we can arrange the α_is such that $\alpha_{g+i} = \frac{q}{\alpha_i}$ for $i = 1, ..., g$. The Riemann Hypothesis for X (proved in the last subsection) then gives us $|\alpha_i| = \sqrt{q}$, so $\alpha_{g+i} = \overline{\alpha}_i$. This gives us the following **explicit formula**, valid for all $d \in \mathbb{N}$:

$$|X(\mathbb{F}_{q^d})| = q^d + 1 - \sum_{i=1}^{g}(\alpha_i^d + \overline{\alpha}_i^d),$$

where each α_i is an algebraic integer such that the absolute value of any conjugate of α_i is $\sqrt{q}$.

Theorem 1.5.5 (Hasse-Weil-Serre)**.** *$|X(\mathbb{F}_q)| \le q + 1 + \lfloor 2\sqrt{q}\rfloor g$.*

Proof. Set $x_i = \lfloor 2\sqrt{q} \rfloor + 1 + \alpha_i + \overline{\alpha}_i$. Then each x_i is a totally positive algebraic integer, so $\prod_{i=1}^g x_i \geq 1$, and then by the AM-GM inequality we have $\sum_{i=1}^g x_i \geq g$. □

When the genus is very large compared to q, strange things start to happen. In this case, the lower bound on the number of points becomes trivial, and the upper bound becomes much smaller than expected. The following bound becomes better than the Weil bound once $g \geq \frac{q-\sqrt{q}}{2}$.

Theorem 1.5.6 (Ihara)**.** $|X(\mathbb{F}_q)| \leq q + 1 + \Big(\sqrt{2q + \frac{1}{4} + \frac{q^2-q}{g}} - \frac{1}{2}\Big)g$.

Proof. Set $t_i = \alpha_i + \overline{\alpha}_i$. Then by the explicit formula,

$$|X(\mathbb{F}_q)| \leq |X(\mathbb{F}_{q^2})| = q^2 + 1 - \sum_{i=1}^g (\alpha_i^2 + \overline{\alpha}_i^2) = q^2 + 1 + 2qg - \sum_{i=1}^g t_i^2,$$

and by the Cauchy-Schwarz inequality the right hand side is

$$\leq q^2 + 1 + 2qg - \frac{1}{g}\Big(\sum_{i=1}^g t_i\Big)^2 = q^2 + 1 + 2qg - \frac{1}{g}\Big(|X(\mathbb{F}_q)| - q - 1\Big)^2.$$

Rearranging and multiplying by g, we have

$$\Big(|X(\mathbb{F}_q)| - q - 1\Big)^2 + g\Big(|X(\mathbb{F}_q)| - q - 1\Big) \leq (q^2 - q)g + 2qg^2,$$

and completing the square finishes the proof. □

Oesterlé Method: Set $\omega_i = \frac{\alpha_i}{\sqrt{q}}$, so $|\omega_i| = 1$. Then from the explicit formula, we get

$$|X(\mathbb{F}_q)|q^{-\frac{d}{2}} \leq |X(\mathbb{F}_{q^d})|q^{-\frac{d}{2}} = q^{\frac{d}{2}} + q^{-\frac{d}{2}} - \sum_{i=1}^g (\omega_i^d + \overline{\omega}_i^d).$$

Multiplying these inequalites by nonnegative constants $c_1, c_2, ...$ and adding them together, we get

$$|X(\mathbb{F}_q)|\lambda(q^{-\frac{d}{2}}) \leq \lambda(q^{\frac{d}{2}}) + \lambda(q^{-\frac{d}{2}}) - \sum_{i=1}^g (\lambda(\omega_i) + \lambda(\overline{\omega}_i)),$$

where

$$\lambda(t) = \sum_{d=1}^\infty c_d t^d.$$

Letting $f(t) = 1 + \lambda(t) + \lambda(\frac{1}{t})$, we see that as long as the c_ds are chosen such that $f(t) \geq 0$ for all t with $|t| = 1$, then we have

$$|X(\mathbb{F}_q)|\lambda(q^{-\frac{d}{2}}) \leq \lambda(q^{\frac{d}{2}}) + \lambda(q^{-\frac{d}{2}}) + g.$$

Theorem 1.5.7 (Drinfeld-Vlădut)**.** $\limsup_{g\to\infty} \frac{|X(\mathbb{F}_q)|}{g} \leq \sqrt{q} - 1$*, that is,* $|X(\mathbb{F}_q)| \leq (\sqrt{q}-1)g + o(g)$ *when q is fixed and g goes to infinity.*

Proof. We want to apply Oesterlé's method with the c_ds as large as possible, in order to maximize $\lambda(q^{-\frac{d}{2}})$. From

$$1 - c_d = \frac{1}{\pi}\int_0^{2\pi} f(e^{i\theta})(1-\cos(n\theta))d\theta \geq 0,$$

we see that each c_d is ≤ 1. If we take

$$f(t) = \frac{1}{N+1}(1+t+\cdots+t^N)(1+t^{-1}+\cdots+t^{-N}),$$

then we see that $f(t) \geq 0$ whenever $|t| = 1$, and $f(t) = 1 + \sum_{d=1}^{N}\frac{N+1-d}{N+1}(t^d + t^{-d})$ gives $c_d = \frac{N+1-d}{N+1} \geq 0$ for $1 \leq d \leq N$. Taking N to ∞, each c_d tends to 1 from below, and

$$\lim_{N\to\infty}\lambda(q^{-\frac{1}{2}}) = q^{-\frac{1}{2}} + q^{-1} + q^{-\frac{3}{2}} + \cdots = \frac{1}{\sqrt{q}-1}. \qquad \square$$

1.6 Dwork's proof of rationality of the zeta function

In this section we follow Dwork's paper [71].

1.6.1 Motivation

Recall the Chevalley-Warning trick:

$$\#\{(x,y)\in\mathbb{F}_p^2 \mid f(x,y)=0\} \equiv \sum_{x,y\in\mathbb{Z}/p}(1-f(x,y)^{p-1}) \pmod{p},$$

$$\sum_{x\in\mathbb{Z}/p} x^i \equiv \begin{cases} 0 & (p-1)\nmid i \text{ or } i=0, \\ -1 & (p-1)\mid i, i>0 \end{cases} \pmod{p}.$$

Together with a crude bound on the number of points, this congruence was often enough to give us an exact point count. We would like to generalize this approach in order to compute zeta functions, so we have to generalize in two different directions at once:

- we need to find a way to count points in $\mathbb{F}_{p^s}$, $s > 1$, and
- we need to find a way to get pount counts modulo p^k, $k > 1$.

Towards the second bullet point, we are lead to wonder what the value of

$$\sum_{x=0}^{p-1} x^i \pmod{p^2}$$

is. While this is a hard question, if we instead look at the sum

$$\sum_{x=0}^{p-1}(x^p)^i \pmod{p^2}$$

it becomes much easier! Generalizing this observation, we see that we want to work with Teichmüller lifts, which are given by

$$[x] = \lim_{n\to\infty} x^{p^n},$$

where the limit is taken p-adically (if x is an integral element of $\overline{\mathbb{Q}}_p$ or $\mathbb{C}_p$, then the limit should be taken over the net of positive integers ordered by divisibility). Teichmüller lifts are always either 0 or roots of unity, we have $[xy] = [x][y]$, and $x \equiv [x] \pmod p$ whenever $|x|_p = 1$. Because of that last point, we can think of Techmüller lifts as a function from $\overline{\mathbb{F}}_p$ to $\overline{\mathbb{Z}}_p$.

Now for the first bullet point: how will we get point counts in $\mathbb{F}_{p^s}$? The strategy is to use either additive or multiplicative characters (Dwork tried both approaches: multiplicative characters almost worked, while additive characters worked perfectly). We will need to have certain compatibilities between our characters for different powers of p. Recall that for complex characters, we made the definitions

$$\psi_q(a) = e^{\frac{2\pi i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(a)}{p}}, \quad \chi_q(a) = \chi(\operatorname{Nm}^{\mathbb{F}_q}_{\mathbb{F}_p}(a)),$$

giving

$$\psi_q(a) = \psi(a + a^p + \cdots + a^{p^{s-1}}) \text{ “} = \psi(a)\psi(a^p)\cdots\psi(a^{p^{s-1}})\text{”}.$$

Dwork's idea is to find a p-adic power series $\theta(x)$ such that for $x \in \mathbb{F}_{p^s}$ we have

$$\zeta_p^{\operatorname{Tr}(x)} = \theta([x])\theta([x]^p)\cdots\theta([x]^{p^{s-1}}),$$

where ζ_p is a primitive pth root of unity in $\overline{\mathbb{Q}}_p$. Then we can evaluate sums of additive characters at points in $\mathbb{F}_{p^s}$ by turning them into sums of power series evaluated at (p^s-1)th roots of unity in $\mathbb{C}_p$.

1.6.2 Combining p-adic congruences with inequalities

Knowing point counts of varieties modulo powers of p is great, but how will we eventually use this to prove that $Z(V,T)$ is rational? Recall that a power series is a rational function if and only if its coefficients eventually satisfy some linear recurrence relation. Thus, our strategy is as follows:

- We know that $Z(V,T)$ is a power series with integer coefficients.
- Trivial bounds on the point counts show that $Z(V,T)$ has a nontrivial radius of convergence, so its coefficients are not too big.
- Since the coefficients are small, if they satisfy a recurrence modulo large enough powers of p, then they actually satisfy that recurrence over the integers.

To make this last bullet point precise, we have the following Lemma, from Chapter 13 of [52].

Lemma 1.6.1. *Suppose that $f(x) = \sum_{i\ge0} f_i x^i$ is a power series with coefficients in some field. Then f is a rational function if and only if there exists some $l \ge 0$ such that for all sufficiently large n, we have*

$$\det\begin{pmatrix} f_n & f_{n+1} & \cdots & f_{n+l} \\ f_{n+1} & f_{n+2} & \cdots & f_{n+l+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+l} & f_{n+l+1} & \cdots & f_{n+2l} \end{pmatrix} = 0.$$

Proof. Let $F(n,l)$ be the determinant corresponding to n and l, so that $F(n,0) = f_n$ and $F(n,1) = f_n f_{n+2} - f_{n+1}^2$, etc.

Suppose first that $f(x)$ is rational, with $f(x) = \frac{p(x)}{q(x)}$, $q(x) = q_l x^l + \cdots + q_1 x + 1$. Then since $f(x)q(x) = p(x)$, we see that

$$f_{k+l} = -q_1 f_{k+l-1} - \cdots - q_l f_k$$

for all $k > \deg p$. Plugging in $k = n, n+1, ..., n+l$, we see that the rigtmost column of the matrix corresponding to n and l is a linear combination of the remaining columns, so the determinant is 0 for all $n > \deg p$.

Now suppose that $l \geq 0$ is chosen to be minimal such that $F(n,l) = 0$ for all sufficiently large n. If $l = 0$ then f is a polynomial and we are done. Using the determinant identity (for a proof of this identity, see [52]: apparently it is a special case of "Jacobi's Theorem on the minors of the adjugate")

$$F(n,l-1)F(n+2,l-1) - F(n+1,l-1)^2 = F(n,l)F(n+2,l-2),$$

we see that for n sufficiently large we have $F(n,l-1)F(n+2,l-1) = F(n+1,l-1)^2$, so the sequence $F(n,l-1)$ is eventually a geometric progression, and so by the minimality of l we must have $F(n,l-1) \neq 0$ for all sufficiently large n.

Thus, for n sufficiently large the matrix corresponding to n and l has rank exactly l (and the first l columns are independent), so there is a unique tuple $q_{1,n}, ..., q_{l,n}$ such that

$$f_{k+l} + q_{1,n} f_{k+l-1} + \cdots + q_{l,n} f_k = 0$$

for $k = n, ..., n+l$. Comparing this system of equations for n and $n+1$, and using the fact that the last l rows of the matrix corresponding to n and l are the same as the first l rows of the matrix corresponding to $n+1$ and l and that these rows are independent, we see that in fact the $q_{i,n}$ are independent of n, so we can write $q_{i,n} = q_i$. Setting $q(x) = q_l x^l + \cdots + q_1 x + 1$, we see that $f(x)q(x)$ is a polynomial $p(x)$, so $f(x) = \frac{p(x)}{q(x)}$ is a rational function, and we are done. □

Recall that a power series $f(x)$ is *meromorphic in a disc of radius R* if and only if there exists a nonzero polynomial $p(x)$ such that the power series $f(x)p(x)$ converges everywhere in the disc of radius R. We also say that a power series is *meromorphic* if it can be written as a ratio of two entire functions, i.e. two power series which converge everywhere. These definitions are compatible since every entire function has only finitely many roots inside any disc. We can make entirely analogous definitions for p-adic meromorphic functions, and this time the compatibility between the definitions relies on a result known as the Weierstrass preparation theorem:

Proposition 1.6.2 (Weierstrass preparation theorem)**.** *Let $f(x) = \sum_i f_i x^i \in \mathbb{C}_p[\![x]\!]$ with $|f_n|_p \to 0$. Let N be defined by $|f_N|_p = \max |f_n|_p$ and $|f_N|_p > |f_n|_p$ for all $n > N$. Then there is a polynomial $g(x) = g_0 + \cdots + g_N x^N \in \mathbb{C}_p[x]$ and a power series $h(x) = 1 + h_1 x + \cdots \in \mathbb{C}_p[\![x]\!]$ with $|h_n|_p \to 0$ such that $f(x) = g(x)h(x)$, $|g_N|_p = \max |g_n|_p$, and $|h_n|_p < 1$ for $n > 0$.*

The proof of the Weierstrass preparation theorem is similar to the proof of Hensel's lemma: first you find a solution that works modulo a small power of p, and then show that you can modify it to make it work modulo larger powers of p (for this last step, you use the division lemma for polynomials).

Theorem 1.6.3 (Borel, Dwork). *Let $f(x) \in \mathbb{Z}[\![x]\!]$ be meromorphic in a disc of radius R_∞, and p-adically meromorphic in a disc of radius R_p. If $R_pR_\infty > 1$, then f is a rational function.*

Proof. Let $g_\infty \in \mathbb{C}[x], g_p \in \mathbb{C}_p[x]$ be polynomials with constant term 1 such that $h_\infty(x) = g_\infty(x)f(x)$ converges in a disc of radius R_∞ and $h_p(x) = g_p(x)f(x)$ converges p-adically in a disc of radius R_p (note that even though f had integral coefficients, g_∞ and g_p might have transcendental coefficients). For $v = p, \infty$ define $t_v, T_v > 0$ such that $T_vR_v > 1$, $T_pT_\infty < 1$, and such that t_∞ is more than the inverse of the radius of convergence of f and t_p is more than the inverse of the p-adic radius of convergence of f, so that $|f_n|_v \ll t_v^n$ and $|h_n|_v \ll T_v^n$. Suppose also that both g_∞, g_p have degree bounded by m.

Since $T_pT_\infty < 1$, by choosing l sufficiently large we can ensure that

$$(t_pt_\infty)^m(T_pT_\infty)^{l+1-m} < 1.$$

Fix such an l with $l \geq m$, and let $F(n, l)$ be the determinant

$$\det \begin{pmatrix} f_n & f_{n+1} & \cdots & f_{n+l} \\ f_{n+1} & f_{n+2} & \cdots & f_{n+l+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+l} & f_{n+l+1} & \cdots & f_{n+2l} \end{pmatrix}$$

considered in the earlier lemma. Then we can replace the fs in the rows after the mth row with the corresponding hs without changing the determinant, by using the recurrences implied by $h(x) = f(x)g(x)$. This gives us the bounds

$$|F(n, l)| \ll (t_\infty^n)^m(T_\infty^n)^{l+1-m}$$

and

$$|F(n, l)|_p \ll (t_p^n)^m(T_p^n)^{l+1-m},$$

where the implied constants depend on l, t_v, T_v, and on the implied constants in the bounds $|f_n|_v \ll t_v^n$ and $|h_n|_v \ll T_v^n$. Combining these, we get

$$|F(n, l)||F(n, l)|_p \ll ((t_pt_\infty)^m(T_pT_\infty)^{l+1-m})^n,$$

and for n sufficiently large the right hand side goes to 0. Since $F(n, l)$ is an integer, the only way to have $|F(n, l)||F(n, l)|_p < 1$ is to have $F(n, l) = 0$. Thus for all sufficiently large n, we have $F(n, l) = 0$, and so $f(x)$ must be a rational function (with denominator of degree at most l). □

Dwork's strategy for proving the rationality of $Z(V, T)$ is now to show that $Z(V, T)$ extends to a p-adic meromorphic function on all of $\mathbb{C}_p$, so we will be able to take R_p as large as we like in the previous theorem.

1.6.3 Summing over roots of unity

We will need to have convenient formulas for sums of the form

$$\sum_{x_1,...,x_n \in \mathbb{F}_{q^s}^\times} F([x_1], ..., [x_n])F([x_1]^q, ..., [x_n]^q) \cdots F([x_1]^{q^{s-1}}, ..., [x_n]^{q^{s-1}}),$$

where F is a power series in n variables, say $F(\vec{x}) = \sum_{\vec{u}} F_{\vec{u}} \vec{x}^{\vec{u}} \in \mathbb{C}_p[\![x]\!]$, with radius of convergence strictly greater than 1.

Since summing over $(q^s - 1)$th roots of unity picks out the monomials with coefficients divisible by $(q^s - 1)$, it's almost natural to define the operator ψ by

$$\psi(\vec{x}^{\vec{u}}) = \begin{cases} \vec{x}^{\frac{\vec{u}}{q}} & q \mid \vec{u}, \\ 0 & q \nmid \vec{u}, \end{cases}$$

which one might call the "left inverse of Frobenius" (it seems at first that powers of ψ will always be "off by one" from what we really want, but this will actually work out nicely later on). What we really care about is not ψ, but the operator $\psi \circ F$, which acts on $\mathbb{C}_p[\![x]\!]$ as follows: first you multiply by F, then you apply ψ to the result of that multiplication. The (infinite) matrix M of this action is given by

$$M_{\vec{u},\vec{v}} = \text{coefficient of } \vec{x}^{\vec{u}} \text{ in } \psi(\vec{x}^{\vec{v}} F(\vec{x})) = F_{q\vec{u}-\vec{v}}.$$

Example 1.6.1. If we take $p = 2$ and $F(x) = \frac{1}{1-2x} = 1 + 2x + 4x^2 + \cdots$, then we get

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 4 & 2 & 1 & 0 & 0 & \cdots \\ 16 & 8 & 4 & 2 & 1 & \cdots \\ 64 & 32 & 16 & 8 & 4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Note that modulo p^n, F is congruent to a polynomial, so M is eventually strictly upper triangular, and therefore $\operatorname{Tr} M^s$ and $\det(1 - tM)$ make sense modulo p^n. We will also write $\operatorname{Tr}(\psi \circ F)^s$ and $\det(1 - t(\psi \circ F))$ for these two quantities.

Lemma 1.6.4. *If $F(\vec{x}) = \sum_{\vec{u}} F_{\vec{u}} \vec{x}^{\vec{u}} \in \mathbb{C}_p[\![x]\!]$ has coefficients going to 0, then for all $s \geq 1$ we have*

$$(q^s - 1)^n \operatorname{Tr}(\psi \circ F)^s = \sum_{\vec{x} \in \mu_{q^s-1}^n} F(\vec{x}) F(\vec{x}^q) \cdots F(\vec{x}^{q^{s-1}}),$$

where μ_{q^s-1} is the set of $(q^s - 1)$th roots of unity in $\mathbb{C}_p$.

Proof. Since $(\psi \circ F)^s = \psi^s \circ (F(\vec{x}) F(\vec{x}^q) \cdots F(\vec{x}^{q^{s-1}}))$, it's enough to prove it for $s = 1$. When $s = 1$ we see that the trace of $\psi \circ F$ is the sum over $\vec{u}$ of $F_{q\vec{u}-\vec{u}} = F_{(q-1)\vec{u}}$, and each of monomial $\vec{x}^{(q-1)\vec{u}}$ contributes $(q-1)^n$ to the sum on the right. □

For the next lemma, we define the *weight* of a vector $\vec{u}$, written $\operatorname{wt}(\vec{u})$, to be the sum of the coefficients of $\vec{u}$. Thus, if $\vec{u}$ is an exponent vector, then $\operatorname{wt}(\vec{u})$ is the total degree.

Lemma 1.6.5. *If there is a constant $c > 0$ such that $v_p(F_{\vec{u}}) \geq c\operatorname{wt}(\vec{u})$ for all $\vec{u} \in \mathbb{N}^n$, then if $M_{\vec{u},\vec{v}} = F_{q\vec{u}-\vec{v}}$ is the matrix of $\psi \circ F$, we have the power series identity*

$$\exp\left(\sum_{s=1}^{\infty} \frac{t^s \operatorname{Tr} M^s}{s}\right) = \frac{1}{\det(1 - tM)}$$

and $\det(1 - tM)$ is entire.

Proof. The identity follows from the corresponding fact for finite dimensional matrices by considering both sides modulo powers of p and the fact that modulo any power of p, M is eventually strictly upper triangular.

Now write $\det(1-tM) = \sum_{m\ge 0} d_m t^m$. By the definition of the determinant in terms of sums of products over permutations, we have

$$\begin{aligned} v_p(d_m) &\ge \min_{\{\vec{u}_1,\ldots,\vec{u}_m\}=\{\vec{v}_1,\ldots,\vec{v}_m\}} \sum_i v_p(F_{q\vec{u}_i-\vec{v}_i}) \\ &\ge c \min_{\{\vec{u}_1,\ldots,\vec{u}_m\}} (\mathrm{wt}(\vec{u}_1)+\cdots+\mathrm{wt}(\vec{u}_m))(q-1) \\ &\gg m^{1+\frac{1}{n}}, \end{aligned}$$

where the last inequality follows from the fact that the number of vectors in $\mathbb{N}^n$ of total weight less than $\frac{1}{2}m^{\frac{1}{n}}$ is at most $\frac{1}{2}m$ for m sufficiently large. Thus p-adic absolute values of the coefficients d_m of $\det(1-tM)$ go to zero faster than any r^m with $r>0$, so $\det(1-tM)$ is entire. □

1.6.4 The additive character as a power series

We need to construct a power series $\theta(x) \in \mathbb{Z}_p[\![x]\!]$ such that

- if $x \in \mathbb{F}_{p^s}$, then
$$\zeta_p^{\mathrm{Tr}(x)} = \prod_{i=0}^{s-1} \theta([x]^{p^i}),$$
where ζ_p is a primitive pth root of unity in $\mathbb{C}_p$, and
- $\theta(x) = \sum_{m\ge 0} \beta_m x^m$ with $v_p(\beta_m) \gg m$.

Let's fix some notation for talking about elements of $\mathbb{C}_p$ such as ζ_p while making as few "choices" as possible. Start by taking your favorite quadratic nonresidue $\alpha \in \mathbb{F}_p^\times \backslash (\mathbb{F}_p^\times)^2$ for $p \ne 2$ (for instance, maybe your favorite is just the least quadratic nonresidue modulo p), and choose a square root $\sqrt{a}$ in $\mathbb{F}_{p^2}$ (this time the choice doesn't really matter). Set $\pi = [\sqrt{\alpha}]p^{\frac{1}{p-1}}$, and if $p=2$ set $\pi=-2$, so that

$$\pi^{p-1} = -p.$$

Then we choose ζ_p to be the primitive pth root of unity satisfying

$$\zeta_p \equiv 1 + \pi + \frac{\pi^2}{2} + \cdots + \frac{\pi^{p-1}}{(p-1)!} \pmod{\pi p}.$$

(Try working out more terms of the expansion of ζ_p in powers of π yourself! It's fun.)

Define a power series $E(x)$, called the *Artin-Hasse exponential*, by

$$E(x) = \exp\Big(\sum_{j\ge 0} \frac{x^{p^j}}{p^j}\Big)$$

for $|x|_p < \frac{1}{p^{\frac{1}{p-1}}}$ (although we will soon show that the power series for $E(x)$ converges for all x with $|x|_p < 1$, the equality above will no longer be valid for x with $|x|_p \geq \frac{1}{p^{\frac{1}{p-1}}}$ - this subtlety has several counterintuitive consequences). Since

$$\frac{E(x)^p}{E(x^p)} = \exp(px) \in 1 + px\mathbb{Z}_p[\![x]\!],$$

we have $E(x) \in \mathbb{Z}_p[\![x]\!]$ by the following easy result:

Lemma 1.6.6 (Dwork's Lemma). *If $f(x) = 1 + f_1x + \cdots \in \mathbb{Q}_p[\![x]\!]$, then $f(x) \in \mathbb{Z}_p[\![x]\!]$ if and only if $\frac{f(x)^p}{f(x^p)} \in 1 + px\mathbb{Z}_p[x]$.*

In the power series identity

$$E(x)^p = \exp\Big(p\sum_{j\geq 0}\frac{x^{p^j}}{p^j}\Big),$$

both sides converge for $|x|_p < 1$, so this identity is valid for all such x.

Let $\eta \in \mathbb{Z}_p[\pi]$ be the unique solution to $E(\eta) = \zeta_p$. One can easily check from the power series expansion of $E(x)$ and the displayed identity above that η exists and is unique, and that we have

$$\eta \equiv \pi \pmod{\pi p^{p-1}}.$$

Finally, define $\theta(x)$ by

$$\theta(x) = E(\eta x).$$

Note that since $E(x) \in \mathbb{Z}_p[\![x]\!]$ and $v_p(\eta) = \frac{1}{p-1}$, if we write $\theta(x) = \sum_{m\geq 0}\beta_m x^m$ then we have $v_p(\beta_m) \geq \frac{m}{p-1}$.

Lemma 1.6.7. *For $x \in \mathbb{F}_{p^s}$, we have*

$$\zeta_p^{\mathrm{Tr}(x)} = \prod_{i=0}^{s-1}\theta([x]^{p^i}).$$

Proof. First we check that the right hand side is a pth root of unity:

$$\Big(\prod_{i=0}^{s-1}\theta([x]^{p^i})\Big)^p = \exp\Big(p\sum_{i=0}^{s-1}\sum_{j\geq 0}\frac{[x]^{p^{i+j}}\eta^{p^j}}{p^j}\Big),$$

and since $[x]^{p^s} = [x]$, we can rewrite the right hand side as

$$\exp\Big(p\sum_{i=0}^{s-1}[x]^{p^i}\sum_{j\geq 0}\frac{\eta^{p^j}}{p^j}\Big).$$

Now, we have $\sum_{i=0}^{s-1}[x]^{p^i} \in \mathbb{Z}_p$ since it is preserved by frobenius and is an integral element of an unramified extension of $\mathbb{Q}_p$, so we can write it as the limit of a sequence of elements of $\mathbb{N}^+$. Since for any element $n \in \mathbb{N}^+$ we have

$$\exp\Big(pn\sum_{j\geq 0}\frac{\eta^{p^j}}{p^j}\Big) = E(\eta)^{pn} = \zeta_p^{pn} = 1,$$

we see that $\prod_{i=0}^{s-1}\theta([x]^{p^i})$ is a pth root of unity. In order to figure out which pth root of unity it is, we just have to compute it modulo π^2:

$$\prod_{i=0}^{s-1}\theta([x]^{p^i}) \equiv \prod_{i=0}^{s-1}(1+\pi[x]^{p^i}) \equiv 1+\pi\operatorname{Tr}(x) \equiv \zeta_p^{\operatorname{Tr}(x)} \pmod{\pi^2}. \qquad \square$$

For general q, we define $\theta_q(x)$ by

$$\theta_q(x) = \theta(x)\theta(x^p)\cdots\theta(x^{\frac{q}{p}}).$$

It turns out that there is actually more than one power series $\theta(x)$ with the properties given above. Dwork's original construction from [71] was the much more complicated:

$$(1+(\zeta_p-1))^x\prod_{j\geq 1}(1+(\zeta_p-1)^{p^j})^{\frac{x^{p^j}-x^{p^{j-1}}}{p^j}},$$

where $(1+y)^x$ was defined to be the binomial series $1+xy+\frac{x(x-1)}{2}y+\cdots$ (proving that this infinite product has a large radius of convergence involved a two-variable version of Dwork's Lemma). Another power series which works is given by $\exp(\pi(x-x^p))$ (is this the same as $\theta(x)$?).

1.6.5 Counting points on hypersurfaces

Let V be the affine hypersurface given by

$$V = \{(x_1,...,x_n)\in\mathbb{A}^n_{\mathbb{F}_q} \mid f(x_1,...,x_n)=0, x_1\cdots x_n\neq 0\},$$

and let $N_s = |V(\mathbb{F}_{q^s})|$. Our goal is to compute

$$Z(V,T) = \exp\Big(\sum_{s\geq 1}\frac{T^sN^s}{s}\Big).$$

We have

$$q^sN_s = \sum_{\vec{x}\in(\mathbb{F}_{q^s}^\times)^n}\sum_{x_0\in\mathbb{F}_{q^s}}\zeta_p^{\operatorname{Tr}(x_0f(\vec{x}))} = (q^s-1)^n + \sum_{(x_0,\vec{x})\in(\mathbb{F}_{q^s}^\times)^{n+1}}\zeta_p^{\operatorname{Tr}(x_0f(\vec{x}))}.$$

Suppose that $x_0f(\vec{x}) = \sum_{\vec{u}\in\mathbb{N}^{n+1}}a_{\vec{u}}(x_0,\vec{x})^{\vec{u}}$, then since $[a_{\vec{u}}]^{q^i} = [a_{\vec{u}}]$ for all i (since $a_{\vec{u}}\in\mathbb{F}_q$), we have

$$q^sN_s = (q^s-1)^n + \sum_{\vec{x}\in(\mathbb{F}_{q^s}^\times)^{n+1}}\prod_{\vec{u}}\prod_{i=0}^{s-1}\theta_q([a_{\vec{u}}][\vec{x}^{\vec{u}}]^{q^i}).$$

Set $F(\vec{x}) = \prod_{\vec{u}}\theta_q([a_{\vec{u}}]\vec{x}^{\vec{u}})$. If we write $F(\vec{x}) = \sum_{\vec{u}}F_{\vec{u}}\vec{x}^{\vec{u}}$, ten from the definition of θ_q we see that

$$v_p(F_{\vec{u}}) \geq \frac{\operatorname{wt}(\vec{u})}{q-1}.$$

Thus we have

$$\begin{aligned} q^s N_s &= (q^s-1)^n + \sum_{\vec{x}\in\mu_{q^s-1}^{n+1}} \prod_{i=0}^{s-1} F(\vec{x}^{q^i}) \\ &= (q^s-1)^n + (q^s-1)^{n+1}\operatorname{Tr}(\psi\circ F)^s, \end{aligned}$$

and this gives us

$$Z(V,T) = \exp\left(\sum_{s\geq 1} \frac{T^s(q^s-1)^n(1+(q^s-1)\operatorname{Tr}(\psi\circ F)^s)}{q^s s}\right).$$

Defining an operator δ by $h(t)^\delta = \frac{h(t)}{h(qt)}$, we can simplify the above to

$$\begin{aligned} Z(V,qT) &= \exp\left(\sum_{s\geq 1}\frac{T^s}{s}\right)^{(-\delta)^n} \exp\left(\sum_{s\geq 1}\frac{T^s\operatorname{Tr}(\psi\circ F)^s}{s}\right)^{(-\delta)^{n+1}} \\ &= (1-t)^{-(-\delta)^n}\det(1-t(\psi\circ F))^{-(-\delta)^{n+1}}, \end{aligned}$$

and this is p-adically meromorphic since $\det(1-t(\psi\circ F))$ is entire (which followed from the fact that F had radius of convergence strictly greater than 1).

1.6.6 General varieties

In the previous section, we completed the proof of the fact that every affine hypersurface has a rational zeta function. Now note that if V_1, V_2 are hypersurfaces, then $V_1 \cup V_2$ is *also* a hypersurface, so

$$Z(V_1\cap V_2, T) = \frac{Z(V_1,T)Z(V_2,T)}{Z(V_1\cup V_2,T)}$$

is also a rational function. Generalizing this in the obvious way, we see that if $V_1, ..., V_k$ are affine hypersurfaces, then $Z(V_1\cap\cdots\cap V_k, T)$ is a rational function. Since every closed affine variety is an intersection of finitely many hypersurfaces, this proves rationality for closed affine varieties. Since every affine variety can be written as the difference of two closed affine varieties, this shows that every affine variety has a rational zeta function.

Finally, since every variety V has an open cover $U_1\cup\cdots\cup U_k$ such that all the intersections $U_{i_1}\cap\cdots\cap U_{i_j}$ with $1\leq i_1<\cdots<i_j\leq k$ are affine, an inclusion-exclusion argument lets us write the zeta function for V in terms of the zeta functions of such intersections, so the zeta function of V is also rational.

1.7 Tony Feng's Notes on Deligne's "La Conjecture de Weil. I"

1.7.1 Introduction

Weil's conjectures

Let X_0 be a smooth projective variety of dimension n over $\mathbf{F}_q$.

Definition 1.7.1. The *zeta function* of X_0 is

$$\zeta(X_0, s) := \prod_{x \in X_0} \left(1 - \frac{1}{q_x^s}\right)^{-1}.$$

This is in obvious analogy to the Riemann zeta function, but it will be more convenient for us to work with the function

$$Z(X_0, t) = \prod_{x \in X_0} \left(1 - t^{-\deg x}\right)^{-1}.$$

We clearly have

$$\zeta(X_0, s) = Z(X_0, q^s).$$

Now we can state Weil's conjectures.

Conjecture 1.7.1 (Weil)**.**

1. $Z(X_0, t)$ is a rational function of t, i.e $Z(X_0, t) \in \mathbf{Q}(t)$, with factorization of the form
$$Z(X_0, t) = \frac{P_1(t) \dots P_{2n-1}(t)}{P_0(t) \dots P_{2n}(t)}.$$
2. $Z(X_0, t)$ satisfies a functional equation.
3. The roots of $P_i(X_0, t)$ have absolute value $q^{-i/2}$.

Cohomological formulation

Weil envisioned these conjectures as a consequence of an appropriate cohomology theory for $X := (X_0)_{\overline{\mathbf{F}}_q}$ which would behave analogously to singular cohomology. In particular, (1) should follow from a "Lefschetz trace formula" in $\overline{X}$, with $X(\mathbf{F}_q)$ interpreted as the "fixed points" of Frobenius. The functional equation predicted in (2) should follow from Poincaré duality. The condition (3) is an analogue of Riemann's hypothesis.

This hypothetical cohomology theory was eventually constructed by Grothendieck, and is now called étale cohomology. The purpose of these notes is to explain the main ideas going into the proof of the proof of (3) in its étale cohomological formulation:

> The eigenvalues of Frobenius on $\mathrm{H}^i_{\text{ét}}(X; \mathbf{Q}_\ell)$ are algebraic over $\mathbf{Q}$, with magnitude $q^{i/2}$ under every complex embedding.

Everything here comes from Deligne's article [69], but I have reorganized the presentation, and focused on the simplest cases in order to highlight the key ideas.

Overview of the proof

By simple reductions, one quickly reduces to checking the eigenvalues of Frobenius on the *middle*-dimensional cohomology. To analyze this, one chooses a Lefschetz pencil $f \colon X \to \mathbf{P}^1$, which always exists after possibly blowing up X (and it is easy to see that blowing up doesn't affect the problem).

The idea is then to study the cohomology of $R^n f_* \mathbf{Q}_\ell$ on $\mathbf{P}^1$. This sheaf will be a local system on a dense open subset of $\mathbf{P}^1$, for general reasons of constructibility of proper pushforwards. There are three main ingredients in the argument.

1. A "big image" result on monodromy for a Lefschetz pencil.
2. A rationality result, showing that a certain characteristic polynomial has coefficients $\mathbf{Q}$ (being a priori in $\overline{\mathbf{Q}}_\ell$). This is achieved by an extremely clever "gcd argument", which is quintessentially Deligne.
3. A very clever analytic estimate, finally establishing the desired bound (in view of the previous two ingredients). This is inspired by the Rankin-Selberg method.

We will actually present (3) first, even though it relies on the first two points, because it is the crux of the argument. Then we will go back and indicate how to verify (1) and (2).

1.7.2 Étale cohomology

The P_i in Weil's conjecture are essentially characteristic polynomials of Frobenius acting on étale cohomology. The intuition to keep in mind is that étale cohomology with coefficients in a *constant* (torsion) sheaf (or more generally, a torsion local system) behaves "like singular cohomology". As we will shortly see, the familiar fundamental results of classical singular cohomology, once phrased invariantly enough, become theorems in étale cohomology.

Remark 1.7.1. For *quasi-coherent* sheaves, étale cohomology coincides with coherent cohomology. These won't come up in our discussion.

The orientation sheaf

Here's an example of what I mean. It's commonly said that complex manifolds are canonically oriented, but from an algebraic perspective that's not quite true - you have to choose an orientation for $\mathbb{C}$. This amounts to a choice of $\pm i$, which can be thought of as a choice of embedding of $\mathbf{Q}/\mathbf{Z}$ into the roots of unity.

We're going to be talking about $\mathbf{Q}_\ell$, the ℓ-adic numbers. The orientation sheaf for $\mathbf{Q}_\ell$ involves a choice of the ℓ-power roots of unity. Such a choice is equivalent to a choice of trivialization

$$\varprojlim \mu_{\ell^n} \simeq \varprojlim \mathbf{Z}/\ell^n \simeq \mathbf{Z}_\ell.$$

In any case $\mathbf{Z}_\ell$ acts on $\varprojlim \mu_{\ell^n}$, and we define

$$\mathbf{Q}_\ell(1) = \mathbf{Q}_\ell \otimes_{\mathbf{Z}_\ell} \varprojlim \mu_{\ell^n}.$$

For any n, we define $\mathbf{Q}_\ell(n) = \mathbf{Q}_\ell(1)^{\otimes n}$. For negative n, this is defined by

$$\mathbf{Q}_\ell(n) := \mathbf{Q}_\ell(-n)^\vee.$$

Remark 1.7.2. For varieties over finite fields, you can think of this in the following way. $\mathbf{Q}_\ell(n)$ is a $\mathbf{Q}_\ell$-vector space with a natural action of $\mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q)$, where Frobenius acts as multiplication by q. However, my Frobenius F will always be the *geometric* Frobenius $x \mapsto x^{q^{-1}}$, which acts as multiplication by q^{-1}.

Properties of étale cohomology

Let X be a smooth variety of pure dimension n over an algebraically closed field. (In terms of earlier notation, think $X = (X_0)_{\overline{\mathbf{F}}_q}$.)

1. (*Fundamental class*) There is a fundamental class
$$\mathrm{Tr}\colon \mathrm{H}^{2n}_c(X, \mathbf{Q}_\ell(n)) \xrightarrow{\sim} \mathbf{Q}_\ell.$$
Equivalently, you can think of this as $\mathrm{Tr}\colon \mathrm{H}^{2n}_c(X, \mathbf{Q}) \xrightarrow{\sim} \mathbf{Q}_\ell(-n)$.

2. (*Cohomological dimension*) X has cohomological dimension $2n$:
$$\mathrm{H}^i(X, \mathbf{Q}_\ell) = 0 \text{ if } i > 2n.$$

3. (*Poincaré duality*) There is a cup product
$$\mathrm{H}^i(X, \mathbf{Q}_\ell) \otimes \mathrm{H}^{2n-i}_c(X, \mathbf{Q}_\ell) \to \mathrm{H}^{2n}_c(X, \mathbf{Q}_\ell) \xrightarrow{\sim} \mathbf{Q}_\ell(-n).$$
which induces a perfect pairing.

4. (*Lefschetz trace formula*) There's a Lefschetz trace formula
$$\mathrm{Fix}(F) = \#X(\mathbf{F}_q) = \sum_i (-1)^i \,\mathrm{Tr}(F, \mathrm{H}^i_c(X, \mathbf{Q}_\ell)).$$

Everything generalizes to a version with coefficients in a more general local system. It may not be clear how to do that for the last one now, but it should become clear later.

Rationality of the zeta function

Because it will actually be important for us later, we derive the rationality of the zeta function from the above properties. Consider

$$\begin{aligned} t\frac{d}{dt}\log Z(X,t) &= t\frac{d}{dt}\sum_x -\log(1 - t^{-\deg x}) \\ &= t\frac{d}{dt}\sum_{n\geq 1}\frac{xt^{-n\deg x}}{n} \\ &= \sum_{n\geq 1} t^{-n}\sum_{\deg x | n}\deg x \end{aligned}$$

Observe that $\sum \deg x \mid n = \#X(\mathbf{F}_{q^n})$, since points of X can be thought of as orbits in $\#X(\mathbf{F}_{q^n})$, of size equal to the their degree. Substituting in the Lefschetz trace formula, we find that this is

$$\sum_{n\geq 1} t^{-n}\sum_i (-1)^i\,\mathrm{Tr}(F, \mathrm{H}^i_c(X, \mathbf{Q}_\ell) = \sum_i (-1)^i \sum_{n\geq 1}\mathrm{Tr}(F^n, \mathrm{H}^i_c(X, \mathbf{Q}_\ell).$$

Now, recall that for an operator F on a vector space V,

$$t\frac{d}{dt}\log\det(1 - tF, V)^{-1} = \sum_{n\geq 1}\mathrm{Tr}(F^n)t^n.$$

Proof: write $\det(1-tF)=\prod(1-t\alpha_i)$, so that this becomes

$$t\frac{d}{dt}\sum_n\sum_i\frac{\alpha_i^n t^n}{n}=\sum_n t^n\sum_i\alpha_i^n.$$

So that tells us that

$$\sum_i(-1)^i\sum_{n\geq 1}\mathrm{Tr}(F^n,\mathrm{H}^i_c(X,\mathbf{Q}_\ell))=t\frac{d}{dt}\log\det(1-Ft,\mathrm{H}^i_c(X,\mathbf{Q}_\ell))^{-1}.$$

Substituting this above, we obtain

$$\prod_x(1-t^{-\deg x})=\prod_i\det(1-Ft,\mathrm{H}^i_c(X,\mathbf{Q}_\ell))^{(-1)^{i+1}}.$$

The right hand side predicts the polynomials appearing in Weil's conjectures.

1.7.3 Some reductions

Let X_0 be a smooth proper variety of dimension n over $\mathbf{F}_q$, and set $X=(X_0)_{\overline{\mathbf{F}}_q}$. Let $RH(\mathrm{H}^i(X))$ denote the statement that

> the eigevalues of F^* on $\mathrm{H}^i(X,\mathbf{Q}_\ell)$ are algebraic with absolute value $q^{i/2}$ under all complex embeddings.

We would like to prove $RH(\mathrm{H}^i(X))$ for $0\leq i\leq 2n$.

Formalities

If we have an embedding

$$\mathrm{H}^i(X)\hookrightarrow\mathrm{H}^i(X')$$

then $RH(\mathrm{H}^i(X'))\implies RH(\mathrm{H}^i(X))$.

Example 1.7.1. If $X'\to X$ the blowup along a closed subvariety $Z\subset X$, then we get such an embedding. We will use the special case where Z is the section by a codimension-2 plane.

If we have a surjection

$$\mathrm{H}^i(X'')\to\mathrm{H}^i(X)$$

then $RH(\mathrm{H}^i(X'))\implies RH(\mathrm{H}^i(X))$.

Poincaré duality

Thanks to the perfect pairing

$$\mathrm{H}^i(X,\mathbf{Q}_\ell)\times\mathrm{H}^{n-i}(X,\mathbf{Q}_\ell)\to\mathbf{Q}_\ell(-n)$$

furnished by Poincaré duality, we automatically know that the $P_i(T)=T^{???}P_{2n-i}(q^n/T)$. In particular, if α is an eigenvalue for F^* on $\mathrm{H}^i(X,\mathbf{Q}_\ell)$ then q^n/α is an eigenvalue for F^* on $\mathrm{H}^i(X,\mathbf{Q}_\ell)$. Therefore,

$$RH(\mathrm{H}^i)\implies RH(\mathrm{H}^{n-i}).$$

The upshot is that it suffices to prove $RH(\mathrm{H}^i)$ for $i=0,\dots,n$.

Weak Lefschetz

Let $Y \subset X$ be a general (smooth) hyperplane section. (Since we're over a finite field, this might not exist a priori. But a smooth *hypersurface* section always exists, so we're okay after passing to some large Veronese embedding first.)

Theorem 1.7.2 (Lefschetz Hyperplane)**.** *The restriction map* $\mathrm{H}^i(X) \to \mathrm{H}^i(Y)$ *is an isomorphism for* $i < n-1$ *and an injection for* $i = n-1$*.*

This will be useful for an inductive proof of the theorem. By the preceding reductions, we get for free that the we only need to worry about the *middle* dimension.

1.7.4 Cohomology of Lefschetz pencils

Introduction to Lefschetz pencils

Most of what we can do for general varieties is bootstrapped from curves, so it is natural to adopt an inductive approach. We've already seen that a hyperplane section of X captures "most" of its cohomology (everything except the middle). To get the rest we'll put X in the"cookie cutter" to get many hyperplane sections. By induction we "know" the cohomology of the hyperplane sections, and then the task is to assemble them together.

A *pencil* of hyperplanes is the set of hyperplanes passing through some codimension-2 plane A, which we call the *axis* of the pencil. This set has a natural structure of a $\mathbf{P}^1$. We have a natural rational map $X \dashrightarrow \mathbf{P}^1$ sending x to the hyperplane spanned by x and A. This is defined away from $A \cap X$. The fibers of this map are points which lie in a common hyperplane through A, i.e. hyperplane sections of X.

We can resolve the indeterminacy of the map by blowing up at the locus $A \cap X$, giving an honest fibration

$$\widetilde{X} \to \mathbf{P}^1.$$

Furthermore,

$$\mathrm{H}^i(X) \hookrightarrow \mathrm{H}^i(\widetilde{X}) = \mathrm{H}^i(X) \oplus \mathrm{H}^{i-2}(X \cap A)(-1)$$

(the last equality by the Thom isomorphism theorem), so by one of reductions it suffices to prove $RH(\mathrm{H}^i(\widetilde{X}))$.

There's an additional technical point in the definition of Lefschetz pencil. The map $\widetilde{X} \to \mathbf{P}^1$ is not smooth, since hyperplane sections can be singular (exactly when the hyperplane becomes tangent to X). I'll want to choose A generally, so that these singularities are as mild as possible, i.e. simple points. You can think of this as asking that the function $f \colon \widetilde{X} \to \mathbf{P}^1$ be a "morse function". A *Lefschetz pencil* is by definition a fibration $\widetilde{X} \to \mathbf{P}^1$, with singularities as mild as possible. As more precise definition will be given when it is needed, in §1.7.6.

Monodromy and the spectral sequence

We're going to try to "fit together" the cohomologies of the different hyperplane sections and see what they tell us about the cohomology of the whole thing. This is an obvious setting for a spectral sequence.

$$E_2^{iq} = \mathrm{H}^i(\mathbf{P}^1, R^q f_* \mathbf{Q}_\ell) \implies \mathrm{H}^{i+q}(X, \mathbf{Q}_\ell).$$

Now, since $\mathbf{P}^1$ is a curve we have that $\mathrm{H}^i(\mathbf{P}^1, R^q f_* \mathbf{Q}_\ell)$ vanishes for $i > 2$. Therefore, there are only three groups that we need to worry about, corresponding to $(i, q) = (0, n), (1, n-1)$, and $(2, n-2)$. However, it is clear that in order to analyze them we need to understand the sheaves $R^q f_* \mathbf{Q}_\ell$.

The basic intuition to keep in mind that is that the "constructible sheaf" $R^q f_* \mathbf{Q}_\ell$ is assembled together from its stalks $(R^q f_* \mathbf{Q}_\ell)_u = \mathrm{H}^q(X_u, \mathbf{Q}_\ell)$ using monodromy. Let me explain.

Let $j \colon U \hookrightarrow \mathbf{P}^1$ be the inclusion of the open set where f is smooth. Over U, $R^q f_* \mathbf{Q}_\ell$ restricts to a local system. This means that it is a locally constant $\mathbf{Q}_\ell$ sheaf for the étale topology (with some finiteness assumptions). There is a monodromy action of $\pi_1(U, u)$ on the fibers which determines the local system - in fact, a $\mathbf{Q}_\ell$-local system is equivalent to the data of a finite-dimensional $\mathbf{Q}_\ell$-representation of $\pi_1(U, u)$.

The key is to understand this monodromy action. Its precise nature will be elaborated upon later, but for now it's enough to emphasize that *the monodromy is only non-trivial on the middle-dimensional groups* $\mathrm{H}^{n-1}(X_{\text{ét}}, \mathbf{Q}_\ell)$. In other words, the local systems $R^i f_* \mathbf{Q}_\ell|_U$ are *trivial* except when $i = n-1$. This fact will be part of the "Picard-Lefschetz" formula for the monodromy to be discussed in the future.

Armed with this knowledge, we can immediately dispose of a couple terms of the spectral sequence. One of them was

$$\mathrm{H}^0(\mathbf{P}^1, R^n f_*(X_u, \mathbf{Q}_\ell) = (\mathrm{H}^n(X_u, \mathbf{Q}_\ell))^{\pi_1} = \mathrm{H}^n(X_u, \mathbf{Q}_\ell).$$

Now, the result follows from induction on the dimension of X.

Remark 1.7.3. Actually, it turns out that we need to induct on *even* dimension (for reasons having to do with the Picard-Lefschetz description of monodromy). We can address this issue by taking another hyperplane section of X_u.

The other term $\mathrm{H}^2(\mathbf{P}^1, R^{n-2} f_* \mathbf{Q}_\ell)$ is basically dual to the one just discussed.

Remark 1.7.4. There is a difference between $\mathrm{H}^i(U, R^n f_* \mathbf{Q}_\ell)$ and $\mathrm{H}^i(\mathbf{P}^1, R^n f_* \mathbf{Q}_\ell)$, and it will typically happen that $R^n f_* \mathbf{Q}_\ell$ is not a local system, while its restriction to U is a local system. But that's not really an issue, because for any $\mathcal{F}$ on X we have a short exact sequence

$$0 \to j_!(\mathcal{F}|_U) \to \mathcal{F} \to (\text{sum of skyscrapers}) \to 0.$$

which induces a surjection

$$\mathrm{H}^1_c(U, \mathcal{F}) \to \mathrm{H}^1(\mathbf{P}^1, j_* \mathcal{F}) \to 0.$$

Therefore, for our purposes is really is enough to consider the restriction to U.

The last case $\mathrm{H}^1(\mathbf{P}^1, R^{n-1} f_* \mathbf{Q}_\ell)$ is the most subtle. For now we'll just say that there is a short exact sequence

$$0 \to j_* \mathcal{E} \to R^{n-1} f_* \mathbf{Q}_\ell \to (\text{constant sheaf}) \to 0 \tag{1.1}$$

with $\mathcal{E}$ a sheaf on U, so it suffices to analyze $\mathrm{H}^1(U, \mathcal{E})$ (since H^1 of $\mathbf{P}^1$ with values in a constant sheaf vanishes). The local system $\mathcal{E}$ contains the "vanishing cycles", which are the cohomology classes that vanish in restriction to some special (singular) fiber. The monodromy action is unipotent, and acts by deforming the cohomology by vanishing cycles, so acts trivially on the quotient sheaf (explaining why it is constant).

We will elaborate on this monodromy theory later, but for present purposes it is only to know the following formal facts:

- The monodromy action preserves the subsheaf $\mathcal{E}$.

- The sheaves $\mathcal{E}^\perp$ (orthogonal for the Poincaré pairing) and $R^{n-1} f_* \mathbf{Q}_\ell / \mathcal{E}$ are constant.

1.7.5 The Fundamental Estimate

Theorem on weights

We are now going to jump into Deligne's estimate on the eigenvalues of Frobenius, *assuming* various auxiliary facts which we have to go back and justifiy later.

We were considering a Lefschetz fibration

$$f \colon X \to \mathbf{P}^1$$

which was smooth over $U \subset \mathbf{P}^1$. This situation is over the algebraic closure $\overline{\mathbf{F}}_q$, but we can assume that everything is defined over $\mathbf{F}_q$, i.e. that the above situation is the base change of

$$f_0 \colon X_0 \to \mathbf{P}^1_0$$

which is smooth over $U_0 \subset \mathbf{P}^1$, with everything defined over $\mathbf{F}_q$.

Definition 1.7.3. A local system $\mathcal{F}_0$ on X_0 is said to have *weight* β if for all $x \in |X_0|$, the (geometric) Frobenius F_x^* acting on $\mathcal{F}_x$ has eigenvalues which are algebraic with absolute value $q_x^{\beta/2}$ under every complex embedding.

Example 1.7.2. In particular, $\mathbf{Q}_\ell(r)$ has weight $-2r$.

Theorem 1.7.4. *Suppose $\mathcal{E}_0$ is a sheaf on U_0 satisfying the following conditions:*

1. *$\mathcal{E}_0$ is equipped with an alternating, non-degenerate bilinear form*
$$\psi \colon \mathcal{E}_0 \otimes \mathcal{E}_0 \to \mathbf{Q}_\ell(-\beta).$$
2. *The image of $\pi_1(U, u)$ in $\mathrm{GL}(\mathcal{E}_u)$ is an open subgroup of $\mathrm{Sp}(\mathcal{E}_u, \psi_u)$.*
3. *For all $x \in U_0$, the polynomial $\det(1 - F_x t, \mathcal{E}_0)$ has rational coefficients.*

Then $\mathcal{E}_0$ has weight β.

Remark 1.7.5. One can imagine that $\mathcal{E}_0$ is essentially sheaf of vanishing cycles as in (1.1). (Then $\beta = n - 1$.) This is not quite how the argument goes, because we don't know a priori that the restriction of the symplectic form to $\mathcal{E}$ is non-degenerate. (This is true, but only by deduction a posteriori.) This can be easily rectified by considering the filtration by the *constant* sheaf $\mathcal{E} \cap \mathcal{E}^\perp$.

The inspiration from the following argument is said to come from ideas of Rankin attacking the Ramanujan conjecture (one of the consequences of Deligne's work).

Recall that

$$t\frac{d}{dt} \log \det(1 - F_x t, \mathcal{E}_0) = \sum_{n \geq 1} \mathrm{Tr}(F_x^n) t^n.$$

In particular, since $\mathrm{Tr}(F_x, \bigotimes^{2k} \mathcal{E}_0) = \mathrm{Tr}(F_x, \mathcal{E}_0)^{2k}$ we have that $t\frac{d}{dt} \log \det(1 - F_x t, \mathcal{E}_0)$ has positive rational coefficients (the positivity would make no sense without knowing that they were rational!). Therefore, the same holds for

$$\det(1 - F_x t, \otimes^{2k} \mathcal{E}_0).$$

Now,

$$Z(U,\otimes^{2k}\mathcal{E}_0,t)=\prod_u \det(1-F_u t,\otimes^{2u}\mathcal{E}_0).$$

The key point is that a product of power series with *positive* coefficients has radius of convergence at most that of any of its factors, since the radius of convergence can be measured by the size of the coefficients of the power series, which *can only increase* by multiplying by a power series with positive coefficients. (If we did not know that the coefficients were positive, then there could be "cancellation of poles" among the factors.)

Now let's consider the Grothendieck-Lefschetz formula for the zeta function:

$$Z(U,\otimes^{2k}\mathcal{E}_0,t)=\frac{P_1(t)}{P_0(t)P_2(t)}.$$

Here $P_0(t)=\det(1-F^*t,\mathrm{H}^0_c(U,\mathcal{E}))$. But a local system on an affine variety has no compactly supported global sections, so $P_0(t)=1$. What about H^2_c? By duality,

$$\mathrm{H}^2_c(\mathcal{E}_0)\simeq \mathrm{H}^0(\mathcal{E}_0^\vee)^\vee(-1)=((\mathcal{E}_u^\vee)^{\pi_1})^\vee=(\mathcal{E}_u)_{\pi_1}(-1)$$

Now, since $\pi_1(U,u)$ is open in $\mathrm{Sp}(\mathcal{E}_u)$ it has the same Lie algebra. This is where we use the "big image" assumption! The coinvariants of representation of $\mathrm{Sp}(\mathcal{E}_u)$ coincide with coinvariants for its Lie algebra, so it is equivalent to understand the coinvariants ofr $\mathrm{Sp}(\mathcal{E}_u)$ on $\pi_1(U,u)$. Then $\mathcal{E}_u$ is just the "standard representation" of the symplectic group. This become a classical question about the coinvariants of tensor powers of the standard representation. It is a theorem that the ring of invariants is generated by the tensor symbols $[x,y]$ corresponding to the symplectic form, and so we find that

$$\left(\otimes^{2k}\mathcal{E}_u\right)_{\pi_1}\simeq\bigoplus_{\mathcal{P}'}\mathbf{Q}_\ell(-k\beta)$$

where $\mathcal{P}'$ is a set of partitions of $[1,2k]$ into pairs, corresponding to $[x_i,x_j]$.

The upshot is that $\mathrm{H}^2_c(U,\otimes^{2k}\mathcal{E})\simeq Q_\ell(-k\beta-1)^N$ for some N. So

$$Z(U_0,\otimes^{2k}\mathcal{E},t)=\frac{P_1}{(1-q^{k\beta+1}t)^N}.$$

In particular, the only pole is at $t=q^{-k\beta-1}$. Since there are no poles of $Z(U_0,\otimes^{2k}\mathcal{E},t)$ with $|t|\le q^{-k\beta-1}$, there are no poles of $\det(1-F_x t,\otimes^{2k}\mathcal{E}_0)^{-1}$ with $|t|\le q^{-k\beta-1}$. In other words, there are no zeros of $\det(1-F_x t,\otimes^{2k}\mathcal{E}_0)$ with absolute value less than $q^{-k\beta-1}$. The zeros are the inverses of the eigenvalues of Frobenius raised to $2k$, so for any such zero α we must have

$$|\alpha|^{-2k}\ge q^{-k\beta-1}.$$

Rearranging we get

$$|\alpha|\le q^{\frac{\beta}{2}+\frac{1}{2k}}.$$

Now we just take $2k\to\infty$ to get the desired upper bound. By Poincaré duality q^β/α is also an eigenvalue, so

$$|q^\beta/\alpha|\le q^{\beta/2}$$

implies the opposite inequality.

Calculation of Frobenius eigenvalues

We now indicate how to complete the calculation of Frobenius eigenvalues. The induction is actually a little subtler than we suggested before, because of the way one needs to use the tensor power trick. The reason is that at some point we need to replace X by a large cartesian power, so we cannot induct on the dimension all at once. Instead, we prove a certain estimate by induction, and then go back and refine it using the tensor power trick.

The statement to be proved by induction is:

> Let $X_0/\mathbf{F}_q$ be a smooth projective variety of even dimension d. Every eigenvalues α of F^* on $\mathrm{H}^d_c(X, \mathbf{Q}_\ell)$ is algebraic and has absolute value
>
> $$q^{\frac{d}{2}-\frac{1}{2}} \leq |\alpha| \leq q^{\frac{d}{2}+\frac{1}{2}}. \tag{1.2}$$

The induction we started will establish this. Then, by considering X^k for large k (the tensor powe trick) and using the Künneth formula, one refines this inequality to the desired equality.

So it remains to establish the bound (1.2) for the eigenvalues of Frobenius on $\mathrm{H}^1_c(\mathbf{P}^1, \mathcal{E}_0)$, where now we take $\mathcal{E}_0$ to be the sheaf of vanishing cycles as in (1.1). The zeta function is

$$\prod_u \det(1 - F_u^* t, \mathcal{E}_u)^{-1} = Z(U, \mathcal{E}_0, t) = P_1(t). \tag{1.3}$$

The zeros of $P_1(t)$ are the inverses of the Frobenius eigenvalues. Now, this is manifestly an ℓ-adic polynomial, but also a power series with *rational* coefficients by our assumptions, hence a rational polynomial. This shows that the eigenvalues are rational.

We want to control the zeros of $P_1(t)$, which are the zeros of $Z(U, \mathcal{E}_0, t)$. We would like to say that by the Euler product (1.3), the zeros of $P_1(t)$ occur at the zeros of $\prod_u \det(1 - F_u^* t, \mathcal{E}_u)^{-1}$. The zeros of this product occur at zeros of the individual factors, but there are none!

The issue is that the product expansion (1.3) only holds for small t. It is valid where it converges, so what we would like is for it to converge for $|t| < q^{-\beta/2}$. In fact it just barely fails; it only converges for $|t| < q^{-\beta/2-1}$. By the tensor power trick and Poincaré duality, we can upgrade this bound to the desired equality.

We have $\det(1 - F_u^* t, \mathcal{E}_u) = \prod (1 - \alpha_{i,u} t)$. Therefore, it suffices to analyze when

$$\sum_{i,u} \alpha_{i,u} t$$

converges. We know that $|\alpha_{i,u}| = q^{\beta \deg u/2}$, so we can regroup the sum as

$$\sum_u \sum_n q^{n\beta/2} \# U(\mathbf{F}_{q^n}) t^n.$$

What is $\#U(\mathbf{F}_{q^n})$? Well U is off from $\mathbf{A}^1$ by just a finite set of points, so $\#U(\mathbf{F}_{q^n}) \leq \mathbf{A}^1(\mathbf{F}_{q^n}) = q^n$. So the conclusion is that the sum is

$$\sum_n q^{n(1+\beta/2)} t^n$$

and thus converges for $|t| < q^{-(1+\beta/2)}$.

We're almost done. We proved that $\mathrm{H}^1(U, \mathcal{E}_0)$ has eigenvalues of magnitude

$$q^{\beta/2-1} \leq |\alpha|.$$

By Poincaré duality, we can conclude for free that

$$q^{\beta/2-1} \leq |\alpha| \leq q^{\beta/2+1}.$$

This is what precisely the estimate (1.2) that we wanted.

1.7.6 Monodromy theory of Lefschetz pencils

We now want to go back and substantiate some of the claims about Lefschetz pencils that we used. The setup of interest is that we have a fibration

$$f \colon X \to \mathbf{P}^1$$

such that

1. X is non-singular of dimension $n+1$
2. f is proper,
3. f has non-degenerate critical points, i.e. the only singular points of the singular fibers are simple double points.

The third condition is essentially that of being a "morse function".

In such a situation, f will be smooth outside a finite set of points $S \subset \mathbf{P}^1$. If U is the open complement, then $R^i f_* \mathbf{Q}_\ell$ will be a loca system on U, and we want to understand the monodromy action of $\pi_1(U, u)$ on $(R^i f_* \mathbf{Q}_\ell)_u = \mathrm{H}^i(f^{-1}(u), \mathbf{Q}_\ell)$.

Existence of Lefschetz pencils

This situation arose from taking a pencil of hyperplane sections of a smooth projective $X \subset \mathbf{P}^N$ along an axis A, and blowing up along $A \cap X$. Why does a pencil of the desired form exist? The picture is clarified by looking at the *dual variety* $X^\vee \subset (\mathbf{P}^N)^\vee$. The points of $(\mathbf{P}^N)^\vee$ are the hyperplanes of $\mathbf{P}^N$, and $X^\vee$ is the subset of hyperplane tangent to some point of X. In other words, it is the image of the incidence correspondence

$$\Sigma = \{(x, H) \subset X \times (\mathbf{P}^N)^\vee \mid H \supset T_x X\}. \tag{1.4}$$

By dimension counting, Σ has dimension $\dim X + (N - \dim X - 1) = N - 1$, so $X^\vee$ has dimension at most $N-1$. A pencil of hyperplanes is the same as a literal pencil $\mathbf{P}^1 \subset (\mathbf{P}^N)^\vee$ (linearly embedded). It turns out that if it avoids the singular locus and intersects $X^\vee$ transversely, then it will be a Lefschetz pencil. This is a local calculation which we leave as an exercise to the reader.

The local theory

Let's consider the classical case first: suppose we have a map $f \colon X^{n+1} \to D$ where is an open unit disc in $\mathbb{C}$, which is smooth outside 0 and such that $X_0 := f^{-1}(0)$ has a double point.

It turns out (but is not obvious) that X deformation retracts to X_0, so we have an isomorphism

$$\mathrm{H}^i(X_0, \mathbb{C}) \simeq \mathrm{H}^i(X, \mathbb{C}).$$

On the other hand, if t denotes some generic non-zero point of D then we have a restriction map

$$\mathrm{H}^i(X_0, \mathbb{C}) \simeq \mathrm{H}^i(X, \mathbb{C}) \to \mathrm{H}^i(X_t, \mathbb{C}).$$

The image consists of the "monodromy invariants" under the monodromy action of $\pi_1(D^*, t) \simeq \mathbf{Z}$ on $\mathrm{H}^i(X_t, \mathbb{C})$. Let γ be a generator of $\pi_1(D^*, t)$.

Definition 1.7.5. We define the *vanishing subspace* to be $\mathrm{H}^n(X_0, \mathbb{C})^\perp \subset \mathrm{H}^n(X_t, \mathbb{C})$ under the pairing induced by Poincaré duality.The elements of $\mathrm{H}^n(X_0, \mathbb{C})^\perp$ will be referred to as *vanishing cycles.*

Here are the essential facts:

- The vanishing subspace is a line, with generator denoted δ.
- γ acts trivially on $\mathrm{H}^i(X_t, \mathbb{C})$ for $i \neq n$.
- For $x \in \mathrm{H}^n(X_t, \mathbb{C})$, γ acts by $x \mapsto x \pm (x, \delta)\delta$.

Remark 1.7.6. The ± depends on n mod 4.

It is straightforward to write down the algebro-geometric analogue. We replace D by the spectrum of a (strictly henselian) DVR, with special point s and generic point η, so we have maps

$$\mathrm{H}^i(X_s, \mathbf{Q}_\ell) \simeq \mathrm{H}^i(X, \mathbf{Q}_\ell) \to \mathrm{H}^i(X_{\overline{\eta}}, \mathbf{Q}_\ell).$$

Now the possibilities are a little complicated. First, they depend on whether n is odd or even. Fortunately we're only going to discuss the even case, so we can ignore that. It is also possible that there is no vanishing cycle, i.e. $\delta = 0$, which makes things easier (no monodromy means everything is a local system). The interesting case is the one where

$$\gamma(x) = x + (x, \delta)\delta \tag{1.5}$$

so this is the one we're going to discuss.

The global theory

We have a Lefschetz pencil

$$f \colon X \to \mathbf{P}^1.$$

This is smooth outside a finite set S. We choose a baseopint $u \notin S$. For each $s \in S$, we get a vanishing cycle δ_s, and a loop γ_s such that for $x \in \mathrm{H}^n(X_u := f^{-1}(u), \mathbf{Q}_\ell)$

$$\gamma_s(x) = x \pm (x, \delta_s)\delta_s.$$

Definition 1.7.6. We define the subspace of (global) *vanishing cycles* $E \subset \mathrm{H}^n(X_u)$ to be the span $\langle \delta_s \colon s \in S \rangle$.

Proposition 1.7.7. *The space E is stable under the monodromy action, and its orthogonal complement (for the Poincaré pairing) $E^\perp$ is the monodromy invariants.*

This obvious from the nature of the Picard-Lefschetz formula. Therefore, we rename $E = E/E^\perp$ and forget that $E^\perp$ exists.

Theorem 1.7.8. *The vanishing cycles δ_s are conjugate under the monodromy action.*

Proof. We give an argument in the classical case, i.e. for varieties over $\mathbb{C}$, and implicitly invoking an equivalence between the étale and analytic settings. (This is also what Deligne does.)

Consider the incidence correspondence $\Sigma \subset X \times \mathbf{P}^\vee$ from (1.4). Let $D \subset \mathbf{P}^\vee$ be the hyperplanes cutting out the Lefschetz pencil. Then $S = D \cap X^\vee$ is precisely the set of points where the Lefschetz pencil is not smooth, and we want to show that the vanishing cycles are all conjugate by $\pi_1(D - S, u)$. By the Lefchetz Hyperplane Theorem, for a general choice of D we have

$$\pi_1(D - S, u) \twoheadrightarrow \pi_1(\mathbf{P}^\vee - X, u).$$

Therefore, it suffices to show that the vanishing cycles are conjugate under $\pi_1(\mathbf{P}^\vee - X, u)$. To do this, we will argue that there is an element in $\pi_1(\mathbf{P}^\vee - X, u)$ taking γ_s to $\gamma_{s'}$. Indeed, we can just draw a loop in $\mathbf{P}^\vee - X$ that follows γ_s until it is very close to s, then moves to s' and winds once around it, and then returns along its original path. □

Proof of "big image"

Corollary 1.7.9. *The representation of $\pi_1(U, u)$ on E is irreducible.*

Proof. Note that $\gamma_s x = x \pm (x, \delta_s)\delta_s$. Take some non-zero $x \in F$. Then $(x, \delta_s) \neq 0$ for some s, so

$$\gamma_s x - x = \pm(x, \delta_s)\delta_s.$$

Therefore, $\delta_s \in F$. But since the δ_s are all conjugate, they must then all lie in F. □

Theorem 1.7.10. *The image of $\rho \colon \pi_1(U, u) \to \mathrm{Sp}(E)$ is open.*

Proof. The image is some compact ℓ-adic Lie group. It suffices to show that its Lie algebra $\mathfrak{L}$ is open.

Note that the $\mathfrak{L}$ is generated by automorphisms of the form

$$d(x \mapsto x \mp (x, \delta_s)\delta_s) = (x \mapsto \pm(x, \delta_s)\delta_s).$$

In slightly more generality, we claim that if V is any irreducible representation of $\mathfrak{L}$, equipped with an invariant non-degenerate symplectic form $(\cdot, \cdot)$, and such that $\mathfrak{L}$ is generated by endomorphisms of the form $x \mapsto \psi(x, \delta)\delta$, then $\mathfrak{L} = \mathrm{Sp}(V, \psi)$.

For any $\delta \in V$, define $N(\delta) \in \mathrm{End}(V)$ by

$$N(\delta)(x) = \psi(x, \delta)\delta.$$

We know that $\mathcal{L}$ is generated by elements of this form. We're going to try to argue that $N(\delta)$ for *every* δ is in $\mathcal{L}$. This at least produces many elements of $\mathcal{L}$, and we leave it as an exercise to verify that they are enough to generate $\mathfrak{sp}(V, \psi)$.

Let W be the set of $\delta \in V$ such that $N(\delta) \in \mathcal{L}$. We know at least that this is non-empty, and we want to show that it is very big. We're going to do that by arguing that it is an invariant subrepresentation of V. Note that at present, is it not even clearly a subspace! However, it is at least obviously closed under scaling.

Since $N(\delta)$ is nilpotent, the endomorphism $\exp(N(\delta))$ makes sense. We want to show that $\exp(N(\delta))$ preserves W for any $\delta \in W$. It will be enough to show that $\exp(\lambda N(\delta))$ preserves ψ and $\mathcal{L}$, since W is defined in terms of these. These statements are familiar (at least by analogy) from classical Lie theory:

- $N(\delta)$ preserves ψ, in the sense that

  $$\psi(N(\delta)x, y) + \psi(x, N(\delta)y) = 0.$$

  This is just what it means for $\mathcal{L} \subset \mathfrak{sp}(V, \psi)$.

- Notice that since $N(\delta)$ has square 0, the automorphism $\exp(N(\delta))$ makes sense. We claim that $\operatorname{Ad}\exp(N(\delta))$ preserves $\mathcal{L}$. You should think of this as analogous to "$\operatorname{Ad} G$ preserves $\mathfrak{g}$", and crank out the calculation if you aren't convinced. (I have done it!)

Now comes an important calculation. For a scalar $\lambda \in \mathbf{Q}_\ell$, we have

$$\exp(\lambda N(\delta'))\delta'' = \delta'' + \lambda\psi(\delta'', \delta')\delta'.$$

This implies that if δ' and δ'' are in W, and $\psi(\delta'', \delta') \neq 0$, then the whole subspace spanned by δ' and δ'' is in W. This almost shows that W is a subspace, but not quite. We know that W is closed under sums of *non-orthogonal* vectors.

What we *can* conclude is that W is the union of its maximal linear subspaces, which must furthermore be pairwise mutually orthogonal. Suppose there is more than one such, say W' and W''. Since neither can be stable under $\mathcal{L}$, by the irreducibility of V, some $N(\delta) \in \mathcal{L}$ doesn't preserve W', say $N(\delta)$ takes $w' \in W'$ out of W'. Then $N(\delta)w'$ is orthogonal to w'. But the image of $N(\delta)$ is the line spanned by δ, so $N(\delta)w' = 0$, a contradiction.

□

1.7.7 The rationality theorem

Finally we are going to justify the rationality assumption of Theorem 1.7.4.

Setup

Let's recall the setup. We have a Lefschetz pencil

$$f \colon X \to \mathbf{P}^1$$

smooth over an open subscheme $U \subset \mathbf{P}^1$ which is the complement of S. We have a local system $\mathcal{E} \subset R^n f_* \mathbf{Q}_\ell|_U$ on U consisting of the "vanishing cycles", where $\dim X = n + 1$. The cup product on the cohomology of the fibers of f induces a pairing

$$\psi \colon R^n f_* \mathbf{Q}_\ell \otimes R^n f_* \mathbf{Q}_\ell \to \mathbf{Q}_\ell(-n).$$

The vanishing cycles are preserved by monodromy, and the pairing restricts to one on $\mathcal{E}$, which is non-degenerate on

$$\psi\colon \mathcal{E}/(\mathcal{E}\cap\mathcal{E}^{\perp})\otimes\mathcal{E}/(\mathcal{E}\cap\mathcal{E}^{\perp})\to\mathbf{Q}_\ell(-n).$$

Lastly, this whole situation is defined over a finite field $\mathbf{F}_q$, and we denote by $X_0, U_0, S_0, \mathcal{E}_0$, etc. the corresponding objects over $\mathbf{F}_q$. What we want to prove is:

Theorem 1.7.11. *For all $u\in|U_0|$, the polynomial $\det(1-F_u^*t,\mathcal{E}_0/(\mathcal{E}_0\cap\mathcal{E}_0^{\perp}))$ has rational coefficients.*

We are going to argue as follows. We know that the zeta function of X_u is rational, and this zeta function is

$$Z(X_u,t)=\prod_{i=0}^{2n}\det(1-F_u^*t,R^if_{0*}\mathbf{Q}_\ell)^{(-1)^{i+1}}.$$

This can be compared to the polynomial in question. The filtrations

$$0\to\mathcal{E}_0\to R^nf_{0*}\mathbf{Q}_\ell\to R^nf_{0*}\mathbf{Q}_\ell/\mathcal{E}_0\to 0$$

and

$$0\to\mathcal{E}_0\cap\mathcal{E}_0^{\perp}\to\mathcal{E}_0\to\mathcal{E}_0/(\mathcal{E}_0\cap\mathcal{E}_0^{\perp})\to 0$$

cut up $R^nf_{0*}\mathbf{Q}_\ell$ into $\mathcal{E}_0$ and pieces which are *constant* on U_0. This gives a factorization

$$Z(X_u,t)=Z_s(t)\cdot Z_b(t)$$

where Z_s contains the factors corresponding to the local systems with **s**mall small monodromy, and Z_b contains the factors corresponding to $\mathcal{E}_0/(\mathcal{E}_0\cap\mathcal{E}_0^{\perp})$ (b for "big monodromy"). More precisely,

$$\begin{aligned}Z_s(t)=\det(1-F_u^*t,\mathcal{E}_0\cap\mathcal{E}_0^{\perp})^{(-1)^{n+1}}&\times\det(1-F_u^*t,R^nf_{0*}\mathbf{Q}_\ell/\mathcal{E}_0)^{(-1)^{n+1}}\\&\times\prod_{i\neq n}\det(1-F_u^*t,R^if_*\mathbf{Q}_\ell)^{(-1)^{i+1}}.\end{aligned}$$

and

$$Z_b(t)=\det(1-F_u^*t,\mathcal{E}_0/(\mathcal{E}_0\cap\mathcal{E}_0^{\perp})).$$

Of course, $Z_b(t)$ is the term that we are interested in showing has rational coefficients. We know that $Z(X_u,t)\in\mathbf{Q}(t)$, so it suffices to show that $Z_s(t)\in\mathbf{Q}(t)$.

Now, the local systems appearing in $Z_s(t)$ are not quite constant, but they are constant after base change to $\overline{\mathbf{F}}_q$. It is worth recording an observation about this situation:

Lemma 1.7.12. *Let $\mathcal{G}_0$ be a $\mathbf{Q}_\ell$-local system on $\epsilon\colon U_0\to\mathbf{F}_q$ whose base change to U is constant. Then there are units $\alpha_i\in\mathbf{Q}_\ell$ such that for all $x\in|U_0|$, we have*

$$\det(1-F_u^*t,\mathcal{G}_0)=\prod_i(1-\alpha_i^{\deg x}t).$$

Proof. Indeed, the hypothesis implies that $\mathcal{G}_0$ is pulled back from a sheaf G_0 on $\operatorname{Spec}\mathbf{F}_q$, namely $G_0:=\epsilon_*\mathcal{G}_0$ (since the hypothesis says that $\epsilon^*\epsilon_*\mathcal{G}_0\to\mathcal{G}_0$ is an isomorphism after base change to $\overline{\mathbf{F}}_q$).

Then F_u acts as $\mathrm{Frob}^{-\deg u}$, and we can take α_i as in

$$\det(1-Ft,G_0)=\prod_i(1-\alpha_it).$$

□

Let $\mathcal{F}_0 = \mathcal{E}_0/(\mathcal{E}_0 \cap \mathcal{E}_0^{\perp})$. Applying the Lemma to the product $Z = Z_s \cdot Z_f$, we find that

$$Z(X_u, t) = \frac{\prod_i (1 - \alpha_i^{\deg u} t)}{\prod_j (1 - \beta_j^{\deg u} t)} \cdot \det(1 - F_u^* t, \mathcal{F}_0).$$

Since the left side is in $\mathbf{Q}(t)$, so is the right side. To complete the proof, we will argue that the α_i and β_j lie in $\mathbf{Q}$.

Overview of the proof

The strategy for proving rationality of α_i, β_j is as follows. First, we may assume that there are no coincidences between the α_i and β_j, since we can just delete them in pairs. We will try to show that the *family* of functions in $\mathbf{Q}_\ell(t)$

$$P_u(t) = \frac{\prod_i (1 - \alpha_i^{\deg u} t)}{\prod_j (1 - \beta_j^{\deg u} t)} \cdot \det(1 - F_u^* t, \mathcal{F}_0)$$

varying with u, allows us to reconstruct the α_i and β_j. Since every member of this family is rational, this will show that the α_i and β_j are rational.

For example, we will try to characterize $\prod (1 - \beta_j^{\deg u} t)$ as being the denominator of $P_u(t)$. The difficulty is that the factors coming from $\det(1 - F_u^* t, \mathcal{F}_0)$ might "accidentally" cancel some of the $(1 - \beta_j^{\deg u} t)$. The key point is that this can only happen to a very limited extent, because $\mathcal{F}_0$ has big monodromy (by Theorem 1.7.10): this suggests that F_u^* behaves like a "random" element of $\mathrm{Sp}((\mathcal{F}_0)_u)$, as u varies. In particular, the eigenvalues of a random family of elements $\{F_u\}$ will behave very differently from the family of eigenvalues $\{\beta_j^{\deg u}\}$.

The fundamental technical lemma, which makes the preceding intuition precise, is the following.

Proposition 1.7.13. *Let $(\gamma_i)_{1 \le i \le P}$ and $(\delta_j)_{1 \le j \le Q}$ be two families of numbers in $\overline{\mathbf{Q}}_\ell$. Assume that $\gamma_i \neq \delta_j$ for any i, j. Then there is a finite exceptional set K of integers $\neq 1$, and an exceptional set L of density 0 in $|U_0|$, such that for $u \in |U_0|$ with $k \nmid \deg u$ for all $k \in K$ and $u \notin L$, the denominator of*

$$\frac{\prod_i (1 - \gamma_i^{\deg u} t)}{\prod_j (1 - \delta_j^{\deg u} t)} \det(1 - F_u^* t, \mathcal{F}_0)$$

written irreducibly is exactly $\prod_j (1 - \delta_j^{\deg u} t)$.

In the next subsection we will complete the proof of rationality *assuming* Proposition 1.7.13. Then we will go back and verify Proposition 1.7.13.

Proof of Theorem 1.7.11

As discussed, Proposition 1.7.13 gives an *intrinsic characterization* of the set $\{(\beta_j^{\deg u})_j\}_u$ in terms of the family $\{P_u(t)\}_u$, which we know to have rational coefficients. A slightly subtle point is to show that this actually pins down $\{\beta_j\}$, which is the content of the following Lemma. Once we know that, it will show that $\beta_j \in \mathbf{Q}$.

Lemma 1.7.14. *Let K be any finite set of integers not containing 1 and $(\delta_j)_{1 \le j \le Q}, (\epsilon_j)_{1 \le j \le Q}$ two families of elements of a field. If for all sufficiently large n not divisible by the members of K we have $\{\delta_j^n\} = \{\epsilon_j^n\}$ then $\{\delta_j\} = \{\epsilon_j\}$.*

Proof. By induction, it suffices to show that $\delta_Q = \epsilon_j$ for some j. Consider the set of exponents n for which we have

$$\delta_Q^n = \epsilon_j^n.$$

This equality is clearly closed under addition and subtraction, hence forms an *ideal* of $\mathbf{Z}$, necessarily of the form (n_j). If none of these ideals (as j varies) is the unit ideal, then we can find an arbitrarily large integer which is not divisible by any n_j or member of K. But hypothesis tells us that such an integer lies in some (n_j), which is a contradiction. □

We next try to give an intrinsic characterization of the α_i.

Proposition 1.7.15. *Let $(\gamma_i)_{1 \leq i \leq P}$ and $(\delta_j)_{1 \leq j \leq Q}$ be two families of numbers in $\overline{\mathbf{Q}}_\ell$. Set*

$$R(t) = \prod_i (1 - \gamma_i t)$$
$$S(t) = \prod_j (1 - \delta_j t).$$

Suppose that for all $u \in |U_0|$ we have the divisibility

$$\prod_i (1 - \delta_j^{\deg u} t) \mid \left[\prod_i (1 - \gamma_i^{\deg u} t) \det(1 - F_u^* t, \mathcal{F}_0) \right].$$

Then $S(t) \mid R(t)$.

Once this is established, it provides the following “recognition principle” for the α_i. Consider the family (varying with u)

$$\prod_i (1 - \alpha_i^{\deg u} t) \det(1 - F_u^* t, \mathcal{F}_0) \in \mathbf{Q}[t].$$

Consider the collection of $(\delta_j \in \mathbf{Q}_\ell)$, possibly with multiplicities, such that for all u

$$\prod_j (1 - \delta_j^{\deg u} t) \mid \left[\prod_i (1 - \alpha_i^{\deg u} t) \det(1 - F_u^* t, \mathcal{F}_0) \right].$$

Proposition 1.7.15 tells us that each $\prod_j (1 - \delta_j t)$ divides a unique maximal such polynomial, which must then be equal to $\prod_i (1 - \alpha_i t)$.

Proof of Proposition 1.7.15. Apply Proposition 1.7.13 to the family of polynomials

$$\frac{\prod_i (1 - \gamma_i^{\deg u} t)}{\prod_j (1 - \delta_j^{\deg u} t)} \det(1 - F_u^* t, \mathcal{F}_0).$$

By hypothesis the denominator is usually 1, so $S(T) \mid R(T)$. □

Proof of Proposition 1.7.13

For a geometric point $\overline{u}$ of U_0, *arithmetic fundamental group* $\pi_1(U_0, \overline{u})$ admits a surjection onto $\mathrm{Gal}(\overline{\mathbf{F}_q}/\mathbf{F}_q)$ whose kernel is the *geometric fundamental group* $\pi_1(U, \overline{u})$:

$$0 \to \pi_1(U, \overline{u}) \to \pi_1(U_0, \overline{u}) \to \mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q) \to 0.$$

The monodromy action of $\pi_1(U_0, \overline{u})$ on $\mathcal{F}_{\overline{u}}$ defines a representation

$$\rho \colon \pi_1(U_0, \overline{u}) \to \mathrm{GSp}(\mathcal{F}_{\overline{u}})$$

which restricts to the previous considered monodromy representation on the geometric fundamental group:

$$\pi_1(U, \overline{u}) \to \mathrm{Sp}(\mathcal{F}_{\overline{u}}).$$

Let μ be the homothety character of $\mathrm{GSp}(\mathcal{F}_{\overline{u}})$. Then we know that the product of projection to $\widehat{\mathbf{Z}}$ and ρ takes $\pi_1(U_0, \overline{u})$ into the subgroup

$$H \subset \widehat{\mathbf{Z}} \times \mathrm{GSp}(\mathcal{F}_{\overline{u}})$$

of (n, g) such that

$$q^n = \mu(g).$$

Let

$$\rho_1 \colon \pi_1(U_0, \overline{u}) \to H$$

denote this representation, and let H_1 be the image of ρ_1.

Lemma 1.7.16. *The image* H_1 *of* ρ_1 *is open in* H*.*

Proof. We know that H_1 surjects onto $\widehat{\mathbf{Z}}$, and by Theorem 1.7.10 the image of the geometric monodromy subgroup in $\mathrm{Sp}(\mathcal{F}_{\overline{u}})$ is open. □

Lemma 1.7.17. *For any* $\delta \in \overline{\mathbf{Q}}_\ell$*, the set* Z *of* $(n, g) \in \mathrm{H}_1$ *such that* δ^n *is an eigenvalue of* g *is closed of measure* 0*.*

Proof. The closedness is obvious. Fix $n \in \widehat{\mathbf{Z}}$, and let $\mathrm{GSp}(\mathcal{F}_{\overline{u}})_n$ denote the subset of g such that $\mu(g) = q^n$. This is a torsor for $\mathrm{Sp}(\mathcal{F}_{\overline{u}})$. It is easily verified that $Z_n := Z \cap \mathrm{GSp}(\mathcal{F}_{\overline{u}})_n$ is the points of a closed algebraic subvariety, which is necessarily of density 0. Thus, we have verified that "fiberwise over $\widehat{\mathbf{Z}}$ the subset Z has density 0. The result then follows from Fubini's Theorem. □

Finally we can complete the proof of Proposition 1.7.13. For each i and j, the set of exponents n such that

$$\gamma_i^n = \delta_j^n$$

is obviously closed under addition and multiplication, hence forms an ideal in $\mathbf{Z}$ of the form (n_{ij}). By hypothesis, $n_{ij} \neq 1$. Take K to be the union of the n_{ij}. By the preceding lemma and Cebotarev's density theorem, the set of $u \in |U_0|$ such that $\delta_j^{\deg u}$ is an eigenvalue of F_u is of density 0. □

Chapter 2

The Sum-Product Theorem

2.1 The Plünnecke-Ruzsa sumset calculus

Definition 2.1.1. If A, B are finite subsets of a semigroup G, A nonempty, define the *magnification ratio* of A, B to be

$$\mu(A,B) = \min_{\emptyset \neq X \subseteq A} \frac{|XB|}{|X|}.$$

Note that if $\emptyset \neq X \subseteq A$ has $\frac{|XB|}{|B|} = \mu(A,B)$ then $\frac{|XB|}{|B|} = \mu(X,B)$.

Theorem 2.1.2 (Petridis). *If X, B are finite subsets of a semigroup G, X nonempty satisfying $\frac{|XB|}{|X|} = \mu(X,B)$, then for all finite subsets C of G such that $|cX| = |X|$ for all $c \in C$, we have*

$$|CXB| \leq \frac{|CX||XB|}{|X|}.$$

Proof. Induct on $|C|$. If C is empty we are done, so suppose $C = C' \cup \{c\}$, $c \notin C'$. Letting $Y = \{x \in X \mid cx \in C'X\}$, we have

$$\begin{aligned}
|CXB| &\leq |C'XB| + |c(XB \setminus YB)| \\
&\leq \frac{|C'X||XB|}{|X|} + |XB| - |YB| \\
&\leq \frac{(|CX| - |X| + |Y|)|XB|}{|X|} + |XB| - \mu(X,B)|Y| \\
&= \frac{|CX||XB|}{|X|}.
\end{aligned}$$

□

Theorem 2.1.3 (Ruzsa triangle inequality). *If X, Y, Z are finite subsets of a group G, then $|X||YZ| \leq |YX^{-1}||XZ|$.*

Theorem 2.1.4 (Ruzsa covering lemma). *If A, B are finite subsets of a group G and A is nonempty, then there is a set $S \subseteq B$ with $|S| \leq \mu(A,B)$ and $B \subseteq A^{-1}AS$.*

Proof. Let $\emptyset \neq X \subseteq A$ be such that $\frac{|XB|}{|X|} = \mu(A,B)$. Take S to be a maximal subset of B such that Xs, Xs' are disjoint for every pair of distinct elements $s, s' \in S$. Then $|X||S| = |XS| \leq |XB|$ and $B \subseteq X^{-1}XS \subseteq A^{-1}AS$. □

Lemma 2.1.5 (Plünnecke tensor power trick)**.** *If A, B are finite subsets of a semigroup G, A', B' are finite subsets of a semigroup G', and A, A' are nonempty, then*

$$\mu(A \times A', B \times B') = \mu(A, B)\mu(A', B').$$

Theorem 2.1.6 (Plünnecke-Ruzsa sumset inequality)**.** *If $A, B_1, ..., B_h$ are finite subsets of an abelian semigroup G with A nonempty, such that for all $b \in (h-1)(B_1 \cup \cdots \cup B_h)$ we have $|A+b| = |A|$, then*

$$\mu(A, B_1 + \cdots + B_h) \leq \frac{|A+B_1|}{|A|} \cdots \frac{|A+B_h|}{|A|}.$$

In particular, if A is cancellative we have $|B_1 + \cdots + B_h| \leq \frac{|A+B_1|}{|A|} \cdots \frac{|A+B_h|}{|A|}|A|$.

Proof. Write $\alpha_i = \frac{|A+B_i|}{|A|}$. Choose a large integer n such that $\frac{n}{\alpha_i}$ is an integer for all i, and set $n_i = \frac{n}{\alpha_i}$. By adding copies of $\mathbb{N}$ to G, we can assume there exist $T_1, ..., T_h \subseteq G$ with $|T_i| = n_i$ such that all sums

$$y + t_1 + \cdots + t_h, \ \ y \in A + B_1 + \cdots + B_h, \ \ \forall 1 \leq i \leq h \ \ t_i \in T_i$$

are distinct. Set $B = \bigcup_i (B_i + T_i)$. We have

$$|A+B| \leq \sum_i |A+B_i||T_i| = \sum_i n_i \alpha_i |A|,$$

so $\mu(A, B) \leq \sum_i n_i \alpha_i = hn$. Let $\emptyset \neq X \subseteq A$ be such that $\frac{|X+B|}{|X|} = \mu(A, B)$. Applying Theorem 2.1.2 h times, we see that $|X + hB| \leq \mu(A, B)^h |X| \leq (hn)^h |X|$. Thus,

$$n_1 \cdots n_h |X + B_1 + \cdots + B_h| = |X + B_1 + \cdots + B_h + T_1 + \cdots + T_h| \leq |X + hB| \leq (hn)^h |X|,$$

so

$$\mu(A, B_1 + \cdots + B_h) \leq \frac{(hn)^h}{n_1 \cdots n_h} = h^h \alpha_1 \cdots \alpha_h.$$

Applying the tensor power trick (Lemma 2.1.5), we have

$$\mu(A, B_1 + \cdots + B_h)^k = \mu(\times^k A, \times^k B_1 + \cdots + \times^k B_h) \leq h^h \alpha_1^k \cdots \alpha_h^k,$$

and taking k to infinity finishes the proof. □

Proposition 2.1.7 (Bourgain)**.** *Let $A_1, ..., A_h, B_1, ..., B_h, C_1, ..., C_h$ be finite subsets of an abelian group G such that for each i $A_i \cap C_i$ is nonempty. Then*

$$|B_1 + \cdots + B_h| \leq \frac{|B_1 + C_1|}{|A_1 \cap C_1|} \cdots \frac{|B_h + C_h|}{|A_h \cap C_h|} |A_1 + \cdots + A_h|.$$

2.1.1 Approximate variants

Lemma 2.1.8. *If A, B are finite subsets of an abelian group G, then there exist $x \in B - A, y \in A + B$ such that*

$$|B \cap (A + x)| \geq \frac{|A||B|}{|A+B|},$$
$$|B \cap (-A + y)| \geq \frac{|A||B|}{|A+B|}.$$

Proof. By Cauchy-Schwarz, we have

$$\#\{(a,b,a',b') \in A \times B \times A \times B \mid a+b = a'+b'\} \geq \frac{|A|^2|B|^2}{|A+B|}.$$

By the pigeonhole principle we can find an x of the form $b-a'$ and a y of the form $a+b$ with the required properties. □

Theorem 2.1.9 (Approximate covering lemma)**.** *If A, B are finite subsets of an abelian group G with A nonempty, then for any $m \geq 1$ there are sets $S_+ \subseteq B - A$, $S_- \subseteq A + B$ such that*

$$|B \cap (A + S_+)| \geq (1 - 1/m)|B|,$$
$$|B \cap (-A + S_-)| \geq (1 - 1/m)|B|,$$

and

$$|S_+|, |S_-| < \log(m)\mu(A,B) + 1.$$

Proof. Assume WLOG that $\mu(A,B) = \frac{|A+B|}{|A|}$. Iteratively apply Lemma 2.1.8 and use the inequality $-\log(1 - \frac{|A|}{|A+B|}) \geq \frac{|A|}{|A+B|}$. □

Theorem 2.1.10 (Approximate Plünnecke-Ruzsa)**.** *If $A, B_1, ..., B_h$ are finite subsets of an abelian semigroup G with A nonempty, such that for all $b \in (h-1)(B_1 \cup \cdots \cup B_h)$ we have $|A+b| = |A|$, then for any $m \geq 1$ there is a set $X \subseteq A$ with*

$$|X| > (1 - 1/m)|A|$$

and

$$|X + B_1 + \cdots + B_h| \leq \frac{hm^{h-1} - 1}{h-1}\frac{|A+B_1|}{|A|}\cdots\frac{|A+B_h|}{|A|}|X|.$$

Proof. We'll show that in fact we can find such X with

$$|X + B_1 + \cdots + B_h| \leq \left(m^h|X| - \left(m^h - \frac{hm^{h-1}-1}{h-1}\right)|A|\right)\frac{|A+B_1|}{|A|}\cdots\frac{|A+B_h|}{|A|}.$$

Suppose for contradiction that there is some $m \geq 1$ for which we can not find such an X. Let n be the infimum of all such m. Since A only has finitely many subsets, we can find a set $\emptyset \neq Y \subseteq A$ with $|Y| \geq (1 - 1/n)|A|$ and

$$|Y + B_1 + \cdots + B_h| \leq \left(n^h|Y| - \left(n^h - \frac{hn^{h-1}-1}{h-1}\right)|A|\right)\frac{|A+B_1|}{|A|}\cdots\frac{|A+B_h|}{|A|}.$$

Note that if $|Y| > (1-1/n)|A|$ then the derivative of the right hand side of the above with respect to n is positive, so by the definition of n we must have $|Y| = (1-1/n)|A|$ for any set Y as above.

By the Plünnecke-Ruzsa inequality (Theorem 2.1.6), we have

$$\mu(A \setminus Y, B_1 + \cdots + B_h) \leq \frac{|A+B_1|}{|A \setminus Y|}\cdots\frac{|A+B_h|}{|A \setminus Y|} \leq n^h\frac{|A+B_1|}{|A|}\cdots\frac{|A+B_h|}{|A|},$$

so there is some $\emptyset \neq X' \subseteq A \setminus Y$ such that

$$|X' + B_1 + \cdots + B_h| \leq n^h \frac{|A+B_1|}{|A|} \cdots \frac{|A+B_h|}{|A|} |X'|.$$

Taking $Y' = Y \cup X'$, we have

$$\begin{aligned} |Y' + B_1 + \cdots + B_h| &\leq |Y + B_1 + \cdots + B_h| + |X' + B_1 + \cdots + B_h| \\ &\leq \left(n^h|Y| + n^h|X'| - \left(n^h - \frac{hn^{h-1}-1}{h-1}\right)|A|\right) \frac{|A+B_1|}{|A|} \cdots \frac{|A+B_h|}{|A|} \\ &= \left(n^h|Y'| - \left(n^h - \frac{hn^{h-1}-1}{h-1}\right)|A|\right) \frac{|A+B_1|}{|A|} \cdots \frac{|A+B_h|}{|A|}, \end{aligned}$$

but $|Y'| > (1-1/n)|A|$, a contradiction. □

Theorem 2.1.11 (Ruzsa)**.** *If A, B, C are finite subsets of a semigroup G with A nonempty, such that for any $b \in B, c \in C$ we have $|cA| = |Ab| = |A|$, then for any $m \geq 1$ there is a set $X \subseteq A$ with*

$$|X| > (1 - 1/m)|A|$$

and

$$|CXB| \leq (2m-1)\frac{|CA|}{|A|}\frac{|AB|}{|A|}|X|.$$

Proof. Since left multiplication by C commutes with right multiplication by B, we can make an auxiliary abelian semigroup G' out of disjoint copies of $A, B, C, CA, AB, B \times C, CAB, \{0\}$ in an obvious way. Now apply Theorem 2.1.10 to G'. □

2.1.2 Energy

Definition 2.1.12. If A, B are finite subsets of a semigroup, define their *energy* to be

$$E(A,B) = \#\{(a,b,c,d) \in A \times B \times A \times B \mid ab = cd\}.$$

When $A = B$, we abbreviate this by $E(A)$.

Proposition 2.1.13 (Cauchy-Schwarz)**.** *If A, B are finite nonempty subsets of a semigroup, then*

$$E(A,B) \geq \frac{|A|^2|B|^2}{|AB|}.$$

Definition 2.1.14. If A, B are finite subsets of an abelian group G and $x \in G$, set

$$\begin{aligned} (A * B)(x) &= \#\{(a,b) \in A \times B \mid a + b = x\}, \\ (A \circ B)(x) &= \#\{(a,b) \in A \times B \mid b - a = x\}. \end{aligned}$$

Lemma 2.1.15 (Sanders, Schoen)**.** *If A is a finite nonempty subset of an abelian group, $0 \leq \alpha < 1$, and $c \geq 0$, then there is a set $X \subseteq A$ with $|X| > \alpha\frac{E(A)}{|A|^2}$ and*

$$\#\left\{(x,y) \in X \times X \mid (A \circ A)(x-y) > c\frac{E(A)}{|A|^2}\right\} \geq \left(1 - \frac{c}{1-\alpha}\right)|X|^2.$$

Proof. We will choose $X = A \cap (A + d)$ for some $d \in A - A$. We have

$$\sum_{(A \circ A)(d) \le \alpha \frac{E(A)}{|A|^2}} (A \circ A)(d)^2 \le \alpha \frac{E(A)}{|A|^2} \sum_d (A \circ A)(d) = \alpha E(A),$$

so

$$\sum_{(A \circ A)(d) > \alpha \frac{E(A)}{|A|^2}} (A \circ A)(d)^2 \ge (1 - \alpha) E(A).$$

Setting

$$S = \left\{ (a, b) \in A \times A \mid (A \circ A)(a - b) \le c \frac{E(A)}{|A|^2} \right\},$$

we have

$$\sum_d \#\{(a, b) \in S \mid a, b \in A + d\} = \sum_{(a,b) \in S} (A \circ A)(a - b) \le c \frac{E(A)}{|A|^2} |S| \le c E(A).$$

Thus

$$\sum_{(A \circ A)(d) > \alpha \frac{E(A)}{|A|^2}} (1 - \alpha) \#\{(a, b) \in S \mid a, b \in A + d\} - c (A \circ A)(d)^2 \le 0,$$

so there must be some d with $(A \circ A)(d) > \alpha \frac{E(A)}{|A|^2}$ and

$$(1 - \alpha) \#\{(a, b) \in S \mid a, b \in A + d\} - c (A \circ A)(d)^2 \le 0.$$

Taking $X = A \cap (A + d)$ for this d, we have $|X| = (A \circ A)(d)$ and

$$\#\left\{ (x, y) \in X \times X \mid (A \circ A)(x - y) > c \frac{E(A)}{|A|^2} \right\} = |X|^2 - \#\{(a, b) \in S \mid a, b \in A + d\}. \qquad \square$$

Theorem 2.1.16 (Balog, Gowers, Schoen, Szemerédi)**.** *If A is a finite nonempty subset of an abelian group, then there is a set $A' \subseteq A$ with $|A'| > \frac{E(A)}{6|A|^2}$ and*

$$|A' - A'| < 486 \frac{|A|^{10}}{E(A)^3}.$$

Proof. Take $\alpha = \frac{1}{2}, c = \frac{1}{9}$ in Lemma 2.1.15 to find a set $X \subseteq A$ with $|X| > \frac{E(A)}{2|A|^2}$ and

$$\#\left\{ (x, y) \in X \times X \mid (A \circ A)(x - y) > \frac{E(A)}{9|A|^2} \right\} \ge \frac{7}{9} |X|^2.$$

Make a graph $\mathcal{H}$ with vertex set X, having an edge between x and y exactly when $(A \circ A)(x - y) > \frac{E(A)}{9|A|^2}$. Letting A' be the set of vertices in $\mathcal{H}$ having degree greater than $\frac{2}{3}|X|$, we see that $|A'| \ge \frac{|X|}{3} > \frac{E(A)}{6|A|^2}$. For any $a, b \in A'$, we can find more than $\frac{1}{3}|X|$ vertices $x \in X$ connected to both a, b in $\mathcal{H}$, and for each such x we can write

$$a - b = (a - x) - (b - x),$$

and we can write the right hand side in the form $(a_1 - a_2) - (a_3 - a_4)$ with $a_1, a_2, a_3, a_4 \in A$, $a_1 - a_2 = a - x$, in at least $\frac{E(A)^2}{81|A|^4}$ different ways. Thus we have

$$|A' - A'| \cdot \frac{1}{3}|X| \cdot \frac{E(A)^2}{81|A|^4} < |A|^4,$$

so

$$|A' - A'| < 486\frac{|A|^{10}}{E(A)^3}.$$ □

2.2 The sum-product theorem

2.2.1 Characteristic Zero

Definition 2.2.1. For any distinct points $a, b \in \mathbb{R}^n$, set

$$D(a, b) = \Big\{p \in \mathbb{R}^n \mid \angle pab \le \frac{\pi}{6}, \angle pba \le \frac{\pi}{6}\Big\}.$$

Lemma 2.2.2. *For any four points $a, b, c, d \in \mathbb{R}^n$ with $a \neq b, c \neq d, \{a, b\} \neq \{c, d\}$, if all of the inequalities*

$$|ab| \le |bc|, \quad |ab| \le |bd|, \quad |cd| \le |ad|, \quad |cd| \le |bd|$$

hold then the interiors of $D(a, b)$ and $D(c, d)$ do not intersect.

Proof. If $|ab| + |cd| \le |bd|$, then since $D(a, b)$ is contained in the sphere of radius $|ab|$ around b and $D(c, d)$ is contained in the sphere of radius $|cd|$ around d, their interiors can't intersect. Otherwise, we can find a point $x \in \mathbb{R}^n$ such that $|bx| = |ab|, |dx| = |cd|$. Since $|ab|, |cd|$ are assumed to be at most $|bd|$, bd is the longest edge of triangle bdx, so we must have $\angle bxd \ge \frac{\pi}{3}$. Thus we can find some point m on the line segment bd with $\angle mxb \ge \frac{\pi}{6}$ and $\angle mxd \ge \frac{\pi}{6}$. Since a is outside the sphere of radius $|cd| = |dx|$ centered at d, we have $\angle abm \ge \angle xbm$, and similarly $\angle cdm \ge \angle xdm$. Thus, if we rotate the ray mx around the line bd we get a cone which separates the interior of $D(a, b)$ from the interior of $D(c, d)$. □

Corollary 2.2.3 (Gilbert, Pollak). *Let P be a finite set of points in $\mathbb{R}^n$, and let T be a minimum spanning tree on P. For any distinct edges $\{a, b\}, \{c, d\}$ of T, the interiors of $D(a, b)$ and $D(c, d)$ do not intersect.*

Proof. Since T is a tree, there is a unique path in T connecting the edge $\{a, b\}$ to the edge $\{c, d\}$. We may assume without loss of generality that this path connects a to c without passing through b or d. Then if we replace edge $\{a, b\}$ with either $\{b, c\}$ or $\{b, d\}$ we again get a spanning tree, so by minimality we must have $|ab| \le |bc|, |bd|$. Similarly we have $|cd| \le |ad|, |bd|$. Now apply Lemma 2.2.2. □

Proposition 2.2.4. *Suppose $a, b, c, d \in \mathbb{H}^\times$ are nonzero quaternions with $\angle b0d \le \frac{\pi}{6}$. Then $(a + c)(b + d)^{-1}$ is in the interior of $D(ab^{-1}, cd^{-1})$.*

Proof. Writing $b = md$, we have

$$(a + c)(b + d)^{-1} = (a + c)d^{-1}(m + 1)^{-1} = ab^{-1} + (cd^{-1} - ab^{-1})(m + 1)^{-1},$$

so it's enough to check that if $\angle m01 \leq \frac{\pi}{6}$ then $(m+1)^{-1}$ is in the interior of $D(0,1)$. Since $\angle(m+1)10 \geq \frac{5\pi}{6}$, we have $\angle 1(m+1)^{-1}0 \geq \frac{5\pi}{6}$, so $(m+1)^{-1}$ is in the interior of $D(0,1)$ by the fact that the angles of a triangle sum to π. □

Theorem 2.2.5 (Konyagin, Rudnev, Solymosi)**.** *Suppose $A \subseteq \mathbb{H}^\times$ is a finite set of nonzero quaternions such that for any $a, b \in A$ we have $\angle a0b \leq \frac{\pi}{6}$. Then*

$$|A+A|^2|AA| \geq \frac{|A|^4 - |A||AA|}{\log \frac{|AA|^2}{|A|} + \gamma},$$

where γ is the Euler-Mascheroni constant.

Proof. By Cauchy-Schwarz, we have

$$\#\{(a,b,c,d) \in A \times A \times A \times A \mid ab = cd\} \geq \frac{|A|^4}{|AA|}.$$

Write $m(x) = \#\{(a,c) \in A \times A \mid c^{-1}a = x\}$, $n(x) = \#\{(b,d) \in A \times A \mid db^{-1} = x\}$. By Cauchy-Schwarz again, we have

$$\sum_x m(x)^2 \sum_y n(y)^2 \geq \Big(\sum_x m(x)n(x)\Big)^2 \geq \frac{|A|^8}{|AA|^2}.$$

Thus we may assume without loss of generality that

$$\sum_x n(x)^2 \geq \frac{|A|^4}{|AA|},$$

since otherwise we may replace A by $\bar{A}$. Choose a numbering $x_1, ..., x_{|AA^{-1}|}$ of the elements of AA^{-1} such that $n(x_1) \geq n(x_2) \geq \cdots$. Choose $1 \leq k \leq |AA^{-1}|$ such that $(k-1)n(x_k)^2$ is maximized. Then by choice of k we have

$$\frac{|A|^4}{|AA|} \leq \sum_{i=1}^{|AA^{-1}|} n(x_i)^2 \leq |A| + (k-1)n(x_k)^2 \sum_{i=2}^{|AA^{-1}|} \frac{1}{i-1},$$

so

$$(k-1)n(x_k)^2 \geq \frac{|A|^4 - |A||AA|}{H_{|AA^{-1}|-1}|AA|},$$

where $H_n = \sum_{i=1}^n \frac{1}{i}$ denotes the nth harmonic number. Note that by the Ruzsa triangle inequality 2.1.3 we have $|AA^{-1}| \leq \frac{|AA|^2}{|A|}$, so

$$H_{|AA^{-1}|-1} \leq \log \frac{|AA|^2}{|A|} + \gamma.$$

Let T be a minimum spanning tree on $\{x_1, ..., x_k\}$. For any edge $\{x_i, x_j\}$ in T, if $a, b, c, d \in A$ have $ab^{-1} = x_i$ and $cd^{-1} = x_j$, then by Proposition 2.2.4 the ratio $(a+c)(b+d)^{-1}$ will be in the interior of $D(ab^{-1}, cd^{-1})$. Thus by Corollary 2.2.3 we have an injection

$$\{(\{x_i, x_j\}, a, b, c, d) \in T \times A \times A \times A \times A \mid ab^{-1} = x_i, cd^{-1} = x_j\} \hookrightarrow (A+A) \times (A+A),$$

taking $(\{x_i, x_j\}, a, b, c, d)$ to $(a+c, b+d)$. Since T has $k-1$ edges and $n(x_i) \geq n(x_k)$ for $1 \leq i \leq k$, we have

$$|A+A|^2 \geq (k-1)n(x_k)^2 \geq \frac{|A|^4 - |A||AA|}{H_{|AA^{-1}|-1}|AA|}.$$ □

2.2.2 Finite fields

Lemma 2.2.6. *If $A, B \subseteq \mathbb{F}_q$, $G \subseteq \mathbb{F}_q^\times$, then there is some $\xi \in G$ with*

$$|A + \xi B| \ge \frac{|A||B||G|}{|A||B| + |G|}.$$

Proof. Define a function $f : G \mapsto \mathbb{N}$ by

$$f(\xi) = \#\{(a, b, a', b') \in A \times B \times A \times B \mid a + \xi b = a' + \xi b'\}.$$

We have

$$\sum_{\xi \in G} f(\xi) \le |A|^2|B|^2 + |A||B||G|,$$

so there must be some $\xi \in G$ with $f(\xi) \le \frac{|A|^2|B|^2}{|G|} + |A||B|$. By Cauchy-Schwarz, we have

$$|A + \xi B| \ge \frac{|A|^2|B|^2}{f(\xi)} \ge \frac{|A||B||G|}{|A||B| + |G|}. \qquad \square$$

Theorem 2.2.7 (Bourgain, Garaev, Katz, Li, Shen, ...)**.** *If p is prime and $A \subseteq \mathbb{F}_p$ then*

$$|A + A|^9|AA|^4 \ge \frac{|A|^{14}}{256} \min\left(1, \frac{p}{|A|^2}\right),$$
$$|A + A|^8|AA|^4 \ge \frac{|A|^{13}}{2^{23}} \min\left(1, \frac{3^7 p}{|A|^2}\right).$$

Proof. We'll prove the second bound (for the first bound, take $X = A$ and $Z = W = Y$ instead of using the approximate variations on the sumset calculus). By the approximate Plünnecke-Ruzsa theorem (Theorem 2.1.10), we can find $X \subseteq A$ with $|X| \ge \frac{3}{4}|A|$ and

$$|X + A + A + A| \le 24\frac{|A + A|^3}{|A|^3}|X|.$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{x \in X, a \in A} |xA \cap Xa| \ge \frac{|X|^2|A|^2}{|XA|},$$

so by the pigeonhole principle there is some $a_0 \in A$ with

$$\sum_{x \in X} |xA \cap Xa_0| \ge \frac{|X|^2|A|}{|XA|}.$$

Let $X = \{x_1, ..., x_{|X|}\}$, set $n_i = |x_iA \cap Xa_0|$, and suppose WLOG that $n_1 \ge \cdots \ge n_{|X|}$. Choose k maximizing the quantity $k^{3/4}n_k$, set $Y = \{x_1, ..., x_k\}$, and set $N = n_k$. We have

$$\frac{|X|^2|A|}{|XA|} \le \sum_{i=1}^{|X|} n_i \le \sum_{i=1}^{|X|} i^{-3/4}k^{3/4}n_k < 4|X|^{1/4}|Y|^{3/4}N,$$

so

$$|Y|^3N^4 \geq \frac{|X|^7|A|^4}{256|XA|^4}.$$

For any $y \in Y$ we have $|yA \cap Xa_0| \geq N$, so by Ruzsa's triangle inequality (Theorem 2.1.3) we have

$$|yA - Xa_0| \leq \frac{|yA + yA \cap Xa_0||yA \cap Xa_0 + Xa_0|}{|yA \cap Xa_0|} \leq \frac{|y(A+A)||(X+X)a_0|}{N} \leq \frac{|A+A|^2}{N},$$

and similarly by Plünnecke-Ruzsa (Theorem 2.1.6) we have

$$|yA + Xa_0| \leq \frac{|yA \cap Xa_0 + yA||yA \cap Xa_0 + Xa_0|}{|yA \cap Xa_0|} \leq \frac{|A+A|^2}{N}.$$

There are now two cases.

Case 1: If $\frac{Y-Y}{(Y-Y)\setminus\{0\}} = \mathbb{F}_p$, then by Lemma 2.2.6 we can find $\xi \in \mathbb{F}_p^\times$ such that $|A + \xi A| \geq \frac{1}{2}\min(|A|^2, p)$. Write $\xi = \frac{c-d}{a-b}$ with $a, b, c, d \in Y$. By Plünnecke-Ruzsa, we have

$$|(a-b)A + (c-d)A| \leq |aA - bA + cA - dA| \leq \frac{|Xa_0 + aA||Xa_0 - bA||Xa_0 + cA||Xa_0 - dA|}{|Xa_0|^3},$$

so

$$|A+A|^8 \geq \frac{|A|^2|X|^3N^4}{2}\min\left(1, \frac{p}{|A|^2}\right).$$

Since $|X|^3N^4 \geq |Y|^3N^4 \geq \frac{|X|^7|A|^4}{256|AA|^4}$ and $|X| \geq \frac{3}{4}|A|$, we have

$$\begin{aligned}|A+A|^8|AA|^4 &\geq \frac{|X|^7|A|^6}{2^9}\min\left(1, \frac{p}{|A|^2}\right)\\ &\geq \frac{3^7|A|^{13}}{2^{23}}\min\left(1, \frac{p}{|A|^2}\right).\end{aligned}$$

Case 2: If $\frac{Y-Y}{(Y-Y)\setminus\{0\}} \neq \mathbb{F}_p$, then we can find $\xi \in \left(\frac{Y-Y}{(Y-Y)\setminus\{0\}} + 1\right) \setminus \frac{Y-Y}{(Y-Y)\setminus\{0\}}$. Writing $\xi = \frac{c-d}{a-b} + 1$ with $a, b, c, d \in Y$, we see that for any $Z, W \subseteq Y$ have

$$|Z||W| = |Z + \xi W| \leq |(a-b)Z + (a-b)W + (c-d)W|.$$

In particular, if $\emptyset \neq Z' \subseteq Z$ is chosen such that $\mu((a-b)Z, (a-b)W + (c-d)W) = \frac{|(a-b)Z' + (a-b)W + (c-d)W|}{|Z'|}$, then by Plünnecke-Ruzsa we have

$$|Z'||W| \leq |(a-b)Z' + (a-b)W + (c-d)W| \leq \frac{|Z+W|}{|Z|}\frac{|(a-b)Z + (c-d)W|}{|Z|}|Z'|,$$

so

$$|Z|^2|W| \leq |A+A||(a-b)Z + (c-d)W|.$$

Applying the approximate covering lemma (Lemma 2.1.9) to $aA \cap Xa_0$, aY, we find a set S with $|S| < 3\frac{|A+A|}{N}$ such that

$$|aY \cap (Xa_0 + aS)| \geq \frac{6}{7}|Y|.$$

Let $Y' = Y \cap (a^{-1}Xa_0 + S)$. Applying it again, we find a set S' with $|S'| < 3\frac{|A+A|}{N}$ such that

$$bY' \cap (-Xa_0 + bS') \geq \frac{6}{7}|Y'|,$$

and let $Z = Y' \cap (-b^{-1}Xa_0 + S)$. Similarly, find sets $W \subseteq Y, S'', S'''$ such that $|W| \geq \frac{6^2}{7^2}|Y|$, $cW \subseteq Xa_0 + cS'', dW \subseteq -Xa_0 + dS'''$, $|S''|, |S'''| \leq 3\frac{|A+A|}{N}$. We have

$$\begin{aligned}|(a-b)Z + (c-d)W| &\leq |aZ - bZ + cW - dW| \\ &\leq |S||S'||S''||S'''||Xa_0 + Xa_0 + Xa_0 + Xa_0| \\ &\leq 3^4\frac{|A+A|^4}{N^4} \cdot 24\frac{|A+A|^3}{|A|^3}|X|,\end{aligned}$$

so

$$|X||A+A|^8 \geq \frac{24|A|^3|Y|^3N^4}{7^6}.$$

By the inequalities $|X| \geq \frac{3}{4}|A|$ and $|Y|^3N^4 \geq \frac{|X|^7|A|^4}{256|AA|^4}$ we have

$$\begin{aligned}|A+A|^8|AA|^4 &\geq \frac{3|X|^6|A|^7}{2^5 \cdot 7^6} \\ &\geq \frac{3^7|A|^{13}}{2^{17} \cdot 7^6} \\ &\geq \frac{|A|^{13}}{2^{23}}.\end{aligned}$$

□

Theorem 2.2.8 (Garaev)**.** *Let q be a prime power. If $A, B \subseteq \mathbb{F}_q$, $C \subseteq \mathbb{F}_q^\times$, then*

$$|A+B||AC| \geq \min\left(\frac{|A|q}{2}, \frac{|A|^2|B||C|}{4q}\right).$$

Proof. Let

$$J = \{(x,b,c,y) \in (A+B) \times B \times C \times AC \mid x = b + yc^{-1}\}.$$

We have an injection $A \times B \times C \hookrightarrow J$ given by $(a,b,c) \mapsto (a+b, b, c, ac)$, so $|J| \geq |A||B||C|$. Let $\phi_0, ..., \phi_{q-1}$ be the additive characters of $\mathbb{F}_q$, ϕ_0 the trivial character. We have

$$\begin{aligned}|J| &= \frac{1}{q}\sum_{n=0}^{q-1}\sum_{x\in A+B}\sum_{b\in B}\sum_{c\in C}\sum_{y\in AC}\phi_n(b - x + yc^{-1}) \\ &\leq \frac{|A+B||B||C||AC|}{q} + \frac{1}{q}\sum_{n=1}^{q-1}\left|\sum_{x\in A+B}\phi_n(x)\right|\left|\sum_{b\in B}\phi_n(b)\right|\sum_{c\in C}\left|\sum_{y\in AC}\phi_n(yc^{-1})\right|.\end{aligned}$$

By Cauchy-Schwarz, for $n \neq 0$ we have

$$\begin{aligned}\left(\sum_{c\in C}\left|\sum_{y\in AC}\phi_n(yc^{-1})\right|\right)^2 &\leq |C|\sum_{d\in\mathbb{F}_q}\left|\sum_{y\in AC}\phi_n(dy)\right|^2 \\ &= q|C||AC|,\end{aligned}$$

and applying Cauchy-Schwarz one more time we have

$$\begin{aligned}\frac{1}{q}\sum_{n=1}^{q-1}\left|\sum_{x\in A+B}\phi_n(x)\right|\left|\sum_{b\in B}\phi_n(b)\right|\sum_{c\in C}\left|\sum_{y\in AC}\phi_n(yc^{-1})\right| &\le \frac{\sqrt{q|C||AC|}}{q}\sum_{n=1}^{q-1}\left|\sum_{x\in A+B}\phi_n(x)\right|\left|\sum_{b\in B}\phi_n(b)\right|\\ &\le \sqrt{q|A+B||B||C||AC|}.\end{aligned}$$

Thus

$$|A||B||C| \le \frac{|A+B||B||C||AC|}{q} + \sqrt{q|A+B||B||C||AC|}. \qquad \square$$

A much better sum-product bound was recently obtained by Rudnev, using a three-dimensional variant of the Szemerédi-Trotter theorem due to Kollár. The proof is sketched below.

Lemma 2.2.9 (Kollár)**.** *Let $\mathcal{L}$ be a set of m distinct lines in $\mathbb{P}^3$.*

1) There exists a surface S of degree at most $\sqrt{6m}-2$ which contains $\mathcal{L}$.

2) For any irreducible surface U of degree $g \le \sqrt{6m}$ there exists a surface T of degree at most $\frac{6m}{g}$ which contains $\mathcal{L}$ and does not contain U.

Proposition 2.2.10 (Kollár)**.** *For $i = 1, ..., n-1$ let H_i be a hypersurface in $\mathbb{P}^n$ of degree a_i, and suppose their intersection $B = H_1 \cap \cdots \cap H_{n-1}$ is 1-dimensional. Let $C \subseteq B$ be a reduced subcurve. Then the arithmetic genus of C satisfies*

$$p_a(C) \le p_a(B) = 1 + \frac{1}{2}\left(\sum_i a_i - n - 1\right)\prod_i a_i.$$

Proof. By induction on n together with the Kodaira vanishing theorem for $\mathbb{P}^n$, one can show that $h^0(B,\mathcal{O}_B) = 1$, so $p_a(B) = h^1(B,\mathcal{O}_B) - h^0(B,\mathcal{O}_B) + 1 = h^1(B,\mathcal{O}_B)$. If J is the ideal sheaf of C on B, we have

$$0 \to J \to \mathcal{O}_B \to \mathcal{O}_C \to 0,$$

so by the long exact sequence of cohomology we have

$$H^1(B,\mathcal{O}_B) \to H^1(C,\mathcal{O}_C) \to H^2(B,J),$$

and $H^2(B,J) = 0$ since B is 1-dimensional. Thus

$$p_a(C) = h^1(C,\mathcal{O}_C) - h^0(C,\mathcal{O}_C) + 1 \le h^1(B,\mathcal{O}_B) = p_a(B).$$

The formula for $p_a(B)$ follows by directly computing the Hilbert polynomial of B. $\square$

Proposition 2.2.11 (Kollár)**.** *Let $S, T \subseteq \mathbb{P}^3$ be surfaces of degrees a, b with no common components, and let C be a reduced curve contained in $S \cap T$. For a point $p \in C$ let $r(p)$ be the multiplicity of C at p.*

1) C has at most ab components.

2) $\sum_{p\in C} r(p) - 1 \le \frac{ab}{2}(a+b-2)$.

Following Rudnev, we give a concrete description of Plücker coordinates for lines in $\mathbb{P}^3$.

Definition 2.2.12. For a line L in $\mathbb{P}^3$ containing points $[q_0 : q_1 : q_2 : q_3], [u_0 : u_1 : u_2 : u_3]$, set

$$P_{ij} = q_i u_j - q_j u_i,$$

and define the Plücker coordinates of L to be $[P_{01} : P_{02} : P_{03} : P_{23} : P_{31} : P_{12}]$. Writing this as $[\omega : \nu]$, if $q_0 = u_0 = 1$ and we set $q = (q_1, q_2, q_3), u = (u_1, u_2, u_3)$ then $\omega = u - q, \nu = q \times \omega$. Define the Klein quadric $\mathcal{K}$ to be the 4-dimensional hypersurface

$$\mathcal{K} = \{[\omega : \nu] \in \mathbb{P}^5 \mid \omega \cdot \nu = 0\}.$$

Proposition 2.2.13. *Two lines with Plücker coordinates $[\omega : \nu], [\omega' : \nu']$ intersect if and only if*

$$\omega \cdot \nu' + \omega' \cdot \nu = 0,$$

and this occurs if and only if the line connecting $[\omega : \nu], [\omega' : \nu']$ is contained in $\mathcal{K}$. Every plane contained in $\mathcal{K}$ is either an α-plane, corresponding to the set of lines through a specific point in $\mathbb{P}^3$, or a β-plane, corresponding to the set of lines contained in a specific plane in $\mathbb{P}^3$. Any two α-planes meet in a point, any two β-planes meet in a point, and an α-plane and a β-plane meet in a line if and only if the point corresponding to the α-plane is contained in the plane corresponding to the β-plane.

Definition 2.2.14. A *ruling* Γ of a surface $S \subset \mathbb{P}^3$ is a closed curve $\Gamma \subset \mathcal{K}$ such that each point of Γ corresponds to a line contained in S. The *degree* of a ruling Γ is defined to be its degree as a curve in $\mathbb{P}^5$. A line contained in S which is not contained in any ruling of S is called *special*.

Proposition 2.2.15. *For any three skew lines $L_1, L_2, L_3 \subset \mathbb{P}^3$, the union of the collection of all lines which intersect all three of L_1, L_2, L_3 is a smooth quadric surface S. Conversely, every smooth quadric surface S has two irreducible rulings Γ_1, Γ_2 of degree 2.*

Corollary 2.2.16. *Every irreducible ruled surface S is either a plane, a cone, a smooth quadric surface, or else has a unique ruling and contains at most two special lines which do not intersect each other. If S is not a plane, the degree d of an irreducible ruling is equal to the degree of S. Any nonspecial line intersects at most $d - 2$ other nonspecial lines.*

Theorem 2.2.17 (Cayley, Monge, Salmon, Voloch)**.** *Let $S \subset \mathbb{P}^3$ be a surface of degree d, with $d < p$ if the characteristic is p. If S has no ruled components, then there is a surface T of degree $11d - 24$ such that S and T have no components in common, and every line contained in S is contained in $S \cap T$.*

Sketch. The surface T is defined by the equation cutting out those points p of S for which there exists a line which is triply tangent to S at p (such a p is called a *flecnodal* point). The equation for T can be computed explicitly using resultants. Next, one shows that if a component of S consists entirely of flecnodal points, then that component must be ruled. □

Theorem 2.2.18 (Kollár)**.** *Let $\mathcal{L}$ be a collection of m distinct lines in $\mathbb{P}^n$ such that for any three distinct lines $L_1, L_2, L_3 \in \mathcal{L}$ the number of lines from $\mathcal{L}$ intersecting all three of L_1, L_2, L_3 is at most $\sqrt{m}$. If the characteristic is p, suppose that $m < \frac{11}{6}p^2$. Then the total number of intersection points between lines in $\mathcal{L}$ is at most*

$$\left(\frac{\sqrt{6}}{2} + \frac{(36 - \frac{1}{2})\sqrt{6}}{\sqrt{11}}\right)m^{\frac{3}{2}} < \sqrt{754}m^{\frac{3}{2}}.$$

Proof. By choosing a generic projection to $\mathbb{P}^3$, we may assume without loss of generality that $n = 3$. We may also assume that $m \geq 754$. Find a surface S of degree $d \leq \sqrt{6m} - 2$ containing $\mathcal{L}$, and assume that the degree of S is minimal. Choose an ordering $S_1, ...$ of the irreducible components of S such that, letting $\mathcal{L}_i = \{l \in \mathcal{L} \mid l \subset S_i \setminus (S_1 \cup \cdots \cup S_{i-1})\}$, we have $\frac{|\mathcal{L}_i|}{\deg S_i}$ nonincreasing in i. Write $m_i = |\mathcal{L}_i|, d_i = \deg S_i$. The number of intersections between lines contained in different sets $\mathcal{L}_i, \mathcal{L}_j$ is at most

$$\sum_{j<i} m_i d_j \leq \sum_{j<i} \frac{m_i d_j + m_j d_i}{2} = \frac{md - \sum_i m_i d_i}{2}.$$

If S_i is a cone, then there is at most 1 intersection point between lines in $\mathcal{L}_i$ (the cone point). If S_i is a plane, then any two lines in S_i intersect, so by assumption $m_i \leq \sqrt{m}$, and the number of intersection points between lines in $\mathcal{L}_i$ is at most

$$\frac{m_i(m_i - 1)}{2} \leq \frac{(m_i - 1)\sqrt{m}}{2}.$$

If S_i is a smooth quadric surface, then either one of the rulings on S_i contains at most two lines from $\mathcal{L}_i$ or by assumption both rulings contain at most $\sqrt{m}$ lines from $\mathcal{L}_i$, so the number of intersection points between lines in $\mathcal{L}_i$ is at most

$$\max\left(m_i - 1, 2(m_i - 2), \frac{m_i\sqrt{m}}{2}\right) \leq \frac{m_i\sqrt{m}}{2}.$$

If S_i is ruled of degree at least 3, then since there are at most two special lines in S_i and since nonspecial lines meet at most $d_i - 2$ other nonspecial lines, the number of intersection points between lines in $\mathcal{L}_i$ is at most

$$\frac{m_i(d_i - 2 + 2) + 2m_i}{2} = \frac{m_i d_i}{2} + m_i.$$

If S_i is not ruled, then by Lemma 2.2.9 and Theorem 2.2.17 we can find a surface T of degree at most $\min\left(11d_i - 24, \frac{6m_i}{d_i}\right)$ which contains $\mathcal{L}_i$ but not S_i (note that if we take $\deg T = 11d_i - 24$ then $d_i \leq \sqrt{\frac{6}{11}m} < p$). Thus by Proposition 2.2.11 the number of intersections between lines in $\mathcal{L}_i$ is at most

$$\min\left(\frac{d_i(11d_i - 24)}{2}(12d_i - 26), 3m_i\Big(d_i + \frac{6m_i}{d_i} - 2\Big)\right) \leq \frac{m_i d_i}{2} + \frac{(36 - \frac{1}{2})\sqrt{6}}{\sqrt{11}} m_i^{\frac{3}{2}}.$$

Putting everything together, we see that the total number of intersection points between lines in $\mathcal{L}$ is at most

$$\frac{md}{2} + \sum_i \frac{(36 - \frac{1}{2})\sqrt{6}}{\sqrt{11}} m_i \sqrt{m} \leq \left(\frac{\sqrt{6}}{2} + \frac{(36 - \frac{1}{2})\sqrt{6}}{\sqrt{11}}\right) m^{\frac{3}{2}}. \qquad \square$$

Corollary 2.2.19 (Rudnev)**.** *Suppose we have n points and n planes in $\mathbb{P}^3$ such that no more than $\sqrt{n}$ points lie on any line and no more than $\sqrt{n}$ planes all contain a common line. Assume further that if the characteristic is p we have $n \leq \frac{11}{12}p^2$. Then the number of point-plane incidences is at most $\sqrt{6032}n^{\frac{3}{2}}$.*

Proof. Taking Plücker coordinates, we get a collection of n α-planes and n β-planes, and every incidence between a point and a plane becomes a pair of an α-plane and a β-plane which intersect in a line. Intersecting the configuration with a general hyperplane which does not contain the intersection of any two α-planes or the intersection of any two β-planes, we get a configuration of $2n$ lines in $\mathbb{P}^4$. Call a line coming from an α-plane an α-line, and similarly define β-lines. Any two α-lines do not intersect, any two β-lines do not intersect, and intersections between α-lines and β-lines correspond to point-plane incidences. For any two α-lines, any β-line intersecting them corresponds to a plane containing the line through the corresponding points, so at most $\sqrt{n}$ lines from the configuration intersect any pair of α-lines. Similarly, at most $\sqrt{n}$ lines from the configuration intersecting any pair of β-lines. Thus we can apply Theorem 2.2.18 to see that the number of incidences is at most

$$\sqrt{754}(2n)^{\frac{3}{2}} = \sqrt{6032}n^{\frac{3}{2}}.$$

□

Theorem 2.2.20 (Roche-Newton, Rudnev, Shkredov)**.** *If A is a finite subset of the nonzero elements of a field with characteristic p satisfying $|A|^2|AA| \le \frac{11}{12}p^2$, then*

$$|A+A|^2|AA|^3 \ge \frac{|A|^6}{6032}.$$

Proof. We estimate the number N of solutions to the equation

$$a + bcd^{-1} = e + fgh^{-1},$$

with $a,b,c,d,e,f,g,h \in A$, in two ways. By taking $c = d, g = h$ and applying Cauchy-Schwarz we see that

$$N \ge \frac{|A|^4}{|A+A|}|A|^2.$$

Now to each tuple $(a,h,bc) \in A \times A \times AA$ we associate the point (a, bc, h^{-1}), and to each tuple $(d,e,fg) \in A \times A \times AA$ we associate the plane $\{(x,y,z) \mid x + d^{-1}y = e + fgz\}$. This gives us a collection of $|A|^2|AA|$ points and $|A|^2|AA|$ planes in $\mathbb{P}^3$ such that at most $|AA| \le \sqrt{|A|^2|AA|}$ points (respectively planes) lie on any line. By Corollary 2.2.19, we see that

$$\sqrt{6032}(|A|^2|AA|)^{\frac{3}{2}} \ge N \ge \frac{|A|^6}{|A+A|}.$$

□

By a similar argument, we obtain the following.

Theorem 2.2.21 (Roche-Newton, Rudnev, Shkredov)**.** *Let A, B, C be finite subsets of a field of characteristic p. If* $\max(|A|,|B|,|C|)^2 \le |A||B||C| \le \frac{11}{12}p^2$*, then*

$$|A+BC|^2 \ge \frac{|A||B||C|}{6032}.$$

2.2.3 General rings

Theorem 2.2.22 (Katz-Tao Lemma)**.** *Let A be a nonempty finite set of non-zero-divisors of a ring R. There is a subset $B \subseteq A$ such that*

$$|B| \ge \frac{|A|^2}{4|AA|}$$

and such that for any natural numbers k, l we have

$$|kBB - lBB| \leq \left(384\frac{|A+A|^3|AA|^7}{|A|^{10}}\right)^{k+l}|kA - lA|.$$

Proof. By Theorem 2.1.11 we can find a subset $X \subseteq A$ with $|X| \geq \frac{|A|}{2}$ and

$$|AXA| \leq 3\frac{|AA|^2}{|A|^2}|X|.$$

By Cauchy-Schwarz we have

$$\sum_{x\in X}\sum_{y\in A}|xA \cap Xy| \geq \frac{|X|^2|A|^2}{|XA|} \geq \frac{|X|^2|A|^2}{|AA|},$$

so we can pick some $y \in A$ such that

$$\sum_{x\in X}|xA \cap Xy| \geq \frac{|X|^2|A|}{|AA|}.$$

Setting

$$B = \left\{x \in X \mid |xA \cap Xy| \geq \frac{|X||A|}{2|AA|}\right\},$$

we have

$$|B| \geq \frac{|X||A|}{2|AA|}.$$

We now show by induction on h that if $b_1, ..., b_k \in B^h$, then

$$|b_1A + \cdots + b_kA| \leq \left(\frac{4|A+A||AA|}{|A|^2}\right)^{hk}|kA|.$$

Suppose that we have shown this already for h. Letting $b_1, ..., b_k \in B^h$ and $x_1, ..., x_k \in B$, since the b_is and x_is are non-zero-divisors we have

$$|b_ix_iA + b_ix_iA| = |A + A|$$

and

$$|b_ix_iA \cap b_iAy| = |x_iA \cap Ay| \geq \frac{|A|^2}{4|AA|},$$

so by Proposition 2.1.7 we have

$$\begin{aligned}|b_1x_1A + \cdots + b_kx_kA| &\leq \frac{|A+A|}{|x_1A \cap Ay|}\cdots\frac{|A+A|}{|x_kA \cap Ay|}|b_1Ay + \cdots + b_kAy|\\ &\leq \left(\frac{4|A+A||AA|}{|A|^2}\right)^{(h+1)k}|kA|,\end{aligned}$$

completing the induction. A similar statement with both additions and subtractions can be proved in the same way.

Now choose an element $m \in BA$ such that, setting

$$C = \{(b, a) \in B \times A \mid ba = m\},$$

we have

$$|C| \geq \frac{|B||A|}{|BA|} \geq \frac{|A|^2}{2|AA|^2}|X|.$$

Fixing a representation $uv + tw$ for each sum in $BB + BB$, we have an injection

$$(BB + BB) \times C \times C \hookrightarrow \{(c, d, s) \mid c, d \in B^3, s \in cA + dA\},$$

sending $(uv + tw, (b, a), (b', a'))$ to $(uvb, twb', (uv + tw)m)$. Thus, using $|B^3| \leq |AXA| \leq 3\frac{|AA|^2}{|A|^2}|X|$, we have

$$\begin{aligned}|BB + BB| &\leq \left(\frac{|B^3|}{|C|}\right)^2 \left(\frac{4|A + A||AA|}{|A|^2}\right)^6 |A + A| \\ &\leq 6^2 \frac{|AA|^8}{|A|^8} \cdot 4^6 \frac{|A + A|^6 |AA|^6}{|A|^{12}} |A + A| \\ &= 384^2 \frac{|A + A|^6 |AA|^{14}}{|A|^{20}} |A + A|.\end{aligned}$$

By the same argument, for any natural numbers k, l we get

$$|kBB - lBB| \leq \left(384 \frac{|A + A|^3 |AA|^7}{|A|^{10}}\right)^{k+l} |kA - lA|.$$

More generally, we even have

$$|kB^h - lB^h| \leq \left(\frac{|B^{h+1}|}{|C|} \left(\frac{4|A + A||AA|}{|A|^2}\right)^{h+1}\right)^{k+l} |kA - lA|. \qquad \square$$

Theorem 2.2.23 (Self-improving property). *Let A be a finite subset of a ring R, and let D be a nonempty subset of $A - A$. If x is an element of R and $r \in R^*$ is a non-zero-divisor such that*

$$|xA + rA| < \frac{|A|^2}{|D|}$$

then there is an element $d \in (A - A) \setminus D$ such that

$$|xAA + rAA| \leq \frac{|2AA - AA|}{|dA|} |3AA - 2AA|.$$

If we take D to be the set of zero-divisors of $A - A$ and we assume that $D \neq A - A$, then we have

$$|xA + rA| \leq \frac{|2AA - 2AA|}{|A|} |3AA - 3AA|.$$

Proof. By Cauchy-Schwarz, we have

$$\#\{(a,b,a',b') \in A\times A\times A\times A \mid xa+rb = xa'+rb'\} \geq \frac{|A|^4}{|xA+rA|},$$

so

$$\#\{(d,e)\in (A-A)\times(A-A) \mid xd = re\} \geq \frac{|A|^2}{|xA+rA|} > |D|.$$

Since r is a non-zero-divisor, each pair (d,e) with $xd = re$ corresponds to a different value of d. Thus we can find $d \in (A-A)\setminus D$ with $xd \in r(A-A)$. By the Ruzsa covering lemma, there is a set $S \subseteq AA$ with

$$|S| \leq \frac{|dA+AA|}{|dA|} \leq \frac{|2AA-AA|}{|dA|}$$

and

$$AA \subseteq dA - dA + S.$$

Thus we have

$$|xAA+rAA| \leq |xdA - xdA + xS + rAA| \leq |S||r(3AA-2AA)| \leq \frac{|2AA-AA|}{|dA|}|3AA-2AA|.$$

For the last claim, we apply the Ruzsa covering lemma to find $S' \subseteq AA-AA$ with

$$AA-AA \subseteq dA - dA + S'$$

to get

$$|xA+rA| \leq |(xA+rA)(A-A)| \leq |xdA-xdA+xS'+rA(A-A)| \leq \frac{|2AA-2AA|}{|A|}|3AA-3AA|. \qquad \square$$

From here on, we take A to be a subset of a ring R such that $A-A$ contains a non-zero-divisor, and we let D be the set of zero-divisors in $A-A$. For any $r \in R$, we define the set S_r to be

$$S_r = \left\{ x\in R \mid |xA+rA| < \frac{|A|^2}{|D|}\right\}.$$

Proposition 2.2.24. $|A-A|, |A+A| \leq |2AA-2AA|$.

Proposition 2.2.25. *If $r\in R^*$ then $|S_r| < |A-A|^2$. If we also have*

$$|D| \leq \frac{|A|^3}{2|2AA-2AA||3AA-3AA|},$$

then

$$|S_r| < \frac{2|A-A|^2|2AA-2AA||3AA-3AA|}{|A|^3}.$$

Proof. Let $x \in S_r$. By the same argument as in Theorem 2.2.23, we have

$$\#\{(d,e)\in((A-A)\setminus D)\times(A-A) \mid xd=re\} \geq \frac{|A|^2}{|xA+rA|} - |D| \geq \frac{|A|^3}{|2AA-2AA||3AA-3AA|} - |D|.$$

Since for each $(d,e)\in((A-A)\setminus D)\times(A-A)$ there is at most one x such that $xd = re$, we see that

$$|S_r| \leq \frac{(|A-A|-|D|)|A-A|}{\frac{|A|^3}{|2AA-2AA||3AA-3AA|} - |D|}. \qquad \square$$

Proposition 2.2.26. *If $r \in R^*$ and*

$$|D| < \frac{|A|^6}{|A+A||2AA-2AA|^2|3AA-3AA|^2},$$

then S_r is closed under addition (and is therefore an additive group).

Proof. For $x, y \in S_r$, we have

$$|(x+y)A+rA| \leq \frac{|xA+rA|}{|A|}\frac{|yA+rA|}{|A|}|A+A| \leq \frac{|A+A||2AA-2AA|^2|3AA-3AA|^2}{|A|^4} < \frac{|A|^2}{|D|}. \quad \square$$

Proposition 2.2.27. *If*

$$|D| < \frac{|A|^8}{|A+A||2AA-2AA|^3|3AA-3AA|^3},$$

then S_1 is closed under multiplication (and is therefore a ring).

Proof. Suppose $x, y \in S_1$. Apply the Ruzsa covering lemma to find $S \subseteq yA$ with

$$|S| \leq \frac{|yA+A|}{|A|}$$

and

$$yA \subseteq A - A + S.$$

Then we have

$$|xyA+A| \leq |xA - xA + xS + A| \leq \frac{|A+A||2AA-2AA|^3|3AA-3AA|^3}{|A|^6} < \frac{|A|^2}{|D|}. \quad \square$$

Proposition 2.2.28. *If $r \in R^*$, $a \in (A-A) \setminus D$, and*

$$|D| < \frac{|A|^{10}}{|A+A||2AA-2AA|^4|3AA-3AA|^4},$$

then $S_rS_a \subseteq S_{ra}$.

Proof. Take $x \in S_r$ and $y \in S_a$. We have

$$|yA+Aa| \leq \frac{|yA+aA|}{|A|}\frac{|Aa+aA|}{|A|}|A| \leq \frac{|yA+aA||2AA-2AA|}{|A|}.$$

Take $S \subseteq yA$ with

$$|S| \leq \frac{|yA+Aa|}{|A|}$$

and

$$yA \subseteq Aa - Aa + S.$$

Take $S' \subseteq xA - xA$ with

$$|S'| \leq \frac{|xA-xA+rA|}{|A|} \leq \frac{|xA+rA|}{|A|}\frac{|-xA+rA|}{|A|}\frac{|A+A|}{|A|}$$

and

$$xA - xA \subseteq rA - rA + S'.$$

Then

$$|xyA + raA| \leq |xAa - xAa + xS + raA| \leq |S||rAa - rAa + S'a + raA|$$
$$\leq |S||S'||Aa - Aa + aA| \leq \frac{|A+A||2AA-2AA|^4|3AA-3AA|^4}{|A|^8} < \frac{|A|^2}{|D|}. \qquad \square$$

Proposition 2.2.29. *If $r, s \in R$ then $sS_r \subseteq S_{sr}$.*

Proposition 2.2.30. *If $r \in R$ and $|D| < \frac{|A|^2}{|A+A|}$, then $r \in S_r$.*

Proposition 2.2.31. *If $r, s \in R$, then $r \in S_s \iff s \in S_r$.*

Proposition 2.2.32. *If $r, s \in R^*$, $S_r \cap S_s \cap R^* \neq \emptyset$, and*

$$|D| < \frac{|A|^7}{|2AA-2AA|^3|3AA-3AA|^3},$$

then $S_r = S_s$.

Proof. Take $t \in S_r \cap S_s \cap R^*$ and $x \in S_r$. We have

$$|rA + sA| \leq \frac{|tA+rA|}{|A|}\frac{|tA+sA|}{|A|}|A|.$$

Then

$$|xA + sA| \leq \frac{|xA+rA|}{|A|}\frac{|rA+sA|}{|A|}|A| \leq \frac{|2AA-2AA|^3|3AA-3AA|^3}{|A|^5} < \frac{|A|^2}{|D|}. \qquad \square$$

Theorem 2.2.33 (Inhomogeneous sum-product theorem)**.** *Let R be a ring, $A \subseteq R$. If*

$$|(A-A) \setminus R^*| < \min\left(\frac{|A|^2}{|A+AA|}, \frac{|A|^8}{2|A+A||2AA-2AA|^3|3AA-3AA|^3}\right),$$

then there is a subring $S \subseteq R$ such that $A \subseteq S$ and

$$|S| < \frac{2|A-A|^2|2AA-2AA||3AA-3AA|}{|A|^3}.$$

Proof. We take $S = S_1$, then $A \subseteq S_1$ by the assumption $|AA + A| < \frac{|A|^2}{|D|}$. Previous propositions show that S_1 is a ring and give the required bound on the size of S_1. $\square$

Theorem 2.2.34 (Homogeneous sum-product theorem with invertible element)**.** *If R has a 1, $A \subseteq R$ has an invertible element a, and*

$$|(A-A) \setminus R^*| \leq \frac{|A|^8}{2|A+A||2AA-2AA|^3|3AA-3AA|^3},$$

then there is a subring $S \subseteq R$ such that

$$A \subseteq aS = Sa$$

and

$$|S| < \frac{2|A-A|^2|2AA-2AA||3AA-3AA|}{|A|^3}.$$

Proof. We take $S = S_1$. As before, we have S_1 a ring with the required size bound. We have

$$|a^{-1}AA + A| = |AA + aA| \leq |AA + AA| < \frac{|A|^2}{|D|}$$

by our assumption, so $a^{-1}A \subseteq S$, that is, $A \subseteq aS$. Since $SS = S$, we have

$$|aSa^{-1}A + A| \leq |aSa^{-1}aS + aS| = |aS| \leq |S| < \frac{2|2AA - 2AA|^3|3AA - 3AA|}{|A|^3} < \frac{|A|^2}{|D|},$$

so $aSa^{-1} \subseteq S$. Since S is finite, this implies that $aS = Sa$. □

Part IV

Constraints and Polymorphisms

0.1 General Outline

These notes were born from a multi-year learning seminar at MIT[1]. Each section corresponds roughly to a one-hour talk from the seminar, with details filled in. The subsections that occur after some of the sections consist of optional extra material that wasn't covered in the learning seminar due to time constraints. The appendices consist of longer portions of optional material - summaries of famous universal algebraic theories that are useful to know about in order to navigate the literature.

In the next section we give a teaser for the sorts of results we'll try to prove here, mainly to convince the reader that there are highly nontrivial results in this area, and that it is not just abstract nonsense. The text proper begins in Chapter 1, which consists of the foundational abstractions we need later, together with several fundamental examples illustrating three different behaviors of CSPs that need to be understood in order to understand the general case.

In Chapter 2, we go over the breakthrough theory of algebras few subpowers, which lead to the first truly nontrivial algorithmic result in this area. In Chapter 3, we go over the more technically challenging theory of absorbing subalgebras and its application to CSPs of "bounded width" - although the algorithms used to solve CSPs in this chapter are much simpler than the ones from the previous chapter, the algebraic machinery necessary to prove that these algorithms always succeed is much more difficult (but more broadly applicable). Chapters 2 and 3 do not necessarily need to be read in order, as the algebraic approaches used are quite different. In Chapter 4 we move to trying to understand the general case of finite Taylor algebras, starting with the simpler case of conservative Taylor algebras to introduce a few of the new ideas that will be necessary to handle the general case.

Currently these notes are in an unfinished state - maybe half way through the material needed for the CSP dichotomy for finite structures, with much more planned if that is ever finished.

0.2 Introduction / Advertisement

In this section we'll state many of the results and motivating questions that we'll try to understand in these notes. If you don't understand something written here right away, don't despair - we'll go over everything in more detail later. The impatient reader can safely skip ahead to Section 1.1.

The story starts with a result of Schaefer [169] on a problem he called "Generalized Satisfiability".

Definition 0.2.1. If Γ is a set of relations on $\{0,1\}$, then $\mathrm{GenSAT}(\Gamma)$ is the decision problem which takes as input a set of variables V and a collection of *constraints*, where each constraint is of the form "the relation $R(v_1,...,v_k)$ must be satisfied" where $(v_1,...,v_k)$ is a tuple of variables of V and R is a relation from Γ of arity k, and where the desired output is whether or not it is possible to assign values in $\{0,1\}$ to the variables such that the assignment satifies all of the given constraints.

Theorem 0.2.2 (Schaefer [169])**.** *If* $\mathrm{GenSAT}(\Gamma)$ *is not NP-complete, then* Γ *is contained in one of the following sets of relations:*

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- *the set of relations containing the all-*0*s vector,*

- *the set of relations containing the all-*1*s vector,*

- *the set of relations which can be written as an intersection of Horn clauses, where a Horn clause is a disjunction of literals such that at most one variable appears positively,*

- *the set of relations which can be written as an intersection of dual-Horn clauses, where a dual-Horn clause is a disjunction of literals such that at most one variable appears negatively,*

- *the set of relations which can be written as an intersection of relations involving at most two variables,*

- *the set of relations which can be written as solution sets to systems of linear equations over* $\mathbb{F}_2$*.*

In each of these cases, $\mathrm{GenSAT}(\Gamma)$ *can be solved in polynomial time.*

The next result of this form is due to Hell and Nešetril [91], on a generalization of n-coloring which they call "H-coloring".

Definition 0.2.3. If H is a graph, then H-coloring is the decision problem which takes a graph G as input, and where the desired output is whether or not there is a graph homomorphism from G to H.

Note that if we take $H = K_n$, then K_n-coloring is equivalent to n-coloring.

Theorem 0.2.4 (Hell, Nešetril [91])**.** *H-coloring is in P if H is bipartite, and it is NP-complete otherwise.*

These two results led Feder and Vardi [77] to ask whether there is a general dichotomy between P and NP. However, any such dichotomy has to avoid Ladner's [131] anti-dichotomy result.

Theorem 0.2.5 (Ladner [131])**.** *If P $\neq$ NP, then there are problems in NP which are neither in P nor NP-complete.*

In order to avoid Ladner's result, Feder and Vardi focused on a special type of problem: "constraint satisfaction problems" (abbreviated as CSPs) with a fixed "template".

Definition 0.2.6. A CSP-*template* T consists of a finite set D together with a finite collection $\Gamma = (R_1, ..., R_n)$ of relations on D - equivalently, we can think of T as a relational structure $(D, R_1, ..., R_n)$. The decision problem $\mathrm{CSP}(T)$ takes as input a list of variables V and for each $i \le n$ a list of tuples C_i of variables of V which are required to satisfy the constraint R_i, and accepts if there exists an assignment of variables to values in the set D satisfying the given constraints.

Example 0.2.1. The problem k-COLORING (given a graph, determine if it can be colored with k colors) is equivalent to $\mathrm{CSP}(\{1, ..., k\}, \neq) = \mathrm{CSP}(K_k)$, where K_k is the complete graph of k vertices (considered as a relational structure).

Example 0.2.2. The problem 2SAT is equivalent to $\mathrm{CSP}(\{0, 1\}, \le, \neq)$. This problem is in P - in fact, it is known to be NL-complete (NL stands for nondeterministic logspace), and it can be solved in linear time.

Example 0.2.3. The problem 3SAT can be thought of as CSP$(\{0,1\}, R_{(0,0,0)}, ..., R_{(1,1,1)})$, where $R_{(i,j,k)} = \{0,1\}^3 \backslash \{(i,j,k)\}$. We can also simplify this to the equivalent problem CSP$(\{0,1\}, \{0,1\}^3 \backslash \{(0,0,0)\}, \neq)$.

Example 0.2.4. The problem NAE-SAT is CSP$(\{0,1\}, NAE)$, where $NAE = \{0,1\}^3 \backslash \{(0,0,0),(1,1,1)\}$ is the relation that states that the three variables in question are not all equal. This CSP template is known to be NP-complete.

Example 0.2.5. The problem 1-IN-3 SAT is CSP$(\{0,1\}, \{(0,0,1),(0,1,0),(1,0,0)\})$. This CSP template is known to be NP-complete.

Example 0.2.6. The problem HORN-SAT is CSP$(\{0,1\}, \{0\}, \{1\}, \{0,1\}^3 \setminus \{(1,1,0)\})$ (the third constraint is $(x \wedge y) \implies z$). This problem is known to be P-complete, and it can be solved in linear time.

Example 0.2.7. The problem XOR-SAT is CSP$(\{0,1\}, \{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}, \neq)$. This problem is in P - in fact, it can be solved in deterministic time $n^{\log_2(7)}$ and randomized quadratic time [186] (it is unknown if it can be solved in linear time).

Generalizing the XOR-SAT example to a larger domain, we have the following very general family of problems which can be thought of as the natural generalization of systems of linear equations, over a possibly noncommutative group.

Example 0.2.8. Let G be a finite group, and consider the CSP template with domain G, and with a relation gH for every subgroup $H \le G^n$ and every element $g \in G^n$, for every n. Note that strictly speaking, this is not a CSP (as we have defined it) since the set of relations is infinite. Feder and Vardi [77] prove that this general subgroup problem is polynomially solvable.

Based on the examples they knew at the time, Feder and Vardi guessed that tractable CSPs fall into two types: "bounded width" problems, which are solved by local propagation of information, and problems with "the ability to count" such as the subgroup problems above. They further divided the bounded width problems into two main subclasses: problems with "width 1" (such as HORN-SAT) and problems with "bounded strict width" (such as 2-SAT).

The bounded width problems can be defined formally in terms of a logic programming language called *Datalog* (a simple subset of the programming language Prolog), where a program consists of rules for updating a database of known facts about tuples of variables by adding new facts if certain preconditions are met. For instance, a program to determine whether a graph is connected might have two predicates, one for the edges of the graph and another for connectivity, and a rule that says "if connected(a,b) and edge(b,c), then add connected(a,c) to the database". This example program maintains facts about pairs of variables, but has rules that involve examining three variables at a time.

Definition 0.2.7. A CSP has *width* (l,k), $k \ge l$ if it can be solved by a Datalog program which keeps track of facts about tuples of at most l variables, and updates its database by using rules that examine at most k variables at a time. We say that it has *width* l if there exists any k such that it has width (l,k).

In some cases we want to consider CSPs with relations of arbitrarily large arities. In these cases, one uses the concept of *relational width*, introduced by Bulatov [43], where our Datalog program is also allowed to update its database of facts about l-tuples of variables by using rules that examine any set of variables which is contained in the scope of some constraint relation, and to shrink our constraint relations based on facts about l-tuples of variables.

As it turns out, there is a canonical Datalog program for solving problems of width (l, k), which correctly solves every instance of a CSP if and only if the CSP has width (l, k). This program just keeps track of the set of all possible assignments to each tuple of at most l variables, and eliminates possibilities from these lists by brute-forcing the set of possible assignments to each k-tuple of variables in turn (checking for consistency with each subset of these variables of size $\le l$), until it can no longer eliminate any further possible assignments from its database. If there are n variables, this runs in time $O(n^k)$ and space $O(n^l)$.

A slight weakening of the above canonical Datalog program with width 1, in which we only consider one relation at a time in order to remove potential values for the variables, is called "arc-consistency", or sometimes "generalized arc-consistency" if the relations have arity greater than 2. CSPs which can be solved by arc-consistency have a special property called "tree-duality", which says that an instance has a solution if and only its "universal cover" has a solution (the universal cover is an instance with variables and constraints forming an infinite tree that corresponds to the universal cover of the (hyper-)graph of variables and constraints of the original instance).

The width of a CSP can also be defined in terms of a two player game (see [5]), in which one player (the Prover) tries to convince the other player (the Verifier) that an instance of the CSP has a solution. The game goes as follows: in each round of the game, the Prover has assigned values to a certain tuple of at most l variables (at the beginning of the game, this tuple is empty). The Verifier then picks a superset of the previous tuple of size at most k, and challenges the Prover to extend their assignment to this larger collection of variables. After this the Verifier selects any subset of the variables of size at most l, restricting the assignment to that subset, and the next round begins. The Verifier wins if at any point the Prover's assignment fails to satisfy some constraint of the CSP. Then a CSP has width (l, k) if the Prover has a winning strategy only when the problem has a valid global solution.

Definition 0.2.8. A CSP has *strict width* l if, whenever a partial solution to an instance of the CSP has no extension to a full solution, there exists a subset of the partial solution of size at most l, such that this subset already has no extension to a full solution. Equivalently, for every instance of the CSP, the projection of the solution set onto any set of $k > l$ variables is completely determined by the projections of the solution set onto subsets of those variables of size l.

As a consequence of the above definition, if a CSP has strict width l, then any constraint having arity greater than l must be expressible as a conjunction of constraints involving at most l variables. Feder and Vardi [77] prove that one can check whether a CSP has strict width l in time polynomial in the size of the domain and the constraints (for a fixed l), and give a necessary and sufficient criterion in terms of the existence of a near-unanimity operation of arity $l + 1$ which "preserves" the constraints of the CSP.

In trying to understand the set of CSPs which do *not* have bounded width, Feder and Vardi [77] introduced the concept of the *ability to count*. Their definition of this concept is quite technical, and it was later realized that it's enough to focus on a simpler case: the affine CSP over an abelian group.

Definition 0.2.9. For every abelian group A, we define the associated *affine CSP* to be the CSP with domain A, with the ternary relation $\{(x, y, z) \mid x+y+z = a\}$ and the unary singleton relation $\{a\}$ for each element $a \in A$.

In case the reader wants to see the general definition of the ability to count, we have reproduced it below.

Definition 0.2.10. A CSP has the *ability to count* if there are elements $0, 1$ in the domain and there are relations C, Z in the library of constraints such that C is ternary, Z is unary, $(0,0,1), (0,1,0), (1,0,0) \in C$, $0 \in Z$, and any instance of the CSP which satisfies the following properties has no solution:

- the instance only uses the constraints C, Z,
- the constraints of the instance can be partitioned into two parts A, B such that each variable of the instance shows up in exactly one constraint from A and exactly one constraint from B, and
- A contains exactly one more copy of the constraint C than B does.

Following an argument of Razborov for bipartite matching, Feder and Vardi prove the following.

Theorem 0.2.11 (Feder, Vardi [77]). *Any CSP with the ability to count can't be solved by polynomial size monotone circuits. A CSP with the ability to count can never have bounded width.*

They then make the following two outrageous conjectures.

Conjecture 0.2.1. Any CSP which can't "simulate" a CSP which has the ability to count *does* have bounded width.

Conjecture 0.2.2. Any CSP which can't "simulate" 1-IN-3 SAT can be solved in polynomial time.

Shockingly, despite seeming hopelessly vague and intractable, both of these conjectures were recently proven to be *correct*! In fact, the conjecture about the ability to count holds even if we only require that our CSP can't simulate any affine CSP.

The examples of subgroup problems given above together with the concept of the ability to count also prompt the following question.

Problem 0.2.1. What is the largest possible generalization of the Gaussian elimination algorithm?

Feder and Vardi [77] made a first attempt at answering this by introducing the concept of *near-subgroups* of a group, and conjectured that they also lead to CSPs that could be solved in polynomial time. Using a result of Aschbacher [4], Feder [76] later succeeded in showing that near-subgroup problems can be solved in polynomial time.

In this case, however, they could have asked for more. Hubie Chen [56] studied the "expressive rate" of a constraint language Γ, which is defined as the function that takes n to the logarithm of the number of n-variable relations which can be defined as solutions sets to CSPs over Γ. He observed that on a two element domain, this expressive rate always either grows as a polynomial or as an exponential function, and that the cases where it grows polynomially are exactly the cases where the class of relations which can be defined from Γ is "polynomially learnable". The same conjecture occurs in chapter 10 of Víctor Dalmau's thesis [60], in an algebraic form.

Conjecture 0.2.3. For any constraint language Γ, the logarithm of the number of distinct n-variable relations which can be defined by primitive positive formulas over Γ always either grows as a polynomial or as an exponential function. In the case of polynomial growth this class of relations is efficiently learnable and the associated CSP can be solved in polynomial time.

This conjecture was resolved via the theory of algebras with “few subpowers”, which classifies CSPs such that the solution sets always have “compact representations”, and gives general procedures for manipulating these compact representations.

In order to approach these questions, the key conceptual ingredient turned out to be a Galois duality from universal algebra, relating a family of relations to the set of operations which “preserve” the relations. This allows us to view CSPs as algebraic structures in disguise, and to use algebraic techniques to study the structure of their solution sets and to design algorithms. However, the algebraic structures we end up studying are much less structured than groups or lattices - they are in a sense the most basic algebraic structures that have any good properties at all.

The new framework was introduced by Jeavons [102], who reinterpreted an instance of a CSP as a homomorphism problem between relational structures.

Definition 0.2.12. An instance of the *general combinatorial problem*, or GCP, is a pair of relational structures $\langle \mathbf{A}, \mathbf{B}\rangle$ having the same signature (a *relational structure* is a set together with a family of named relations on that set, and the *signature* of a relational structure is a list of names of relations together with specifications of their arities). The question is whether there exists a homomorphism from $\mathbf{A}$ to $\mathbf{B}$.

Example 0.2.9. Suppose that $\mathbf{T}$ is a CSP template (in the sense of Feder and Vardi above), interpreted as a relational structure $(D, R_1, ..., R_n)$. To any instance of the CSP, we can associate a relational structure $\mathbf{X} = (V, C_1, ..., C_n)$, where V is the set of variables of the instance, and each C_i is a list of those tuples of variables of V which are required to satisfy the constraint R_i. Then a homomorphism of relational structures $\mathbf{X} \to \mathbf{T}$ is the same as an assignment of values in D to each variable in V, such that each tuple of variables in each C_i is mapped to an element of R_i. In other words, the GCP instance $\langle \mathbf{X}, \mathbf{T}\rangle$ is equivalent to the instance of CSP($\mathbf{T}$) corresponding to $\mathbf{X}$.

Jeavons also gives a few ways for other well-known problems (not CSPs) to be realized as instances of his general combinatorial problem.

Example 0.2.10. If G is a graph and K_q is a clique with q vertices, then the GCP instance $\langle K_q, G\rangle$ is the q-CLIQUE problem. Note that in this case, the *target* of the homomorphism is the main variable, while the source stays fixed (aside from the parameter q).

Example 0.2.11. Let $G = (V, E)$ be a graph on n vertices, and let $C_n = (W, F)$ be a cycle on n vertices. Then the GCP instance $\langle (W, F, \neq), (V, E, \neq)\rangle$ is the problem of determining whether G has a Hamiltonian circuit.

These other sorts of problems, where the target of the homomorphism varies arbitrarily and the source varies according to some parameter can be studied from the point of view of *parametrized complexity* and *fixed parameter tractability*. It turns out that hardness and easiness in this alternative setting is determined by the *treewidth* of the source structures [88]. We won’t discuss this research direction much.

After demonstrating the generality of the framework, Jeavons [102] restricts to studying homomorphism problems with a fixed target structure $\mathbf{T}$. He calls this GCP(Γ), where Γ is the list of relations of $\mathbf{T}$, but we will call it CSP($\mathbf{T}$) in these notes. Note that this is the same problem as the CSP defined in the sense of Feder and Vardi above, but the instances are now treated as relational structures (which is useful notationally), and the new perspective in terms of homomorphisms gives a hint of a more algebraic approach. For instance, the homomorphism point of view prompts the following definition.

Definition 0.2.13. Two relational structures $\mathbf{A}, \mathbf{B}$ with the same signature are *homomorphically equivalent* if there exist homomorphisms $\mathbf{A} \to \mathbf{B}, \mathbf{B} \to \mathbf{A}$.

The homomorphism point of view now makes it obvious that if $\mathbf{A}$ and $\mathbf{B}$ are homomorphically equivalent, then CSP($\mathbf{A}$) and CSP($\mathbf{B}$) are equivalent problems - that is, a "yes" instance of one will always be a "yes" instance of the other. For instance, every bipartite graph H having at least one edge is homomorphically equivalent to the complete graph K_2 on two vertices, so if H is bipartite then the H-coloring problem is trivial.

Jeavons [102] points out that for a given CSP template, one can introduce new relations without changing the complexity of the CSP so long as these new relations are built out of the old relations in certain ways. Specifically, Jeavons shows that up to logspace reductions, we may as well assume that the collection of relations Γ contains the equality relation, and is closed under the following four operations:

- permutation of inputs,
- adding dummy variables (extra variables which are ignored by the relation),
- existential projection onto a subset of the variables, and
- intersection.

Note that any new relation which can be built out of these four operations can be viewed as the solution set to some instance of CSP(Γ), projected onto some subset of the variables. We can also think of the new relation as being defined by a *primitive positive formula*, that is, a formula built out of the existential quantifier $\exists$, the relations R_i of Γ (and equality), and conjunctions $\wedge$, but which does not involve negation, disjunction, implication, or universal quantification (such a formula is called a *conjunctive query* in database theory).

Example 0.2.12. The template we gave for HORN-SAT did not contain all possible Horn clauses - it stopped at the 3-ary Horn clause $x \wedge y \implies z$. The 4-ary Horn clause $x \wedge y \wedge z \implies w$ can be represented by the following primitive positive formula:

$$\exists u \ (x \wedge y \implies u) \wedge (u \wedge z \implies w).$$

The Horn clauses of higher arity can be represented by primitive positive formulas over HORN-SAT in a similar way.

Definition 0.2.14. A set of relations Γ on a fixed domain D is called a *relational clone* if it contains the equality relation, and is closed under permutations, adding dummy variables, projection, and intersections. Equivalently, a relational clone is a set of relations which is closed under defining new relations via primitive positive formulas.

The connection to algebra comes from the following fundamental result.

Theorem 0.2.15. *There is a Galois duality between relational clones and clones. In particular, a relational clone is completely determined by its set of "polymorphisms", that is, the set of functions that "preserve" all of the relations of* Γ.

In order to understand this result we must define clones, polymorphisms, and the concept of a function preserving a relation.

Definition 0.2.16. A set of functions $D^k \to D, k \in \mathbb{N}$ is called a *clone* if it contains the *projections* $\pi_i^k : D^k \to D$ which satisfy $\pi_i^k(x_1, ..., x_k) = x_i$ (generally the superscript k is omitted when it is clear), and is closed under *composition*, the operation which takes a k-ary function f and k l-ary functions $g_1, ..., g_k$ to the function

$$(f \circ (g_1, ..., g_k)) : (x_1, ..., x_l) \mapsto f(g_1(x_1, ..., x_l), ..., g_k(x_1, ..., x_l)).$$

The reader should play with the above definition in order to convince themself that every natural method of building new functions from old functions can be described in terms of the composition operation given above together with the projections π_i^k. For instance, the function $f(x, g(y, x))$ can be built out of f and g as follows:

$$(f \circ (\pi_1, g \circ (\pi_2, \pi_1)))(x, y) = f(x, g(y, x)).$$

Definition 0.2.17. A k-ary function f is said to *preserve* an m-ary relation R, written $f \rhd R$, if for every choice of k m-tuples in R, applying f componentwise produces a new m-tuple which is also in R. If we think of elements of R as column vectors, we can write this as

$$\begin{bmatrix} x_{11} \\ \vdots \\ x_{1m} \end{bmatrix}, ..., \begin{bmatrix} x_{k1} \\ \vdots \\ x_{km} \end{bmatrix} \in R \implies f\left(\begin{bmatrix} x_{11} \\ \vdots \\ x_{1m} \end{bmatrix}, ..., \begin{bmatrix} x_{k1} \\ \vdots \\ x_{km} \end{bmatrix}\right) = \begin{bmatrix} f(x_{11}, ..., x_{k1}) \\ \vdots \\ f(x_{1m}, ..., x_{km}) \end{bmatrix} \in R.$$

A function f is a *polymorphism* of a relational structure (D, Γ) or of a relational clone Γ if f preserves R_i for each relation $R_i \in \Gamma$.

The concept of preservation can be understood in two different ways. From the relational point of view, we have $f \rhd R$ iff $f : D^k \to D$ is a homomorphism of relational structures $(D, R)^k \to (D, R)$, where $(D, R)^k$ is the categorical kth power of the relational structure (D, R) (the kth power of (D, R) has underlying set D^k and relation R^k given by listing all m-tuples of k-tuples such that the m-tuple of ith coordinates is in R for each $i \le k$). From the algebraic point of view, we have $f \rhd R$ iff the subset $R \subseteq D^m$ is a subalgebra of the algebraic structure $(D, f)^m$, where $(D, f)^m$ is the categorical mth power of the algebraic structure (D, f), where the basic operation is simply f acting componentwise on D^m.

The Galois duality between relational clones and clones prompts a shift in ones way of thinking about CSPs. Instead of studying a CSP template, one studies an algebraic structure whose operations are the polymorphisms of the original CSP template. Constraints that can be expressed in terms of the original library of relations become *subalgebras* of powers of this algebraic structure. Instances of a CSP become questions about whether intersections of various subalgebras of a power of the original algebra are empty or not.

Example 0.2.13. Suppose $\mathbb{A} = (\mathbb{Z}/p, f)$ is the algebraic structure with basic operation $f : (x, y, z) \mapsto x - y + z \pmod p$ for some prime p. Then a subalgebra of $\mathbb{A}^n$ - that is, a subset which is closed under f - is exactly the same as an *affine linear subspace* of $(\mathbb{Z}/p)^n$ (recall that affine linear subspaces are like vector subspaces, but that they might not pass through the origin). Checking whether a collection of affine linear subspaces has a nonempty intersection is equivalent to solving a system of linear equations $\pmod p$.

By using an old result classifying the minimal (nontrivial) clones on the domain $\{0,1\}$ (under the Galois duality, a minimal clone of functions corresponds to a maximal relational clone), Jeavons [102] was able to give a new and relatively simple proof of Schaefer's dichotomy theorem [169]. The algebraic structures corresponding to the basic polynomial time solvable problems are as follows.

Example 0.2.14. If $\Gamma = (\{0\}, \{1\}, \{0,1\}^3 \setminus \{1,1,0\})$ is the template corresponding to HORN-SAT, then the clone of polymorphisms is generated by the function $\min : \{0,1\}^2 \to \{0,1\}$. This operation is an example of a *semilattice* operation.

Example 0.2.15. If $\Gamma = (\le, \ne)$ is the template corresponding to 2SAT, then the clone of polymorphisms is generated by the *majority* (or *median*) function $\mathrm{maj} : \{0,1\}^3 \to \{0,1\}$.

Example 0.2.16. If $\Gamma = (\{(0,0,0),(0,1,1),(1,0,1),(1,1,0)\}, \ne)$ is the template corresponding to XOR-SAT, then the clone of polymorphisms is generated by the *affine linear* function $(x,y,z) \mapsto x - y + z \pmod 2$ (this function is sometimes referred to as the *minority* function).

Early results focused on generalizing these basic examples, and developing the algebraic perspective further:

- If all polymorphisms of Γ are unary, then CSP(Γ) is NP-hard by a gadget reduction from NAE-SAT (if the domain has size 2) or k-coloring (if the domain has size $k \ge 3$).

- Generalized arc-consistency solves any CSP which has an associative, commutative, idempotent polymorphism. These types of operations were called ACI operations at the time, but are now generally referred to as semilattice operations.

- Later, Dalmau and Pearson [66] showed that generalized arc-consistency solves a CSP *iff* it has a "set" polymorphism, also known as a family of "totally symmetric" polymorphisms, where the output depends only on the set of inputs and not on their order or multiplicity.

- Already in Feder and Vardi's work [77], it was shown that a CSP has strict width l iff it has an $l+1$-ary "near-unanimity" polymorphism, that is, an operation such that whenever all but one of the inputs are equal, their common value is the output. This fact is closely connected to a result in universal algebra known as the Baker-Pixley theorem [7].

- Bulatov and Dalmau [41] gave an algorithm generalizing Gaussian elimination as well as the algorithm for the general subgroup problem introduced by Feder and Vardi to the case of CSPs with a *Mal'cev polymorphism* $p(x,y,z)$, that is, a polymorphism satisfying the identity $p(x,y,y) = x = p(y,y,x)$ for all x,y. In the case of groups such an operation is given by $(x,y,z) \mapsto xy^{-1}z$, but such operations also exist in quasigroups, making this a very wide generalization.

- One can restrict to the case where Γ contains a unary relation for every singleton subset of the domain. On the algebraic side, this corresponds to restricting to the case of idempotent algebras (that is, algebras where every singleton subset forms a subalgebra).

- At this point multiple authors started to realize that whether a CSP is hard or not doesn't depend on the particular polymorphisms, but rather on the *identities* that are satisfied by the polymorphisms. One of the first papers to point this out was a paper by Bulatov and Jeavons [49] which also introduced a notion of polymorphisms for *multisorted* relations, as well as the use of "tame congruence theory" from universal algebra.

- In particular, it was shown that if no finite subset of the identities satisfied by the polymorphisms imply that the polymorphisms can't be unary, then CSP(Γ) is NP-hard by a gadget reduction from NAE-SAT or k-coloring. It was conjectured that this is an if and only if, that is, if a CSP has a family of polymorphisms that satisfy a nontrivial set of identities, then the CSP can always be solved in polynomial time.

- Identities which do not involve composing functions with each other were singled out as special (such identities are called *linear* identities, or identities of *height at most* 1). In [103], a simple procedure was given to transform the search for polymorphisms of Γ into an equivalent *indicator instance* of CSP(Γ) - and a simple modification of this procedure can be used to search for the polymorphisms which satisfy a given collection of linear identities. In order to attack the *meta-problem* (which takes the set of relations Γ as input, and determines whether a given type of algorithm can solve CSP(Γ) as output), one then only has to find a way to solve the corresponding type of indicator instance.

The first big result was a comprehensive generalization of Gaussian elimination, generalizing the algorithm for Mal'cev operations as far as was reasonably possible. The basic idea here is to represent the solution space of all the constraints processed so far by giving a small generating set for that solution space, considered as a subalgebra of a power of the domain. In order for any algorithm along these lines to exist, there must first be a guarantee that every subalgebra of any power of the domain actually *has* a small generating set.

Theorem 0.2.18 (Few Subpowers [26], [101])**.** *The following are equivalent for an algebraic structure $\mathbb{A}$ on a finite domain:*

- *the number of subalgebras of $\mathbb{A}^n$ grows like $2^{O(n^k)}$ for some fixed k,*

- *every subalgebra of $\mathbb{A}^n$ has a (nice) generating set of size $O(n^k)$ for some fixed k, called a* compact representation*,*

- *$\mathbb{A}$ has a k-*edge term *for some k, that is, a term satisfying the "shepherd's crook" identity*

$$f\left(\begin{bmatrix} y & y & x & x & \cdots & x \\ y & x & y & x & \cdots & x \\ x & x & x & y & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & x & x & \cdots & y \end{bmatrix}\right) = \begin{bmatrix} x \\ x \\ x \\ \vdots \\ x \end{bmatrix},$$

where all but the first column has exactly one y and $k-1$ xs, and

- *a Gaussian-elimination-like algorithm solves* CSP($\mathbb{A}$) *in polynomial time (the degree of the polynomial may depend on k).*

Note that when $k = 2$, a k-edge term is the same as a Mal'cev operation (up to permuting the inputs). Additionally, if a k-edge term ignores its first input, then it is a k-ary near-unanimity term. So few subpowers algebras generalize both subgroup CSPs and CSPs with bounded strict width. There is still an important open question connected to few subpowers algebras, related to the following algebraic problem.

Problem 0.2.2 (Subpower Membership Problem)**.** Given a finite subset $S \subseteq \mathbb{A}^n$ and an element $x \in \mathbb{A}^n$, determine if x is in the subalgebra of $\mathbb{A}^n$ generated by S.

A result of Kozik [125] shows that for general algebraic structures, this problem can be EXPTIME-complete. A recent result of Shriner [172] has upgraded this hardness result to most situations where algebras do not automatically have few subpowers, such as the case of congruence distributive algebras.

Conjecture 0.2.4. If $\mathbb{A}$ has few subpowers, then the subpower membership problem can be solved in polynomial time.

Peter Mayr [141] has shown that this conjecture holds for nilpotent Mal'cev algebras of prime power order (and for expansions of such algebras). In a different direction, a recent result of Bulatov, Mayr, and Szendrei [42] has proved the conjecture in the special case that the algebra $\mathbb{A}$ is "residually small" (for those with a little universal algebra background, this means that every subdirectly irreducible algebra in the variety it generates has size bounded by some fixed cardinal - in the case of groups, it is equivalent to all Sylow subgroups being abelian). In the same paper, they also show that the subpower membership problem for algebras with few subpowers is always in NP. As far as I know, the above conjecture is still open even for the special case of quasigroups.

The second big result in this story was the classification of CSPs with bounded width, together with a surprising "collapse" of the bounded width hierarchy. The ideas used in the proof of this result - especially the theory of absorbing subalgebras - led to a number of breakthrough results in universal algebra.

Theorem 0.2.19 (Bounded Width Algebras [44], [13], [19], [133], [20], [128], [126])**.** *For an idempotent algebra on a finite domain, the following are equivalent:*

- CSP($\mathbb{A}$) *has bounded width,*
- CSP($\mathbb{A}$) *can't simulate any CSP which has the ability to count,*
- CSP($\mathbb{A}$) *has relational width* $(2, 3)$*,*
- CSP($\mathbb{A}$) *can be solved by a "cycle consistency" algorithm, which checks arc-consistency and checks that every "cycle" of constraints has a valid solution extending each possible value of every variable in the cycle,*
- $\mathbb{A}$ *has terms* f, g *of arity* 3 *satisfying the identities*

$$g(x, x, y) = g(x, y, x) = g(y, x, x) = f(x, x, y) = f(x, y, x) = f(x, y, y),$$

- CSP($\mathbb{A}$) *can be "robustly" solved by the basic semidefinite programming relaxation (i.e., if an* ϵ*-portion of the constraints are garbled, then the basic SDP can be used to find an assignment that satisfies all but an* $f(\epsilon)$*-portion of the constraints, where* $f(\epsilon) \to 0$ *as* $\epsilon \to 0$*).*

Furthermore, there is a polynomial time algorithm for checking whether a relational structure (which has all unary singleton relations) has bounded width, and to find terms f, g as in the above theorem if it does. This algorithm leverages the fact that the canonical width-$(2, 3)$ Datalog program will correctly solve any instance of any bounded width CSP in polynomial time, and constructs a

CSP whose solution corresponds to a pair of polymorphisms satisfying a nice set of identities. A similar algorithm is not known to exist for checking that a CSP has width 1.

The third big result of the subject was a nice classification of the algebras which were conjectured to correspond to CSPs with polynomial time algorithms. This result was proved with the absorption theory that had been developed for the study of bounded width algebras.

Theorem 0.2.20 (Cyclic Terms for Taylor Algebras [18])**.** *For an idempotent algebraic structure $\mathbb{A}$ on a finite domain, the following are equivalent:*

- *there is a finite set of identities satisfied by the operations of $\mathbb{A}$ which can't be satisfied by essentially unary functions,*
- *for every prime $p > |\mathbb{A}|$, $\mathbb{A}$ has a "cyclic term" f of arity p, that is, a term which satisfies the identity*
$$f(x_1, ..., x_{p-1}, x_p) = f(x_2, ..., x_p, x_1),$$
- *$\mathbb{A}$ has a 4-ary term t which satisfies the identity*
$$t(x, x, y, z) = t(y, z, z, x).$$

With this in hand, the main conjecture of the subject was finally possible to state simply: "CSP($\mathbb{A}$) is in P iff $\mathbb{A}$ has a cyclic term."

The fourth big result of the subject concerns a generalization of CSPs to "valued constraints".

Definition 0.2.21. A *valued constraint* on k variables is a cost function from D^k to $(-\infty, \infty]$. An instance of a valued CSP (abbreviated VCSP) consists of a sum of valued constraints applied to various tuples of the variables, possibly with nonnegative coefficients. The goal is to minimize the sum of the cost functions. The associated CSP to a VCSP is the problem of finding an assignment that makes all of the costs finite.

The Galois duality between clones and relational clones can be generalized to a duality between VCSP templates and "fractional polymorphisms" - essentially just formal convex combinations of ordinary polymorphisms, with the property that when they are applied to any tuple of elements of D^k, on average they decrease the cost assigned by any cost function from the VCSP template.

The standard example of a valued constraint with an interesting fractional polymorphism is a *submodular function*, that is, a cost function c defined on a lattice (or a power of a lattice) which satisfies the inequality
$$\tfrac{1}{2}c(A) + \tfrac{1}{2}c(B) \ge \tfrac{1}{2}c(A \vee B) + \tfrac{1}{2}c(A \wedge B).$$
It is well known that submodular cost functions can be minimized using a linear programming relaxation.

Theorem 0.2.22 (VCSP Dichotomy [129], [122])**.** *If a VCSP has its associated CSP in P and has a cyclic fractional polymorphism, then by using the algorithm for the associated CSP as a black box to get the set of "feasible" values for each variable and applying the basic linear programming relaxation to the restriction of the VCSP to the feasible values we get the minimum cost solution. If the VCSP has no cyclic fractional polymorphisms, then it is NP-hard.*

Finally, the biggest result of all was recently proved independently by Bulatov and by Zhuk.

Theorem 0.2.23 (CSP Dichotomy [48], [190])**.** *A finite algebra* $\mathbb{A}$ *has a cyclic term iff* $\mathrm{CSP}(\mathbb{A})$ *is in P (assuming P* $\neq$ *NP).*

A major open problem is whether one can test if $\mathrm{CSP}(\Gamma)$ is in P in time polynomial in the size of the description of the constraints of Γ (given that Γ contains all singleton unary relations). Zhuk tells me that he conjectures it to be NP-hard to test for the existence of cyclic terms. If so, perhaps this could lead to a new form of public key cryptography, where the private key is a cyclic term, and the public key is a CSP template which is preserved by that cyclic term...

The story has not ended with the proof of the main conjecture. There are at least six interesting research directions that are still being actively investigated: qualitative CSPs, counting complexity, promise problems, quantified CSPs, "hybrid" tractability (combining restrictions on both the source and the target relational structures), and planar CSPs.

Qualitative CSPs come from allowing the domain of the CSP to be infinite. Of course, this immediately leads to problems, for instance, how can one even specify a set of relations on an infinite domain? The idea, capturing good old fashioned AI intuition about "qualitative" reasoning, is to require that the specific values of the solutions are not important, just the qualitative relationships between them. To make this more precise, we require our domain (and the relations on it) to have a very large automorphism group.

Definition 0.2.24. A permutation group G acting on a set S is *oligomorphic* if S^n has finitely many G-orbits for every $n \geq 1$.

A standard example of an oligomorphic group is the group of order-preserving bijections on the rational numbers. Relations invariant under this group give rise to "temporal" CSPs, where the goal is to find some assignment of variables to times satisfying constraints about their relative ordering.

Bodirsky, in his habilitation thesis [32] introduced qualitative CSPs and gave a number of classification results. Before beginning a classification, he first chooses an oligomorphic group G acting on a countable set S. He then uses results from model theory (specifically, ω-categorical theories and Fraïssé limits) as well as structural Ramsey theory (and the theory of extremely amenable groups) to understand the relations which are invariant under G and their polymorphisms, and for several groups G he succeeds in finding a complete classification of problems into "easy" and "hard". The main three cases considered by Bodirsky [32] are the following:

- the automorphism group of $(\mathbb{Q}, <)$, corresponding to temporal CSPs,
- the automorphism group of the random graph, for which he proves "Schaefer's Theorem for graphs" (such CSPs can be interpreted as problems where the variables correspond to decisions about whether certain pairs of vertices of an unknown graph are connected by an edge or not), and
- the automorphism group of an infinite branching tree structure $(L, |)$, where $|$ is a 3-ary relation where $ab|c$ means that the youngest common ancestor of a, b lies below the youngest common ancestor of b, c - the invariant relations correspond to "branching time constraints", or "phylogeny constraints", and the associated CSPs could in principle be of interest to biologists.

Recent results on QCSPs indicate that the difficulty of the classification results seems to be related to the orbit growth function of the oligomorphic group G, which takes n to the number of orbits of n-tuples under G [33]. For sufficiently small orbit growth functions, a dichotomy result has been proven (using the finite case as a black-box). The main conjecture in this field is the following somewhat technical statement.

Conjecture 0.2.5 ([25])**.** Let $\mathbf{A}$ be the core of a reduct of a finitely bounded homogeneous structure. Then CSP($\mathbf{A}$) is in P iff $\mathbf{A}$ has a 6-ary polymorphism s and unary polymorphisms α, β satisfying the "pseudo-Siggers" identity:

$$\alpha \circ s(x, y, x, z, y, z) = \beta \circ s(y, x, z, x, z, y).$$

Otherwise, CSP($\mathbf{A}$) is NP-complete.

The most recent development in the study of CSPs is the study of promise problems. Promise problems are similar in spirit to approximation algorithms, but much more amenable to an algebraic approach. A promise problem is defined here to be a pair of problems, one more restrictive than the other, where the goal is to give an algorithm which correctly says "yes" if the less restrictive problem has a solution and says "no" if the more restrictive problem has no solution (if neither case holds, any output is allowable).

Definition 0.2.25. If $\mathbf{A}, \mathbf{B}$ are relational structures such that a homomorphism $\mathbf{A} \rightarrow \mathbf{B}$ exists, then PCSP($\mathbf{A}, \mathbf{B}$) is the following problem. The input is a relational structure $\mathbf{C}$ s.t. there exists a homomorphism $\mathbf{C} \rightarrow \mathbf{A}$ (the promise), although this map is not revealed to us. The desired output is a homomorphism from $\mathbf{C}$ to $\mathbf{B}$.

A typical strategy for proving tractability of PCSP($\mathbf{A}, \mathbf{B}$) is to find a relational structure $\mathbf{X}$ such that there exist homomorphisms $\mathbf{A} \rightarrow \mathbf{X} \rightarrow \mathbf{B}$ and such that CSP($\mathbf{X}$) is in P.

Example 0.2.17. Let $\mathbf{A}$ be 1-IN-3 SAT and let $\mathbf{B}$ be NAE-SAT (where the 1-IN-3 relation and the NAE relation have the same name in the signature). The identity map on the domain gives a homomorphism $\mathbf{A} \rightarrow \mathbf{B}$ since the 1-IN-3 relation is contained in the NAE relation. Although both problems are NP-complete, the PCSP associated to the pair is tractable: let $\mathbf{X} = (\mathbb{Z}, x+y+z = 1)$, note that the inclusion map $\mathbf{A} \rightarrow \mathbf{X}$ is a homomorphism, and that the map $\operatorname{sgn} : \mathbb{Z} \rightarrow \{0, 1\}$ given by

$$\operatorname{sgn}(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x \geq 1 \end{cases}$$

defines a homomorphism $\mathbf{X} \rightarrow \mathbf{B}$. The CSP associated to $\mathbf{X}$ is tractable (even though it is defined over an infinite domain), since it boils down to solving a system of linear equations over the integers. It is not possible to find a *finite* relational structure $\mathbf{X}$ with polynomial time CSP that fits between 1-IN-3 SAT and NAE-SAT (see [10] for a proof).

The relevant algebraic object in this context is Pol($\mathbf{A}, \mathbf{B}$), the set of homomorphisms $\mathbf{A}^k \rightarrow \mathbf{B}$. At first this structure doesn't seem algebraic at all, since there is no way to compose elements of Pol($\mathbf{A}, \mathbf{B}$). However, one can still write down "minor identities" between the functions in Pol($\mathbf{A}, \mathbf{B}$) such as $f(x, x, y) = g(y, x)$, and compare the set of minor identities obtained to the identities that occur in polymorphism algebras of tractable CSPs. This approach of studying minor identities has been surprisingly useful, and has led to the proposal to call sets of functions such as Pol($\mathbf{A}, \mathbf{B}$) "minions" (a competing proposed name is "clonoid").

Unlike the situation for CSPs, it is still quite hard to prove hardness results for PCSPs. The following basic problem is still wide open.

Conjecture 0.2.6. For any $k \geq l \geq 3$, the promise problem $\mathrm{PCSP}(K_l, K_k)$ is NP-hard (this problem is the problem of k-coloring a graph which is promised to be l-colorable).

One of the first results in this direction concerned a PCSP called $(2+\epsilon)$-SAT, where one is given clauses of $2k+1$ variables and wants to satisfy the associated instance of SAT given the promise that it is possible to find an assignment in which every clause has at least k satisfied literals. The $(2+\epsilon)$-SAT problem was proven to be NP-hard [6], and this result was slightly generalized and put into the PCSP framework in [38].

Recent results in the study of PCSPs include a result of Barto, Bulin, Opršal, and Krokhin [10] in which they used minion techniques to show that $\mathrm{PCSP}(K_d, K_{2d-1})$ is NP-hard for every $d \geq 3$, reducing from the hypergraph promise problem $\mathrm{PCSP}(\mathrm{NAE}_2, \mathrm{NAE}_k)$. The hypergraph coloring problem $\mathrm{PCSP}(\mathrm{NAE}_2, \mathrm{NAE}_k)$ was itself shown to be hard via a reduction from a variant of the PCP theorem [70]. The big result in PCSPs is the following result which connects computational complexity to height 1 identities satisfied by the minion of polymorphisms.

Theorem 0.2.26 (Barto, Bulin, Opršal, Krokhin [10])**.** *If there is a "minion homomorphism" from* $\mathrm{Pol}(\mathbf{A}_1, \mathbf{B}_1)$ *to* $\mathrm{Pol}(\mathbf{A}_2, \mathbf{B}_2)$*, then* $\mathrm{PCSP}(\mathbf{A}_2, \mathbf{B}_2)$ *has a logspace reduction to* $\mathrm{PCSP}(\mathbf{A}_1, \mathbf{B}_1)$*.*

Remark 0.2.1. For those who like category theory, an abstract minion is just a covariant functor from the category of (finite) sets to the category of sets, and a minion homomorphism is just a natural transformation of functors. We could say that a "representation" of an abstract minion over A, B is a natural transformation to the functor $I \mapsto \mathrm{Hom}(A^I, B)$.

$\mathrm{PCSP}(\mathbf{A}, \mathbf{B})$ ends up being logspace equivalent to the problem of distinguishing between diagrams in the category of sets of size at most N (N any fixed large enough number) which have a nonempty limit ("yes" instances), and diagrams such that the image under the minion $\mathrm{Pol}(\mathbf{A}, \mathbf{B})$ has an empty limit ("no" instances) (this is the "promise satisfaction of a minor condition" problem of [10]).

0.3 Incomplete list of Notation and Definitions

Most of the notation is either standard, or will be defined as it is introduced. In this section we record some of the definitions so that the reader can refer back to it if necessary. (Much more comprehensive background on structures can be found in [96] or [97].)

Definition 0.3.1. A (first order) *structure* $\mathbf{A}$ is a tuple $(A, \{f_i\}, \{R_j\})$ such that A is a set (called the *underlying set* of $\mathbf{A}$), each f_i is a function of some arity n_i on A, i.e. $f_i : A^{n_i} \to A$, and each R_j is a relation of some arity m_j on A, i.e. $R_j \subseteq A^{m_j}$.

The *signature* of a first order structure is the assignment of each function symbol f_i to an arity n_i together with an assignment of each relation symbol R_j to some arity m_j. If two structures $\mathbf{A}, \mathbf{B}$ share the same signature, then we sometimes write $f_i^{\mathbf{A}}$, $R_j^{\mathbf{A}}$ to refer to the interpretations of the function and relation symbols in $\mathbf{A}$, and $f_i^{\mathbf{B}}, R_j^{\mathbf{B}}$ to refer to the interpretations of the same function and relation symbols in $\mathbf{B}$.

If $\mathbf{A}, \mathbf{B}$ are structures with the same signature, then a *homomorphism* $\varphi : \mathbf{A} \to \mathbf{B}$ is a map $\varphi : A \to B$ of the underlying sets, such that for each function symbol f_i we have

$$\varphi(f_i^{\mathbf{A}}(a_1, ..., a_{n_i})) = f_i^{\mathbf{B}}(\varphi(a_1), ..., \varphi(a_{n_i}))$$

for all $a_1, ..., a_{n_i} \in A$, and such that for each relation symbol R_j we have

$$(a_1, ..., a_{m_j}) \in R_j^{\mathbf{A}} \quad \implies \quad (\varphi(a_1), ..., \varphi(a_{m_j})) \in R_j^{\mathbf{B}}$$

for all $a_1, ..., a_{m_j} \in A$. A homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$ is called an *isomorphism* if there is a homomorphism $\psi : \mathbf{B} \to \mathbf{A}$ such that $\psi \circ \varphi = \mathrm{id}_{\mathbf{A}}$ and $\varphi \circ \psi = \mathrm{id}_{\mathbf{B}}$.

A structure is called a *relational structure* if it has no functions in its signature, and a structure is called an *algebraic structure* or an *algebra* if it is nonempty and has no relations in its signature. We usually write a relational structure with the bold font, i.e. $\mathbf{A}$, while we usually write an algebraic structure with the blackboard bold font, i.e. $\mathbb{A}$ (note that many authors reverse this convention).

We define the *total size* of a relational structure $\mathbf{A} = (A, \{R_j\})$, written $\|\mathbf{A}\|$, to be

$$\|\mathbf{A}\| \coloneqq \sum_j |R_j|.$$

Note that the number of bits needed to describe $\mathbf{A}$ is larger than $\|\mathbf{A}\|$ by a factor of about $\log_2 |A|$, ignoring the overhead needed to describe the signature of $\mathbf{A}$.

When convenient, we often abuse notation to treat a structure $\mathbf{A}$ like its underlying set A: we write $a \in \mathbf{A}$ to mean that $a \in A$, we write $|\mathbf{A}|$ for $|A|$, etc.

Definition 0.3.2. A *subalgebra* of an algebra $\mathbb{A}$ is an algebraic structure $\mathbb{B}$ with the same signature such that the underlying set B of $\mathbb{B}$ is a subset of the underlying set A of $\mathbb{A}$, and such that the inclusion map $\iota : B \hookrightarrow A$ defines a homomorphism $\iota : \mathbb{B} \to \mathbb{A}$. In this case we write $\mathbb{B} \le \mathbb{A}$. If $S \subseteq A$, then we define $\mathrm{Sg}_{\mathbb{A}}(S)$, the *subalgebra generated by* S, to be the smallest subalgebra of $\mathbb{A}$ whose underlying set contains S.

If $\mathbb{A}, \mathbb{B}$ are structures with the same signature, then we define their *product* $\mathbb{A} \times \mathbb{B}$ to be the structure with underlying set $A \times B$ where A, B are the underlying sets of $\mathbb{A}, \mathbb{B}$, with each function symbol f_i interpreted by

$$f_i^{\mathbb{A}\times\mathbb{B}}\Big(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, ..., \begin{bmatrix} a_{n_i} \\ b_{n_i} \end{bmatrix} \Big) = \begin{bmatrix} f_i^{\mathbb{A}}(a_1, ..., a_{n_i}) \\ f_i^{\mathbb{B}}(b_1, ..., b_{n_i}) \end{bmatrix},$$

and with each relation symbol R_j interpreted by

$$\Big(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, ..., \begin{bmatrix} a_{m_j} \\ b_{m_j} \end{bmatrix} \Big) \in R_j^{\mathbb{A}\times\mathbb{B}} \quad \iff \quad (a_1, ..., a_{m_j}) \in R_j^{\mathbb{A}} \wedge (b_1, ..., b_{m_j}) \in R_j^{\mathbb{B}}.$$

Arbitrarily large products $\prod_{i\in I} \mathbb{A}_i$ of structures $\mathbb{A}_i$ all having the same signature are defined similarly. (This definition matches with the category-theoretic definition of the product, in the category of structures with a fixed signature.)

A *homomorphic image* of $\mathbb{A}$ is defined to be any $\mathbb{B}$ of the same signature such that there is a surjective homomorphism $\varphi : \mathbb{A} \twoheadrightarrow \mathbb{B}$. An algebraic structure $\mathbb{A}$ is called *simple* if every surjective homomorphism $\mathbb{A} \twoheadrightarrow \mathbb{B}$ is either an isomorphism, or has $|\mathbb{B}| = 1$.

Definition 0.3.3. A *congruence* on an algebraic structure $\mathbb{A}$ is an equivalence relation θ on $\mathbb{A}$ which is also a subalgebra: $\theta \le \mathbb{A}^2$. The set of all congruences of $\mathbb{A}$ is written as $\mathrm{Con}(\mathbb{A})$. If S is some collection of ordered pairs $(a, b) \in \mathbb{A}^2$, then the *congruence generated by* S, written $\mathrm{Cg}_{\mathbb{A}}(S)$, is the least congruence of $\mathbb{A}$ which contains S. We generally use greek letters for congruences.

If $\theta \in \mathrm{Con}(\mathbb{A})$ and $a \in \mathbb{A}$, then we write a/θ for the *congruence class* of a with respect to θ, that is, $a/\theta = \{b \mid (a,b) \in \theta\}$. We say that a, b are *congruent* with respect to θ if $(a,b) \in \theta$, and we may write this in symbols in several different ways:

$$(a,b) \in \theta \iff b \in a/\theta \iff a \equiv b \pmod{\theta} \iff a \equiv_\theta b \iff a\ \theta\ b.$$

If $\theta \in \mathrm{Con}(\mathbb{A})$, then we write $\mathbb{A}/\theta$ for the set of equivalence classes of θ considered as an algebraic structure, with

$$f_i^{\mathbb{A}/\theta}(a_1/\theta, ..., a_{n_i}/\theta) = f_i^{\mathbb{A}}(a_1, ..., a_{n_i})/\theta.$$

The algebra $\mathbb{A}/\theta$ is called a *quotient* of $\mathbb{A}$, and there is a canonical *quotient map* $\mathbb{A} \twoheadrightarrow \mathbb{A}/\theta$ which takes each element of $\mathbb{A}$ to its equivalence class in θ.

If $\varphi : \mathbb{A} \to \mathbb{B}$ is a homomorphism, then we define the *kernel* of φ, written $\ker\varphi$, as the equivalence relation on $\mathbb{A}$ given by $(x,y) \in \ker\varphi$ iff $\varphi(x) = \varphi(y)$. The kernel of a homomorphism is always a congruence, and if φ is surjective then $\mathbb{A}/\ker\varphi$ is isomorphic to $\mathbb{B}$. In particular, every homomorphic image of $\mathbb{A}$ is isomorphic to a quotient of $\mathbb{A}$.

For $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$, we define their *meet* $\alpha \wedge \beta$ to be their intersection, and we define their *join* $\alpha \vee \beta$ to be the congruence generated by $\alpha \cup \beta$. We define the least congruence $0_{\mathbb{A}}$ to be the equivalence relation on $\mathbb{A}$ where each equivalence class is a singleton, and we define the greatest congruence $1_{\mathbb{A}}$ to be the equivalence relation on $\mathbb{A}$ consisting of just one equivalence class.

Definition 0.3.4. If $R \subseteq A_1 \times \cdots \times A_n$ is a (possibly multisorted) relation, then for any $I \subseteq \{1, ..., n\}$ we define the projection map $\pi_I : R \to \prod_{i\in I} A_i$ in the obvious way. A relation $R \subseteq A_1 \times \cdots \times A_n$ is *subdirect* if $\pi_i(R) = A_i$ for all i.

If $\mathbb{A}_1, ..., \mathbb{A}_n$ are algebraic structures with the same signature, then any subalgebra $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is called a (multisorted) *relation* on the $\mathbb{A}_i$s. If $\pi_i(\mathbb{R}) = \mathbb{A}_i$ for all i then we say that $\mathbb{R}$ is subdirect, and we write this in symbols as $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$.

Definition 0.3.5. A *tolerance* $\mathbb{S}$ on an algebraic structure $\mathbb{A}$ is a binary relation $\mathbb{S} \le \mathbb{A} \times \mathbb{A}$ which is symmetric and contains the diagonal. Note that the transitive closure of any tolerance is automatically a congruence. We say that a tolerance is *connected* if its transitive closure is the full congruence $1_{\mathbb{A}}$.

If $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, then we define the ith *link tolerance* of $\mathbb{R}$ to be the binary relation

$$\{(b,c) \mid \exists a_j \text{ s.t. } (a_1, ..., a_{i-1}, b, a_{i+1}, ..., a_n), (a_1, ..., a_{i-1}, c, a_{i+1}, ..., a_n) \in \mathbb{R}\}$$

on $\mathbb{A}_i$. A binary relation is called *linked* if its link tolerances are connected.

Definition 0.3.6. If $R \subseteq A \times B$ is a binary relation, then we define its *reverse* $R^- \subseteq B \times A$ by

$$R^- = \{(b,a) \mid (a,b) \in R\}.$$

If $R \subseteq A \times B$ and $S \subseteq B \times C$, then we define their *relational composition* $R \circ S \subseteq A \times C$ by

$$R \circ S = \{(a,c) \mid \exists b \in B \text{ s.t. } (a,b) \in R \wedge (b,c) \in S\}.$$

If $R \subseteq A \times B$ and $U \subseteq A$, then we define the *sum* $U + R \subseteq B$ by

$$U + R = \{b \mid \exists a \in U \text{ s.t. } (a,b) \in R\},$$

and for $V \subseteq B$ we define the difference $V - R \subseteq A$ by $V - R = V + R^-$. Note that we have $(U + R) + S = U + (R \circ S)$ for $U \subseteq A$, $R \subseteq A \times B$, $S \subseteq B \times C$.

Definition 0.3.7. If $\{R_j\}$ is some collection of relations, then a *primitive positive formula* over the R_js is defined to be a formula of the form

$$\exists y_1 \cdots \exists y_n \text{ s.t. } \bigwedge_{i \in I} \varphi_i(x_1, ..., x_m, y_1, ..., y_n),$$

where I is a finite set, and each φ_i is either some relation R_j applied to some of the variables $x_1, ..., x_m, y_1, ..., y_n$, or is the equality relation applied to some pair of variables.

A relation R is *primitively positively definable* over the R_js if there is a primitive positive formula φ such that the elements of R are exactly the tuples of values $(x_1, ..., x_m)$ that satisfy φ. For instance, the relational composition $R \circ S$ is primitively positively definable over R and S.

A *relational clone* is a collection of relations (on a common domain) which is closed under primitive positive definitions. The smallest relational clone which contains $\{R_i\}$ is written as $\langle \{R_i\} \rangle$.

The algebraic analogue of primitive positive formulas is the concept of *terms*.

Definition 0.3.8. If $\{f_i\}$ is a collection of function symbols in some fixed signature, then we define a *term* inductively as follows:

- each variable x_j is a term, corresponding to the operation π_j, and
- if f_i is function symbol of arity k and $t_1, ..., t_k$ are terms, then $f_i(t_1, ..., t_k)$ is also a term, corresponding to the operation $f_i \circ (t_1, ..., t_k)$.

For instance, if f is binary and g is ternary, then the expression

$$f(g(x, y, f(x, z)), f(u, v))$$

is a term. Every term can be visualized as a labeled ordered tree, where every leaf is labeled by a variable and where every internal vertex is labeled by a function whose arity is equal to the number of children of that vertex. The *height* of a term is the largest distance between the root of this tree and any leaf.

In some cases, it may be more efficient to visualize terms via directed acyclic graphs, to avoid repeatedly drawing many copies of a common subtree - to distinguish these points of view, a tree representation of a term is called a *formula*, while a directed acyclic graph representation of a term is called a *circuit*.

The collection of all terms in a given signature σ defines the *term algebra* $\mathcal{F}_\sigma(\{x_j\})$, with each function symbol f_i of arity k interpreted as the operation

$$f_i^{\mathcal{F}_\sigma(\{x_j\})} : (t_1, ..., t_k) \mapsto f_i(t_1, ..., t_k),$$

where the right hand side is interpreted as an abstract term. The term algebra is also called the *absolutely free algebra* in the signature σ.

Definition 0.3.9. An *identity* is just a pair of terms s, t with the symbol $\approx$ in between them:

$$s \approx t.$$

The identity $s \approx t$ has *height 1* if both terms s and t have height 1, and the identity $s \approx t$ is called *linear* if both s and t have height at most 1.

An algebraic structure *satisfies* the identity $s \approx t$, written $\mathbb{A} \models s \approx t$, if

$$\forall x_1, ..., x_k \in \mathbb{A},\ \ s^{\mathbb{A}}(x_1, ..., x_k) = t^{\mathbb{A}}(x_1, ..., x_k),$$

where $x_1, ..., x_k$ is a list of all of the variables which occur in s or t.

Definition 0.3.10. If $\mathcal{T}$ is a set of identities in the signature σ, then we define the *variety* $\mathcal{V}(\mathcal{T})$ to be the collection of all algebraic structures $\mathbb{A}$ with signature σ such that $\mathbb{A} \models \mathcal{T}$.

If $\{\mathbb{A}_i\}$ is a collection of algebraic structures in the signature σ, then the *variety generated by* $\{\mathbb{A}_i\}$, written $\mathcal{V}(\{\mathbb{A}_i\})$, is the variety $\mathcal{V}(\mathcal{T})$ where $\mathcal{T}$ is the collection of all identities $s \approx t$ which are satisfied in every single $\mathbb{A}_i$.

Definition 0.3.11. If $\mathcal{V} = \mathcal{V}(\mathcal{T})$ is a variety with signature σ and defining identities $\mathcal{T}$, then we define the congruence $\approx_{\mathcal{V}}$ on the absolutely free algebra $\mathcal{F}_\sigma(\{x_j\})$ to be the congruence generated by the set of pairs of terms $(s \circ (u_1, ..., u_k), t \circ (u_1, ..., u_k))$ such that $s \approx t \in \mathcal{T}$, where $u_1, ..., u_k$ is an arbitrary list of terms which we use to replace the variables $x_1, ..., x_k$ which occur in s and t. The algebraic structure

$$\mathcal{F}_\sigma(\{x_j\})/{\approx_{\mathcal{V}}}$$

is called the *free algebra on the generators* $\{x_j\}$ in the variety $\mathcal{V}$, and is written as $\mathcal{F}_{\mathcal{V}}(\{x_j\})$. By construction, the free algebra $\mathcal{F}_{\mathcal{V}}(\{x_j\})$ satisfies every identity in $\mathcal{T}$. In the language of category theory, the functor

$$\mathcal{F}_{\mathcal{V}} : S \mapsto \mathcal{F}_{\mathcal{V}}(\{x_j\}_{j \in S})$$

is adjoint to the forgetful functor from algebras in $\mathcal{V}$ to their underlying sets:

$$\mathrm{Hom}_{\mathrm{Set}}(S, A) = \mathrm{Hom}_{\mathcal{V}}(\mathcal{F}_{\mathcal{V}}(S), \mathbb{A})$$

when $\mathbb{A} \in \mathcal{V}$ is an algebraic structure with underlying set A.

We define a *term operation* of a variety $\mathcal{V}$ to be an equivalence class $t/{\approx_{\mathcal{V}}}$ together with a set of variables containing all variables which occur in t. If $\mathbb{A}$ is an algebraic structure, then we define the *term operations* of $\mathbb{A}$ to be the collection of interpretations $t^{\mathbb{A}}$ of terms t as operations on $\mathbb{A}$ - these are easily seen to correspond with the term operations of the variety $\mathcal{V}(\mathbb{A})$ generated by $\mathbb{A}$.

Viewing each k-ary term operation $t^{\mathbb{A}} : \mathbb{A}^k \to \mathbb{A}$ as an element of $\mathbb{A}^{\mathbb{A}^k}$, we get an alternative construction of the free algebra $\mathcal{F}_{\mathcal{V}(\mathbb{A})}(x_1, ..., x_k)$:

$$\begin{aligned}\mathcal{F}_{\mathcal{V}(\mathbb{A})}(x_1, ..., x_k) &= \{k\text{-ary term operations } t \text{ of } \mathcal{V}(\mathbb{A})\}\\ &\cong \{k\text{-ary term operations } t^{\mathbb{A}} \text{ of } \mathbb{A}\}\\ &\cong \mathrm{Sg}_{\mathbb{A}^{\mathbb{A}^k}}\{\pi_1, ..., \pi_k\}.\end{aligned}$$

More generally, if $\mathcal{V} = \mathcal{V}(\{\mathbb{A}_i\}_{i \in I})$, then $\mathcal{F}_{\mathcal{V}}(x_1, ..., x_k)$ is isomorphic to a subalgebra of $\prod_{i \in I} \mathbb{A}_i^{\mathbb{A}_i^k}$.

Definition 0.3.12. If $\mathbb{A}$ is an algebraic structure, then the *clone* of $\mathbb{A}$, written as $\mathrm{Clo}(\mathbb{A})$, is the collection of all term operations of $\mathbb{A}$. If $\{f_i\}$ is a collection of operations on the domain A, then we write $\langle\{f_i\}\rangle$ for the clone of the algebraic structure $(A, \{f_i\})$. Alternatively, a *clone* on a domain A is just a collection of operations of A which is closed under composition.

If $\mathcal{V}$ is a variety, then the *clone* of $\mathcal{V}$, written as $\mathrm{Clo}(\mathcal{V})$, is defined to be the collection of free algebras

$$\mathcal{F}_{\mathcal{V}}(x_1, ..., x_k),$$

together with the rules for composing a k-ary term operation $t \in \mathcal{F}_\mathcal{V}(x_1, ..., x_k)$ with k-tuples of m-ary term operations $u_1, ..., u_k \in \mathcal{F}_\mathcal{V}(x_1, ..., x_m)$ to produce the m-ary term operation $t \circ (u_1, ..., u_k)$.

An *abstract clone* is a collection of function symbols $\{f_i\}$ in a signature σ which is closed under a composition law $\circ$ which takes a function symbol f of arity k and a k-tuple of function symbols $g_1, ..., g_k$ of arity m as input and produces a function symbol $f \circ (g_1, ..., g_k)$ of arity m as output, satisfying a generalized associativity law:

$$(f \circ (g_1, ..., g_k)) \circ (h_1, ..., h_m) = f \circ (g_1 \circ (h_1, ..., h_m), ..., g_k \circ (h_1, ..., h_m)),$$

with special "projection" function symbols π_i^k of every arity k which satisfy

$$\pi_i^k \circ (g_1, ..., g_k) = g_i$$

and

$$f \circ (\pi_1^k, ..., \pi_k^k) = f.$$

Note that in any abstract clone, the collection of function symbols of arity 1 always forms a semigroup under $\circ$ with identity element π_1^1.

A *clone homomorphism* is a map ξ from the function symbols of one abstract clone to the function symbols of another abstract clone which preserves arities, sends the projections π_i^k to themselves, and respects composition:

$$\xi(f \circ (g_1, ..., g_k)) = \xi(f) \circ (\xi(g_1), ..., \xi(g_k)).$$

A *height 1 clone homomorphism*, also known as a *minion homomorphism*, is a map ξ which preserves arities and respects composition with projections:

$$\xi(f \circ (\pi_{i_1}^m, ..., \pi_{i_k}^m)) = \xi(f) \circ (\pi_{i_1}^m, ..., \pi_{i_k}^m).$$

Definition 0.3.13. A *poset* is a relational structure $(P, \le)$ such that $\le$ is a *partial order* - that is, a reflexive and transitive relation which satisfies $a \le b \wedge b \le a \implies a = b$. More generally, a *quasiorder* (also called a *preorder*) is any reflexive and transitive relation. We usually use $\preceq$ to denote a quasiorder and $\le$ to denote a partial order. If $\preceq$ is a quasiorder, then we define an *associated equivalence relation* $\sim$ by

$$a \sim b \iff a \preceq b \wedge b \preceq a.$$

For any $a \le b \in P$, we define the *interval* between a and b, written $[\![a, b]\!]$, to be the set of $c \in P$ such that $a \le c \le b$. (We use this notation to avoid confusion with the commutator, which is written as $[\cdot, \cdot]$.)

We say that b is a *cover* of a, written $a \prec b$ (if there is no danger of confusion with a strict quasiorder), if $a < b$ and there is no $c \in P$ such that $a < c < b$, that is, if $[\![a, b]\!] = \{a, b\}$.

Definition 0.3.14. A *lattice* is either an algebraic structure $\mathcal{L} = (L, \wedge, \vee)$ which satisfies the identities

$$\begin{aligned} &x \wedge x \approx x, && x \vee x \approx x,\\ &x \wedge y \approx y \wedge x, && x \vee y \approx y \vee x,\\ &x \wedge (y \wedge z) \approx (x \wedge y) \wedge z, && x \vee (y \vee z) \approx (x \vee y) \vee z,\\ &x \wedge (x \vee y) \approx x, && x \vee (x \wedge y) \approx x,\end{aligned}$$

or it is a poset $\mathcal{L} = (L, \leq)$ such that every pair of elements $\{a, b\} \subseteq \mathcal{L}$ has a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$, or it is a first-order structure $\mathcal{L} = (L, \wedge, \vee, \leq)$ which satisfies

$$a \leq b \iff a = a \wedge b \iff a \vee b = b$$

as well as the algebraic identities above. Note that the operations $\wedge, \vee$ are determined by $\leq$, and the partial order $\leq$ is determined by either of the operations $\wedge$ or $\vee$. A $0, 1$-*lattice* is a lattice with named constants $0, 1$ which are repectively the least and greatest elements of $\mathcal{L}$.

A *lattice homomorphism* is a homomorphism of the algebraic structure $(L, \wedge, \vee)$. Note that there may be some homomorphisms of the relational structure $(L, \leq)$ which do not count as lattice homomorphisms - a map which respects the partial order $\leq$ is just called a *monotone* map.

Chapter 1

Initial Intuition

1.1 The Inv-Pol Galois connection

We begin by recalling some definitions from the introduction.

Definition 1.1.1. A set of relations Γ on a fixed domain D is called a *relational clone* if it contains the equality relation, and is closed under permutations, adding dummy variables, existential projection, and intersections. Equivalently, a relational clone is a set of relations which is closed under defining new relations via primitive positive formulas.

Definition 1.1.2. A set of functions $D^k \to D, k \in \mathbb{N}$ is called a *clone* if it contains the *projections* $\pi_i^k : D^k \to D$ which satisfy $\pi_i^k(x_1, ..., x_k) = x_i$ (generally the superscript k is omitted when it is clear), and is closed under *composition*, the operation which takes a k-ary function f and k l-ary functions $g_1, ..., g_k$ to the function

$$(f \circ (g_1, ..., g_k)) : (x_1, ..., x_l) \mapsto f(g_1(x_1, ..., x_l), ..., g_k(x_1, ..., x_l)).$$

Definition 1.1.3. A k-ary function f is said to *preserve* an m-ary relation R, written $f \rhd R$, if for every choice of k m-tuples in R, applying f componentwise produces a new m-tuple which is also in R. If we think of elements of R as column vectors, we can write this as

$$\begin{bmatrix} x_{11} \\ \vdots \\ x_{1m} \end{bmatrix}, ..., \begin{bmatrix} x_{k1} \\ \vdots \\ x_{km} \end{bmatrix} \in R \implies f\left(\begin{bmatrix} x_{11} \\ \vdots \\ x_{1m} \end{bmatrix}, ..., \begin{bmatrix} x_{k1} \\ \vdots \\ x_{km} \end{bmatrix}\right) = \begin{bmatrix} f(x_{11}, ..., x_{k1}) \\ \vdots \\ f(x_{1m}, ..., x_{km}) \end{bmatrix} \in R.$$

A function f is a *polymorphism* of a relational structure (D, Γ) or of a relational clone Γ if f preserves R_i for each relation $R_i \in \Gamma$.

We can write the condition for $f \rhd R$ more compactly as $M \in R^k \implies f(M) \in R$, where $M \in R^k$ means that M is a matrix with k columns, each of which belongs to R, and $f(M)$ is the column vector obtained by applying f to the rows of M.

In order to state the Galois connection, we need a few additional definitions.

Definition 1.1.4. If Γ is any set of relations on a domain D, then we define $\langle\Gamma\rangle$ to be the relational clone generated by Γ (that is, $\langle\Gamma\rangle$ is the smallest relational clone which contains Γ). Similarly, if

$\mathcal{O}$ is any set of operations on D, we define $\langle\mathcal{O}\rangle$ to be the clone generated by $\mathcal{O}$. If $\mathbb{A} = (D, \mathcal{O})$ is an algebraic structure, we let $\mathrm{Clo}(\mathbb{A})$ be the clone generated by the basic operations of $\mathbb{A}$, so $\mathrm{Clo}(\mathbb{A}) = \langle\mathcal{O}\rangle$.

Definition 1.1.5. If Γ is any set of relations on a domain D, then we define $\mathrm{Pol}(\Gamma)$ to be the the set of operations on D that preserve every relation of Γ. If $\mathcal{O}$ is any set of operations on D, we define $\mathrm{Inv}(\mathcal{O})$ to be the set of relations which are preserved by every operation in $\mathcal{O}$. If we want to restrict to operations or relations of a specific arity, we use the notations

$$\begin{aligned}\mathrm{Pol}_k(\Gamma) &= \{f : D^k \to D \mid \forall R \in \Gamma,\ f \rhd R\},\\ \mathrm{Inv}_m(\mathcal{O}) &= \{R \subseteq D^m \mid \forall f \in \mathcal{O},\ f \rhd R\}.\end{aligned}$$

It is worth thinking about what sort of information about an algebraic structure $(D, \mathcal{O})$ can be found in $\mathrm{Inv}(\mathcal{O})$.

Example 1.1.1. If $\mathbb{A} = (D, \mathcal{O})$ is an algebraic structure, then $\mathrm{Inv}_2(\mathcal{O})$ determines (among other things)

- the lattice of subalgebras of $\mathbb{A}$,
- $\mathrm{Aut}(\mathbb{A})$, the automorphism group of $\mathbb{A}$,
- $\mathrm{End}(\mathbb{A})$, the semigroup of endomorphisms of $\mathbb{A}$,
- $\mathrm{Con}(\mathbb{A})$, the lattice of congruences on $\mathbb{A}$,
- the set of partial orders on D which are compatible with the operations of $\mathbb{A}$, and
- $\mathrm{Inv}_2(\mathbb{B})$ for any subalgebra $\mathbb{B} \subset \mathbb{A}$ or quotient $\mathbb{B} = \mathbb{A}/{\sim}$.

It is easy to see that for all Γ, $\mathrm{Pol}(\Gamma)$ will be a clone, and that for all $\mathcal{O}$, $\mathrm{Inv}(\mathcal{O})$ will be a relational clone. As a consequence, we have $\langle\Gamma\rangle \subseteq \mathrm{Inv}(\mathrm{Pol}(\Gamma))$ and $\langle\mathcal{O}\rangle \subseteq \mathrm{Pol}(\mathrm{Inv}(\mathcal{O}))$. The next two results show that these inclusions are actually equalities.

Before diving into the proof, the following concrete example will be useful for understanding the notation. Consider what it means for a ternary function f to preserve the binary relation $\le$ (functions which preserve $\le$ are often called *monotone*). Since $0 \le 0$, $0 \le 1$, and $1 \le 1$, we have

$$\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \in \le \quad\implies\quad \begin{bmatrix}f(0,0,1)\\f(0,1,1)\end{bmatrix} \in \le,$$

that is, $f(0,0,1) \le f(0,1,1)$. It's convenient to abbreviate the above as follows:

$$\begin{bmatrix}0 & 0 & 1\\0 & 1 & 1\end{bmatrix} \in \le^3 \quad\implies\quad f\left(\begin{bmatrix}0 & 0 & 1\\0 & 1 & 1\end{bmatrix}\right) \in \le .$$

Theorem 1.1.6. *If* Γ *is a set of relations on a finite domain* D*, then* $\mathrm{Inv}(\mathrm{Pol}(\Gamma)) = \langle\Gamma\rangle$*. In fact, if a relation* $S \subseteq D^m$ *is preserved by* $\mathrm{Pol}(\Gamma)$ *and can be generated by* k *elements of* D^m *(using operations of* $\mathrm{Pol}(\Gamma)$*), then* S *can be defined by a primitive positive formula over* Γ *which involves at most* $|D|^k$ *auxiliary variables.*

Proof. Suppose that S is generated by elements $x_1, ..., x_k \in D^m$, and let X be the matrix having the x_is as columns. Then $S = \{f(X) \mid f \in \mathrm{Pol}_k(\Gamma)\}$, so as a starting point we will construct a primitive positive formula Φ that describes $\mathrm{Pol}_k(\Gamma)$.

Note that D^{D^k} is naturally interpreted as the set of functions $f : D^k \to D$: if $f \in D^{D^k}$, then the $(a_1, ..., a_k)$-coordinate of f is $f(a_1, ..., a_k)$. We can now give a positive primitive formula for $\mathrm{Pol}_k(\Gamma) \subseteq D^{D^k}$:

$$\Phi(f) \coloneqq \bigwedge_{R \in \Gamma} \bigwedge_{M \in R^k} f(M) \in R.$$

If Γ is infinite, the outer $\bigwedge$ will be an infinite conjunction. However, since there are only finitely many possible subsets of D^{D^k}, some finite subset Φ' of the inner conjunctions will define the same subset of D^{D^k}.

Finally, to define S we use the primitive positive formula

$$S(a) \coloneqq \exists f \in D^{D^k}\ \Phi'(f) \wedge (f(X) = a). \qquad \square$$

Theorem 1.1.7. *If $\mathcal{O}$ is a set of operations on a finite domain D, then* $\mathrm{Pol}(\mathrm{Inv}(\mathcal{O})) = \langle \mathcal{O} \rangle$.

Proof. Suppose that $f \in \mathrm{Pol}(\mathrm{Inv}(\mathcal{O}))$ is a k-ary function. Let $\mathcal{F}(k) \subseteq D^{D^k}$ be the subalgebra of the algebraic structure $(D, \mathcal{O})^{D^k}$ generated by the functions $\pi_i : D^k \to D$, $\pi_i(x_1, ..., x_k) = x_i$. Then $\mathcal{F}(k)$, interpreted as a set of functions from D^k to D, is exactly the set of k-ary functions in $\langle \mathcal{O} \rangle$.

Since $f \in \mathrm{Pol}(\mathrm{Inv}(\mathcal{O}))$ and $\mathcal{F}(k) \in \mathrm{Inv}(\mathcal{O})$, we must have $f \rhd \mathcal{F}(k)$, so in particular we must have $f(\pi_1, ..., \pi_k) \in \mathcal{F}(k)$. But $f(\pi_1, ..., \pi_k)$ is exactly f thought of as an element of D^{D^k}, so this means that $f \in \langle \mathcal{O} \rangle$. $\square$

Corollary 1.1.8. *There is an order reversing bijection between clones and relational clones, given by the operations* Inv *and* Pol.

Remark 1.1.1. The map $\{1, ..., k\} \to D^{D^k}$ given by $i \mapsto \pi_i$, where $\pi_i : D^k \to D$ is given by $\pi_i(x_1, ..., x_k) = x_i$, shows up in the theory of approximation algorithms as the *long code*, which is the longest way of encoding $\{1, ..., k\}$ over the alphabet D which doesn't have any redundant coordinates.

Example 1.1.2. In the next section we will prove the following three correspondences between clones and relational clones on the domain $\{0, 1\}$:

- $\langle \mathrm{2SAT} \rangle = \langle \leq, \neq \rangle$ corresponds to $\langle \mathrm{maj} \rangle$ (the majority function on three inputs),
- $\langle \text{HORN-SAT} \rangle = \langle \{0\}, \{1\}, x \wedge y \implies z \rangle$ corresponds to $\langle \min \rangle$ (the minimum function on two inputs), and
- $\langle \text{XOR-SAT} \rangle = \langle \{1\}, x + y + z \equiv 0 \pmod 2 \rangle$ corresponds to $\langle x - y + z \pmod 2 \rangle$.

Definition 1.1.9. If $\mathbb{A}, \mathbb{A}'$ are two algebraic structures on the same domain such that every basic operation of $\mathbb{A}'$ is in $\mathrm{Clo}(\mathbb{A})$, then we say that $\mathbb{A}'$ is a *reduct* of $\mathbb{A}$ and that $\mathbb{A}$ is an *expansion* of $\mathbb{A}'$. If $\mathrm{Clo}(\mathbb{A}) = \mathrm{Clo}(\mathbb{A}')$, then $\mathbb{A}$ and $\mathbb{A}'$ are called *term equivalent*.

The lattice of clones on the domain $\{0, 1\}$ has been completely described - it has countably many elements, and is known as Post's lattice [158] (see also chapter II.3 of [134]). It is known that on a domain of size ≥ 3, there are uncountably many clones [188], [189] (see also chapter II.8 of [134]). In particular, we see that most clones and relational clones can't be generated by finitely many functions or relations.

Definition 1.1.10. A clone $\mathcal{O}$ is said to be *finitely generated* if there is a finite set S of operations such that $\mathcal{O} = \langle S\rangle$. It is said to be *finitely related* if there is a finite set of relations Γ such that $\mathcal{O} = \mathrm{Pol}(\Gamma)$.

Example 1.1.3. The clone on $\{0,1\}$ generated by the binary implication function $\rightarrow$, given by $\rightarrow(x,y) = \neg x \vee y$, is finitely generated but not finitely related. One quick way to prove this is to show that for every $n \geq 3$, the n-ary threshold function t_2^n defined by

$$t_2^n(x_1,...,x_n) = \begin{cases} 1 & \sum_i x_i \geq 2 \\ 0 & \sum_i x_i \leq 1 \end{cases}$$

is not in $\langle\rightarrow\rangle$, but every way of identifying two coordinates of t_2^n gives a function which is in $\langle\rightarrow\rangle$ (exercise: why does this prove that $\langle\rightarrow\rangle$ can not be finitely related?). $\mathrm{Inv}(\rightarrow)$ is generated by the infinite sequence of relations $R_1, R_2, ...$ given by $R_n = \{0,1\}^n \setminus \{(0,...,0)\}$, and $\langle\rightarrow\rangle$ consists of all functions of the form $f(x_1,...,x_n) \vee x_i$.

Matthew Moore [146] has shown that determining whether a given finitely generated clone is finitely related is a Turing-complete problem, and therefore undecidable in general. It is conjectured that determining whether a given finitely related clone is finitely generated is also undecidable in general.

Remark 1.1.2. The Galois connection between relational clones and clones on a finite set was originally discovered by Geiger in 1968 [84], and Reinhard Pöschel investigated the general case (where the domain may be infinite) in [157] - in the infinite case, the main difference is that clones must also be taken to be closed in the topology of pointwise convergence. (Jeavons reproved one direction of the connection - that relational clones on a finite domain are determined by their polymorphisms - in [102].)

Remark 1.1.3. The Galois connection presented here, between operations and relations, can be viewed as being induced by the two-sorted preservation relation $\rhd$. In general, whenever one has a two-sorted binary relation R on a pair of sets A, B, one can define operations F, G on the power sets of A, B respectively by

$$F(S) = \{b \in B \mid \forall a \in S\ aRb\},$$
$$G(T) = \{a \in A \mid \forall b \in T\ aRb\}.$$

The abstract order-theoretic properties of such a pair F, G are

- F and G are antitone: $S \subseteq S' \implies F(S) \supseteq F(S')$, and similarly for G, and
- for any $S \in \mathcal{P}(A)$ and $T \in \mathcal{P}(B)$, we have
$$T \subseteq F(S) \iff S \subseteq G(T).$$

Actually the first property listed is redundant, as we have

$$(S \subseteq S') \wedge (F(S') \subseteq F(S')) \implies S \subseteq S' \subseteq G(F(S')) \implies F(S') \subseteq F(S),$$

and either of F, G is determined by the other together with the second property: $F(S) = \bigcup_{S \subseteq G(T)} T$. Additionally, the second property follows from the first property together with $S \subseteq G(F(S))$ and $T \subseteq F(G(T))$: for one direction, we have

$$T \subseteq F(S) \implies S \subseteq G(F(S)) \subseteq G(T).$$

Any such pair F, G determines the binary relation R, since

$$(a,b) \in R \iff b \in F(\{a\}) \iff a \in G(\{b\}),$$

and F, G are both determined by the second property and their restrictions to singletons, since

$$b \in F(S) \iff \forall a \in S,\ a \in G(\{b\}) \iff \forall a \in S,\ b \in F(\{a\}).$$

Then one can define closure operators $G \circ F, F \circ G$ on subsets of A and B. When we say these are "closure operators", we mean that the images of these operators form collections of "closed" sets, such that any intersection of closed sets is closed, and for $S \subseteq A$, $G \circ F(S)$ is equal to the smallest closed set which contains S. All of these properties are easy to show directly in terms of the binary relation R, but they can also be proved order theoretically.

For the order theoretic proof, note that

$$F(S) \subseteq F(S) \implies S \subseteq G \circ F(S),$$

and similarly for $F \circ G$, and so we have

$$F(S) \subseteq F \circ G(F(S)) = F(G \circ F(S)) \subseteq F(S),$$

and we see that a set in $X \subseteq B$ is closed iff it is of the form $F(S)$ for some $S \subseteq A$. For the intersection property, note that

$$S \subseteq G(X) \cap G(Y) \iff X \cup Y \subseteq F(S) \iff S \subseteq G(X \cup Y),$$

and for the characterization of the closure of S we have

$$G \circ F(S) \subseteq G(Y) \iff Y \subseteq F \circ G \circ F(S) = F(S) \iff S \subseteq G(Y).$$

Then F and G will provide a Galois correspondence between the closed subsets of A and the closed subsets of B. The nontrivial thing to do is to describe the closure operators explicitly.

In our case, the relation R was given by $\rhd$, and the sets A, B were the sets of operations and relations on a given domain. Our main difficulty was in proving that the closure operators $G \circ F = \mathrm{Pol} \circ \mathrm{Inv}$ and $F \circ G = \mathrm{Inv} \circ \mathrm{Pol}$ were concretely described by the closure operators $\mathcal{O} \mapsto \langle \mathcal{O} \rangle$ for clones and $\Gamma \mapsto \langle \Gamma \rangle$ for relational clones, respectively. In ordinary Galois theory, the sets A, B are taken to be a field and a group of automorphisms of the field, and the relation R determines whether a given element of the field is fixed by a given automorphism (exercise: find the corresponding closure operations).

1.2 Three basic examples

We start with the correspondence between 2SAT and majority.

Theorem 1.2.1. *Suppose that a relation* $R \subseteq \{0,1\}^m$ *is preserved by the majority function* $\mathrm{maj} : \{0,1\}^3 \to \{0,1\}$. *Then R is bijunctive, that is, R can be written as a conjunction of binary and unary relations.*

Proof. We prove this by induction on m. If $m \le 2$ then there is nothing to prove. Otherwise, for each $i \le 3$ let R_i be the existential projection of R onto all variables except for the ith. We will show that R is equivalent to

$$\Phi(x_1, ..., x_m) \coloneqq R_1(x_2, x_3, ..., x_m) \wedge R_2(x_1, x_3, ..., x_m) \wedge R_3(x_1, x_2, ..., x_m),$$

and the result will then follow by the induction hypothesis. It is clear that $R \subseteq \Phi$, so suppose $(x_1, ..., x_m) \in \Phi$. Then by the definitions of R_1, R_2, R_3, there exist x_1', x_2', x_3' such that $(x_1', x_2, x_3, ..., x_m), (x_1, x_2', x_3, ..., x_m), (x_1, x_2, x_3', ..., x_m) \in R$, and applying maj to these three tuples we see that $(x_1, ..., x_m) \in R$ as well. □

Definition 1.2.2. An operation $f : \{0,1\}^k \to \{0,1\}$ is called *monotone* if it preserves the relation $\le$. It is called *self-dual* if it preserves the relation $\neq$.

Theorem 1.2.3. *Suppose that a function $f : \{0,1\}^k \to \{0,1\}$ is monotone and self-dual. Then $f \in \langle \mathrm{maj} \rangle$.*

Proof. We prove this by induction on k. It's easy to check that there are no monotone self-dual functions of arity ≤ 2 other than the projections, so assume that $k \ge 3$. By the induction hypothesis, any function we can make by identifying two variables of f is in $\langle \mathrm{maj} \rangle$. We claim that we have

$$f(x, y, z, ...) = \mathrm{maj}(f(x, y, y, ...), f(z, y, z, ...), f(x, x, z, ...)),$$

where the ... always represent the remaining $k - 3$ variables. To see this, note that the formula is trivially true when $x = y = z$, so we only need to check it when one of the variables is different from the other two. We will check it in the case $(x, y, z) = (0, 1, 0)$, since every other case is analogous (via cyclically permuting x, y, z or swapping 0s and 1s throughout). In this case, since f is monotone we have

$$f(0, 1, 1, ...) \ge f(0, 1, 0, ...) \ge f(0, 0, 0, ...),$$

so the median of these three values will be $f(0, 1, 0, ...) = f(x, y, z, ...)$, and the majority is equal to the median on $\{0, 1\}$. □

Examining the proof, we see that every n-ary monotone self-dual function f can be written in terms of maj as a term of depth at most $n - 2$, such that every subterm is obtained by identifying some of the variables of f.

Corollary 1.2.4. *For any odd n, the n-ary function m_n given by*

$$m_n(x_1, ..., x_n) \coloneqq \begin{cases} 1 & \sum_i x_i > \frac{n}{2}, \\ 0 & \sum_i x_i < \frac{n}{2} \end{cases}$$

is in the clone generated by maj. *In fact, we may write m_n as a term of depth at most $n - 2$, such that every subterm is also a linear threshhold function, where for $a \in \mathbb{N}^n$ with $\sum_i a_i = n$ we define the n-ary linear threshhold function t_a by*

$$t_a(x_1, ..., x_n) \coloneqq \begin{cases} 1 & \sum_i a_i x_i > \frac{n}{2}, \\ 0 & \sum_i a_i x_i < \frac{n}{2}. \end{cases}$$

For the majority function m_n, we can actually find a substantially smaller term using a probabilistic construction. (A deterministic construction, based on sorting networks, can be found in [124].)

Proposition 1.2.5 (Valiant [184])**.** *For any odd n, the majority function m_n can be represented by a term of depth $O(\log(n))$ and size $O(n^{4.3})$.*

Proof. We'll follow Goldreich's exposition [85]. Consider the completely generic formula $f_\ell(y_1, ..., y_{3^\ell})$ of depth ℓ, defined recursively by $f_0 = \pi_1$, $f_1 = \text{maj}$, and

$$f_{\ell+1}(y) \coloneqq \text{maj}(f_\ell(y_1, ..., y_{3^\ell}), f_\ell(y_{3^\ell+1}, ..., y_{2\cdot 3^\ell}), f_\ell(y_{2\cdot 3^\ell+1}, ..., y_{3^{\ell+1}})).$$

Then define a random function $g_\ell(x_1, ..., x_n)$ by replacing each y_i in f_ℓ with a random choice of x_{j_i}, where the j_i are independently and uniformly randomly chosen from the set $\{1, ..., n\}$. For any particular input $x \in \{0,1\}^n$, if p_i is the probability that $g_i(x) = m_n(x)$, then we have

$$p_0 \geq \frac{1}{2} + \frac{1}{2n}$$

and

$$\begin{aligned} p_{i+1} &= 3(1-p_i)p_i^2 + p_i^3 \\ &= 0.5 + (1.5 - 2(p_i - 0.5)^2)(p_i - 0.5) \\ &= 1 - (3 - 2(1-p_i))(1-p_i)^2. \end{aligned}$$

A little computation then shows that for $\ell \approx (1 + 1/\log_2(1.5))\log_2(n) \approx 2.71\log_2(n)$ we have $1 - p_\ell < 2^{-n}$, so a union bound shows that for this choice of ℓ at least one assignment to the y_is has $g_\ell(x) = m_n(x)$ for all $x \in \{0,1\}^n$. □

Monotone self-dual functions can be interpreted as voting functions. They also have a combinatorial interpretation in terms of maximal "intersecting families" of sets.

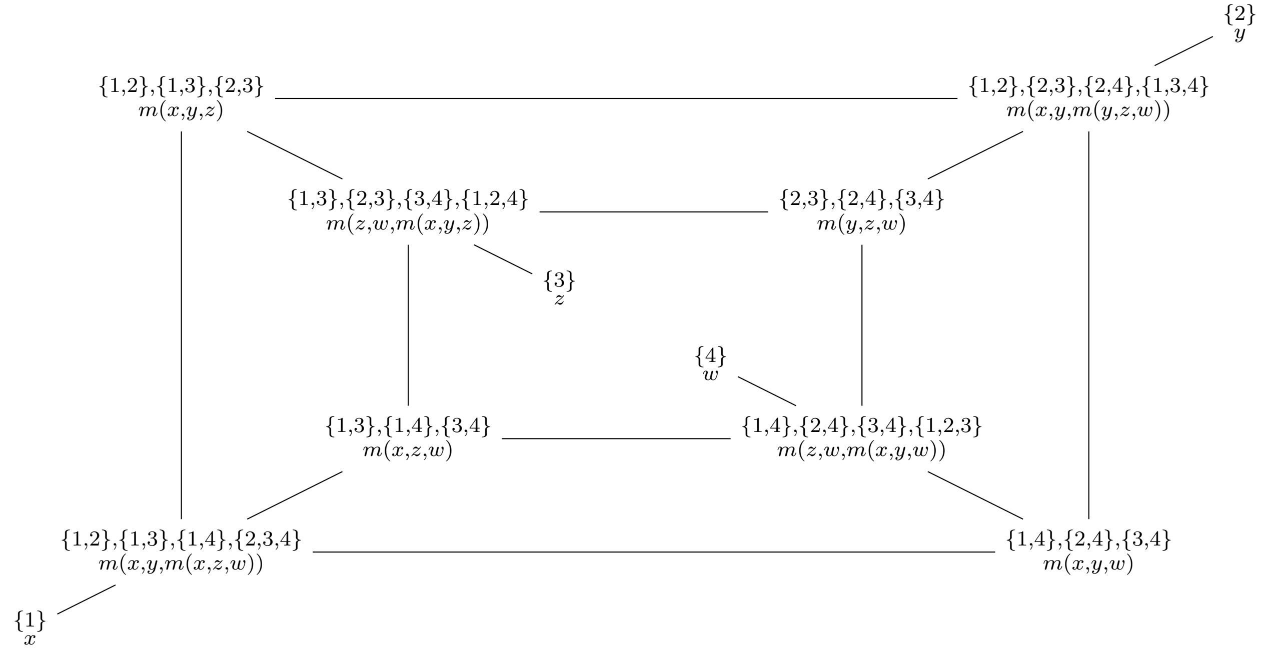
Definition 1.2.6. Let S be a set. A family $\mathcal{F} \subseteq \mathcal{P}(S)$ is called an *intersecting family* of subsets of S if $A, B \in \mathcal{F}$ implies $A \cap B \neq 0$.

Proposition 1.2.7. *An intersecting family of subsets of a set S is maximal (with respect to containment) if and only if for every set $A \subseteq S$ we have either $A \in \mathcal{F}$ or $(S \setminus A) \in \mathcal{F}$. For every $n \geq 1$ there is a bijection between maximal intersecting families $\mathcal{F}$ of subsets of $\{1, ..., n\}$ and monotone self-dual boolean functions $f : \{0,1\}^n \to \{0,1\}$.*

We can describe a maximal intersecting family of subsets of a set more compactly by describing its collection of minimal elements. We can mutate an intersecting family by taking one of its minimal elements A, deleting it, and replacing it with its complement - this is called "switching" the subset A with its complement.

Definition 1.2.8. For every n, we define an undirected graph $\mathcal{M}_n$ whose vertices are the maximal intersecting families of subsets of $\{1, ..., n\}$, and whose edges are the pairs of families $\mathcal{F}, \mathcal{G}$ such that $|\mathcal{F} \setminus \mathcal{G}| = 1$.

The graph $\mathcal{M}_4$ is depicted below, with vertices labeled by the minimal elements of the corresponding intersecting families as well as the corresponding monotone self-dual functions (written in terms of the majority function, which we abbreviate as m).



The graph $\mathcal{M}_n$ is always connected: given two maximal intersecting families $\mathcal{F}, \mathcal{G}$, there will always be some minimal element of $\mathcal{F}$ which is not contained in $\mathcal{G}$, and switching this set with its complement gives us a maximal intersecting family $\mathcal{F}'$ which is adjacent to $\mathcal{F}$ and has one more element in common with $\mathcal{G}$ than $\mathcal{F}$ does. For more about maximal intersecting families of sets, see [144].

Next we move to the correspondence between HORN-SAT and the minimum operation.

Theorem 1.2.9. *Suppose that a relation $R \subseteq \{0,1\}^m$ is preserved by the minimum function $\min : \{0,1\}^2 \to \{0,1\}$. Then R can be written as a conjunction of Horn clauses.*

Proof. Write $R = \bigwedge_i C_i$ in conjunctive normal form, such that each clause C_i is minimal. Note that this means that for each literal l in C_i, there is some assignment to the variables that satisfies R, and has the rest of the literals in C_i other than l set to 0.

Suppose, for a contradiction, that some clause C_i has at least two non-negated variables in it, and assume without loss of generality that $C_i = x_1 \vee \cdots \vee x_p \vee \bar{x}_{p+1} \vee \cdots \vee \bar{x}_{p+q}$, $p \ge 2$. By the minimality of C_i, there are assignments a, b which satisfy R and such that $a_2 = \cdots = a_p = \bar{a}_{p+1} = \cdots = \bar{a}_{p+q} = 0$ and $b_1 = \cdots = b_{p-1} = \bar{b}_{p+1} = \cdots = \bar{b}_{p+q} = 0$. But then $\min(a, b)$ fails to satisfy C_i, and hence fails to satisfy R. □

Theorem 1.2.10. *Suppose that a function $f : \{0,1\}^k \to \{0,1\}$ preserves the relations $\{0\}, \{1\}$, and $x \wedge y \implies z$. Then there is a nonempty subset $I \subseteq \{1, ..., k\}$ such that $f(x_1, ..., x_k) = \min_{i \in I} x_i$.*

Proof. Since f preserves $\{0\}$ and $\{1\}$, we have $f(0,...,0) = 0$ and $f(1,...,1) = 1$. Since $\leq$ is in the relational clone generated by $x \wedge y \implies z$, f must be monotone.

For each subset $I \subseteq \{1,...,k\}$, let χ_I be the indicator vector of I. Suppose that I, J have $f(\chi_I) = f(\chi_J) = 1$, then from $\chi_I \wedge \chi_J \implies \chi_{I\cap J}$ (coordinatewise) we see that we must have $f(\chi_{I\cap J}) = 1$ as well. Thus, there is a unique minimum subset I^* satisfying $f(\chi_{I^*}) = 1$. Since f is monotone, we have $f(\chi_J) = 1 \iff J \supseteq I^*$. □

Remark 1.2.1. The fact that min-closed relations on the domain $\{0,1\}$ can always be written as intersections of Horn clauses has the following useful consequence in logic.

Suppose that $P_1,...,P_m$ is a list of logical statements about some type of structure M in some collection of structures $\mathcal{M}$. Suppose that for every pair of structures $M_1, M_2 \in \mathcal{M}$ there is a structure $M' \in \mathcal{M}$ such that for each i, P_i holds in M' iff P_i holds in both M_1 and M_2. Then there is a collection of Horn clauses $\phi_1,...,\phi_n$ in the propositions $P_1,...,P_m$ such that an assignment of true/false values to the P_is can be realized by some $M \in \mathcal{M}$ iff the assignment satisfies the collection of Horn-clauses $\phi_1,...,\phi_n$.

Finally, we come to the affine linear case. We leave the proofs of the following two results to the reader.

Theorem 1.2.11. *Suppose that a relation $R \subseteq (\mathbb{Z}/p)^m$ is preserved by the ternary operation $x-y+z \pmod p$. Then R is an affine linear subspace of $(\mathbb{Z}/p)^m$ - that is, a vector subspace of $(\mathbb{Z}/p)^m$ offset by a fixed vector - and $R \in \langle\{1\}, x+y \equiv z \pmod p\rangle$.*

Theorem 1.2.12. *Suppose that a function $f : (\mathbb{Z}/p)^k \to \mathbb{Z}/p$ preserves the relations $\{1\}$ and $x + y \equiv z \pmod p$. Then f is an affine linear function - that is, a linear function such that the sum of the coefficients is 1 - and $f \in \langle x - y + z \pmod p\rangle$. If p is odd, we have $\langle x - y + z \pmod p\rangle = \langle \frac{x+y}{2} \pmod p\rangle$.*

1.3 Varieties, Birkhoff's HSP theorem, and the hardness proof

From here on we switch over to the algebraic language. To a relational structure $\mathbf{A} = (D, \Gamma)$ we associate an algebraic structure $\mathbb{A} = (D, \mathcal{O})$ with $\langle\mathcal{O}\rangle = \mathrm{Pol}(\Gamma)$. We let CSP($\mathbb{A}$) be another name for CSP($\mathbf{A}$) = CSP(Inv($\mathbb{A}$)).

Remark 1.3.1. Suppose $\mathbb{A}, \mathbb{B}$ are two algebraic structures with associated relational structures $\mathbf{A}, \mathbf{B}$. It is tempting to think that a homomorphism $\mathbb{A} \to \mathbb{B}$ will correspond to a homomorphism $\mathbf{A} \to \mathbf{B}$, or vice versa. Unfortunately, this is total nonsense - if the (functional) signatures of $\mathbb{A}$ and $\mathbb{B}$ match, the (relational) signatures of $\mathbf{A}$ and $\mathbf{B}$ will likely have nothing to do with each other!

In a similar vein, the automorphism groups Aut($\mathbb{A}$) and Aut($\mathbf{A}$) have almost nothing to do with each other. A trivial but illuminating example is the case where $\mathbb{A}$ has no functions at all, so that Aut($\mathbb{A}$) is the full symmetric group - in this case, $\mathbf{A}$ has every possible relation in its signature, including named singleton unary relations for every element of the domain. Thus, if $\mathbb{A}$ is trivial, then $\mathbf{A}$ is *rigid*, with Aut($\mathbf{A}$) = $\{1\}$.

We will now use the algebraic language to relate the complexities of CSPs with different domains. This will finally clarify what we meant by one CSP "simulating" another CSP in the introduction (well, there is one more method of simulation that will be introduced in the next section).

Theorem 1.3.1. *If $\mathbb{A}$ is an algebraic structure, and $\mathbb{B}$ is either*

- *a subalgebra of* $\mathbb{A}$*,*
- *a power of* $\mathbb{A}$*, or*
- *a quotient of* $\mathbb{A}$*,*

then there is a logspace reduction from $\mathrm{CSP}(\mathbb{B})$ *to* $\mathrm{CSP}(\mathbb{A})$*.*

Proof. If $\mathbb{B}$ is a subalgebra of $\mathbb{A}$, we can convert any instance of $\mathrm{CSP}(\mathbb{B})$ into an instance of $\mathrm{CSP}(\mathbb{A})$ by simply adding an extra unary constraint for each variable corresponding to the relation $\mathbb{B} \subseteq \mathbb{A}^1$.

If $\mathbb{B} = \mathbb{A}^n$ for some n, then we can convert an instance of $\mathrm{CSP}(\mathbb{B})$ to an instance of $\mathrm{CSP}(\mathbb{A})$ by replacing each variable with an n-tuple of variables, and using the fact that every subalgebra of $(\mathbb{A}^n)^m$ is a subalgebra of $\mathbb{A}^{mn}$.

If $\mathbb{B} = \mathbb{A}/{\sim}$ for some congruence ${\sim} \subseteq \mathbb{A}^2$ on $\mathbb{A}$, then every relation $R \subseteq \mathbb{B}^m$ lifts to a relation $\tilde{R} \subseteq \mathbb{A}^m$ by the rule $x \in \tilde{R} \iff x/{\sim} \in R$. □

More generally, if we have several algebras $\mathbb{A}_1, \mathbb{A}_2, ...$ in the same (functional) signature, we can define $\mathrm{CSP}(\{\mathbb{A}_1, \mathbb{A}_2, ...\})$ to be the problem where each variable comes with a *sort* - that is, a specific algebra $\mathbb{A}_i$ that it lives in - and each relation is *multisorted*, where a multisorted relation is “allowed” if it cuts out a subalgebra of the relevant product of the $\mathbb{A}_i$s. This sort of multisorted relation was considered by Bulatov and Jeavons [49]. In this framework, there is a logspace equivalence between $\mathrm{CSP}(\mathbb{A}_1 \times \mathbb{A}_2)$ and $\mathrm{CSP}(\{\mathbb{A}_1, \mathbb{A}_2\})$.

So we see that we are naturally led to study families of finite algebras (all sharing a signature) which are closed under taking finite products, subalgebras, and quotients. This leads us to the concept of a *variety* (or *pseudovariety*, if the family of finite algebras is not finitely generated). Lurking in the background here is a new Galois connection, this time between families of *identities* and families of algebras.

Definition 1.3.2. A *term* (in a given functional signature) is either a variable name or a k-ary function symbol applied to a k-tuple of previously constructed terms. An *identity* is a formal expression $s \approx t$, where s and t are terms. An algebra $\mathbb{A}$ *satisfies* an identity $s \approx t$, written $\mathbb{A} \models s \approx t$, if

$$\forall x_1, ..., x_n \in \mathbb{A}\ s(x_1, ..., x_n) = t(x_1, ..., x_n)$$

(here we are assuming that the variables of s and t are drawn from $x_1, ..., x_n$).

The $\approx$ notation is confusing at first, since in the context of universal algebra it is viewed as a statement which is *stronger* than ordinary equality. The idea here is that approximate equality is never considered in universal algebra, so there should be no confusion in repurposing the symbol $\approx$ into an abbreviation for universal quantifiers. For instance, the intended meaning of the expression “$f(x, y) \approx f(y, x)$” is “$\forall x, y\ f(x, y) = f(y, x)$”. An alternate point of view is that $\approx$ refers to the congruence on the absolutely free algebra corresponding to the identities which are satisfied by the algebras we are interested in.

Definition 1.3.3. The *variety* $\mathcal{V}(\mathcal{T})$ determined by a set of identities $\mathcal{T}$ is the set of algebras that satisfy all of the identities in $\mathcal{T}$. If $\mathbb{A}_1, \mathbb{A}_2, ...$ is a collection of algebras, then $\mathcal{V}(\mathbb{A}_1, \mathbb{A}_2, ...)$ is the variety associated to the set of all identities that hold simultaneously in all of the algebras $\mathbb{A}_i$.

Birkhoff [27] introduced a convenient notation for manipulating sets (strictly speaking these are classes, not sets) of algebras: if $\mathcal{S}$ is a set of algebras, then $P\mathcal{S}$ is the set of products of algebras from $\mathcal{S}$ (possibly infinite - P_{fin} is the notation if one restricts to finite products), $S\mathcal{S}$ is the set of subalgebras of algebras from $\mathcal{S}$, and $H\mathcal{S}$ is the set of quotients (homomorphic images) of algebras from $\mathcal{S}$.

Theorem 1.3.4 (Birkhoff's HSP Theorem [27])**.** *For any collection $\mathcal{S}$ of algebras, we have $\mathcal{V}(\mathcal{S}) = HSP(\mathcal{S})$, that is, an algebra $\mathbb{A}$ satisfies every identity which is satisfied in every element of $\mathcal{S}$ if and only if it is the homomorphic image of a subalgebra of a product of elements of $\mathcal{S}$. Furthermore, if $\mathcal{S}$ is a finite collection of finite algebras, then the set of finite algebras in $\mathcal{V}(\mathcal{S})$ is equal to $HSP_{fin}(\mathcal{S})$.*

Proof. It is easy to check that if $\mathbb{A}, \mathbb{B} \models s \approx t$, then $\mathbb{A} \times \mathbb{B} \models s \approx t$, and similarly every subalgebra and quotient of $\mathbb{A}$ also satisfies $s \approx t$. Thus $HSP(\mathcal{S}) \subseteq \mathcal{V}(\mathcal{S})$.

For the other containment, suppose that $\mathbb{A} \in \mathcal{V}(\mathcal{S})$, and suppose that $\mathbb{A}$ is generated by a subset $I \subseteq \mathbb{A}$. We let $\mathbb{P}$ be the product of all the algebras of $\mathcal{S}$, and define the "free algebra" $\mathcal{F}(I)$ to be the subalgebra of $\mathbb{P}^{\mathbb{P}^I}$ which is generated by the projection functions π_i for $i \in I$, given by $\pi_i(x) = x_i$. We claim that there is a surjective homomorphism $h : \mathcal{F}(I) \twoheadrightarrow \mathbb{A}$ with $h(\pi_i) = i$.

Suppose not. Then there are two terms s, t with $s(\pi_{i_1}, ..., \pi_{i_n}) = t(\pi_{i_1}, ..., \pi_{i_n})$ in $\mathbb{P}^{\mathbb{P}^I}$, but $s(i_1, ..., i_n) \neq t(i_1, ..., i_n)$ in $\mathbb{A}$. But then $s \approx t$ is satisfied by $\mathbb{P}$, and hence by every algebra in $\mathcal{S}$, and is not satisfied in $\mathbb{A}$, contradicting our assumption that $\mathbb{A} \in \mathcal{V}(\mathcal{S})$.

For the last claim, note that if $\mathbb{A}, \mathcal{S}$, and every element of $\mathcal{S}$ are finite, then so are $I, \mathbb{P}, \mathbb{P}^I$, and $\mathbb{P}^{\mathbb{P}^I}$. □

Birkhoff's HSP theorem gives one half of the Galois connection between identities and algebras. The other half is a result from model theory, which explains why elementary results in algebra can always be proved by writing down a long string of equalities.

Theorem 1.3.5. *If $\mathcal{T}$ is a family of identities, then the set of identities which hold in $\mathcal{V}(\mathcal{T})$ is equal to the closure of $\mathcal{T} \cup \{x \approx x\}$ under:*

- *substituting a term for a variable in an identity,*
- *applying a k-ary function to both sides of a k-tuple of identities,*
- *deducing $s \approx t$ from $t \approx s$, and*
- *deducing $s \approx u$ from $s \approx t$ and $t \approx u$.*

Proof. Define the free algebra $\mathcal{F}_{\mathcal{T}}(x_1, ...)$ on countably many variables by taking the set of all terms on these variables, and then taking the quotient of this term algebra by the congruence generated by the images under all possible substitutions of the identities in $\mathcal{T}$. The result will be an algebra satisfying all of the identities of $\mathcal{T}$, and one can check directly from the definition of a congruence that the identities that hold in this free algebra are exactly the ones described in the theorem statement. □

Using Birkhoff's theorem, we can give a criterion for NP-completeness.

Theorem 1.3.6. *If* CSP($\mathbb{A}$) *is not NP-complete, then there is a finite set of identities $s_i \approx t_i$ which are satisfied by $\mathbb{A}$, which can't be satisfied by assigning each function symbol to a projection of the same arity.*

Proof. If $\mathcal{V}(\mathbb{A})$ contains an algebra $\mathbb{B}$ of size at least 2 where each function symbol acts as a projection, then CSP($\mathbb{B}$) is NP-complete and has a logspace reduction to CSP($\mathbb{A}$). Such an algebra $\mathbb{B}$ will exist if there is a way to assign the function symbols to projections that satisfies *every* identity satisfied by $\mathbb{A}$. To see that we only have to consider finite sets of identities, we use a compactness argument: each function symbol has only a finite number of projections it can be assigned to, so we can apply Kőnig's Lemma. □

Example 1.3.1. Consider the algebra $\mathbb{A} = (\{0,1\}, \min)$, and use the binary function symbol s to abbreviate min. Then $\mathcal{V}(\mathbb{A}) = SP(\mathbb{A}) = \mathcal{V}(\mathcal{T}_{semi})$, where $\mathcal{T}_{semi}$ is the following set of identities:

$$s(x,x) \approx x, \quad s(x,y) \approx s(y,x), \quad s(x,s(y,z)) \approx s(s(x,y),z).$$

The second identity above, $s(x,y) \approx s(y,x)$, can't be satisfied by assigning s to either of the projections π_1, π_2.

An algebra in $\mathcal{V}(\mathcal{T}_{semi})$ is called a *semilattice*, and can be visualized as a poset where every nonempty finite subset has a greatest lower bound (if we visualize it this way, we often call it a *meet* semilattice).

Any finite meet semilattice which has a greatest element can be extended to a lattice, since every finite subset will also have a least upper bound (just take the greatest lower bound of the collection of all upper bounds, which is nonempty by the assumption that there is a greatest element). Since we can always adjoin a new "top" element to any finite meet semilattice, we see that every finite semilattice is isomorphic to a subalgebra of the meet semilattice reduct of some lattice (in fact, this is also true for infinite semilattices).

Alternatively, a semilattice can be thought of as a poset where every nonempty finite subset has a least upper bound, if we are thinking in terms of an operation like max - if we are visualizing it in this way, we call it a *join* semilattice. (I generally prefer to visualize semilattices as join semilattices, but most authors prefer to visualize semilattices as meet semilattices.)

Since it is often confusing when people who think of semilattices as meet semilattices try to talk to people who think of them as join semilattices (i.e. minimal elements in one language become maximal elements in the other language), it is useful to have some vocabulary which is agnostic to the meet/join distinction. We say that an element a is *absorbing* with respect to s if it satisfies

$$s(a,x) = s(x,a) = a$$

for all x, and we say that an element b is *neutral* with respect to s if it satisfies

$$s(b,x) = s(x,b) = x$$

for all x. Every two-element semilattice has a neutral element and an absorbing element, and knowing which is which determines the semilattice operation. In general, every finite semilattice has an absorbing element, but might not have a neutral element (for instance, the free semilattice on two generators has absorbing element $s(x,y)$ and has no neutral element). In a meet semilattice, the absorbing element will be the bottom and the neutral element (if it exists) will be the top, while in a join semilattice, the absorbing element will be the top and the neutral element (if it exists) will be the bottom.

Example 1.3.2. Consider the algebra $\mathbb{A} = (\{0,1\}, \mathrm{maj})$, and use the ternary function symbol m to abbreviate maj. Then $\mathcal{V}(\mathbb{A}) = SP(\mathbb{A}) = \mathcal{V}(\mathcal{T}_{med})$, where $\mathcal{T}_{med}$ is the following set of identities:

$$\begin{aligned} m(x,y,z) &\approx m(y,z,x) \approx m(x,z,y), \\ m(x,x,y) &\approx x, \\ m(m(x,y,z),u,v) &\approx m(x,m(y,u,v),m(z,u,v)). \end{aligned}$$

The identity $m(x,y,z) \approx m(y,z,x)$ can't be satisfied by assigning m to one of the projections π_1, π_2, π_3.

An algebra in $\mathcal{V}(\mathcal{T}_{med})$ is called a *median algebra*. A finite median algebra corresponds to a *median graph*, that is, a graph with the property that for every three vertices x, y, z there exists a unique vertex which lies on a shortest path connecting every pair of x, y, z (to recover the graph structure, we draw an edge from x to y whenever $m(x,y,z) \in \{x,y\}$ for all z). Examples of median graphs are paths, trees, planar grids, "squaregraphs", hypercubes, Hasse diagrams of distributive lattices, and the graph $\mathcal{M}_n$ of maximal intersecting families from the last section. For more about the theory of median algebras, see [164] or [35].

Median algebras are very closely connected to distributive lattices. It isn't hard to show that in any distributive lattice, the following identity holds:

$$(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) \approx (x \vee y) \wedge (y \vee z) \wedge (z \vee x),$$

and in fact this identity is *equivalent* to the lattice being distributive. The common value of both sides is called the median operation $m(x,y,z)$ on the lattice - the reader can easily check that it satisfies the identities $\mathcal{T}_{med}$. In fact, if a median algebra has two elements $0, 1$ with $m(0,x,1) = x$ for all x, then it forms a distributive lattice under the operations $x \wedge y = m(0,x,y)$ and $x \vee y = m(x,y,1)$, and the median operation m can be recovered from $\wedge, \vee$ via the above formula [30].

Example 1.3.3. The operation $m(x,y,z) = x - y + z \pmod p$ satisfies the identity $m(x,y,y) \approx x \approx m(y,y,x)$, and this identity can't be satisfied by assigning m to one of the projections π_1, π_2, π_3.

Similarly, if p is odd, the operation $m(x,y) = \frac{x+y}{2} \pmod p$ satisfies the identity $m(x,y) \approx m(y,x)$, which can't be satisfied by projections.

As with clones and relational clones, there are several natural finiteness questions that come up with varieties.

Definition 1.3.7. A variety $\mathcal{V}$ is *finitely generated* if there is a finite list of finite algebras $\mathbb{A}_1, ..., \mathbb{A}_n$ such that $\mathcal{V} = \mathcal{V}(\mathbb{A}_1, ..., \mathbb{A}_n)$. A variety $\mathcal{V}$ is *locally finite* if the free algebra on n generators $\mathcal{F}_{\mathcal{V}}(x_1, ..., x_n)$ is finite for every n. A variety $\mathcal{V}$ is *finitely based* if there is a finite set of equations $\mathcal{T}$ such that $\mathcal{V} = \mathcal{V}(\mathcal{T})$.

A variety $\mathcal{V}$ is locally finite iff for all $\mathbb{A} \in \mathcal{V}$ and for all finite subsets $\{a_1, ..., a_n\} \subseteq \mathbb{A}$, the subalgebra of $\mathbb{A}$ generated by $a_1, ..., a_n$ is finite. Every finitely generated variety is locally finite (by the proof of the HSP Theorem). In general, determining whether a given finitely generated variety is finitely based, or vice versa, is a very difficult problem. For instance, the famous Burnside problem is the problem of determining whether the variety of groups satisfying the identity $x^n \approx e$ is locally finite.

Remark 1.3.2. Sometimes we want to consider infinite families of finite algebras with a finite functional signature, closed under *finite* products, subalgebras, and homomorphisms. Such a family of algebras is called a *pseudovariety*. There are two different ways to describe pseudovarieties in terms of identities.

Eilenberg and Schützenbérger [73] show that a pseudovariety is determined by an infinite sequence of identities, such that a finite algebra is contained in the pseudovariety iff it satisfies *all but finitely many* of the identities in the sequence. The trick is to sort the isomorphism classes of finite algebras by their sizes, and for each size k write down a finite set of identities in k variables which characterizes the free algebra on k generators in the subvariety generated by the set of algebras of size at most k.

Reiterman [162] shows that a pseudovariety is determined by identities between "implicit operations": operations which aren't defined from terms directly, but which can still be defined on any particular finite algebra in a way that is compatible with homomorphisms. Examples of implicit operations in the language of a unary function f are

$$f^\infty = \lim_{n\to\infty} f^{\circ n!}, \quad f^{\infty-1} = \lim_{n\to\infty} f^{\circ(n!-1)},$$

where the limits are taken pointwise (note that the functions $f^{\circ n!}$ stabilize once n exceeds the size of the domain). For any function f on a finite domain, f^∞ will always satisfy the identity $f^\infty(f^\infty(x)) \approx f^\infty(x)$, while the pseudovariety of *invertible* functions on finite sets is cut out by the identities

$$f(f^{\infty-1}(x)) \approx f^{\infty-1}(f(x)) \approx x.$$

For those who like category theory, a k-ary implicit operation of a pseudovariety $\mathcal{V}$ with underlying set functor $S : \mathcal{V} \to \mathsf{Set}$ is a natural transformation from S^k to S. If a free algebra on k elements exists in $\mathcal{V}$, then a standard argument shows that every k-ary implicit operation of $\mathcal{V}$ is actually *explicit*, that is, a term of $\mathcal{V}$. In general, every finite subset of $\mathcal{V}$ will generate a locally finite subvariety of $\mathcal{V}$, which shows that the restriction of any implicit operation to this subset agrees with some term of $\mathcal{V}$. Reiterman [162] puts a metric structure on the set of implicit operations of a pseudovariety such that the collection of implicit operations becomes the completion of the collection of explicit operations.

1.4 Cores and Idempotent Reducts

In this section we briefly return to the relational point of view, and the concept of homomorphic equivalence, to provide one last algebraic ingredient: the restriction to *idempotent* algebraic operations.

Definition 1.4.1. Two relational structures $\mathbf{A}, \mathbf{B}$ with the same signature are *homomorphically equivalent* if there exist homomorphisms $\mathbf{A} \to \mathbf{B}, \mathbf{B} \to \mathbf{A}$.

The prototypical example of homomorphic equivalence is a (non-surjective) endomorphism from a relational structure to itself, providing a homomorphic equivalence between the original relational structure and the restriction of the relational structure to a proper subset of its domain. On the algebraic side, this manifests as a unary operation which is not invertible. The algebraic implications of such unary operations in the polynomial clone of an algebra are at the heart of the subject called "tame congruence theory", which was introduced to give the first structure theory for finite algebras in the book by Hobby and McKenzie [95].

Example 1.4.1. Consider the relational structure $\mathbf{A}$ corresponding to the binary implication algebra $\mathbb{A} = (\{0,1\}, \to)$. This relational structure has as basic relations $R_n = \{0,1\}^n \setminus \{(0,...,0)\} = x_1 \vee \cdots \vee x_n$. The unary algebraic operation $\to(x,x)$ of $\mathbb{A}$ takes every element to 1, and defines an endomorphism of relational structures $\mathbf{A} \to \mathbf{A}$ whose image is $\{1\}$. Together with the inclusion relation, we get a homomorphic equivalence between $\mathbf{A}$ and the one-element relational structure with domain $\{1\}$ and relations $R_n \mid_{\{1\}^n} = \{1\}^n$, whose CSP is clearly trivial.

As the example shows, non-surjective endomorphisms provide trivial ways to simplify CSPs.

Definition 1.4.2. A relational structure on a finite domain $\mathbf{A}$ is called a *core* if every endomorphism of $\mathbf{A}$ is also an automorphism of $\mathbf{A}$. If $\mathbf{A}$ is not a core, then $\mathbf{B}$ is called a *core of* $\mathbf{A}$ if $\mathbf{B}$ is a core and $\mathbf{B}$ is homomorphically equivalent to $\mathbf{A}$.

Example 1.4.2. Every complete graph $K_n = ([n], \neq)$ is a core.

Example 1.4.3. If G is a bipartite graph with at least one edge, then the core of G is K_2, the complete graph on two vertices.

Remark 1.4.1. In the infinite case, the definition of a core must be modified: an infinite relational structure is called a core if every endomorphism is an *embedding*, i.e. an injective map that is an isomorphism onto the restriction of the target relational structure to its image. An example of an infinite core is $(\mathbb{Q}, <)$. See section 3.6 of [32] for more information about cores of infinite structures.

Proposition 1.4.3. *Every relational structure on a finite domain* $\mathbf{A}$ *has a core. Any two cores of* $\mathbf{A}$ *are isomorphic.*

Proof. The first statement follows directly from induction on the size of $\mathbf{A}$: if $\mathbf{A}$ is not a core, then it is homomorphically equivalent to its restriction to some proper subset of itself. For the second statement, note that if $\mathbf{B}, \mathbf{B}'$ are two cores of $\mathbf{A}$ then they are homomorphically equivalent, and composing the maps $\mathbf{B} \to \mathbf{B}'$, $\mathbf{B}' \to \mathbf{B}$ gives us endomorphisms of $\mathbf{B}, \mathbf{B}'$ which must both be invertible by the definition of a core. □

Note that although restricting our attention to cores seems like a trivial step, we are sweeping the following problem under the rug.

Problem 1.4.1. Given a finite relational structure $\mathbf{A}$ as input, determine whether or not $\mathbf{A}$ is a core.

Obviously there is a brute-force approach to checking if $\mathbf{A}$ is a core: simply write down every possible endomorphism, and go through them one by one. Since we only have to do this brute force once for a given CSP template, this is not as bad as it sounds, but it is still far from ideal. Unfortunately, as it turns out, a brute force approach is pretty much the best one can do.

Theorem 1.4.4 (Hell, Nešetřil [92])**.** *Determining whether a given undirected graph is a core is NP-complete, even if the graph is assumed to be* 3*-colorable (with a given* 3*-coloring).*

The next main idea comes from "self-reducibility": often, when solving a CSP, one makes a guess (or deduces) that a certain variable should have a certain value. We would like to be able to express a CSP together with some constraints stating that certain variables have certain values using the language of the original CSP. If this is possible, then an algorithm for deciding whether the CSP has a solution can be directly converted into an algorithm for *finding* a solution to the CSP.

Definition 1.4.5. A relational structure $\mathbf{A}$ is a *rigid core* if it has no endomorphisms other than the identity. (In general, a structure is called *rigid* if it has no automorphisms.)

Theorem 1.4.6. *A relational structure* $\mathbf{A}$ *on a finite domain* D *is a rigid core if and only if it has the following property: for every element* $a \in D$*, the unary relation* $\{a\}$ *is contained in the relational clone generated by the relations of* $\mathbf{A}$*.*

Proof. This follows directly from the Inv-Pol Galois connection: $\{a\} \in \langle \mathbf{A} \rangle$ iff $\{a\}$ is closed under $\mathrm{Pol}(\mathbf{A})$, and since $\{a\}$ is generated by a single element, we only need to check that it is closed under $\mathrm{Pol}_1(\mathbf{A})$, which is exactly the set of endomorphisms of $\mathbf{A}$.

We can also give a direct proof, by unraveling the proof of the Inv-Pol connection in this special case, as follows. Define a CSP with a variable f_a for each $a \in D$. For every relation $R \subseteq D^m$ of $\mathbf{A}$ and every tuple $(a_1, ..., a_m) \in R$, we impose the constraint $R(f_{a_1}, ..., f_{a_m})$ on our CSP. Now the solution-set to our CSP exactly corresponds to the set of endomorphisms of $\mathbf{A}$, and if $\mathbf{A}$ is a rigid core then existentially projecting onto the variable f_a produces the unary relation $\{a\}$. □

So it is very desirable to restrict our attention to rigid cores. Most of the example CSPs from the introduction were rigid cores, with the notable exceptions of k-coloring and NAE-SAT. The k-coloring problem is an excellent toy example: the reader may be already be aware of the fact that $\mathrm{CSP}(\{1, ..., k\}, \neq)$ (the k-coloring problem) is logspace equivalent to $\mathrm{CSP}(\{1, ..., k\}, \neq, \{1\}, ..., \{k\})$ - the rigid core obtained by adjoining the unary singleton relations to k-coloring. It is worth examining the proof of that equivalence and understanding how the next result generalizes it.

Theorem 1.4.7. *Suppose that* $\mathbf{A} = (D, \Gamma)$ *is a core on a finite domain* D*, and let* $\mathbf{A}^{rig}$ *be the rigid core obtained by adjoining all singleton unary relations to* $\mathbf{A}$*. Then* $\mathrm{CSP}(\mathbf{A})$ *is equivalent to* $\mathrm{CSP}(\mathbf{A}^{rig})$ *under logspace reductions.*

Proof. We need to find a way to convert an instance of $\mathrm{CSP}(\mathbf{A}^{rig})$ to an instance of $\mathrm{CSP}(\mathbf{A})$ without changing whether it has a solution. As in the previous result, introduce a set of variables f_a for each element $a \in D$, and define a primitive positive formula Φ by

$$\Phi(f) \coloneqq \bigwedge_{R \in \Gamma} \bigwedge_{(a_1, ..., a_m) \in R} R(f_{a_1}, ..., f_{a_m}).$$

Suppose that our instance of $\mathrm{CSP}(\mathbf{A}^{rig})$ has the form

$$\Psi(x) = \exists x_{n+1}, ..., x_{n+m}\ \Psi_0(x) \wedge \bigwedge_{(i,a) \in E} x_i \in \{a\},$$

where Ψ_0 is a primitive positive formula using the relations of Γ, and E is a set describing the additional unary singleton constraints of Ψ. Let Ψ' be the following formula:

$$\Psi'(x) \coloneqq \exists f\ \exists x_{n+1}, ..., x_{n+m}\ \Phi(f) \wedge \Psi_0(x) \wedge \bigwedge_{(i,a) \in E} x_i = f_a.$$

We claim that the instance Ψ' of $\mathrm{CSP}(\mathbf{A})$ has a solution iff the instance Ψ of $\mathrm{CSP}(\mathbf{A}^{rig})$ has a solution. Suppose that f, x solves Ψ', then by the construction of $\Phi(f)$ f describes an endomorphism $f : \mathbf{A} \to \mathbf{A}$, and since $\mathbf{A}$ is a core this endomorphism must have an inverse f^{-1}. Then $f^{-1}(x)$ satisfies Ψ_0 (since f^{-1} is an endomorphism of $\mathbf{A}$), and for $(i, a) \in E$ we have $f^{-1}(x_i) = f^{-1}(f_a) = a$, so $f^{-1}(x)$ is a solution to $\Psi(x)$. □

Now we look at what the restriction to rigid cores means on the algebraic side.

Definition 1.4.8. A function $f : D^k \to D$ is *idempotent* if it satisfies the identity $f(x, x, ..., x) \approx x$. An algebraic structure $\mathbb{A} = (D, \mathcal{O})$ is *idempotent* if every $f \in \mathcal{O}$ is idempotent. Equivalently, $\mathbb{A}$ is idempotent if every singleton subset of D is a subalgebra of $\mathbb{A}$.

Definition 1.4.9. If $\mathbb{A} = (D, \mathcal{O})$ is an algebraic structure, then the *idempotent reduct* $\mathbb{A}^{id}$ of $\mathbb{A}$ has the same domain, and has as its operations the set of all idempotent functions $f \in \langle \mathcal{O} \rangle$ (or, alternatively, some smaller generating set of idempotent functions).

Example 1.4.4. If $\mathbb{A} = (\mathbb{Z}/p, +, 0, 1)$, then $\mathbb{A}^{id}$ has as its operations the set of all affine linear functions on $\mathbb{Z}/p$, and one can take $\{x - y + z \pmod p\}$ as a generating set of basic operations (or, if p is odd, one can alternatively take $\{\frac{x+y}{2} \pmod p\}$ as a generating set of basic operations).

Proposition 1.4.10. *If* $\mathbf{A}$ *is a core corresponding to the algebraic structure* $\mathbb{A}$*, then the rigid core* $\mathbf{A}^{rig}$ *corresponds to the idempotent reduct* $\mathbb{A}^{id}$*. In particular, every CSP is equivalent up to logspace reductions to* $\mathrm{CSP}(\mathbb{A})$ *for some idempotent algebra* $\mathbb{A}$*.*

The reader might be worried that there is no obvious way to generate the collection of all idempotent operations contained in a given clone. For core structures this is not an issue: the polymorphisms of a core structure always decompose neatly into idempotent parts and invertible unary parts.

Proposition 1.4.11. *Suppose that* $\mathcal{O}$ *is a clone such that all of the unary operations in* $\mathcal{O}$ *are invertible. Then for every* k*-ary function* $f \in \mathcal{O}$*, if we define the unary function* f_{un} *by*

$$f_{un}(x) \coloneqq f(x, ..., x)$$

and the k*-ary function* f_{id} *by*

$$f_{id}(x_1, ..., x_k) \coloneqq f_{un}^{-1}(f(x_1, ..., x_k)),$$

then f_{id} *is idempotent and*

$$f = f_{un} \circ f_{id}.$$

In particular, if G *is the group of unary operations in* $\mathcal{O}$*, then for every* k *there are precisely* $|G|$ *times as many* k*-ary operations in* $\mathcal{O}$ *as there are* k*-ary idempotent operations in* $\mathcal{O}$*.*

If $\mathcal{O}$ *is generated by the functions* $f_1, ..., f_m$ *of arities* $k_1, ..., k_m$*, then the set of idempotent operations of* $\mathcal{O}$ *is generated by the functions*

$$(f_i \circ (g_1, ..., g_{k_i}))_{id},$$

over all choices of i *and all choices of* $g_1, ..., g_{k_i} \in G$*. In particular, the set of idempotent operations of* $\mathcal{O}$ *is finitely generated if and only if the full clone* $\mathcal{O}$ *is finitely generated.*

Example 1.4.5. There is an example of a core structure $\mathbf{A}$ which has polymorphisms satisfying a nontrivial system of identities, but such that its rigidification $\mathbf{A}^{rig}$ has no such polymorphisms and is therefore NP-complete. This example is due to Ross Willard and can be found in [25].

The underlying set of $\mathbf{A}$ is the set of expressions a_i with $a \in \{1,2,3\}$ and $i \in \{0,1\}$. The relations of $\mathbf{A}$ are given by

$$\begin{aligned} R(a_i, b_j) &\coloneqq (i = j) \wedge (a \neq b), \\ S(a_i, b_j) &\coloneqq i \neq j. \end{aligned}$$

It is easy to check that this structure is a core.

Polymorphisms of $\mathbf{A}$ include the unary automorphism $\alpha(a_i) = a_{1-i}$ and the ternary function s given by

$$s(a_i, b_j, c_k) = \begin{cases} c_k & i = j, \\ a_i & i \neq j. \end{cases}$$

These polymorphisms satisfy the identity

$$s(x, x, y) \approx s(y, \alpha(y), x) \approx y,$$

which can't be satisfied by projections.

Since the unary relation $\{a_i \mid i = 0\}$ is definable in $\mathbf{A}^{rig}$, we see that polymorphisms of $\mathbf{A}^{rig}$ restrict to idempotent polymorphisms of the triangle K_3. We will show that K_3 has no nontrivial idempotent polymorphisms: in fact, we'll show that every polymorphism of K_3 is the composition of a projection with an automorphism of $\{1,2,3\}$.

To see that all polymorphisms of K_3 are essentially unary, suppose that $f : K_3^n \to K_3$ depends nontrivially on its first coordinate, that is, that there are $x, y \in K_3^n$ with $x_i = y_i$ for all $i > 1$ with $f(x) \neq f(y)$. By composing with automorphisms of $\{1,2,3\}$, we may assume without loss of generality that

$$f(1,1,...,1) = 1, f(2,1,...,1) = 2.$$

Since f preserves the $\neq$ relation, we must then have

$$f(3,2,...,2) = f(3,3,...,3) = 3.$$

These imply that

$$f(\{1,2\}^n) \subseteq \{1,2\}, f(\{1,2\} \times \{1,3\}^{n-1}) \subseteq \{1,2\}.$$

For any $z_2, ..., z_n$, we can find $x_2, ..., x_n \in \{1,2\}$ and $y_2, ..., y_n \in \{1,3\}$ with x_i, y_i, z_i all distinct. Thus we must have

$$f(3, z_2, ..., z_n) = 3$$

for all $z_2, ..., z_n$, and in particular $f(3,1,...,1) = 3$. Now we can repeat the argument with 1 or 2 in place of 3 to see that $f(x_1, ..., x_n) = x_1$ for all $x_1, ..., x_n$, that is, $f = \pi_1$.

Alternatively, we could have shown that $\mathrm{Pol}(K_3^{rig})$ is trivial by instead showing that every relation on $\{1,2,3\}$ is primitively positively definable from the singleton relations together with $\neq$. We leave this as an exercise for the reader (hint: once you have all ternary relations of the form $(x = a) \wedge (y = b) \implies (z = c)$, it's easy to construct the rest).

1.4.1 Reflections and Height 1 Identities

Let's recap the various methods we have used to reduce different CSPs to each other:

- Reduce the set of basic relations Γ of a relational structure $\mathbf{A} = (A, \Gamma)$ to some collection of relations $\Gamma' \subseteq \langle \Gamma \rangle$. Equivalently, expand the collection of basic operations $\mathcal{O}$ in an algebraic structure $\mathbb{A} = (A, \mathcal{O})$ to a collection of operations $\mathcal{O}'$ with $\mathcal{O} \subseteq \langle \mathcal{O}' \rangle$. The collection of algebraic structures $\mathbb{A}' = (A, \mathcal{O}')$ with $\mathcal{O} \subseteq \mathcal{O}'$ is called the collection of *expansions* of $\mathbb{A}$, and we use the notation $E(\{\mathbb{A}\})$ for it in analogy with Birkhoff's HSP operations.

- Each of Birkhoff's algebraic HSP operations, on the algebraic side: we can replace an algebraic structure $\mathbb{A}$ by any power $\mathbb{A}^n$, any subalgebra $\mathbb{B} \le \mathbb{A}$, or any quotient $\mathbb{A}/{\sim}$ to get a CSP which is no harder than CSP($\mathbb{A}$).

- Homomorphic equivalence of relational structures: when there are homomorphisms $\mathbf{A} \to \mathbf{B}$ and $\mathbf{B} \to \mathbf{A}$, then CSP($\mathbf{A}$) = CSP($\mathbf{B}$), since a relational structure $\mathbf{X}$ will have a homomorphism to $\mathbf{A}$ iff $\mathbf{X}$ has a homomorphism to $\mathbf{B}$. We mainly use this to reduce the general case to the case where $\mathbf{A}$ is a core relational structure.

- Starting from a core relational structure $\mathbf{A}$, we showed that the rigidification $\mathbf{A}^{rig}$ which we get by adding each singleton unary relation to the basic relations of $\mathbf{A}$ has CSP($\mathbf{A}^{rig}$) no harder than CSP($\mathbf{A}$). On the algebraic side, this lets us reduce the general case to the case where every basic operation of $\mathbb{A}$ is idempotent.

Barto, Opršal, and Pinsker [24] find it unsatisfactory to have so many unrelated methods of proving reductions between CSPs, and looked for a single framework which could encompass all known techniques for proving reductions. They show that every single method of proving a reduction between CSP($\mathbf{A}$) and CSP($\mathbf{B}$) introduced so far can be described by combining just two basic cases:

- if $\mathbf{B}$ is a "pp-power" (defined below) of $\mathbf{A}$, then CSP($\mathbf{B}$) has a logspace reduction to CSP($\mathbf{A}$), and

- if $\mathbf{B}$ is homomorphically equivalent to $\mathbf{A}$ then CSP($\mathbf{B}$) = CSP($\mathbf{A}$).

Furthermore, they show that even if we chain several such reductions together, we can always find an equivalent reduction where the pp-power step is taken before the homomorphic equivalence step.

Definition 1.4.12. A *pp-power* of a relational structure $\mathbf{A}$ is a relational structure $\mathbf{B}$ with domain $\mathbf{A}^n$ for some n, such that every relation of $\mathbf{B}$ can be defined by a primitive positive formula using the relations of $\mathbf{A}$ (note that the signatures of $\mathbf{A}$ and $\mathbf{B}$ will generally be different).

Proposition 1.4.13. *If $\mathbf{B}$ is homomorphically equivalent to a pp-power of $\mathbf{A}$, then there is a reduction from* CSP($\mathbf{B}$) *to* CSP($\mathbf{A}$) *which can be computed in linear time and logarithmic space.*

Definition 1.4.14. We say that $\mathbf{A}$ *pp-constructs* $\mathbf{B}$ if $\mathbf{B}$ is homomorphically equivalent to some pp-power of $\mathbf{A}$.

For most of the reductions we have described so far, it is easy to see how we can express them in the pp-constructability framework. For instance, in order to simulate the relational structure corresponding to $\mathbb{A}/{\sim}$, we first construct the relational structure $\mathbf{A}'$ whose basic relations consist of all subpowers $\mathbb{R} \le \mathbb{A}^m$ which are compatible with the congruence $\sim$ (using the special case of the pp-power construction where the power n is equal to 1), and then we follow up with a homomorphic equivalence from $\mathbf{A}'$ to a relational structure where each equivalence class of $\sim$ is collapsed to a single element. The only really tricky reduction is the last one: adding singleton unary relations to a core structure.

Here is how we can go about adding a singleton unary relation $\{a\}$ to a core $\mathbf{A}$ in the pp-constructability framework. Let $\mathbf{B}$ be the relational structure which has the new unary relation $\{a\}$ (along with all of the original relations which $\mathbf{A}$ had). We will define a relational structure $\mathbf{C}$ which will be a pp-power of $\mathbf{A}$ having domain $\mathbf{A}^2$, and show that $\mathbf{C}$ is homomorphically equivalent to $\mathbf{B}$.

Let O be the orbit of a under $\mathrm{Aut}(\mathbf{A})$ - note that O is in the relational clone defined by $\mathbf{A}$ - and for every m-ary relation R of $\mathbf{A}$, make a corresponding relation $\tilde{R}$ of $\mathbf{C}$ by

$$((x_1, y_1), ..., (x_m, y_m)) \in \tilde{R} \iff (x_1, ..., x_m) \in R \wedge y_1 = \cdots = y_m \in O.$$

For the relation $\{a\}$ of $\mathbf{B}$, we make a corresponding relation S of $\mathbf{C}$ given by

$$(x, y) \in S \iff x = y \in O.$$

To show that $\mathbf{B}$ and $\mathbf{C}$ are homomorphically equivalent, we just need to exhibit a pair of homomorphisms between them. The homomorphism $\mathbf{B} \to \mathbf{C}$ is given by $x \mapsto (x, a)$. To define the homomorphism from $\mathbf{C}$ to $\mathbf{B}$, we need to choose an automorphism g_y of $\mathbf{A}$ with $g_y(y) = a$ for every $y \in O$. Then the homomorphism $\mathbf{C} \to \mathbf{B}$ is given by $(x, y) \mapsto g_y(x)$ if $y \in O$ (and (x, y) maps to an arbitrary element if $y \notin O$).

Now let's check that pp-constructability is transitively closed.

Proposition 1.4.15 (From [24]). *If* $\mathbf{A}$ *pp-constructs* $\mathbf{B}$ *and* $\mathbf{B}$ *pp-constructs* $\mathbf{C}$*, then* $\mathbf{A}$ *pp-constructs* $\mathbf{C}$*.*

Proof. It's easy to check that homomorphic equivalence is an equivalence relation, and that a pp-power of a pp-power is a pp-power of the original structure. We just need to check that if $\mathbf{A}$ is homomorphically equivalent to $\mathbf{B}$ and $\mathbf{C}$ is a pp-power of $\mathbf{B}$, then there is some $\mathbf{C}'$ such that $\mathbf{C}'$ is a pp-power of $\mathbf{A}$ and $\mathbf{C}'$ is homomorphically equivalent to $\mathbf{C}$.

Suppose that $\mathbf{C}$ is a pp-power of $\mathbf{B}$ with power n, so that the underlying set C of $\mathbf{C}$ is equal to B^n. We will construct the pp-power $\mathbf{C}'$ of $\mathbf{A}$ using the same power n, with the same relational signature as $\mathbf{C}$, as follows. For each m-ary relation symbol R of $\mathbf{C}$, by the definition of a pp-power there is some primitive positive formula in terms of relations S_i of $\mathbf{B}$ which defines R as an mn-ary relation on B:

$$x = (x_1, ..., x_m) = ((x_{11}, ..., x_{1,n}), ..., (x_{m1}, ..., x_{mn})) \in R \iff \exists y_1, ..., y_k \in B \text{ s.t. } \bigwedge_i \pi_{I_i}(x, y) \in S_i.$$

Then since $\mathbf{A}$ and $\mathbf{B}$ have the same relational signature (this is part of our assumption that they are homomorphically equivalent), we can interpret each relation symbol S_i in $\mathbf{A}$ to define an mn-ary relation on A, which we will use to give an interpretation of the relation symbol R in $\mathbf{C}'$:

$$x = (x_1, ..., x_m) = ((x_{11}, ..., x_{1,n}), ..., (x_{m1}, ..., x_{mn})) \in R^{\mathbf{C}'} \iff \exists y_1, ..., y_k \in A \text{ s.t. } \bigwedge_i \pi_{I_i}(x, y) \in S_i^{\mathbf{A}}.$$

It is now easy to check that any homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$ defines a homomorphism $\varphi^n : \mathbf{C}' \to \mathbf{C}$ by simply letting φ act componentwise on elements of $C' = A^n$. Similarly, any homomorphism $\mathbf{B} \to \mathbf{A}$ defines a homomorphism $\mathbf{C} \to \mathbf{C}'$, so $\mathbf{C}$ and $\mathbf{C}'$ are homomorphically equivalent. □

Barto, Opršal, and Pinsker [24] also characterize what happens on the algebraic side of the picture when one relates two relational structures by a pp-power or a homomorphic equivalence. The new thing here is really the homomorphic equivalence: if $g : \mathbf{A} \to \mathbf{B}$ and $h : \mathbf{B} \to \mathbf{A}$, then there is a relationship between $\mathrm{Pol}(\mathbf{A})$ and $\mathrm{Pol}(\mathbf{B})$ which they call a *reflection*, which takes a function $f \in \mathrm{Pol}_k(\mathbf{A})$ to the operation

$$\xi(f) : (x_1, ..., x_k) \mapsto g(f(h(x_1), h(x_2), ..., h(x_k)))$$

in $\mathrm{Pol}_k(\mathbf{B})$. Note that ξ does not respect composition: $\xi(f_0 \circ (f_1, ..., f_k))$ is not in general equal to $\xi(f_0) \circ (\xi(f_1), ..., \xi(f_k))$. However, ξ *does* preserve *height 1 identities.*

Definition 1.4.16. An identity is called a *height 1 identity*, or a *minor identity*, if it has the form $f(x_1, ..., x_k) \approx g(y_1, ..., y_l)$, where the x_is and y_js are (not necessarily distinct) variables. A map $\mathrm{Pol}(\mathbf{A}) \to \mathrm{Pol}(\mathbf{B})$ (taking functions to functions) which respects height 1 identities is called a *height 1 clone homomorphism* or a *minion homomorphism.*

Definition 1.4.17. If $\mathbb{A} = (A, \mathcal{O})$ is an algebraic structure and B is a set, and maps $g : A \to B$, $h : B \to A$ are given, then the *reflection* of $\mathbb{A}$ induced by g, h is defined to be the algebraic structure $\mathbb{B}$ with domain B and the same signature as $\mathbb{A}$, with the operation $g \circ f \circ (h, ..., h)$ on B corresponding to the operation $f \in \mathcal{O}$.

Proposition 1.4.18. $\mathbf{B}$ *is homomorphically equivalent to a pp-power of* $\mathbf{A}$ *iff* $\mathrm{Pol}(\mathbf{B})$ *contains a reflection of* $\mathrm{Pol}(\mathbf{A})^n$ *for some* n *(by* $\mathrm{Pol}(\mathbf{A})^n$ *we mean the clone of operations of* $\mathrm{Pol}(\mathbf{A})$ *acting on a power of the* domain*).*

Proof. We prove the non-obvious direction. Let A, B be the underlying sets of $\mathbf{A}, \mathbf{B}$, and suppose that $g : A^n \to B$ and $h : B \to A^n$ induce a reflection $\xi : \mathrm{Pol}(\mathbf{A})^n \to \mathrm{Pol}(\mathbf{B})$. We will construct a pp-power $\mathbf{C}$ of $\mathbf{A}$ with underlying set A^n which is homomorphically equivalent to $\mathbf{B}$. For every relation R of $\mathbf{B}$, let $\tilde{R}$ be the relation

$$\tilde{R} \coloneqq \{f(h(r_1), ..., h(r_k)) \mid f \in \mathrm{Pol}_k(\mathbf{A}), r_1, ..., r_k \in R\}.$$

By definition, $\tilde{R}$ is the closure of $h(R)$ under $\mathrm{Pol}(\mathbf{A})$, so $\tilde{R}$ is defined by a primitive positive formula over $\mathbf{A}$. We use $\tilde{R}$ as the relation corresponding to R in $\mathbf{C}$. Finally, we just need to check that $g : \mathbf{C} \to \mathbf{B}$ and $h : \mathbf{B} \to \mathbf{C}$ are homomorphisms. That h is a homomorphism follows from $h(R) \subseteq \tilde{R}$. To check that g is a homomorphism, note that if $x = f(h(r_1), ..., h(r_k)) \in \tilde{R}$ with $r_1, ..., r_k \in R$, then $g(x) = \xi(f)(r_1, ..., r_k)$ is an element of R since $\xi(f) \in \mathrm{Pol}(\mathbf{B})$ by assumption. □

Theorem 1.4.19 (ERP Theorem [24])**.** $\mathrm{Pol}(\mathbf{B})$ *contains a reflection of* $\mathrm{Pol}(\mathbf{A})^n$ *for some* n *iff there is a height 1 clone homomorphism* $\mathrm{Pol}(\mathbf{A}) \to \mathrm{Pol}(\mathbf{B})$*.*

Proof. We prove the non-obvious direction. Let $\mathbb{A} = (A, \mathrm{Pol}(\mathbf{A}))$ be the algebraic structure corresponding to $\mathbf{A}$, and suppose $\xi : \mathrm{Pol}(\mathbf{A}) \to \mathrm{Pol}(\mathbf{B})$ is a height 1 clone homomorphism. Let $\mathcal{F}$ be the subalgebra of $\mathbb{A}^{\mathbb{A}^B}$ generated by the operations $\pi_b : \mathbb{A}^B \to \mathbb{A}$ given by $\pi_b : x \mapsto x_b$. Note that $\mathcal{F}$ is secretly the free algebra over $\mathbb{A}$ on $|B|$ generators.

Define maps $g : \mathbb{A}^{\mathbb{A}^B} \to B$ and $h : B \to \mathbb{A}^{\mathbb{A}^B}$ by $h(b) = \pi_b$ and

$$g(f(\pi_{b_1}, ..., \pi_{b_k})) = \xi(f)(b_1, ..., b_k)$$

for $f \in \mathrm{Pol}_k(\mathbf{A})$, and define $g(x)$ arbitrarily for $x \not\in \mathcal{F}$. To see that g is well-defined, note that if $f_0(\pi_{b_1}, ..., \pi_{b_k}) = f_1(\pi_{c_1}, ..., \pi_{c_l})$, then f_0, f_1 are related by a height 1 identity in $\mathbb{A}$ which implies that

$$\xi(f_0)(b_1, ..., b_k) = \xi(f_1)(c_1, ..., c_l).$$

Finally, we see that g, h induce ξ as a reflection from $\mathbb{A}^{\mathbb{A}^B}$:

$$\xi(f)(b_1, ..., b_k) = g(f(\pi_{b_1}, ..., \pi_{b_k})) = g(f(h(b_1), ..., h(b_k))). \qquad \square$$

As a consequence, we see that the complexity of a CSP only depends on the set of height 1 identities satisfied by its polymorphisms, and that identities involving composition of functions are in a sense superfluous. We also have the following result.

Corollary 1.4.20. *Let* $\mathbf{A}$ *be a relational structure with core* $\mathbf{B}$*, and let* $\mathbf{B}^{rig}$ *be* $\mathbf{B}$ *together with any finite collection of singleton unary relations. Then a system of height 1 identities can be satisfied in* $\mathrm{Pol}(\mathbf{A})$ *iff it can be satisfied in* $\mathrm{Pol}(\mathbf{B}^{rig})$*.*

Remark 1.4.2. A height 1 clone homomorphism $\mathrm{Pol}(\mathbf{A}) \to \mathrm{Pol}(\mathbf{B})$ is completely determined by its restriction to polymorphisms of $\mathbf{A}$ of arity at most $|B|$, since every operation $f : B^k \to B$ is determined by its $|B|$-ary minors. So there are only finitely many candidates for height 1 clone homomorphisms from $\mathrm{Pol}(\mathbf{A})$ to $\mathrm{Pol}(\mathbf{B})$: if the underlying sets are A, B, then there are at most

$$\left|\mathrm{Pol}_{|B|}(\mathbf{B})^{\mathrm{Pol}_{|B|}(\mathbf{A})}\right| \le |B|^{|B|^{|B|}\cdot|A|^{|A|^{|B|}}}$$

candidates. Less obviously, requiring that these candidates respect minors of arity $\le |B|$ brings the number of candidates down to just

$$\left|B^{\mathrm{Pol}_{|B|}(\mathbf{A})}\right| \le |B|^{|A|^{|A|^{|B|}}}.$$

Unwinding the proofs of the ERP Theorem 1.4.19 and Proposition 1.4.18, each of these corresponds to a candidate pp-construction of $\mathbf{B}$.

More explicitly, suppose that

$$\xi_{|B|} : \mathrm{Pol}_{|B|}(\mathbf{A}) \to \mathrm{Pol}_{|B|}(\mathbf{B})$$

defines our candidate height 1 homomorphism and respects minors. Then we define maps $g : \mathbb{A}^{\mathbb{A}^B} \to B$ and $h : B \to \mathbb{A}^{\mathbb{A}^B}$ as in the proof of the ERP Theorem 1.4.19, but with the definition of g slightly modified: we set

$$g(f(\pi_{b_1}, ..., \pi_{b_{|B|}})) = \xi_{|B|}(f)(b_1, ..., b_{|B|})$$

for $|B|$-ary polymorphisms $f \in \mathrm{Pol}_{|B|}(\mathbf{A})$, with $b_1, ..., b_{|B|}$ a fixed enumeration of the elements of B. For each basic m-ary relation R of $\mathbf{B}$, we define the corresponding relation

$$\begin{aligned}\tilde{R} \subseteq \left(\mathbb{A}^{\mathbb{A}^B}\right)^m &= \{f(h(r_1), ..., h(r_k)) \mid f \in \mathrm{Pol}_k(\mathbf{A}), r_1, ..., r_k \in R\} \\ &= \mathrm{Sg}_{(\mathbb{A}^{\mathbb{A}^B})^m}\{h(r) \mid r \in R\},\end{aligned}$$

exactly as in the proof of Proposition 1.4.18. Each $\tilde{R}$ is pp-definable from the basic relations of $\mathbf{A}$, so the relational structure $\mathbf{C}$ with underlying set $\mathbb{A}^{\mathbb{A}^B}$ and basic relations given by the $\tilde{R}$s is a pp-power of $\mathbf{A}$ with the same relational signature as $\mathbf{B}$. By construction, the map $h : \mathbf{B} \to \mathbf{C}$ will be a homomorphism, so the only challenge is to check whether or not $g : \mathbf{C} \to \mathbf{B}$ is actually a homomorphism, that is, to check whether or not

$$g(\tilde{R}) \overset{?}{\subseteq} R$$

for each basic relation R of $\mathbf{B}$. In fact, we can forget about the height 1 homomorphism $\xi_{|B|}$ and just search for $g : \mathbf{C} \to \mathbf{B}$ - in other words, we treat $\mathbf{C}$ as an instance of CSP($\mathbf{B}$) with $|\operatorname{Pol}_{|B|}(\mathbf{A})|$ variables that actually participate in any constraints, and $\mathbf{A}$ will pp-construct $\mathbf{B}$ if and only if the instance $\mathbf{C}$ has a solution.

If $\mathbf{B}$ has only finitely many basic relations, then we can test each candidate pp-construction in finite time, so in this case there is an effective procedure to decide whether or not $\mathbf{A}$ pp-constructs $\mathbf{B}$. (Note that each $\tilde{R}$ is determined by the collection of $|R|$-ary polymorphisms of $\mathbf{A}$, so this argument also shows that in order to check that $\xi_{|B|}$ extends to a height 1 clone homomorphism from Pol($\mathbf{A}$) to Pol($\mathbf{B}$), we just need to check that it extends to a height 1 clone homomorphism from $\operatorname{Pol}_k(\mathbf{A})$ to $\operatorname{Pol}_k(\mathbf{B})$ for $k = \max |R|$ over the basic relations R of $\mathbf{B}$.)

1.5 Taylor Algebras

Once we restrict to idempotent algebras, we can start playing games with identities involving nesting functions to simplify our criterion for NP-completeness.

Definition 1.5.1. An algebra $\mathbb{A}$ is a *Taylor algebra* if it has an idempotent term t that satisfies a system of identities of the form

$$t\left(\begin{bmatrix} x & ? & \cdots & ? \\ ? & x & \cdots & ? \\ \vdots & \vdots & \ddots & \vdots \\ ? & ? & \cdots & x \end{bmatrix}\right) \approx t\left(\begin{bmatrix} y & ? & \cdots & ? \\ ? & y & \cdots & ? \\ \vdots & \vdots & \ddots & \vdots \\ ? & ? & \cdots & y \end{bmatrix}\right),$$

where the ?s are filled in somehow with xs and ys. Such an operation t is called a *Taylor term*, and a variety with a Taylor term is called a *Taylor variety.*

Note that by the defining identities of any Taylor term t, t can't be any projection (unless the algebra in question has only one element).

Theorem 1.5.2 (Taylor [181])**.** *If an idempotent algebra $\mathbb{A}$ satisfies any set of identities that can't be satisfied by projections, then it has a Taylor term. Equivalently, an idempotent variety is Taylor iff it does not contain a two element algebra having no nontrivial operations.*

Before we prove Taylor's theorem, we will work through an example.

Example 1.5.1. Let f be an idempotent ternary term satisfying the identity

$$f(f(y,x,z),x,f(z,y,y)) \approx f(x,y,z).$$

Then

$$t(x_1, ..., x_9) \coloneqq f(f(x_1, x_2, x_3), f(x_4, x_5, x_6), f(x_7, x_8, x_9))$$

is a Taylor term, since it satisfies the identities

$$t(y, x, z, x, x, x, z, y, y) \approx t(x, x, x, y, y, y, z, z, z) \approx t(x, y, z, x, y, z, x, y, z),$$

and by specializing these identities (substituting $x = y$, $y = z$, or $z = x$) we can get a system of Taylor identities for t.

Definition 1.5.3. If $f : D^k \to D$ and $g : D^l \to D$, we define the *star composition* $f * g : D^{kl} \to D$ to be $f \circ (g, g, ..., g)$.

Proposition 1.5.4. *If f, g are idempotent, then $f, g \in \langle f * g \rangle$.*

Proof. $f(x_1, ..., x_k) \approx f(g(x_1, ..., x_1), ..., g(x_k, ..., x_k))$ and $g(x_1, ..., x_l) \approx f(g(x_1, ..., x_l), ..., g(x_1, ..., x_l))$. □

Definition 1.5.5. An identity is called a *height 1 identity*, or a *minor identity*, if it has the form $f(x_1, ..., x_k) \approx g(y_1, ..., y_l)$, where the x_is and y_js are (not necessarily distinct) variables.

Proposition 1.5.6. *If an idempotent term satisfies a system of height 1 identities which can't be satisfied by projections, then it is a Taylor term.*

Proof of Taylor's Theorem. By a compactness argument, there is a finite set $\mathcal{T}$ of identities satisfied by a finite set of operations $f_1, ..., f_n$ of $\mathbb{A}$ which can't be satisfied by projections. Let $s = f_1 * \cdots * f_n$. Then each $f_i \in \langle s \rangle$, so we can convert $\mathcal{T}$ into a collection $\mathcal{T}'$ of identities in s which can't be satisfied by projections either.

The identities of $\mathcal{T}'$ might involve some amount of nesting of s within itself, that is, they may not be height 1 identities. Let m be the greatest nesting depth occuring in $\mathcal{T}'$, and let $t = s * \cdots * s$, with m copies of s. Let $\mathcal{T}''$ be the set of height 1 identities involving t only which are satisfied by $\mathbb{A}$. We claim that $\mathcal{T}''$ can't be satisfied by projections.

To see this, note first that for every $k \le m$, if we index the variables of t by m-tuples $(i_1, ..., i_m)$ of indices for coordinates of s, and if we let x^k be the tuple of variables given by

$$x^k_{(i_1,...,i_m)} = y_{i_k}$$

for all $(i_1, ..., i_m)$, then we have

$$t(x^1) \approx \cdots \approx t(x^m) \quad (\approx s(y)).$$

If this system of height 1 identities in t is satisfied by a projection $\pi_{(i_1,...,i_m)}$, then we see that we must have $i_1 = \cdots = i_m = i$, say, for some index i of the variables of s. But then there is some identity of $\mathcal{T}'$ which is incompatible with $s = \pi_i$, and this identity of $\mathcal{T}'$ can be modified by replacing variables z by expressions $s(z, ..., z)$ repeatedly until it becomes a height 1 identity involving only t, which will then be incompatible with $t = \pi_{(i,...,i)}$. □

Corollary 1.5.7. *If $\mathbb{A}$ is an idempotent algebra and* CSP($\mathbb{A}$) *is not NP-complete, then $\mathbb{A}$ has a Taylor term.*

When $\mathbb{A}$ is not Taylor, the above result lets us conclude that there is some two element algebra $\mathbb{B} \in HSP(\mathbb{A})$ with no nontrivial operations, but it doesn't give us a good bound on how large a power of $\mathbb{A}$ we will need to take to find $\mathbb{B}$. It turns out that, in fact, if such a $\mathbb{B}$ exists then it already can be found inside $HS(\mathbb{A})$. We will prove a slight generalization of this fact, which applies to *strictly simple* algebras.

Definition 1.5.8. An algebra $\mathbb{A}$ is called *strictly simple* if $\mathbb{A}$ is simple and every subalgebra of $\mathbb{A}$ either has size 1 or is equal to $\mathbb{A}$.

Lemma 1.5.9. *If $\mathbb{A}$ is an idempotent algebra and $\mathbb{B} \in HSP_{fin}(\mathbb{A})$ is strictly simple, then $\mathbb{B} \in HS(\mathbb{A})$. Note that if $\mathbb{A}, \mathbb{B}$ are both finite, then $\mathbb{B} \in HSP_{fin}(\mathbb{A})$ iff $\mathbb{B} \in HSP(\mathbb{A})$.*

More generally, if $\mathbb{A}_i$ are idempotent for all $i \in I$ and if $\mathbb{B} \in HSP_{fin}(\{\mathbb{A}_i\}_{i\in I})$ is strictly simple, then $\mathbb{B} \in HS(\mathbb{A}_i)$ for some $i \in I$.

Proof. (Following Zhuk [191]) We prove the first statement - the proof of the general case is nearly identical, just slightly more notationally involved. Pick n minimal such that there is some $\mathbb{S} \le \mathbb{A}^n$ and some $\sigma \in \mathrm{Con}(\mathbb{S})$ with $\mathbb{S}/\sigma \cong \mathbb{B}$. If there is any pair $r, s \in \mathbb{S}$ such that $\pi_1(r) = \pi_1(s)$ but $r/\sigma \ne s/\sigma$, then if we set $a = \pi_1(r)$ and

$$\mathbb{S}' = \pi_{\{2,...,n\}}(\mathbb{S} \cap (\{a\} \times \mathbb{A}^{n-1})) \le \mathbb{A}^{n-1},$$

and define $\sigma' \in \mathrm{Con}(\mathbb{S}')$ by restricting σ in the obvious way, then $r' = \pi_{\{2,...,n\}}(r)$ and $s' = \pi_{\{2,...,n\}}(s)$ have $r'/\sigma' \ne s'/\sigma'$. Thus $\mathbb{S}'/\sigma'$ is isomorphic to a subalgebra of a quotient of $\mathbb{B}$ of size at least 2, so $\mathbb{S}'/\sigma' \cong \mathbb{B}$, contradicting the minimality of n.

Otherwise, if there is no such pair r, s, then there is a congruence $\sigma_1 \in \mathrm{Con}(\pi_1(\mathbb{S}))$ such that for all $r \in \mathbb{S}$, the congruence class r/σ is completely determined by $\pi_1(r)/\sigma_1$. But then we have

$$\mathbb{B} \cong \mathbb{S}/\sigma \cong \pi_1(\mathbb{S})/\sigma_1 \in HS(\mathbb{A}). \qquad \square$$

Corollary 1.5.10. *If $\mathbb{A}$ is finite and idempotent, then either $\mathbb{A}$ has a Taylor term, or there is some two element algebra $\mathbb{B} \in HS(\mathbb{A})$ with no nontrivial operations.*

Corollary 1.5.11. *If $\mathbb{A}$ is finite, idempotent, and has no Taylor term, then there are nonempty subalgebras $\mathbb{B}, \mathbb{C} \le \mathbb{A}$ such that $\mathbb{B} \cap \mathbb{C} = \emptyset$ and $(\mathbb{B} \cup \mathbb{C})^3 \setminus (\mathbb{B}^3 \cup \mathbb{C}^3) \le \mathbb{A}^3$. In particular,* CSP($\mathbb{A}$) *can simulate NAE-SAT in a trivial way.*

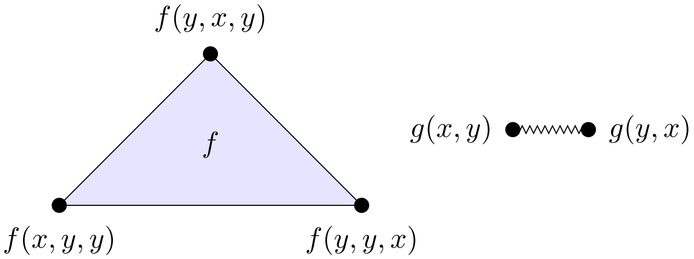
Remark 1.5.1. A recent result of Olšák simplifies the identities we need to consider even further. Olšák [147] proves that in any Taylor algebra, whether finite or infinite, there is always a 6-ary *weak 3-cube* term t, that is, an idempotent term satisfying the identity

$$t(x, y, y, y, x, x) \approx t(y, x, y, x, y, x) \approx t(y, y, x, x, x, y).$$

The weak 3-cube term may be understood as saying that the ternary relation on the free algebra $\mathcal{F}_{\mathcal{V}}(x, y)$ which is generated by the ternary Not-All-Equal relation on $\{x, y\}$ has a diagonal element.

Olšák's proof that such a term exists first uses the theory of absorbing subalgebras to produce a 12-ary term which he calls a double loop term, and then simplifies it down to a weak 3-cube term by using an intricate collection of identities which are satisfied by binary idempotent operations.

Remark 1.5.2. There is a curious connection between systems of two-variable height 1 identities on ternary functions and the problem 1-IN-3 SAT. Suppose that you are given such a system of identities $\mathcal{T}$ on ternary functions $f_1, ..., f_n$, and that you want to determine whether these identities rule out projections.



Define a set of binary functions $f_i^j(x,y)$, $j \le 3$, by

$$\begin{aligned}f_i^1(x,y) &= f_i(x,y,y),\\ f_i^2(x,y) &= f_i(y,x,y),\\ f_i^3(x,y) &= f_i(y,y,x),\end{aligned}$$

and identify any pair of f_i^js which are indentified by $\mathcal{T}$. Make a drawing of a hypergraph with a vertex for every equivalence class of f_i^js, with a zigzag edge connecting any pair of vertices g, h with $g(x,y) \approx h(y,x)$ under $\mathcal{T}$, and with a hyperedge for each f_i connecting it to f_i^1, f_i^2, f_i^3. An assignment of projections π_j to the functions f_i is the same as a choice j of 1-IN-3 of the vertices on the hyperedge f_i to be granted the value π_1, while every zigzag edge of the hypergraph corresponds to a $\ne$ constraint. (Olšák's paper [147] has one such picture, and I've found the technique enormously helpful for visualizing large systems of identities on ternary functions.)

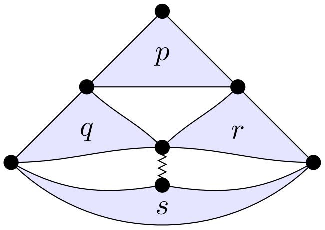
As a concrete example, take following collection of four ternary terms p, q, r, s defined in terms of Olšák's weak 3-cube term:

$$\begin{aligned}p(x,y,z) &= t(y,z,z,x,x,x),\\ q(x,y,z) &= t(x,z,y,z,y,z),\\ r(x,y,z) &= t(y,x,x,z,y,y),\\ s(x,y,z) &= t(x,x,y,z,y,x).\end{aligned}$$

Then the definitions together with the weak 3-cube identities imply the following system of identities on p, q, r, s:

$$\begin{aligned}p(x,y,x) &\approx q(y,x,x),\\ p(x,x,y) &\approx r(y,x,x),\\ q(x,y,x) &\approx s(x,y,x),\\ r(x,x,y) &\approx s(x,x,y),\\ s(x,y,y) &\approx q(x,x,y) \approx r(x,y,x).\end{aligned}$$

It may not be apparent, at a glance, whether or not this system of identities can be satisfied by projections. If we draw the associated 1-IN-3 SAT instance, we find that it has 7 vertices (corresponding to binary terms), 4 occurences of the 1-IN-3 SAT constraint (for the four ternary terms p, q, r, s), and one occurence of the $\neq$ constraint (coming from the fact that the last identity above relates $s(x, y, y)$ to $q(x, x, y)$):



It is now easy (well, as easy as solving a small instance of 1-IN-3 SAT) to verify that the associated 1-IN-3 SAT instance has no solution, so this system of identities can't be satisfied by projections.

Remark 1.5.3. Taylor's original reason for studying Taylor algebras was to try to deeply understand the reason that π_1 of a topological group is always abelian. Taylor [181] considers, for any variety $\mathcal{V}$, the category of topological $\mathcal{V}$-objects, that is, topological algebraic structures satisfying the identities of $\mathcal{V}$. Taylor showed that the π_1s of topological $\mathcal{V}$-objects will share a nontrivial property iff $\mathcal{V}$ has a Taylor term, and that this occurs iff π_1 is always abelian. The fact that a Taylor term must be taken to be idempotent is related to the fact that the fundamental group is really a groupoid (in the sense of category theory), rather than a group, so only the idempotent operations of $\mathcal{V}$ can constrain its structure (I'm slightly fuzzy on the details).

Aside from the topological details, this can be viewed as an analogue of the Eckmann-Hilton principle [72] which states that a unital magma object in the category of unital magmas is necessarily commutative and associative. In fact, the following result holds for Taylor algebras: if $\mathbb{A}$ is a Taylor algebra, and $m : \mathbb{A}^2 \to \mathbb{A}$ is a homomorphism such that there exists an element $0 \in \mathbb{A}$ with $m(0, x) = m(x, 0) = x$ for all x, then m is commutative and associative.

Note that our assumption on m implies that $m * m$ satisfies the identities

$$m(x, y) \approx m * m(x, y, 0, 0) \approx m * m(x, 0, y, 0) \approx \cdots \approx m * m(0, 0, x, y),$$

where in each $m * m$ we always have the x occuring to the left of the y. Additionally, since $m * m : \mathbb{A}^4 \to \mathbb{A}$ is a homomorphism, for any n-ary operation t of $\mathbb{A}$ we can evaluate $(m * m) * t$ on a $4 \times n$ matrix of variables in two different ways: we may either start by applying t to the rows and then apply $m * m$ to the resulting column vector, *or* we may first apply $m * m$ to the columns and then apply t to the resulting row vector - either way gives the same result.

Using these two observations together with the Taylor identities for an n-ary Taylor term t, we prove that m is commutative by writing $m(x, y)$ as $(m * m) * t$ applied to a $4 \times n$ matrix of 0s, xs, and ys where every column has an x above a y, and manipulate this expression until every y is above an x. The strategy is to always keep the xs in the middle two rows and the ys in the top or bottom, and to move a y up a column whenever that column is free of xs. To temporarily move xs out of the way, we apply the Taylor identities for t to swap them with 0s, possibly shifting the xs up and down between the middle two rows to get to a configuration where the Taylor identities will apply. A similar argument with $m * m * m$ in the place of $m * m$ can be used to prove associativity. If t is a 6-ary weak 3-cube term, for instance, then a portion of the proof of the commutativity of

m goes as follows:

$$\begin{aligned}
m\left(\begin{bmatrix}x\\y\end{bmatrix}\right) &= (m*m)*t\left(\begin{bmatrix}0&0&0&0&0&0\\x&0&0&0&x&x\\0&x&x&x&0&0\\y&y&y&y&y&y\end{bmatrix}\right) = (m*m)*t\left(\begin{bmatrix}0&0&0&0&0&0\\0&0&x&x&x&0\\0&x&x&x&0&0\\y&y&y&y&y&y\end{bmatrix}\right)\\
&= (m*m)*t\left(\begin{bmatrix}y&0&0&0&0&y\\0&0&x&x&x&0\\0&x&x&x&0&0\\0&y&y&y&y&0\end{bmatrix}\right) = (m*m)*t\left(\begin{bmatrix}y&0&0&0&0&y\\x&0&0&0&x&x\\0&x&x&x&0&0\\0&y&y&y&y&0\end{bmatrix}\right)\\
&= \cdots = (m*m)*t\left(\begin{bmatrix}y&y&y&y&y&y\\0&0&x&x&x&0\\x&x&0&0&0&x\\0&0&0&0&0&0\end{bmatrix}\right) = m\left(\begin{bmatrix}y\\x\end{bmatrix}\right),
\end{aligned}$$

where we have used the Taylor identity $t(x,0,0,0,x,x) \approx t(0,0,x,x,x,0)$ satisfied by a weak 3-cube term to temporarily move the first and last x out of the way.

1.6 Two simple algorithms (width 1 and bounded strict width)

Definition 1.6.1. A CSP template $\mathbf{A} = (D, \Gamma)$ has *relational width 1* if the relational width $(1, k)$ algorithm below solves it for some k.

Algorithm 1 Relational width $(1, k)$ algorithm

```
Set $S_v \leftarrow D$ for each variable $v$.
repeat
    for all $v_1, ..., v_k$ do
        Let $X$ be the set of solutions to the restriction of the CSP to the variables $v_1, ..., v_k$ (projecting each constraint onto this subset of variables).
        Set $S_{v_i} \leftarrow \pi_i(X \cap (S_{v_1} \times \cdots \times S_{v_k}))$ for each $i \le k$.
        For each constraint $R$ which involves some $v_i$, remove all tuples of $R$ which are incompatible with $S_{v_i}$.
until the sets $S_v$ stop changing.
If any $S_v = \emptyset$, there is no solution.
```

Compare this to the generalized arc-consistency algorithm, which is more popular (and more efficient!) in practice. (After this section, I'll usually refer to generalized arc-consistency as just "arc-consistency" to save space.)

Theorem 1.6.2 (Feder, Vardi [77])**.** *A CSP template* $\mathbf{A}$ *has relational width 1 iff it is solved by the generalized arc-consistency algorithm.*

Sketch. Suppose $\mathbf{A}$ has width $(1, k)$, and let $\mathbf{B}$ be an instance of CSP($\mathbf{A}$). By a generalization of the randomized construction of graphs with large girth and large chromatic number, there is a relational structure $\mathbf{B}'$ which has a map to $\mathbf{B}$, has girth larger than k, and which has a map to

Algorithm 2 Generalized arc-consistency algorithm

```
1: Set $S_v \leftarrow D$ for each variable $v$.
2: while some constraint $R$ on variables $(v_1, ..., v_m)$ has $\pi_j(R \cap (S_{v_1} \times \cdots \times S_{v_m})) \neq S_{v_j}$ do
3:     Set $S_{v_j} \leftarrow \pi_j(R \cap (S_{v_1} \times \cdots \times S_{v_m}))$.
4: If any $S_v = \emptyset$, there is no solution.
```

A iff **B** has a map to **A** (alternatively, if Γ contains the equality relation, we can cheat by adding long chains of equalities). Since $\mathbf{B}'$ locally looks like a tree, the width $(1, k)$ algorithm and the generalized arc-consistency algorithm give the same results for $\mathbf{B}'$, so if there is no homomorphism from **B** to **A**, then generalized arc-consistency applied to $\mathbf{B}'$ will correctly find that there is no solution. But then generalized arc-consistency applied to **B** will also find that there is no solution, since every deduction on $\mathbf{B}'$ can be mimicked on **B**. □

Definition 1.6.3. A connected relational structure is a *tree* if every collection of occurences of relations with arities $r_1, ..., r_k$ involves at least $1 + \sum_i (r_i - 1)$ distinct elements. A relational structure **A** has *tree duality* if for every **B**, there is a map $\mathbf{B} \to \mathbf{A}$ iff every tree which maps to **B** has a map to **A**.

Proposition 1.6.4. **A** *has width 1 iff it has tree duality.*

Proof. If generalized arc-consistency shows that there is no homomorphism $\mathbf{B} \to \mathbf{A}$, then we can make a proof tree that shows that some set S_v eventually becomes empty. Each node of the proof tree corresponds to the fact that some variable w of **B** takes values from a set S_w, and the hyperedges of the proof tree are labeled by relations of **B**. So the proof tree is actually a relational structure with a map to **B**, and the same sequence of generalized arc-consistency deductions apply to the proof tree to show that it has no map to **A**. □

Remark 1.6.1. Essentially the same arguments apply for any width (l, k), with "trees" replaced by "(l, k)-trees" (definition left as an exercise to the reader). Note that (l, k)-trees have tree-width $k - 1$. When studying *relational* width, we replace "trees" by "(l, k)-reltrees" (defined in [62] for $k = l$).

Remark 1.6.2. Dalmau has shown that any CSP with relational width $(2, 2)$ is also solved by generalized arc-consistency [62]. 2SAT is an example of a CSP with width $(2, 3)$ which is *not* solved by arc-consistency, so Dalmau's result is best possible.

Generalized arc-consistency has a close connection with the algebraic concept of a "subdirect product".

Definition 1.6.5. A subalgebra $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is called a *subdirect product*, written $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, if $\pi_i(\mathbb{R}) = \mathbb{A}_i$ for all i.

So an algebraic way of thinking of arc-consistency is that we shrink the domains of the variables until we get to a situation where every relation is a subdirect product. It's worth noting that as we shrink our domains and relations, the new domains and relations we obtain will always be preserved by any polymorphisms which preserved the original relations, since the new domains and relations can be defined by primitive positive formulas from the original ones.

We now find an algebraic characterization of CSP templates with width 1. The main idea is to consider the "most generic" problem which arc-consistency requires to have a solution, and to ask what such a solution must look like. This most generic problem will have a different variable for each possible nonempty set $S \subseteq D$, and will have all relations which are consistent with these sets imposed.

Definition 1.6.6. For $\mathbf{A} = (D, \Gamma)$ a relational structure, define $\mathcal{P}_\emptyset(\mathbf{A})$ to be the structure with ground set $\mathcal{P}(D) \setminus \{\emptyset\}$, and for every m-ary relation $R \in \Gamma$ let the corresponding relation $\mathcal{P}_\emptyset(R)$ be the set of all m-tuples $S_1, ..., S_m \in \mathcal{P}(D) \setminus \{\emptyset\}$ such that there is some nonempty $X \subseteq R$ with $\pi_i(X) = S_i$ for each i.

Note that $\mathcal{P}_\emptyset(R)$ can be equivalently defined as the set of m-tuples $(S_1, ..., S_m)$ such that $\pi_i(R \cap (S_1 \times \cdots \times S_m)) = S_i$ for each i.

Definition 1.6.7. A homomorphism $\mathcal{P}_\emptyset(\mathbf{A}) \to \mathbf{A}$ is called a *set polymorphism* of $\mathbf{A}$.

Definition 1.6.8. A function $f : D^k \to D$ is called *totally symmetric* if the value of $f(a_1, ..., a_k)$ only depends on $\{a_1, ..., a_k\}$. Note that this is stronger than being symmetric, since the multiplicity of the a_is is also ignored.

Theorem 1.6.9. *The following are equivalent:*

- $\mathbf{A}$ *has width 1,*
- $\mathbf{A}$ *has a set polymorphism, and*
- $\mathbf{A}$ *has totally symmetric polymorphisms of every arity.*

Proof. If $\mathbf{A}$ has width 1, then generalized arc-consistency applied to $\mathcal{P}_\emptyset(\mathbf{A})$ shows that there is a homomorphism $f : \mathcal{P}_\emptyset(\mathbf{A}) \to \mathbf{A}$, since at every step the set associated to the variable $S \subseteq D$ will contain S (by induction on the number of steps and the definition of $\mathcal{P}_\emptyset(R)$). So suppose that f is a set polymorphism, and for every $k \geq 1$, let f_k be the totally symmetric function

$$f_k(a_1, ..., a_k) = f(\{a_1, ..., a_k\}).$$

We need to check that f_k is a polymorphism of $\mathbf{A}$. Suppose that $x_1, ..., x_k \in R$, then if $X = \{x_1, ..., x_k\}$, $f_k(x_1, ..., x_k)$ has ith coordinate equal to $f(\pi_i(X))$. Since $(\pi_1(X), ..., \pi_m(X)) \in \mathcal{P}_\emptyset(R)$ by the definition of $\mathcal{P}_\emptyset(R)$, we see that $f_k(x_1, ..., x_k) = (f(\pi_1(X)), ..., f(\pi_m(X))) \in R$.

Finally, suppose that $\mathbf{A}$ has totally symmetric polymorphisms f_k of every arity, and let $\mathbf{B}$ be a (finite) instance such that generalized arc-consistency stops after finding nonempty sets S_v for every variable $v \in \mathbf{B}$. Choose k at least as large as the largest number of tuples in any relation that shows up in $\mathbf{B}$, and let f be the function on sets of size $\leq k$ associated to f_k. We claim that the map $v \mapsto f(S_v)$ defines a homomorphism from $\mathbf{B}$ to $\mathbf{A}$. To see this, let $(v_1, ..., v_m)$ be a tuple with the constraint R imposed, and let $X = R \cap (S_{v_1}, ..., S_{v_m}) = \{x_1, ..., x_k\}$ (possibly with repeated x_is if $|X| < k$). Then $f_k(x_1, ..., x_k) = (f(\pi_1(X)), ..., f(\pi_m(X))) = (f(S_{v_1}), ...f(S_{v_m})) \in R$ since f_k is a polymorphism. □

Corollary 1.6.10. *A relational structure* $\mathbf{A}$ *has width 1 iff it is homomorphically equivalent to a pp-power of HORN-SAT.*

Proof. Let f be a set polymorphism of $\mathbf{A}$, and let f_k be the associated totally symmetric polymorphism of arity k. We define a height 1 clone homomorphism from $\langle\min\rangle \to \mathrm{Pol}(\mathbf{A})$ by sending $\min(x_1, ..., x_k)$ to $f_k(x_1, ..., x_k)$. Now apply the ERP Theorem 1.4.19 and Proposition 1.4.18 from the subsection on reflections. □

Example 1.6.1. Suppose that $\mathbf{A}$ has a binary polymorphism s which is associative, commutative, and idempotent (such an s is called a *semilattice operation*). Then we can define n-ary polymorphisms s_n inductively by $s_n(x_1, ..., x_n) = s(s_{n-1}(x_1, ..., x_{n-1}), x_n)$, and s_n will be totally symmetric for every n. Thus, every relational structure with a semilattice polymorphism has width 1.

Example 1.6.2. We give an example of a width 1 algebra which is not a semilattice. Let f be the idempotent set operation on $\{a, b, c\}$ given by

$$f(\{a, b\}) = b,\ f(\{b, c\}) = c,\ f(\{c, a\}) = a,\ f(\{a, b, c\}) = a,$$

and let f_k be the associated totally symmetric polymorphism of arity k. We have $f_k \in \langle f_3\rangle$ for every k, and in fact a k-ary function g which depends on all its inputs is in $\langle f_3\rangle$ iff its restriction to every two element subset of $\{a, b, c\}$ is equal to the corresponding restriction of f_k (tricky exercise). The relational clone $\mathrm{Inv}(f_3)$ is generated by the ternary relations R_{ab}, R_{bc}, R_{ca}, where R_{ab} is defined by

$$R_{ab}(x, y, z) \coloneqq (x \in \{a, b\}) \wedge (x = a \implies y = z),$$

and R_{bc}, R_{ca} are defined similarly.

Example 1.6.3. Here we give a more surprising example, of a width 1 clone such that no finitely generated subclone has width 1. Let f be the idempotent set operation on $\{-1, 0, 1\}$ (which we stylize as $\{-, 0, +\}$) given by

$$f(\{0, -\}) = -,\ f(\{0, +\}) = +,\ f(\{-, +\}) = f(\{-, 0, +\}) = 0,$$

and let f_k be the associated totally symmetric polymorphism of arity k. The clone $\mathcal{O}$ generated by the collection of all f_k then has width 1. Every finitely generated subclone of $\mathcal{O}$ is contained in $\langle f_k\rangle$ for some k. To see that $\mathcal{O} \neq \langle f_k\rangle$, consider the $k+1$-ary relation R_k given by

$$R_k(x_0, ..., x_k) \coloneqq \bigwedge_{i<j} (x_i + x_j \geq 0) \wedge (x_0, ..., x_k) \neq (0, ..., 0).$$

Then it is easy to check that R_k is preserved by f_k, but is not preserved by f_{k+1}. To see that $\langle f_k\rangle$ does not have width 1, define R_k^- similarly to R_k, but with each $x_i + x_j \geq 0$ replaced by $x_i + x_j \leq 0$. Then for $k \geq 2$ the instance

$$R_k(x_0, ..., x_k) \wedge R_k^-(x_0, ..., x_k)$$

of $\mathrm{CSP}(\mathrm{Inv}(\langle f_k\rangle))$ is arc-consistent (since both R_k and R_k^- are subdirect) but has no solution.

The relational clone $\mathrm{Inv}(\mathcal{O})$ corresponding to this example is generated by the unary relation $\{+\}$, the binary relations $x = -y$ and $x \leq y$, and the ternary relation $(x \geq 0) \wedge (x = 0 \implies y = z)$. The clone $\mathcal{O}$ is an example of a clone which is finitely related but not finitely generated.

Note that one doesn't need to *know* what the set polymorphism of $\mathbf{A}$ is to apply the arc-consistency algorithm. If $\mathbf{A}$ is a rigid core, we can use the self-reducibility of $\mathrm{CSP}(\mathbf{A})$ to find a solution to every solvable instance $\mathbf{B}$ of $\mathrm{CSP}(\mathbf{A})$ in polynomial time. By applying this to $\mathcal{P}_\emptyset(\mathbf{A})$, we can then *find* a set polymorphism of $\mathbf{A}$ - in time polynomial in the size of $\mathcal{P}_\emptyset(\mathbf{A})$, which is sadly exponential in the size of $\mathbf{A}$. The following problem is currently open.

Problem 1.6.1. Given a rigid core $\mathbf{A}$, can we determine whether it has width 1 in time polynomial in the size of the description of $\mathbf{A}$?

Now we move to the case of bounded strict width. This has a connection to an intriguing paper of Dechter [67] which predates the algebraic approach to the CSP. The next definition follows Dechter [67].

Definition 1.6.11. A partial assignment of values to variables is *locally consistent* if it satisfies every constraint which only involves those variables. A CSP instance is *strong i-consistent* if every locally consistent partial assignment to less than i variables can always be extended to a locally consistent partial assignment of any containing set of i variables. An instance is *globally consistent* if every locally consistent partial assignment extends to a global solution.

There is a straightforward polynomial time algorithm to enforce strong i-consistency for any fixed i, introducing new constraints of arity $< i$ by intersecting and existentially projecting old constraints until no changes occur. It is desirable to have globally consistent problems, because then a solution may be found greedily. Can we check if a given problem is globally consistent?

Theorem 1.6.12 (Dechter [67])**.** *If a CSP with domain sizes bounded by n and all constraint arities bounded by m is strong $(n(m-1)+1)$-consistent, then it is globally consistent.*

Proof. Suppose for contradiction that some locally consistent partial assignment $a_1, ..., a_k$ to $v_1, ..., v_k$ can't be extended to v_{k+1}, $k \ge n(m-1)+1$. Then for every possible value a of v_{k+1}, there is some constraint C_a involving at most $m-1$ of the variables $v_1, ..., v_k$ which is inconsistent with this choice of a and whichever of the a_is are relevant. Thus, there is a collection of at most n constraints C_a involving at most $n(m-1)$ of the variables from $v_1, ..., v_k$ together with the variable v_{k+1}, for which a locally consistent partial assignment of all but one of the variables can't be extended. But this contradicts the assumption of strong $(n(m-1)+1)$-consistency. □

The trouble with applying Dechter's result is that as we enforce strong consistency, we may need to add constraints of higher and higher arities. To avoid this, we want to find situations in which the newly introduced constraints can always be written as intersections of constraints of low arity.

Definition 1.6.13. A CSP template $\mathbf{A} = (D, \Gamma)$ has *strict width l* if every strong $(l+1)$-consistent instance of $\mathrm{CSP}(D, \langle\Gamma\rangle)$ which contains the projections of its relations onto subsets of size at most l is globally consistent, and has its solution-set determined by the collection of relations of arity at most l.

Note that the definition of strict width only makes sense in terms of the whole relational clone generated by Γ, a hint that it is properly viewed as an algebraic condition. Algebraically, the relevant result is the Baker-Pixley theorem [7].

Theorem 1.6.14 (Baker, Pixley [7])**.** *The following are equivalent for an algebraic structure $\mathbb{A}$:*

- *every subalgebra of $\mathbb{A}^n$ is equal to the intersection of its projections onto sets of at most l coordinates, and*

- $\mathbb{A}$ *has an* $(l+1)$*-ary* near-unanimity *term, that is, a term* t *satisfying the identities*

$$x \approx t(y, x, ..., x) \approx t(x, y, ..., x) \approx t(x, x, ..., y),$$

  where in each case all but one of the inputs to t *is* x*.*

If $\mathbb{A}$ *is idempotent, then these are both equivalent to every subalgebra of* $\mathbb{A}^{l+1}$ *being equal to the intersection of its projections onto sets of* l *coordinates.*

Proof. Note that if $|\mathbb{A}| \geq 2$, then either condition implies $l > 1$ (consider the equality relation as a subalgebra of $\mathbb{A}^2$). Suppose that the first condition holds, and consider the free algebra on $l+1$ generators $\mathcal{F}_{\mathbb{A}}(l+1) \subseteq \mathbb{A}^{\mathbb{A}^{l+1}}$ which is generated by the projections $\pi_i : \mathbb{A}^{l+1} \to \mathbb{A}$. Let A_{nu}^{l+1} be the set of tuples of elements in $\mathbb{A}^{l+1}$ which have all but at most one entry equal to each other, and let $X \subseteq \mathbb{A}^{A_{nu}^{l+1}}$ be the projection of $\mathcal{F}_{\mathbb{A}}(l+1)$ onto these coordinate tuples.

We claim that X contains the tuple t of near-unanimous votes of the entries of the coordinate tuples. By assumption, we just have to check that for every projection $\pi_{x_1,...,x_l}(X)$ onto at most l coordinates $x_1, ..., x_l \in A_{nu}^{l+1}$, there is some element $f \in \mathcal{F}_{\mathbb{A}}(l+1)$ with $\pi_{x_1,...,x_l}(f) = \pi_{x_1,...,x_l}(t)$. But each tuple x_i has at most one dissenting coordinate, so there must be some coordinate $j \leq l+1$ such that each $(x_i)_j$ is equal to the vote $t(x_i)$. Thus we can take $f = \pi_j$ to see that $\pi_{x_1,...,x_l}(\pi_j) = \pi_{x_1,...,x_l}(t)$.

Now suppose that t is an $(l+1)$-ary near-unanimity term, and suppose that $\mathbb{B} \subseteq \mathbb{A}^n$. Let $b \in \mathbb{A}^n$ be such that $\pi_I(b) \in \pi_I(\mathbb{B})$ for every $I \subseteq \{1, ..., n\}$ with $|I| \leq l$, we will show by induction on $|J|$ that $\pi_J(b) \in \pi_J(\mathbb{B})$ for *every* subset $J \subseteq \{1, ..., n\}$. For the inductive step, if $|J| \geq l+1$ then we may set $J_1, ..., J_{l+1}$ to be subsets of J formed by deleting different elements of J, and for each J_i there is some $b_{J_i} \in \mathbb{B}$ with $\pi_{J_i}(b_{J_i}) = \pi_{J_i}(b)$ by induction. But then $b_J = t(b_{J_1}, ..., b_{J_{l+1}}) \in \mathbb{B}$ and has $\pi_J(b_J) = \pi_J(b)$ by the near-unanimity equations.

For the last claim, if $\mathbb{A}$ is idempotent and $\mathbb{B} \subseteq \mathbb{A}^n$ with $n > l+1$ and $b \in \bigcap_{|I|=l} \pi_I(\mathbb{B})$, then $\mathbb{B}' = \pi_{\{1,...,n-1\}}(\mathbb{B} \cap (\mathbb{A}^{n-1} \times \{b_n\}))$ is a subalgebra of $\mathbb{A}^{n-1}$, and we may induct on n to see that $\mathbb{B}' = \bigcap_{|I|=l} \pi_I(\mathbb{B}')$, while the assumption on subalgebras of $\mathbb{A}^{l+1}$ gives $\pi_I(b) \in \pi_I(\mathbb{B}')$ for every I with $|I| = l$. □

Theorem 1.6.15. *A relational structure* $\mathbf{A}$ *has strict width* l *iff it has an* $(l+1)$*-ary near-unanimity polymorphism.*

Proof. Let $\mathbb{A}$ be the associated algebraic structure. For any n and any $\mathbb{B} \subseteq \mathbb{A}^n$, the strong $(l+1)$-consistent instance formed via the relations $\mathbb{B}$ and $\pi_I(\mathbb{B})$ for all $I \subseteq \{1, ..., n\}$ with $|I| \leq l$ together with the definition of strict width l imply that $\mathbb{B} = \bigcap_{|I| \leq l} \pi_I(\mathbb{B})$, so by the Baker-Pixley Theorem $\mathbb{A}$ has an $(l+1)$-ary near unanimity term.

For the other direction, suppose that t is an $(l+1)$-ary near-unanimity term of $\mathbb{A}$ and that we have a strong $(l+1)$-consistent instance of $\mathrm{CSP}(\mathbb{A})$, which we may assume by the Baker-Pixley Theorem to only involve relations of arity at most l. Suppose that we have a locally consistent partial solution which assigns the values $a_1, ..., a_k$ to the variables $v_1, ..., v_k$ which we want to extend to the variable v_{k+1}. By strong $(l+1)$-consistency, we can assume that $k \geq l+1$. By induction on k, we can assume that for each $i \leq l+1$ there is some value a_{k+1}^i that we can assign the the variable v_{k+1} such that if we ignore v_i, we get a locally consistent partial solution.

We claim that assigning the value $a_{k+1} = t(a_{k+1}^1, ..., a_{k+1}^{l+1})$ to v_{k+1} gives a locally consistent partial solution. To see this, consider some constraint C which involves the variable v_{k+1} and some variables from $v_1, ..., v_k$. For each $i \leq l+1$, by l-consistency and the fact that C has arity at most l we can find a value a_i' such that $(a_1, ..., a_i', ..., a_{k+1}^i)$ satisfies the constraint C. Applying t to these $l+1$ tuples, we see that the tuple $(a_1, ..., a_{l+1}, ..., t(a_{k+1}^1, ..., a_{k+1}^{l+1}))$ also satisfies C, by the near-unanimity identities and the fact that t is a polymorphism of C. □

Algorithm 3 Strict width l algorithm

```
1: Replace each constraint with its projections onto all subsets of at most $l$ variables.
2: repeat
3:     for all sets $\{v_1, ..., v_k\}$ of variables with $k \leq l+1$ do
4:         Let $X$ be the set of solutions to the restriction of the CSP to the variables $v_1, ..., v_k$.
5:         If $\pi_I(X)$ is not implied by the restriction of the CSP to the variables in $I$ for some
           $I \subset \{v_1, ..., v_k\}$, add it as a new constraint.
6: until no new constraints are added.
7: Greedily assign values to variables until we find a global solution.
```

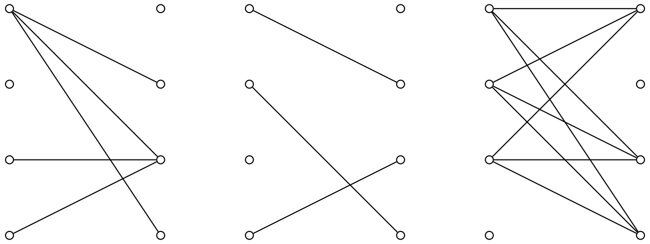
Example 1.6.4. 2SAT has the ternary polymorphism maj, which is a near-unanimity operation. Therefore 2SAT has strict width 2, a fact which also follows from Dechter's result above [67].

Example 1.6.5. Generalizing 2SAT, let D be any domain, and let $d : D^3 \to D$ be given by

$$d(x, y, z) = \begin{cases} x & \text{if } y \neq z, \\ y & \text{if } y = z. \end{cases}$$

This function d is known as the *dual discriminator*, and for $|D| \neq 4$ it is the only majority function (up to permuting inputs) on D which preserves the graph of every bijection from D to itself.

A binary relation $R \subseteq D^2$ is preserved by the dual discriminator iff it is a "0/1/all constraint", that is, a constraint such that when viewed as a bipartite graph on the disjoint union $D \sqcup D$, every vertex which doesn't have degree 0 or 1 connects to all vertices on the other side which have positive degree. Typical 0/1/all constraints are displayed below.



For any a in D, a generating set of binary relations for $\mathrm{Inv}(d)$ is given by the graphs of a pair of bijections which generate the symmetric group on $|D|$ elements, the unary relation $D \setminus \{a\}$, and the binary relation $x = a \vee y = a$.

Example 1.6.6. For every n, the relational structure $(\{0,1\}, \{0\}, \leq, \{0,1\}^n \setminus \{(0, ..., 0)\})$ has strict width exactly n. A near-unanimity term for it is given by the threshold function

$$t_2^{n+1}(x_1, ..., x_{n+1}) = \begin{cases} 1 & \sum_i x_i \geq 2, \\ 0 & \sum_i x_i \leq 1. \end{cases}$$

To see that it doesn't have strict width less than n, note that the relation $\{0,1\}^n \setminus \{(0,...,0)\}$ is not the intersection of its projections onto $n-1$ coordinates. Note that this template also has width 1 (it is preserved by the semilattice operation max), so the strict width algorithm is far from being the best way to solve it for n large.

Note that the existence of an $(l+1)$-ary near-unanimity operation in $\mathrm{Pol}(\mathbf{A})$ is equivalent to the solvability of the CSP instance Φ (of $\mathbf{A}$ together with singleton unary relations) with variables indexed by elements of $\mathbf{A}^{l+1}$ described by the primitive positive formula

$$\Phi(t) \coloneqq \bigwedge_{R\in\Gamma} \bigwedge_{M\in R^{l+1}} t(M) \in R \wedge \bigwedge_{a,b\in\mathbf{A}} t(b,a,...,a) = t(a,b,...,a) = \cdots = t(a,a,...,b) \in \{a\}.$$

This instance may be solved in polynomial time by the strict width l algorithm, giving us an $(l+1)$-ary near-unanimity term t as output. Note, however, that the number of variables is exponential in l - what if we just want to know whether the structure $\mathbf{A}$ has bounded strict width, allowing l to be arbitrarily large?

Problem 1.6.2. Given a relational structure $\mathbf{A}$, determine whether it has bounded strict width.

The good news is that whether the structure is given as a finite relational structure or a finite algebraic structure, the existence of a near unanimity term is at least *decidable* [138], [12], [192]. The bad news is that the minimal arity of a near-unanimity term may be very large.

Theorem 1.6.16 (Zhuk, Barto, Draganov [15], [192])**.** *For any relational structure $\mathbf{A}$ with $|\mathbf{A}| = n$ such that every basic relation of $\mathbf{A}$ has arity at most m, if $\mathbf{A}$ has bounded strict width, then $\mathbf{A}$ has strict width at most*

$$\frac{1}{2}(2m-2)^{3^n}.$$

Conversely, for each $m \geq 3$ and $n \geq 2$, there is an example of a relational structure with bounded strict width such that every basic relation of $\mathbf{A}$ has arity at most m, which has no near-unanimity polymorphism of arity at most

$$(m-1)^{2^{n-2}},$$

and for $m = 2, n \geq 3$ there is an example with no near-unanimity polymorphism of arity at most

$$2^{2^{n-3}}.$$

Luckily, it is possible to determine whether a relational structure has bounded strict width without actually exhibiting a near-unanimity polymorphism. For instance, in [14] a nondeterministic polynomial time algorithm, which only tests for the existence of certain chains of ternary polymorphisms of $\mathbf{A}$, is given for deciding whether a given subset of $\mathbf{A}$ is an absorbing subalgebra (defined later). Using the fact that cycle consistency solves CSPs which have bounded width (which we will prove later), this can be converted into a polynomial time algorithm for testing whether $\mathbf{A}$ has bounded strict width.

1.6.1 The Basic LP relaxation of a CSP

Another simple algorithm for solving CSPs, which is closely related to generalized arc-consistency, is the basic LP relaxation. If the domain of each variable v is D_v, we replace the set of potential values D_v with its formal convex hull, which we can think of as the set of *probability distributions* on D_v. We represent the probability distribution corresponding to a variable v as a tuple of real numbers $p_{v,a}$, one for each $a \in D_v$, satisfying

$$0 \le p_{v,a} \le 1, \sum_{a \in D_v} p_{v,a} = 1.$$

We also replace each constraint with its convex hull. That is, if the constraint C imposes the relation $R = R_C$ on the variables $v_1, ..., v_m$, then we require the existence of a probability distribution $p_{C,r}$, on the tuples r of R such that

$$0 \le p_{C,r} \le 1, \sum_r p_{C,r} = 1,$$

and which is compatible with the probability distributions on the individual variables in the sense that

$$p_{v_i,a} = \sum_{r_i = a} p_{C,r}.$$

If a problem is known not to be fully satisfiable, we can relax it further by extending the probability distributions over relations $R \subseteq D_{v_1} \times \cdots \times D_{v_m}$ to probability distributions over all of $D_{v_1} \times \cdots \times D_{v_m}$, and then try to maximize the sum of the probabilities that tuples which are supposed to be in R are actually in r:

$$\tfrac{1}{\#C} \sum_C \sum_{r \in R_C} p_{C,r}.$$

This system of linear equations and inequalities, with the optimization target above, is known as the *basic LP* relaxation of a given CSP instance.

Theorem 1.6.17 (Kun, O'Donnell, Tamaki, Yoshida, Zhou [130])**.** *For any relational structure* $\mathbf{A}$*, the following are equivalent:*

- *the basic LP relaxation correctly solves every instance of* $\mathrm{CSP}(\mathbf{A})$*,*
- $\mathbf{A}$ *has symmetric polymorphisms of every arity.*

Furthermore, if $\mathbf{A}$ *has width 1 then the basic LP relaxation can be used to* robustly *solve* $\mathrm{CSP}(\mathbf{A})$*, that is, if we are given an instance which is* $1-\epsilon$ *satisfiable, then we can find a solution which satisfies a* $1 - O(1/\log(1/\epsilon))$ *fraction of the constraints.*

Proof. Suppose first that the basic LP solves $\mathrm{CSP}(\mathbf{A})$, and consider the (by now standard) instance Φ that describes the existance of a symmetric polymorphism of arity n:

$$\Phi(s) \coloneqq \bigwedge_{R \in \Gamma} \bigwedge_{M \in R^n} s(M) \in R \wedge \bigwedge_{a_1,...,a_n \in \mathbb{A}} \bigwedge_{\sigma \in S_n} s(a_1, ..., a_n) = s(a_{\sigma(1)}, ..., a_{\sigma(n)}).$$

By the assumption that the basic LP decides $\mathrm{CSP}(\mathbf{A})$, we just need to exhibit a fractional solution to this CSP. This is achieved by taking $s = \frac{1}{n}\pi_1 + \cdots + \frac{1}{n}\pi_n$: as a convex combination of polymorphisms, it satisfies the relaxation of the first collection of constraints, and since it is a symmetric convex combination of its inputs it satisfies the second collection of constraints.

For the other direction, suppose that an instance of the CSP has a fractional solution to its basic LP relaxation, with probability distributions $p_{v,a}$ for each variable/value and $p_{C,r}$ for each constraint/tuple. We may assume that these probabilities are all rational (since the defining system of linear equations and inequalities had rational coefficients), and that they have a common denominator n. By assumption $\mathbf{A}$ has a symmetric polymorphism s of arity n, which we can think of as a function from probability distributions with denominator n over the domain of $\mathbf{A}$ to elements of $\mathbf{A}$.

Applying s to each $p_{v,\cdot}$ gives an element $a_v \in \mathbf{A}$, and applying it to each probability distribution $p_{C,\cdot}$ gives a tuple r_C in the associated relation R (since s is a polymorphism). Furthermore, the compatibility equations between the distributions $p_{v_i,\cdot}$ and $p_{C,\cdot}$ that we get when v_i is the ith coordinate of the constraint C, together with the symmetry of s, imply that $a_{v_i} = (r_C)_i$ for each i, so $(a_{v_1}, ..., a_{v_m}) = r_C \in R$. Thus the a_vs form a valid solution to the CSP instance.

Finally, assume that $\mathbf{A}$ has width 1, with set polymorphism f, and suppose that our original instance was $1-\epsilon$ satisfiable. Then the basic LP finds a fractional solution with value $\ge 1-\epsilon$. We will use the polymorphism f to make a randomized rounding scheme. First, we immediately give up on any constraints C that the LP only satisfies with value $\le 1-\sqrt{\epsilon}$ - these can form at most a $\sqrt{\epsilon}$ fraction of the constraints by Markov's inequality. Second, we will choose a threshold $\theta \le \frac{1}{|\mathbf{A}|}$, and for each variable v we assign the value

$$a_v = f(\{a \in \mathbf{A} \mid p_{v,a} \ge \theta\}).$$

Note that the restriction $\theta \le \frac{1}{|\mathbf{A}|}$ ensures that the sets on the right hand side are nonempty. We will show that if θ is chosen from a certain probability distribution, then on average we will obtain a good solution to the CSP, and deduce from this that some specific choice of θ works at least as well. For this we need the following claim.

Claim. If C is the constraint $(v_1, ..., v_m) \in R$ which is satisfied with value $\ge 1-\sqrt{\epsilon}$, and if $2\sqrt{\epsilon} \le \theta \le \frac{1}{|\mathbf{A}|}$ is such that

$$\theta \notin (p_{v_i,a}/(2|R|), p_{v_i,a}]$$

for any pair $i \le m, a \in \mathbf{A}$, then $(a_{v_1}, ..., a_{v_m})$ satisfies C.

Proof of Claim. For each v, let $S_v = \{a \mid p_{v,a} \ge \theta\}$, so $a_v = f(S_v)$. In order to show that $(a_{v_1}, ..., a_{v_m})$ satisfies R, we just need to check that this colection of sets S_{v_i} together with R form a generalized arc-consistent instance. Let $a \in S_{v_i}$ for some i, then we have $p_{v_i,a} \ge \theta \ge 2\sqrt{\epsilon}$ by the definition of S_{v_i}. From

$$\sum_{r \in R, r_i = a} p_{C,r} \ge p_{v_i,a} - \sqrt{\epsilon} \ge p_{v_i,a}/2,$$

we see that there must be some $r \in R$ with $r_i = a$ and $p_{C,r} \ge p_{v_i,a}/(2|R|)$. Since $p_{v_i,a} \ge \theta$, by the assumption on θ we have $p_{v_i,a}/(2|R|) \ge \theta$, so $p_{C,r} \ge \theta$. But then $p_{v_j,r_j} \ge p_{C,r} \ge \theta$ for all j, so $r_j \in S_{v_j}$ for all j, and we see that a extends to a solution of $R \cap (S_{v_1} \times \cdots \times S_{v_m})$.

To finish the proof, we choose θ uniformly at random from the set $\{\frac{1}{|\mathbf{A}|}, \frac{1}{|\mathbf{A}|T}, \cdots \frac{1}{|\mathbf{A}|T^b}\}$, where T is twice the maximum number of tuples in any relation R and $b = \lfloor \log(1/2|A|\sqrt{\epsilon})/\log(T) \rfloor$. Note that b grows like $\log(1/\epsilon)$, that's the only important thing to keep track of in the mess. Then every constraint of arity m which we hadn't given up on is satisfied with probability at least $1 - m|A|/b$ (since there are at most $m|A|$ bad choices of θ where the claim doesn't apply), and asymptotically that looks like $1 - O(1/\log(1/\epsilon))$. □

Remark 1.6.3. The dependence of the error in $1 - O(1/\log(1/\epsilon))$ on ϵ in the previous theorem is best possible in the case of HORN-SAT: Guruswami and Zhou [90] show that there are integrality gap instances even for the SDP relaxation (see Example 3.16.3), and by a fundamental result of Raghavendra [160] they deduce that under the Unique Games conjecture it is NP-hard to find an assignment satisfying a $1 - o(1/\log(1/\epsilon))$ fraction of the constraints.

Remark 1.6.4. In [130], it is also claimed that the basic LP solves every instance of CSP($\mathbf{A}$) if and *only if* $\mathbf{A}$ has width 1. The proof has a subtle error, however. The following counterexample, due to Kun, can be found in [65].

Example 1.6.7. Let $\mathbf{A} = (\{-1, 0, 1\}, R_+, R_-)$, where $R_+ = \{(a, b, c) \mid a + b + c \ge 1\}$ and $R_- = \{(a, b, c) \mid a + b + c \le -1\}$. Then for every h, n with $h < \frac{n}{3}$, the function

$$s_{h,n}(x_1, ..., x_n) = \begin{cases} 1 & \sum_i x_i > h \\ 0 & -h \le \sum_i x_i \le h \\ -1 & \sum_i x_i < -h \end{cases}$$

is a symmetric polymorphism of $\mathbf{A}$. Thus CSP($\mathbf{A}$) is solved by the basic LP relaxation. However, $\mathbf{A}$ has no totally symmetric polymorphism of arity 3, since such a polymorphism would necessarily map the matrices

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \in R_+^3, \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \in R_-^3$$

to the same diagonal tuple, so $\mathbf{A}$ does not have width 1.

Example 1.6.8. The previous example can be generalized to a much larger relational structure on $\{-1, 0, 1\}$ as follows. Set $s_n = s_{0,n}$, then it isn't hard to show that $s_n \in \mathrm{Clo}(s_2)$ for all n (hint: start by defining $t_n(x_1, ..., x_n) = s_2(x_1, s_{n-1}(x_2, ..., x_n))$), so $\mathrm{Inv}(s_2)$ also defines a CSP template which is solved by the basic LP relaxation.

s_2	$-$	0	$+$
$-$	$-$	$-$	0
0	$-$	0	$+$
$+$	0	$+$	$+$

$\mathrm{Inv}(s_2)$ is generated by the relations $\{1\}$, $x = -y$, and the set of *odd cycle relations*, where the m-th odd cycle relation R_m is defined by

$$R_m(x_1, ..., x_{2m-1}, y, z) \coloneqq (x_1 + x_2 \ge 0) \wedge \cdots \wedge (x_{2m-1} + x_1 \ge 0) \wedge (x_1 = \cdots = x_{2m-1} = 0 \implies y = z).$$

(I found this set of generating relations by a technique I learned from Zhuk [193], in which we search for "key" relations R, for which there is some "key tuple" $x \notin R$ such that the relation R is maximal among those relations of $\mathrm{Inv}(s_2)$ which do not contain x. It isn't hard to show that any key tuple must consist mostly of 0s, and using the negation symmetry we can assume that R contains all tuples in $\{0, 1\}^n$ aside from the key tuple. Then we look at the set of pairs of coordinates that can't simultaneously be set to -1, and prove that the resulting graph can't be bipartite...)

The clone $\langle s_2 \rangle$ is not finitely related. To see this, define an operation s'_n for n odd by the rule

$$s'_n(x_1, ..., x_n) = \begin{cases} s_{0,n}(x_1, ..., x_n) & \text{if some } x_i = 0, \\ s_{1,n}(x_1, ..., x_n) & \text{if all } x_i \in \{-1, 1\}. \end{cases}$$

For every odd $n = 2m - 1$, the operation $s'_n \notin \langle s_2 \rangle$ - since it does not preserve the relation R_m - but the function $s'_n(x, x, y_3, ..., y_n)$ we get by identifying two of its inputs *is* in $\langle s_2 \rangle$ (exercise for the reader), so it preserves every relation in $\mathrm{Inv}(s_2)$ which contains strictly less than n tuples.

The clone $\langle s_2 \rangle$ is strictly contained in the width 1 clone from Example 1.6.3, and corresponds to a strictly larger relational clone with a tractable CSP. Later we will see that this relational clone can be enlarged further, such that the CSP remains solvable by bounded width reasoning.

Currently it is unknown if the following problem is decidable.

Problem 1.6.3. Given a finite relational structure $\mathbf{A}$, determine if it has symmetric polymorphisms of every arity.

An interesting result in this direction is proved in [51]: an algebraic structure $\mathbb{A}$ has symmetric polymorphisms of all arities iff there is no $\mathbb{B} \in HSP(\mathbb{A})$ which has a pair of automorphisms in $\mathrm{Aut}(\mathbb{B})$ having no common fixed point (in fact, if $\mathbb{A}$ has no symmetric polymorphism of arity n, we can take $\mathbb{B}$ to be the free algebra on n variables in the variety generated by $\mathbb{A}$). If $HSP(\mathbb{A})$ could be replaced by $HS(\mathbb{A})$ in their result, then this would imply that it is enough to check for the existence of symmetric polymorphisms of arities up to $|\mathbf{A}|$.

Later we will prove that any Taylor algebra has cyclic polymorphisms of all arities which have no small prime factors, so we might hope that we could use these to help construct symmetric polymorphisms of higher arity. More ingredients are likely needed for such an argument, however: in [51], an example is given of a relational structure which has cyclic polymorphisms of every arity, but which has no symmetric polymorphism of arity 5.

1.7 Mal'cev algebras

The goal in this section and the next is to generalize group theoretic algorithms (such as the algorithm for solving XOR-SAT) by isolating the special feature of groups which makes them so nice. First we should connect groups to CSPs, by defining the correct analogue of "affine spaces" for general groups.

Proposition 1.7.1. *If G is a group, then a nonempty subset $H \subseteq G^n$ is preserved by the ternary operation $(x, y, z) \mapsto xy^{-1}z$ iff H is a coset of a subgroup of G^n.*

Proof. Let U be the subgroup of G^n generated by expressions of the form $y^{-1}z$ for $y, z \in H$. Then H is preserved under $(x, y, z) \mapsto xy^{-1}z$ iff H is closed under the right action of U, so H is a union of left cosets of U. To see that H is just a single coset, note that for $x, y \in H$, we have $x^{-1}y \in U$ and $x(x^{-1}y) = y$.

Conversely, if $H = hU$ for some subgroup U of G^n, then $HH^{-1}H = hU(hU)^{-1}hU = hUUh^{-1}hU = hUUU = hU = H$. □

The idempotent operation $(x, y, z) \mapsto xy^{-1}z$ was isolated by universal algebraists who wanted to understand the underlying reason for the fact that normal subgroups commute: if $K, N \lhd G$ are normal subgroups of a group G, then $KN = NK$ and KN is also a normal subgroup of G. Of course this is easy to verify in the context of groups, but from the point of view of universal algebra it is really saying something interesting about *congruences* of groups. If K, N correspond to congruences α, β on G, then we can view this equality as the statement that $\alpha \circ \beta = \beta \circ \alpha = \alpha \vee \beta$, where composition of binary relations is defined as follows.

Definition 1.7.2. Let R, S be binary relations $R \subseteq A \times B, S \subseteq B \times C$. Then we define their *composition* $R \circ S$ to be the subset of $A \times C$ consisting of pairs (a, c) such that there exists a $b \in B$ with aRb and bSc. As a primitive positive formula, we can write this as

$$R \circ S(a, c) \coloneqq \exists b \in B \ R(a, b) \wedge S(b, c).$$

In general, it is not the case that congruences commute. In order to find the smallest congruence containing a pair of congruences in a general algebraic structure, one uses the following fact.

Proposition 1.7.3. *If α, β are congruences on an algebraic structure $\mathbb{A}$, then their least upper bound $\alpha \vee \beta$ is the transitive closure of $\alpha \circ \beta$, that is,*

$$\alpha \vee \beta = \bigcup_{n \geq 0} (\alpha \circ \beta)^{\circ n}.$$

If α, β do commute, then the above formula simplifies to $\alpha \vee \beta = \alpha \circ \beta$. So it is natural to try to understand the collection of all algebraic structures with commuting congruences. Of course, a structure with no congruences at all has this property - but we want to understand algebraic structures that have a *reason* for their congruences to commute, so rather than studying algebras in isolation we study varieties with this property.

Definition 1.7.4. We say that a variety $\mathcal{V}$ is *congruence permutable* if for all $\mathbb{A} \in \mathcal{V}$ and all $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$ we have $\alpha \circ \beta = \beta \circ \alpha$.

Theorem 1.7.5. *A variety $\mathcal{V}$ is congruence permutable iff $\mathcal{V}$ has a ternary term p which satisfies the identity*

$$p(x, y, y) \approx p(y, y, x) \approx x.$$

Proof. Suppose first that $\mathcal{V}$ is congruence permutable. Let $\mathcal{F} = \mathcal{F}_{\mathcal{V}}(x, y, z)$ be the free algebra on three generators in $\mathcal{V}$. Define a congruence α on $\mathcal{F}$ to be the least congruence with $x/\alpha = y/\alpha$, that is, α is the kernel of the homomorphism $\mathcal{F}_{\mathcal{V}}(x, y, z) \to \mathcal{F}_{\mathcal{V}}(x, z)$ given by $x, y \mapsto x, z \mapsto z$. Similarly, let β be the least congruence on $\mathcal{F}$ with $y/\beta = z/\beta$.

Then $(x, z) \in \alpha \circ \beta$, so if $\mathcal{V}$ has commuting congruences, then there must be some $p(x, y, z) \in \mathcal{F}$ such that $x/\beta = p(x, y, z)/\beta$ and $p(x, y, z)/\alpha = z/\alpha$. But this is equivalent to the pair of identities $x \approx p(x, y, y)$, $p(x, x, z) \approx z$.

Conversely, suppose such a term p exists, and let $\mathbb{A} \in \mathcal{V}$ and $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$. Then for any a, b, c with $a/\alpha = b/\alpha$ and $b/\beta = c/\beta$ we have

$$p(a, b, c)/\beta = p(a, b, b)/\beta = a/\beta$$

and

$$p(a, b, c)/\alpha = p(a, a, c)/\alpha = c/\alpha,$$

so $(a, c) \in \beta \circ \alpha$. □

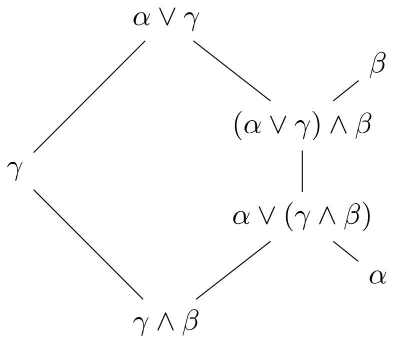
Definition 1.7.6. A ternary term p is called a *Mal'cev term* if it satisfies the identity $p(x, y, y) \approx p(y, y, x) \approx x$. An algebra with a Mal'cev term is called a *Mal'cev algebra*, and a variety with a Mal'cev term is called a *Mal'cev variety.*

One reason universal algebraists like congruence permutability is that it implies that the congruence lattice is *modular*, a property first isolated by Dedekind in his investigation of the lattice of submodules of a module over a ring.

Definition 1.7.7. A lattice $\mathcal{L}$ is *modular* if for all $\alpha, \beta, \gamma \in \mathcal{L}$, we have

$$\alpha \le \beta \implies \alpha \vee (\gamma \wedge \beta) = (\alpha \vee \gamma) \wedge \beta.$$

Equivalently, a lattice $\mathcal{L}$ is modular if it has no five element sublattice isomorphic to the lattice $\mathcal{N}_5$ whose Hasse diagram is a pentagon: consider the sublattice generated by $\alpha' = \alpha \vee (\gamma \wedge \beta), \beta' = (\alpha \vee \gamma) \wedge \beta$, and γ, with top element $\alpha \vee \gamma = \alpha' \vee \gamma$ and bottom element $\gamma \wedge \beta = \gamma \wedge \beta'$, and note that we always have $\alpha' \le \beta'$.


$\alpha \vee \gamma$
β
$(\alpha \vee \gamma) \wedge \beta$
γ
$\alpha \vee (\gamma \wedge \beta)$
α
$\gamma \wedge \beta$

Proposition 1.7.8. *If* $\mathbb{A}$ *has permuting congruences, then* $\mathrm{Con}(\mathbb{A})$ *is a modular lattice.*

Proof. We just have to check that if $\alpha \le \beta$, then $\alpha \circ (\gamma \wedge \beta) \ge (\alpha \circ \gamma) \wedge \beta$. Suppose that $(x, z) \in (\alpha \circ \gamma) \wedge \beta$, and choose y such that $(x, y) \in \alpha, (y, z) \in \gamma$. Then $(y, x) \in \alpha \subseteq \beta$ and $(x, z) \in \beta$, so $(y, z) \in \beta \circ \beta = \beta$, so $(y, z) \in \gamma \wedge \beta$, so $(x, z) \in \alpha \circ (\gamma \wedge \beta)$. □

An unexpectedly large example of a Mal'cev variety is the variety of *quasigroups*.

Definition 1.7.9. A binary operation on a finite set is called a *quasigroup* if its multiplication table is a Latin square (i.e. each element appears exactly once in each row and column). The *variety of quasigroups* has three basic operations $\cdot, /, \backslash$, which satisfy the following identities:

$$(a \cdot b)/b \approx a, \quad (a/b) \cdot b \approx a, \quad b\backslash(b \cdot a) \approx a, \quad b \cdot (b\backslash a) \approx a.$$

Note that in the finite case, if $\cdot$ is a quasigroup operation, then the operations $/, \backslash$ can be defined in terms of $\cdot$ by an iteration argument (for any invertible unary function f on an n element set, $f^{-1} = f^{\circ(n!-1)}$). For infinite quasigroups, they have to be introduced into the language explicitly.

Proposition 1.7.10. *If* $\mathbb{A} = (A, \cdot, /, \backslash)$ *is a quasigroup, then* $p : (x, y, z) \mapsto (x/y) \cdot ((y/y)\backslash z)$ *is a Mal'cev term.*

Proof. Plugging in $x = y$ we get

$$p(y, y, z) = (y/y) \cdot ((y/y)\backslash z) \approx z,$$

and plugging in $z = y$ we get

$$p(x, y, y) = (x/y) \cdot ((y/y)\backslash y) \approx (x/y) \cdot ((y/y)\backslash((y/y) \cdot y)) \approx (x/y) \cdot y \approx x. \qquad \square$$

The corresponding property on the CSP side of the picture is something known as the *parallelogram property* (some authors call this *rectangularity*, although the definition of rectangularity is often slightly weaker in the case of relations of higher arity).

Definition 1.7.11. A binary relation $R \subseteq A \times B$ has the *parallelogram property* if whenever $(a,b),(c,b),(c,d) \in R$, we also have $(a,d) \in R$. A relation of higher arity is said to have the parallelogram property if every way of grouping its coordinates into two groups gives a binary relation with the parallelogram property.

Theorem 1.7.12. *A finite algebraic structure $\mathbb{A}$ has a Mal'cev term p iff every relation $\mathbb{R} \in \mathrm{Inv}(\mathbb{A})$ has the parallelogram property.*

Proof. Suppose first that $\mathbb{A}$ has a Mal'cev term p, let $\mathbb{B},\mathbb{C} \in \mathcal{V}(\mathbb{A})$ and let $\mathbb{R} \le \mathbb{B}\times\mathbb{C}$ be a subalgebra of their product. Suppose that $(a,b),(c,b),(c,d) \in \mathbb{R}$. Then

$$\begin{bmatrix} a \\ d \end{bmatrix} = p\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) \in \mathbb{R},$$

so $\mathbb{R}$ has the parallelogram property.

Conversely, suppose that every relation in $\mathrm{Inv}(\mathbb{A})$ has the parallelogram property. Let $\pi_1,\pi_2 \in \mathbb{A}^{\mathbb{A}^2}$ be the elements corresponding to the functions $\pi_i : (a_1,a_2) \mapsto a_i$. Let $\mathbb{R} \le (\mathbb{A}^{\mathbb{A}^2})^2$ be the subalgebra generated by the three pairs $(\pi_1,\pi_1),(\pi_2,\pi_1),(\pi_2,\pi_2)$. Then since $\mathbb{R}$ has the parallelogram property, we must have $(\pi_1,\pi_2) \in \mathbb{R}$, so there must be a ternary term p such that

$$\begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = p\left(\begin{bmatrix} \pi_1 \\ \pi_1 \end{bmatrix}, \begin{bmatrix} \pi_2 \\ \pi_1 \end{bmatrix}, \begin{bmatrix} \pi_2 \\ \pi_2 \end{bmatrix}\right).$$

But then this p is a Mal'cev term for $\mathbb{A}$. □

If we want to test whether an algebra has a Mal'cev term, then the above result would make it seem like we need to test whether relations of arbitrarily large arity have the parallelogram property. As it turns out, for idempotent algebras we only need to test whether all binary relations have the parallelogram property.

Theorem 1.7.13 ([98], [112], [182], [111], [191])**.** *A finite idempotent algebra $\mathbb{A}$ has a Mal'cev term if and only if every binary relation $\mathbb{R} \in \mathrm{Inv}_2(\mathbb{A})$ has the parallelogram property. More explicitly, this occurs if and only if we have*

$$\begin{bmatrix} a \\ d \end{bmatrix} \in \mathrm{Sg}_{\mathbb{A}^2}\left\{\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right\}$$

for all $a,b,c,d \in \mathbb{A}$.

Proof. (Following [191]) Suppose that $\mathbb{A}$ is not Mal'cev, and consider a relation $\mathbb{R} \in \mathrm{Inv}(\mathbb{A})$ of minimal arity n among those which do not have the parallelogram property. If $n = 2$, we are done. Otherwise, we will try to use $\mathbb{R}$ to define a relation of lower arity which also fails to have the parallelogram property.

Suppose that $\mathbb{R}$ fails to have the parallelogram property when considered as a binary relation on $\mathbb{A}^k \times \mathbb{A}^{n-k}$, and assume without loss of generality that $k \ge 2$. Since $\mathbb{R}$ fails to have the parallelogram property, there are tuples $a,c \in \mathbb{A}^k$ and tuples $b,d \in \mathbb{A}^{n-k}$ such that $(a,b),(c,b),(c,d) \in \mathbb{R}$ but

$(a,d) \not\in \mathbb{R}$. Write $a_1 = \pi_1(a)$ and $a' = \pi_{2,...,k}(a)$, and define c_1, c' similarly. Define $\mathbb{R}' \le \mathbb{A}^{k-1} \times \mathbb{A}^{n-k}$ by

$$(x', y) \in \mathbb{R}' \iff \exists x_1 \in \mathbb{A}\ ((x_1, x'), y) \in \mathbb{R} \wedge ((x_1, x'), b) \in \mathbb{R}.$$

Then we have $(a', b), (c', b), (c', d) \in \mathbb{R}'$, so if $\mathbb{R}'$ has the parallelogram property then we must have $(a', d) \in \mathbb{R}'$. Thus there is some $e_1 \in \mathbb{A}$ such that $((e_1, a'), b), ((e_1, a'), d) \in \mathbb{R}$. Now define $\mathbb{R}'' \le \mathbb{A} \times \mathbb{A}^{n-k}$ by

$$(x_1, y) \in \mathbb{R}'' \iff ((x_1, a'), y) \in \mathbb{R}.$$

Then we have $(a_1, b), (e_1, b), (e_1, d) \in \mathbb{R}''$, so if $\mathbb{R}''$ has the parallelogram property then we must have $(a_1, d) \in \mathbb{R}''$, which means that $(a, d) \in \mathbb{R}$, a contradiction. □

Remark 1.7.1. In [111], the authors give an explicit polynomial time procedure to construct a Mal'cev term out of a collection of idempotent "local Mal'cev terms" t_{abcd} satisfying

$$t_{abcd}(a,b,b) = a, \ \ t_{abcd}(c,c,d) = d.$$

The construction consists of two stages. In the first stage we construct, for each a, b, a term t_{ab} which satisfies

$$t_{ab}(a,b,b) = a, \ \ t_{ab}(y,y,x) \approx x.$$

To do this, we pick an ordering (c_i, d_i) of the set of ordered pairs (c, d), and inductively define terms t_{ab}^i by $t_{ab}^0(x,y,z) \coloneqq x$ and

$$t_{ab}^{i+1}(x,y,z) \coloneqq t_{abu_id_i}(t_{ab}^i(x,y,z), t_{ab}^i(y,y,z), z),$$

where $u_i = t_{ab}^i(c_i, c_i, d_i)$. These terms will satisfy $t_{ab}^i(a,b,b) = a$ and $t_{ab}^i(c_j, c_j, d_j) = d_j$ for all $j < i$. We finish the first stage by taking $t_{ab} \coloneqq t_{ab}^{n^2}$, where n is the number of elements in our algebra.

The second stage of the construction is similar. We first pick an ordering (a_i, b_i) of the set of ordered pairs (a, b), and then inductively define terms t_i by $t_0(x,y,z) \coloneqq z$ and

$$t_{i+1}(x,y,z) \coloneqq t_{a_iv_i}(x, t_i(x,y,y), t_i(x,y,z)),$$

where $v_i = t_i(a_i, b_i, b_i)$. These terms will satisfy $t_i(y,y,x) \approx x$ and $t_i(a_j, b_j, b_j) = a_j$ for all $j < i$. The term $p(x,y,z) \coloneqq t_{n^2}(x,y,z)$ will then be a Mal'cev term.

Definition 1.7.14. If $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$ is a subdirect binary relation, then the *linking congruence* of $\mathbb{R}$ can refer to any of the following three congruences: the congruence $\ker \pi_1 \vee \ker \pi_2$ on $\mathbb{R}$, the congruence α on $\mathbb{A}$ generated by pairs $a, a' \in \mathbb{A}$ such that there exists a $b \in \mathbb{B}$ with $(a, b), (a', b) \in \mathbb{R}$, or the similar congruence β defined on $\mathbb{B}$. The relation $\mathbb{R}$ is called *linked* if these congruences are full.

Note that in the above definition, we have $\alpha = \pi_1(\ker \pi_1 \vee \ker \pi_2), \beta = \pi_2(\ker \pi_1 \vee \ker \pi_2)$, and

$$\mathbb{A}/\alpha \cong \mathbb{R}/(\ker \pi_1 \vee \ker \pi_2) \cong \mathbb{B}/\beta.$$

A more *visual* way to understand the linking congruence is to think of the relation $\mathbb{R}$ as a bipartite graph on $\mathbb{A} \sqcup \mathbb{B}$, and to define the congruence classes to be the connected components of this graph. In particular, $\mathbb{R}$ is linked iff this bipartite graph is connected.

Proposition 1.7.15 (Goursat's Lemma). *A subdirect binary relation $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$ has the parallelogram property iff there are congruences α, β on $\mathbb{A}, \mathbb{B}$ respectively and an isomorphism $f : \mathbb{A}/\alpha \to \mathbb{B}/\beta$ such that, writing π_α, π_β for the quotient maps, we have $\mathbb{R} = \pi_\alpha^{-1} \circ f^{-1} \circ \pi_\beta$ (treating π_α, π_β, f as binary relations with inputs on the right and outputs on the left).*

Proof. Thinking of $\mathbb{R}$ as a bipartite graph on $\mathbb{A} \sqcup \mathbb{B}$, we just have to prove that every connected component of $\mathbb{R}$ is a complete bipartite graph. Suppose $a \in \mathbb{A}$ and $b \in \mathbb{B}$ are in the same connected component of $\mathbb{R}$, and let $a = a_1, b_1, ..., a_k, b_k = b$ be a path from a to b with $(a_i, b_i) \in \mathbb{R}$ and $(a_{i+1}, b_i) \in \mathbb{R}$ for each i. We will show that $(a_1, b_i) \in \mathbb{R}$ by induction on i:

$$\begin{bmatrix} a_1 \\ b_i \end{bmatrix}, \begin{bmatrix} a_{i+1} \\ b_i \end{bmatrix}, \begin{bmatrix} a_{i+1} \\ b_{i+1} \end{bmatrix} \in \mathbb{R} \implies \begin{bmatrix} a_1 \\ b_{i+1} \end{bmatrix} \in \mathbb{R}. \qquad \square$$

Despite the trivial nature of binary relations with the parallelogram property, higher arity relations can encode more complicated global information.

Example 1.7.1. Consider the affine algebra $\mathbb{A} = (\mathbb{Z}/p, x - y + z)$, and let $\mathbb{R} \le_{sd} \mathbb{A}^n$ be the relation $x_1 + \cdots + x_n \equiv 0 \pmod{p}$. Then if we think of $\mathbb{R}$ as a (subdirect) binary relation on $\mathbb{A} \times \mathbb{A}^{n-1}$, it is the graph of the homomorphism $\mathbb{A}^{n-1} \to \mathbb{A}$ given by $(x_2, ..., x_n) \mapsto -x_2 - \cdots - x_n \pmod{p}$.

More generally, for any i, if we think of $\mathbb{R}$ as a subdirect binary relation on $\mathbb{A}^i \times \mathbb{A}^{n-i}$, then the linking congruence gives homomorphisms $\mathbb{A}^i \to \mathbb{A} \leftarrow \mathbb{A}^{n-i}$: $(x_1, ..., x_i) \mapsto x_1 + \cdots + x_i \pmod{p}$ and $(x_{i+1}, ..., x_n) \mapsto -x_{i+1} - \cdots - x_n \pmod{p}$.

Ternary relations on simple Mal'cev algebras have a particularly interesting structure.

Proposition 1.7.16. *Let $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ be simple idempotent Mal'cev algebras, and suppose that $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3$ has $\pi_{i,j}(\mathbb{R}) = \mathbb{A}_i \times \mathbb{A}_j$ for each $i \ne j$ but that $\mathbb{R} \ne \mathbb{A}_1 \times \mathbb{A}_2 \times \mathbb{A}_3$. Then for each $a \in \mathbb{A}_1$, the relation*

$$\mathbb{R}_a = \pi_{2,3}(\mathbb{R} \cap (\{a\} \times \mathbb{A}_2 \times \mathbb{A}_3))$$

is the graph of an isomorphism between $\mathbb{A}_2$ and $\mathbb{A}_3$, and for every $b \in \mathbb{A}_2, c \in \mathbb{A}_3$ there is a unique $a \in \mathbb{A}_1$ such that $(b, c) \in \mathbb{R}_a$.

Proof. Consider $\mathbb{R}$ as a subdirect relation on $\mathbb{A}_1 \times (\mathbb{A}_2 \times \mathbb{A}_3)$. Since the linking congruence on $\mathbb{A}_1$ is not full (else $\mathbb{R}$ would be the full relation by the parallelogram property), it must be trivial (since $\mathbb{A}_1$ is simple), so $\mathbb{R}$ is the graph of a homomorphism from $\mathbb{A}_2 \times \mathbb{A}_3$ to $\mathbb{A}_1$, which proves the last assertion.

Similarly, $\mathbb{R}$ may be viewed as the graph of a homomorphism from $\mathbb{A}_1 \times \mathbb{A}_2$ to $\mathbb{A}_3$, so $\mathbb{R}_a$ is the graph of a surjective homomorphism from $\mathbb{A}_2$ to $\mathbb{A}_3$ for any $a \in \mathbb{A}_1$ (surjective because $\pi_{1,3}(\mathbb{R}) = \mathbb{A}_1 \times \mathbb{A}_3$), and by simplicity of $\mathbb{A}_2$ this homomorphism must be an isomorphism. □

If we fix an isomorphism $\mathbb{A}_1 \cong \mathbb{A}_2 \cong \mathbb{A}_3 \cong \mathbb{A}$ coming from the above proposition, then the kernel of the associated homomorphism $\mathbb{A} \times \mathbb{A} \cong \mathbb{A}_2 \times \mathbb{A}_3 \to \mathbb{A}_1$ contains the diagonal of $\mathbb{A} \times \mathbb{A}$ as a congruence class. In this case - that is, the case where $\mathbb{A} \times \mathbb{A}$ has the diagonal as a congruence class of some congruence - $\mathbb{A}$ is called an *abelian* algebra.

Example 1.7.2. Let $\mathbb{A}_n = (\{0, ..., n-1\}, p)$, where p is the ternary Mal'cev operation defined by

$$p(x, y, z) = \begin{cases} z & \text{if } x = y, \\ y & \text{if } x = z, \\ x & \text{if } x \notin \{y, z\}. \end{cases}$$

For $n \geq 3$, $\mathbb{A}_n$ is simple and non-abelian (i.e. the diagonal is not a congruence class of any congruence on $\mathbb{A}_n^2$). $\mathrm{Inv}(\mathbb{A}_n)$ is generated by a pair of graphs of permutations of $\{0, ..., n-1\}$ which generate the full symmetric group, the unary relation $x \neq 0$, and the ternary relation

$$(x, y, z \in \{0,1\}) \wedge (x + y + z \equiv 0 \pmod 2).$$

It is a good exercise to prove that the above relations generate $\mathrm{Inv}(\mathbb{A}_n)$.

Example 1.7.3. Here we describe an example of a three element Mal'cev algebra which is "solvable", but which is not abelian. Let $\mathbb{A} = (\{0, 1, *\}, p)$, where p is the ternary Mal'cev operation defined by

$$p(x,y,z) = \begin{cases} x & \text{if } y = z, \\ y & \text{if } x = z, \\ z & \text{if } x = y, \\ * & \text{if } \{x,y,z\} = \{0,1,*\}. \end{cases}$$

Every two element subset of $\mathbb{A}$ is a subalgebra isomorphic to the idempotent reduct of $\mathbb{Z}/2$, and $\mathbb{A}$ has a congruence θ corresponding to the partition $\{0,1\}, \{*\}$ such that $\mathbb{A}/\theta$ is also isomorphic to the idempotent reduct of $\mathbb{Z}/2$.

Along with the obvious relations on $\mathbb{A}$, there is also the ternary relation

$$(x = y = z = *) \vee (x, y, z \in \{0,1\} \wedge x + y + z \equiv 0 \pmod 2),$$

whose elements correspond to the columns of the matrix

$$\begin{bmatrix} * & 0 & 0 & 1 & 1 \\ * & 0 & 1 & 0 & 1 \\ * & 0 & 1 & 1 & 0 \end{bmatrix}.$$

That this relation forms a subalgebra of $\mathbb{A}^3$ is related to the fact that θ can be considered to be an "abelian congruence" (in a sense we will define later).

1.8 Mal'cev algorithm and compact representations

The algorithm for solving CSPs invariant under a Mal'cev operation, due to Bulatov and Dalmau [41], is based on the fact that any Mal'cev constraint has a small generating set. More specifically, we will show that any subset of a relation $\mathbb{R}$ which has the same projection to each factor and contains representatives of all of the "forks" of $\mathbb{R}$ actually generates $\mathbb{R}$.

Definition 1.8.1. If $R \subseteq \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, then we define the *signature* of R, written $\mathrm{Sig}(R)$, to be the set of triples (i, a, b) with $i \in \{1, ..., n\}, a, b, \in \mathbb{A}_i$ such that there are some $t_a, t_b \in R$ with $\pi_{1,...,i-1}(t_a) = \pi_{1,...,i-1}(t_b)$ and $\pi_i(t_a) = a, \pi_i(t_b) = b$. In this case we say that the pair t_a, t_b *witnesses* the triple (i, a, b).

Theorem 1.8.2. *Suppose that a relation $\mathbb{R} \leq \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is preserved by a Mal'cev term p, and that $S \subseteq \mathbb{R}$ is a subset with $\mathrm{Sig}(S) = \mathrm{Sig}(\mathbb{R})$. Then $\mathbb{R}$ is generated by S (using only p).*

Proof. Let $\mathbb{S}$ be the subset of $\mathbb{R}$ generated by S using p. We will prove by induction on i that $\pi_{1,...,i}(\mathbb{S}) = \pi_{1,...,i}(\mathbb{R})$.

Suppose that $t \in \mathbb{R}$. By the induction hypothesis, there is some $t' \in \mathbb{S}$ with $\pi_{1,...,i-1}(t) = \pi_{1,...,i-1}(t')$. Let $a = \pi_i(t'), b = \pi_i(t)$. Since $\mathbb{S} \subseteq \mathbb{R}$, we have $(i, a, b) \in \mathrm{Sig}(\mathbb{R}) = \mathrm{Sig}(S)$, so there must be a pair $t_a, t_b \in S$ witnessing the triple (i, a, b). Define $t'' \in \mathbb{S}$ by

$$t'' = p(t', t_a, t_b).$$

Then from $\pi_{1,...,i-1}(t_a) = \pi_{1,...,i-1}(t_b)$ and the fact that p is Mal'cev, we have

$$\pi_{1,...,i-1}(t'') = \pi_{1,...,i-1}(p(t', t_a, t_a)) = \pi_{1,...,i-1}(t') = \pi_{1,...,i-1}(t).$$

Additionally, from $\pi_i(t') = \pi_i(t_a) = a$ and the fact that p is Mal'cev, we have

$$\pi_i(t'') = p(a, a, b) = b = \pi_i(t),$$

so $\pi_{1,...,i}(t'') = \pi_{1,...,i}(t)$. □

Definition 1.8.3. A subset $S \subseteq \mathbb{R}$ is called a *compact representation* of a Mal'cev relation $\mathbb{R}$ if $\mathrm{Sig}(S) = \mathrm{Sig}(\mathbb{R})$ and $|S| \le 2|\mathrm{Sig}(\mathbb{R})|$.

Proposition 1.8.4. *Every Mal'cev relation $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ has a compact representation S. We always have $|S| \le 2n \cdot \max_i |A_i|^2$.*

Now we need some subroutines for manipulating compact representations. The first such procedure is called `Nonempty`: it takes as input a compact representation R of a relation $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ and any description of a relation $\mathbb{S} \le \mathbb{A}_{i_1} \times \cdots \times \mathbb{A}_{i_k}$ on a small subset $\{i_1, ..., i_k\}$ of the indices, and it tells us whether $\mathbb{R} \cap \mathbb{S} \ne \emptyset$. In the case $\mathbb{R} \cap \mathbb{S} \ne \emptyset$, `Nonempty` returns an element of the intersection.

Algorithm 4 `Nonempty`$(R, i_1, ..., i_k, \mathbb{S})$, p a Mal'cev term, R a compact representation of $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $\mathbb{S} \le \mathbb{A}_{i_1} \times \cdots \times \mathbb{A}_{i_k}$.

```
Set $R' \leftarrow R$.
while $\pi_{i_1,...,i_k}(R')$ is not closed under $p$ and $R' \cap \mathbb{S} = \emptyset$ do
    Pick $t_1, t_2, t_3 \in R'$ with $\pi_{i_1,...,i_k}(p(t_1, t_2, t_3)) \notin \pi_{i_1,...,i_k}(R')$.
    Set $R' \leftarrow R' \cup \{p(t_1, t_2, t_3)\}$.
if $R' \cap \mathbb{S} \ne \emptyset$ then
    return any element of $R' \cap \mathbb{S}$.
else
    return $\emptyset$.
```

Proposition 1.8.5. ***Nonempty** correctly determines whether $\mathbb{R} \cap \mathbb{S} \ne \emptyset$ in time polynomial in n, $|R|$, and $|\pi_{i_1,...,i_k}(\mathbb{R})| \le \prod_{j \le k} |\mathbb{A}_{i_j}|$.*

Proof. Since $\mathbb{R}$ is generated by R using p, we also have $\pi_{i_1,...,i_k}(\mathbb{R})$ generated by $\pi_{i_1,...,i_k}(R)$ using p. To see the bound on the running time, note that in each iteration of the while loop, the set $\pi_{i_1,...,i_k}(R')$ gains a new element, and its size is clearly bounded by $|\pi_{i_1,...,i_k}(\mathbb{R})|$. □

The next subroutine for manipulating compact representations is `Fix-values`. `Fix-values` converts a compact representation R of $\mathbb{R} \leq \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ to a compact representation of

$$\mathbb{R} \wedge (x_1 = a_1) \wedge \cdots \wedge (x_m = a_m),$$

for any choice of $m \leq n$ and $a_i \in \mathbb{A}_i$ for all i. `Fix-values` is really the core of the algorithm, the other steps are mostly formal (in fact, `Nonempty` and `Fix-values` are the only two subroutines which use the Mal'cev term p).

Algorithm 5 `Fix-values`$(R, a_1, ..., a_m)$, p a Mal'cev term, R a compact representation of $\mathbb{R} \leq \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$.

```
Set $R_0 \leftarrow R$.
for $j$ from 1 to $m$ do
    if $(j, a_j, a_j) \notin \mathrm{Sig}(R_{j-1})$ then
        return $\emptyset$.
    else
        Set $R_j \leftarrow \{t\}$, where $t \in R_{j-1}$ and the pair $t, t$ witnesses the triple $(j, a_j, a_j)$.
    for all $(i, a, b) \in \mathrm{Sig}(R_{j-1})$ with $i > j$ do
        Let $t_a, t_b \in R_{j-1}$ witness the triple $(i, a, b)$.
        Let $t \leftarrow$ Nonempty$(R_{j-1}, j, i, \{(a_j, a)\})$.
        if $t \neq \emptyset$ then
            Set $R_j \leftarrow R_j \cup \{t, p(t, t_a, t_b)\}$.
return $R_m$.
```

Proposition 1.8.6. *`Fix-values` correctly returns a compact representation of $\mathbb{R}_m = \mathbb{R} \wedge (x_1 = a_1) \wedge \cdots \wedge (x_m = a_m)$ in polynomial time.*

Proof. We prove by induction on j that R_j is a compact representation of $\mathbb{R}_j$ for each $j \leq m$. Note that for any $(i, a, b) \in \mathrm{Sig}(R_j)$, if $a \neq b$ then we must have $i > j$. For $i \leq j$, we have $(i, a_i, a_i) \in \mathrm{Sig}(R_j)$ iff $\mathbb{R}_j \neq \emptyset$ by how we initialize R_j.

If $i > j$, then $(i, a, b) \in \mathrm{Sig}(\mathbb{R}_j)$ implies $(i, a, b) \in \mathrm{Sig}(\mathbb{R}_{j-1})$, witnessed by some pair $t_a, t_b \in R_{j-1}$. Additionally, if $(i, a, b) \in \mathrm{Sig}(\mathbb{R}_j)$, then there is certainly some $t \in \mathbb{R}_j$ with $\pi_i(t) = a$, so the call to `Nonempty` inside the loop will succeed. Then

$$\pi_{1,\ldots,i-1}(p(t, t_a, t_b)) = \pi_{1,\ldots,i-1}(p(t, t_a, t_a)) = \pi_{1,\ldots,i-1}(t),$$

so from $i > j$ we have $p(t, t_a, t_b) \in \mathbb{R}_j$. From $\pi_i(t) = a, \pi_i(p(t, t_a, t_b)) = p(a, a, b) = b$, we see that the pair $t, p(t, t_a, t_b)$ witnesses the triple (i, a, b).

To see that `Fix-values` runs in polynomial time, note that every call to `Nonempty` involves a constraint on two variables. □

Corollary 1.8.7. *Given a compact representation R of a relation $\mathbb{R} \leq \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ which is preserved by a given Mal'cev operation p, and given a tuple $t \in \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, we can determine whether $t \in \mathbb{R}$ in polynomial time.*

The next subroutine will give a compact representation for the intersection of a relation $\mathbb{R}$ given by a compact representation R and a relation $\mathbb{S}$ of small arity. In [41] this subroutine was called `Next-beta`, so we will copy that notation here.

Algorithm 6 `Next-beta`$(R, i_1, ..., i_k, \mathbb{S})$, R a compact representation of $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $\mathbb{S} \le \mathbb{A}_{i_1} \times \cdots \times \mathbb{A}_{i_k}$.

```
Set $R' \leftarrow \emptyset$.
for all $(i, a, b) \in \mathrm{Sig}(R)$ do
    Set $t_a \leftarrow$ Nonempty$(R, i_1, ..., i_k, i, \mathbb{S} \times \{a\})$.
    if $t_a \ne \emptyset$ then
        Set $t_b \leftarrow$ Nonempty(Fix-values$(R, \pi_1(t_a), ..., \pi_{i-1}(t_a)), i_1, ..., i_k, i, \mathbb{S} \times \{b\})$.
        if $t_b \ne \emptyset$ then
            Set $R' \leftarrow R' \cup \{t_a, t_b\}$.
return $R'$.
```

Proposition 1.8.8. *`Next-beta` correctly finds a compact representation of $\mathbb{R} \cap \mathbb{S}$ in time polynomial in n, $|R|$, and $|\pi_{i_1,...,i_k}(\mathbb{R})| \cdot \max_i |\mathbb{A}_i| \le \prod_{j \le k} |\mathbb{A}_{i_j}| \cdot \max_i |\mathbb{A}_i|$.*

Bulatov and Dalmau [41] then go on to define a subroutine `Next` which calls `Next-beta` on larger and larger projections of $\mathbb{S}$, ensuring that $|\pi_{i_1,...,i_k}(\mathbb{R})| \le |\mathbb{S}| \cdot \max_i |\mathbb{A}_i|$ every time that `Next-beta` is called. A better approach, leading to a more powerful algorithm, was found by Maróti [139]. The subroutine `Intersect` takes two compact representations R, S of relations $\mathbb{R}, \mathbb{S}$ as input and returns a compact representation of $\mathbb{R} \cap \mathbb{S}$ as output.

Algorithm 7 `Intersect`$(R, i_1, ..., i_k, S)$, R a compact representation of $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, S a compact representation of $\mathbb{S} \le \mathbb{A}_{i_1} \times \cdots \times \mathbb{A}_{i_k}$.

```
Let $t_R \in R$ and $t_S \in S$ be any tuples.
Set $R' \leftarrow (R \times \{t_S\}) \cup (\{t_R\} \times S) \subseteq \mathbb{A}_1 \times \cdots \mathbb{A}_n \times \mathbb{A}_{i_1} \times \cdots \times \mathbb{A}_{i_k}$.
for $j \le k$ do
    Set $R' \leftarrow$ Next-beta$(R', i_j, n + j, =_{\mathbb{A}_{i_j}})$.
return a minimal subset of $\pi_{1,...,n}(R')$ which witnesses every triple $(i, a, b) \in \mathrm{Sig}(\pi_{1,...,n}(R'))$.
```

Theorem 1.8.9. *Any CSP which is preserved by a Mal'cev operation, where the relations are given by their compact representations, can be solved in time polynomial in the number of variables, the number of relations, and the size of the largest domain. In fact, we can find a compact representation of the solution set in polynomial time.*

Proof. We start with any compact representation of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, and simply apply the subroutine `Intersect` repeatedly to find a compact representation of the intersection of all the constraint relations. To see that `Intersect` works correctly and efficiently, note that R' is initialized as a compact representation of $\mathbb{R} \times \mathbb{S}$ and ends as a compact representation of $\mathbb{R} \cap \mathbb{S}$ followed by k repeated coordinates. To see that `Intersect` runs in polynomial time, note that each call of `Next-beta` involves a relation of arity 2. □

Corollary 1.8.10. *For any primitive positive formula φ in a collection of relations $\mathbb{R}_i$, if we are given compact representations of each $\mathbb{R}_i$ then we can efficiently find a compact representation of the relation described by φ.*

Proof. If we are given a compact representation of a relation and we permute its variables, we can efficiently find a compact representation for the permuted relation by using the `Intersect` subroutine with $\mathbb{R}$ equal to a full relation. To handle projections, note that we can project onto any initial segment of the variables by just projecting our compact representation and pruning it. □

While this might appear to be a fully satisfactory theory, there is still one big question remaining: what happens if instead of having relations described by compact representations, we have relations which are instead described by an arbitrary set of generators? It's clear that we just need to find a way to compute a compact representation of $\mathrm{Sg}_{\mathbb{A}^n}(S)$ for any small set $S \subseteq \mathbb{A}^n$, and a little thought shows that this can be reduced to the following problem.

Problem 1.8.1. Let $\mathbb{A}$ be a fixed Mal'cev algebra. Given a subset $S \subseteq \mathbb{A}^n$, and given a tuple $t \in \mathbb{A}^n$, can we determine whether $t \in \mathrm{Sg}_{\mathbb{A}^n}(S)$ in time polynomial in $|S|$ and n?

This is a special case of the Subpower Membership Problem 2.4.1. Even this special case is open (the answer is conjectured to be yes). In the case of groups, the famous Schreier-Sims algorithm gives a positive solution (see [81] for a straightforward exposition).

Remark 1.8.1. The proof of correctness of the subroutine `Nonempty` and the algorithm for `Fix-values` are both directly connected to the proof of Theorem 1.8.2. The subroutines `Next-beta` and `Intersect` use the subroutines `Nonempty` and `Fix-values` as black boxes and don't involve the algebraic structure at all. Thus, in order to generalize the Mal'cev algorithm to more general algebraic structures, the only new ingredient needed is a proof of a generalization of Theorem 1.8.2.

1.8.1 Near-subgroups

In this subsection, we will describe the maximal polynomial-time solvable extension $\mathbf{G}^*$ of the relational clone $\mathbf{G}$ of cosets of subgroups of $\mathbb{G}^m$, where $\mathbb{G}$ is a finite group. The relational clone $\mathbf{G}^*$ will turn out to have a Mal'cev polymorphism, so the algorithm for Mal'cev algebras can be used to prove the dichotomy for extensions of $\mathbf{G}$.

First, consider the simple case where $\mathbb{G} = \mathbb{Z}/n$ is cyclic of order n at least 3. It's easy to see that if we add the unary relation $\{0, 1\}$ to $\mathbb{Z}/n$, then we can simulate 1-IN-3 SAT via the primitive positive formula

$$x + y + z = 1 \ \wedge \ x, y, z \in \{0, 1\}.$$

Using an inductive argument with this as the base case, Feder and Vardi [77] show that if we adjoin any unary relation to $\mathbb{Z}/n$ which isn't a coset of a subgroup, then we can simulate 1-IN-3 SAT as well.

Proposition 1.8.11 (Feder, Vardi [77])**.** *If we adjoin any unary relation K to the relational structure $\mathbf{G} = (\mathbb{Z}/n, \{1\}, x + y = z)$, then the resulting CSP is NP-complete unless K is a coset of a subgroup of $\mathbb{Z}/n$.*

Proof. Using the binary relation $y = x+i$ for constants $i \in \mathbb{Z}/n$, we see that $K-i$ is in the relational clone generated by K and $\mathbf{G}$. Thus we may assume without loss of generality that $0 \in K$, and by possibly restricting to a subgroup we may assume that $\langle K \rangle = \mathbb{Z}/n$. By applying an automorphism of $\mathbb{Z}/n$, we may also assume that $1 \in K$.

We induct on $|K|, n$. If there is an $i \neq 0$ with $i, i+1 \in K$, then $K \cap (K - i)$ also contains $0, 1$, and will be strictly smaller than K unless $K = K - i$, in which case we may take the quotient by $\langle i \rangle$. Thus we may assume that $i, i+1$ are not both in K for any $i \neq 0$.

If K contains some i with neither of $i, i-1$ relatively prime to n, then by induction $K \cap \langle i \rangle$ and $(K-1) \cap \langle i-1 \rangle$ are subgroups, so $2i, 2i-1 \in K$ and we must have $2i - 1 \equiv 0 \pmod{n}$, contradicting the assumption that i has a common factor with n.

If K contains $i \neq 1$ with i relatively prime to n, then $K \cap (i - K)$ contains $0, i$ but not 1, and we may apply the induction hypothesis to get a contradiction. Similarly, if K contains $i \neq 0$ with $i - 1$ relatively prime to n, then $(K-1) \cap (i - K)$ contains 0 and $i-1$ but not -1, and we may apply the induction hypothesis.

Thus the only case to consider is the case $K = \{0, 1\}$, and we have already seen that in this case we can simulate 1-IN-3 SAT (unless $n = 2$, in which case $K = \mathbb{Z}/n$). □

Next, consider the case where $\mathbb{G}$ is the Klein four-group $(\mathbb{Z}/2)^2$. The only unary relations which aren't already cosets of subgroups of $(\mathbb{Z}/2)^2$ are the relations with three elements. If we adjoin any three element unary relation to $(\mathbb{Z}/2)^2$, then we can again simulate 1-IN-3 SAT: if we adjoin the relation $K = \{(0,0), (0,1), (1,0)\}$, for instance, then we can use the primitive positive formula

$$\exists t\ (x, y, z \in \{(0,0), (0,1)\} \wedge x+y+z = (0,1) \wedge (x,t) \in \{((0,0),(0,0)), ((0,1),(1,0))\} \wedge y+t \in K),$$

which is satisfied iff exactly one of x, y, z is $(0,1)$ and the other two are $(0,0)$.

Now consider the case where $\mathbb{G}$ is any finite abelian group, and $K \subseteq \mathbb{G}$ is a unary relation which can be added without creating NP-completeness. Then if any $a, a+b \in K$, we must have $a + ib \in K$ for all $i \in \mathbb{Z}$ by the cyclic case. By the Klein four-group case, if we have subgroups $\mathbb{N} \leq \mathbb{M} \leq \mathbb{G}$ with $\mathbb{M}/\mathbb{N} \cong (\mathbb{Z}/2)^2$, then if K meets any three elements of $\mathbb{M}/\mathbb{N}$ it must also meet the fourth.

Proposition 1.8.12. *Suppose that $\mathbb{G}$ is an abelian group and that $K \subseteq \mathbb{G}$ has $0 \in K$, has the property that if $a, a+b \in K$ then $a + \langle b \rangle \subseteq K$, and the property that for any subgroups $\mathbb{N} \leq \mathbb{M} \leq \mathbb{G}$ with $\mathbb{M}/\mathbb{N} \cong (\mathbb{Z}/2)^2$, we have $|(K \cap \mathbb{M})/\mathbb{N}| \neq 3$. Then K must be a subgroup of $\mathbb{G}$.*

Proof. From the first assumption, for any $a, b \in K$ we must have $-ia, jb \in K$, so

$$ia + 2jb = jb - (-ia - jb) \in jb + \langle -ia - jb \rangle \subseteq K,$$

and similarly $2ja + ib \in K$ for all $i, j \in \mathbb{Z}$.

Thus, if we take $\mathbb{M} = \langle a, b \rangle$ and $\mathbb{N} = \langle 2a, 2b \rangle$, we see that either $|\mathbb{M}/\mathbb{N}| < 4$ in which case $a + b \in \langle a, b \rangle \subseteq K$, or $\mathbb{M}/\mathbb{N} \cong (\mathbb{Z}/2)^2$ and $|(K \cap \mathbb{M})/\mathbb{N}| \geq 3$. In the latter case, the second assumption implies that there are i, j such that $(2i+1)a + (2j+1)b \in K$. Then $a+b = (2i+1)a + (2j+1)b - 2ia - 2jb \in K$ by repeated application of the first assumption. Either way, $a + b \in K$, so K is closed under addition. □

Corollary 1.8.13. *If $\mathbb{G}$ is a finite abelian group and $\mathbf{G}$ the associated relational structure, then for any m-ary relation K which is not a coset of a subgroup of $\mathbb{G}^m$, the CSP we get by adding K to $\mathbf{G}$ is NP-complete.*

Proof. Apply the previous proposition to the abelian group $\mathbb{G}^m$. □

In the case of nonabelian groups, however, we may be able to adjoin interesting new constraints. Note that if we adjoin any constraint, then we automatically adjoin all of its cosets, since for any constant $b \in \mathbb{G}$ the relation $y = bx$ is a left coset of the diagonal subgroup of $\mathbb{G}^2$. So by the abelian case, the only possibilities for new relations are those described by the following definition.

Definition 1.8.14. A subset $K \subseteq \mathbb{G}$ is a *near subgroup* of $\mathbb{G}$ if it contains 1, and for any $b \in K^{-1}$, any $\mathbb{M} \le \mathbb{G}$ and any $\mathbb{N} \lhd \mathbb{M}$ with $\mathbb{M}/\mathbb{N}$ abelian, the quotient set $(bK \cap \mathbb{M})/\mathbb{N}$ is a subgroup of $\mathbb{M}/\mathbb{N}$.

Proposition 1.8.15. *If $\mathbb{H} \le \mathbb{G}$ is a subgroup and $K \subseteq \mathbb{H}$ is a near subgroup of $\mathbb{H}$, then K is a near subgroup of $\mathbb{G}$. Similarly, if $\varphi : \mathbb{G} \twoheadrightarrow \mathbb{H}$ is a surjective group homomorphism and $K \subseteq \mathbb{H}$ is a near subgroup of $\mathbb{H}$, then $\varphi^{-1}(K)$ is a near subgroup of $\mathbb{H}$.*

Theorem 1.8.16 (Aschbacher [4])**.** *The intersection of two near subgroups of a finite group is a near subgroup.*

Corollary 1.8.17 (Feder [76])**.** *Let $\mathbb{G}$ be a finite group, and let $\mathbf{G}^*$ be the relational structure on the underlying set of $\mathbb{G}$ having as relations all cosets of all near subgroups of $\mathbb{G}^n$. Then $\mathbf{G}^*$ has a Mal'cev polymorphism.*

Proof. Consider the "free near subgroup generated by two elements", that is, the smallest near subgroup K of $\mathbb{G}^{\mathbb{G}^2}$ which contains π_1, π_2 (a smallest such near subgroup exists since the intersection of all of them is guaranteed to be a near subgroup as well). Let $\mathbb{N}$ be the commutator subgroup of the group generated by π_1, π_2. Since $\langle \pi_1, \pi_2 \rangle/\mathbb{N}$ is abelian, there must be some $c \in \mathbb{N}$ with $\pi_1\pi_2 c \in K$ by the definition of a near subgroup.

We define a binary operation g by $g = \pi_1\pi_2 c$, that is, $g(x, y) = xyc(x, y)$, where $c \in \mathbb{G}^{\mathbb{G}^2}$ is interpreted as a function $c : \mathbb{G}^2 \to \mathbb{G}$. Since for all x, y we know that $c(x, y)$ is contained in the commutator subgroup of $\langle x, y\rangle$, we have $c(x, 1) = c(1, x) = 1$ for all x, so $g(x, 1) = g(1, x) = x$. Now we define a Mal'cev operation p by

$$p(x, y, z) = yg(y^{-1}x, y^{-1}z) = xy^{-1}zc(y^{-1}x, y^{-1}z).$$

That p is Mal'cev follows directly from the fact that g satisfies the identities $g(1, x) \approx g(x, 1) \approx x$.

To see that p is really a polymorphism of $\mathbf{G}^*$, let X be any coset of any near subgroup of $\mathbb{G}^n$, and let $x, y, z \in X$. Then $y^{-1}X$ is a near subgroup of $\mathbb{G}^n$. Since $g = \pi_1\pi_2 c \in K$, g preserves every near subgroup of $\mathbb{G}^n$ (since for any $a, b \in \mathbb{G}^n$, K is contained in the near subgroup of $\mathbb{G}^{\mathbb{G}^2}$ obtained by taking the preimage of the map φ from the subgroup generated by π_1, π_2 to $\mathbb{G}^n$ which sends $\pi_1 \mapsto a, \pi_2 \mapsto b$). Thus from $y^{-1}x, y^{-1}z \in y^{-1}X$ we have $g(y^{-1}x, y^{-1}z) \in y^{-1}X$, and $p(x, y, z) = yg(y^{-1}x, y^{-1}z) \in X$, so p does indeed preserve X. □

In order to prove Aschbacher's Theorem 1.8.16, we first need a more convenient characterization of near-subgroups.

Definition 1.8.18. A subset K of a finite group $\mathbb{G}$ is a *twisted subgroup* if $1 \in K$ and $x, y \in K \implies xyx \in K$.

Proposition 1.8.19. *If K is a twisted subgroup and $x \in K$, then $\langle x\rangle \subseteq K$, so in particular $K = K^{-1}$. If $b \in K$, then bK is also a twisted subgroup.*

Proof. For the first statement, for any $x \in K$ we have $x^k{\cdot}1{\cdot}x^k, x^k{\cdot}x{\cdot}x^k \in K$ for all $k \ge 0$, so $\langle x\rangle \subseteq K$. For the second statement, if $x, y \in bK$ and $b \in K$, then $b^{-1}(x{\cdot}y{\cdot}x) = (b^{-1}x){\cdot}(b{\cdot}b^{-1}y{\cdot}b){\cdot}(b^{-1}x) \in K$, so $xyx \in bK$. □

Proposition 1.8.20. *A subset $K \subseteq \mathbb{G}$ is a near-subgroup iff it is a twisted subgroup such that for any $b \in K^{-1}$, any $\mathbb{M} \le \mathbb{G}$ and any $\mathbb{N} \lhd \mathbb{M}$ with $\mathbb{M}/\mathbb{N}$ isomorphic to the Klein four-group, $|(bK \cap \mathbb{M})/\mathbb{N}| \ne 3$.*

Proof. We just need to check that K being a twisted subgroup is equivalent to $\langle x\rangle \subseteq bK$ for all x, b with $x \in bK, b \in K^{-1}$. The previous proposition proves one direction of the equivalence. For the other direction, if $x, y \in K$, then $yx \in yK$ and $y^{-1} \in \langle y\rangle \subseteq K$, so $(yx)^2 \in \langle yx\rangle \subseteq yK$, which is equivalent to $xyx = y^{-1}(yx)^2 \in K$. □

Example 1.8.1. An explicit example of a near subgroup which is not a subgroup is given in [77]. Let $\mathbb{G}$ be the Heisenberg group of order p^3 (p odd):

$$\mathbb{G} = \left\{\begin{bmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{bmatrix} \text{ s.t. } a, b, c \in \mathbb{Z}/p\right\} \le \mathrm{SL}_3(\mathbb{Z}/p).$$

Let $K \subseteq \mathbb{G}$ be given by

$$K = \left\{\begin{bmatrix}1 & a & \frac{ab}{2}\\ 0 & 1 & b\\ 0 & 0 & 1\end{bmatrix} \text{ s.t. } a, b \in \mathbb{Z}/p\right\}.$$

Since $\mathbb{G}$ has odd order, to check that K is a near subgroup we just need to check that it is a twisted subgroup, i.e. that it contains the identity and is closed under the binary operation $x, y \mapsto xyx$. This can be checked by direct calculation: for any $a, b, c, d \in \mathbb{Z}/p$ we have

$$\begin{bmatrix}1 & a & \frac{ab}{2}\\ 0 & 1 & b\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & c & \frac{cd}{2}\\ 0 & 1 & d\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & a & \frac{ab}{2}\\ 0 & 1 & b\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2a+c & \frac{(2a+c)(2b+d)}{2}\\ 0 & 1 & 2b+d\\ 0 & 0 & 1\end{bmatrix}.$$

That K is not a subgroup follows from

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix} \notin K.$$

That we needed to take p odd in the above example is no coincidence, as the next proposition shows.

Proposition 1.8.21. *If $\mathbb{G}$ is a 2-group, then any near-subgroup of $\mathbb{G}$ is a subgroup of $\mathbb{G}$.*

Proof. We prove this by induction on the order of $\mathbb{G}$. Let K be a near-subgroup of $\mathbb{G}$, and assume without loss of generality that $\langle K\rangle = \mathbb{G}$.

Let $z \in Z(\mathbb{G})$ be a nontrivial involution in the center of $\mathbb{G}$, which must exist since every 2-group has a nontrivial center, and every nontrivial element of the center has a power which is a nontrivial involution. By induction we have $K/\langle z\rangle = \mathbb{G}/\langle z\rangle$.

If $z \in K$ then for any $g \in \mathbb{G} \setminus K$, we have $gz \in K$, and from $\langle gz, z\rangle$ abelian and $gz, z \in K$ we have $\langle gz, z\rangle \subseteq K$, so in particular $g = gz \cdot z \in K$. Thus if $z \in K$ we have $\mathbb{G} = K$.

Thus we may assume that $z \notin K$, and in fact that $Z(\mathbb{G}) \cap K = 1$. Let $\mathbb{M}$ be a maximal subgroup of $\mathbb{G}$ containing $\langle z\rangle$, then by induction we have $\mathbb{M} \cap K$ a subgroup of $\mathbb{M}$. Since $(\mathbb{M} \cap K)/\langle z\rangle = \mathbb{M}/\langle z\rangle$, we have $\mathbb{M} \cong (\mathbb{M} \cap K) \times \langle z\rangle$.

Let $\Phi(\mathbb{M})$ be the Frattini subgroup of $\mathbb{M}$, which for 2-groups is given by $\Phi(\mathbb{M}) = \mathbb{M}^2[\mathbb{M}, \mathbb{M}]$ (where by $\mathbb{M}^2$ we mean the collection of all squares a^2 for $a \in \mathbb{M}$). Then from $\Phi(\langle z\rangle) = 1$ we have $\Phi(\mathbb{M}) = \Phi(\mathbb{M} \cap K)$, and since $\mathbb{M} \lhd \mathbb{G}$ we have $\Phi(\mathbb{M}) \lhd \mathbb{G}$. Thus if $\Phi(\mathbb{M}) \neq 1$ then by considering parities of the sizes of the orbits of elements of $\Phi(\mathbb{M})$ under conjugation we see that $\Phi(\mathbb{M} \cap K) = \Phi(\mathbb{M})$ contains a nontrivial element of $Z(\mathbb{G})$, contradicting $K \cap Z(\mathbb{G}) = 1$. Thus $\Phi(\mathbb{M}) = 1$, so $\mathbb{M}$ has exponent 2. Since this holds for every maximal subgroup of $\mathbb{G}$ which contains $\langle z\rangle$, we see that $\mathbb{G}$ has exponent 2, so $\mathbb{G}$ is abelian. □

Next we show that we can reduce to the situation where $\langle K\rangle$ has an automorphism of order two which sends k to k^{-1} for all $k \in K$.

Definition 1.8.22. If K is a twisted subgroup, we define the *K-radical* Ξ_K to be the set of elements of the form $k_1 \cdots k_n$ with $k_i \in K$ such that $k_1^{-1} \cdots k_n^{-1} = 1$.

Proposition 1.8.23. *If K is a twisted subgroup and Ξ_K is the K-radical, then Ξ_K is a normal subgroup of $\langle K\rangle$, and for any $x \in K$ we have $x\Xi_K \subseteq K$.*

Proof. To see that Ξ_K is normal in $\langle K\rangle$, just note that for any $b \in K$ we have

$$k_1^{-1} \cdots k_n^{-1} = 1 \iff b^{-1}k_1^{-1} \cdots k_n^{-1}b = 1 \implies bk_1 \cdots k_n b^{-1} \in \Xi_K,$$

so $b\Xi_K b^{-1} \subseteq \Xi_k$.

For the second statement, note that $k_1^{-1} \cdots k_n^{-1} = 1 \iff k_n \cdots k_1 = 1$, so if $x \in K$ then we have

$$x(k_1 \cdots k_n) = (k_n \cdots k_1)x(k_1 \cdots k_n) = k_n(\cdots(k_1xk_1)\cdots)k_n \in K.$$ □

Proposition 1.8.24. *If K is a twisted subgroup with $\Xi_K = 1$, and if τ satisfies $\tau^2 = 1$, $\tau k\tau = k^{-1}$ for $k \in K$, then τK is preserved under conjugation by elements of $\langle K, \tau\rangle$.*

Proof. If $x, y \in K$, then $x^{-1}\tau yx = \tau xyx \in \tau K$, and $\tau^{-1}\tau y\tau = \tau y^{-1} \in \tau K$. □

Proposition 1.8.25. *A twisted subgroup $K \subseteq \mathbb{G}$ is a near-subgroup of $\mathbb{G}$ iff the intersection $bK \cap \mathbb{S}$ is a subgroup of $\mathbb{S}$ for every 2-Sylow subgroup $\mathbb{S}$ of $\mathbb{G}$ and every $b \in K^{-1}$.*

Proof. Suppose for contradiction that $\mathbb{M} \le \mathbb{G}$, $\mathbb{N} \lhd \mathbb{M}$ with $\mathbb{M}/\mathbb{N}$ isomorphic to the Klein four-group, and $b \in K^{-1}$ with $|(bK \cap \mathbb{M})/\mathbb{N}| = 3$. We may assume without loss of generality that $\mathbb{M} = \langle K\rangle \cap \mathbb{M}$ and $\mathbb{N} = \langle K\rangle \cap \mathbb{N}$, that $\mathbb{G} = \langle K\rangle$, that $b = 1$, and that $\Xi_K = 1$. From $\Xi_K = 1$, we see that there is an order 2 automorphism τ of $\mathbb{G} = \langle K\rangle$ with $k^\tau = k^{-1}$ for all $k \in K$, so we work in the semidirect product of $\mathbb{G}$ and $\langle\tau\rangle$, with $\tau^2 = 1$ and $\tau g\tau = g^\tau$ for $g \in \mathbb{G}$.

Let $x, y \in K$ be representatives of the nontrivial elements of $(K \cap \mathbb{M})/\mathbb{N}$. We may assume without loss of generality that x, y have orders equal to powers of 2, since otherwise we may replace them with odd powers of themselves. Let $\mathbb{S}_x, \mathbb{S}_y$ be 2-Sylow subgroups of $\mathbb{M}\langle\tau\rangle$ containing $\langle x, \tau\rangle, \langle y, \tau\rangle$, respectively, then by the Sylow theorems there is some $g \in \mathbb{M}\langle\tau\rangle$ with $g^{-1}\mathbb{S}_y g = \mathbb{S}_x$. Then $x, \tau, g^{-1}yg, g^{-1}\tau g \in \mathbb{S}_x$, and our strategy is to show that $x, g^{-1}yg \in K \cap \mathbb{S}_x$.

We have $g^{-1}\tau yg \in \tau K$ by the previous proposition, so $\tau g^{-1}\tau yg \in K \cap \mathbb{S}_x$, and similarly $\tau g^{-1}\tau g \in K \cap \mathbb{S}_x$. Since $K \cap \mathbb{S}_x$ is assumed to be a subgroup, we have

$$xg^{-1}yg = x(\tau g^{-1}\tau g)^{-1}(\tau g^{-1}\tau yg) \in K \cap \mathbb{S}_x.$$

Then since $\mathbb{M}/\mathbb{N}$ is abelian and $\tau y\tau = y^{-1} \equiv_{\mathbb{N}} y$, we have $xg^{-1}yg \equiv_{\mathbb{N}} xy$, contradicting the assumption $|(K \cap \mathbb{M})/\mathbb{N}| = 3$. □

Proof of Theorem 1.8.16. If K, K' are near-subgroups of $\mathbb{G}$, then they are both twisted subgroups and so their intersection $K \cap K'$ is also a twisted subgroup. Now let $\mathbb{S}$ be any 2-group contained in $\mathbb{G}$, then for any $b \in K \cap K'$ we see that $bK \cap bK' \cap \mathbb{S} = (bK \cap \mathbb{S}) \cap (bK' \cap \mathbb{S})$ is an intersection of subgroups of $\mathbb{S}$, so it is a subgroup of $\mathbb{S}$, and the previous proposition shows that this implies that $K \cap K'$ is a near-subgroup of $\mathbb{G}$. □

1.9 Abelian Mal'cev algebras are affine

In this section we will prove that abelian Mal'cev algebras are affine. This is an important step in the proof that problems which do not have the "ability to count" have bounded width. First we will carefully define what an affine algebra is, starting with the more basic concept of a quasi-affine algebra.

Definition 1.9.1. An algebra $\mathbb{A}$ is called *quasi-affine* if there is an abelian group $\mathbb{G} = (G, 0, +, -)$ with underlying set G containing the underlying set of $\mathbb{A}$, such that the restriction of the 4-ary relation $x + y = z + w$ to $\mathbb{A}$ is preserved by all the operations of $\mathbb{A}$.

We want to relate this to the more familiar concept of a module over a ring.

Definition 1.9.2. If $\mathbb{R}$ is a ring and $\mathbb{M}$ is a module over $\mathbb{R}$ with underlying group $(M, 0, +, -)$, then we consider $\mathbb{M}$ to be a universal algebraic object $(M, 0, +, -, \{\phi_r\}_{r\in\mathbb{R}})$, where for each $r \in \mathbb{R}$ the unary operation $\phi_r : \mathbb{M} \to \mathbb{M}$ is given by $\phi_r : m \mapsto rm$.

In general, a universal algebraic object is called a *module* if it is an expansion of an abelian group by any collection of unary operations that distribute over addition.

The way these concepts are related is a coarser notion than term equivalence, known as *polynomial equivalence* (warning: in some older references, "polynomial equivalence" means term equivalence and "functional equivalence"/"algebraic equivalence" means polynomial equivalence).

Definition 1.9.3. If $\mathcal{O}$ is any set of operations, then the *polynomial clone* generated by $\mathcal{O}$ is the clone generated by $\mathcal{O}$ together with the constant functions (one for each element of the underlying set). Two algebras or clones on the same underlying set are called *polynomially equivalent* if they have the same polynomial clones.

Proposition 1.9.4. *An algebra $\mathbb{A}$ is quasi-affine iff it is a subalgebra of a reduct of the polynomial clone of a module.*

Proof. Let $\mathbb{A}$ be a quasi-affine algebra, and let $\mathbb{G} = (G, 0, +, -)$ be the corresponding group. We may assume without loss of generality that $0 \in \mathbb{A}$, and that $\mathbb{G}$ is the abelian group with the following presentation: the generators are the elements of $\mathbb{A} \setminus \{0\}$, and the relations are given by $x + y - z - w = 0$ for every quadruple of elements $x, y, z, w \in \mathbb{A}$ such that $x + y = z + w$ in $\mathbb{G}$.

Suppose that f is any n-ary operation of $\mathbb{A}$, and for each $i \le n$ let $\phi_i : \mathbb{A} \to G$ be the unary operation given by

$$\phi_i(x) = f(0, ..., 0, x, 0, ..., 0) - f(0, ..., 0),$$

with the x in the ith position. Since f preserves the relation $x + y = z + w$ on $\mathbb{A}$, we have

$$\phi_i(x) + \phi_i(y) = \phi_i(z) + \phi_i(w)$$

for any $x, y, z, w \in \mathbb{A}$ such that $x+y = z+w$ in $\mathbb{G}$. Thus ϕ_i is compatible with the defining relations of $\mathbb{G}$, so the map $\phi_i : \mathbb{A} \to G$ extends to a unique homomorphism $\phi_i : \mathbb{G} \to \mathbb{G}$.

To finish, we just need to prove that

$$f(x_1, ..., x_n) = \phi_1(x_1) + \cdots + \phi_n(x_n) + f(0, ..., 0)$$

for all $x_1, ..., x_n \in \mathbb{A}$, since $f(0, ..., 0)$ is a constant operation.

We prove this by induction on the number k of nonzero values among $x_1, ..., x_n$. The base cases $k = 0, 1$ follow from the definition of the ϕ_i. For the inductive step, assume without loss of generality that the nonzero values of the x_is are $x_1, ..., x_{k+1}$. Since f preserves the relation $x + y = z + w$, we have

$$f(x_1, ..., x_{k+1}, 0, ..., 0) + f(0, ..., 0) = f(x_1, ..., x_k, 0, 0, ..., 0) + f(0, ..., 0, x_{k+1}, 0, ..., 0),$$

so by the inductive hypothesis and the definition of ϕ_{k+1} we have

$$f(x_1, ..., x_{k+1}, 0, ..., 0) = \phi_1(x_1) + \cdots + \phi_k(x_k) + f(0, ..., 0) + \phi_{k+1}(x_{k+1}). \qquad \square$$

Definition 1.9.5. An algebra $\mathbb{A}$ is called *affine* if it is polynomially equivalent to a module.

Proposition 1.9.6. *An algebra is affine iff it is quasi-affine and has a Mal'cev term.*

Proof. The hardest step is showing that every affine algebra $\mathbb{A}$ has a Mal'cev term. Since $\mathbb{A}$ is polynomially equivalent to a module, there must be some $n + 3$-ary term t and some constants $a_1, ..., a_n \in \mathbb{A}$ such that

$$t(x, y, z, a_1, ..., a_n) = x - y + z$$

for all x, y, z. Since any affine algebra is quasi-affine, we can write t in the form

$$t(x, y, z, u_1, ..., u_n) = x - y + z + \sum_i \phi_i(u_i) + c$$

for some unary ϕ_i and some constant c. Define $p(x, y, z)$ by

$$p(x, y, z) = t(x, t(y, y, y, x, ..., x), z, x, ..., x).$$

Then p is a term of $\mathbb{A}$, and we have

$$p(x, y, z) = x - \big(y - y + y + \sum_i \phi_i(x) + c\big) + z + \sum_i \phi_i(x) + c = x - y + z,$$

so p is Mal'cev.

For the converse, if $\mathbb{A}$ is quasi-affine and has a Mal'cev term p, then $p(x, y, y) \approx p(y, y, x) \approx x$ imply that $p(x, 0, 0) = x, p(0, 0, z) = z$, and $p(y, y, 0) = y + p(0, y, 0) = 0$, so we must have $p(x, y, z) = x - y + z$. Thus $x + z = p(x, 0, z)$ and $x - y = p(x, y, 0)$ are polynomial operations of $\mathbb{A}$, and therefore for each term f of $\mathbb{A}$ the unary function $\phi(x) = f(x, 0, ..., 0) - f(0, ..., 0)$ is a polynomial operation of $\mathbb{A}$ as well. $\square$

It is less trivial to give a universal algebraic definition of what it means to be *abelian*. We will give several different definitions and prove that they are equivalent to each other, and that they restrict to the right concept in the special case of groups.

Definition 1.9.7. An algebraic structure $\mathbb{A}$ is called *abelian* if there is a congruence Θ on $\mathbb{A} \times \mathbb{A}$ such that the diagonal $\Delta_{\mathbb{A}} = \{(a, a) \mid a \in \mathbb{A}\}$ is one of the congruence classes of Θ.

Proposition 1.9.8. *A group is abelian iff it is commutative.*

Proof. A group $\mathbb{G}$ is abelian iff the diagonal $\Delta_{\mathbb{G}}$ is a normal subgroup of $\mathbb{G} \times \mathbb{G}$. To check that $\Delta_{\mathbb{G}}$ is normal, we just need to check that it is closed under conjugation by elements of the form $(1, b)$ for all $b \in \mathbb{G}$. Since

$$(1, b)(a, a)(1, b)^{-1} = (a, bab^{-1}),$$

the normality of $\Delta_{\mathbb{G}}$ is equivalent to the identity $a \approx bab^{-1}$, which is equivalent to $ab \approx ba$.

Alternatively, we can argue as follows. The group $\mathbb{G}$ is commutative iff the map $\mathbb{G} \to \mathbb{G}$ given by $x \mapsto x^{-1}$ is a homomorphism, and if this occurs then there is a homomorphism $\mathbb{G} \times \mathbb{G} \to \mathbb{G}$ such that the restriction $\mathbb{G} \times \{1\} \to \mathbb{G}$ is the identity, and such that the diagonal maps to $\{1\}$. Conversely, if the diagonal is a normal subgroup, then every coset intersects $\mathbb{G} \times \{1\}$ and $\{1\} \times \mathbb{G}$ exactly once, so the quotient $\mathbb{G} \times \mathbb{G}/\Delta_{\mathbb{G}}$ is isomorphic to $\mathbb{G}$ in two different ways, and composing these isomorphisms we obtain the map $x \mapsto x^{-1}$, so $\mathbb{G}$ is commutative. □

Now we give a second definition of abelian, which is phrased in a way which is closely related to the concept of a "commutator" of congruences in a general algebraic structure.

Definition 1.9.9. We say that an algebraic structure $\mathbb{A}$ satisfies the *term condition* if for all terms $t \in \mathrm{Clo}_{n+1}(\mathbb{A})$ and all $u, v \in \mathbb{A}$, $a_i, b_i \in \mathbb{A}$ for $i \le n$, we have

$$t(u, a_1, ..., a_n) = t(u, b_1, ..., b_n) \iff t(v, a_1, ..., a_n) = t(v, b_1, ..., b_n).$$

Proposition 1.9.10. *An algebra $\mathbb{A}$ is abelian iff it satisfies the term condition.*

Proof. We think of congruences on $\mathbb{A}^2$ as subalgebras of $\mathbb{A}^{2\times 2}$, the set of 2×2 matrices with entries in $\mathbb{A}$ (here elements of $\mathbb{A}^2$ are visualized as column vectors, and an element of $\mathbb{A}^{2\times 2}$ is viewed as a row vector of column vectors). To understand the smallest congruence on $\mathbb{A}^2$ with $\Delta_{\mathbb{A}}$ contained in a congruence class, we consider the relation $\mathbb{M} \le \mathbb{A}^{2\times 2}$ generated by matrices of the form

$$\begin{bmatrix} u & v \\ u & v \end{bmatrix}, \quad \begin{bmatrix} a & a \\ b & b \end{bmatrix},$$

where the first type of matrix corresponds to the fact that any two elements of $\Delta_{\mathbb{A}}$ are congruent, while the second type of matrix corresponds to the fact that every element of $\mathbb{A}^2$ is congruent to itself. Then considering $\mathbb{M}$ as a binary relation on $\mathbb{A}^2$, the transitive closure of $\mathbb{M}$ is a congruence Θ on $\mathbb{A}^2$, and it is clearly as small as possible given that $\Delta_{\mathbb{A}}$ is contained in a congruence class of Θ.

To understand whether $\Delta_{\mathbb{A}}$ is a congruence class of Θ, it's enough to check whether $\Delta_{\mathbb{A}}$ meets any element of $\mathbb{A}^2 \setminus \Delta_{\mathbb{A}}$ in $\mathbb{M}$. This occurs (that is, $\mathbb{A}$ is nonabelian) iff there is some term $t \in \mathrm{Pol}_{m+n}(\mathbb{A})$ and some $u_i, v_i \in \mathbb{A}$ for $i \le m$, $a_i, b_i \in \mathbb{A}$ for $i \le n$ such that

$$t(u_1, ..., u_m, a_1, ..., a_n) = t(u_1, ..., u_m, b_1, ..., b_n)$$

but

$$t(v_1, ..., v_m, a_1, ..., a_n) \neq t(v_1, ..., v_m, b_1, ..., b_n).$$

So if $\mathbb{A}$ is abelian, then it certainly satisfies the term condition (just take $m = 1$ in the above). Conversely, if $\mathbb{A}$ satisfies the term condition, then we will show that the above situation can't happen by induction on m. We just note that by the induction hypothesis, we have

$$t(u_1, ..., u_m, a_1, ..., a_n) = t(u_1, ..., u_m, b_1, ..., b_n) \implies t(v_1, ..., v_{m-1}, u_m, a_1, ..., a_n) = t(v_1, ..., v_{m-1}, u_m, b_1, ..., b_n),$$

and then by the term condition applied to a version of t with variables permuted so that the mth variable becomes the first, this implies that

$$t(v_1, ..., v_m, a_1, ..., a_n) = t(v_1, ..., v_m, b_1, ..., b_n).$$ □

Proposition 1.9.11. *Every quasi-affine algebra satisfies the term condition and is therefore abelian.*

Proof. If t is an $n + 1$-ary term of a quasi-affine algebra, then we can write t in the form

$$t(x_0, ..., x_n) = \phi_0(x_0) + \cdots + \phi_n(x_n) + c,$$

where the ϕ_i are unary and c is a constant. Then for any $u \in \mathbb{A}, a_i, b_i \in \mathbb{A}$, we have

$$t(u, a_1, ..., a_n) = t(u, b_1, ..., b_n) \iff \phi_1(a_1) + \cdots + \phi_n(a_n) = \phi_1(b_1) + \cdots + \phi_n(b_n),$$

and this is a condition which does not depend on the value of u. □

Example 1.9.1. If a group is commutative, then it is affine, so it satisfies the term condition. Conversely, if a group satisfies the term condition for the binary term $t(x, y) = yxy^{-1}$, then the group is commutative, since we have $t(1, 1) = t(1, y) \iff t(x, 1) = t(x, y)$, that is, $1 = yy^{-1} \iff x = yxy^{-1}$.

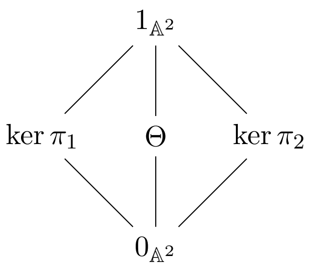
Example 1.9.2. A ring is abelian in the sense of universal algebra iff it is a *zero ring*, that is, a ring satisfying the identity $xy \approx 0$. To see the necessity, we apply the term condition with the term $t(x, y) = xy$ and the pairs $(u, v) = (0, x)$ and $(a, b) = (0, y)$, to see that $0 \cdot 0 = 0 \cdot y \iff x \cdot 0 = x \cdot y$. To see the sufficiency, note that every zero ring is affine.

Example 1.9.3. The quasigroup with multiplication table

$\cdot$	0	1	2	3
0	3	2	0	1
1	2	3	1	0
2	1	0	2	3
3	0	1	3	2

is abelian, but is neither commutative nor associative. In fact it is affine, with underlying group equal to the Klein four-group: the multiplication can be written as $x \cdot y = x \oplus \phi(y) \oplus 3$, where ϕ is the transposition (2 3). This example is from [80].

In terms of congruence lattices, the main important feature of an affine algebra $\mathbb{A}$ is that $\mathrm{Con}(\mathbb{A} \times \mathbb{A})$ contains the following five element sublattice.



The abstract five element lattice corresponding to this picture is known as the diamond lattice $\mathcal{M}_3$. The lattice $\mathcal{M}_3$ has a special role in lattice theory: every modular lattice which isn't distributive contains a sublattice which is isomorphic to $\mathcal{M}_3$ (see Proposition A.4.2 in the appendix).

Theorem 1.9.12. *If $\mathbb{A}$ is an abelian Mal'cev algebra, and if Θ is any congruence of $\mathbb{A}^2$ which contains the diagonal $\Delta_{\mathbb{A}}$ as a congruence class, then the congruences $\Theta, \ker\pi_1, \ker\pi_2$ generate a five element sublattice of $\mathrm{Con}(\mathbb{A}^2)$ isomorphic to $\mathcal{M}_3$.*

Proof. In general, we always have $\ker\pi_1 \vee \ker\pi_2 = 1_{\mathbb{A}^2}$ and $\ker\pi_1 \wedge \ker\pi_2 = 0_{\mathbb{A}^2}$. Since every element of $\mathbb{A}$ is congruent under $\ker\pi_1$ to an element of the diagonal $\Delta_{\mathbb{A}}$, we have $\ker\pi_1 \vee \Theta = 1_{\mathbb{A}^2}$, and similarly $\ker\pi_2 \vee \Theta = 1_{\mathbb{A}^2}$.

All that remains is to check that $\Theta \wedge \ker\pi_1 = \Theta \wedge \ker\pi_2 = 0_{\mathbb{A}^2}$, and this is where we will use the assumption that $\mathbb{A}$ has a Mal'cev term p. If (a,b) is congruent to (c,d) modulo $\Theta \wedge \ker\pi_1$, then we must have $a = c$. Then

$$\begin{bmatrix} b \\ d \end{bmatrix} = p\left(\begin{bmatrix} b \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a \\ d \end{bmatrix}\right) \equiv_\Theta p\left(\begin{bmatrix} b \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} b \\ b \end{bmatrix} \in \Delta_{\mathbb{A}},$$

so $(b,d) \in \Delta_{\mathbb{A}}$, that is, $b = d$. So from $(a,b) \equiv_{\Theta\wedge\ker\pi_1} (c,d)$ we have shown $(a,b) = (c,d)$, that is, we have $\Theta \wedge \ker\pi_1 = 0_{\mathbb{A}^2}$. □

The idea now is to study the *equivalence class geometry* on $\mathbb{A}^2$, where points are elements of $\mathbb{A}^2$, lines correspond to congruence classes of congruences, and two lines are considered *parallel* if they are both congruence classes of the same congruence. The three congruences $\ker\pi_1, \Theta, \ker\pi_2$ on an abelian Mal'cev algebra give us a particularly nice type of combinatorial geometry.

Definition 1.9.13. An *S-3-system* is a set of points S together with three parallel classes of lines $\Theta_1, \Theta_2, \Theta_3$ on S, which satisfy the following properties:

- for any point $p \in S$ and any $i \le 3$, there is exactly one line l_i of Θ_i which contains p, and
- if l_i, l_j are lines of Θ_i, Θ_j, respectively, with $i \ne j$, then their intersection $l_i \cap l_j$ contains exactly one point $p \in S$.

Equivalently, an S-3-system is a relational structure $(S, \Theta_1, \Theta_2, \Theta_3)$ such that:

- each Θ_i is an equivalence relation on S,
- for $i \ne j$ we have $\Theta_i \wedge \Theta_j = 0_S$, and
- for $i \ne j$ we have $\Theta_i \circ \Theta_j = 1_S$.

The assumption $\Theta_i \wedge \Theta_j = 0_S$ says that any pair of non-parallel lines intersect in *at most* one point, while the assumption $\Theta_i \circ \Theta_j = 1_S$ says that any pair of non-parallel lines intersect in *at least* one point.

Corollary 1.9.14. *If $\mathbb{A}$ is an abelian Mal'cev algebra and Θ is any congruence of $\mathbb{A}^2$ with the diagonal as a congruence class, then $(\mathbb{A}^2, \ker\pi_1, \ker\pi_2, \Theta)$ is an S-3-system with a Mal'cev polymorphism.*

From here on we will classify S-3-systems which have Mal'cev polymorphisms, following Gumm's approach [152]. As a preliminary result, we will show that every S-3-system has a coordinate system which describes the three parallel classes of lines in terms of a *loop* (recall that a loop is just a quasigroup which has an identity).

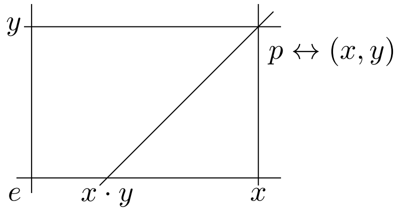
Lemma 1.9.15. *If $(S, \Theta_1, \Theta_2, \Theta_3)$ is an S-3-system, and e is any point of S, then there is a loop $\mathbb{L} = (L, \cdot, 1)$ and a bijection $L \times L \to S$ with $(1, 1) \mapsto e$, such that for any $x, y, x', y' \in L$ we have*

$$\begin{aligned} (x,y) \equiv_{\Theta_1} (x',y') \quad &\iff \quad x = x', \\ (x,y) \equiv_{\Theta_2} (x',y') \quad &\iff \quad y = y', \\ (x,y) \equiv_{\Theta_3} (x',y') \quad &\iff \quad x \cdot y = x' \cdot y', \end{aligned}$$

where we have implicitly identified S with $L \times L$.

Proof. Take L to be the line l_1 through e in the parallel class Θ_1, and take $1 = e$. Let l_2 be the line through e in the parallel class Θ_2. Then there is a bijection between elements of l_1 and elements of l_2, taking $x \in l_1$ to $y \in l_2$ when x, y are on a line l_3 in the parallel class Θ_3: each x is in a unique such line l_3, and each l_3 intersects l_2 in a unique y. Using this bijection, we identify the elements of l_2 with L as well.

Now we note that for any point $p \in S$, there is a unique pair of lines $l_1' \in \Theta_1, l_2' \in \Theta_2$ with $l_1' \cap l_2' = \{p\}$. So we can uniquely identify the point p by describing the point $x \in l_1 \cap l_2'$ and the point $y \in l_2 \cap l_1'$ - this gives us the desired bijection between $L \times L$ and S.

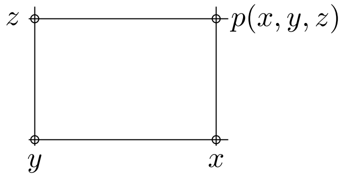


Finally, to define the multiplication $\cdot$ on L, note that for every $x, y \in L$ there is a point $p \in S$ corresponding to (x, y), and this point p is in a unique line $l_3 \in \Theta_3$. We then define $x \cdot y$ to be the element of L corresponding to the point $l_3 \cap l_1$, or alternatively to the point $l_3 \cap l_2$ (which corresponds to the same element of L by the way we identified points of l_2 with points of l_1). □

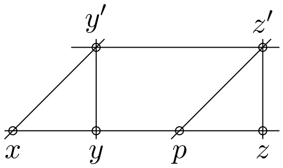
The key observation is that the Mal'cev operation is completely determined by the geometry of the configuration.

Lemma 1.9.16. *If an S-3-system* $\mathbf{S} = (S, \Theta_1, \Theta_2, \Theta_3)$ *has a Mal'cev polymorphism p, then p is completely determined by* $\mathbf{S}$*. In fact, $p(x, y, z)$ can be "geometrically constructed" from the points x, y, z.*

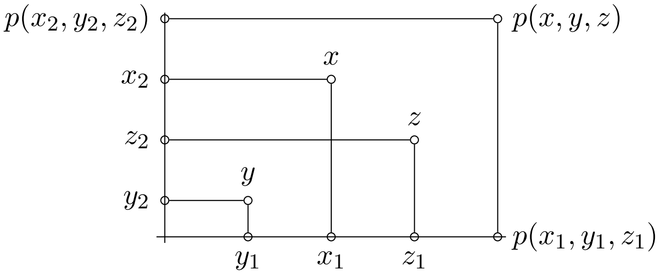
Proof. First consider the special case where x, y lie on a line l_1 and y, z lie on a different line l_2. Suppose that $l_1 \in \Theta_1$ and $l_2 \in \Theta_2$. Then $p(x, y, z) \equiv_{\Theta_1} p(y, y, z) = z$ and $p(x, y, z) \equiv_{\Theta_2} p(x, y, y) = x$, so if we draw the line $l_2' \in \Theta_2$ through x and the line $l_1' \in \Theta_1$ through z, we see that $p(x, y, z)$ is the intersection point $l_1' \cap l_2'$.



Next consider the special case where x, y, z lie on a line l_1, and suppose $l_1 \in \Theta_1$. Draw the line $l_2 \in \Theta_2$ through y and the line $l_3 \in \Theta_3$ through x, and let $y' \in l_2 \cap l_3$ be their point of intersection. Draw the line l_1' through y' parallel to l_1, draw the line l_2' through z parallel to l_2, and let $z' \in l_1' \cap l_2'$ be their point of intersection. Finally, draw the line l_3' through z' parallel to the line l_3, and let p be the intersection point of l_1 and l_3'.



We claim that $p = p(x, y, z)$. To see this, note that $x \equiv_{\Theta_3} y'$, so $p(x, y, z) \equiv_{\Theta_3} p(y', y, z)$, and $p(y', y, z) = z'$ by the first case we considered. Thus $p(x, y, z) \equiv_{\Theta_3} z'$, i.e. $p(x, y, z) \in l_3'$, and since $x \equiv_{\Theta_1} y \equiv_{\Theta_1} z$, we have $p(x, y, z) \equiv_{\Theta_1} p(x, x, x) = x$, i.e. $p(x, y, z) \in l_1$. Thus $p(x, y, z) \in l_1 \cap l_3'$, so $p(x, y, z) = p$. (Alternatively, we could have used $p(x, y, z) \equiv_{\Theta_2} p(x, y', z') = p$, by the first case.)



For the general case, we can pick any lines $l_1 \in \Theta_1, l_2 \in \Theta_2$, set x_1, y_1, z_1 to be the projections of x, y, z onto l_1 via lines in Θ_2 and define $x_2, y_2, z_2 \in l_2$ similarly, and note that $p(x, y, z) \equiv_{\Theta_2} p(x_1, y_1, z_1)$ and $p(x, y, z) \equiv_{\Theta_1} p(x_2, y_2, z_2)$, and we can construct $p(x_1, y_1, z_1), p(x_2, y_2, z_2)$ using the second case considered. □

Corollary 1.9.17. *If p is a Mal'cev polymorphism of an S-3-system, then $p(x, y, z) \approx p(z, y, x)$.*

Proof. The term $p(z, y, x)$ is also a Mal'cev polymorphism, so by the Lemma it must be identical to $p(x, y, z)$. □

Corollary 1.9.18. *If p is a Mal'cev polymorphism of an S-3-system $(S, \Theta_1, \Theta_2, \Theta_3)$, then the graph Γ_p of p, considered as a 4-ary relation on S, is primitively positively definable from $\Theta_1, \Theta_2, \Theta_3$.*

Corollary 1.9.18 can also be interpreted as saying that the map $p : \mathbb{S}^3 \to \mathbb{S}$ is a homomorphism of the *algebraic* structure $\mathbb{S}$ consisting of all polymorphisms of the relational structure $\mathbf{S}$. In particular, p "commutes with itself", that is, the two ways of computing $p * p$ on a 3×3 grid of variables (columns first or rows first) agree with each other. We can summarize this fact by saying that the Mal'cev operation p is *central*.

Definition 1.9.19. An n-ary term t of an algebraic structure $\mathbb{A}$ is called *central* if the map $t : \mathbb{A}^n \to \mathbb{A}$ is a homomorphism.

Now we relate the Mal'cev polymorphism to the coordinate loop $\mathbb{L}$. First we will show that $\mathbb{L}$ is associative.

Lemma 1.9.20. *If* $\mathbf{S} = (S, \Theta_1, \Theta_2, \Theta_3)$ *is an S-3-system with a Mal'cev polymorphism* p*, and if* $\mathbb{L}$ *is a coordinate loop of* $\mathbf{S}$*, then* $\mathbb{L}$ *satisfies*

$$(x_1 \cdot y_1 = x_2 \cdot y_2) \wedge (x_1 \cdot y_3 = x_2 \cdot y_4) \wedge (x_3 \cdot y_1 = x_4 \cdot y_2) \implies (x_3 \cdot y_3 = x_4 \cdot y_4).$$

In particular, $\mathbb{L}$ *is associative, that is,* $\mathbb{L}$ *is a group.*

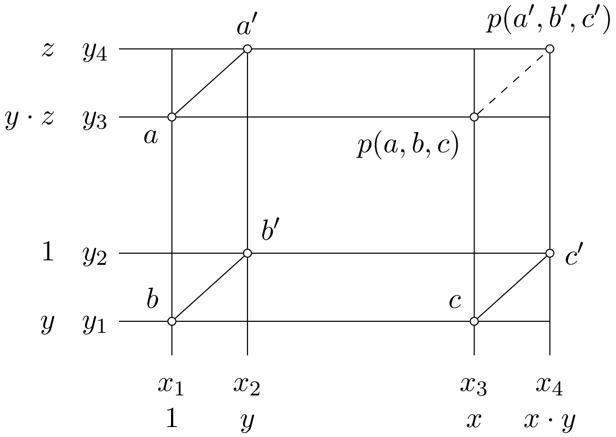
Proof. For those who prefer a purely algebraic proof, this follows from

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = p\left(\begin{bmatrix} x_1 \\ y_3 \end{bmatrix}, \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_3 \\ y_1 \end{bmatrix}\right) \equiv_{\Theta_3} p\left(\begin{bmatrix} x_2 \\ y_4 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_4 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} x_4 \\ y_4 \end{bmatrix}.$$

To see that this implies the associativity of $\mathbb{L}$, let x, y, z be any elements of L, and plug in $(x_1, x_2, x_3, x_4) = (1, y, x, x \cdot y), (y_1, y_2, y_3, y_4) = (y, 1, y \cdot z, z)$. Then we get

$$(1 \cdot y = y \cdot 1) \wedge (1 \cdot (y \cdot z) = y \cdot z) \wedge (x \cdot y = (x \cdot y) \cdot 1) \implies (x \cdot (y \cdot z) = (x \cdot y) \cdot z).$$

For a geometric way to visualize the proof, note that the stated property of $\mathbb{L}$ corresponds to the existence of the dashed line in the following picture.

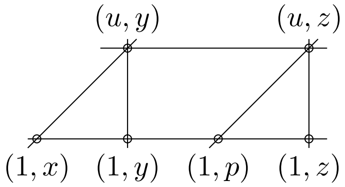


If we set $a = (x_1, y_3)$, etc. as in the picture, then the existence of the dashed line follows from the fact that p preserves the congruence Θ_3 and the fact that $p(a, b, c)$ completes the parallelogram through a, b, c and $p(a', b', c')$ completes the parallelogram through a', b', c'. □

Lemma 1.9.21. *If* $\mathbf{S} = (S, \Theta_1, \Theta_2, \Theta_3)$ *is an S-3-system with a Mal'cev polymorphism* p*, and if* $\mathbb{L}$ *is a coordinate group of* $\mathbf{S}$*, then for* $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in S$*,* $p(x, y, z)$ *is given by*

$$p\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \cdot y_1^{-1} \cdot z_1 \\ x_2 \cdot y_2^{-1} \cdot z_2 \end{bmatrix}.$$

Proof. It's enough to consider the case where x, y, z are along the line $l_1 \in \Theta_1$ with Θ_1-coordinate 1. Consider the diagram



which we used to construct $p(x, y, z)$. Then from $(1, x) \equiv_{\Theta_3} (u, y)$ we have $1 \cdot x = u \cdot y$, and from $(1, p) \equiv_{\Theta_3} (u, z)$ we have $1 \cdot p = u \cdot z$. Solving for u we get $u = xy^{-1}$, and solving for p we get $p = xy^{-1}z$. □

Corollary 1.9.22. *If* $\mathbf{S} = (S, \Theta_1, \Theta_2, \Theta_3)$ *is an S-3-system with a Mal'cev polymorphism* p*, and if* $\mathbb{L}$ *is a coordinate group of* $\mathbf{S}$*, then* $\mathbb{L}$ *is commutative.*

Proof. From $p(x, y, z) \approx p(z, y, x)$ we get $xy^{-1}z \approx zy^{-1}x$ in $\mathbb{L}$, and plugging in $y = 1$ gives $xz \approx zx$, so $\mathbb{L}$ is commutative. □

Putting all of this together, we have the main result of this section.

Theorem 1.9.23. *Any abelian Mal'cev algebra* $\mathbb{A}$ *is affine.*

Proof. By Theorem 1.9.12 and its corollary, $\mathbf{S} = (\mathbb{A}^2, \ker \pi_1, \ker \pi_2, \Theta)$ is an S-3-system with Mal'cev polymorphism p, where p is the Mal'cev term of $\mathbb{A}$ and Θ is a congruence on $\mathbb{A}^2$ with the diagonal as a congruence class. By Lemma 1.9.15, there is a loop structure $\mathbb{L}$ on the underlying set of $\mathbb{A}$ which describes $\mathbf{S}$. By Lemma 1.9.20, Lemma 1.9.21, and its corollary, $\mathbb{L}$ is an abelian group and p is given by $p(x, y, z) = x - y + z$ (writing the abelian group operation additively).

By Corollary 1.9.18, the relation $x - y + z = p$ is primitively positively definable from $\ker \pi_1, \ker \pi_2, \Theta$, so the relation $x + z = y + p$ is preserved by all operations of $\mathbb{A}$, that is, $\mathbb{A}$ is quasi-affine. Since $\mathbb{A}$ was assumed to be Mal'cev, this means that $\mathbb{A}$ is affine. □

We have proved the hardest part of the Fundamental Theorem of Abelian Algebras. For the sake of completeness, we include the rest of it.

Theorem 1.9.24 (Fundamental Theorem of Abelian Algebras)**.** *For an algebraic structure* $\mathbb{A}$*, the following are equivalent:*

(1) $\mathbb{A}$ *is affine,*

(2) $\mathbb{A}$ *is abelian and has a Mal'cev polynomial,*

(3) $\mathbb{A}$ *has a central Mal'cev polynomial.*

Proof. That (1) implies (2) and (3) is clear. For (2) implies (1) and (3), note that any polynomial of $\mathbb{A}$ preserves every congruence of $\mathbb{A}^2$, so the polynomial clone of $\mathbb{A}$ is also abelian and we may apply the previous theorem. For (3) $\implies$ (1), we just need to show that any Mal'cev operation p which commutes with itself comes from an abelian group, since then the fact that $p(x, y, z) = x - y + z$ is central will imply that $\mathbb{A}$ is quasi-affine.

So suppose that p is a Mal'cev operation which commutes with itself, and pick any element to call 0 in $\mathbb{A}$. We define addition and negation on $\mathbb{A}$ by

$$x + y \coloneqq p(x, 0, y), \quad -x \coloneqq p(0, x, 0).$$

That 0 is an identity element for + follows from the Mal'cev identities $p(x, 0, 0) = p(0, 0, x) = x$.

To see that + is associative, we evaluate the expression

$$p * p\left(\begin{bmatrix} x & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{bmatrix}\right)$$

in two ways: evaluating it by rows first, we get $(x + y) + z$, and evaluating it by columns first, we get $x + (y + z)$.

To see that $-$ computes the inverse, we evaluate the expression

$$p * p\left(\begin{bmatrix} x & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{bmatrix}\right)$$

in two ways: by rows we get $p(x, x, 0) = 0$, and by columns we get $x + (-x)$. A similar argument shows that $(-x) + x = 0$.

For commutativity of +, we evaluate the expression

$$p * p\left(\begin{bmatrix} y & 0 & x \\ y & y & x \\ x & y & y \end{bmatrix}\right)$$

in two ways: by rows we get $p(y + x, x, x) = y + x$, and by columns we get $p(x, 0, y) = x + y$.

Finally, to express p in terms of the group operations $+, -$, we evaluate the expression

$$p * p\left(\begin{bmatrix} x & y & z \\ 0 & y & 0 \\ -y & 0 & 0 \end{bmatrix}\right)$$

in two ways: by rows we get $p(p(x, y, z), -y, -y) = p(x, y, z)$, and by columns we get $p(x - y, 0, z) = x - y + z$. □

The method of visualizing algebraic arguments via the geometry of equivalence classes was extended to congruence modular varieties by Gumm in his book "Geometrical methods in congruence modular algebras" [89], where he used it to show that any abelian algebra in a congruence modular variety is affine. This was extended further by Hobby and McKenzie [95], who used tame congruence theory to show that any finite abelian algebra in a Taylor variety is affine (in the infinite case, Kearnes and Szendrei [117] show that any abelian Taylor algebra is quasi-affine - the example $(\mathbb{R}, \frac{x+y}{2})$ shows that an additional assumption is needed for it to be affine). Later we will go over a simpler proof of the fact that finite abelian Taylor algebras are affine, from [22].

Remark 1.9.1. If we leave the context of Taylor varieties, we can no longer expect abelian algebras to be affine, since they could fail to have any interesting operations at all. But we can still ask whether abelian algebras are quasi-affine. The following problem is open.

Problem 1.9.1. Under what conditions are abelian algebras quasi-affine? Is it true that every idempotent abelian algebra is quasi-affine?

It is known that if we drop idempotence, then some extra condition is needed: Quackenbush [159] gives an example of an infinite, non-idempotent algebra which is abelian but not quasi-affine. Quackenbush's example is a slight modification of the completely free algebra on 8 elements with a single binary operation, where the modification is that $x_1 \cdot x_2 = x_5 \cdot x_6, x_3 \cdot x_4 = x_7 \cdot x_8, x_1 \cdot x_4 = x_5 \cdot x_8$, but $x_3 \cdot x_2 \neq x_7 \cdot x_6$. Another example with just five elements is given in Example B.3.2.

Kearnes [116] has shown that any *simple* idempotent abelian algebra is quasi-affine - in fact, he shows that any simple idempotent algebra which has a skew congruence (that is, a congruence on some power $\mathbb{A}^n$ which is not the kernel of some projection) either has an absorbing element (that is, an element a such that every term t which depends on its first variable has $t(a, ...) = a$) or is a subalgebra of a simple reduct of a module.

There are a few other contexts in which it is known that abelian implies quasi-affine. In [114], Kearnes shows that any abelian algebra with a central binary polynomial which is cancellative is quasi-affine, and in [177] this is extended to the result that any abelian algebra with a commutative cancellative polynomial is quasi-affine. In [104], it is shown that abelian quandles are quasi-affine.

1.9.1 Commutators

In this subsection we define an extension of the commutator from group theory to a commutator on congruences of general algebraic structures. The purpose of the commutator is to detect situations where the operations of an algebraic structure behave linearly. The theory of the commutator works best in congruence modular varieties, but it still has some use in general Taylor varieties, although slight differences in the technical details of the definition become important outside the world of congruence modular varieties. The commutator we will be discussing is called the *term condition* commutator.

Definition 1.9.25. If $\alpha, \beta, \delta \in \mathrm{Con}(\mathbb{A})$, we say that α *centralizes* β modulo δ, written $C(\alpha, \beta; \delta)$ (or $C(\alpha, \beta)$ if $\delta = 0_{\mathbb{A}}$), if for every $n+1$-ary term $t \in \mathrm{Clo}_{n+1}(\mathbb{A})$, for any $(u, v) \in \alpha$, and for any $(a_1, b_1), ..., (a_n, b_n) \in \beta$, we have

$$t(u, a_1, ..., a_n) \equiv_\delta t(u, b_1, ..., b_n) \iff t(v, a_1, ..., a_n) \equiv_\delta t(v, b_1, ..., b_n).$$

The smallest δ which satisfies $C(\alpha, \beta; \delta)$ is called the *commutator* of α, β, and is written as $[\alpha, \beta]$. If $\theta \leq \alpha, \beta$, then we also define the *relative commutator* $[\alpha, \beta]_\theta$ to be the least $\delta \geq \theta$ which satisfies $C(\alpha, \beta; \delta)$.

As with the criterion for abelianness, the term condition implies a seemingly stronger version where more variables change at once.

Proposition 1.9.26. *If α centralizes β modulo δ, then for every $m+n$-ary term $t \in \mathrm{Clo}_{m+n}(\mathbb{A})$, for any $(u_1, v_1), ..., (u_m, v_m) \in \alpha$, and for any $(a_1, b_1), ..., (a_n, b_n) \in \beta$, we have*

$$t(u_1, ..., u_m, a_1, ..., a_n) \equiv_\delta t(u_1, ..., u_m, b_1, ..., b_n) \iff t(v_1, ...v_m, a_1, ..., a_n) \equiv_\delta t(v_1, ..., v_m, b_1, ..., b_n).$$

Before we go on, let's check that this matches the usual commutator from group theory.

Proposition 1.9.27. *If $\mathbb{M}, \mathbb{N}$ are normal subgroups of a group $\mathbb{G}$, $[\mathbb{M}, \mathbb{N}]$ is the (normal) subgroup generated by commutators $[m, n] = mnm^{-1}n^{-1}$ for $m \in \mathbb{M}, n \in \mathbb{N}$, and $\theta_{\mathbb{M}}, \theta_{\mathbb{N}}, \theta_{[\mathbb{M},\mathbb{N}]}$ are the associated congruences, then $\theta_{[\mathbb{M},\mathbb{N}]} = [\theta_{\mathbb{M}}, \theta_{\mathbb{N}}]$.*

Proof. We will show that $\theta_{\mathbb{M}}$ centralizes $\theta_{\mathbb{N}}$ iff every element of $\mathbb{M}$ commutes with every element of $\mathbb{N}$ - this will finish the proof, since $[\mathbb{M}, \mathbb{N}]$ is the smallest normal subgroup $\mathbb{K}$ of $\mathbb{G}$ such that every element of $\mathbb{M}/\mathbb{K}$ commutes with every element of $\mathbb{N}/\mathbb{K}$ in $\mathbb{G}/\mathbb{K}$.

First suppose that $\theta_{\mathbb{M}}$ centralizes $\theta_{\mathbb{N}}$. Let t be the binary term $t(x, y) = yxy^{-1}$, then for any $m \in \mathbb{M}, n \in \mathbb{N}$, by the term condition applied to $(1, m) \in \theta_{\mathbb{M}}, (1, n) \in \theta_{\mathbb{N}}$, we have

$$1 = nn^{-1} \iff m = nmn^{-1},$$

so m and n commute.

Now suppose that every element of $\mathbb{M}$ commutes with every element of $\mathbb{N}$, and consider an arbitrary $n + 1$-ary term $t \in \mathrm{Clo}_{n+1}(\mathbb{G})$ and any $(u, v) \in \theta_{\mathbb{M}}, (a_1, b_1), ..., (a_n, b_n) \in \theta_{\mathbb{N}}$ with

$$t(u, a_1, ..., a_n) = t(u, b_1, ..., b_n).$$

Thinking of $t(ux, a_1y_1, ..., a_ny_n)$ as a function of $x, y_1, ..., y_n$ with parameters $u, a_1, ..., a_n$, we may rearrange it into the form

$$t(ux, a_1y_1, ..., a_ny_n) = t'(x, y_1, ..., y_n)t(u, a_1, ..., a_n)$$

for some t' in the clone generated by the group operations together with the unary conjugation operations $\phi_c : x \mapsto cxc^{-1}$, so we may rewrite our assumption as

$$t'(1, 1, ..., 1) = t'(1, a_1^{-1}b_1, ..., a_n^{-1}b_n).$$

To show that

$$t(v, a_1, ..., a_n) = t(v, b_1, ..., b_n),$$

we just need to show that

$$t'(u^{-1}v, 1, ..., 1) = t'(u^{-1}v, a_1^{-1}b_1, ..., a_n^{-1}b_n),$$

which follows from the assumed equality together with the fact that for each $c, d \in \mathbb{G}$ and each i, $\phi_c(u^{-1}v) \in \mathbb{M}$ commutes with $\phi_d(a_i^{-1}b_i) \in \mathbb{N}$. □

Example 1.9.4. In the case of rings, the term condition commutator applied to a pair of ideals I, J gives $[I, J] = IJ + JI$. Note that this is a bit different from what we might have expected (it has nothing to do with the Lie bracket), but it makes more sense when we remember that we only consider a ring to be abelian if it is a zero ring.

Example 1.9.5. In a majority algebra, the commutator is given by $[\alpha, \beta] = \alpha \wedge \beta$. To see this, suppose $(a, b) \in \alpha \wedge \beta$, and apply the term condition to the majority operation to see that

$$m(\boxed{a}, a, a) = m(\boxed{a}, a, b) \implies m(\boxed{b}, a, a)\ [\alpha, \beta]\ m(\boxed{b}, a, b),$$

so $(a, b) \in [\alpha, \beta]$. A similar argument shows that the commutator is given by intersection in any variety with a near-unanimity term.

Example 1.9.6. In a semilattice, the commutator is given by $[\alpha, \beta] = \alpha \wedge \beta$. Let s be the semilattice operation, and let s_3 be the term given by $s_3(x, y, z) = s(x, s(y, z))$. Then for $(a, b) \in \alpha \wedge \beta$, we have

$$s_3(\boxed{a}, a, b) = s_3(\boxed{a}, b, b) \implies s_3(\boxed{b}, a, b)\ [\alpha, \beta]\ s_3(\boxed{b}, b, b),$$

so $s(a, b)\ [\alpha, \beta]\ b$, and similarly $s(a, b)\ [\alpha, \beta]\ a$, so $(a, b) \in [\alpha, \beta]$.

Sometimes it is helpful to visualize the term condition via 2×2 matrices.

Definition 1.9.28. For $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$, we define the algebra $\mathbb{M}(\alpha, \beta) \le \mathbb{A}^{2\times 2}$ to be the subalgebra of 2×2 matrices which is generated by the matrices of the form

$$\begin{bmatrix} u & u \\ v & v \end{bmatrix} \text{ with } (u, v) \in \alpha, \quad \begin{bmatrix} a & b \\ a & b \end{bmatrix} \text{ with } (a, b) \in \beta.$$

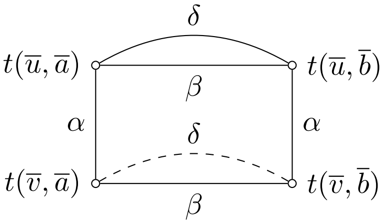
Proposition 1.9.29. *If $\alpha, \beta, \delta \in \mathrm{Con}(\mathbb{A})$, then α centralizes β modulo δ iff for all*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}(\alpha, \beta)$$

we have

$$a \equiv_\delta b \iff c \equiv_\delta d.$$

The usual picture which is drawn to represent the term condition for $C(\alpha, \beta; \delta)$ is this:


δ
$t(\overline{u}, \overline{a})$
$t(\overline{u}, \overline{b})$
β
α
δ
α
$t(\overline{v}, \overline{a})$
$t(\overline{v}, \overline{b})$
β

where the positioning of the four corners matches with the way we have laid out the 2×2 matrices in $\mathbb{M}(\alpha, \beta)$. A mnemonic for remembering where the δ edges go is that in the term condition $C(\alpha, \beta; \delta)$, "δ is next to β".

We now list a few elementary properties of the commutator which hold in general, which are given as exercises in Hobby and McKenzie's book [95].

Proposition 1.9.30. *For $\alpha, \beta, \delta \in \mathrm{Con}(\mathbb{A})$, we have*

(a) if $C(\alpha, \beta; \delta_i)$ for $i \in I$, then $C(\alpha, \beta; \bigwedge_{i\in I} \delta_i)$, so $[\alpha, \beta]$ and $[\alpha, \beta]_\theta$ are well-defined,

(b) if $(\alpha \vee (\beta \wedge \delta)) \wedge \beta \le \delta$ then $C(\alpha, \beta; \delta)$ holds, so $[\alpha, \beta] \le \alpha \wedge \beta$,

(c) if $\alpha' \le \alpha, \beta' \le \beta$, then $C(\alpha, \beta; \delta) \implies C(\alpha', \beta'; \delta)$, so $[\alpha', \beta'] \le [\alpha, \beta]$,

(d) for any γ we have $C(\alpha, \beta; \delta) \implies C(\alpha \wedge \gamma, \beta; \delta \wedge \gamma)$,

(e) if $C(\alpha_i, \beta; \delta)$ holds for all $i \in I$ then $C(\bigvee_{i\in I} \alpha_i, \beta; \delta)$ holds,

(f) if $\theta \le \alpha, \beta, \delta$ then $C(\alpha, \beta; \delta)$ holds iff $C(\alpha/\theta, \beta/\theta; \delta/\theta)$ holds in $\mathbb{A}/\theta$, so $[\alpha/\theta, \beta/\theta] = [\alpha, \beta]_\theta/\theta$,

(g) if $\mathbb{B} \le \mathbb{A}$ then $C(\alpha, \beta; \delta) \implies C(\alpha|_{\mathbb{B}}, \beta|_{\mathbb{B}}; \delta|_{\mathbb{B}})$, so $[\alpha|_{\mathbb{B}}, \beta|_{\mathbb{B}}] \le [\alpha, \beta]|_{\mathbb{B}}$,

(h) if $[\alpha, \alpha] = 0_{\mathbb{A}}$, then any congruence class of α which is also a subalgebra of $\mathbb{A}$ is an abelian subalgebra.

Proof. Parts (a), (c), (d), (f), (g), (h) follow immediately from the definitions. For (b), note that for any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}(\alpha, \beta)$ with $a \equiv_\delta b$, we have $c \equiv_\alpha a \equiv_{\beta\wedge\delta} b \equiv_\alpha d$ and $c \equiv_\beta d$, so $(c, d) \in (\alpha \circ (\beta \wedge \delta) \circ \alpha) \wedge \beta$, which is a subset of δ by assumption.

For (e), we string together several instances of the term condition: if $(u, v) \in \bigvee_i \alpha_i$, $(a_i, b_i) \in \beta$, and $t(u, \overline{a}) \equiv_\delta t(u, \overline{b})$, then if we let $u = u_0, u_1, ..., u_n = v$ be a sequence of elements of $\mathbb{A}$ with $(u_i, u_{i+1}) \in \alpha_{j_i}$ for some $j_i \in I$, then by the term condition $C(\alpha_{j_i}, \beta; \delta)$ we have

$$t(u_i, \overline{a}) \equiv_\delta t(u_i, \overline{b}) \implies t(u_{i+1}, \overline{a}) \equiv_\delta t(u_{i+1}, \overline{b}),$$

so by inducting on i we get $t(v, \overline{a}) \equiv_\delta t(v, \overline{b})$. □

Corollary 1.9.31. *If an idempotent algebra $\mathbb{A}$ has any congruences $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$ with $[\alpha, \beta] \ne \alpha \wedge \beta$, then some subalgebra of some quotient of $\mathbb{A}$ is a nontrivial abelian algebra.*

Proof. Let $\delta = \alpha \wedge \beta$, then from $\delta \le \alpha, \beta$ we have $[\delta, \delta] \le [\alpha, \beta] < \alpha \wedge \beta = \delta$. Thus $\delta' = \delta/[\delta, \delta]$ is a nontrivial congruence on $\mathbb{A}/[\delta, \delta]$ with $[\delta', \delta'] = [\delta, \delta]/[\delta, \delta] = 0_{\mathbb{A}/[\delta,\delta]}$, so there is some nontrivial congruence class $\mathbb{B}$ of δ' and $\mathbb{B}$ is an abelian subalgebra of $\mathbb{A}/[\delta, \delta]$. □

Proposition 1.9.32. *If $[\alpha, \beta] = \alpha \wedge \beta$ for all $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$, then $\mathrm{Con}(\mathbb{A})$ satisfies the meet-semidistributive law:*

$$\alpha \wedge \beta = \alpha \wedge \gamma \implies \alpha \wedge (\beta \vee \gamma) = \alpha \wedge \beta.$$

Proof. If $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbb{A})$ satisfy $\alpha \wedge \beta = \alpha \wedge \gamma$, then $C(\beta, \alpha; \alpha \wedge \beta)$ and $C(\gamma, \alpha; \alpha \wedge \beta)$ hold, so $C(\beta \vee \gamma, \alpha; \alpha \wedge \beta)$ holds, so $\alpha \wedge (\beta \vee \gamma) = [\beta \vee \gamma, \alpha] \le \alpha \wedge \beta$. □

Definition 1.9.33. An algebra $\mathbb{A}$ is *congruence meet-semidistributive*, written SD($\wedge$) for short, if for all $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbb{A})$ with $\alpha \wedge \beta = \alpha \wedge \gamma$, we have $\alpha \wedge (\beta \vee \gamma) = \alpha \wedge \beta$. A variety $\mathcal{V}$ is SD($\wedge$) if every algebra $\mathbb{A} \in \mathcal{V}$ is SD($\wedge$).

The next corollary is the key to classifying CSPs which do not have the “ability to count” - as we will see later, a finite idempotent algebra generates an SD($\wedge$) variety if and only if the associated CSP has bounded width.

Corollary 1.9.34. *If an idempotent variety does not contain any nontrivial abelian algebras, then it is congruence meet-semidistributive. Conversely, a congruence meet-semidistributive variety does not contain any nontrivial affine algebra.*

Proof. For the converse statement, note that if $\mathbb{A}$ is affine, then $\mathrm{Con}(\mathbb{A}^2)$ contains a copy of the diamond lattice $\mathcal{M}_3$, and $\mathcal{M}_3$ doesn't satisfy the meet-semidistributive law. □

Now we consider some definitions which are useful in the case where the commutator is not trivial (i.e., not given by $[\alpha, \beta] = \alpha \wedge \beta$).

Definition 1.9.35. Suppose that $\alpha \le \beta \in \mathrm{Con}(\mathbb{A})$. We say that β is *abelian* over α if the term condition $C(\beta, \beta; \alpha)$ holds. We say that β is *solvable* over α if there is a chain of congruences $\alpha = \alpha_0 \le \cdots \le \alpha_n = \beta$ such that α_{i+1} is abelian over α_i for each i.

A congruence α is called abelian if it is abelian over $0_\mathbb{A}$ (equivalently $[\alpha, \alpha] = 0_\mathbb{A}$), and similarly α is called solvable if α is solvable over $0_\mathbb{A}$. An algebra $\mathbb{A}$ is called solvable if $1_\mathbb{A}$ is solvable.

The *center* of an algebra $\mathbb{A}$ is defined to be the largest ζ such that $C(\zeta, 1_\mathbb{A})$ holds (equivalently, the largest ζ with $[\zeta, 1_\mathbb{A}] = 0_\mathbb{A}$). For β a congruence, we define the *centralizer* of β, written $(0 : \beta)$, to be the largest congruence α such that $[\alpha, \beta] = 0$, and more generally for any δ we define the *relative centralizer* $(\delta : \beta)$ to be the largest α such that $C(\alpha, \beta; \delta)$ holds.

Proposition 1.9.36. *For congruences on $\mathbb{A}$, we have the following:*

(a) for any β, δ there exists a largest α such that $C(\alpha, \beta; \delta)$ holds, so $(\delta : \beta)$ (and, in particular, the center of $\mathbb{A}$) is well-defined,

(b) if γ is solvable over β and β is solvable over α, then γ is solvable over α,

(c) if β is solvable (abelian) over α, then $\beta \wedge \gamma$ is solvable (abelian) over $\alpha \wedge \gamma$ for any γ,

(d) if $\theta \le \alpha \le \beta$, then β is solvable (abelian) over α iff β/θ is solvable (abelian) over α/θ,

(e) $\mathbb{A}/\theta$ is solvable (abelian) iff $1_\mathbb{A}$ is solvable (abelian) over θ.

Proof. Part (a) follows from Proposition 1.9.30(e), part (b) is obvious, part(c) follows from Proposition 1.9.30(d), part (d) follows from Proposition 1.9.30(f), and part (e) is part (d) specialized to the case $\beta = 1_\mathbb{A}, \alpha = \theta$. □

If our algebra is finite, then solvability has a surprisingly simple alternative characterization based on tame congruence theory, which is described in Appendix B.6. To take the general theory further, we need to make an additional assumption on our variety, such as congruence modularity. The interested reader can find the (surprisingly deep) theory of commutators in congruence modular varieties in Appendix A.

A weaker assumption which is still good enough to prove most of the basic properties of commutators is the existence of a ternary term known as a *difference term*, generalizing the Gumm difference term found in congruence modular varieties, which acts like a Mal'cev term on abelian algebras.

Definition 1.9.37. A ternary term p is called a *difference term* for a variety, if it satisfies the identity $p(y, y, x) \approx x$, and for every $(x, y) \in \theta$ for θ a congruence, we always have $p(x, y, y) \equiv_{[\theta,\theta]} x$.

Example 1.9.7. Any SD($\wedge$) variety has a difference term: just take $p(x, y, z) = z$. That this works relies on the fact that $[\alpha, \beta] = \alpha \wedge \beta$ in SD($\wedge$) varieties, which we haven't proved - this can be found in [117].

One property of a difference term is that it forces several alternative commutators to match with the term condition commutator, and one of these commutators is clearly symmetric.

Definition 1.9.38. For any $n \ge 1$, we define the n-cycle commutator $[\alpha, \beta]_n$ to be the least congruence δ such that for any cycle of n matrices

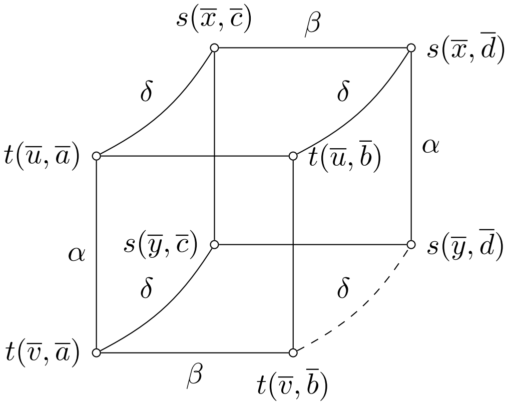
$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \ldots, \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \in \mathbb{M}(\alpha, \beta)$$

such that $b_i \equiv_\delta a_{i+1}$ for all $i < n$, $b_n \equiv_\delta a_1$, and $d_i \equiv_\delta c_{i+1}$ for all $i < n$, we have additionally that $d_n \equiv_\delta c_1$.

If $\mathbb{A}$ is affine, then it is easy to check that $[1_\mathbb{A}, 1_\mathbb{A}]_n = 0_\mathbb{A}$ for every n. Note that for $n = 1$, we have $[\alpha, \beta]_1 = [\alpha, \beta]$. Additionally, since we can take the nth matrix in the cycle to have a pair of equal columns, we have $[\alpha, \beta]_i \le [\alpha, \beta]_{i+1}$ for all i.

Quackenbush's famous example of an abelian algebra which is not quasi-affine from [159] is an example of an algebra where $[1_\mathbb{A}, 1_\mathbb{A}]_1 = 0_\mathbb{A}$ but $[1_\mathbb{A}, 1_\mathbb{A}]_2 \ne 0_\mathbb{A}$.

For $n = 2$, the 2-cycle commutator is clearly symmetric: $[\alpha, \beta]_2 = [\beta, \alpha]_2$. Since it is defined via two matrices in $\mathbb{M}(\alpha, \beta)$, and since each matrix comes from some term, the commutator $[\alpha, \beta]_2$ is also called the *two term commutator*. The two term condition is illustrated in the following diagram.


$s(\overline{x},\overline{c})$
β
$s(\overline{x},\overline{d})$
δ
δ
$t(\overline{u},\overline{a})$
$t(\overline{u},\overline{b})$
α
α
$s(\overline{y},\overline{c})$
$s(\overline{y},\overline{d})$
δ
δ
$t(\overline{v},\overline{a})$
β
$t(\overline{v},\overline{b})$

If we have a difference term, then all of the n-cycle commutators turn out to be equal.

Theorem 1.9.39 (Lipparini [135]). *In a variety with a difference term, we have $[\alpha, \beta]_n = [\alpha, \beta]$ for all n. In particular, we have $[\alpha, \beta] = [\beta, \alpha]$.*

Proof. Suppose that p is a difference term. We will show that $[\alpha, \beta]$ satisfies the n-cycle term condition by induction on n. Suppose that matrices $\begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in \mathbb{M}(\alpha, \beta)$ for $i \le n$ are as in the definition of the n-cycle condition for $\delta = [\alpha, \beta]$. Applying the difference term, we have

$$p\left(\begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}, \begin{bmatrix} b_1 & b_1 \\ d_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_1 & a_1 \\ c_1 & c_1 \end{bmatrix}\right) = \begin{bmatrix} p(a_i, b_1, a_1) & p(b_i, b_1, a_1) \\ p(c_i, d_1, c_1) & p(d_i, d_1, c_1) \end{bmatrix} \in \mathbb{M}(\alpha, \beta)$$

for $2 \le i \le n-1$, and

$$p\left(\begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}, \begin{bmatrix} b_1 & a_1 \\ d_1 & c_1 \end{bmatrix}, \begin{bmatrix} a_1 & a_1 \\ c_1 & c_1 \end{bmatrix}\right) = \begin{bmatrix} p(a_n, b_1, a_1) & p(b_n, a_1, a_1) \\ p(c_n, d_1, c_1) & p(d_n, c_1, c_1) \end{bmatrix} \in \mathbb{M}(\alpha, \beta).$$

The reader can check that these form a system of matrices as in the definition of the $n-1$-cycle condition for $\delta = [\alpha, \beta]$, so by the inductive hypothesis we have

$$p(c_2, d_1, c_1) \equiv_{[\alpha,\beta]} p(d_n, c_1, c_1).$$

From $c_2 \equiv_{[\alpha,\beta]} d_1$ and the fact that p is a difference term, the left hand side is congruenct to c_1 modulo $[\alpha,\beta]$. From the fact that $c_1 \equiv_\beta d_n$ and $(c_1, d_n) \in \alpha \circ [\alpha,\beta] \circ \alpha = \alpha$, we have $(c_1, d_n) \in \alpha \wedge \beta$, so from the fact that p is a difference term we have $p(d_n, c_1, c_1) \equiv_{[\alpha\wedge\beta,\alpha\wedge\beta]} d_n$, and from $[\alpha \wedge \beta, \alpha \wedge \beta] \le [\alpha, \beta]$ we get $c_1 \equiv_{[\alpha,\beta]} d_n$. □

In fact, substantially more is true in varieties with a difference term. Kearnes [115] shows that almost all properties of the commutator which hold in congruence modular varieties generalize to varieties with a difference term, other than $[\alpha_1 \vee \alpha_2, \beta] = [\alpha_1, \beta] \vee [\alpha_2, \beta]$. This property must be weakened, but it is at least true that if $[\alpha_1, \beta] = [\alpha_2, \beta]$ then $[\alpha_1 \vee \alpha_2, \beta] = [\alpha_1, \beta]$ in varieties with difference terms.

If we go beyond varieties with a difference term, the commutator may no longer be symmetric. For instance, in the algebra $\mathbb{A} = (\{0, 1, 2, *\}, \cdot)$ with $\cdot$ given by

$$\begin{array}{c|cccc} \cdot & 0 & 1 & 2 & * \\ \hline 0 & 0 & 2 & 1 & * \\ 1 & 2 & 1 & 0 & * \\ 2 & 1 & 0 & 2 & * \\ * & * & * & * & * \end{array},$$

if we let $\theta \in \operatorname{Con}(\mathbb{A})$ be the congruence corresponding to the partition $\{0, 1, 2\}, \{*\}$, then we have

$$[\theta, 1_\mathbb{A}] = 0_\mathbb{A}, \quad [1_\mathbb{A}, \theta] = \theta.$$

The simplest way to fix this asymmetry is to make the following definition from [117].

Definition 1.9.40. If $\alpha, \beta \in \operatorname{Con}(\mathbb{A})$, then we define their *symmetric commutator*, written $[\alpha,\beta]_s$, to be the least congruence δ such that both $C(\alpha, \beta; \delta)$ and $C(\beta, \alpha; \delta)$ hold.

To see that $[\alpha,\beta]_s$ is well-defined, we use Proposition 1.9.30(a) to see that the intersection of any collection of congruences that simultaneously satisfy $C(\alpha, \beta; \delta)$ and $C(\beta, \alpha; \delta)$ will also satisfy this pair of term conditions. Since the two-term commutator satisfies $C(\alpha, \beta; [\alpha,\beta]_2)$ and $C(\beta, \alpha; [\alpha, \beta]_2)$, we always have $[\alpha,\beta]_s \le [\alpha,\beta]_2$.

We can also go in the other direction, and define a more general commutator by trying to directly think about what an algebra needs to satisfy to be quasi-affine. This leads to the following definition.

Definition 1.9.41. If $\alpha, \beta \in \operatorname{Con}(\mathbb{A})$, then we define their *linear commutator*, written $[\alpha,\beta]_\ell$, as follows. Define a group $\mathbb{G}$ with the following presentation: the generators of $\mathbb{G}$ are the elements of $\mathbb{A}$, and the relations are given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}(\alpha, \beta) \quad \implies \quad a + d = b + c \text{ in } \mathbb{G}.$$

Then we define the equivalence relation $[\alpha,\beta]_\ell$ to be the kernel of the natural map $\mathbb{A} \to \mathbb{G}$.

Proposition 1.9.42. *The linear commutator $[\alpha,\beta]_\ell$ always defines a congruence of $\mathbb{A}$, we have $[\alpha,\beta]_n \le [\alpha,\beta]_\ell$ for each n and $[\alpha,\beta]_\ell \le \alpha \wedge \beta$, and $[1_\mathbb{A}, 1_\mathbb{A}]_\ell = 0_\mathbb{A}$ iff $\mathbb{A}$ is quasi-affine.*

Proof. The linear commutator can be defined combinatorially as follows. For each matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}(\alpha,\beta)$, call a, d the "positive" corners of the matrix M, and b, c the "negative" corners of M. Then $(x,y) \in [\alpha,\beta]_\ell$ iff there is some collection of matrices $M_i \in \mathbb{M}(\alpha,\beta)$ and a way to pair off values in the positive corners of the matrices M_i to equal values in negative corners of the matrices, so that the only unpaired values are x and y, with one occuring in a positive corner of some matrix and the other occuring in a negative corner of some matrix. This defines the linear commutator $[\alpha,\beta]_\ell$ as the union of a directed limit of relations defined by primitive positive formulas in $\mathbb{M}(\alpha,\beta)$, so the equivalence relation $[\alpha,\beta]_\ell$ is compatible with the operations of $\mathbb{A}$.

The inequality $[\alpha,\beta]_n \le [\alpha,\beta]_\ell$ follows from the combinatorial description of $[\alpha,\beta]_n$ above (the n-cycle condition is a special case of the general setup of matching corners of matrices together). To prove that $[\alpha,\beta]_\ell \le \alpha \wedge \beta$, we just need to check that $[\alpha,\beta]_\ell \le \alpha$ by symmetry, and this follows by chasing equalities and congruences through the matrices in the combinatorial description of $[\alpha,\beta]_\ell$.

If $\mathbb{A}$ is quasi-affine, then it is easy to see that $[1_\mathbb{A}, 1_\mathbb{A}]_\ell = 0_\mathbb{A}$. Finally, if $[1_\mathbb{A}, 1_\mathbb{A}]_\ell = 0_\mathbb{A}$, then $\mathbb{A}$ embeds injectively into the group $\mathbb{G}$ from the definition of $[1_\mathbb{A}, 1_\mathbb{A}]_\ell$, and we can generalize the combinatorial description of $[\alpha,\beta]_\ell$ to get a combinatorial description of the restriction of the 4-ary relation $a + d = b + c$ to $\mathbb{A}$, so this 4-ary relation is preserved by the operations of $\mathbb{A}$. □

We have the following relationship between the various commutators which have been defined so far:

$$[\alpha,\beta] \le [\alpha,\beta]_s \le [\alpha,\beta]_2 \le [\alpha,\beta]_3 \le \cdots \le [\alpha,\beta]_\ell \le \alpha \wedge \beta,$$

and among these, the commutators $[\alpha,\beta]_s, [\alpha,\beta]_2$, and $[\alpha,\beta]_\ell$ are symmetric by construction. In [117], Kearnes and Szendrei prove that in every Taylor variety we always have $[\alpha,\beta]_s = [\alpha,\beta]_\ell$, so almost all of the commutators collapse into a single concept in Taylor varieties (and they *all* collapse in varieties with difference terms). They also give an alternative characterization of the linear commutator by showing that it is equivalent to the commutator obtained by first "freely" extending your variety to make the basic operations multinear inside some larger abelian group, and then computing commutators in the multilinear setting, which has a Mal'cev operation $x - y + z$.

Chapter 2

Compact Representations and algebras with Few Subpowers

2.1 Generalized Majority-Minority operations (motivating Few Subpowers)

The Few Subpowers algorithm was heavily influenced by Dalmau's paper on generalized majority-minority operations [61]. Dalmau's motivation was that in both near-unanimity algebras and Mal'cev algebras, every subalgebra of $\mathbb{A}^n$ has a nice generating set: in the Mal'cev case, we can use a compact representation, while in the near-unanimity case, if the arity is $l+1$, we can use any set of elements which has the same projection onto every subset of the coordinates of size at most l. The goal was to unify these two cases.

Definition 2.1.1. An operation φ is a *generalized majority-minority* operation (abbreviated as *gmm* operation) if for each pair a, b we either have

$$\varphi(x, y, ..., y) = \varphi(y, x, ..., y) = \cdots = \varphi(y, y, ..., x) = y \quad \text{for all } x, y \in \{a, b\},$$

or

$$\varphi(x, y, ..., y) = \varphi(y, y, ..., x) = x \quad \text{for all } x, y \in \{a, b\}.$$

In the second case we say that a, b is a *minority pair* for φ.

Definition 2.1.2. If $R \subseteq \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, then we define the *signature* of R, written $\mathrm{Sig}(R)$, to be the set of triples (i, a, b) with $i \in \{1, ..., n\}$, a, b a minority pair in $\mathbb{A}_i$, such that there are some $t_a, t_b \in R$ with $\pi_{1,...,i-1}(t_a) = \pi_{1,...,i-1}(t_b)$ and $\pi_i(t_a) = a, \pi_i(t_b) = b$. In this case we say that the pair t_a, t_b *witnesses* the triple (i, a, b).

Theorem 2.1.3. *If $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is preserved by an $l+1$-ary gmm operation φ and $S \subseteq \mathbb{R}$ has $\mathrm{Sig}(S) = \mathrm{Sig}(\mathbb{R})$ and $\pi_I(S) = \pi_I(\mathbb{R})$ for all $I \subseteq \{1, ..., n\}$ with $|I| \le l$, then $\mathbb{R}$ is generated by S (using only φ).*

Proof. We prove this by induction on the arity n of $\mathbb{R}$. Suppose that $a = (a_1, ..., a_n) \in \mathbb{R}$, by the induction hypothesis there is some b_n with $(a_1, ..., a_{n-1}, b_n)$ in the subalgebra generated by S. We have two cases, based on whether a_n, b_n is a majority pair or a minority pair.

Case 1: a_n, b_n is a majority pair. In this case we show that for every $I \subseteq \{1, ..., n\}$, we have $\pi_I a$ in the subalgebra generated by $\pi_I S$, by induction on $|I|$. We already know it for $|I| \le l$ and for $n \notin I$. Suppose $I = \{i_1, ..., i_m\}$ with $i_1 < \cdots < i_m = n$ and $m \ge l + 1$. By the inductive hypothesis, there are elements $b_{i_1}, ...$ such that

$$(b_{i_1}, a_{i_2}, ..., a_n), (a_{i_1}, b_{i_2}, ..., a_n), ..., (a_{i_1}, a_{i_2}, ..., b_n) \in \mathrm{Sg}_\varphi(S).$$

If some $b_i = a_i$ then we are done. If some pair a_i, b_i is minority then - assuming WLOG that a_{i_1}, b_{i_1} is minority - we have

$$\varphi\left(\begin{bmatrix} b_{i_1} & \cdots & b_{i_1} & a_{i_1} \\ a_{i_2} & \cdots & a_{i_2} & a_{i_2} \\ \vdots & \ddots & \vdots & \vdots \\ a_n & \cdots & a_n & b_n \end{bmatrix}\right) = \begin{bmatrix} a_{i_1} \\ a_{i_2} \\ \vdots \\ a_n \end{bmatrix} \in \mathrm{Sg}_\varphi(\pi_I S),$$

where all but the last column of the displayed matrix are equal. Otherwise, if all pairs a_i, b_i are majority, then we have

$$\varphi\left(\begin{bmatrix} b_{i_1} & a_{i_1} & \cdots & a_{i_1} \\ a_{i_2} & b_{i_2} & \cdots & a_{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & \cdots & b_n \end{bmatrix}\right) = \begin{bmatrix} a_{i_1} \\ a_{i_2} \\ \vdots \\ a_n \end{bmatrix} \in \mathrm{Sg}_\varphi(\pi_I S),$$

where all of the columns of the displayed matrix are distinct, which is possible because $m \ge l + 1$.

Case 2: a_n, b_n is a minority pair. In this case, by the assumption $\mathrm{Sig}(S) = \mathrm{Sig}(\mathbb{R})$, there are $c, d \in S$ witnessing the triple (n, a_n, b_n). Set $b = (a_1, ..., a_{n-1}, b_n)$, then we claim that

$$a = \varphi(b, b, ..., b, \varphi(b, d, ..., d, c)).$$

First consider the last coordinate: since a_n, b_n is a minority pair and $c_n = a_n, d_n = b_n$, we have

$$\varphi(b_n, ..., b_n, \varphi(b_n, d_n, ..., d_n, c_n)) = \varphi(b_n, ..., b_n, \varphi(b_n, ..., b_n, a_n)) = a_n,$$

so the last coordinates agree. For $i < n$, we have $a_i = b_i$ and $c_i = d_i$, so

$$\varphi(b_i, ..., b_i, \varphi(b_i, d_i, ..., d_i, c_i)) = \varphi(a_i, ..., a_i, \varphi(a_i, c_i, ..., c_i, c_i)) = a_i,$$

where the last equality holds regardless of whether a_i, c_i is a majority pair or a minority pair. □

Definition 2.1.4. A subset $S \subseteq \mathbb{R}$ is called a *compact representation* of a relation $\mathbb{R}$ preserved by an $l + 1$-ary gmm operation if $\mathrm{Sig}(S) = \mathrm{Sig}(\mathbb{R})$, $\pi_I(S) = \pi_I(\mathbb{R})$ for every I with $|I| \le l$, and $|S| \le 2|\mathrm{Sig}(\mathbb{R})| + \sum_{|I| \le l} |\pi_I(\mathbb{R})|$.

In order to manipulate compact representations of relations, we again define subroutines `Nonempty`, `Fix-values`, `Next-beta`, and `Intersect`:

- `Nonempty`$(R, i_1, ..., i_k, \mathbb{S})$ takes R a compact representation of $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $\mathbb{S} \le \mathbb{A}_{i_1} \times \cdots \times \mathbb{A}_{i_k}$, computes the subalgebra generated by $\pi_{i_1,...,i_k}(R)$ under φ, and if this intersects with $\mathbb{S}$, then it returns an element of $\mathbb{R}$ which maps to an element of the intersection,

- `Fix-values`$(R, a_1, ..., a_m)$ takes R a compact representation of $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ and returns a compact representation of the relation $x \in \mathbb{R} \wedge (x_1 = a_1) \wedge \cdots \wedge (x_m = a_m)$ by inductively fixing one coordinate x_i to a_i at a time, and for each new coordinate that is fixed we compute a new compact representation by computing projections onto at most l coordinates using `Nonempty` and computing witnesses for triples in the signature using the proof of Case 2 of Theorem 2.1.3,

- `Next-beta`$(R, i_1, ..., i_k, \mathbb{S})$ takes R a compact representation of $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, $\mathbb{S} \le \mathbb{A}_{i_1} \times \cdots \times \mathbb{A}_{i_k}$, and returns a compact representation of $\mathbb{R} \cap \mathbb{S}$ by computing all projections onto at most l coordinates using `Nonempty` and computing witnesses for triples in the signature using `Fix-values` and `Nonempty`, and

- `Intersect`$(R, i_1, ..., i_k, S)$ takes R a compact representation of $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, S a compact representation of $\mathbb{S} \le \mathbb{A}_{i_1} \times \cdots \times \mathbb{A}_{i_k}$, and computes a compact representation for $\mathbb{R} \cap \mathbb{S}$ by first making a compact representation of $\mathbb{R} \times \mathbb{S}$ and then repeatedly calling `Next-beta` to intersect this with the equality relation on the pair of coordinates $i_j, n + j$.

The only subroutine which has changed substantially from the Mal'cev case is the `Fix-values` subroutine.

Algorithm 8 `Fix-values`$(R, a_1, ..., a_m)$, φ an $l+1$-ary gmm term, R a compact representation of $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$.

```
Set $R_0 \leftarrow R$.
for $j$ from 1 to $m$ do
    Let $R_j \leftarrow \emptyset$.
    for all $I = \{i_1, ...\} \subseteq \{1, ..., n\}$ with $|I| \le l$ and $(b_{i_1}, ...) \in \pi_I(R_{j-1})$ do
        Set $R_j \leftarrow R_j \cup$ Nonempty$(R_{j-1}, j, i_1, ..., i_{|I|}, \{(a_j, b_{i_1}, ..., b_{i_{|I|}})\})$.
    for all $(i, a, b) \in \mathrm{Sig}(R_{j-1})$ with $i > j$ and $a, b$ a minority pair do
        Let $t_a, t_b \in R_{j-1}$ witness the triple $(i, a, b)$.
        Let $t \leftarrow$ Nonempty$(R_{j-1}, j, i, \{(a_j, a)\})$.
        if $t \ne \emptyset$ then
            Set $R_j \leftarrow R_j \cup \{t, \varphi(t, t, ..., t, \varphi(t, t_a, ..., t_a, t_b))\}$.
return $R_m$.
```

Reviewing what we've done, we have a procedure for converting proofs that compact representations generate relations into algorithms for computing compact representations of intersections for relations. The most critical step of the algorithm is the step of the `Fix-values` subroutine in which we convert a pair that witnesses a triple (i, a, b) in R_{j-1} to a pair that witnesses a triple (i, a, b) in R_j.

Before we go on, we can use this algorithm to settle the dichotomy conjecture for constraint languages which contain "swap" relations $\{(a, b), (b, a)\}$ for every pair of elements a, b.

Theorem 2.1.5. *Suppose that $\mathbf{A} = (A, \Gamma)$ is a relational structure where Γ is a set of relations which contains the swap relation $S_{ab} = \{(a, b), (b, a)\}$ for every pair $a, b \in \mathbf{A}$. Then either* $\mathrm{CSP}(\Gamma)$ *is NP-complete, or* $\mathbf{A}$ *has a ternary generalized majority-minority polymorphism. In the second case,* $\mathrm{CSP}(\Gamma)$ *can be solved in polynomial time by Dalmau's algorithm.*

Proof. Note that Γ is automatically core, since any unary polymorphism of S_{ab} must send a, b to distinct values in $\{a, b\}$. Thus if CSP(Γ) is not NP-complete, then it must have a Taylor polymorphism t.

First we will show that this implies that for all $a, b \in \mathbf{A}$ there is a ternary polymorphism f_{ab} such that the restriction of f_{ab} to $\{a, b\}$ is either the majority operation or the minority operation. Since $\pi_1(S_{ab}) = \{a, b\}$, the set $\{a, b\}$ is closed under t. Let $t' \in \mathrm{Clo}(t)$ have minimal arity such that the restriction of t' to $\{a, b\}$ is not a projection. An elementary combinatorial argument known as Świerczkowski's Lemma [178] shows that if t' has arity at least four, then there is some way of identifying two variables of t' to get a term t'' of smaller arity such that the restriction of t'' to $\{a, b\}$ is also not a projection. Thus the arity of t' is at most three. The arity of t' can't be one or two since t' is idempotent and preserves S_{ab}.

Since every way of identifying two variables of $t'|_{\{a,b\}}$ gives a projection, up to reordering the variables of t' there are just three cases. In two of these cases, t' already restricts to a majority or minority operation on $\{a, b\}$. In the remaining case, after reordering the variables we may assume that $t'(x, y, y) = t'(y, y, x) = t'(x, y, x) = x$ for $x, y \in \{a, b\}$, and taking $f_{ab}(x, y, z) = t'(x, t'(x, y, z), z)$ gives a function f_{ab} which restricts to a majority operation on $\{a, b\}$.

Now we choose any ordering of the collection of pairs $\{a, b\}$, with the ith pair given by $\{a_i, b_i\}$. We inductively define functions $f_i \in \mathrm{Clo}(t)$ by $f_0 = \pi_1$, and for $i \geq 0$ we set

$$f_{i+1}(x, y, z) = f_{a_i b_i}(f_i(x, y, z), f_i(y, z, x), f_i(z, x, y)).$$

We claim that the final function f_n (with $n = \binom{|A|}{2}$) is a generalized majority-minority polymorphism of $\mathbf{A}$. Since each f_{ab} is idempotent, it's enough to check that the restriction of f_{i+1} to $\{a_i, b_i\}$ is either a pure majority or pure minority function.

From the fact that f_i preserves the unary relation $\pi_1(S_{a_i b_i}) = \{a_i, b_i\}$ and the fact that the restriction of $f_{a_i b_i}$ to $\{a_i, b_i\}$ is invariant under cyclically permuting its input variables, we see that f_{i+1} also restricts to a cyclic term on $\{a_i, b_i\}$. Since f_{i+1} preserves S_{ab}, it must therefore either restrict to the pure majority or pure minority function on $\{a_i, b_i\}$. □

There are two examples of generalized majority-minority algebras on a three element domain which do not come from majority or Mal'cev operations, and correspond to maximal tractable constraint languages.

Example 2.1.1. The first example is $\mathbb{A}_1 = (\{a, b, c\}, \varphi_1)$, where φ_1 is a ternary gmm such that $\{a, x\}$ is a pure minority subalgebra of $\mathbb{A}_1$ for all x, $\{b, c\}$ is a majority subalgebra of $\mathbb{A}_1$, and the equivalence relation corresponding to the partition $\{a\}, \{b, c\}$ is a congruence α on $\mathbb{A}_1$ such that the quotient $\mathbb{A}_1/\alpha$ is a pure minority algebra. Explicitly, φ_1 is the symmetric idempotent function of its inputs which is given by

$$\varphi_1(a, a, x) = x, \varphi_1(a, x, x) = a, \varphi_1(b, b, c) = b, \varphi_1(b, c, c) = c, \varphi_1(a, b, c) = a.$$

The corresponding relational clone is generated by the partial order $\{(a, a), (b, b), (b, c), (c, c)\}$, the order two automorphism $\{(a, a), (b, c), (c, b)\}$, and the affine ternary relation $\{(a, a, b), (a, b, a), (b, a, a), (b, b, b)\}$.

Example 2.1.2. The second example is $\mathbb{A}_2 = (\{a, b, c\}, \varphi_2)$, where φ_2 is a ternary gmm such that $\{a, x\}$ is a majority subalgebra of $\mathbb{A}_2$ for all x, $\{b, c\}$ is a pure minority subalgebra of $\mathbb{A}_2$, the equivalence relation corresponding to the partition $\{a\}, \{b, c\}$ is a congruence α on $\mathbb{A}_2$ such that the quotient $\mathbb{A}_2/\alpha$ is a majority algebra, and the permutation $(b\ c)$ is an automorphism of $\mathbb{A}_2$. Explicitly, φ_2 is the cyclically symmetric idempotent function of its inputs which is given by

$$\varphi_2(a,a,x) = a, \varphi_2(a,x,x) = x, \varphi_2(b,b,c) = c, \varphi_2(b,c,c) = b, \varphi_2(a,b,c) = b, \varphi_2(a,c,b) = c.$$

The corresponding relational clone is generated by the binary relations $\{(a,b),(b,a)\}$, $\{(a,a),(a,b),(b,b)\}$, $\{(a,a),(b,c),(c,b)\}$ and the ternary relation $\{(a,a,a),(b,b,b),(b,c,c),(c,b,c),(c,c,b)\}$.

The reader might notice that generalized majority-minority operations are *not* defined in terms of satisfying a system of identities. So we should be able to immediately generalize Dalmau's result to the variety of algebras generated by algebras with a gmm operation, by finding the identities which are satisfied by a gmm operation that were critical to the correctness of the algorithm. How did we apply the operation φ, throughout the algorithm `Fix-values` and the proof of Theorem 2.1.3?

The first thing to note is that we often set almost all of the entries of φ to the same value. So define auxiliary binary and ternary terms p, d by

$$\begin{aligned} d(x,y) &= \varphi(x,y,...,y,y),\\ p(x,y,z) &= d(\varphi(x,y,...,y,z),z). \end{aligned}$$

The important property of d is that we have $d(a,b) = a$ when a, b are a minority pair. For p, the important property is that when a, b are a minority pair, then we have $p(a,b,b) = a$, and in every case we always have

$$p(y,y,z) = z.$$

We can express the fact that $p(a,b,b) = a$ when a, b are a minority pair by the equation

$$p(x,y,y) = d(x,y),$$

which also holds for majority pairs.

Where did we actually use the function φ? It is only called directly in the subroutine `Nonempty`. It is crucial that it is actually used there, because the full function φ was necessary for Case 1 of Theorem 2.1.3. The proof of that case does not immediately appear to generalize, as there was substantial casework within it, based on whether there was a minority pair a_i, b_i or not. However, clever use of the function $d(x,y)$ can mimic the casework that appeared there. For each a_i, b_i, the expression $d(a_i, b_i)$ has the nice property that $a_i, d(a_i, b_i)$ automatically forms a majority pair (or an equal pair, which we can think of as a degenerate case of a majority pair). So if we define a function $s(x_0, x_1, ..., x_l)$ by

$$s(x_0,x_1,...,x_l) = \varphi(x_0, d(x_0,x_1), ..., d(x_0,x_l)),$$

then we find that

$$\begin{aligned} s(y,x,x,...,x) &= \varphi(y,d(y,x),...,d(y,x)) = d(y,x),\\ s(x,y,x,...,x) &= \varphi(x,d(x,y),x,...,x) = x,\\ s(x,x,y,...,x) &= \varphi(x,x,d(x,y),...,x) = x,\\ &\vdots\\ s(x,x,x,...,y) &= \varphi(x,x,x,...,d(x,y)) = x. \end{aligned}$$

This function s lets us generalize Case 1 of Theorem 2.1.3, the case where $d(a_n, b_n) = b_n$, while the function p was necessary to generalize Case 2. To unify them, we should slightly modify our construction of s to create the following term e:

$$e(u, v, x_1, ..., x_l) = \varphi(v, d(u, x_1), ..., d(u, x_{l-1}), d(x_1, x_l)).$$

Then s is related to e by

$$s(x_0, x_1, ..., x_l) = e(x_1, x_0, x_1, ..., x_l) \text{ if all but one of the } x_i \text{ are equal,}$$

p is related to e by

$$p(x, y, z) = e(y, x, z, ..., z) \text{ if } x = y \text{ or } y = z,$$

and e satisfies the identities

$$\begin{aligned}
e(y, y, x, x, ..., x) &= \varphi(y, d(y, x), ..., d(y, x), x) = x,\\
e(y, x, y, x, ..., x) &= \varphi(x, y, d(y, x), ..., d(y, x)) = x,\\
e(x, x, x, y, ..., x) &= \varphi(x, x, d(x, y), ..., x) = x,\\
&\vdots\\
e(x, x, x, x, ..., y) &= \varphi(x, x, x, ..., d(x, y)) = x.
\end{aligned}$$

Can we use this system of identities to prove an analogue of Theorem 2.1.3? Yes! The trick is to plug things back into e, to make the following term t:

$$t(u, v, w, x_1, ..., x_l) = e(p(v, u, x_1), s(w, x_1, ..., x_l), x_1, ..., x_l).$$

Now if we have a tuple $a = (a_1, ..., a_n)$ which we want to prove is in the subalgebra generated by S, and if this subalgebra already contains $(a_1, ..., a_{i-1}, b_i, a_{i+1}, ..., a_n)$ for each i, as well as a pair $(c_1, ..., c_{n-1}, a_n), (c_1, ..., c_{n-1}, b_n)$ which witnesses the triple (n, a_n, b_n), then we have

$$t\left(\begin{bmatrix} c_1 & c_1 & a_1 & b_1 & a_1 & \cdots \\ c_2 & c_2 & a_2 & a_2 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ a_n & b_n & b_n & a_n & a_n & \cdots \end{bmatrix}\right) = e\left(\begin{bmatrix} b_1 & a_1 & b_1 & a_1 & \cdots \\ a_2 & a_2 & a_2 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ d_n & d_n & a_n & a_n & \cdots \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix},$$

where $d_n = d(b_n, a_n)$.

While playing these sorts of games with identities may yield more and more general examples of algebraic structures where relations have compact representations, we are not being very systematic here. So perhaps we should work backwards: what absolutely needs to be true for something like compact representations to exist?

Proposition 2.1.6. *If every subpower $\mathbb{R} \le \mathbb{A}^n$ has a compact representation S consisting of at most $p(n)$ tuples, then the number of different subalgebras of $\mathbb{A}^n$ is at most $|\mathbb{A}^n|^{p(n)} = |\mathbb{A}|^{np(n)}$.*

Corollary 2.1.7. *No analogue of compact representations can exist for subpowers of a nontrivial semilattice.*

Proof. It's enough to consider the case $\mathbb{A} = (\{0,1\}, \max)$, since every semilattice contains a subalgebra isomorphic to it. The number of subpowers of $\mathbb{A}^n$ is at least the number of subsets on $\{0,1\}^n$ which are generated by subsets $S \subseteq \{x \in \{0,1\}^n, \sum_i x_i = n/2\}$ (suppose n is even). Any two distinct subsets S, S' of the set of tuples with weight $n/2$ will generate different subalgebras of $\mathbb{A}^n$, so the number of subalgebras of $\mathbb{A}^n$ is at least

$$2^{\binom{n}{n/2}} \geq 2^{2^n/n},$$

and $2^n/n$ clearly grows faster than any polynomial. □

What makes the semilattice case so different from the Mal'cev case and the near-unanimity case? The main difference is that the identities satisfied by a semilattice do not allow us to get back to x once we start combining it with other values, while the identities for Mal'cev and near-unanimity terms all have xs on the right hand sides.

So we should start by trying to prove that having few subpowers implies that there are terms satisfying a nontrivial system of identities which have xs on the right hand sides of each identity, such as the system of identities satisfied by the term e constructed earlier. The trick, as we will see, is to apply the existence of compact representations to the case of a power of the free algebra on two generators, considered as a subalgebra of $(\mathbb{A}^{\mathbb{A}^2})^n$.

2.2 Algebras with Few Subpowers

First we define an invariant of an algebraic structure and the variety it generates, which is slightly more well-behaved than the function that takes n to the number of subalgebras of $\mathbb{A}^n$.

Definition 2.2.1. If $\mathbb{A}$ is an algebraic structure and $a_1, ..., a_k \in \mathbb{A}$, we say that $a_1, ..., a_k$ are *independent* if no a_i is in the subalgebra generated by the rest of the a_js. For every n, we define $i_{\mathbb{A}}(n)$ to be the size of the largest independent set in $\mathbb{A}^n$.

Proposition 2.2.2. *If $\mathbb{A}$ is a finite algebra, then any subalgebra of $\mathbb{A}^n$ can be generated by at most $i_{\mathbb{A}}(n)$ elements, so the number of subalgebras of $\mathbb{A}^n$ is bounded above by $|\mathbb{A}^n|^{i_{\mathbb{A}}(n)} = 2^{n\lg(|\mathbb{A}|)i_{\mathbb{A}}(n)}$. The number of subalgebras of $\mathbb{A}^n$ is also bounded below by $2^{i_{\mathbb{A}}(n)}$.*

Proof. Since $\mathbb{A}$ is finite, every subalgebra of $\mathbb{A}^n$ has a minimal generating set, and this minimal generating set is necessarily independent. The upper bound on the number of subalgebras follows from counting the number of possible minimal generating sets.

For the lower bound on the number of subalgebras, suppose that $a_1, ..., a_k$ are independent in $\mathbb{A}^n$. Then every subset S of $\{a_1, ..., a_k\}$ generates a distinct subalgebra of $\mathbb{A}^n$, since $\mathrm{Sg}_{\mathbb{A}^n}(S) \cap \{a_1, ..., a_k\} = S$ by the definition of independence. Thus $\mathbb{A}^n$ has at least 2^k distinct subalgebras. □

Proposition 2.2.3. *If $\mathbb{B} \in HSP(\mathbb{A})$ is also finite, then $i_{\mathbb{B}}(n) \leq i_{\mathbb{A}}(cn)$ for some constant c depending only on $\mathbb{B}$.*

Proof. If $\mathbb{A}, \mathbb{B}$ are both finite, then there is some finite number c such that $\mathbb{B} \in HS(\mathbb{A}^c)$, that is, there is a subalgebra $\mathbb{C} \leq \mathbb{A}^c$ and a surjective homomorphism $f : \mathbb{C} \to \mathbb{B}$. Then every independent set in $\mathbb{B}^n$ lifts to an independent set in $(\mathbb{A}^c)^n = \mathbb{A}^{cn}$ by choosing any section of f and applying it coordinate-wise. □

We will apply the above result to the free algebra on two generators $\mathcal{F}_{\mathcal{V}(\mathbb{A})}(x,y) \le \mathbb{A}^{\mathbb{A}^2}$ to prove that if an algebra has few subpowers, then it has a *cube term*. Since cube terms have exponentially high arity, it's necessary to develop some notation to define them properly.

Definition 2.2.4. For every subset $S \subseteq \{1,...,k\}$, we define the k-dimensional column vector v^S by

$$v_i^S = \begin{cases} y & i \in S, \\ x & i \not\in S. \end{cases}$$

A *k-cube term* is a term t with variables indexed by nonempty subsets of $\{1,...,k\}$, such that if we fix an enumeration $S_1,...,S_{2^k-1}$ of these subsets, we have the identity

$$t(v^{S_1},...,v^{S_{2^k-1}}) \approx v^{\emptyset}.$$

For instance, if $k = 3$ then (with one possible choice of variable ordering) a 3-cube term is a 7-ary term t satisfying the identity

$$t\left(\begin{bmatrix} y & y & y & x & y & x & x \\ y & y & x & y & x & y & x \\ y & x & y & y & x & x & y \end{bmatrix}\right) \approx \begin{bmatrix} x \\ x \\ x \end{bmatrix}.$$

Note that a Mal'cev term is the same as a 2-cube term (up to reordering variables).

Theorem 2.2.5 (Few subpowers implies cube term [26])**.** *Let $\mathbb{F} = \mathcal{F}_{\mathcal{V}(\mathbb{A})}(x,y) \le \mathbb{A}^{\mathbb{A}^2}$ be the free algebra on two generators in the variety generated by $\mathbb{A}$.*

- *If $i_{\mathbb{F}}(k) < 2^k$ for any k, then $\mathbb{A}$ has a k-cube term.*
- *If $i_{\mathbb{F}}(m) < \binom{m}{k}$ for any m, k, then $\mathbb{A}$ has a k-cube term.*

In particular, if $i_{\mathbb{A}}(n) = o(n^k)$ then $\mathbb{A}$ has a k-cube term, and if $i_{\mathbb{A}}(n) = 2^{o(n)}$ then there exists some k such that $\mathbb{A}$ has a k-cube term.

Proof. For the first statement, if $i_{\mathbb{F}}(k) < 2^k$, then the vectors v^S for $S \subseteq \{1,...,k\}$ can't be independent, so some v^S is in the subalgebra generated by the others. By applying an automorphism of $\mathbb{F}^k$ which swaps xs and ys in the coordinates belonging to S, we may assume without loss of generality that $S = \emptyset$. From $v^{\emptyset} \in \mathrm{Sg}_{\mathbb{F}^k}\{v^S \mid S \ne \emptyset\}$, we see that there is a term t such that $t(v^{S_1},...) = v^{\emptyset}$, and since $\mathbb{F}$ is the free algebra on two generators, this implies the k-cube term identities.

For the second statement, consider the set of vectors v^S with $S \in \binom{\{1,...,m\}}{k}$. By assumption, these are not independent, so some v^S is in the subalgebra generated by the others. Then if we project onto the coordinates of S and use the fact that for $S \ne T$ with $|S| = |T|$ we never have $S \subseteq T$, we get the situation of the previous paragraph inside $\mathbb{F}^S \cong \mathbb{F}^k$. □

Next, we upgrade the k-cube term by repeatedly plugging it into itself to produce simpler terms, finally arriving at the k-edge term.

Definition 2.2.6. If $\Delta \subseteq \mathcal{P}(\{1,...,k\}) \setminus \{\emptyset\}$, then we say that t is a Δ-*cube* term if it has variables indexed by elements of Δ and satisfies the identity $t(v^{S_1},...) = v^{\emptyset}$, where $S_1,...$ is an enumeration of the elements of Δ.

If we set $\Delta^e = \{\{1,2\},\{1\},\{2\},...,\{k\}\}$, then a Δ^e-cube term is called a *k-edge* term.

A k-edge term is simple enough that we can write out the identities it satisfies explicitly: a $k+1$-ary term e is a k-edge term iff it satisfies

$$e\left(\begin{bmatrix} y & y & x & x & \cdots & x \\ y & x & y & x & \cdots & x \\ x & x & x & y & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & x & x & \cdots & y \end{bmatrix}\right) \approx \begin{bmatrix} x \\ x \\ x \\ \vdots \\ x \end{bmatrix}.$$

Theorem 2.2.7 (Cube term implies edge term [26])**.** *If $\mathbb{A}$ has a k-cube term, then it also has a k-edge term.*

Proof. Since it is hard to deal with terms having exponentially many variables, we will do the last step of the proof first, and show that if $\mathbb{A}$ has a Δ^*-cube term t^* then it has a k-edge term, where

$$\Delta^* = \{\{1,2\}, ..., \{1,k\}, \{1\}, \{2\}, ..., \{k\}\}$$

only has $2k-1$ elements. The Δ^*-cube term identities for t^* state that

$$t^*\left(\begin{bmatrix} y & y & \cdots & y & y & x & x & \cdots & x \\ y & x & \cdots & x & x & y & x & \cdots & x \\ x & y & \cdots & x & x & x & y & \cdots & x \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & \cdots & y & x & x & x & \cdots & y \end{bmatrix}\right) \approx \begin{bmatrix} x \\ x \\ x \\ \vdots \\ x \end{bmatrix}.$$

In order to show that there is a k-edge term, we just need to show that $v^\emptyset$ can be generated from $\{v^S \mid S \in \Delta^e\}$ using the Δ^*-cube term t^*.

Let $a = t^*(x, ..., x, y, x, ..., x)$, where the only y occurs at the index corresponding to $\{1\}$ (this is the middle index if we order the variables of t as in the displayed indentities above). First we will use t to generate vectors $v^{S,a}$ for $S \in \Delta^*$ which look just like the vectors v^S, except ys in the first coordinate are replaced by as. If $S \in \Delta^*$ and $1 \not\in S$, then S is already in Δ^e and $v^{S,a} = v^S$, so we don't have to worry about these. If $S = \{1\}$, then we use

$$t^*\left(\begin{bmatrix} x & x & \cdots & x & y & x & x & \cdots & x \\ y & x & \cdots & x & x & y & x & \cdots & x \\ x & y & \cdots & x & x & x & y & \cdots & x \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & \cdots & y & x & x & x & \cdots & y \end{bmatrix}\right) = \begin{bmatrix} a \\ x \\ x \\ \vdots \\ x \end{bmatrix},$$

and note that every column of the matrix on the left hand side is v^S for some $S \in \Delta^e$. If $S = \{1,2\}$, then we use

$$t^*\left(\begin{bmatrix} x & x & \cdots & x & y & x & x & \cdots & x \\ y & y & \cdots & y & y & y & y & \cdots & y \\ x & x & \cdots & x & x & x & x & \cdots & x \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & \cdots & x & x & x & x & \cdots & x \end{bmatrix}\right) = \begin{bmatrix} a \\ y \\ x \\ \vdots \\ x \end{bmatrix},$$

again noting that every column corresponds to an element of Δ^e. Finally, if $S = \{1, i\}$, say $S = \{1, 3\}$ without loss of generality, then we use

$$t^* \left(\begin{bmatrix} x & x & \cdots & x & y & x & x & \cdots & x \\ y & y & \cdots & y & y & x & x & \cdots & x \\ x & x & \cdots & x & x & y & y & \cdots & y \\ x & x & \cdots & x & x & x & x & \cdots & x \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & \cdots & x & x & x & x & \cdots & x \end{bmatrix} \right) = \begin{bmatrix} a \\ x \\ y \\ x \\ \vdots \\ x \end{bmatrix},$$

where every row other than the first three (or other than the first, second, and ith in the general case) is all xs, and again every column belongs to Δ^e.

Now that we've constructed the $v^{S,a}$s for all $S \in \Delta^*$, we use t^* to put them all together:

$$t^* \left(\begin{bmatrix} a & a & \cdots & a & a & x & x & \cdots & x \\ y & x & \cdots & x & x & y & x & \cdots & x \\ x & y & \cdots & x & x & x & y & \cdots & x \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & \cdots & y & x & x & x & \cdots & y \end{bmatrix} \right) = \begin{bmatrix} x \\ x \\ x \\ \vdots \\ x \end{bmatrix}.$$

Thus if $\mathbb{A}$ has a Δ^*-cube term, then it has a k-edge term. Explicitly, the construction we just worked through corresponds to the formula

$$\begin{aligned} e(x_0, x_1, ..., x_k) = t^*(&t^*(x_2, ..., x_2, x_0, x_2, ..., x_2), t^*(x_2, ..., x_2, x_0, x_3, ..., x_3), ..., \\ &t^*(x_2, ..., x_2, x_0, x_k, ..., x_k), t^*(x_2, ..., x_k, x_1, x_2, ..., x_k), x_2, ..., x_k). \end{aligned}$$

Now that we have the general idea down, we work through the inductive argument needed to prove that if we have a k-cube term, then we have a Δ^*-cube term. Let $\Delta^{\ell *} = \Delta^* \cup \mathcal{P}(\{1, ..., \ell\}) \setminus \emptyset$. Note that a k-cube term is the same as a Δ^{k*}-cube term, and a Δ^*-cube term is the same as a Δ^{0*}-cube term.

Claim: If $\mathbb{A}$ has a $\Delta^{\ell *}$-cube term t^ℓ, then it also has a $\Delta^{(\ell-1)*}$-cube term.

Proof of Claim: We argue as before, this time taking $a = t^\ell(x, ..., x, y, x, ..., x)$, where the lone y occurs in the index corresponding to $\{\ell\}$. For $S \in \Delta^{\ell *}$, we let $v^{S,a}$ be the vector similar to v^S, but with any y in the ℓth coordinate replaced with an a. We just need to generate each $v^{S,a}$ for $S \in \Delta^{\ell *}$ using the vectors coming from $\Delta^{(\ell-1)*}$. Again, if $\ell \notin S$ then $v^{S,a} = v^S$ and $S \in \Delta^{(\ell-1)*}$ already.

If $S = \{\ell\}$, then we plug in the matrix M to t^ℓ which looks just like the matrix which gives the defining identities for t^ℓ, but has the ℓth row replaced by the sequence of xs and ys we used to define a. Explicitly, M is given by

$M_{i,T}$	$T \neq \{\ell\}$	$T = \{\ell\}$
$i \neq \ell$	v_i^T	$v_i^T = x$
$i = \ell$	x	$y.$

Then $t^\ell(M) = v^{\{\ell\},a}$, and the Tth column of M is $v^{T \setminus \{\ell\}}$ if $T \neq \{\ell\}$ and is $v^{\{\ell\}}$ if $T = \{\ell\}$.

If $\ell \in S$ but $S \neq \{\ell\}$, then we plug in a matrix M^S such that each of its columns is equal to one of $v^{S\setminus\{\ell\}}, v^{\{1\}}, v^{\{1,\ell\}}$: if $\ell \notin T$, then the Tth column of M^S is $v^{S\setminus\{\ell\}}$, if $\ell \in T$ but $T \neq \{\ell\}$ then the Tth column is $v^{\{1\}}$, and if $T = \{\ell\}$ then the Tth column is $v^{\{1,\ell\}}$. Explicitly, M^S is given by

$M^S_{i,T}$	$\ell \notin T$	$\ell \in T \neq \{\ell\}$	$T = \{\ell\}$
$i = 1 \in S$	y	y	y
$i = 1 \notin S$	x	y	y
$i \neq 1, \ell,\ i \in S$	y	x	x
$i \neq 1, \ell,\ i \notin S$	x	x	x
$i = \ell$	x	x	$y.$

These choices ensure that $t^\ell(M^S) = v^{S,a}$.

To finish, we apply t^ℓ to the set of vectors $v^{S,a}$ for $S \in \Delta^{\ell *}$, and see that the defining identities for t^ℓ imply that the resulting vector is $v^\emptyset$. Thus there is a $\Delta^{(\ell-1)*}$-cube term $t^{\ell-1}$ which can in principle be written explicitly by plugging in variables to the star composition $t^\ell * t^\ell$. □

From a k-edge term e, we can now construct terms s, p that act like near-unanimity and Mal'cev terms which have been "glued together" by a binary term d. I've rearranged the variables of these terms from the notation used in [26], for the sake of readability and for consistency with the notation used in Appendix A.

Theorem 2.2.8 (Edge terms imply terms s, p, d [26])**.** *If e is a k-edge term on a finite algebra $\mathbb{A}$, then there are terms $s, p, d \in \mathrm{Clo}(e)$ with s k-ary which satisfy the system of identities*

$$\begin{aligned}
s(y, x, x, ..., x) &\approx d(y, x),\\
s(x, y, x, ..., x) &\approx x,\\
&\vdots\\
s(x, x, x, ..., y) &\approx x,\\
p(y, y, x) &\approx x,\\
p(x, y, y) &\approx d(x, y),\\
d(d(x, y), y) &\approx d(x, y).
\end{aligned}$$

Furthermore, these terms can be computed from e in time $O(|\mathbb{A}|^k)$. If $\mathbb{A}$ is infinite, then we can find terms $s, p, d \in \mathrm{Clo}(e)$ satisfying all but the last displayed identity.

Proof. If we ignore the last identity involving d, we can find terms s_1, p_1, d_1 satisfying the other identities as follows:

$$\begin{aligned}
s_1(x_1, x_2, ..., x_k) &= e(x_2, x_1, x_2, ..., x_k),\\
p_1(x, y, z) &= e(y, x, z, ..., z),\\
d_1(x, y) &= e(y, x, y, ..., y).
\end{aligned}$$

We can get the last identity by an iteration argument. For each i, we set

$$\begin{aligned}
s_{i+1}(x_1, x_2, ..., x_k) &= s_1(s_i(x_1, x_2, ..., x_k), x_2, ..., x_k),\\
p_{i+1}(x, y, z) &= p_1(d_i(x, y), y, z),\\
d_{i+1}(x, y) &= d_1(d_i(x, y), y).
\end{aligned}$$

Then for each i, the terms s_i, p_i, d_i satisfy the desired identities aside from the last one. Since $\mathbb{A}$ is finite, we can take $i = |\mathbb{A}|!$ to find that

$$d_{|\mathbb{A}|!}(d_{|\mathbb{A}|!}(x,y),y) = d_{|\mathbb{A}|!}(x,y)$$

for all $x, y \in \mathbb{A}$.

To compute $s_{|\mathbb{A}|!}$ efficiently from e, first we compute s_1, and then for each choice of $a_2, ..., a_k \in \mathbb{A}$ we find the induced unary polynomial $f_{a_2,...,a_k} : x_1 \mapsto s_1(x_1, a_2, ..., a_k)$. To finish, we note that for every unary function $f : \mathbb{A} \to \mathbb{A}$ we can compute $f^\infty := \lim_{n\to\infty} f^{\circ n!}$ in time $O(|\mathbb{A}|)$ using a clever algorithm which we will go over later, but which the reader may enjoy trying to discover now as an exercise. □

Now we can use the binary term d to define minority pairs and signatures.

Definition 2.2.9. If s, p, d are terms as in the Theorem 2.2.8, then we say that $a, b \in \mathbb{A}$ are a *minority pair* if $d(b,a) = b$. If $R \subseteq \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, then we say that (i, a, b) is a *minority index* of R which is *witnessed* by a pair $t_a, t_b \in R$ if:

- a, b are a minority pair, i.e. $d(b,a) = b$,
- the pair t_a, t_b agree up to coordinate i: $\pi_{1,...,i-1}(t_a) = \pi_{1,...,i-1}(t_b)$, and
- we have $\pi_i(t_a) = a, \pi_i(t_b) = b$.

We define the *signature* of R, written $\mathrm{Sig}(R)$, to be the set of minority indices which are witnessed by pairs in R.

Definition 2.2.10. If $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ and the $\mathbb{A}_i$ are in a variety with a k-edge term, then we say that a set $S \subseteq \mathbb{R}$ is a *compact representation* of $\mathbb{R}$ if:

- $\mathrm{Sig}(S) = \mathrm{Sig}(\mathbb{R})$,
- for every $I \subseteq \{1, ..., n\}$ with $|I| \le k-1$ we have $\pi_I(S) = \pi_I(\mathbb{R})$, and
- $|S| \le 2|\mathrm{Sig}(\mathbb{R})| + \sum_{I\subseteq\{1,...,n\}, |I|\le k-1} |\pi_I(\mathbb{R})|$.

Theorem 2.2.11 (Subpowers with edge terms are generated by compact representations [26])**.** *If $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ and the $\mathbb{A}_i$ are finite algebras in a variety with a k-edge term e, then for any compact representation S of $\mathbb{R}$, we have $\mathbb{R} = \mathrm{Sg}_e(S)$.*

Proof. Let s, p, d be terms as in Theorem 2.2.8. We induct on n. Suppose $a = (a_1, ..., a_n) \in \mathbb{R}$, then by the induction hypothesis there is $b_n \in \mathbb{A}_n$ with $(a_1, ..., a_{n-1}, b_n) \in \mathrm{Sg}_e(S)$. Then if we let $d_n = d(b_n, a_n)$ then we see that a_n, d_n is a minority pair and $(a_1, ..., a_n, d_n) \in \mathbb{R}$, so $(n, a_n, d_n) \in \mathrm{Sig}(\mathbb{R})$, and from the definition of a compact representation we see that there must be some $c_1, ..., c_{n-1}$ such that

$$(c_1, ..., c_{n-1}, a_n), (c_1, ..., c_{n-1}, d_n) \in S.$$

We show by an inner induction on subsets $I \subseteq \{1, ..., n\}$ that for each I, we have $\pi_I(a) \in \pi_I(\mathrm{Sg}_e(S))$. If $|I| \le k-1$ this follows from the definition of a compact representation, while if $n \notin I$ then this follows from the outer inductive hypothesis. For the sake of notational simplicity we will assume that $I = \{1, ..., n\}$. Then by the inductive hypothesis, there are $b_1, ..., b_{n-1}$ such that for each i, we have

$$(a_1, ..., a_{i-1}, b_i, a_{i+1}, ..., a_n) \in \mathrm{Sg}_e(S).$$

Then we have

$$s\left(\begin{bmatrix} a_1 & b_1 & a_1 & \cdots \\ a_2 & a_2 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ b_n & a_n & a_n & \cdots \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ d_n \end{bmatrix} \in \mathrm{Sg}_e(S).$$

Additionally, we have

$$p\left(\begin{bmatrix} c_1 & c_1 & b_1 \\ c_2 & c_2 & a_2 \\ \vdots & \vdots & \vdots \\ d_n & a_n & a_n \end{bmatrix}\right) = \begin{bmatrix} b_1 \\ a_2 \\ \vdots \\ d_n \end{bmatrix} \in \mathrm{Sg}_e(S).$$

Now we can apply the k-edge term e to see that

$$e\left(\begin{bmatrix} b_1 & a_1 & b_1 & a_1 & \cdots \\ a_2 & a_2 & a_2 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ d_n & d_n & a_n & a_n & \cdots \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathrm{Sg}_e(S). \qquad \square$$

Corollary 2.2.12. *For a fixed finite algebra $\mathbb{A}$:*

- *$\mathbb{A}$ has a k-edge term but no $k-1$-edge term iff $i_{\mathbb{A}}(n) = \Theta(n^{k-1})$, and*
- *$\mathbb{A}$ has no k-edge term for any k iff $i_{\mathbb{A}}(n) = 2^{\Theta(n)}$.*

Proof. We only need to check that if $\mathbb{A}$ has a k-edge term, then $i_{\mathbb{A}}(n) = O(n^{k-1})$. Suppose that $a_1, ..., a_m \in \mathbb{A}^n$ are independent, and consider the relations $\mathbb{R}_i = \mathrm{Sg}_{\mathbb{A}^n}\{a_1, ..., a_i\}$. We can easily find a sequence of compact representations $S_1, ..., S_m$ of $\mathbb{R}_1, ..., \mathbb{R}_m$ with $S_i \subseteq S_{i+1}$ for each i. From the independence of the a_is, we have $\mathbb{R}_i \ne \mathbb{R}_{i+1}$ for all i, so by induction we see that $|S_i| \ge i$ for all i. Then from the fact that S_m is a compact representation, we have

$$m \le |S_m| \le 2n|\mathbb{A}|^2 + \sum_{I \subseteq \{1,...,n\}, |I| \le k-1} |\mathbb{A}|^{k-1} = O(n^{k-1}). \qquad \square$$

We can now generalize Dalmau's generalized majority-minority algorithm to an algorithm for computing compact representations of intersections of two relations which are both described by compact representations. The only changes we need to make are to use the edge term e in the `Nonempty` subroutine in the place of the gmm term φ, and to modify the `Fix-values` subroutine to use the ternary term p from Theorem 2.2.8.

That the modified `Fix-values` subroutine works follows from the following Proposition.

Proposition 2.2.13. *If the pair of tuples t_a, t_b witness the minority index (i, a, b), then for any t with $\pi_i(t) = a$ the pair of tuples $t, p(t_b, t_a, t)$ also witnesses the minority index (i, a, b).*

Algorithm 9 `Fix-values`$(R, a_1, ..., a_m)$, p, d terms as in Theorem 2.2.8, R a compact representation of $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$.

```
Set $R_0 \leftarrow R$.
for $j$ from 1 to $m$ do
    Let $R_j \leftarrow \emptyset$.
    for all $I = \{i_1, ...\} \subseteq \{1, ..., n\}$ with $|I| < k$ and $(b_{i_1}, ...) \in \pi_I(R_{j-1})$ do
        Set $R_j \leftarrow R_j \cup$ Nonempty$(R_{j-1}, j, i_1, ..., i_{|I|}, \{(a_j, b_{i_1}, ..., b_{i_{|I|}})\})$.
    for all $(i, a, b) \in \mathrm{Sig}(R_{j-1})$ with $i > j$ and $a, b$ a minority pair (i.e. $d(b, a) = b$) do
        Let $t_a, t_b \in R_{j-1}$ witness the triple $(i, a, b)$.
        Let $t \leftarrow$ Nonempty$(R_{j-1}, j, i, \{(a_j, a)\})$.
        if $t \neq \emptyset$ then
            Set $R_j \leftarrow R_j \cup \{t, p(t_b, t_a, t)\}$.
return $R_m$.
```

Proof. From the identity $p(y, y, x) \approx x$ we have

$$\pi_{<i}(p(t_b, t_a, t)) = \pi_{<i}(p(t_a, t_a, t)) = \pi_{<i}(t),$$

and since (a, b) is a minority pair, we have

$$\pi_i(p(t_b, t_a, t)) = p(b, a, a) = d(b, a) = b.$$

□

Example 2.2.1. There is an example of an algebra $\mathbb{A} = (\{a, b, c\}, g)$ with g a ternary operation such that $\mathbb{A}$ has a 3-edge term, but is not in the variety generated by generalized majority-minority algebras of any arity (up to term equivalence). The ternary operation g is the idempotent symmetric function given by

$$g(a, b, b) = b, g(a, a, b) = a, g(a, c, c) = a, g(a, a, c) = c, g(b, c, c) = a, g(b, b, c) = c, g(a, b, c) = c.$$

You can understand this as follows: the subset $\{a, b\}$ is a majority subalgebra, the subset $\{a, c\}$ is a pure minority subalgebra, and there is a congruence with equivalence classes $\{a, b\}, \{c\}$ so that the quotient is a pure minority algebra. Also, the only way to get b out of an application of g is if at least two of the inputs are bs (this property is called “absorption”: the subalgebra $\{a, c\}$ absorbs $\{a, b, c\}$ with respect to g).

To see that this isn't in the variety generated by generalized majority-minority algebras, recall that in any gmm algebra there are functions s, p, d as in Theorem 2.2.8, where d satisfies the additional identity $d(x, d(y, x)) \approx x$ since d either acts as first or second projection for any particular pair x, y. Since the quotient corresponding to $\{a, b\}, \{c\}$ is a pure minority algebra, we must have $d(c, b) = c$, so by the extra identity we have $d(b, c) = d(b, d(c, b)) = b$. Then the function p would satisfy

$$p\left(\begin{bmatrix} b & c & c \\ c & c & b \end{bmatrix}\right) = \begin{bmatrix} d(b, c) \\ b \end{bmatrix} \overset{?}{=} \begin{bmatrix} b \\ b \end{bmatrix}.$$

But this is impossible: the subalgebra of $\mathbb{A}^2$ generated by $(b, c), (c, c), (c, b)$ doesn't contain (b, b), because of the absorption property of $\{a, c\}$ with respect to g.

To see that $\mathbb{A}$ has a 3-edge term, we define an auxiliary 4-ary term f by

$$f(u, x, y, z) = g(g(u, x, z), g(u, y, z), g(u, z, z)),$$

and then define our 3-edge term by

$$e(u,x,y,z) = g(g(f(u,x,y,z),x,x),g(f(u,x,y,z),y,y),g(f(u,x,y,z),z,z)).$$

If we define functions s,p,d from the 3-edge term e as in Theorem 2.2.8, then d is given by

$d(x,y)$	a	b	c
a	a	b	a
b	a	b	a
c	c	c	c

and the minority pairs are $(a,c),(c,a),(b,c)$. The fact that (c,b) is *not* a minority pair is witnessed by the fact that the relation $\mathrm{Sg}_{\mathbb{A}^2}\{(b,c),(c,c),(c,b)\}$ contains (b,c) but does not contain (b,b), even though it has $(2,b,c)$ in its signature.

The associated relational clone is generated by the order two automorphism $\{(a,b),(b,a)\}$ of $\{a,b\}$, the partial order $\{(a,a),(a,b),(b,b),(c,c)\}$, the binary relation $\{(a,a),(a,b),(a,c),(b,a),(b,c)\}$ which witnesses the fact that $\{a,c\}$ is a "central" subalgebra in Zhuk's terminology [190] (which is closely related to $\{a,c\}$ being a ternary absorbing subalgebra), and the affine ternary relation $\{(a,a,c),(a,c,a),(c,a,a),(c,c,c)\}$.

For an idempotent algebra $\mathbb{A}$ with a nontrivial congruence $\theta \in \mathrm{Con}(\mathbb{A})$, such as the previous example, we can test whether $\mathbb{A}$ has few subpowers by checking that $\mathbb{A}/\theta$ has few subpowers and that each congruence class of θ has few subpowers separately. This follows from the following easy results from [137].

Proposition 2.2.14. *Suppose $\mathbb{A}$ is an idempotent algebra, $\theta \in \mathrm{Con}(\mathbb{A})$, and that there are terms t_1, t_2 such that t_1 acts as a Δ_1-cube term on $\mathbb{A}/\theta$ and t_2 acts as a Δ_2-cube term on each congruence class of θ. Then $t_2 * t_1$ is a Δ-cube term for $\mathbb{A}$, where $\Delta = \{S \times T \mid S \in \Delta_2, T \in \Delta_1\}$.*

Corollary 2.2.15. *If $\mathbb{A}_1, ..., \mathbb{A}_n$ are idempotent algebras with the same signature such that each $\mathbb{A}_i$ has a Δ_i-cube term t_i, then $t_1 * \cdots * t_n$ is a Δ-cube term for $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, where $\Delta = \{S_1 \times \cdots \times S_n \mid S_i \in \Delta_i\}$.*

Corollary 2.2.16. *Suppose $\mathbb{A}$ is a finite idempotent algebra and $\theta \in \mathrm{Con}(\mathbb{A})$. Then $\mathbb{A}$ has few subpowers iff $\mathbb{A}/\theta$ has few subpowers and each congruence class of θ has few subpowers.*

2.2.1 Some connections with congruence modularity

Theorem 2.2.17. *If an algebra has an edge term, then it generates a congruence modular variety.*

Proof. By Theorem A.4.8 from Appendix A, we just need to check that an algebra with an edge term has directed Gumm terms, that is, terms $f_1, ..., f_k, p$ satisfying the system of identities

$$\begin{aligned} f_1(x,x,y) &\approx x, \\ f_i(x,y,x) &\approx x \text{ for all } i, \\ f_i(x,y,y) &\approx f_{i+1}(x,x,y) \text{ for all } i, \\ f_k(x,y,y) &\approx p(x,y,y), \\ p(x,x,y) &\approx y. \end{aligned}$$

If the reader wants to understand why this system of identities implies congruence modularity *without* reading all of Appendix A, then they can take the following path: first, read the discussion before Theorem A.4.8 to see why the existence of directed Gumm terms implies the existence of Gumm terms, then read part of the proof of Theorem A.4.7 to see how to construct Day terms from Gumm terms, and finally, read Section A.1 of Appendix A to see why the existence of Day terms is equivalent to congruence modularity.

Suppose that e is a k-edge term. Define terms $f_i(x,y,z)$ for $i < k$ by

$$f_i(x,y,z) = e(x,...,x,y,z,...,z),$$

such that there are $i-1$ zs, a single y, and $k+1-i$ xs. Then we have

$$f_1(x,x,y) = e(x,...,x,x) = x,$$

and for $i < k$ we have

$$f_i(x,y,x) = e(x,...,x,y,x,...,x) = x.$$

From the construction of the f_is we have

$$f_i(x,y,y) = e(x,...,x,y,y,...,y) = f_{i+1}(x,x,y)$$

for $i+1 < k$. Finally, if we define $f_k(x,y,z)$ by

$$f_k(x,y,z) = e(y,x,y,z,...,z)$$

and $p(x,y,z)$ by

$$p(x,y,z) = e(y,x,z,z,...,z),$$

then

$$f_{k-1}(x,y,y) = e(x,x,x,y,...,y) = f_k(x,x,y)$$

and

$$f_k(x,y,x) = e(y,x,y,x,...,x) = x$$

by the k-edge identities, while

$$f_k(x,y,y) = e(y,x,y,y,...,y) = p(x,y,y)$$

and

$$p(x,x,y) = e(x,x,y,y,...,y) = y$$

by the k-edge identities again. Thus $f_1,...,f_k,p$ are a sequence of directed Gumm terms. □

Theorem 2.2.18. *For $k \geq 3$, an algebra has a k-edge term and generates a congruence distributive variety iff it has a k-ary near-unanimity term.*

Proof. First the easy direction. If an algebra $\mathbb{A}$ has a k-ary near-unanimity term t, then adding an extra variable at the beginning of t produces a k-edge term. Additionally, the discussion before Theorem A.4.8 shows that we can construct a sequence of Jónsson terms from t, and then Theorem A.4.4 shows that $\mathbb{A}$ generates a congruence distributive variety.

Now the harder direction: assume that $\mathbb{A}$ generates a congruence distributive variety and has a k-edge term e. By Theorem A.4.8, there is a sequence of directed Jónsson terms $f_1, ..., f_m$, that is, a sequence satisfying the system of identities

$$\begin{aligned} f_1(x,x,y) &\approx x, \\ f_i(x,y,x) &\approx x \text{ for all } i, \\ f_i(x,y,y) &\approx f_{i+1}(x,x,y) \text{ for all } i, \\ f_m(x,y,y) &\approx y. \end{aligned}$$

Let $\mathcal{F} = \mathcal{F}_{\mathcal{V}(\mathbb{A})}(x,y) \le \mathbb{A}^{\mathbb{A}^2}$ be the free algebra on two generators in the variety generated by $\mathbb{A}$. Let $\mathbb{S} \le \mathcal{F}^k$ be generated by the vectors $(x, ..., x, y, x, ..., x)$ with all but one entry equal to x and the remaining entry equal to y. Note that $\mathbb{S}$ is symmetric under permuting its coordinates. We just need to prove that $(x, ..., x) \in \mathbb{S}$.

Claim: For all i, we have $(f_i(y,x,x), x, ..., x) \in \mathbb{S}$.

Proof of Claim: We induct on i, taking $(y, x, ..., x) \in \mathbb{S}$ as our base case. By the induction hypothesis, we have

$$(f_i(y,y,x), x, ..., x) = (f_{i-1}(y,x,x), x, ..., x) \in \mathbb{S}.$$

Additionally, the tuples

$$\begin{bmatrix} f_i(y,x,x) \\ f_i(x,x,y) \\ x \\ \vdots \\ x \end{bmatrix} = f_i\left(\begin{bmatrix} y & x & x \\ x & x & y \\ x & y & x \\ \vdots & \vdots & \vdots \\ x & x & x \end{bmatrix}\right)$$

and

$$\begin{bmatrix} f_i(y,y,x) \\ f_i(x,x,y) \\ x \\ \vdots \\ x \end{bmatrix} = f_i\left(\begin{bmatrix} y & y & x \\ x & x & y \\ x & x & x \\ \vdots & \vdots & \vdots \\ x & x & x \end{bmatrix}\right)$$

are both in $\mathbb{S}$. Now we apply the k-edge term e:

$$e\left(\begin{bmatrix} f_i(y,y,x) & f_i(y,y,x) & f_i(y,x,x) & f_i(y,x,x) & \cdots & f_i(y,x,x) \\ f_i(x,x,y) & x & f_i(x,x,y) & x & \cdots & x \\ x & x & x & f_i(x,x,y) & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & x & x & \cdots & f_i(x,x,y) \end{bmatrix}\right) = \begin{bmatrix} f_i(y,x,x) \\ x \\ x \\ \vdots \\ x \end{bmatrix}.$$

To finish the proof, we apply the Claim with $i = m$ to see that $(x, x, ..., x) = (f_m(y, x, x), x, ..., x) \in \mathbb{S}$. □

Example 2.2.2. We give an example of a congruence distributive algebra without few subpowers. Recall from Example 1.6.6 that for each n, the relational structure $(\{0,1\}, \{0\}, \le, \{0,1\}^n \setminus \{(0, ..., 0)\})$

has strict width exactly n. The limiting relational clone on $\{0,1\}$ generated by the relations $\{0\}, \leq$, and $\{0,1\}^n \setminus \{(0,...,0)\}$ for all $n \in \mathbb{N}$ corresponds to the clone generated by the ternary operation

$$f(x,y,z) = x \vee (y \wedge z).$$

Since the n-ary critical relation $\{0,1\}^n \setminus \{(0,...,0)\}$ doesn't have the parallelogram property and is preserved by f for all n, the clone generated by f can't have few subpowers by Theorem 2.3.4.

To check that the algebra $\mathbb{A} = (\{0,1\}, x \vee (y \wedge z))$ generates a congruence distributive variety, consider the sequence of ternary terms given by

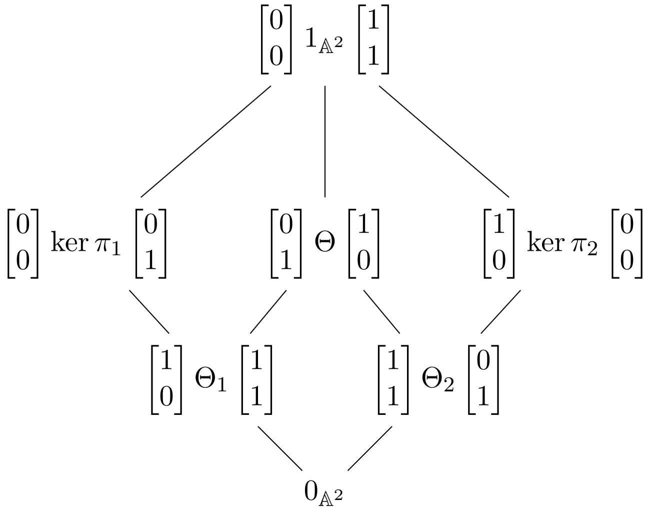
$$f_1(x,y,z) = x \vee (y \wedge z), \quad f_2(x,y,z) = (x \wedge y) \vee z.$$

To see that this is a sequence of directed Jónsson terms, note that they satisfy $f_i(x,y,x) = x \vee (y \wedge x) = x$, are connected by

$$f_1(x,y,y) = x \vee (y \wedge y) = x \vee y = (x \wedge x) \vee y = f_2(x,x,y),$$

and have $f_1(x,x,y) = x, f_2(x,y,y) = y$. By Theorem A.4.4 and the discussion before Theorem A.4.8, this implies that $\mathbb{A}$ is congruence distributive.

Example 2.2.3. We've seen earlier that the two-element semilattice $\mathbb{A} = (\{0,1\}, \max)$ does not have few subpowers. Here we will check that the two-element semilattice does not generate a congruence modular variety. In fact, the congruence lattice $\mathrm{Con}(\mathbb{A}^2)$ already fails to be modular. It turns out that every congruence on $\mathbb{A}^2$ is generated (as a congruence) by just one pair of elements a, b of $\mathbb{A}^2$, so we can label the nontrivial congruences on $\mathbb{A}^2$ by pairs of elements $a, b \in \mathbb{A}^2$, yielding the following congruence lattice.


$\begin{bmatrix}0\\0\end{bmatrix} 1_{\mathbb{A}^2} \begin{bmatrix}1\\1\end{bmatrix}$
$\begin{bmatrix}0\\0\end{bmatrix} \ker \pi_1 \begin{bmatrix}0\\1\end{bmatrix}$
$\begin{bmatrix}0\\1\end{bmatrix} \Theta \begin{bmatrix}1\\0\end{bmatrix}$
$\begin{bmatrix}1\\0\end{bmatrix} \ker \pi_2 \begin{bmatrix}0\\0\end{bmatrix}$
$\begin{bmatrix}1\\0\end{bmatrix} \Theta_1 \begin{bmatrix}1\\1\end{bmatrix}$
$\begin{bmatrix}1\\1\end{bmatrix} \Theta_2 \begin{bmatrix}0\\1\end{bmatrix}$
$0_{\mathbb{A}^2}$

To see that this isn't modular, note that the sublattice generated by $\ker \pi_1, \ker \pi_2, \Theta_2$ is isomorphic to the pentagon lattice $\mathcal{N}_5$. Considered as an abstract lattice, $\mathrm{Con}(\mathbb{A}^2)$ is the standard example of a lattice which is meet-semidistributive (recall from Example 1.9.6 and Proposition 1.9.32 that the variety of semilattices is $\mathrm{SD}(\wedge)$) but not join-semidistributive (we have $\Theta \vee \ker \pi_1 = \Theta \vee \ker \pi_2 = 1_{\mathbb{A}^2}$, but $\Theta \vee (\ker \pi_1 \wedge \ker \pi_2) = \Theta \neq 1_{\mathbb{A}^2}$).

Although congruence modularity is slightly weaker than having few subpowers, the concepts are quite close. One hint at the connection between them comes from counting *congruences* on subpowers of $\mathbb{A}$.

Definition 2.2.19. If $\mathbb{A}$ is an algebra, then we define the function $c_{\mathbb{A}}(n)$ to be the base-2 logarithm of the maximum size of $\operatorname{Con}(\mathbb{R})$ over all $\mathbb{R} \le \mathbb{A}^n$.

Proposition 2.2.20. *A variety $\mathcal{V}$ is congruence distributive iff for all subdirect products $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ in $\mathcal{V}$, every congruence on $\mathbb{R}$ can be written as a product of congruences on the $\mathbb{A}_i$s.*

Proof. Suppose first that $\mathcal{V}$ is congruence distributive. Then for any congruence θ on $\mathbb{R}$, by distributivity and $\bigwedge_i \ker \pi_i = 0_{\mathbb{R}}$ we have

$$\bigwedge_i (\theta \vee \ker \pi_i) = \theta \vee \bigwedge_i \ker \pi_i = \theta \vee 0_{\mathbb{R}} = \theta,$$

so θ is the product of the congruences $\pi_i(\theta \vee \ker \pi_i) \in \operatorname{Con}(\mathbb{A}_i)$.

Conversely, suppose that $\mathbb{A} \in \mathcal{V}$, and suppose that $\alpha, \beta, \gamma \in \operatorname{Con}(\mathbb{A})$. Then $\mathbb{A}/(\beta \wedge \gamma)$ is a subdirect product of $\mathbb{A}/\beta$ and $\mathbb{A}/\gamma$, so the congruence

$$\alpha \vee (\beta \wedge \gamma),$$

considered as a congruence on $\mathbb{A}/(\beta \wedge \gamma)$, is a product congruence iff it is equal to

$$(\alpha \vee \beta) \wedge (\alpha \vee \gamma).$$

□

Corollary 2.2.21. *If $\mathcal{V}(\mathbb{A})$ is congruence distributive, then $c_{\mathbb{A}}(n) = nc_{\mathbb{A}}(1)$.*

If a variety is congruence modular but *not* congruence distributive, then it necessarily contains a (finitely generated) nontrivial affine algebra. So we need to understand $c_{\mathbb{A}}(n)$ for $\mathbb{A}$ a finite affine algebra, and since the congruence lattice only depends on the polynomial clone, we may assume that $\mathbb{A}$ is a module over a ring. In this case, there is a bijection between congruences on $\mathbb{A}^n$ and submodules of $\mathbb{A}^n$.

Proposition 2.2.22. *If $\mathbb{A}$ is a nontrivial finite module over a ring, then $c_{\mathbb{A}}(n) \ge \frac{n^2-1}{4}$.*

Proof. We may as well assume that $\mathbb{A}$ is simple. Let c be any nonzero element of $\mathbb{A}$. For $n = 2m$, the span of the columns of the $n \times m$ matrix $\begin{bmatrix} cI \\ M \end{bmatrix}$ completely determines the $m \times m$ matrix M, so $c_{\mathbb{A}}(2m) \ge m^2 \log_2(|\mathbb{A}|) \ge m^2$. □

Corollary 2.2.23. *If $\mathbb{A}$ is finite and $\mathcal{V}(\mathbb{A})$ is congruence modular but not congruence distributive, then $c_{\mathbb{A}}(n) = \Omega(n^2)$.*

How can we get an upper bound on $c_{\mathbb{A}}(n)$ when $\mathbb{A}$ is congruence modular? The trick is to use the fact that in modular lattices, the *height* of the lattice is well-behaved. We can relate the height of a congruence lattice to its size using the following elementary bound.

Proposition 2.2.24. *If $\mathbb{A}$ is a finite algebra such that $\operatorname{Con}(\mathbb{A})$ has height h, then*

$$|\operatorname{Con}(\mathbb{A})| \le \sum_{i=0}^{h} \binom{|\mathbb{A}|}{2}^i \le |\mathbb{A}|^{2h}.$$

Proof. Consider any congruence $\alpha \in \mathrm{Con}(\mathbb{A})$. Since every cover of α is generated (as a congruence) by α together with some pair $(a,b) \not\in \alpha$, the number of covers of α is bounded by $\binom{|\mathbb{A}|}{2}$. Since every element of $\mathrm{Con}(\mathbb{A})$ can be reached from $0_{\mathbb{A}}$ by repeatedly choosing covers at most h times, we get the stated bound on $|\mathrm{Con}(\mathbb{A})|$.

We can get a slightly better bound as follows: the above argument shows that every congruence can be generated (as a congruence) by at most h pairs in $\binom{\mathbb{A}}{2}$. Additionally, there is only one congruence at height h, since $\mathrm{Con}(\mathbb{A})$ has a top element $1_{\mathbb{A}}$. So we have

$$|\mathrm{Con}(\mathbb{A})| \le 1 + \sum_{i=0}^{h-1} \binom{\binom{|\mathbb{A}|}{2}}{i}. \qquad \square$$

Corollary 2.2.25. *If $\mathbb{A}$ is finite and generates a congruence modular variety, then $c_{\mathbb{A}}(n) \le n^2 \cdot 2|\mathbb{A}|\log_2(|\mathbb{A}|)$.*

Proof. Let c be the maximum height of $\mathrm{Con}(\mathbb{B})$ over all subalgebras $\mathbb{B} \le \mathbb{A}$ (c is automatically bounded by $|\mathbb{A}|$). We claim that for any $\mathbb{R} \le \mathbb{A}^n$, the height of $\mathrm{Con}(\mathbb{R})$ is bounded by cn. Since $\mathrm{Con}(\mathbb{R})$ is modular, we can compute its height by looking at the size of *any* maximal chain in $\mathrm{Con}(\mathbb{R})$.

We will choose our maximal chain to be any maximal extension of the chain

$$0_{\mathbb{R}} \le \ker \pi_{[n-1]} \le \cdots \le \ker \pi_{[2]} \le \ker \pi_1 \le 1_{\mathbb{R}}.$$

By the Diamond Isomorphism Theorem A.2.5, the interval $[\![\ker \pi_{[i]}, \ker \pi_{[i-1]}]\!]$ is isomorphic to the interval $[\![\ker \pi_i, \ker \pi_{[i-1]} \vee \ker \pi_i]\!]$, so its height is bounded by the height of the interval $[\![\ker \pi_i, 1_{\mathbb{R}}]\!]$, which is isomorphic to $\mathrm{Con}(\mathbb{R}/\ker \pi_i)$. Since $\mathbb{R}/\ker \pi_i \cong \pi_i(\mathbb{R}) \le \mathbb{A}$, the height of $\mathrm{Con}(\mathbb{R}/\ker \pi_i)$ is bounded by c, and putting these intervals together we see that the height of $\mathrm{Con}(\mathbb{R})$ is bounded by cn.

Using the previous bound, we get

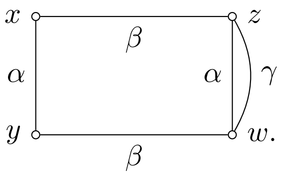
$$\log_2(|\mathrm{Con}(\mathbb{R})|) \le \log_2(|\mathbb{R}|^{2cn}) \le 2cn \log_2(|\mathbb{A}|^n) = 2cn^2 \log_2(|\mathbb{A}|). \qquad \square$$

Theorem 2.2.26 (Few congruences on subpowers iff congruence modular [26])**.** *Let $\mathbb{A}$ be a finite algebra with at least two elements, and let $\mathcal{V}(\mathbb{A})$ be the variety it generates.*

- *If $\mathcal{V}(\mathbb{A})$ is congruence distributive, then $c_{\mathbb{A}}(n) = \Theta(n)$.*
- *If $\mathcal{V}(\mathbb{A})$ is congruence modular but not congruence distributive, then $c_{\mathbb{A}}(n) = \Theta(n^2)$.*
- *If $\mathcal{V}(\mathbb{A})$ is not congruence modular, then $c_{\mathbb{A}}(n) = 2^{\Theta(n)}$.*

Proof. By the previous results, all we need to check is that if $\mathcal{V}(\mathbb{A})$ is not congruence modular, then $c_{\mathbb{A}}(n) = 2^{\Omega(n)}$. Let $\mathbb{F} = \mathcal{F}_{\mathcal{V}(\mathbb{A})}(x,y,z,w) \le \mathbb{A}^{\mathbb{A}^4}$ be the free algebra on four generators. We will show that if $c_{\mathbb{F}}(2n) < 2^n$ for any n, then $\mathbb{A}$ has Day terms, and is therefore congruence modular by Appendix A.1.

Define congruences on $\mathbb{F}$ as in Corollary A.1.2: let θ_{ab} be the congruence generated by the pair (a,b) for any pair of variables a,b, set $\alpha = \theta_{xy} \vee \theta_{zw}, \beta = \theta_{xz} \vee \theta_{yw}$, and $\gamma = (\alpha \wedge \beta) \vee \theta_{zw}$. This is the generic Shifting Lemma configuration:



To show the existence of Day terms, we just need to show that $(x, y) \in \gamma$.

Pick an n such that $c_{\mathbb{F}}(2n) < 2^n$, and consider the subalgebra $\mathbb{R} \le \mathbb{F}^{2n}$ consisting of tuples such that every pair of coordinates are related by β (it helps to imagine elements of $\mathbb{R}$ written out horizontally as row vectors, following the convention that variables which are related by β are laid out on horizontal lines). We will define a family of 2^n pairs of elements of $\mathbb{R}$ as follows.

First, we define elements $x^0, x^1, y^0, y^1 \in \mathbb{F}^2$ by $x^0 = (x, z), x^1 = (z, x)$ and similarly $y^0 = (y, w), y^1 = (w, y)$. Then, for any $i = (i_1, ..., i_n) \in \{0, 1\}^n$, we define $f_i, g_i \in \mathbb{R}$ by

$$f_i = (x^{i_1}, ..., x^{i_n}),$$
$$g_i = (y^{i_1}, ..., y^{i_n}).$$

For each $i \in \{0, 1\}^n$, we define a congruence $\Theta(i)$ to be the congruence of $\mathbb{R}$ generated by the pair (f_i, g_i). Since $c_{\mathbb{F}}(2n) < 2^n$, there must be some $i \in \{0, 1\}^n$ such that

$$\Theta(i) \le \bigvee_{j \ne i} \Theta(j),$$

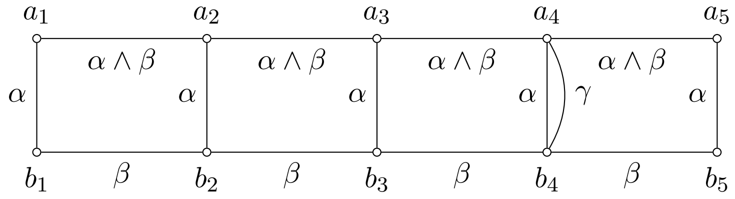
and by permuting the coordinates of $\mathbb{R}$, we see that in fact this must hold for every i, and in particular for $i = (0, ..., 0)$. By dropping half of the coordinates of $\mathbb{R}$ to get a similar algebra $\mathbb{R}' \le \mathbb{F}^n$ such that f_0 becomes the vector $f_0' = (x, ..., x)$ and g_0 becomes the vector $g_0' = (y, ..., y)$, and defining elements f_j', g_j' by dropping half the coordinates of f_j, g_j, we see that

$$(f_0', g_0') \in \bigvee_{j \ne (0,...,0)} \Theta'(j),$$

where $\Theta'(j)$ is the congruence of $\mathbb{R}'$ generated by the pair (f_j', g_j').

Each $\Theta'(j)$ has the following property: if $(a, b) \in \Theta'(j)$ and every pair of coordinates of a are related by $\alpha \wedge \beta$, then every pair of coordinates of b are also related by $\alpha \wedge \beta$. To see this, just note that for each coordinate $i \le n$ we have $(a_i, b_i) \in \alpha$, since this holds in the case where $(a, b) = (f_j', g_j')$.

For $j \ne (0, ..., 0)$, $\Theta'(j)$ has the following additional property: there exists some coordinate $i \le n$ such that if $(a, b) \in \Theta'(j)$, then $(a_i, b_i) \in \gamma$. In fact, we can take the coordinate i to be the first coordinate of j such that $j_i = 1$, and note that the ith coordinates of f_j', g_j' are z, w respectively, with $(z, w) \in \gamma$ by the definition of γ.



Putting the above properties together, and using $\alpha\wedge\beta \le \gamma$, we see that $(f_0', b) \in \bigvee_{j\ne(0,\ldots,0)} \Theta'(j)$ implies that every coordinate of f_0' is congruent modulo γ to every coordinate of b, and taking $b = g_0'$ we see that $(x, y) \in \gamma$, which completes the proof. $\square$

Example 2.2.4. Consider the two-element semilattice $\mathbb{A} = (\{0, 1\}, \max)$ once again. In this case, we can check directly that $c_{\mathbb{A}}(n) \ge \binom{n}{n/2}$. To see this, note that for every nonempty upwards closed subset $U \le \mathbb{A}^n$, there is a congruence θ_U which collapses all elements of U into a single top element of $\mathbb{A}^n/\theta_U$, and which does not identify any pair of elements $a \ne b$ such that $\{a, b\} \not\subseteq U$. In other words, $\theta_U = U^2 \cup \Delta_{\mathbb{A}^n}$.

We just need to check that the number of distinct nonempty upwards closed subsets U of $\{0, 1\}^n$ is at least $2^{\binom{n}{n/2}}$: for this, note that upwards closed sets U are in a one-to-one correspondence with antichains (every upwards closed set U is determined by its antichain of minimal elements), and every set of elements of $\mathbb{A}^n$ which each have exactly $n/2$ coordinates equal to 1 forms an antichain.

2.3 Parallelogram terms

Examining the proof of Theorem 2.2.11, we can extract useful terms known as *parallelogram terms*, which we can use to give a better description of the relational clone corresponding to an algebra with few subpowers.

Definition 2.3.1. If $k = m + n$, then an m, n*-parallelogram term* is a $k + 3$-ary term r which satisfies the identities

$$r\left(\begin{bmatrix} y & y & x & z & \cdots & x & x & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y & y & x & x & \cdots & z & x & \cdots & x \\ x & y & y & x & \cdots & x & z & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x & y & y & x & \cdots & x & x & \cdots & z \end{bmatrix}\right) = \begin{bmatrix} x \\ \vdots \\ x \\ x \\ \vdots \\ x \end{bmatrix},$$

where the upper left $m \times 3$ block has all rows given by y, y, x, the lower left $n \times 3$ block has all rows given by x, y, y, and the right $k \times k$ block has zs on the diagonal and xs elsewhere.

Theorem 2.3.2 (Edge term implies parallelogram terms [118])**.** *For any $m, n > 0$ with $m + n = k$, a variety has a k-edge term e iff it has an m, n-parallelogram term r.*

Proof. It's clear that every m, n-parallelogram term is a Δ-cube term for

$$\Delta = \{\{1, ..., m\}, \{1, ..., k\}, \{m + 1, ..., m + n\}, \{1\}, ..., \{k\}\},$$

so by Theorem 2.2.7 if $\mathbb{A}$ has a parallelogram term then it has an edge term.

Now suppose that e is a k-edge term. We will build m, n-parallelogram terms r_m by induction on m. For $m = 1$, we need to show that the vector in $\mathcal{F}(x, y, z)^k$ of all xs is in the subalgebra generated by the columns of the matrix defining a $1, k - 1$-parallelogram term. These vectors are the vectors where all entries other than one are xs and the last is a z, the vector of all ys, and the vectors $(x, y, ..., y), (y, x, ..., x)$.

Letting $d = d(y, x) = e(x, y, x, ..., x)$, we have

$$e\left(\begin{bmatrix} x & y & x & x & \cdots & x \\ z & x & z & x & \cdots & x \\ x & x & x & z & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & x & x & \cdots & z \end{bmatrix}\right) = \begin{bmatrix} d \\ x \\ x \\ \vdots \\ x \end{bmatrix}$$

and

$$e\left(\begin{bmatrix} x & y & x & x & \cdots & x \\ y & y & z & z & \cdots & z \\ y & y & x & x & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y & y & x & x & \cdots & x \end{bmatrix}\right) = \begin{bmatrix} d \\ z \\ x \\ \vdots \\ x \end{bmatrix},$$

so the vectors $(d, x, x, ..., x)$ and $(d, z, x, ..., x)$ are in the subalgebra of $\mathcal{F}(x, y, z)^k$ generated by the columns of the matrix defining a $1, k-1$-parallelogram term. Note that the previous two applications of the edge term e correspond to applications of the terms

$$s(x_1, ..., x_k) = e(x_2, x_1, x_2, ..., x_k) \text{ and } p(x, y, z) = e(y, x, z, ..., z)$$

which act like near-unanimity and Mal'cev terms, respectively. To get the vector of all xs, we apply e one more time:

$$e\left(\begin{bmatrix} d & d & x & x & \cdots & x \\ z & x & z & x & \cdots & x \\ x & x & x & z & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x & x & x & x & \cdots & z \end{bmatrix}\right) = \begin{bmatrix} x \\ x \\ x \\ \vdots \\ x \end{bmatrix}.$$

Explicitly, our $1, k-1$-parallelogram term r_1 is defined from the edge term e by

$$\begin{aligned} r_1(x, y, z, u_1, ..., u_k) &= e(p(y, z, u_2), s(x, u_2, ..., u_k), u_2, ..., u_k) \\ &= e(e(z, y, u_2, ..., u_2), e(u_2, x, u_2, ..., u_k), u_2, ..., u_k). \end{aligned}$$

For $m > 1$, we construct the $m, k-m$-parallelogram term r_m using the previous term r_{m-1}. Here we focus on the mth rows of our matrices. Let

$$a = r_{m-1}(y, y, x, x, ..., x, z, x, ..., x),$$

where the z occurs in the $m+3$rd entry. We want to construct tuples $(x, ..., x, a, x, ..., x)$ and $(x, ..., x, a, y, ..., y)$ from the columns of the defining matrix for an $m, k-m$-parallelogram term. We construct these tuples via

$$r_{m-1}\left(\begin{bmatrix} y & y & x & z & \cdots & x & x & x & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y & y & x & x & \cdots & z & x & x & \cdots & x \\ y & y & x & x & \cdots & x & z & x & \cdots & x \\ x & y & y & x & \cdots & x & x & z & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x & y & y & x & \cdots & x & x & x & \cdots & z \end{bmatrix}\right) = \begin{bmatrix} x \\ \vdots \\ x \\ a \\ x \\ \vdots \\ x \end{bmatrix}$$

and

$$r_{m-1}\left(\begin{bmatrix} y & y & x & x & \cdots & x & x & x & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y & y & x & x & \cdots & x & x & x & \cdots & x \\ y & y & x & x & \cdots & x & z & x & \cdots & x \\ y & y & y & y & \cdots & y & x & y & \cdots & y \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y & y & y & y & \cdots & y & x & y & \cdots & y \end{bmatrix}\right) = \begin{bmatrix} x \\ \vdots \\ x \\ a \\ y \\ \vdots \\ y \end{bmatrix}.$$

To get to the vector of all xs, we use

$$r_{m-1}\left(\begin{bmatrix} x & x & x & x & x & \cdots & z & x & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ x & x & x & x & z & \cdots & x & x & \cdots & x \\ a & a & x & z & x & \cdots & x & x & \cdots & x \\ x & y & y & x & x & \cdots & x & z & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \ddots & \vdots \\ x & y & y & x & x & \cdots & x & x & \cdots & z \end{bmatrix}\right) = \begin{bmatrix} x \\ \vdots \\ x \\ x \\ x \\ \vdots \\ x \end{bmatrix},$$

where the middle row works out because $m > 1$. Explicitly, r_m is defined in terms of r_{m-1} by

$$r_m(x, y, z, u_1, ..., u_k) = t_{m-1}(t_{m-1}(x, y, z, u_1, ..., u_k), t_{m-1}(y, y, z, z, ..., u_m, ..., z), z, u_m, ..., u_1, u_{m+1}, ..., u_k).$$

□

To understand what parallelogram terms tell us, it is necessary to restrict to certain special relations, known as *critical* relations.

Definition 2.3.3. A subalgebra $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is *critical* if it is $\cap$-irreducible, that is, if it can't be written as an intersection of strictly larger subalgebras, and if furthermore the relation $\mathbb{R}$ has no dummy variables (that is, it depends on all of its inputs).

A standard result in the theory of algebraic lattices (Proposition A.5.6 from Appendix A) shows that every relation can be written as an intersection of critical relations (possibly of lower arity). The following result shows that every relation in an algebra with k-parallelogram terms can be written as an intersection of relations of arity less than k and relations with the parallelogram property.

Theorem 2.3.4 (Parallelogram terms constrain critical relations [118])**.** *A variety $\mathcal{V}$ has k-parallelogram terms iff for all critical $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ with $\mathbb{A}_i \in \mathcal{V}$, either $n < k$ or $\mathbb{R}$ has the parallelogram property.*

Proof. First suppose that $\mathcal{V}$ has k-parallelogram terms, and let $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ be a critical relation. Let $\mathbb{R}^*$ be the cover of $\mathbb{R}$, i.e., $\mathbb{R}^*$ is the intersection of all relations which properly contain $\mathbb{R}$, and let $a = (a_1, ..., a_n) \in \mathbb{R}^* \setminus \mathbb{R}$. Then a relation $\mathbb{S}$ which contains $\mathbb{R}$ will properly contain $\mathbb{R}$ iff $\mathbb{S}$ contains a. Following Zhuk [193], we call a a *key tuple* for the critical relation $\mathbb{R}$.

Since $\mathbb{R}$ is critical, $\mathbb{R}$ is properly contained in its existential projections onto any proper subset of the coordinates $1, ..., n$. Thus, there must exist elements $b_1, ..., b_n$ such that the tuples $(b_1, a_2, ..., a_n), (a_1, b_2, ..., a_n), ..., (a_1, a_2, ..., b_n)$ are all in $\mathbb{R}$.

Now suppose, for contradiction, that $n \geq k$ and that $\mathbb{R}$ does not have the parallelogram property when considered as a binary relation on $(\mathbb{A}_1 \times \cdots \times \mathbb{A}_i) \times (\mathbb{A}_{i+1} \times \cdots \times \mathbb{A}_n)$. Then the are $x_1, ..., x_n, y_1, ..., y_n$ such that the three tuples $(y_1, ..., y_n), (y_1, ..., y_i, x_{i+1}, ..., x_n), (x_1, ..., x_i, y_{i+1}, ..., y_n)$ are in $\mathbb{R}$, but $(x_1, ..., x_n)$ is not in $\mathbb{R}$. Since $x = (x_1, ..., x_n)$ is not in $\mathbb{R}$, the subalgebra generated by $\mathbb{R} \cup \{x\}$ must properly contain $\mathbb{R}$, so

$$a \in \mathrm{Sg}(\mathbb{R} \cup \{x\}).$$

Thus there are tuples $c^1, ..., c^m \in \mathbb{R}$ and an $m+1$-ary term t such that

$$t(x, c^1, ..., c^m) = a.$$

Defining a tuple d by

$$t(y, c^1, ..., c^m) = d,$$

we see that the three tuples $(d_1, ..., d_n), (d_1, ..., d_i, a_{i+1}, ..., a_n), (a_1, ..., a_i, d_{i+1}, ..., d_n)$ are all in $\mathbb{R}$. But then we can use an $i, n-i$-parallelogram term r (which exists because $n \geq k$) to see that

$$\begin{bmatrix} a_1 \\ \vdots \\ a_i \\ a_{i+1} \\ \vdots \\ a_n \end{bmatrix} = r\left(\begin{bmatrix} d_1 & d_1 & a_1 & b_1 & \cdots & a_1 & a_1 & \cdots & a_1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ d_i & d_i & a_i & a_i & \cdots & b_i & a_i & \cdots & a_i \\ a_{i+1} & d_{i+1} & d_{i+1} & a_{i+1} & \cdots & a_{i+1} & b_{i+1} & \cdots & a_{i+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_n & d_n & d_n & a_n & \cdots & a_n & a_n & \cdots & b_n \end{bmatrix}\right) \in \mathbb{R},$$

contradicting the assumption that $a \notin \mathbb{R}$.

For the converse direction, suppose that $\mathcal{V}$ is a variety such that every critical k-ary relation has the parallelogram property, and suppose that $m+n = k$. Let $\mathcal{F} = \mathcal{F}_{\mathcal{V}}(x, y, z)$ be the free algebra on three generators in $\mathcal{V}$. Suppose for contradiction that $\mathcal{V}$ doesn't have an m, n-parallelogram term. Then

$$\begin{bmatrix} x \\ \vdots \\ x \\ x \\ \vdots \\ x \end{bmatrix} \notin \mathrm{Sg}_{\mathcal{F}^k}\left\{\begin{bmatrix} y & y & x & z & \cdots & x & x & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y & y & x & x & \cdots & z & x & \cdots & x \\ x & y & y & x & \cdots & x & z & \cdots & x \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x & y & y & x & \cdots & x & x & \cdots & z \end{bmatrix}\right\},$$

so by Zorn's Lemma there exists a maximal k-ary relation $\mathbb{R}$ on $\mathcal{F}$ which contains the right hand side but does not contain the tuple $(x, ..., x)$. The relation $\mathbb{R}$ is then a critical k-ary relation on $\mathcal{F}$, since every relation which properly contains $\mathbb{R}$ must contain $\mathbb{R}^* = \mathrm{Sg}(\mathbb{R} \cup \{(x, ..., x)\})$ and since every existential projection of $\mathbb{R}$ onto a proper subset of the coordinates contains a vector of all xs (by the last k columns of the matrix of generators above). However, $\mathbb{R}$ does not have the parallelogram property when considered as a binary relation on $\mathcal{F}^m \times \mathcal{F}^n$, by the first three columns of the matrix of generators above, contradicting our assumption on $\mathcal{V}$. □

Corollary 2.3.5. *A variety $\mathcal{V}$ has k-parallelogram terms iff for every relation $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ with $\mathbb{A}_i \in \mathcal{V}$, there exists a relation $\mathbb{R}' \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ such that $\mathbb{R}'$ has the parallelogram property and*

$$\mathbb{R} = \mathbb{R}' \cap \bigcap_{I \subseteq [n], |I| < k} \pi_I(\mathbb{R}).$$

The relation $\mathbb{R}'$ from the corollary need not be so mysterious: we can take it to be the *least* relation $\mathbb{R}'$ which contains $\mathbb{R}$ and has the parallelogram property, since any intersection of relations which have the parallelogram property also has the parallelogram property. This choice of $\mathbb{R}'$ can also be "generated" from $\mathbb{R}$, by repeatedly adjoining tuples which are required to be inside in order for the parallelogram property to hold.

More explicitly, for any $I \subseteq [n]$, we can find the least relation $\mathbb{R}^I$ which contains $\mathbb{R}$ and has the (binary) parallelogram property when considered as a subalgebra of

$$\Big(\prod_{i \in I} \mathbb{A}_i\Big) \times \Big(\prod_{j \not\in I} \mathbb{A}_j\Big),$$

by finding the linking congruence of $\mathbb{R}$ when considered as a subalgebra of the above, which restricts to a congruence $\alpha_I \in \mathrm{Con}(\pi_I(\mathbb{R}))$, and taking $\mathbb{R}^I$ to be the relation $\alpha_I \circ \mathbb{R}$. We can then take

$$\mathbb{R}' = \bigcup_{I_1, I_2, \ldots \subseteq [n]} \mathbb{R}^{I_1 I_2 \cdots}.$$

In particular, if all of the algebras $\mathbb{A}_i$ are finite, then $\mathbb{R}'$ is contained in the (multisorted) relational clone generated by $\mathbb{R}$.

2.3.1 Critical rectangular relations in congruence modular varieties

Using the commutator theory for congruence modular varieties, we can give a more detailed structure theory for the high-arity critical relations preserved by algebras with few subpowers. In fact, this structure theory applies more generally in congruence modular varieties, so long as we restrict our attention to critical relations with a weak form of the parallelogram property.

Definition 2.3.6. A relation $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_k$ is said to have the $1, k-1$-*parallelogram property*, or alternatively is called *rectangular*, if for any $i \le k$, when we regard $\mathbb{R}$ as a binary relation on

$$(\mathbb{A}_1 \times \cdots \times \mathbb{A}_{i-1} \times \mathbb{A}_{i+1} \times \cdots \times \mathbb{A}_k) \times \mathbb{A}_i,$$

it has the (binary) parallelogram property.

The main property of subdirect rectangular relations which we need - and which holds in complete generality, not just in the context of congruence modularity - is that if we define a congruence θ_i on $\mathbb{A}_i$ from the linking congruence of $\mathbb{R}$ (considered as a binary relation on $(\cdots) \times \mathbb{A}_i$), then we have $x \in \mathbb{R}$ iff $x/\prod_i \theta_i \in \mathbb{R}/\prod_i \theta_i$. Thus we may as well study the relation

$$\mathbb{R}/\prod_i \theta_i \le_{sd} \mathbb{A}_1/\theta_1 \times \cdots \times \mathbb{A}_k/\theta_k$$

instead of studying $\mathbb{R}$. The reduced relation is critical if the original $\mathbb{R}$ is critical, is still rectangular, and has trivial linking congruences on each $\mathbb{A}_i/\theta_i$, so it can be viewed as the graph of a surjective homomorphism

$$\pi_{[k]\setminus\{i\}}\Big(\mathbb{R}/\prod_i \theta_i\Big) \twoheadrightarrow \mathbb{A}_i/\theta_i$$

for each i.

Definition 2.3.7. A subdirect rectangular relation $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_k$ is called *reduced* if for each $i \le k$, $\mathbb{R}$ is the graph of a surjective homomorphism

$$\pi_{[k]\setminus\{i\}}(\mathbb{R}) \twoheadrightarrow \mathbb{A}_i,$$

or equivalently, for each i the map

$$\pi_{[k]\setminus\{i\}} : \mathbb{R} \to \pi_{[k]\setminus\{i\}}(\mathbb{R})$$

is an isomorphism, i.e. $\ker \pi_{[k]\setminus\{i\}} = 0_{\mathbb{R}}$.

Proposition 2.3.8. *If $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_k$ is a reduced subdirect critical rectangular relation, then each $\mathbb{A}_i$ is subdirectly irreducible.*

Proof. Let $\mathbb{R}^*$ be the cover of $\mathbb{R}$ in the lattice of subalgebras of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_k$, and let $a = (a_1, ..., a_k)$ be a key tuple for $\mathbb{R}$, that is, an element of $\mathbb{R}^* \setminus \mathbb{R}$. Since $\mathbb{R}$ is critical, for every i there is some $b_i \in \mathbb{A}_i$ such that $(a_1, ..., a_{i-1}, b_i, a_{i+1}, ..., a_k) \in \mathbb{R}$ (and this b_i is unique, since $\mathbb{R}$ is reduced). The claim is that for each i, every nontrivial congruence on $\mathbb{A}_i$ contains the pair (a_i, b_i) - that is, each $\mathbb{A}_i$ is subdirectly irreducible with monolith equal to the congruence generated by the pair (a_i, b_i).

Let $\psi_i \in \mathrm{Con}(\mathbb{A}_i)$ be any nontrivial congruence. Then the relation

$$\exists y_i\ ((x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_k) \in \mathbb{R}) \wedge (x_i \equiv_{\psi_i} y_i)$$

strictly contains $\mathbb{R}$ (since $\mathbb{R}$ is reduced), so it contains $\mathbb{R}^*$, and in particular contains the key tuple a. Using the fact that $\mathbb{R}$ is reduced again, we see that the pair (a_i, b_i) must be contained in ψ_i. □

As it turns out, reduced critical rectangular relations are closely related to the concept of *similarity* between subdirectly irreducible algebras (see Appendix A.5.1). We won't need the full theory of similarity, just the following definition.

Definition 2.3.9. If $\mathbb{A}_1, ..., \mathbb{A}_k$ are subdirectly irreducible algebras, then we say that an algebra $\mathbb{R}$ is the *graph of a joint similarity* between the $\mathbb{A}_i$s if for each i, $\mathbb{R}$ has a (critical) congruence α_i with $\mathbb{R}/\alpha_i \cong \mathbb{A}_i$, and for each pair i, j there are congruences $\gamma_{ij}, \delta_{ij} \in \mathrm{Con}(\mathbb{R})$ such that

$$[\![\alpha_i, \alpha_i^*]\!] \searrow [\![\gamma_{ij}, \delta_{ij}]\!] \nearrow [\![\alpha_j, \alpha_j^*]\!].$$

More explicitly, this means that $\alpha_i \vee \delta_{ij} = \alpha_i^*$, $\alpha_j \vee \delta_{ij} = \alpha_j^*$, and $\alpha_i \wedge \delta_{ij} = \alpha_j \wedge \delta_{ij}$.

Note that by Proposition A.5.36, $\mathbb{R}/(\alpha_1 \wedge \cdots \wedge \alpha_k)$ is also a graph of a joint similarity, so there is no real loss in restricting to the case where $\mathbb{R}$ is a subdirect product of the $\mathbb{A}_i$s, with $\alpha_i = \ker \pi_i$.

Theorem 2.3.10 (Kearnes, Szendrei [118]). *If $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_k$ is a reduced subdirect critical rectangular relation of arity $k \ge 3$ in a congruence modular variety, then* *(a) $\mathbb{R}$ is the graph of a joint similarity between the $\mathbb{A}_i$s,*

(b) for each i, j, the image of $\pi_{i,j}(\mathbb{R})$ in $\mathbb{A}_i/(0_{\mathbb{A}_i} : 0^*_{\mathbb{A}_i}) \times \mathbb{A}_j/(0_{\mathbb{A}_j} : 0^*_{\mathbb{A}_j})$ is the graph of an isomorphism

$$\mathbb{A}_i/(0_{\mathbb{A}_i} : 0^*_{\mathbb{A}_i}) \xrightarrow{\sim} \mathbb{A}_j/(0_{\mathbb{A}_j} : 0^*_{\mathbb{A}_j}),$$

(c) each monolith $0^*_{\mathbb{A}_i}$ is abelian, and

(d) the cover $\mathbb{R}^*$ is also rectangular, and the linking congruence of $\mathbb{R}^*$ on $\mathbb{A}_i$ is the monolith $0^*_{\mathbb{A}_i}$.

If $\mathbb{R}$ has the parallelogram property, then so does its cover $\mathbb{R}^*$.

Proof. (a) Let $a = (a_1, ..., a_k) \in \mathbb{R}^* \setminus \mathbb{R}$ be a key tuple for $\mathbb{R}$, and for each i let $b_i \in \mathbb{A}_i$ such that $(a_1, ..., a_{i-1}, b_i, a_{i+1}, ..., a_k) \in \mathbb{R}$. Let $a^i = (a_1, ..., a_{i-1}, b_i, a_{i+1}, ..., a_k)$. Then for any $i \neq j$, if we let δ_{ij} be the congruence generated by the pair (a^i, a^j), we claim that

$$[\![\ker \pi_i, (\ker \pi_i)^*]\!] \searrow [\![0_{\mathbb{R}}, \delta_{ij}]\!].$$

The equality $\ker \pi_i \vee \delta_{ij} = (\ker \pi_i)^*$ was proved in the previous proposition. For the equality $\ker \pi_i \wedge \delta_{ij} = 0_{\mathbb{R}}$, note that

$$\ker \pi_i \wedge \delta_{ij} \leq \ker \pi_i \wedge \ker \pi_{[k]\setminus\{i,j\}} = \ker \pi_{[k]\setminus\{j\}} = 0_{\mathbb{R}},$$

where the last equality follows from the fact that $\mathbb{R}$ is reduced.

(b) This follows directly from (a) and the Diamond Isomorphism Theorem A.2.5 - for details, see Proposition A.5.36.

(c) By Proposition A.5.36 again, if $\pi_{i,j}(\mathbb{R})$ is not the graph of an isomorphism for any pair i, j, then each monolith $0^*_{\mathbb{A}_i}$ must be abelian.

(d) Suppose that $u, v, w \in \mathbb{R}$ with $\pi_{[k]\setminus\{i\}}(u) = \pi_{[k]\setminus\{i\}}(v)$ and $v_i = w_i$. We need to show that there is some element $t \in \mathbb{R}$ with $\pi_{[k]\setminus\{i\}}(t) = \pi_{[k]\setminus\{i\}}(w)$ and $t_i = u_i$.

Since $\mathbb{R}^*$ is contained in the relation

$$\exists y_i \ ((x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_k) \in \mathbb{R}) \wedge (x_i \equiv_{0^*_{\mathbb{A}_i}} y_i)$$

and $\mathbb{R}$ is reduced, we have $(u_i, v_i) \in 0^*_{\mathbb{A}_i}$. Let $p(x, y, z)$ be a Gumm difference term as in Theorem A.3.1, i.e. a term such that $p(y, y, x) \approx x$, and such that for $(x, y) \in \theta$ and θ any congruence we have $p(x, y, y)\ [\theta, \theta]\ x$. Then taking $\theta = 0^*_{\mathbb{A}_i}$, we have $p(u_i, v_i, v_i) = u_i$ by part (c), so we can take $t = p(u, v, w)$.

For the last claim, suppose that we view $\mathbb{R}^*$ as a binary relation on $\mathbb{A}_I \times \mathbb{A}_{[n]\setminus I}$, where we set $\mathbb{A}_I = \prod_{i \in I} \mathbb{A}_i$, and that we have $(a, b), (c, b), (c, d) \in \mathbb{R}^*$. Pick some $i \in I$ and $j \notin I$. Then there is some a' such that $\pi_{I\setminus\{i\}}(a') = \pi_{I\setminus\{i\}}(a)$, $a'_i \equiv_{0^*_{\mathbb{A}_i}} a_i$, and $(a', b) \in \mathbb{R}$. Similarly find c' which only differs from c in the ith coordinate, has $c'_i \equiv_{0^*_{\mathbb{A}_i}} c_i$, and has $(c', b) \in \mathbb{R}$. Then $(c', d) \in \mathbb{R}^*$ by part (d), so we can find d' which only differs from d in the jth coordinate, has $d'_i \equiv_{0^*_{\mathbb{A}_j}} d_i$, and has $(c', d') \in \mathbb{R}$. Then by the parallelogram property for $\mathbb{R}$, we have $(a', d') \in \mathbb{R}$, so by part (d) we have $(a, d) \in \mathbb{R}^*$. □

Example 2.3.1. Consider the generalized majority-minority algebra $\mathbb{A} = (\{a,b,c\}, \varphi_2)$ from Example 2.1.2, which is subdirectly irreducible with abelian monolith $0_{\mathbb{A}}^*$ corresponding to the partition $\{a\}, \{b,c\}$ of its elements, and has $\mathbb{A}/0_{\mathbb{A}}^*$ isomorphic to a two element majority algebra. We can check that the monolith $0_{\mathbb{A}}^*$ of $\mathbb{A}$ is equal to its own centralizer by verifying that $[1_{\mathbb{A}} : 0_{\mathbb{A}}^*] = 0_{\mathbb{A}}^*$ and $[0_{\mathbb{A}}^*, 0_{\mathbb{A}}^*] = 0_{\mathbb{A}}$: to see this, note that

$$\varphi_2\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}, \begin{bmatrix} a & a \\ b & b \end{bmatrix}, \begin{bmatrix} b & c \\ b & c \end{bmatrix}\right) = \begin{bmatrix} a & a \\ b & c \end{bmatrix} \in \mathbb{M}(1_{\mathbb{A}}, 0_{\mathbb{A}}^*),$$

so $(b,c) \in [1_{\mathbb{A}}, 0_{\mathbb{A}}^*]$, while every element of $\mathbb{M}(0_{\mathbb{A}}^*, 0_{\mathbb{A}}^*)$ either has all entries equal to a, or has all entries in $\{b,c\}$ with an even number of bs and an even number of cs.

The ternary relation $\mathbb{R} \le_{sd} \mathbb{A}^3$ corresponding to the columns of the matrix

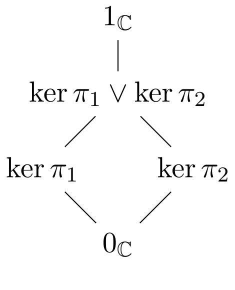
$$\begin{bmatrix} a & b & b & c & c \\ a & b & c & b & c \\ a & b & c & c & b \end{bmatrix}$$

is a reduced subdirect critical rectangular relation of arity 3 (with key tuple (c,c,c)), so by the structure theorem it is the graph of a joint similarity between three copies of $\mathbb{A}$. Every two-coordinate projection $\pi_{i,j}(\mathbb{R})$ is equal to the congruence $0_{\mathbb{A}}^* = (0_{\mathbb{A}} : 0_{\mathbb{A}}^*)$, and the cover $\mathbb{R}^*$ of $\mathbb{R}$ in $\mathrm{Inv}_3(\mathbb{A})$ is the relation $x\ 0_{\mathbb{A}}^*\ y\ 0_{\mathbb{A}}^*\ z$.

More generally, for any k we can define a relation $\mathbb{R}_k \le_{sd} \mathbb{A}^k$ which contains the tuple $(a, ..., a)$ together with the 2^{k-1} tuples in $\{b,c\}^k$ such that the total number of cs is even, and we see that $\mathbb{R}_k$ is a reduced critical rectangular relation for each k. We claim that for every $k \ge 3$, there are exactly four critical relations in $\mathrm{Inv}_k(\mathbb{A})$: $\mathbb{R}_k$, $\mathbb{R}_k \setminus \{(a, ..., a)\}$, and the two relations we get from these by swapping bs and cs in the last coordinate.

To prove the claim, we first note that the only algebras in $HS(\mathbb{A})$ which have abelian monoliths are $\mathbb{A}$ and $\{b,c\}$, and that these two algebras are not similar to each other (since $\mathbb{A}/0_{\mathbb{A}}^*$ is not isomorphic to any quotient of $\{b,c\}$). Thus by the structure theorem, we only need to consider relations which are either subdirect in $\mathbb{A}^k$ or subdirect in $\{b,c\}^k$. The interesting case is the case of relations which are subdirect in $\mathbb{A}^k$.

The next thing we need to check is that no graph of a similarity $\mathbb{C} \le_{sd} \mathbb{A}^2$ from $\mathbb{A}$ to $\mathbb{A}$ induces the isomorphism $\mathbb{A}/0_{\mathbb{A}}^* \to \mathbb{A}/0_{\mathbb{A}}^*$ which corresponds to swapping the equivalence classes $\{a\}$ and $\{b,c\}$ of $0_{\mathbb{A}}^*$. Note that the only candidate for $\mathbb{C}$ is the relation $\{(a,b),(a,c),(b,a),(c,a)\}$, and for this choice of $\mathbb{C}$ the congruence lattice $\mathrm{Con}(\mathbb{C})$ is given by the following picture.



As the reader can see, there is no pair $\gamma, \delta \in \mathrm{Con}(\mathbb{C})$ such that $[\![\ker\pi_1, \ker\pi_1 \vee \ker\pi_2]\!] \searrow [\![\gamma,\delta]\!] \nearrow [\![\ker\pi_2, \ker\pi_1 \vee \ker\pi_2]\!]$, so $\mathbb{C}$ is not the graph of a similarity. Alternatively, we can see that $\mathbb{C}$ can't be the graph of a similarity using the characterization in Corollary A.5.38, since the corresponding congruence classes of $0_{\mathbb{A}}^*$ which are linked by $\mathbb{C}$ do not have the same sizes.

Thus, in any subdirect critical relation $\mathbb{R} \le_{sd} \mathbb{A}^k$ of arity $k > 2$, each $\pi_{i,j}(\mathbb{R})$ must be the congruence $0_{\mathbb{A}}^*$, so $\mathbb{R}$ will consist of the tuple $(a, ..., a)$ together with some subalgebra of $\{b, c\}^k$. Since for any $\mathbb{S} \le \{b, c\}^k$ the set $\mathbb{S} \cup \{(a, ..., a)\}$ will always be closed under φ_2, if $\mathbb{R}$ is critical then so is $\mathbb{R} \setminus \{(a, ..., a)\}$, and it's easy to check that there are only two critical relations $\mathbb{S} \le_{sd} \{b, c\}^k$. This completes the classification of critical relations in $\mathrm{Inv}_k(\mathbb{A})$ for $k > 2$.

Remark 2.3.1. Using the structure theorem 2.3.10 and the fact that the centralizer of the monolith $(0 : 0^*)$ is automatically abelian for subdirectly irreducible algebras in residually small congruence modular varieties (Corollary A.5.30), one can easily reduce the subpower membership problem 2.4.1 for residually small congruence modular varieties to the subpower membership problem for abelian groups by taking advantage of the properties of the Gumm difference term (see Corollary A.3.9). For details of the reduction, see [42].

Example 2.3.2. We give an example of a minimal algebra with few subpowers which does not generate a residually small variety. Let $\mathbb{A} = (\{a, b, c, d\}, g)$, where g is the idempotent ternary symmetric operation which is determined by that fact that it commutes with the cyclic permutation $\sigma = (a\ b\ c\ d)$ and satisfies

$$\begin{aligned} g(a, a, b) &= a, \\ g(a, a, c) &= c, \\ g(a, a, d) &= c, \\ g(a, b, c) &= c. \end{aligned}$$

Then $\mathbb{A}$ has a unique nontrivial congruence $0_{\mathbb{A}}^*$ corresponding to the partition $\{a, c\}, \{b, d\}$, and $\mathbb{A}/0_{\mathbb{A}}^*$ is isomorphic to a two element majority algebra. The congruence classes of $0_{\mathbb{A}}^*$ are affine over $\mathbb{Z}/2$, and the algebra $\mathbb{S} = \mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\}$ has a congruence ψ corresponding to the partition

$$\left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ c \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} d \\ a \end{bmatrix} \right\}, \left\{ \begin{bmatrix} a \\ d \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix}, \begin{bmatrix} c \\ b \end{bmatrix}, \begin{bmatrix} d \\ c \end{bmatrix} \right\},$$

such that $\mathbb{S}/\psi$ is isomorphic to a two element affine algebra over $\mathbb{Z}/2$ (which is isomorphic to $\{a, c\}$). In fact, we have an isomorphism $\mathbb{S} \cong \mathbb{A} \times \{a, c\}$.

To see that $\mathbb{A}$ has few subpowers, let e be the term

$$e(u, x, y, z) = g(x, g(u, y, y), g(y, g(x, y, z), g(x, y, z))).$$

Then e acts like the majority operation $g(x, y, z)$ on $\mathbb{A}/0_{\mathbb{A}}^*$, acts like the minority operation $g(x, u, y)$ on $\{a, c\}$, and has

$$\begin{aligned} e\left(\begin{bmatrix} b & b & a & a \\ b & a & b & a \\ a & a & a & b \end{bmatrix}\right) &= g\left(\begin{bmatrix} b & a & a \\ a & b & a \\ a & a & a \end{bmatrix}\right) = \begin{bmatrix} a \\ a \\ a \end{bmatrix}, \\ e\left(\begin{bmatrix} d & d & a & a \\ d & a & d & a \\ a & a & a & d \end{bmatrix}\right) &= g\left(\begin{bmatrix} d & c & a \\ a & d & c \\ a & a & a \end{bmatrix}\right) = \begin{bmatrix} a \\ a \\ a \end{bmatrix}. \end{aligned}$$

Thus e is a 3-edge term.

Note that applying σ to the second coordinate of $\mathbb{S}$ turns it into $0_{\mathbb{A}}^*$, and under the isomorphism $(1,\sigma) : \mathbb{S} \xrightarrow{\sim} 0_{\mathbb{A}}^*$, one of the congruence classes of ψ becomes the diagonal $\{(x,x) \mid x \in \mathbb{A}\}$. Thus $0_{\mathbb{A}}^*$ is the center of $\mathbb{A}$, and $\mathbb{A}$ is similar to the idempotent reduct of $\mathbb{Z}/2$. Since $1_{\mathbb{A}} = (0_{\mathbb{A}} : 0_{\mathbb{A}}^*)$ is not abelian, we see that $\mathbb{A}$ can't generate a residually small variety.

We can check that

$$\begin{bmatrix} a & b \\ a & b \\ a & b \end{bmatrix} \not\in \mathrm{Sg}_{\mathbb{A}^{3\times 2}} \left\{ \begin{bmatrix} a & b \\ a & b \\ b & a \end{bmatrix}, \begin{bmatrix} a & b \\ b & a \\ a & b \end{bmatrix}, \begin{bmatrix} b & a \\ a & b \\ a & b \end{bmatrix} \right\}$$

by taking the rows modulo ψ. Thus none of the subsets $\{a,b\},\{b,c\},\{c,d\},\{d,a\}$ (which are taken to each other by powers of the automorphism σ) are closed under any term which acts nontrivially on $\mathbb{A}/0_{\mathbb{A}}^*$. Using this, one can show that $\mathrm{Clo}(g)$ does not contain any proper Taylor subclones.

What do critical relations on $\mathbb{A}$ look like? Suppose that $\mathbb{R} \le_{sd} \mathbb{A}^m \times \{a,c\}^n$ is critical and subdirect for some m,n with $m+n \ge 3$. By Theorem 2.3.4, $\mathbb{R}$ has the parallelogram property. All we can conclude from Theorem 2.3.10 is that $\mathbb{R}^*$ has the parallelogram property and has linking congruence $(0_{\mathbb{A}}^*)^m \times 1_{\{a,c\}}^n$, so the reduction $\mathbb{R}_{red}^*$ of $\mathbb{R}^*$ is a subdirect m-ary relation on the two element majority algebra $\mathbb{A}/0_{\mathbb{A}}^*$ which has the parallelogram property.

Luckily, it turns out that any such $\mathbb{R}_{red}^*$ has $\pi_{ij}(\mathbb{R}_{red}^*)$ either a full relation or the graph of an automorphism of $\mathbb{A}/0_{\mathbb{A}}^*$ for any $i,j \in [m]$. This can be proved directly by reasoning about globally consistent instances of 2-SAT whose solution sets have the parallelogram property, or it can be proved more abstractly by using the fact that the two element majority algebra is subdirectly irreducible and generates a congruence distributive variety.

However we prove the claim about $\mathbb{R}_{red}^*$, we see that if we assume without loss of generality that $(a,...,a) \in \mathbb{R}$ (by applying powers of σ to coordinates of $\mathbb{R}$), then we can group the coordinates of $\mathbb{R}$ into groups of size $m_1,...,m_k$,

$$\mathbb{R} \le_{sd} \mathbb{A}^{m_1} \times \cdots \times \mathbb{A}^{m_k} \times \{a,c\}^n,$$

such that $\pi_{ij}(\mathbb{R})$ is full for coordinates i,j coming from separate groups, and $\pi_{ij}(\mathbb{R}) = 0_{\mathbb{A}}^*$ for coordinates i,j coming from the same group.

Since we have assumed $(a,...,a) \in \mathbb{R}$, $\mathbb{R}$ must be closed under the unary polynomial $\phi : x \mapsto g(a,x,x)$. Since $\phi(a) = \phi(c) = a$ and $\phi(b) = \phi(d) = d$, we see that any vector of as and ds which is constant on each group of coordinates will be contained in $\mathbb{R}$. From this we see that in fact, any piecewise-constant vector

$$((x_1,...,x_1),(x_2,...,x_2),...,(x_k,...,x_k),(a,...,a)) \in \mathbb{A}^{m_1} \times \cdots \times \mathbb{A}^{m_k} \times \{a,c\}^n$$

must be contained in $\mathbb{R}$. If we now consider the restriction $\mathbb{R} \cap \{a,c\}^{m+n}$, then we find that it is an affine space defined by a system of linear equations over $\mathbb{Z}/2$, where the number of coordinates from any single group which show up in any equation must be even, since we may swap $(a,...,a),(c,...,c) \in \mathbb{A}^{m_i}$ in any element of $\mathbb{R}$. Thus we see that $\mathbb{R}$ can be written as an intersection of relations $\mathbb{R}'$ where the coordinates pair up in groups $\{i,j\}$ of size two, such that $\pi_{ij}(\mathbb{R}') = 0_{\mathbb{A}}^*$ and the relation $\mathbb{R}'$ factors through the map $0_{\mathbb{A}}^* \twoheadrightarrow \{a,c\}$ for each such pair of coordinates.

Using the above analysis, we see that the relational clone corresponding to $\mathbb{A}$ is generated by the graph of the automorphism σ, which is $\mathrm{Sg}_{\mathbb{A}^2}\{(a,b),(d,a)\}$, the critical binary relation

$\text{Sg}_{\mathbb{A}^2}\{(a,a),(a,b),(b,b)\}$, which corresponds to a partial order on the majority algebra $\mathbb{A}/0^*_{\mathbb{A}}$, and the ternary relation $\text{Sg}_{\mathbb{A}^3}\{(a,a,a),(a,c,c),(b,b,a)\}$, which is the graph of the homomorphism $0^*_{\mathbb{A}} \twoheadrightarrow \{a,c\}$.

2.4 Learnability of relations encoded by compact representations

We'll start off by reviewing some of the standard definitions of learning theory.

Definition 2.4.1. Fix a universe U. We call a collection $\mathcal{C}$ of subsets of U, together with a rule for encoding the elements of $\mathcal{C}$, a *concept class*. An encoding of an element $C \in \mathcal{C}$ is called a *concept* (from $\mathcal{C}$). The encoding scheme is called *polynomially evaluable* if there is an algorithm which takes an encoding of a concept $C \in \mathcal{C}$ and an element $x \in U$, and determines whether $x \in C$ in polynomial time.

Generally we imagine a situation in which a teacher knows a target concept $C \in \mathcal{C}$, and a student tries to learn the target concept C from the teacher, either by seeing (random) examples of elements in U and being told whether or not they are in the target concept C, or by asking the teacher certain types of questions. The teacher is modeled as an oracle which can be queried by the learner.

The main model which we will be examining in this section is the model of *exact learning with (improper) equivalence queries* from [3]. Learnability results in the equivalence query model can be converted directly into learnability results in the *probably approximately correct* model (which is often abbreviated as PAC-learning).

Definition 2.4.2. Let $\mathcal{C}'$ be a concept class which contains $\mathcal{C}$, and call $\mathcal{C}'$ the *hypothesis* class. We define an *equivalence oracle* O_C with *target concept* $C \in \mathcal{C}$ to be the function which takes as input a hypothesis $C' \in \mathcal{C}'$, returns "true" if $C = C'$, and otherwise returns an (arbitrary) element of the symmetric difference $C \Delta C'$.

Definition 2.4.3. An algorithm which makes calls to an oracle O is said to *learn* the concept class $\mathcal{C}$ in the exact model with equivalence queries if, when the oracle O is the equivalence oracle O_C with target concept $C \in \mathcal{C}$, the algorithm makes finitely many calls to the oracle O with encodings of hypotheses $C' \in \mathcal{C}'$ before finally discovering the concept C. The learning algorithm is called *proper* if $\mathcal{C}' = \mathcal{C}$, and *improper* otherwise. If there is an algorithm which learns $\mathcal{C}$ in time polynomial in $\log |U|$, then we say that $\mathcal{C}$ is *polynomially learnable*.

We are interested in the case where the universe U is $\mathbb{A}^n$ for n large and $\mathbb{A}$ a fixed algebraic structure, and where the concept class $\mathcal{C}$ consists of the set of subalgebras of $\mathbb{A}^n$, i.e. $\mathcal{C} = \text{Inv}_n(\mathbb{A})$ (recall $\text{Inv}_n(\mathbb{A})$ is the set of n-ary relations which are preserved by the basic operations of $\mathbb{A}$). In order for polynomial (in n) length encodings of the concepts in $\mathcal{C}$ to exist, we need $\log |\mathcal{C}|$ to be bounded by a polynomial in n, that is, we need $\mathbb{A}$ to have few subpowers.

Suppose that $\mathbb{A}$ has a k-edge term, and fix a particular k-edge term e. In this case, n-ary relations on $\mathbb{A}$ are naturally encoded by compact representations, so we will use compact representations as our encoding scheme for the concept class $\mathcal{C} = \text{Inv}_n(\mathbb{A})$.

For the sake of definiteness, we will slightly modify the definition of a compact representation R by requiring that for each element x_I of $\pi_I(R)$ (where $|I| < k$), a specific element $x \in R$ with $\pi_I(x) = x_I$ has been marked (by x_I), and similarly for each minority index (i,a,b) of R, a particular ordered pair $(u_a, u_b) \in R^2$ witnessing this index has been marked (by (i, a, b)). We will also require that each element x of the compact representation R is marked at least once (i.e., either x is part of a marked witness to a minority index of R, or x is a marked witness for some element of a projection of R onto a small set of coordinates).

One feature which we would like this encoding scheme to satisfy is that there should be a polynomial time procedure to check whether an element $a \in \mathbb{A}^n$ is contained in the relation $\mathbb{R}$ encoded by the compact representation R. In other words, we want our encoding scheme to be *polynomially evaluable*. The next lemma can be used to show that our encoding scheme is polynomially evaluable. We use the notation $[i]$ for the set $\{1, ..., i\}$.

Lemma 2.4.4. *Suppose that $R \subseteq \mathbb{A}^n$ is a compact representation of $\mathbb{R} \le \mathbb{A}^n$, $i \le n$, $a \in \mathbb{A}^n$, $b \in \mathbb{R}$ with $\pi_{[i-1]}(a) = \pi_{[i-1]}(b)$, and set $c_i = d(b_i, a_i)$. Suppose that*

- *for each $I \subseteq [i]$ with $|I| < k$ and $i \in I$, the element $x^I \in R$ is the marked element of R witnessing $\pi_I(x^I) = \pi_I(a)$, and*
- *the pair $(u_a, u_c) \in R^2$ is the marked witness of the minority index (i, a_i, c_i).*

Then there is a term $t^{[i]}$ of $\mathbb{A}$ which can be built out of the terms e, s, p, d of Theorem 2.2.8 in time polynomial in n, such that $b^{[i]} = t^{[i]}(b, u_a, u_c, x^{I_1}, ...) \in \mathbb{R}$ satisfies $\pi_{[i]}(a) = \pi_{[i]}(b^{[i]})$.

Proof. The proof is a modification of the proof of Theorem 2.2.11, with the induction over subsets of $[i]$ modified to only involve polynomially many subsets of $[i]$. The trick is to consider sets I of the form $[j] \cup J$, where $j \le i$, $|J| = k - 1$, and $i \in J$. There are only polynomially many such sets I, and we can induct on j to handle them.

So we will show by induction on j that for every set $I = [j] \cup J$ with $|J| = k - 1$ and $i \in J$, there is a term t^I such that $b^I = t^I(b, u_a, u_c, x^{I_1}, ...)$ satisfies $\pi_I(b^I) = \pi_I(a)$. The base case $j = 0$ is handled by taking $t^I = x^I$ for $|I| = k - 1$.

For the inductive step, note that if $I = [j] \cup J$, then we can also write $I = [j-1] \cup (\{j\} \cup J)$. Let $\{j\} \cup J = \{l_1, ..., l_{k-1}, i\}$, and define sets $I_1, ..., I_{k-1}$ by deleting $l_1, ..., l_{k-1}$, respectively, from I, and note that each of the sets I_m has the form $I_m = [j-1] \cup J_m$, where $J_m = (\{j\} \cup J) \setminus \{l_m\}$ and $i \in J_m$. By the induction hypothesis, we have already constructed terms t^{I_m} and corresponding elements $b^{I_m} \in \mathbb{R}$ with $\pi_{I_m}(b) = \pi_{I_m}(a)$. Then if we consider

$$s(b, b^{I_1}, ..., b^{I_{k-1}}),$$

we see that if we restrict to the coordinates in $\{j\} \cup J$, we have

$$s\left(\begin{bmatrix} a_{l_1} & b^{I_1}_{l_1} & a_{l_1} & \cdots & a_{l_1} \\ a_{l_2} & a_{l_2} & b^{I_2}_{l_2} & \cdots & a_{l_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{l_{k-1}} & a_{l_{k-1}} & a_{l_{k-1}} & \cdots & b^{I_{k-1}}_{l_{k-1}} \\ b_i & a_i & a_i & \cdots & a_i \end{bmatrix}\right) = \begin{bmatrix} a_{l_1} \\ a_{l_2} \\ \vdots \\ a_{l_{k-1}} \\ c_i \end{bmatrix}.$$

Additionally, if we consider

$$p(u_c, u_a, b^{I_1}),$$

then if we restrict to the coordinates in $\{j\} \cup J$, we have

$$p\left(\begin{bmatrix} u_{l_1} & u_{l_1} & b_{l_1}^{I_1} \\ u_{l_2} & u_{l_2} & a_{l_2} \\ \vdots & \vdots & \vdots \\ u_{l_{k-1}} & u_{l_{k-1}} & a_{l_{k-1}} \\ c_i & a_i & a_i \end{bmatrix}\right) = \begin{bmatrix} b_{l_1}^{I_1} \\ a_{l_2} \\ \vdots \\ a_{l_{k-1}} \\ c_i \end{bmatrix}.$$

Thus, if we take t^I to be given by

$$t^I = e(p(u_c, u_a, t^{I_1}), s(b, t^{I_1}, ..., t^{I_{k-1}}), t^{I_1}, ..., t^{I_{k-1}}),$$

then when we restrict to the coordinates in $\{j\} \cup J$, we get

$$e\left(\begin{bmatrix} b_{l_1}^{I_1} & a_{l_1} & b_{l_1}^{I_1} & a_{l_1} & \cdots & a_{l_1} \\ a_{l_2} & a_{l_2} & a_{l_2} & b_{l_2}^{I_2} & \cdots & a_{l_2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{l_{k-1}} & a_{l_{k-1}} & a_{l_{k-1}} & a_{l_{k-1}} & \cdots & b_{l_{k-1}}^{I_{k-1}} \\ c_i & c_i & a_i & a_i & \cdots & a_i \end{bmatrix}\right) = \begin{bmatrix} a_{l_1} \\ a_{l_2} \\ \vdots \\ a_{l_{k-1}} \\ a_i \end{bmatrix},$$

which completes the induction step. □

We can now check if an element $a \in \mathbb{A}^n$ is in the relation encoded by the compact representation R as follows. First we check that $\pi_I(a) \in \pi_I(S)$ for each I with $|I| < k$, and let x^I be the marked element of R with $\pi_I(x^I) = \pi_I(a)$. Then we try to construct elements $b^{[i]} \in \mathrm{Sg}_{\mathbb{A}^n}(R)$ iteratively with $\pi_{[i]}(b^{[i]}) = \pi_{[i]}(a)$. We start by taking $b^{[k-1]} = x^{[k-1]}$, and repeatedly invoke Lemma 2.4.4 to see that if $(i, a_i, c_i) \in \mathrm{Sig}(R)$, where $c_i = d(b_i^{[i-1]}, a_i)$, then we can construct $b^{[i]} \in \mathrm{Sg}_{\mathbb{A}^n}(R)$ in polynomial time. We formalize this procedure as a subroutine which I will call `Approximate`(R, a) (this is almost the same as the combination of the subroutines `Interpolate` and `New-Fix-values` from [101]).

The running time of `Approximate` is $O(n^{k+1})$: there are less than n^k choices for $j = l_1 < \cdots < l_{k-1} < i$, and for each choice, computing the new b^I takes $O(n)$ steps (since b^I has n coordinates). By only maintaining the values of $b^{[i-1]}$ and b^I with $i \in I$ in the ith step through the outer loop, the memory required is reduced to $O(n^k)$, which is the same as the space required to store a typical compact representation R.

Proposition 2.4.5. *If $R \subseteq \mathbb{A}^n$ is a compact representation and $a \in \mathbb{A}^n$ with $\pi_I(a) \in \pi_I(R)$ for all I with $|I| < k$, then either `Approximate`(R, a) returns a and $a \in \mathrm{Sg}_{\mathbb{A}^n}(R)$, or `Approximate`$(R, a)$ returns $b \neq a$ such that $b \in \mathrm{Sg}_{\mathbb{A}^n}(R)$, and such that if i is minimal with $b_i \neq a_i$, then the minority index $(i, a_i, d(b_i, a_i))$ is not witnessed in R.*

At this point everything seems wonderful, but there is one major wrinkle: we have no idea how to (efficiently) test whether a given "compact representation" $R \subseteq \mathbb{A}^n$ is actually a compact representation of the subalgebra $\mathbb{R} = \mathrm{Sg}_{\mathbb{A}^n}(R)$ it generates - in other words, we don't know how to test whether R is a valid encoding of a concept from the concept class $\mathcal{C} = \mathrm{Inv}_n(\mathbb{A})$. While it's easy to test whether R and $\mathbb{R}$ have the same projections onto small subsets of the coordinates (just

Algorithm 10 $\texttt{Approximate}(R, a)$, e, s, p, d terms as in Theorem 2.2.8, $R \subseteq \mathbb{A}^n$ a compact representation such that $\pi_I(a) \in \pi_I(R)$ for all I with $|I| < k$.

```
for all $I \subseteq [n]$ with $|I| < k$ do
    Let $x^I$ be the marked element of $R$ with $\pi_I(x^I) = \pi_I(a)$.
Set $b^{[k-1]\cap[n]} = x^{[k-1]\cap[n]}$.
for $i$ from $k$ to $n$ do
    Set $c_i \leftarrow d(b_i^{[i-1]}, a_i)$.
    if $(i, a_i, c_i) \not\in \mathrm{Sig}(R)$ then
        return $b^{[i-1]}$.
    else
        Let $(u_a^i, u_c^i)$ be the marked witness of the minority index $(i, a_i, c_i)$ in $R$.
    for $j$ from 1 to $i - k + 1$ do
        for all $l_1, ..., l_{k-1}$ with $j = l_1 < l_2 < \cdots < l_{k-1} < i$ do
            Set $I \leftarrow [j] \cup \{l_2, ..., l_{k-1}, i\}$.
            for $m$ from 1 to $k - 1$ do
                Set $I_m \leftarrow I \setminus \{l_m\}$.
            Set $b^I \leftarrow e(p(u_c^i, u_a^i, b^{I_1}), s(b^{[i-1]}, b^{I_1}, ..., b^{I_{k-1}}), b^{I_1}, ..., b^{I_{k-1}})$.
return $b^{[n]}$.
```

check whether $\pi_I(R)$ is closed under the operations of $\mathbb{A}$ for all I with $|I| < k$), what is missing is a way to test whether R witnesses every minority index which is witnessed in $\mathbb{R}$.

Let's think for a moment about the problem of checking whether R and $\mathrm{Sg}(R)$ witness the same minority indices. Since there are only $n|\mathbb{A}|^2$ possible minority indices, we may as well focus on one particular minority index (i, a, b). By replacing R with $\pi_{[i]}(R)$ and n with i, we may reduce to the case $i = n$.

Proposition 2.4.6. *Suppose that the minority index (i, a, b) is witnessed by some pair (u_a, u_b) in a relation $\mathbb{R} \leq \mathbb{A}^n$. Then for any tuple $t_a \in \mathbb{R}$ with $\pi_i(t_a) = a$, there is a tuple $t_b \in \mathbb{R}$ such that the pair (t_a, t_b) also witnesses the minority index (i, a, b). If $i = n$, then t_b is uniquely determined by t_a.*

Proof. Take $t_b = p(u_b, u_a, t_a)$. Then the identity $p(y, y, x) \approx x$ implies that $\pi_{[i-1]}(t_b) = \pi_{[i-1]}(t_a)$, and the fact that a, b are a minority pair (that is, that $d(b, a) = b$) and the identity $p(x, y, y) \approx d(x, y)$ imply that $\pi_i(t_b) = p(b, a, a) = b$. □

So we can check whether a minority index (i, a, b) is witnessed by $\mathrm{Sg}(R)$ as follows. First we pick any tuple $t_a \in R$ with $\pi_i(t_a) = a$. Then we modify it to make a tuple t_b, by replacing the ith coordinate with a b. Finally, we check whether $\pi_{[i]}(t_b) \in \mathrm{Sg}(\pi_{[i]}(R))$. By the above results, we have $\pi_{[i]}(t_b) \in \mathrm{Sg}(\pi_{[i]}(R))$ if and only if $(i, a, b) \in \mathrm{Sig}(\mathrm{Sg}(R))$. We find ourselves naturally led to consider the *subpower membership problem.*

Problem 2.4.1 (Subpower Membership Problem)**.** Given a finite subset $S \subseteq \mathbb{A}^n$ and an element $x \in \mathbb{A}^n$, determine if x is in the subalgebra of $\mathbb{A}^n$ generated by S.

Theorem 2.4.7 (Bulatov, Mayr, Szendrei [42])**.** *For a fixed finite algebra $\mathbb{A}$ with few subpowers, the following problems are polynomial time reducible to each other:*

- *The subpower membership problem for $\mathbb{A}$: determine if $x \in \mathrm{Sg}_{\mathbb{A}^n}(S)$, given $x \in \mathbb{A}^n$ and $S \subseteq \mathbb{A}^n$.*
- *Find a compact representation for $\mathrm{Sg}_{\mathbb{A}^n}(S)$, given a subset $S \subseteq \mathbb{A}^n$.*
- *The subpower intersection problem for $\mathbb{A}$: given subsets $R, S \subseteq \mathbb{A}^n$, find a set of generators for $\mathrm{Sg}_{\mathbb{A}^n}(R) \cap \mathrm{Sg}_{\mathbb{A}^n}(S)$.*

If $\mathbb{A}$ has few subpowers and has a finite number of basic operations, then the subpower membership problem for $\mathbb{A}$ is in NP.

Proof. Left as an exercise to the reader. The hardest bit is the claim that the subpower membership problem is in NP: for this, imagine that we have a set R which looks like a compact representation, and consider the set C of all $a \in \mathbb{A}^n$ such that $\texttt{Approximate}(R, a)$ returns a. If C is not closed under the basic operations of $\mathbb{A}$, then there should be a *witness* to the fact that C is not closed, and a nondeterministic algorithm can guess such a witness, verify that it works, and use it to enlarge R. □

Unfortunately, whether the subpower membership problem is in P for algebras with few subpowers is currently an open problem (even in the special case of quasigroups). So we need to find a workaround for this issue.

The workaround is to enlarge the concept class $\mathcal{C} = \mathrm{Inv}_n(\mathbb{A})$ to a larger concept class $\mathcal{C}'$, where concepts in $\mathcal{C}'$ are encoded by "compact representations" $R \subseteq \mathbb{A}^n$, where we allow sets R which are not compact representations of the subalgebra $\mathbb{R} = \mathrm{Sg}_{\mathbb{A}^n}(R)$ which they generate. In order to be precise about exactly what concept C is encoded by R, we use the $\texttt{Approximate}$ subroutine.

Definition 2.4.8. If $R \subseteq \mathbb{A}^n$ is a "compact representation", then the corresponding concept $C \subseteq \mathbb{A}^n$ encoded by R is defined by the following rule. An element $a \in \mathbb{A}^n$ is in C iff the following two conditions are satisfied:

- for every $I \subseteq [n]$ with $|I| < k$, we have $\pi_I(a) \in \pi_I(R)$, and
- the subroutine $\texttt{Approximate}(R, a)$ returns a.

The penalty we will pay for this workaround is that since the new concept class $\mathcal{C}'$ is larger than $\mathcal{C} = \mathrm{Inv}_n(\mathbb{A})$, our learning algorithm will now be making *improper* equivalence queries. If the subpower membership problem for $\mathbb{A}$ can be proved to be in P, then we will be able to upgrade to a learning algorithm which makes only proper equivalence queries.

Now we can finally describe the learning algorithm, which is remarkably simple.

Proposition 2.4.9. *For a fixed algebra $\mathbb{A}$ with few subpowers, the algorithm $\texttt{Learn}(O)$ takes time polynomial in n to find an encoding R for the target concept $C \in \mathrm{Inv}_n(\mathbb{A})$.*

Proof. At every step of the algorithm, we have $\mathrm{Sg}(R) \subseteq C$: this is true at the beginning, and if it is true before we call $O(R)$, then since the concept C' encoded by R has $C' \subseteq \mathrm{Sg}(R) \subseteq C$, the value a returned by $O(R)$ will be contained in $C \Delta C' = C \setminus C' \subseteq C$, so $\mathrm{Sg}(R \cup \{a\}) \subseteq C$.

Furthermore, every time we process a new $a \in C \setminus C'$, we strictly enlarge R to make the new concept encoded by R contain a, either by adding a as a designated witness to $\pi_I(a) \in \pi_I(R)$ for some I, or by adding new minority indices which were not present in the original R. Since R can only increase in size polynomially many times (as a compact representation has size bounded by a polynomial in n), we can only call the oracle polynomially many times before the process must terminate. □

Algorithm 11 `Learn`(O), O an equivalence oracle for an unknown target concept $C \in \mathrm{Inv}_n(\mathbb{A})$.

```
Set $R \leftarrow \emptyset$.
while $O(R)$ does not return “true” do
    Set $a \leftarrow O(R)$.
    for all $I \subseteq [n]$ with $|I| < k$ such that $\pi_I(a)$ has no designated witness in $R$ do
        Set $R \leftarrow R \cup \{a\}$.
        Mark $a$ as the designated witness for $\pi_I(a)$.
    while Approximate$(R, a)$ does not return $a$ do
        Set $b \leftarrow$ Approximate$(R, a)$.
        Let $i$ be minimal such that $a_i \neq b_i$.
        Set $R \leftarrow R \cup \{a, d(b, a)\}$.
        Mark the pair $(a, d(b, a))$ as the designated witness for the minority index $(i, a_i, d(b_i, a_i))$.
    Optionally, enlarge $R$ further to make it closer to a compact representation of $\mathrm{Sg}(R)$.
```

Remark 2.4.1. If we did not insist on polynomial evaluability of the encoding scheme (or if we could solve the subpower membership problem), then we could instead encode relations via generating sets. The learning algorithm would then be even simpler: at every step, the learner guesses that the target concept is the relation generated by all the examples it has seen so far. This learning algorithm is known as the *closure algorithm*. The issue is that now the equivalence oracle becomes hard to implement, as the teacher is forced to determine whether a given set generates the target relation they have in mind.

Now we will explain how all of this relates to Valiant's PAC-learning model [183]. In the PAC-learning model, the teacher (oracle) has access to both a target concept $C \in \mathcal{C}$ and a probability distribution μ over the universe U, both of which are unknown to the learner. The learner is allowed to request random classified examples, sampled from the distribution μ (by a “classified” example, I mean that the learner is given an example and told whether or not it is in the target concept C).

Definition 2.4.10. If $C \in \mathcal{C}$ is a target concept and μ is a probability distribution on the universe U, then the *sampling oracle* for the pair C, μ is a randomized oracle which samples a random element $a \in U$ drawn from the distribution μ, and returns the ordered pair $(a, a \in C)$, where by “$a \in C$” we mean either “true” or “false” based on whether a is in the target concept C.

In the PAC-learning model, the goal of a learning algorithm is to output an encoding of a concept C' in the hypothesis class $\mathcal{C}'$, such that the μ-measure $\mu(C \Delta C')$ of the symmetric difference between C and C' is small. We can't hope to do better than this, since the chance of seeing an example which lets us distinguish between C and C' is at most $\mu(C \Delta C')$ times the number of classified examples we request.

Definition 2.4.11. We say that an algorithm with access to a sampling oracle *learns* a concept class $\mathcal{C}$ in the *probably approximately correct* model with *error* ϵ and *confidence* $1 - \delta$ if for any target concept $C \in \mathcal{C}$ and any probability distribution μ over the universe, the algorithm eventually returns a hypothesis $C' \in \mathcal{C}'$ such that

$$\mathbb{P}[\mu(C \Delta C') \leq \epsilon] \geq 1 - \delta.$$

The probability here is taken over the random choices made by the oracle (and possibly the learning algorithm) - the target concept C is *not* being randomized here, we require this for *all* $C \in \mathcal{C}$ and *all* μ.

We say that a concept class $\mathcal{C}$ is *efficiently PAC-learnable* if there is an algorithm which learns $\mathcal{C}$ in the PAC-model and takes time polynomial in $\log(|U|)$, $\frac{1}{\epsilon}$, and $\log(\frac{1}{\delta})$ for $\epsilon, \delta > 0$.

The standard learning algorithm in the PAC model is to request a large number of classified examples, and then choose *any* hypothesis $C' \in \mathcal{C}'$ which is consistent with all of the classified examples we have seen so far. For this to work, it is necessary that the hypothesis class $\mathcal{C}'$ is in some sense "small", and we also need to have a way to efficiently find at least one hypothesis which is consistent with the examples. First we will define a measure of the "size" a the concept class $\mathcal{C}$, known as the VC-dimension.

Definition 2.4.12. If $\mathcal{C}$ is a collection of subsets of some universe U, then we say that a set S is *shattered* by $\mathcal{C}$ if for all $X \subseteq S$, there is some $C \in \mathcal{C}$ with $C \cap S = X$. We define the *Vapnik-Chervonenkis dimension* of $\mathcal{C}$, written $\mathrm{VC}(\mathcal{C})$, to be the size of the largest set S which is shattered by $\mathcal{C}$.

To see that the VC-dimension is a good measure of the complexity of a concept class, we recall the Sauer-Shelah Lemma.

Lemma 2.4.13 (Sauer-Shelah Lemma). *If $\mathcal{C}$ is a collection of subsets of U with VC-dimension d, then*

$$|\mathcal{C}| \le \sum_{i=0}^{d} \binom{|U|}{i}.$$

In fact, we have the stronger result that the number of sets $S \subseteq U$ which are shattered by $\mathcal{C}$ is at least $|\mathcal{C}|$.

Proof. We show that $\mathcal{C}$ shatters at least $|\mathcal{C}|$ sets by induction on $|\mathcal{C}|$. For the base case, note that the empty set is shattered by $\mathcal{C}$ as long as $|\mathcal{C}| \ge 1$. For the inductive step, let $x \in U$ be an element which is in some of the sets in $\mathcal{C}$ but not all of them, and let $\mathcal{C}_x$ be the collection of $C \in \mathcal{C}$ with $x \in C$ and $\mathcal{C}'_x = \mathcal{C} \setminus \mathcal{C}_x$. Inductively, $\mathcal{C}_x$ shatters at least $|\mathcal{C}_x|$ sets and $\mathcal{C}'_x$ shatters at least $|\mathcal{C}'_x|$ sets, and any set shattered by $\mathcal{C}_x$ or $\mathcal{C}'_x$ must not contain x.

To finish the induction, we just need to check that for any set S which is shattered by both $\mathcal{C}_x$ and $\mathcal{C}'_x$, the set $S \cup \{x\}$ is shattered by $\mathcal{C}$. □

If the set S is shattered by $\mathcal{C}$, then the sampling oracle could sample from a uniform distribution on S, and in this case the learner is faced with the problem of learning an arbitrary subset $X = C \cap S$ of S given an oracle which returns uniformly random classified examples. If the learner examines $o(|S|)$ classified examples, then clearly they can't hope to succeed. The following result makes this precise.

Proposition 2.4.14. *If an algorithm learns a concept class $\mathcal{C}$ with error ϵ and confidence $1 - \delta$ after requesting at most m classified examples, then*

$$m \ge (2(1-\epsilon)(1-\delta) - 1)\,\mathrm{VC}(\mathcal{C}).$$

Proof. Let S be a set with $|S| = \mathrm{VC}(\mathcal{C})$ which is shattered by $\mathcal{C}$, and for each $X \subseteq S$ consider the sampling oracle O_X which samples from the uniform distribution μ on S, and has target concept some $C_X \in \mathcal{C}$ with $C_X \cap S = X$. If we average the performance of the learning algorithm over the sampling oracles O_X (with X chosen as a uniformly random subset of S), we see that if it outputs a hypothesis C', then

$$\mathbb{E}[\mu(C_X \Delta C')] \geq \frac{1}{2}\Big(1 - \frac{m}{|S|}\Big).$$

By Markov's inequality, this implies that

$$(1-\epsilon)\mathbb{P}[\mu(C_X \Delta C') \leq \epsilon] \leq \frac{1}{2} + \frac{m}{2|S|},$$

so

$$\frac{1}{2} + \frac{m}{2|S|} \geq (1-\epsilon)(1-\delta). \qquad \square$$

Conversely, if the VC-dimension of $\mathcal{C}$ is small, then the standard learning algorithm in the PAC model performs well, so long as it can be implemented.

Theorem 2.4.15 (VC-dimension determines sample-complexity [31])**.** *If* $\mathrm{VC}(\mathcal{C}') = d$*, then any algorithm which takes*

$$m \geq \max\left(\frac{4}{\epsilon}\log\left(\frac{2}{\delta}\right),\ \frac{8d}{\epsilon}\log\left(\frac{13}{\epsilon}\right)\right).$$

samples from a sampling oracle and outputs any hypothesis $C' \in \mathcal{C}'$ *consistent with the data will learn* $\mathcal{C}$ *with error* ϵ *and confidence* $1-\delta$*.*

Sketch. Consider the following process: pick $2m$ samples from the probability distribution μ, permute them randomly, feed the first m samples (after permuting) to the learning algorithm, and count how many of the last m samples are classified incorrectly by the hypothesis C' chosen by the learning algorithm.

If the learning algorithm fails to learn $\mathcal{C}$ with error ϵ and confidence $1-\delta$, then for some choice of target concept C and distribution μ, the process described will incorrectly classify at least $\frac{\epsilon m}{2}$ of the last m samples with probability at least $\frac{\delta}{2}$, by Chebyshev's inequality (at least for $m \geq \frac{8}{\epsilon}$). Thus there will be some specific set X of size $2m$, such that at least a $\frac{\delta}{2}$ fraction of its permutations lead to an incorrect classification of at least $\frac{\epsilon m}{2}$ of its last m elements.

By the Sauer-Shelah Lemma 2.4.13, the number of distinct subsets of X which can be written as $C' \cap X$ for some $C' \in \mathcal{C}'$ is bounded by $\sum_{i \leq d}\binom{2m}{i}$. For each possible intersection $C' \cap X$, the chance of the first m samples from X being consistent with C' and the last m samples from X having at least $\frac{\epsilon m}{2}$ inconsistencies with C' is at most $2^{-\epsilon m/2}$. Thus if the learning algorithm fails, then by the union bound we must have

$$2^{-\epsilon m/2}\sum_{i \leq d}\binom{2m}{i} \geq \frac{\delta}{2},$$

and plugging in the assumed bounds on m and chugging through the inequalities gives a contradiction. $\square$

Note that if $\mathbb{A}$ is an algebraic structure and we take $U = \mathbb{A}^n$, $\mathcal{C} = \mathrm{Inv}_n(\mathbb{A})$, then a set S is shattered by $\mathrm{Inv}_n(\mathbb{A})$ iff S is an *independent* subset of $\mathbb{A}^n$. Thus the VC-dimension of $\mathrm{Inv}_n(\mathbb{A})$ is exactly the same thing as the number $i_{\mathbb{A}}(n)$, so if the concept classes $\mathrm{Inv}_n(\mathbb{A})$ are efficiently PAC-learnable as n varies, then $\mathbb{A}$ must have few subpowers.

We can convert learning algorithms in the equivalence query model into learning algorithms in the PAC model by using the sampling oracle to simulate an equivalence oracle.

Proposition 2.4.16 (Angluin [3])**.** *If a concept class $\mathcal{C}$ is efficiently learnable in the (improper) equivalence query model using a hypothesis class $\mathcal{C}'$ which has a polynomially evaluable encoding scheme, then $\mathcal{C}$ is also efficiently learnable in the PAC model.*

Proof. Given a sampling oracle O, we simulate an equivalence oracle as follows. The ith time the equivalence oracle is called by the learner, say to determine whether the target concept C is equivalent to a hypothesis $C' \in \mathcal{C}'$, we call the sampling oracle O some number q_i times to get q_i random classified examples, and we check whether the way they are classified agrees with the hypothesis C' (here is where we are using polynomial evaluability). If their classifications do agree with C', then we pretend that the equivalence oracle returned "true", and otherwise we pick one of the examples a whose classification does not agree with C' and return a as the counterexample in $C\Delta C'$.

By the union bound, the probability that the simulated equivalence oracle *ever* returns "true" for a hypothesis C' with $\mu(C\Delta C') \ge \epsilon$ is at most

$$\sum_i (1-\mu(C\Delta C'))^{q_i} \le \sum_i (1-\epsilon)^{q_i}.$$

If we take

$$q_i \ge \frac{1}{\epsilon}(\ln(1/\delta) + i\ln(2)),$$

for instance, then we get

$$\sum_i (1-\epsilon)^{q_i} \le \sum_i e^{-\epsilon q_i} \le \sum_i e^{\ln\delta - i\ln(2)} = \sum_i \frac{\delta}{2^i} \le \delta. \qquad \square$$

Remark 2.4.2. Another learning model is the on-line learning model described by Littlestone [136]. In this model, the learner is repeatedly presented with examples, and for each example must guess its classification before being told whether its guess is correct. The goal of the learner is to have an upper bound on the number of incorrect guesses it makes, even if the sequence of examples is chosen adversarially. It is easy to convert a learnability result in the (improper) equivalence query model into an algorithm for on-line learning.

Remark 2.4.3. There is a variant of the PAC learning model in which the learner is also allowed to use *membership queries*: in a membership query, the learner picks an element $x \in U$, and asks the teacher (oracle) whether x is in the target concept.

In [2], several situations are given where the addition of membership queries can be shown not to help with learning, under some standard cryptographic assumptions. In [40], there is a claim that some of the impossibility results for PAC learning of $\mathrm{Inv}_n(\mathbb{A})$ when $\mathbb{A}$ doesn't have few subpowers can be generalized to impossibility results in the model of PAC learning with membership queries (under cryptographic assumptions), but the exact statement and the proof are left to a "full version" of the paper which I have been unable to track down. The more recent paper [58] by Chen and Valeriote proves such a hardness result for algebraic structures which are not congruence modular, and for finitely related structures congruence modularity is equivalent to few subpowers.

2.5 Algebras with few subpowers are finitely related

Suppose a clone $\mathcal{O}$ on a finite domain A has a k-edge term e. We want to show that there exists some finite set of relations $R_1, ..., R_m$ which generate the relational clone which is dual to $\mathcal{O}$. This is equivalent to $\mathcal{O}$ being exactly the set of operations $\mathrm{Pol}(R_1, ..., R_m)$ which preserve the relations $R_1, ..., R_m$. If $R_1, ..., R_m$ are all preserved by $\mathcal{O}$, then the clone $\mathrm{Pol}(R_1, ..., R_m)$ will certainly contain $\mathcal{O}$, but might end up being too large. In this case, $\mathrm{Pol}(R_1, ..., R_m)$ will still contain the k-edge term e, and we can use this to our advantage.

To understand the structure of a clone $\mathcal{O}$ with a k-edge term, we go back to the explicit representation of the set $\mathcal{O}_n$ of n-ary operations of $\mathcal{O}$ as the free algebra over $\mathbb{A} = (A, \mathcal{O})$ on n generators, which is concretely given by the subalgebra

$$\mathcal{O}_n = \mathcal{F}_{\mathbb{A}}(x_1, ..., x_n) \le \mathbb{A}^{\mathbb{A}^n}$$

generated by the elements $\pi_i : \mathbb{A}^n \to \mathbb{A}$ given by $\pi_i(a_1, ..., a_n) = a_i$, where $x_i \in \mathcal{F}_{\mathbb{A}}(x_1, ..., x_n)$ is identified with the element $\pi_i \in \mathbb{A}^{\mathbb{A}^n}$. Similarly, recall that the set of n-ary operations $f \in \mathrm{Pol}_n(R_1, ..., R_m)$, considered as a subalgebra of $\mathbb{A}^{\mathbb{A}^n}$, is given by the primitive positive formula

$$f \in \mathrm{Pol}_n(R_1, ..., R_m) \iff \bigwedge_{i \le m} \bigwedge_{M \in R_i^n} f(M) \in R_i.$$

To check that these two subalgebras of $\mathbb{A}^{\mathbb{A}^n}$ are equal, by Theorem 2.2.11 and the fact that one is contained within the other, it suffices to check that they have the same projections onto subsets $I \subseteq \mathbb{A}^n$ of the coordinates with $|I| < k$, and to check that they have the same forks. If $R_1, ..., R_m$ generate all relations of $\mathrm{Inv}(\mathbb{A})$ with arity less than k, then the first condition will be satisfied. The hard part is dealing with the forks.

In order to make precise statements about the set of forks in $\mathbb{A}^{\mathbb{A}^n}$, we first need to choose an ordering on the coordinates of $\mathbb{A}^{\mathbb{A}^n}$, that is, an ordering on the elements of A^n. A natural choice is to first fix any total order $\le$ on the set A, and to extend this to the *lexicographic order* on A^n.

Definition 2.5.1. If $(A, \le)$ is a set with a total order, then we define the *lexicographic order* $\le_{lex}$ on A^n by $a \le_{lex} b$ iff either $a = b$ or there is some $i \le n$ such that $a_j = b_j$ for $j < i$ and $a_i < b_i$. In other words, $a <_{lex} b$ if $a_i < b_i$ at the first coordinate i where a and b differ.

Definition 2.5.2. If $(I, \le)$ is a totally ordered set and $R \subseteq A^I$ is a relation on A, then for $i \in I$ we define the set of *forks* of R at the ith coordinate to be the set of pairs $(a, b) \in A^2$ given by

$$\mathrm{Forks}(R, i) \coloneqq \{(a, b) \mid \exists t_a, t_b \in R,\ \pi_{<i}(t_a) = \pi_{<i}(t_b), \pi_i(t_a) = a, \pi_i(t_b) = b\}.$$

So in order to understand a clone $\mathcal{O}$ with a k-edge term, we need to understand the relations of arity less than k, together with the set of forks $\mathrm{Forks}(\mathcal{O}_n, a)$ for all $a \in A^n$ and all n. The issue is that while each set $\mathrm{Forks}(\mathcal{O}_n, a)$ is given by a finite collection of pairs of elements, there are infinitely many elements $a \in A^n, n \in \mathbb{N}^+$ to consider. So we need a way to relate $\mathrm{Forks}(\mathcal{O}_n, a)$ to $\mathrm{Forks}(\mathcal{O}_m, b)$ for some choices of $a \in A^n, b \in A^m$.

Proposition 2.5.3. *Suppose $a \in A^n, b \in A^m$. If there is a map $\phi : [m] \to [n]$ such that the associated function $\phi^* : A^n \to A^m$ given by $\phi^*(x_1, ..., x_n) = (x_{\phi(1)}, ..., x_{\phi(m)})$ satisfies the conditions*

- *$\phi^*(a) = b$ and*

- *for all $c <_{lex} a$ we have $\phi^*(c) <_{lex} b$,*

then for any clone $\mathcal{O}$ on A, we have $\text{Forks}(\mathcal{O}_m, b) \subseteq \text{Forks}(\mathcal{O}_n, a)$.

Proof. Letting $\mathbb{A} = (A, \mathcal{O})$, ϕ induces a natural map of free algebras $\mathcal{O}_m \to \mathcal{O}_n$ given by $x_i \mapsto x_{\phi(i)}$. We will write this natural map as $f \mapsto f_\phi$. Considering $\mathcal{O}_m, \mathcal{O}_n$ as subalgebras of $\mathbb{A}^{A^m}, \mathbb{A}^{A^n}$, respectively, we see that for $c \in A^n$ and $f \in \mathcal{O}_m$, the cth coordinate of the image f_ϕ of f under this map is given by

$$f_\phi(c) = f(\phi^*(c)).$$

In particular, if $t, t' \in \mathcal{O}_m$ with $\pi_{<_{lex}b}(t) = \pi_{<_{lex}b}(t')$, then

$$\pi_{<_{lex}a}(t_\phi) = \pi_{<_{lex}a}(t'_\phi),$$

and

$$\begin{bmatrix} t_\phi(a) \\ t'_\phi(a) \end{bmatrix} = \begin{bmatrix} t(\phi^*(a)) \\ t'(\phi^*(a)) \end{bmatrix} = \begin{bmatrix} t(b) \\ t'(b) \end{bmatrix},$$

so every fork in $\text{Forks}(\mathcal{O}_m, b)$ is also a fork in $\text{Forks}(\mathcal{O}_n, a)$. □

Proposition 2.5.4. *A map ϕ as in the previous proposition exists if there is a strictly increasing function $h : [n] \to [m]$ such that*

- *the same elements of A occur in both a and b,*
- *$h^*(b) = a$, that is, $a_i = b_{h(i)}$ for all $i \in [n]$, and*
- *for all $s \in A$, if the index of the first occurence of s in a is i, then $h(i)$ is the index of the first occurence of s in b.*

If no coordinate a_i of a is minimal or maximal with respect to the order $<$ on $\mathbb{A}$, then the converse is true: such a ϕ exists iff such an h exists.

Proof. Given such an h, we define ϕ as follows. We set $\phi(h(i)) = i$, and for j not in the image of h let $\phi(j)$ be the first index i such that $a_i = b_j$, so that $h(\phi(j)) \le j$ for all j. Then for any $c <_{lex} a$, if i is the first index where $a_i \ne c_i$, and if j is the first coordinate where $\phi^*(a)$ and $\phi^*(c)$ differ, then we have $a_{\phi(j)} \ne c_{\phi(j)}$, so $i \le \phi(j)$, so $h(i) \le h(\phi(j)) \le j$, so we must have $h(i) = j$ since $\phi^*(a)$ and $\phi^*(c)$ also differ at $h(i)$. Thus $\phi^*(c) <_{lex} \phi^*(a) = b$.

Now suppose that no coordinate a_i of a is minimal or maximal with respect to the order $<$ on $\mathbb{A}$. Then the map ϕ in the previous proposition must be surjective: if i is not in the image of ϕ, then we can define $c <_{lex} a$ which only differs from a on the ith coordinate, and $\phi^*(c) = \phi^*(a) \not<_{lex} b$, contradicting the choice of ϕ. Thus we can define $h : [n] \to [m]$ by

$$h(i) = \min\{j \in [m] \mid \phi(j) = i\},$$

so $h^*(b) = a$ and we see that a and b have the same set of symbols.

For any i, if we define $c <_{lex} a$ which matches a up to the ith coordinate, has $c_i < a_i$, and $c_j > a_j$ for all $j > i$, then from $\phi^*(c) < b = \phi^*(a)$, we see that $h(i) < h(j)$ for all $j > i$. Thus h must be strictly increasing. Finally, from the definition of h, we see that if i is the index of the first occurence of s in a, then $h(i)$ must be the index of the first occurence of s in b. □

Definition 2.5.5. Let $A^+ = \bigcup_{n \ge 1} A^n$, and define the partial order $\le_E$ on A^+ by $a \le_E b$ iff there exists a map h as in the previous proposition. Equivalently, $a \le_E b$ iff the same set of elements of A occur in a and b, and b can be formed from a by inserting elements $s \in A$ after their first occurences in a.

Note that the partial ordering $\le_E$ on A^+ has no dependence on the arbitrary choice of ordering $<$ we introduced on the elements of A (of course, the set $\mathrm{Forks}(\mathcal{O}, a)$ still depends on the choice of $<$). The partial order $\le_E$ is a refinement of the embeddability partial ordering that occurs in Higman's Lemma [94]. We can now simplify the description of the sets $\mathrm{Forks}(\mathcal{O}, a)$ using the ordering $\le_E$.

Definition 2.5.6. For any pair $(c, d) \in A^2$, we define the set $\lambda(\mathcal{O}, (c, d)) \subseteq A^+$ to be the set of $a \in A^+$ such that $(c, d) \notin \mathrm{Forks}(\mathcal{O}, a)$.

Corollary 2.5.7. *For any clone $\mathcal{O}$ on a set A, the set $\lambda(\mathcal{O}, (c, d))$ is upwards closed in A^+ with respect to $\le_E$, that is, if $a \in \lambda(\mathcal{O}, (c, d))$ and $a \le_E b$, then $b \in \lambda(\mathcal{O}, (c, d))$.*

To describe an upwards closed subset (also called an *upset*) of a *finite* poset, it is enough to describe its minimal elements. We want to show that $\lambda(\mathcal{O}, (c, d))$ can be described in terms of its minimal elements, but for this to work, it's necessary to show that $(A^+, \le_E)$ is a *well partial order.*

Definition 2.5.8. A partial order $(X, \le)$ is a *well partial order* if for every infinite sequence $x_1, x_2, ...$ of elements of X, there exists an infinite increasing subsequence $i_1 < i_2 < \cdots$ such that $x_{i_1} \le x_{i_2} \le \cdots$.

Proposition 2.5.9. *A partial order $(X, \le)$ is a well partial order iff it has no infinite descending chains and no infinite antichains.*

Proof. Let $x_1, x_2, ...$ be any infinite sequence of elements of X. Color the edges of the complete graph on $\mathbb{N}^+$ with three colors, as follows: for $i < j$, the edge $\{i, j\}$ is colored red if $x_i > x_j$, colored blue if x_i, x_j are incomparable, and colored green if $x_i \le x_j$. By Ramsey's Theorem, there must be some infinite monochromatic clique in this graph, so either there is an infinite descending chain, an infinite antichain, or an infinite subsequence $i_1 < i_2 < \cdots$ with $x_{i_1} \le x_{i_2} \le \cdots$. □

Proposition 2.5.10. *A partial order $(X, \le)$ is a well partial order iff for all upsets $U \subseteq X$, every element of U is $\ge$ some minimal element of U and U has finitely many minimal elements, that is, there exists a finite set of elements $u_1, ..., u_k \in U$ such that $U = \{x \mid x \ge u_i$ for some $i\}$.*

Proof. Suppose first that $(X, \le)$ is a well partial order. If some element $u \in U$ is not above any minimal element of U, then we can find an infinite descending chain in U. Since any pair of distinct minimal elements of U are incomparable, the number of minimal elements of U must be finite. The converse follows from the previous proposition. □

So the last ingredient of the argument will be the proof that $\le_E$ is a well partial order. While the tools we have available are capable of proving this directly, it is useful to reduce this to the fact that the simpler (and more well-known) embeddability partial ordering $\le_e$, due to Higman, is a well partial order - this allows us to transfer other results about Higman's ordering to the partial order $\le_E$.

Definition 2.5.11. Define the partial order $\leq_e$ on A^+ by $a \leq_e b$ if b can be formed from a by inserting elements of A.

Proposition 2.5.12. *If B is the disjoint union of A with the two-element set of symbols $\{\#, '\}$, then there is an embedding of partial orders*

$$F : (A^+, \leq_E) \hookrightarrow (B^+, \leq_e),$$

i.e. a function F such that for $x, y \in A^+$, we have $x \leq_E y$ iff $F(x) \leq_e F(y)$.

Proof. We define $F : A^+ \to B^+$ to be the function which modifies $x \in A^+$ by inserting a $'$ after the first occurence of each symbol within x, inserting a # at the end of x, and then following that with a $'$ for each symbol which doesn't occur within x. For instance, if $A = \{a, b, c, d, e\}$, then

$$F(adaadca) = a'd'aadc'a\#'',$$

where the two $'$s at the end keeps track of the fact that b, e did not occur within the word $adaadca$.

Note that $F(x)$ is always formed from x by inserting exactly 1 copy of # and exactly $|A|$ copies of the symbol $'$. Thus, if $F(x) \leq_e F(y)$, then $F(y)$ must be obtained by inserting only symbols from A into the word $F(x)$. If any symbol $s \in A$ is inserted before its first occurence in $F(x)$, or inserted directly in front of a $'$, then we can see that the resulting word can't be of the form $F(y)$, by considering the first location with an invalid insertion. □

Theorem 2.5.13. *If A is a finite set, then the partial order $\leq_e$ on A^+ is a well partial order.*

Proof. We prove this by induction on $|A|$. Since $\leq_e$ clearly has no infinite descending chains (as $a <_e b$ implies $|a| < |b|$), we just need to prove that $\leq_e$ has no infinite antichains. Suppose for contradiction that $\leq_e$ has an infinite antichain, and let $x_1, x_2, ...$ be a lexicographically minimal infinite antichain, that is, suppose that x_1 is minimal such that there exists an infinite antichain containing x_1, that x_2 is minimal such that there exists an infinite antichain containing $\{x_1, x_2\}$, etc.

By the infinite pigeonhole principle, we see that there is an infinite subsequence $i_1 < i_2 < \cdots$ such that every element x_{i_j} ends in the same element of A, say a. Let x'_{i_j} be the element of A^+ we obtain by deleting the a in the last coordinate of x_{i_j}, then from the definition of $\leq_e$ we see that $x_{i_j} \leq_e x_{i_k} \iff x'_{i_j} \leq_e x'_{i_k}$. Let j be minimal such that $x_j >_e x'_{i_k}$ for some k, and note that $j \leq i_1$ so j is well-defined. Then the sequence

$$x_1, x_2, ..., x_{j-1}, x'_{i_k}, x'_{i_{k+1}}, ...$$

is also an infinite antichain, and is lexicographically smaller than $x_1, x_2, ..., x_{j-1}, x_j, ...$, a contradiction. □

Corollary 2.5.14. *If A is a finite set, then the partial order $\leq_E$ on A^+ is a well partial order.*

Theorem 2.5.15 (Few subpowers implies inherently finitely related [1])**.** *If a clone $\mathcal{O}$ contains a k-edge term, then it is finitely related. In fact, a set $\Gamma \subseteq$ Inv($\mathcal{O}$) generates* Inv($\mathcal{O}$) *iff the following two conditions are satisfied:*

- *every relation of arity strictly less than k in* Inv($\mathcal{O}$) *is contained in $\langle\Gamma\rangle$, and*

- *for each minority pair $(c,d) \in A^2$ and each minimal element $a \in \lambda(\mathcal{O},(c,d))$, if we set $n = |a|$, then the relation $\mathrm{Pol}_n(\Gamma)$ on $\mathbb{A}^{\mathbb{A}^n}$ defined by the primitive positive formula*

$$\bigwedge_{R\in\Gamma}\bigwedge_{M\in R^n} f(M) \in R$$

has $(c,d) \notin \mathrm{Forks}(\mathrm{Pol}_n(\Gamma), a)$.

Proof. By the fact that $\le_E$ is a well partial order, we see that there is a finite set $\Gamma \subseteq \mathrm{Inv}(\mathcal{O})$ which satisfies the conditions given: for instance, we may take Γ to consist of the collection of all relations in $\mathrm{Inv}(\mathcal{O})$ of arity less than k, together with the relations $\mathcal{O}_n \le \mathbb{A}^{\mathbb{A}^n}$ for every n such that some minimal element $a \in \lambda(\mathcal{O},(c,d))$ has $|a| = n$ for some $(c,d) \in A^2$.

Now suppose that Γ satisfies the given conditions. Then for any $(c,d) \in A^2$ and any $b \in \lambda(\mathcal{O},(c,d))$, there exists some minimal $a \in \lambda(\mathcal{O},(c,d))$ with $a \le_E b$. Thus if $|a| = n, |b| = m$, then $\mathrm{Forks}(\mathrm{Pol}_m(\Gamma), b) \subseteq \mathrm{Forks}(\mathrm{Pol}_n(\Gamma), a)$, and by the second condition on Γ we have $(c,d) \notin \mathrm{Forks}(\mathrm{Pol}_n(\Gamma), a)$. Thus for any $b \in A^+$ with $|b| = m$, we have

$$(c,d) \notin \mathrm{Forks}(\mathcal{O}_m, b) \implies (c,d) \notin \mathrm{Forks}(\mathrm{Pol}_m(\Gamma), b),$$

so $\mathrm{Forks}(\mathrm{Pol}_m(\Gamma), b) \subseteq \mathrm{Forks}(\mathcal{O}_m, b)$. Since $\mathrm{Pol}_m(\Gamma)$ contains $\mathcal{O}_m$ (by $\Gamma \subseteq \mathrm{Inv}(\mathcal{O})$), and every projection of $\mathrm{Pol}_m(\Gamma)$ onto fewer than k coordinates of $\mathbb{A}^{\mathbb{A}^m}$ is contained in the corresponding projection of $\mathcal{O}_m$ (by the first condition on Γ), we can apply Theorem 2.2.11 to see that $\mathrm{Pol}_m(\Gamma) = \mathcal{O}_m$. □

Corollary 2.5.16. *The number of clones on a finite set which contain an edge term is countable.*

Remark 2.5.1. There is a converse to Theorem 2.5.15: if $\mathcal{O}$ is a clone on a finite set such that every clone $\mathcal{O}'$ with $\mathcal{O}' \supseteq \mathcal{O}$ is finitely related, then $\mathcal{O}$ has an edge term. The proof of this relies on the theory of *cube term blockers*, which roughly states that a clone $\mathcal{O}$ fails to contain a cube term iff there is an infinite sequence of invariant relations which look like the relations $\{0,1\}^n \setminus \{(0,...,0)\}$ - recall that the clone corresponding to this sequence of relations on $\{0,1\}$ was our basic example of a clone which was not finitely related (Example 1.1.3).

Example 2.5.1. Consider the algebra $\mathbb{A} = (\{a,b,c\}, g)$ from Example 2.2.1, which has $\{a,b\}$ a majority subalgebra and $\{a,c\}$ an absorbing minority subalgebra. Recall that the minority pairs of $\mathbb{A}$ were $(a,c),(c,a),(b,c)$. Since $a \in \mathrm{Sg}_{\mathbb{A}}\{b,c\}$, for any $s \in A^+$ we have

$$(b,c) \in \mathrm{Forks}(\langle g\rangle, s) \implies (a,c) \in \mathrm{Forks}(\langle g\rangle, s).$$

Take the standard alphabetical ordering $<$ on $\{a,b,c\}$. It's easy to check that $\lambda(\langle g\rangle,(a,c))$ contains $a, b, c, ab, ba, bc, ca, cb, abc, acb, acc, bac, bca, cab, cba$ and that $\lambda(\langle g\rangle,(b,c)) = A^+$: for the strings of length 2, the free algebra $\mathcal{F}_{\mathbb{A}}(x,y)$ only has six elements so we may compute the forks directly, for permutations of abc we note that a corresponding permutation of aac comes before it and g preserves the congruence corresponding to the partition $\{a,b\},\{c\}$, and for acc we note that aac and aca come before it and that g preserves the affine ternary relation $\{(a,a,a),(a,c,c),(c,a,c),(c,c,a)\}$.

To complete the description of $\lambda(\langle g\rangle,(a,c))$, we just need to check that for all $2 \le i \le n$, the word $s_{in} = a\cdots aca\cdots a \in A^+$ of length n with a c in the ith position and as elsewhere has $(a,c) \in \mathrm{Forks}(\langle g\rangle, s_{in})$. For this, we take the terms x_1 and $g(x_1,x_1,x_i)$ in the free algebra, and check that they make a fork at s_{in}. For $s' <_{lex} s_{in}$, we have $s_1' = a$ and $s_i' < c$, so

$$g(s_1', s_1', s_i') = g(a,a,a) \text{ or } g(a,a,b) = a = s_1',$$

so x_1 and $g(x_1, x_1, x_i)$ agree on tuples which come lexicographically before s_{in}. At s_{in}, we get the fork $(a, g(a, a, c)) = (a, c)$.

Example 2.5.2. Consider the gmm algebra $\mathbb{A}_2 = (\{a, b, c\}, \varphi_2)$ from Example 2.1.2, which had majority subalgebras $\{a, b\}, \{a, c\}$ and minority subalgebra $\{b, c\}$. The only minority pair to worry about is (b, c), and under the standard alphabetical ordering $<$ on $\{a, b, c\}$, we find that $\lambda(\langle\varphi_2\rangle, (b, c))$ contains the following 16 elements of A^+:

$$a, b, c, ab, ac, ba, ca, cb, acb, bcc, cab, cba, abcc, bacc, bcac, bcca.$$

Again, it is easy to check the strings of length 2 as $\mathcal{F}_{\mathbb{A}_2}(x, y)$ only has 4 elements, strings which have a c preceding the first b such as acb don't work because the corresponding word with bs and cs swapped (i.e. abc in this case) comes before it and φ_2 preserves order two the automorphism swapping b and c, and strings containing bcc such as $abcc$ don't work because the two strings where one of the cs is replaced by a b (i.e. $abbc$ and $abcb$ in this case) come before it and φ_2 preserves the ternary relation corresponding to the columns of the matrix

$$\begin{bmatrix} a & b & b & c & c \\ a & b & c & b & c \\ a & b & c & c & b \end{bmatrix}.$$

It's much harder to show that the remaining elements s which are not $\ge_E$ to one of the 16 strings displayed above all have $(b, c) \in \operatorname{Forks}(\langle\varphi_2\rangle, s)$. Each such s has at least one b, exactly one c, and has its first b before its c. We may assume without loss of generality that s begins with a b, and suppose s has its only c at the ith position for some $i \ge 2$. We need to show that there is some term $t \in \mathcal{F}_{\mathbb{A}_2}(x_1, ..., x_n)$ such that the pair (x_1, t) gives us a fork at s. In other words, we need to show that we can find a term t such that for each $s' <_{lex} s$ we have $t(s') = s'_1$ and $t(s) = c$.

The only way I know to show the existence of such a term t is to use the analysis of critical relations in $\operatorname{Inv}_k(\mathbb{A}_2)$ carried out in Example 2.3.1. By that analysis, we see that every relation $\mathbb{R} \le \mathbb{A}_2^k$ is the intersection of some family of binary relations and some family of relations $\mathbb{R}_I \le \mathbb{A}^I$ such that for each I and each $i, j \in I$, we have $\pi_{i,j}(\mathbb{R}_I) \subseteq 0^*_{\mathbb{A}_2}$, where $0^*_{\mathbb{A}_2}$ is the congruence corresponding to the partition $\{a\}, \{b, c\}$. Thus, if the term t we are looking for does not exist, then either there is some $s' <_{lex} s$ such that

$$\begin{bmatrix} s'_1 \\ c \end{bmatrix} \not\in \operatorname{Sg}_{\mathbb{A}_2^2} \begin{bmatrix} s' \\ s \end{bmatrix},$$

or there is some family $s^1, ..., s^k <_{lex} s$ such that for each j, l, we have $(s^l_j, s_j) \in 0^*_{\mathbb{A}_2}$ but

$$\begin{bmatrix} s^1_1 \\ \vdots \\ s^k_1 \\ c \end{bmatrix} \not\in \operatorname{Sg}_{\mathbb{A}_2^2} \begin{bmatrix} s^1 \\ \vdots \\ s^k \\ s \end{bmatrix}.$$

To rule out the first possibility, we note that if $s' <_{lex} s$ then $s'_1 \in \{a, b\}$, and if $(s'_1, s'_i) \neq (b, c)$, then (s'_1, s'_i) is a majority pair and taking $\varphi_2(x_1, x_1, x_i)$ does the trick, while if $(s'_1, s'_i) = (b, c)$, then at the first location j where s' and s differ we must have $s'_j = a, s_j = b$, so taking $\varphi_2(x_j, x_1, x_i)$ does the trick:

$$\varphi_2\left(\begin{bmatrix} a & b & c \\ b & b & c \end{bmatrix}\right) = \begin{bmatrix} b \\ c \end{bmatrix}.$$

To rule out the second possibility, note that if $s^l <_{lex} s$ and $(s_j^l, s_j) \in 0_{\mathbb{A}_2}^*$ for all j, then the first coordinate where s^l and s can differ is at the coordinate i with $s_i = c$, so we must have $s_i^l = b$, $s_i = c$ and $s_1^l = s_1 = b$. Thus the term x_i rules out the second possibility.

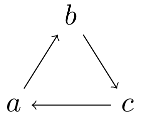
Chapter 3

Absorption and Bounded Width

3.1 Fourth basic example: the Rock-Paper-Scissors algebra

We're going to start building intuition for the bounded width case with a detailed investigation of a fourth basic algebra on three elements, which is sometimes called the "rock-paper-scissors" algebra. This algebra is $\mathbb{A}_{rps} = (\{a,b,c\},\cdot)$, where $\cdot$ is the binary, commutative, idempotent operation described by the following table.

$\cdot$	a	b	c
a	a	b	a
b	b	b	c
c	a	c	c



The algebra $\mathbb{A}_{rps}$ is not a semilattice, but every two-element subset of $\mathbb{A}_{rps}$ is a semilattice. Thus, the binary operation $\cdot$ satisfies the following identities:

$$xx \approx x, \quad xy \approx yx, \quad x(xy) \approx xy.$$

Any binary operation satisfying the above identities is known as a 2-*semilattice* operation, and the algebra $\mathbb{A}_{rps}$ is the smallest 2-semilattice which is not a semilattice.

As we will see, the corresponding relational clone is generated by the binary relation $\{(a,b),(b,c),(c,a)\}$ (which corresponds to the fact that the algebra has a cyclic automorphism) and the ternary relation $R_{a,b}$ given by the formula

$$R_{a,b}(x,y,z) \coloneqq (x \in \{a,b\}) \wedge (x = a \implies y = z).$$

The ternary relation $R_{a,b}$ has a special role, which is closely connected to the fact that $\{a,b\}$ is a semilattice subalgebra of $\mathbb{A}_{rps}$.

Proposition 3.1.1. *If $\mathbb{R},\mathbb{S} \subseteq \mathbb{A}^n$ are any n-ary relations with $\mathbb{S} \subseteq \mathbb{R}$, then the $n+1$-ary relation*

$$((x,y) \in \mathbb{R} \times \{a,b\}) \wedge (y = a \implies x \in \mathbb{S})$$

can be defined by a primitive positive formula in $\mathbb{R},\mathbb{S}$, and $R_{a,b}$.

Proof. Just use the following primitive positive formula:

$$\exists z \in \mathbb{A}^n \quad x \in \mathbb{R} \wedge z \in \mathbb{S} \wedge \bigwedge_{i \le n} R_{a,b}(y, x_i, z_i). \qquad \square$$

Proposition 3.1.2. *If $\mathbb{R} \le_{sd} \mathbb{A}_{rps}^n$ is a subdirect n-ary relation, then $\mathbb{R}$ is the intersection of its two-variable projections, each of which is either a full relation or the graph of an automorphism of $\mathbb{A}_{rps}$ which is either the identity or is cyclic. In particular, there is some subset of the coordinates $I \subseteq \{1, ..., n\}$ such that the projection π_I is an isomorphism from $\mathbb{R}$ to $\mathbb{A}_{rps}^I$.*

Proof. We prove this by induction on n. The base case, $n = 2$, is easily verified: since $\mathbb{A}_{rps}$ is simple, every subdirect binary relation on $\mathbb{A}_{rps}$ is either the graph of an automorphism or is linked, and we can check that every connected subgraph of the complete bipartite graph $K_{3,3}$ either contains a bipartite matching or is a tree with two leaves on both parts (e.g. using Hall's Marriage Lemma). Therefore up to automorphisms of $\mathbb{A}_{rps}$ we just need to consider relations which contain $\{(a,a),(a,b),(b,b),(c,c)\}$, $\{(a,a),(b,c),(c,b)\}$, or $\{(a,b),(a,c),(b,a),(c,a)\}$, and all three of these generate $\mathbb{A}_{rps}^2$.

Now consider the case $n > 2$. By the induction hypothesis, we may assume without loss of generality that $\pi_{[n]\setminus\{i\}}(\mathbb{R}) = \mathbb{A}_{rps}^{n-1}$ for every $i \le n$. Suppose for contradiction that $\mathbb{R} \ne \mathbb{A}_{rps}^n$.

Since the automorphism group of $\mathbb{A}_{rps}$ is transitive, we may assume without loss of generality that $(a, ..., a) \notin \mathbb{R}$. Since $\mathbb{A}_{rps}$ is idempotent, the set $\mathbb{R}'$ of triples (x, y, z) such that $(x, y, z, a, ..., a) \in \mathbb{R}$ is a subalgebra of $\mathbb{A}_{rps}^3$, and by the inductive hypothesis every projection of $\mathbb{R}'$ onto any pair of coordinates is full. So we can reduce to the case $n = 3$.

If any two of $(a,a,c),(a,c,a),(c,a,a)$ are in $\mathbb{R}$, then we can combine them to obtain (a,a,a). So we may suppose that $(a,a,b) \in \mathbb{R}$. If we consider the binary relation consisting of pairs (y,z) with $(a,y,z) \in \mathbb{R}$, then by the $n = 2$ case, we must have $(a,c,a) \in \mathbb{R}$. Similar reasoning with the roles of the first and second coordinates reversed then shows that we must have $(c,a,a) \in \mathbb{R}$, a contradiction. □

Proposition 3.1.3. *If $\mathbb{R} \le_{sd} \mathbb{A}_{rps}^n \times \{a,b\}^k$ has full projection onto $\mathbb{A}_{rps}^n$, then we have $\mathbb{A}_{rps}^n \times \{(b, ..., b)\} \subseteq \mathbb{R}$.*

Proof. For any $x \in \mathbb{A}_{rps}^n$, let x^- be the tuple obtained from x by applying the cyclic permutation $(a\ c\ b)$ componentwise. Then it's easy to check that for any $x, y \in \mathbb{A}_{rps}^n$, we have

$$(xy^-)y = y.$$

By multiplying all of the elements of $\mathbb{R}$ together in any order (with parentheses placed arbitrarily), we see that there is some $x \in \mathbb{A}_{rps}^n$ such that $(x_1, ..., x_n, b, ..., b) \in \mathbb{R}$. For any $y \in \mathbb{A}_{rps}^n$, there are tuples $c, d \in \{a,b\}^n$ such that $(y,c),(y^-,d) \in \mathbb{R}$ by the assumption $\pi_{[n]}(\mathbb{R}) = \mathbb{A}_{rps}^n$. Thus

$$(y,b) = ((x,b) \cdot (y^-,d)) \cdot (y,c) \in \mathbb{R}. \qquad \square$$

The previous two propositions are enough to describe an algorithm which solves $\mathrm{CSP}(\mathbb{A}_{rps})$. The algorithm first establishes arc-consistency, reducing some of the domains of the variables until every constraint relation becomes subdirect. Then for each variable with a two element domain, the last proposition shows that we may as well take that variable equal to the top/absorbing element of that domain. After this restriction, if we consider the remaining variables, each relation decomposes into binary relations, each of which is either an equality relation or the graph of a cyclic automorphism. This final problem can be solved by checking that no cycle of binary relations implies that any variable is related to itself by a nontrivial cyclic automorphism.

Definition 3.1.4. An instance of a CSP is *cycle-consistent* if for every sequence of variables $v_1, ..., v_n$ and relations $R_1, ..., R_n$ and pairs of coordinates (i_k, j_k) such that v_k, v_{k+1} are related by $\pi_{(i_k,j_k)}(R_k)$ for each k (indices taken modulo n), the composition

$$\pi_{(i_1,j_1)}(R_1) \circ \cdots \circ \pi_{(i_n,j_n)}(R_n)$$

contains the equality relation on the domain of the variable v_1.

Corollary 3.1.5. *Any cycle-consistent instance of* $\mathrm{CSP}(\mathbb{A}_{rps})$ *has a solution.*

If we want to understand the complete structure of a general relation $\mathbb{R} \le \mathbb{A}_{rps}^n$, things become more complicated. A typical relation we need to consider has the form

$$\begin{aligned} x_1 \in &\{a,b\} \wedge (x_1 = a \implies x_2 \in \{a,b\}) \wedge (x_1 = x_2 = a \implies x_3 \in \{a,b\}) \\ &\wedge \cdots \wedge (x_1 = \cdots = x_k = a \implies y = z). \end{aligned}$$

The final $y = z$ in the last implication can also be replaced with any unary relation on y, and for any subset of the variables we can apply a cyclic automorphism of $\mathbb{A}_{rps}$. We call any such relation a *basic relation* on $\mathbb{A}_{rps}$.

Theorem 3.1.6. *Suppose* $\mathbb{R} \le \mathbb{A}_{rps}^n$. *Then* $x \in \mathbb{R}$ *iff* x *satisfies every basic relation on* $\mathbb{A}_{rps}$ *which contains* $\mathbb{R}$. *In particular,* $\mathbb{R}$ *is contained in the relational clone generated by* $\{(a,b),(b,c),(c,a)\}$ *and* $R_{a,b}$.

Proof. Suppose x satisfies every basic relation which contains $\mathbb{R}$. Let $I = \{i_1, ..., i_k\} \subseteq [n]$ be maximal such that, after applying cyclic automorphisms to coordinates in I, we have $x_{i_j} = a$ for all $j \le k$, and such that the basic relation

$$\begin{aligned} y_{i_1} \in &\{a,b\} \wedge (y_{i_1} = a \implies y_{i_2} \in \{a,b\}) \wedge (y_{i_1} = y_{i_2} = a \implies y_{i_3} \in \{a,b\}) \\ &\wedge \cdots \wedge (y_{i_1} = \cdots = y_{i_{k-1}} = a \implies y_{i_k} \in \{a,b\}) \end{aligned}$$

contains $\mathbb{R}$. Assume without loss of generality that the coordinates are ordered such that $I = \{n-k+1, ..., n\}$ and such that the $n-k$-ary relation $\mathbb{R}'$ defined by

$$(y_1, ..., y_{n-k}) \in \mathbb{R}' \iff (y_1, ..., y_{n-k}, a, ..., a) \in \mathbb{R}$$

has $\mathbb{R}' \le_{sd} \mathbb{A}_{rps}^m \times \{a,b\}^{n-m-k}$ for some m (possibly after further applications of cyclic automorphisms). Then by the maximality of I, we have $x = (x_1, ..., x_m, b, ..., b, a, ..., a)$. By Propositions 3.1.2, 3.1.3, and our assumption that x satisfies all basic relations containing $\mathbb{R}$, we have $(x_1, ..., x_m) \in \pi_{[m]}(\mathbb{R}')$ and $\pi_{[m]}(\mathbb{R}') \times \{(b, ..., b)\} \subseteq \mathbb{R}'$, so $(x_1, ..., x_m, b, ..., b) \in \mathbb{R}'$, so $x \in \mathbb{R}$. □

Remark 3.1.1. The intricate yet understandable structure of the basic relations considered above is at the heart of the uncountable region found by Zhuk [189] in the lattice of clones on a three-element domain. Each of the clones in Zhuk's uncountable region properly contains the clone of the rock-paper-scissors algebra, so the generating relations for the corresponding relational clones can be written in terms of the basic relations considered above.

Proposition 3.1.7. *Suppose that* $f : \mathbb{A}^n \to \mathbb{A}$ *is any idempotent operation which depends on all of its inputs and preserves the relation* $R_{a,b}$. *Then the restriction of* f *to* $\{a,b\}$ *must be the* n*-ary semilattice operation on* $\{a,b\}$, *that is, for any* $(x_1, ..., x_n) \in \{a,b\}^n \setminus \{(a, ..., a)\}$, *we have* $f(x_1, ..., x_n) = b$.

Proof. Suppose for contradiction that there is some $(x_1, ..., x_n) \in \{a, b\}^n \setminus \{(a, ..., a)\}$ with $f(x_1, ..., x_n) \neq b$. Since $\{a, b\} = \pi_1(R_{a,b})$ is preserved by f, we must then have $f(x_1, ..., x_n) = a$. We will show that for all i with $x_i = b$, f does not depend on its ith input.

Let $y, z \in \mathbb{A}^n$ be any pair of tuples with $y_i = z_i$ whenever $x_i = a$. Then each $(x_i, y_i, z_i) \in R_{a,b}$, so

$$\begin{bmatrix} a \\ f(y) \\ f(z) \end{bmatrix} = f\left(\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n \end{bmatrix}\right) \in R_{a,b},$$

so $f(y) = f(z)$. □

Theorem 3.1.8. *An n-ary operation f is contained in $\mathrm{Clo}_n(\mathbb{A}_{rps})$ iff it preserves the relations $\{(a, b), (b, c), (c, a)\}$ and $R_{a,b}$. If f depends on all its inputs, this occurs iff f preserves the cyclic automorphism of $\mathbb{A}_{rps}$ and $f|_{\{a,b\}}$ is the n-ary semilattice operation on $\{a, b\}$.*

Proof. We just need to check this in the case where f depends on all of its inputs. Let $\mathcal{F} = \mathcal{F}_{\mathcal{V}(\mathbb{A}_{rps})}(x_1, ..., x_n) \le \mathbb{A}_{rps}^{\mathbb{A}_{rps}^n}$ be the subalgebra generated by $\pi_1, ..., \pi_n : \mathbb{A}_{rps}^n \to \mathbb{A}_{rps}$. The projection $\pi_x(\mathcal{F})$ of $\mathcal{F}$ onto the coordinate of $\mathbb{A}_{rps}^{\mathbb{A}_{rps}^n}$ corresponding to $x \in \mathbb{A}_{rps}^n$ is the subalgebra of $\mathbb{A}_{rps}$ generated by $\{\pi_1(x), ..., \pi_n(x)\} = \{x_1, ..., x_n\}$.

If x is a diagonal tuple, say $x = (a, ..., a)$, then $\pi_x(\mathcal{F}) = \{a\}$, corresponding to the fact that any $f \in \mathcal{F}$ must be idempotent, with $f(a, ..., a) = a$. If exactly two elements of $\mathbb{A}_{rps}$ occur in x, say $x \in \{a, b\}^n$, then $\pi_x(\mathcal{F}) = \{a, b\}$, and if f depends on all its inputs and preserves $R_{a,b}$, this implies that we must have $f(x) = b$, i.e. $\pi_x(f) = b$. Thus, if $I \subseteq \mathbb{A}_{rps}^n$ is the set of x such that all three of a, b, c show up in the coordinates of x, we see that $\pi_I(\mathcal{F}) \le_{sd} \mathbb{A}_{rps}^I$, and by Proposition 3.1.3 we have $f \in \mathcal{F} \iff \pi_I(f) \in \pi_I(\mathcal{F})$.

By Proposition 3.1.2, $\pi_I(\mathcal{F})$ is the intersection of its two-variable projections, each of which is either full or the graph of a cyclic automorphism of $\mathbb{A}_{rps}$. A two variable projection $\pi_{x,y}(\mathcal{F})$ will only be the graph of a cyclic automorphism $\sigma \in \mathrm{Aut}(\mathbb{A}_{rps})$ if $(\pi_i(x), \pi_i(y))$ is in the graph of σ for all i, that is, if $y_i = \sigma(x_i)$ for all i. Thus, $\pi_I(f) \in \pi_I(\mathcal{F})$ iff whenever $y = \sigma(x)$, we have $f(y) = \sigma(f(x))$. □

Note that one of the key steps behind the analysis of the rock-paper-scissors algebra was Proposition 3.1.2 which classified the subdirect powers of the algebra, and that the method of proof depended only on checking properties of subdirect binary and ternary relations on $\mathbb{A}_{rps}$. The general pattern behind this is best understood in terms of a property of the polynomial clone known as *polynomial completeness.*

Definition 3.1.9. An algebra is *polynomially complete* if its polynomial clone is the clone of all operations, that is, if every operation on the underlying set can be expressed using the basic operations of the algebra together with the constant operations.

Theorem 3.1.10. *A finite idempotent algebra $\mathbb{A}$ is polynomially complete if every binary relation on $\mathbb{A}$ which contains the diagonal is either the equality relation or the full relation, and every ternary relation $\mathbb{R} \le_{sd} \mathbb{A}^3$ such that every two variable projection of $\mathbb{R}$ is full is equal to the full relation $\mathbb{A}^3$.*

Proof. We will show by induction on n that every n-ary relation $\mathbb{R} \le \mathbb{A}^n$ which contains the subalgebra of diagonal tuples $(x, ..., x)$, $x \in \mathbb{A}$ is given by a conjunction of equalities between pairs of coordinates. The base case $n = 2$ follows from our assumption on $\mathbb{A}$. By the inductive hypothesis, we may assume without loss of generality that $\pi_{[n]\setminus\{i\}}\mathbb{R} = \mathbb{A}^{n-1}$ for each i.

If $n = 3$, then our assumption on $\mathbb{A}$ implies that $\mathbb{R} = \mathbb{A}^3$. Otherwise, suppose for contradiction that $(x_1, ..., x_n) \notin \mathbb{A}^n$, and consider the ternary relation $\mathbb{R}'$ consisting of triples (u, v, w) such that $(u, v, w, x_4, ..., x_n) \in \mathbb{R}$. Since $\mathbb{A}$ is idempotent, $\mathbb{R}'$ is a subalgebra of $\mathbb{A}^3$, and every two-variable projection of $\mathbb{R}'$ is full, so by the $n = 3$ case we must have $(x_1, x_2, x_3) \in \mathbb{R}'$, a contradiction.

Note that we have shown that the relational clone corresponding to the polynomial clone of $\mathbb{A}$ is generated by the equality relation. The general $\mathrm{Inv} - \mathrm{Pol}$ Galois duality now shows that $\mathbb{A}$ is polynomially complete. To see this concretely, consider the subalgebra of $\mathbb{A}^{\mathbb{A}^n}$ generated by the functions π_i and the constant (diagonal) tuples. Then this subalgebra is described by a conjunction of equalities between pairs of coordinates. But no two-variable projection of this subalgebra can be an equality relation: if $x \neq y \in \mathbb{A}^n$, then there is always some i such that $\pi_i(x) \neq \pi_i(y)$. Thus this subalgebra of $\mathbb{A}^{\mathbb{A}^n}$ must be the full set of operations $\mathbb{A}^n \to \mathbb{A}$. □

Corollary 3.1.11. *The rock-paper-scissors algebra is polynomially complete.*

As far as relations go, the main impact of polynomial completeness is that it strongly constrains subdirect relations where each factor is polynomially complete. As we have seen, if some factors are not polynomially complete, then the structure of an arbitrary relation can be quite intricate. In the case of the rock-paper-scissors algebra, we are able to side-step this intricacy by restricting each factor which is a proper subalgebra of $\mathbb{A}_{rps}$ to its top/absorbing element. This is a general strategy that can be used in the study of bounded width algebras, as well as finite Taylor algebras.

We conclude this section with a few classical results about polynomial completeness.

Definition 3.1.12. The ternary *discriminator* function is the function t defined by

$$t(x, y, z) = \begin{cases} z & x = y, \\ x & x \neq y. \end{cases}$$

Proposition 3.1.13. *A finite algebra is polynomially complete iff it has the ternary discriminator as a polynomial operation.*

Proof. One direction is obvious. For the other direction, it's enough to show that the idempotent algebra $\mathbb{A} = (A, t)$ whose only basic operation is the ternary discriminator t is polynomially complete. We may assume that the underlying set A contains at least two distinct elements a, b. Suppose first that $\mathbb{R} \leq_{sd} \mathbb{A}^2$ is a relation properly containing the diagonal of $\mathbb{A}^2$, and assume without loss of generality that $(a, b) \in \mathbb{R}$ with $a \neq b$. Then for any $c \in \mathbb{A}$, we have

$$\begin{bmatrix} a \\ c \end{bmatrix} = t\left(\begin{bmatrix} a & b & c \\ b & b & c \end{bmatrix}\right) \in \mathbb{R},$$

and similarly $(d, b) \in \mathbb{R}$ for any $d \in \mathbb{A}$. Then for any $c, d \in \mathbb{A}$ we have

$$\begin{bmatrix} d \\ c \end{bmatrix} = t\left(\begin{bmatrix} a & a & d \\ c & b & b \end{bmatrix}\right) \in \mathbb{R},$$

so $\mathbb{R} = \mathbb{A}^2$.

To finish, we just need to show that any ternary relation $\mathbb{R} \leq_{sd} \mathbb{A}^3$ such that every two variable projection is full must be the full relation $\mathbb{A}^3$. Since $\mathbb{A}$ has full automorphism group, if

$\mathbb{R} \neq \mathbb{A}^3$ then we may assume without loss of generality that $(a,a,a) \not\in \mathbb{R}$, while all three of $(a,a,b),(a,b,a),(b,a,a)$ are in $\mathbb{R}$. Then we have

$$\begin{bmatrix} b \\ a \\ b \end{bmatrix} = t\left(\begin{bmatrix} a & a & b \\ a & b & a \\ b & a & a \end{bmatrix}\right) \in \mathbb{R},$$

so

$$\begin{bmatrix} a \\ a \\ a \end{bmatrix} = t\left(\begin{bmatrix} a & b & b \\ a & a & a \\ b & b & a \end{bmatrix}\right) \in \mathbb{R},$$

contradicting the assumption $(a,a,a) \not\in \mathbb{R}$. □

Example 3.1.1. We can give an alternative proof of the fact that the rock-paper-scissors algebra is polynomially complete by expressing the ternary discriminator as a polynomial. First, we can define the unary polynomial x^+ corresponding to the cyclic permutation $(a\ b\ c)$ by

$$x^+ = ((xa)c)(xb),$$

and we can define the inverse of this by $x^- = (x^+)^+$. Note that we now have

$$xy^+ = \begin{cases} x^+ & x = y, \\ x & x \neq y. \end{cases}$$

Thus if we set $u(x,y,z) = (z(xy^+)^-)x$, then we have

$$u(x,y,z) = (z(xy^+)^-)x = \begin{cases} xz & x = y, \\ x & x \neq y, \end{cases}$$

so we may take

$$t(x,y,z) = ((u(x,y,z)u(x,y,z^+)^-)u(x,y,z^-)^+)^-.$$

To see that this works, note that if $x = y$, then two of $xz, (xz^+)^-, (xz^-)^+$ are equal to z while the third is equal to z^+, so since $\{z, z^+\}$ is a semilattice we see that in this case $t(x,y,z)$ is given by

$$(((xz)(xz^+)^-)(xz^-)^+)^- = (zzz^+)^- = (z^+)^- = z,$$

while if $x \neq y$ then $u(x,y,?) = x$, so $t(x,y,z)$ is given by

$$((xx^-)x^+)^- = (xx^+)^- = (x^+)^- = x.$$

The ternary discriminator t satisfies the system of identities

$$\begin{aligned} t(x,y,y) &\approx x, \\ t(x,y,x) &\approx x, \\ t(y,y,x) &\approx x. \end{aligned}$$

Any ternary term satisfying this system of identities is known as a *Pixley term*. Note that any Pixley term is automatically a Mal'cev term, and that the term d defined from t by

$$d(x,y,z) = t(x, t(x,y,z), z)$$

is automatically a majority term. In the case where t is the ternary discriminator, d becomes the dual discriminator of Example 1.6.5.

Theorem 3.1.14 (Pixley [155])**.** *An algebra $\mathbb{A}$ generates a variety which is both congruence permutable and congruence distributive iff it has a Pixley term. If $\mathbb{A}$ is also simple, then it is polynomially complete.*

Proof. If $\mathbb{A}$ has a Pixley term, then it has both a Mal'cev term and a majority term, so it generates a congruence permutable and congruence distributive variety. Conversely, suppose that $\mathbb{A}$ generates a congruence permutable and congruence distributive variety. Let $\mathcal{F} = \mathcal{F}_{\mathcal{V}(\mathbb{A})}(x, y, z)$ be the free algebra on three generators in this variety, and for $a, b \in \{x, y, z\}$ let θ_{ab} be the smallest congruence with $a \equiv_{\theta_{ab}} b$. Then $(x, z) \in \theta_{xz} \wedge (\theta_{xy} \circ \theta_{yz})$, so by congruence distributivity and permutability, we have

$$(x, z) \in \theta_{xz} \wedge (\theta_{xy} \vee \theta_{yz}) = (\theta_{xz} \wedge \theta_{xy}) \vee (\theta_{xz} \wedge \theta_{yz}) = (\theta_{xz} \wedge \theta_{yz}) \circ (\theta_{xz} \wedge \theta_{xy}).$$

Thus there is some $t \in \mathcal{F}$ such that

$$x \ (\theta_{xz} \wedge \theta_{yz}) \ t \ (\theta_{xz} \wedge \theta_{xy}) \ z.$$

Thus t is a ternary term which satisfies the Pixley identities.

Now suppose that $\mathbb{A}$ is simple. Since $\mathbb{A}$ is Mal'cev, every binary relation on $\mathbb{A}$ is the graph of an isomorphism modulo the linking congruence, and the linking congruence is necessarily either $0_{\mathbb{A}}$ or $1_{\mathbb{A}}$. Thus every binary relation on $\mathbb{A}$ which contains the diagonal is either full or equal to the diagonal. Since $\mathbb{A}$ has a majority term, every ternary relation on $\mathbb{A}$ whose two variable projections are all full must itself be a full relation. Thus $\mathbb{A}$ is polynomially complete. □

Varieties which are both congruence distributive and congruence permutable are known as *arithmetical* varieties. The name arithmetical comes from the theory of arithmetical rings, which are rings where the "Chinese remainder condition" holds: for any ideals $I_1, ..., I_n$ and elements $a_1, ..., a_n$ with $a_i \equiv a_j \pmod{I_i + I_j}$ for all i, j, there exists some x with $x \equiv a_i \pmod{I_i}$ for all i.

3.2 Partial semilattice operations and the digraph of semilattice subalgebras

In this section we will go over a binary analogue of a standard result about iterating unary functions to make (compositionally) idempotent functions, that is, functions satisfying $e \circ e = e$. First we review the case of unary iteration.

Definition 3.2.1. If $f : A \to A$ is a unary function, we define $f^{\circ n}$ to be $f \circ \cdots \circ f$, with n copies of f. If (A, f) is either finite or profinite, we define f^∞ by

$$f^\infty(x) \coloneqq \lim_{n\to\infty} f^{\circ n!}(x).$$

Alternatively, we can define f^∞ as the limit of $f^{\circ n}$ over the net of positive integers n, ordered by divisibility. Similarly, we define $f^{\infty-1}$ by

$$f^{\infty-1}(x) \coloneqq \lim_{n\to\infty} f^{\circ(n!-1)}(x).$$

Proposition 3.2.2. *If (A, f) is profinite, then the limit defining f^∞ exists, and f^∞ satisfies the identity*

$$f^\infty(f^\infty(x)) \approx f^\infty(x).$$

Furthermore, if A is finite, then

$$f^\infty = f^{\circ \operatorname{lcm}\{1,...,|A|\}},$$

and the graph of f^∞ can be computed from the graph of f in time linear in $|A|$.

Proof. It's enough to prove this in the case where A is finite. Let m, m' be any positive multiples of $\operatorname{lcm}\{1, ..., |A|\}$, we will show that $f^{\circ m} = f^{\circ m'}$: this will show that the limit is equal to $f^{\circ m}$, and taking $m' = 2m$ will show that $f^\infty \circ f^\infty = f^\infty$. To see that $f^{\circ m} = f^{\circ m'}$, note that for any x, the sequence $x, f(x), f(f(x)), ..., f^{\circ k}(x), ...$ must be eventually periodic with period p at most $|A|$, and the periodic behavior must begin within the first $|A|$ steps, so for any $k \ge |A|$ we have $f^{\circ k}(x) = f^{\circ(k+p)}(x)$. Since $|m-m'|$ is a multiple of p and $m, m' \ge |A|$, this implies that $f^{\circ m} = f^{\circ m'}$.

In order to compute the graph of f^∞ efficiently, we will also compute the function $f^{\infty-1}$ simultaneously. First, make a list of elements of A, and mark all of them as "unprocessed". In each round, we pick the next unprocessed element x from the list, and compute the sequence of iterates $x, f(x), f(f(x)), ...,$ marking each one as "processed" as we compute it, until the first time we compute $f^{\circ k}(x)$ and find that it has already been marked as "processed". There are two cases: either $f^{\circ k}(x)$ is equal to $f^{\circ i}(x)$ for some $i < k$, or $f^{\circ k}(x)$ was processed in some previous round. We can distinguish between the two cases by checking whether the value of $f^\infty(f^{\circ k}(x))$ has already been computed.

In the case where $f^{\circ k}(x) = f^{\circ i}(x)$ for some $i < k$, we first set $f^\infty(f^{\circ j}(x)) \coloneqq f^{\circ j}(x)$ and $f^{\infty-1}(f^{\circ j}(x)) \coloneqq f^{\circ(j-1)}(x)$ for $i < j \le k$. For $j < i$, we iterate downwards, setting

$$f^\infty(f^{\circ j}(x)) \coloneqq f^{\infty-1}(f^{\circ(j+1)}(x))$$

and

$$f^{\infty-1}(f^{\circ j}(x)) \coloneqq f^{\infty-1}(f^\infty(f^{\circ j}(x))).$$

In the case where $f^{\circ k}(x)$ was processed in a previous round, we iterate downwards using the above rules to handle all $j < k$.

Since the number of steps needed for each round is linear in the number of elements which are marked as processed in that round, and since each element of A is marked as processed at most once, the entire procedure for computing f^∞ and $f^{\infty-1}$ runs in time linear in $|A|$. □

In the context of CSPs, the reduction to the case of core structures was based on the observation than any non-surjective unary polymorphism $f : \mathbf{A} \to \mathbf{A}$ allows us to replace the underlying set A by the smaller set $f(A)$ to obtain a homomorphically equivalent CSP on a smaller domain. In this case, the map $f^\infty : \mathbf{A} \to \mathbf{A}$ will also be non-surjective, and in fact we have the guarantee that

$$f^\infty(A) \subseteq f^{\circ n}(A)$$

for all $n \ge 0$. So whenever we shrink the domain of a non-core CSP using a unary polymorphism, we may as well assume that the unary polymorphism in question is (compositionally) idempotent.

On the algebraic side, if $e \circ e = e$ and $e \in \operatorname{Clo}_1(\mathbb{A})$, we can define a reduct $\mathbb{A}_e$ of $\mathbb{A}$ as follows. For every n-ary operation $f \in \operatorname{Clo}_n(\mathbb{A})$, we define the corresponding operation $f_e : A^n \to A$ by

$$f_e(x_1, ..., x_n) = e(f(e(x_1), ..., e(x_n))).$$

Then we define $\mathbb{A}_e$ to be the algebraic structure $(A, \{f_e \mid f \in \mathrm{Clo}(\mathbb{A})\})$ having a basic operation f_e for each term f of $\mathbb{A}$.

Each operation f_e only depends on the restriction of f to $e(A)$, and takes values in $e(A)$. Also, if f preserves $e(A)$, then f_e and f agree when they are restricted to $e(A)$. The reduct $\mathbb{A}_e$ has $e(A)$ as a subalgebra, and is completely determined by its restriction to the subalgebra $e(A)$ together with the description of the map $e : A \to e(A)$. So the reduct $\mathbb{A}_e$ and its subalgebra $e(A)$ are essentially interchangeable, and the subalgebra $e(A)$ of $\mathbb{A}_e$ has as its basic operations the terms of $\mathbb{A}$ which preserve $e(A)$.

As a special case of the general result relating reflections to height 1 identities, we have the following basic result.

Proposition 3.2.3. *If a system of height* 1 *identities is satisfied by terms* $f^1, ..., f^k$ *of* $\mathbb{A}$*, then the same system of height* 1 *identities is satisfied by the corresponding operations* $f_e^1, ..., f_e^k$ *of* $\mathbb{A}_e$ *(defined as above).*

Note that identities which involve nesting functions may not survive the process of passing from $\mathbb{A}$ to the reduct $\mathbb{A}_e$.

Now we return to the world of idempotent operations, and describe a surprisingly powerful binary analogue of unary iteration. Rather than (compositionally) idempotent operations, we will produce a type of binary operation which I call a *partial semilattice* operation.

Definition 3.2.4. We say that an idempotent binary operation s is a *partial semilattice* if it satisfies the identity

$$s(x, s(x, y)) \approx s(s(x, y), x) \approx s(x, y).$$

Equivalently, s is a partial semilattice if for all x, y, the set $\{x, s(x, y)\}$ is closed under s, and acts like a semilattice subalgebra with absorbing element $s(x, y)$ under s.

Note that unlike semilattices and 2-semilattices, partial semilattices are *not necessarily* Taylor operations. The binary projection π_1 is an extreme example of a partial semilattice operation which is not Taylor. This is a necessary feature of the definition, since we will show that *any* idempotent binary operation can be used to produce a partial semilattice operation (in a nontrivial way).

In order to produce partial semilattice operations, we will start by treating our binary operation as a unary function of the second variable, with the first variable treated as a (constant) parameter.

Definition 3.2.5. If $t : \mathbb{A}^2 \to \mathbb{A}$ is a binary function and $\mathbb{A}$ is finite (or profinite), then we define t^∞ to be the pointwise limit

$$t^\infty(x, y) := \lim_{n\to\infty} t^{n!}(x, y),$$

where $t^1 := t$ and $t^{n+1}(x, y) := t(x, t^n(x, y))$.

Proposition 3.2.6. *For any binary term* t*, we have*

$$t^\infty(x, t^\infty(x, y)) \approx t^\infty(x, y).$$

If t *is idempotent, then so is* t^∞*.*

The function t^∞ now satisfies one of the two defining identities for a partial semilattice. Note that t^∞ can be computed from t in time linear in $|A|^2$. To find a term u which satisfies the second identity $u(u(x, y), x) \approx u(x, y)$, we plug t^∞ into itself in a surprisingly counterintuitive way.

Proposition 3.2.7. *If f is an idempotent binary term which satisfies the identity*

$$f(x, f(x, y)) \approx f(x, y),$$

and if we define a term u by

$$u(x, y) \coloneqq f(x, f(y, x)),$$

then u satisfies the identity

$$u(u(x, y), x) \approx u(x, y).$$

Proof. We have

$$f(x, u(x, y)) \approx f(x, f(x, f(y, x))) \approx f(x, f(y, x)) \approx u(x, y),$$

so

$$u(u(x, y), x) \approx f(u(x, y), f(x, u(x, y))) \approx f(u(x, y), u(x, y)) \approx u(x, y). \qquad \square$$

Finally, to get a term which satisfies *both* defining identities of a partial semilattice, we iterate the function u on its second variable.

Proposition 3.2.8. *If u is an idempotent binary term which satisfies the identity*

$$u(u(x, y), x) \approx u(x, y),$$

then $s \coloneqq u^\infty$ satisfies the identity

$$s(x, s(x, y)) \approx s(s(x, y), x) \approx s(x, y).$$

Proof. Define u^n as in the definition of u^∞. Then for any m we have

$$u^m(u(x, y), x) \approx u(x, y),$$

and on replacing y by $u^{n-1}(x, y)$, we get

$$u^m(u^n(x, y), x) \approx u^n(x, y)$$

for any m, n. $\square$

The full process, going from t to $f = t^\infty$ to $u(x, y) = f(x, f(y, x))$ to $s = u^\infty$, is functorial, and the final function $s : A^2 \to A$ can be computed from t in time linear in $|A|^2$. Since s was defined from t in a nontrivial way, we get the following result.

Proposition 3.2.9. *If t is a binary idempotent term and a, b are such that $t(a, b) = t(b, a) = b$, then the partial semilattice term $s \in \mathrm{Clo}(t)$ defined by the above process also satisfies $s(a, b) = s(b, a) = b$.*

More generally, if B, C are subsets of $\mathbb{A}$ such that for any $x \in B \cup C$ and any $y \in C$ we have $t(x, y), t(y, x) \in C$, then the same holds for s.

Corollary 3.2.10. *If $(b, b) \in \mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\}$, then there is a partial semilattice term $s \in \mathrm{Clo}(\mathbb{A})$ such that $s(a, b) = s(b, a) = b$.*

Once we have a partial semilattice term s with $s(a,b) = s(b,a) = b$, we can use it to preprocess the inputs to other n-ary functions to force them to preserve the subset $\{a,b\}$ and act like the n-ary semilattice operation on this subset. To do this, we first need to find terms $s_n \in \mathrm{Clo}(s)$ which act like the n-ary semilattice operation.

Proposition 3.2.11. *If s is a partial semilattice operation, then for all n there are terms $s_n \in \mathrm{Clo}(s)$ of arity n such that if $\{x, x_2, ..., x_n\} = \{x, y\}$, then*

$$s_n(x, x_2, ..., x_n) \approx s(x, y).$$

Proof. If $\{x, x_2, ..., x_n\} = \{x, y\}$, then the expressions $s(x, x_2), ..., s(x, x_n)$ are all equal to either x or $s(x,y)$, and at least one of them is equal to $s(x,y)$, so since $\{x, s(x,y)\}$ acts like a semilattice oriented from x to $s(x,y)$ under s, we can combine these expressions in any order to produce such a term s_n.

For concreteness, we define s_n inductively, as follows: $s_1(x) = x, s_2(x,y) = s(x,y)$ and

$$s_n(x_1, ..., x_n) = s(s_{n-1}(x_1, ..., x_{n-1}), s(x_1, x_n)). \qquad \square$$

Now we can use the terms s_n to preprocess the inputs to n-ary functions. If f is an n-ary term of $\mathbb{A}$, define the term f_s by

$$f_s(x_1, ..., x_n) = f(s_n(x_1, ..., x_n), s_n(x_2, ..., x_n, x_1), ..., s_n(x_n, x_1, ..., x_{n-1})).$$

As in the case of unary operations, we will consider the reduct $\mathbb{A}_s$ with basic operations f_s for every term f of $\mathbb{A}$. This reduct will be simpler in the sense that for any a, b with $s(a,b) = s(b,a) = b$, each term f_s will act like the n-ary semilattice operation on $\{a, b\}$. Additionally, every two-variable height 1 identity which holds in $\mathbb{A}$ will also hold in $\mathbb{A}_s$.

Proposition 3.2.12. *Let $\mathbb{A} = (A, (f^i)_{i\in I})$ be a finite idempotent algebra, and let Σ be the set of all two-variable height 1 identities which involve both variables on each side and are satisfied in $\mathbb{A}$. Then the operations $(f_s^i)_{i\in I}$ of $\mathbb{A}_s$ will also satisfy the identities in Σ.*

Additionally, if $\mathbb{B}, \mathbb{C}$ are subalgebras of $\mathbb{A}$ such that for any $x \in \mathbb{B}$ and any $y \in \mathbb{C}$ we have $s(x,y), s(y,x) \in \mathbb{C}$, then for any n-ary term f of $\mathbb{A}$ and any $x_1, ..., x_n \in \mathbb{B} \cup \mathbb{C}$ such that at least one $x_i \in \mathbb{C}$, we have $f_s(x_1, ..., x_n) \in \mathbb{C}$.

Proof. Suppose we have an identity

$$f^i(a_1, ..., a_m) \approx f^j(b_1, ..., b_n),$$

with $\{a_1, ..., a_m\} = \{b_1, ..., b_n\} = \{x, y\}$. Define $a_1', ..., a_m'$ by $a_k' = s(x,y)$ if $a_k = x$ and $a_k' = s(y,x)$ if $a_k = y$, and define $b_1', ..., b_n'$ similarly. Then for each k, we have

$$s_m(a_k, ..., a_m, a_1, ..., a_{k-1}) \approx a_k',$$

and similarly for the b_l's, so

$$f_s^i(a_1, ..., a_m) \approx f^i(a_1', ..., a_m') \approx f^j(b_1', ..., b_n') \approx f_s^j(b_1, ..., b_n).$$

For the last statement, we just need to check that for any $x_1, ..., x_n \in \mathbb{B} \cup \mathbb{C}$ with at least one of the x_is in $\mathbb{C}$ we have $s_n(x_1, ..., x_n) \in \mathbb{C}$ (since $\mathbb{C}$ is closed under each term f of $\mathbb{A}$). This follows from the fact that s_n is defined from s in a way that involves all of its variables. $\square$

Since an algebra $\mathbb{A}$ is Taylor iff it satisfies a nontrivial system of two-variable height 1 identities, if $\mathbb{A}$ is Taylor then $\mathbb{A}_s$ will also be Taylor. Later, we will see that algebras with bounded width are also characterized by two-variable height 1 identities, so the same sort of implication (i.e. $\mathbb{A}$ has bounded width implies $\mathbb{A}_s$ has bounded width) will hold in that case as well. Algebras with few subpowers are *not* characterized by height 1 identities, essentially because no semilattice can have few subpowers, so such an implication fails in that case.

There are two other interesting cases which are not characterized by two-variable height 1 identities: algebras of width 1, and algebras such that the associated CSP is solved by the linear programming relaxation. It turns out that we can still prove a similar result in these cases.

Proposition 3.2.13. *If $\mathbb{A}$ has symmetric terms f_n of every arity, then it has symmetric terms f_n^s which act like the semilattice operation on each set $\{a, b\}$ with $s(a, b) = s(b, a) = b$.*

Proof. Let f_n be a symmetric term of arity n, for each n. Then for any n, let $\sigma_1, ..., \sigma_{n!}$ be an enumeration of the permutations of $\{1, ..., n\}$, and define f_n^s by

$$f_n^s(x_1, ..., x_n) \coloneqq f_{n!}(s_n(x_{\sigma_1(1)}, ..., x_{\sigma_1(n)}), ..., s_n(x_{\sigma_{n!}(1)}, ..., x_{\sigma_{n!}(n)})).$$

Then f_n^s is a symmetric term of arity n. □

Proposition 3.2.14. *If $\mathbb{A}$ has totally symmetric terms f_n of every arity, then it has totally symmetric terms f_n^s which act like the semilattice operation on each set $\{a, b\}$ with $s(a, b) = s(b, a) = b$.*

Proof. Fix n. For every $m \geq 1$, let S_m^n be the set of n-ary terms t of $\mathbb{A}$ such that there exist variables $y_1, ..., y_l$ with $\{y_1, ..., y_l\} = \{x_1, ..., x_n\}$ and such that for each i, the number of j with $y_j = x_i$ is at least m, and

$$t(x_1, ..., x_n) = s_l(y_1, ..., y_l).$$

Note that for $m' > m$ we have $S_{m'}^n \subseteq S_m^n$, and each S_m^n is finite and nonempty, so the intersection $S^n = \bigcap_m S_m^n$ is also finite and nonempty. Furthermore, for any $a_1, ..., a_n \in \mathbb{A}$, the set of values

$$\{t(a_1, ..., a_n) \mid t \in S^n\}$$

depends only on the set $\{a_1, ..., a_n\}$. Thus we can take

$$f_n^s(x_1, ..., x_n) \coloneqq f_{|S^n|}(\{t(x_1, ..., x_n) \mid t \in S^n\}).$$ □

Remark 3.2.1. The previous two propositions only used the fact that the restrictions of the s_ns to $\{a, b\}$ are symmetric and totally symmetric, respectively. So they can be generalized to show that if an algebra $\mathbb{A}$ has symmetric/totally symmetric operations of each arity, then for every subset X of $\mathbb{A}$ such that some collection of terms t_n of $\mathbb{A}$ preserve X and have symmetric/totally symmetric restrictions to X, we can find symmetric/totally symmetric operations of $\mathbb{A}$ which preserve X and such that their restrictions to X agree with the restrictions of the terms t_n. It turns out that a similar general result holds for Taylor clones and clones of bounded width, but the proof of that will need to wait until we show that Taylor algebras have cyclic terms.

Recall that for any a, b, the set $\{a, b\}$ is a semilattice subalgebra of $\mathbb{A}$ iff the ternary relation $(x \in \{a, b\}) \wedge (x = a \implies y = z)$ defines a subalgebra of $\mathbb{A}^3$. We can generalize this somewhat.

Proposition 3.2.15. *If B, C are subsets of $\mathbb{A}$, then the ternary relation*

$$(x \in B \cup C) \wedge (x \not\in C \implies y = z)$$

defines a subalgebra of $\mathbb{A}^3$ iff $B \cup C$ is a subalgebra of $\mathbb{A}$, and for any n, any n-ary term $f \in \mathrm{Clo}_n(\mathbb{A})$ which depends on all of its inputs, and any $x_1, ..., x_n \in B \cup C$ such that at least one $x_i \in C$, we have $f(x_1, ..., x_n) \in C$.

Definition 3.2.16. If $\mathbb{C} \le \mathbb{B}$ are subalgebras of $\mathbb{A}$ such that there exists a term t with $t(\mathbb{B}, \mathbb{C}), t(\mathbb{C}, \mathbb{B}) \subseteq \mathbb{C}$, then we say that $\mathbb{C}$ *binary absorbs* $\mathbb{B}$, and write $\mathbb{C} \lhd_{bin} \mathbb{B}$. If for any n, any n-ary term $f \in \mathrm{Clo}_n(\mathbb{A})$ which depends on all of its inputs, and any $x_1, ..., x_n \in \mathbb{B}$ such that at least one $x_i \in \mathbb{C}$, we have $f(x_1, ..., x_n) \in \mathbb{C}$, then we say that $\mathbb{C}$ *strongly absorbs* $\mathbb{B}$, and write $\mathbb{C} \lhd_{str} \mathbb{B}$.

We can summarize the previous results in the following proposition, which shows that binary absorption and strong absorption are very nearly the same thing.

Proposition 3.2.17. *If $\mathbb{C} \lhd_{bin} \mathbb{B}$, then there is a partial semilattice term s with $s(\mathbb{B}, \mathbb{C}), s(\mathbb{C}, \mathbb{B}) \subseteq \mathbb{C}$, and in the reduct $\mathbb{A}_s$ the subalgebras $\mathbb{B}_s, \mathbb{C}_s$ satisfy $\mathbb{C}_s \lhd_{str} \mathbb{B}_s$. Furthermore, $\mathbb{C} \lhd_{str} \mathbb{B}$ iff the ternary relation $(x \in \mathbb{B}) \wedge (x \not\in \mathbb{C} \implies y = z)$ defines a subalgebra of $\mathbb{A}^3$ (and $\mathbb{C} \le \mathbb{B}$).*

In general, a binary absorbing subalgebra of a binary absorbing subalgebra might not be binary absorbing (consider the 4 element lattice $(\{0,1\}^2, \wedge, \vee)$ and the sequence $\{(0,1)\} \lhd_{bin} \{(0,0),(0,1)\} \lhd_{bin} \{0,1\}^2$), and similarly for strongly absorbing subalgebras (consider the idempotent commutative groupoid $(\{a,b,c\}, \cdot)$ given by $ab = ac = b, bc = c$ and the sequence $\{c\} \lhd_{str} \{b,c\} \lhd_{str} \{a,b,c\}$). However, we can always chain together binary and strong absorption in one particular order.

Proposition 3.2.18. *If $\mathbb{C} \lhd_{bin} \mathbb{B} \lhd_{str} \mathbb{A}$, then $\mathbb{C} \lhd_{bin} \mathbb{A}$. Applying this repeatedly, we see that if*

$$\mathbb{C} \lhd_{bin} \mathbb{B}_n \lhd_{str} \cdots \lhd_{str} \mathbb{B}_1 \lhd_{str} \mathbb{A},$$

then $\mathbb{C} \lhd_{bin} \mathbb{A}$.

Proof. Suppose that $\mathbb{C}$ absorbs $\mathbb{B}$ with respect to the binary term t. Define a term u by

$$u(x,y) \coloneqq t(t(x, t(x,y)), t(y, t(y,x))).$$

Then for any $a \in \mathbb{A}, c \in \mathbb{C}$, we have $t(a,c) \in \mathbb{B}$ and $t(a, t(a,c)) \in \mathbb{B}$ since $c \in \mathbb{B} \lhd_{str} \mathbb{A}$, so

$$u(a,c) \in t(\mathbb{B}, t(c, \mathbb{B})) \subseteq t(\mathbb{B}, \mathbb{C}) \subseteq \mathbb{C},$$

and similarly $u(c,a) \in \mathbb{C}$. □

By iteratively replacing $\mathbb{A}$ with reducts $\mathbb{A}_s$ for partial semilattice terms s quadratically many times, we can reduce to the case where for all a, b, we have $(b,b) \in \mathrm{Sg}_{\mathbb{A}^2}\{(a,b),(b,a)\}$ iff $\{a,b\}$ is a semilattice subalgebra of $\mathbb{A}$ with absorbing element b.

Definition 3.2.19. We say that an idempotent algebra $\mathbb{A}$ has been *prepared* if for every pair a, b such that $(b,b) \in \mathrm{Sg}_{\mathbb{A}^2}\{(a,b),(b,a)\}$, the set $\{a,b\}$ is a semilattice subalgebra of $\mathbb{A}$.

For algebras which have been prepared, it makes sense to define a digraph whose edges correspond to semilattice subalgebras of $\mathbb{A}$.

Definition 3.2.20. If s is a partial semilattice operation and a, b have $s(a, b) = b$, then we write $a \to_s b$, or just $a \to b$ if s is understood (or if the algebra has been prepared).

Theorem 3.2.21. *Let s be a fixed nontrivial partial semilattice term of an idempotent algebra $\mathbb{A}$. If $\mathbb{A}$ is prepared, then the following are equivalent.*

(a) $s(a, b) = b$, that is, $a \to b$,

(b) the restriction of s to $\{a, b\}$ acts like the semilattice operation on $\{a, b\}$ with absorbing element b,

(c) there exists c such that $s(a, c) = b$,

(d) $\begin{bmatrix} b \\ b \end{bmatrix} \in \mathrm{Sg}_{\mathbb{A}^2}\left\{\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix}\right\}$

(e) there is a binary term t of $\mathbb{A}$ with $t(a, b) = t(b, a) = b$,

(f) there is a partial semilattice term s' of $\mathbb{A}$ with $s'(a, b) = b$,

(g) for every n and every n-ary term $f \in \mathrm{Clo}_n(\mathbb{A})$ which depends on all its inputs, the restriction of f to $\{a, b\}$ acts like the n-ary semilattice operation on $\{a, b\}$ with absorbing element b,

(h) the ternary relation $(x \in \{a, b\}) \wedge (x = a \implies y = z)$ defines a subalgebra of $\mathbb{A}^3$.

If $\mathbb{A}$ has not been prepared, then (a), (b), (c) are equivalent to each other, (d), (e), (f) are equivalent to each other, (g), (h) are equivalent to each other, and (g) implies (a) implies (d).

Proposition 3.2.22. *If $\mathbb{A}$ is prepared, then the following hold:*

(a) for $\mathbb{B} \lhd_{bin} \mathbb{A}$ and any $a \in \mathbb{A}$, there is some $b \in \mathbb{B}$ such that $a \to b$,

(b) if $\mathbb{B} \lhd_{bin} \mathbb{A}$ and $a \in \mathbb{A}$, $b \in \mathbb{B}$ have $b \to a$, then $a \in \mathbb{B}$,

(c) if $\mathbb{C} \lhd_{bin} \mathbb{B} \lhd_{bin} \mathbb{A}$, then $\mathbb{C} \lhd_{bin} \mathbb{A}$,

(d) if $\mathbb{B}_1, \mathbb{B}_2 \lhd_{bin} \mathbb{A}$, then $\mathbb{B}_1 \cap \mathbb{B}_2 \neq \emptyset$ and $\mathbb{B}_1 \cap \mathbb{B}_2 \lhd_{bin} \mathbb{A}$.

In particular, there is a unique minimal binary absorbing subalgebra $\mathbb{B} \lhd_{bin} \mathbb{A}$, and this $\mathbb{B}$ has no proper binary absorbing subalgebra.

Proof. Part (a) follows from the existence of a partial semilattice term s with $s(\mathbb{A}, \mathbb{B}) \subseteq \mathbb{B}$ and part (b) follows from part (g) of the previous proposition.

For part (c), choose a partial semilattice term s with $s(\mathbb{B}, \mathbb{C}), s(\mathbb{C}, \mathbb{B}) \subseteq \mathbb{C}$, and choose any binary term t with $t(\mathbb{A}, \mathbb{B}), t(\mathbb{B}, \mathbb{A}) \subseteq \mathbb{B}$. Define a binary term u by

$$u(x, y) \coloneqq s(s(t(x, y), y), s(t(y, x), x)).$$

Then for $a \in \mathbb{A}, c \in \mathbb{C}$ we have $t(a, c), t(c, a) \in \mathbb{B}$, and we have $t(c, a) \to s(t(c, a), a)$, so by part (b) we have $s(t(c, a), a) \in \mathbb{B}$. Thus

$$u(a, c) \in s(s(\mathbb{B}, c), \mathbb{B}) \subseteq s(\mathbb{C}, \mathbb{B}) \subseteq \mathbb{C},$$

and similarly $u(c, a) \in \mathbb{C}$.

For part (d), pick $b_1 \in \mathbb{B}_1$, then by part (a) there is some $b_2 \in \mathbb{B}_2$ with $b_1 \to b_2$, and then by part (b) we have $b_2 \in \mathbb{B}_1$, so $b_2 \in \mathbb{B}_1 \cap \mathbb{B}_2$. Then from $\mathbb{B}_2 \lhd_{bin} \mathbb{A}$ we have $\mathbb{B}_1 \cap \mathbb{B}_2 \lhd_{bin} \mathbb{B}_1$, and we can apply part (c) to finish. □

Proposition 3.2.23. *If $\mathbb{A}$ has been prepared and $a, b, c \in \mathbb{A}$ have $c \in \mathrm{Sg}\{a, b\}$ with $a \to c$, then $\mathbb{A}$ has a partial semilattice term s with $s(a, b) = c$.*

Proof. Let s' be an arbitrary nontrivial partial semilattice term of $\mathbb{A}$, and choose p a binary term of $\mathbb{A}$ with $p(a, b) = c$. Then take $s(x, y) = s'(x, p(x, y))$. We clearly have $s(a, b) = s'(a, p(a, b)) = s'(a, c) = c$, so we just have to check that s is a partial semilattice.

If p is second projection then $s = s'$ and we are done. Otherwise, since $\mathbb{A}$ has been prepared, p and s' both act as the semilattice operation on $\{x, s'(x, p(x, y))\} = \{x, s(x, y)\}$, so s also acts as the semilattice operation on $\{x, s(x, y)\}$. □

In any digraph, the strongly connected components have a natural partial order.

Definition 3.2.24. We say that b is *reachable* from a if there is a sequence $a = a_0, a_1, ..., a_k = b$ such that $a_i \to a_{i+1}$ for $i = 0, ..., k - 1$.

Proposition 3.2.25. *If $\mathbb{A}$ is prepared and $s^1, ..., s^k$ are partial semilattice terms of $\mathbb{A}$, then for any n-ary term $f \in \langle s^1, ..., s^k \rangle$, $f(x_1, ..., x_n)$ is always reachable from at least one of the variables $x_1, ..., x_n$.*

Definition 3.2.26. We say that a subset S of an algebra $\mathbb{A}$ which has a partial semilattice operation s is *upwards closed* if whenever $a \in S$ and $a' \in \mathbb{A}$ have $a \to_s a'$, we also have $a' \in S$.

Definition 3.2.27. We say that a set A is *strongly connected* if for every subset $S \subset A$ with $S \neq \emptyset, A$ there is an $a \in S$ and a $b \in A \setminus S$ such that $a \to b$. We say that a set A is a *maximal strongly connected component* of an algebra $\mathbb{A}$ if A is a strongly connected subset which is upwards closed (note that every finite upwards closed set contains at least one maximal strongly connected component). Finally, we call an element of an algebra $\mathbb{A}$ *maximal* if it is contained in any maximal strongly connected component of $\mathbb{A}$.

The main application of partial semilattice terms to CSPs is the following general idea: if a solvable instance of a CSP is arc-consistent (i.e. all relations are subdirect), then it probably has a solution where each variable is assigned a value in a maximal strongly connected component of the corresponding domain. So a basic case to try to understand is the case where every domain is a strongly connected algebra.

Remark 3.2.2. The digraph considered in this section is the same as the set of "thin red edges" of Andrei Bulatov's colored graph [47] attached to any Taylor algebra. Bulatov has a different construction of a partial semilattice operation s from a binary term t, which is still based on the couterintuitive idea of taking a function f which satisfies $f(x, f(x, y)) \approx f(x, y)$ and plugging in $f(x, f(y, x))$.

3.3 Maximal strongly connected components and polynomial completeness

In this section we prove a few results of Andrei Bulatov [44] about the way maximal strongly connected components of partial semilattice algebras interact with binary and ternary relations. A consequence of the results of this section is that simple, strongly connected algebras are always polynomially complete. Throughout this section, we will always fix a partial semilattice operation s.

Theorem 3.3.1. *Fix a partial semilattice operation s. Suppose $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$ is subdirect and A, B are maximal strongly connected subsets of $\mathbb{A}, \mathbb{B}$, respectively.*

- *(a) The set of a such that $(\{a\} \times B) \cap \mathbb{R} \ne \emptyset$ is upwards closed. In particular, if $(A \times B) \cap \mathbb{R}$ is nonempty, then it is subdirect in $A \times B$.*
- *(b) The set of a such that $\{a\} \times B \subseteq \mathbb{R}$ is upwards closed.*
- *(c) If A is contained in a linked component of $\mathbb{R}$ (that is, a connected component of $\mathbb{R}$ considered as a bipartite graph on $\mathbb{A} \sqcup \mathbb{B}$), $(A \times B) \cap \mathbb{R} \ne \emptyset$, and A, B are finite, then $A \times B \subseteq \mathbb{R}$.*

Additionally, the product $A \times B$ is a maximal strongly connected subset of $\mathbb{A} \times \mathbb{B}$.

Proof. For part (a), suppose that $(a, b) \in \mathbb{R}$ and $b \in B$, and let $a \to a'$. Since $\mathbb{R}$ is subdirect, there is some b' with $(a', b') \in \mathbb{R}$. Then

$$\begin{bmatrix} a' \\ s(b, b') \end{bmatrix} = s\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a' \\ b' \end{bmatrix}\right) \in \mathbb{R},$$

and $b \to s(b, b')$, so $s(b, b') \in B$.

For part (b), suppose that $\{a\} \times B \subseteq \mathbb{R}$ and $a \to a'$. Let S be the set of $b \in B$ such that $(a', b) \in \mathbb{R}$, that is, $S = \pi_2((\{a'\} \times B) \cap \mathbb{R})$. By part (a), S is nonempty. To finish, we just have to show that S is upwards closed. Suppose $b \in S$ and $b \to b'$. Then by assumption we have $(a, b') \in \mathbb{R}$, so

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = s\left(\begin{bmatrix} a' \\ b \end{bmatrix}, \begin{bmatrix} a \\ b' \end{bmatrix}\right) \in \mathbb{R}.$$

For part (c), suppose first that $A \times A \subseteq \mathbb{R} \circ \mathbb{R}^-$, where $\mathbb{R}^- \le \mathbb{B} \times \mathbb{A}$ is the reverse of $\mathbb{R}$ (we will later reduce the general case to this case). Let a be any element of A, and let X be the set of $b \in \mathbb{B}$ such that $(a, b) \in \mathbb{R}$, that is, $X = \pi_2((\{a\} \times \mathbb{B}) \cap \mathbb{R})$. By part (a), $X \cap B \ne \emptyset$, and by the finiteness of B, the intersection $X \cap B$ has a maximal strongly connected component S. Since B is a maximal strongly connected component of $\mathbb{B}$, S is a maximal strongly connected component of X.

By the assumption $A \times A \subseteq \mathbb{R} \circ \mathbb{R}^-$ and the definition of X, we see that $(A \times X) \cap \mathbb{R}$ is subdirect in $A \times X$. Thus by part (b) and the fact that $\{a\} \times S \subseteq (A \times X) \cap \mathbb{R}$, we see that $A \times S \subseteq (A \times X) \cap \mathbb{R}$, so $A \times S \subseteq \mathbb{R}$. Then by part (b) applied to $\mathbb{R}^-$, we see that $A \times B \subseteq \mathbb{R}$.

Now suppose that $A \times A \not\subseteq \mathbb{R} \circ \mathbb{R}^-$. From the finiteness of A we see that there is some k such that $A \times A \subseteq (\mathbb{R} \circ \mathbb{R}^-)^{\circ k}$. Choose k minimal, and let $\mathbb{R}' = (\mathbb{R} \circ \mathbb{R}^-)^{\circ (k-1)} \le_{sd} \mathbb{A}^2$. Then $\mathbb{R}'$ is equal to its own reverse $\mathbb{R}'^-$, and $A \times A \subseteq \mathbb{R}' \circ \mathbb{R}'$ since $2(k-1) \ge k$ for $k \ge 2$. Thus the previous paragraphs applied to $\mathbb{R}'$ (using $\mathbb{R}' = \mathbb{R}'^-$) show that $A \times A \subseteq \mathbb{R}'$, contradicting the minimality of k. □

Corollary 3.3.2. *If $\pi : \mathbb{A} \twoheadrightarrow \mathbb{B}$ is a surjective homomorphism of finite algebras, then the subalgebra of $\mathbb{A}$ generated by the maximal elements of $\mathbb{A}$ maps surjectively onto the subalgebra of $\mathbb{B}$ generated by the maximal elements of $\mathbb{B}$.*

Corollary 3.3.3. *If we start with any arc-consistent instance of $\mathrm{CSP}(\mathbb{A}_1, ..., \mathbb{A}_n)$ and replace every domain and every relation by the subalgebra generated by its maximal elements, then the resulting instance will still be arc-consistent.*

Corollary 3.3.4. *Fix a partial semilattice operation s. Suppose that $\mathbb{R} \leq_{sd} \mathbb{A} \times \mathbb{B}$ is a subdirect product of finite algebras $\mathbb{A}, \mathbb{B}$, and that $\mathbb{B}$ is simple and $\mathbb{B} = \mathrm{Sg}(B)$, with B a maximal strongly connected component of $\mathbb{B}$. Then:*

(a) if $\mathbb{A}$ is also simple and $\mathbb{A} = \mathrm{Sg}(A)$ with A a maximal strongly connected component of $\mathbb{A}$, and if $\mathbb{R} \cap (A \times B) \neq \emptyset$, then $\mathbb{R}$ is either the graph of an isomorphism or is $\mathbb{A} \times \mathbb{B}$, and

(b) if $\mathbb{A}$ is arbitrary and $\mathbb{R}$ is not the graph of a homomorphism from $\mathbb{A}$ to $\mathbb{B}$, then there is an $a \in \mathbb{A}$ with $\{a\} \times \mathbb{B} \subseteq \mathbb{R}$.

Proof. If $\mathbb{B}$ is simple, then the linking congruence of $\mathbb{R}$ on $\mathbb{B}$ must either be the trivial congruence $0_{\mathbb{B}}$, in which case $\mathbb{R}$ is the graph of a homomorphism from $\mathbb{A}$ to $\mathbb{B}$, or the full congruence $1_{\mathbb{B}}$, in which case $\mathbb{R}$ is linked. In the second case, the results follow from Theorem 3.3.1(c). □

Theorem 3.3.5. *Fix a partial semilattice operation s. Suppose $R \subseteq A \times B \times C$ is closed under s, A is strongly connected, $\pi_{23}(R)$ is strongly connected, $\pi_{12}(R) = A \times B$, $\pi_{13}(R) = A \times C$, and A, B, C are finite. Then $R = A \times \pi_{23}(R)$.*

Proof. By Theorem 3.3.1(c), we just need to show that R is linked as a subset of $A \times \pi_{23}(R)$. We will do this by showing that for any $a \to a'$ in A, some fork of R links a to a' in one step.

Since $\pi_1(R) = A$, there exist $b \in B, c \in C$ such that $(a, b, c) \in R$. Since $\pi_{13}(R) = A \times C$, there exists some $b' \in B$ such that $(a', b', c) \in R$. Since

$$\begin{bmatrix} a' \\ s(b,b') \\ c \end{bmatrix} = s\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} a' \\ b' \\ c \end{bmatrix}\right) \in R,$$

we may assume without loss of generality that $b' = s(b, b')$, that is, that $b \to b'$.

Since $\pi_{12}(R) = A \times B$, there exists some $c' \in C$ such that $(a, b', c') \in R$. Since

$$\begin{bmatrix} a \\ b' \\ s(c,c') \end{bmatrix} = s\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} a \\ b' \\ c' \end{bmatrix}\right) \in R,$$

we may assume without loss of generality that $c' = s(c, c')$, that is, that $c \to c'$.

Since (a', b', c) and (a, b', c') are in R, we have

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = s\left(\begin{bmatrix} a' \\ b' \\ c \end{bmatrix}, \begin{bmatrix} a \\ b' \\ c' \end{bmatrix}\right) \in R.$$

Thus both a and a' meet $(b', c') \in \pi_{23}(R)$. □

Remark 3.3.1. The proof of Theorem 3.3.5 actually proves something slightly more general: if $R \subseteq A \times B \times C$ is closed under s, $\pi_{12}(R) = A \times B, \pi_{13}(R) = A \times C$, and A is weakly connected, then R is linked when considered as a subalgebra of $A \times \pi_{23}(R)$.

Corollary 3.3.6. *Fix a partial semilattice operation s. Suppose $R \subseteq A_1 \times \cdots \times A_n$ is closed under s, A_1 is strongly connected, $\pi_{[2,n]}(R)$ is strongly connected, $\pi_{1i}(R) = A_1 \times A_i$ for $i \in [2, n]$, and A_i are finite for all i. Then $R = A_1 \times \pi_{[2,n]}(R)$.*

Corollary 3.3.7. *Fix a partial semilattice operation s. Suppose $R \subseteq A_1 \times \cdots \times A_n$ is closed under s, all A_i are strongly connected, $\pi_{ij}(R) = A_i \times A_j$ for all $i \neq j$, and A_i are finite for all i. Then $R = A_1 \times \cdots \times A_n$.*

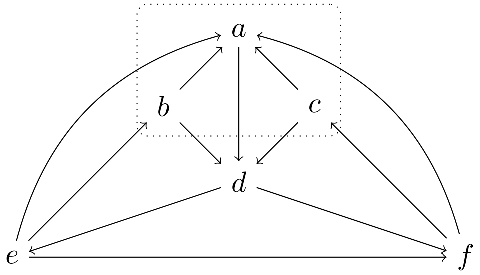
Corollary 3.3.8. *Fix a partial semilattice operation s. If $\mathbb{A}$ is simple and is generated by a finite maximal strongly connected component A, then $\mathbb{A}$ is polynomially complete.*

Proof. We just need to show that every relation $\mathbb{R} \leq \mathbb{A}^n$ which contains the set of constant tuples $\Delta_n = \{(a, ..., a) \mid a \in \mathbb{A}\}$ is an intersection of equality relations. First consider the case $n = 2$. From the assumption that $\mathbb{A}$ is simple we see that either $\mathbb{R}$ is the equality relation, or $\mathbb{R}$ is linked. If $\mathbb{R}$ is linked, then Theorem 3.3.1(c) and the fact that $\mathbb{R}$ contains Δ_2 implies that $A \times A \subseteq \mathbb{R}$, and from the assumption $\mathbb{A} = \mathrm{Sg}(A)$ we see that $\mathbb{R} = \mathbb{A} \times \mathbb{A}$.

Now consider the case $n \geq 3$. If any two-variable projection $\pi_{ij}(\mathbb{R})$ is the equality relation, then we can ignore one of the coordinates i, j, so we may assume without loss of generality that $\pi_{i,j}(\mathbb{R}) = \mathbb{A} \times \mathbb{A}$ for all i, j. Let a be any element of A, and let R be a maximal strongly connected component of $\mathbb{R}$ which is reachable from $(a, ..., a)$. Then for any i, j we must have $\pi_{i,j}(R) = A \times A$, so by the previous corollary we have $R = A^n$. Thus $A^n \subseteq \mathbb{R}$, and from the assumption $\mathbb{A} = \mathrm{Sg}(A)$ we see that $\mathbb{R} = \mathbb{A}^n$. □

Example 3.3.1. The reader may be wondering whether we can weaken the assumption that $\pi_{23}(R)$ is strongly connected from Theorem 3.3.5 to the assumption that B, C are strongly connected. It seems plausible that if B, C are both strongly connected and $\pi_{23}(R)$ is a subdirect product of B and C, $\pi_{23}(R)$ might automatically be strongly connected.

However, there is an example of a strongly connected 2-semilattice $\mathbb{A}$ and a subdirect product $\mathbb{R} \leq_{sd} \mathbb{A}^2$ which is *not* strongly connected. The 2-semilattice $\mathbb{A}$ is pictured below.

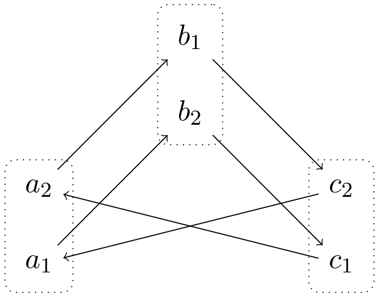


The missing values are given by $s(b, c) = s(b, f) = s(e, c) = a$.

If we let $\theta \leq_{sd} \mathbb{A}^2$ be the smallest congruence containing (b, c), then θ corresponds to the partition $\{a, b, c\}, \{d\}, \{e\}, \{f\}$, and $\mathbb{A}/\theta$ is a four element tournament. Considering θ as an algebra, we find that θ is *not* strongly connected: (b, c) and (c, b) are incomparable minimal elements of θ, and the remaining elements of θ form a maximal strongly connected component.

Example 3.3.2. Here we will give an example of a subdirect product of strongly connected algebras which has two maximal strongly connected components (such an example is necessarily not a 2-semilattice, since every 2-semilattice has a unique maximal strongly connected component).

As in the previous example, we will consider a congruence θ on a six-element algebra $\mathbb{A}$. This time $\mathbb{A}/\theta$ will be the three-element rock-paper-scissors algebra, and every congruence class of $\mathbb{A}$ will have two elements, with s acting as π_1 on the congruence class. As a digraph, $\mathbb{A}$ is just a directed six-cycle, pictured below.



Given the above digraph structure and the assumption that there is a congruence θ corresponding to the partition $\{a_1, a_2\}, \{b_1, b_2\}, \{c_1, c_2\}$, there is only one way to fill in the values of the partial semilattice operation s. The reader can check that the congruence θ, considered as a subalgebra of $\mathbb{A}^2$, has two maximal strongly connected components which are both isomorphic to $\mathbb{A}$.

Despite the above examples, we do at least have the following result, which is important for understanding how restricting to maximal strongly connected components interacts with cycle-consistency.

Theorem 3.3.9. *Fix a partial semilattice operation s. Suppose that $R \subseteq A \times A$ is closed under s, A is finite and strongly connected, and R contains the diagonal $\Delta_A = \{(a, a) \mid a \in A\}$. Then R has a maximal strongly connected component which contains Δ_A.*

Proof. Since Δ_A is strongly connected, it's enough to show that if (a, b) is reachable from (a, a) in R, then some element (c, c) of Δ_A is reachable from (a, b) in R. We will define a unary polynomial ϕ of R such that $\phi((a, a)) = (a, b)$ and such that $\phi(x)$ is reachable from x in R for all $x \in R$.

To construct ϕ, choose some sequence $(a_i, b_i) \in R$ such that $(a, a) = (a_0, b_0)$, $(a_i, b_i) \to (a_{i+1}, b_{i+1})$ for all i, and $(a_k, b_k) = (a, b)$ for some k. Then define ϕ by

$$\phi(x) = s\left(s\left(\cdots s\left(s\left(x, \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\right), \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right), \cdots\right), \begin{bmatrix} a_k \\ b_k \end{bmatrix}\right).$$

Note that since $\phi((a, a)) = (a, b)$, we have $\pi_1(\phi((a, x))) = a$ for all $x \in A$.

Since A is finite, we can find $m \geq 1$ such that $\phi^{\circ 2m} = \phi^{\circ m}$. Define another unary polynomial ϕ_Δ of R by

$$\phi_\Delta(x) = s\left(s\left(\cdots s\left(s\left(x, \begin{bmatrix} b_1 \\ b_1 \end{bmatrix}\right), \begin{bmatrix} b_2 \\ b_2 \end{bmatrix}\right), \cdots\right), \begin{bmatrix} b_k \\ b_k \end{bmatrix}\right),$$

that is, by replacing each (a_i, b_i) in the definition of ϕ by (b_i, b_i). Then if $\phi^{\circ m}((a, a)) = (a, c)$, we have

$$\phi_\Delta^{\circ m}\left(\phi^{\circ(m-1)}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)\right) = \phi_\Delta^{\circ m}\left(\begin{bmatrix} a \\ c \end{bmatrix}\right) = \begin{bmatrix} c \\ c \end{bmatrix}.$$

Thus (c, c) is reachable from (a, b) in R. □

Corollary 3.3.10. *If we start with any cycle-consistent instance of* $\mathrm{CSP}(\mathbb{A}_1, ..., \mathbb{A}_n)$ *and replace every domain and every relation by the subalgebra generated by its maximal elements, then the resulting instance will still be cycle-consistent.*

Proof. By Theorem 3.3.1(a), we just need to check this in the special case where our cycle-consistent instance is a cycle of binary relations $\mathbb{R}_i \le_{sd} \mathbb{A}_i \times \mathbb{A}_{i+1}$ with indices taken modulo n. Let $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n \times \mathbb{A}_1$ be the relation given by the formula

$$(x_1, x_2) \in \mathbb{R}_1 \wedge \cdots \wedge (x_n, x_{n+1}) \in \mathbb{R}_n.$$

The assumption that the instance is cycle-consistent implies that $\Delta_{\mathbb{A}_1} \subseteq \pi_{1,n+1}\mathbb{R}$. Set $\mathbb{R}_\Delta = \pi_{1,n+1}\mathbb{R}$.

For any algebra $\mathbb{A}$, let $\mathbb{A}^{\max}$ denote the subalgebra of $\mathbb{A}$ generated by the maximal elements of $\mathbb{A}$. We see from Theorem 3.3.1(a) that $\pi_{i,i+1}(\mathbb{R}^{\max}) = \mathbb{R}_i^{\max}$ for each i and that $\pi_{1,n+1}(\mathbb{R}^{\max}) = \mathbb{R}_\Delta^{\max}$. By Theorem 3.3.9 we have $\Delta_{\mathbb{A}_1^{\max}} \subseteq \mathbb{R}_\Delta^{\max}$, so the new instance is cycle-consistent at the first variable. □

3.4 2-semilattices, spirals, and ancestral algebras

In this section we'll discuss a pretty general class of partial semilattice algebras which are nice enough for the associated CSP to have bounded width, due to Bulatov [39]. Following the strategy of replacing domains of variables with the subalgebras generated by their maximal elements, and noting that many of the structural results proved in the preceeding section apply best to strongly connected algebras, we see that it would be quite convenient if every domain of every variable in our CSP has a unique maximal strongly connected component. The most straightforward examples of algebras with this property are 2-semilattices.

Definition 3.4.1. A binary operation s is a 2-*semilattice* operation if it satisfies the identities

$$s(x,y) \approx s(y,x), \quad s(x, s(x,y)) \approx s(x,y), \quad s(x,x) \approx x.$$

In other words, a 2-semilattice is a partial semilattice operation which is also commutative.

Proposition 3.4.2. *An algebra $\mathbb{A} = (A, s)$ is a 2-semilattice iff for all $a, b \in \mathbb{A}$, the subalgebra $\mathrm{Sg}_{\mathbb{A}}\{a,b\}$ is a semilattice under s.*

Proposition 3.4.3. *If $\mathbb{A} = (A, s)$ is a finite 2-semilattice, then $\mathbb{A}$ has a unique maximal strongly connected component.*

Proof. If a, b are any two maximal elements of $\mathbb{A}$, then $s(a,b) = s(b,a)$ is reachable from both a and b, so a and b must be in the same maximal strongly connected component. □

The first difficult results about bounded width CSPs were proved for 2-semilattices. However, the proofs only depended on the fact that every 2-semilattice has a unique maximal strongly connected component. Bulatov [39] calls this the "maximal red component condition". I've chosen to call such algebras "ancestral" instead, because they can be equivalently defined as follows.

Definition 3.4.4. An idempotent algebra $\mathbb{A}$ with a fixed partial semilattice operation s is called *ancestral* if for all $a, b \in \mathbb{A}$, there is some $c \in \mathrm{Sg}_{\mathbb{A}}\{a,b\}$ which is reachable from both a and b. We call any such c a *common ancestor* of a and b.

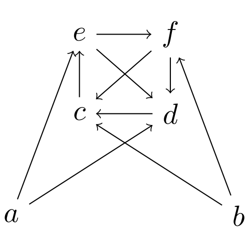
Proposition 3.4.5. *A finite idempotent algebra $\mathbb{A}$ is ancestral iff every proper subalgebra of $\mathbb{A}$ has a unique maximal strongly connected component.*

A nice generalization of 2-semilattices is the collection of algebras which I call "spirals". Spirals are defined in terms of a single commutative binary operation, so they can be described more rapidly than general ancestral algebras. As we will see later, a minimal Taylor clone is ancestral if and only if it is a minimal spiral, so we would not lose too much generality by restricting the study of ancestral algebras to the study of spirals.

Definition 3.4.6. An algebra $\mathbb{A} = (A, f)$ is a *spiral* if f is a commutative idempotent binary operation and every subalgebra of $\mathbb{A}$ which is generated by two elements either has size two or has a surjective homomorphism to the free semilattice on two generators.

Example 3.4.1. Here we give an example of a minimal spiral $\mathbb{A}_6$ which is not a 2-semilattice.

$\mathbb{A}_6$	a	b	c	d	e	f
a	a	c	e	d	e	d
b	c	b	c	c	f	f
c	e	c	c	c	e	c
d	d	c	c	d	d	d
e	e	f	e	d	e	f
f	d	f	c	d	f	f



Every proper subalgebra of $\mathbb{A}_6$ is a 2-semilattice - in fact, every pair of elements other than $\{a, b\}$ generates a two or three element semilattice subalgebra of $\mathbb{A}_6$. The pair $\{a, b\}$ generates $\mathbb{A}_6$, and $\mathbb{A}_6$ has a congruence θ corresponding to the partition $\{a\}, \{b\}, \{c, d, e, f\}$ such that $\mathbb{A}_6/\theta$ is isomorphic to the free semilattice on two generators.

The reader may check that any nonempty subset S of $\mathbb{A}_6$ which is closed under multiplication by a and by b must necessarily contain all four of c, d, e, f - using this observation, it is easy to check that $\mathrm{Clo}(\mathbb{A}_6)$ contains no nontrivial proper subclones.

Theorem 3.4.7. *If $\mathbb{A} = (A, f)$ is a spiral, then for any partial semilattice term $s \in \mathrm{Clo}(f)$ which is defined nontrivially in terms of f, the reduct $\mathbb{A}_s = (A, s)$ is ancestral.*

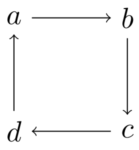
Proof. We prove this by induction on the size of A. Let a, b be any two elements of A. If $\mathrm{Sg}_{\mathbb{A}}\{a, b\}$ has size two, then since f is commutative we must either have $a \rightarrow b$ or $b \rightarrow a$, so one of a, b is a common ancestor of a and b.

Otherwise, by the definition of a spiral, there is a surjective homomorphism α from $\mathrm{Sg}_{\mathbb{A}}\{a, b\}$ to the free semilattice on two generators. Clearly a and b must be sent to the two generators of the free semilattice by α, say $\alpha(a) = x$ and $\alpha(b) = y$, and every nontrivial binary term $t \in \mathrm{Clo}(f)$ must have $\alpha(t(a, b)) = t(x, y) = f(x, y)$. Thus the kernel of α has congruence classes $\{a\}, \{b\}$, and $S = \mathrm{Sg}_{\mathbb{A}}\{a, b\} \setminus \{a, b\}$, and S is a binary absorbing subalgebra of $\mathbb{A}$ with respect to f.

Since S is a binary absorbing subalgebra of $\mathbb{A}$ with respect to f and $s \in \mathrm{Clo}(f)$ is defined nontrivially, we must have $s(a, b), s(b, a) \in S$. Since $|S| \le |A| - 2$, we can apply the inductive hypothesis to see that $s(a, b), s(b, a)$ have a common ancestor in $\mathrm{Sg}_{(S,s)}\{s(a, b), s(b, a)\} \subseteq \mathrm{Sg}_{\mathbb{A}_s}\{a, b\}$. □

Example 3.4.2. An example of an ancestral algebra which is not a 2-semilattice or a spiral is the algebra $\mathbb{A}_4 = (\{a, b, c, d\}, s)$, where s is the partial semilattice operation described below.

s	a	b	c	d
a	a	b	b	a
b	b	b	c	c
c	d	c	c	d
d	a	a	d	d



The algebra $\mathbb{A}_4$ has the cyclic automorphism $(a\ b\ c\ d)$, and is generated by the pair a, c, since $s(a,c) = b, s(c,a) = d$. The binary term s' given by

$$s'(x,y) \coloneqq s(x, s(y,x))$$

is another (nontrivial) partial semilattice term of $\mathbb{A}_4$, such that $s'(a,c) = a, s'(c,a) = c$. So the reduct $(\{a,b,c,d\}, s')$ of $\mathbb{A}_4$ is *not* an ancestral algebra, as it has the subalgebra $(\{a,c\}, s')$ which has the two maximal strongly components $\{a\}$ and $\{c\}$.

It is easy to check that $\mathbb{A}_4$ is simple, and every proper subalgebra of $\mathbb{A}_4$ is a two element semilattice. By Corollary 3.3.8, $\mathbb{A}_4$ is polynomially complete, and in fact Theorem 3.3.1 and Theorem 3.3.5 imply that every subdirect relation $\mathbb{R} \le_{sd} \mathbb{A}_4^n$ can be written as an intersection of two variable relations, each of which is the graph of an automorphism of $\mathbb{A}_4$. In particular, if we consider the ternary relation

$$\mathbb{R}_{ac} = \mathrm{Sg}_{\mathbb{A}^3}\left\{ \begin{bmatrix} a \\ a \\ c \end{bmatrix}, \begin{bmatrix} a \\ c \\ a \end{bmatrix}, \begin{bmatrix} c \\ a \\ a \end{bmatrix} \right\},$$

we find that $\mathbb{R}_{ac} = \mathbb{A}_4^3$. Since there is an automorphism of $\mathbb{A}_4$ which interchanges a and c, we see that there are ternary terms $g, g' \in \mathrm{Clo}(\mathbb{A}_4)$ such that $\{a,c\}$ is closed under g and g', with $(\{a,c\}, g)$ a two element majority algebra and $(\{a,c\}, g')$ a two element affine algebra. Either of the reducts $(\{a,b,c,d\}, g)$ or $(\{a,b,c,d\}, g')$ defines a Taylor algebra, since g satisfies the identity

$$g(x,x,y) \approx g(x,y,x) \approx g(y,x,x) \approx s'(x,y),$$

and g' satisfies the similar identity

$$g'(x,x,y) \approx g'(x,y,x) \approx g'(y,x,x) \approx s'(y,x).$$

Proposition 3.4.8. *Every quotient of an ancestral algebra is ancestral.*

Theorem 3.4.9. *If $\mathbb{A}_1, ..., \mathbb{A}_n$ are ancestral algebras with partial semilattice operation s, then so is $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$.*

Proof. We prove this by induction on n. Let $a, b \in \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$. Since $\mathbb{A}_1$ is ancestral, there is some $c_1 \in \mathrm{Sg}_{\mathbb{A}_1}\{a_1, b_1\}$ which is reachable from both a_1 and b_1. Lifting the path from a_1 to c_1 to a path from a to some element $c' \in \mathrm{Sg}\{a,b\}$ with $c'_1 = c_1$, and lifting the path from b_1 to c_1 to a path from b to some $c'' \in \mathrm{Sg}\{a,b\}$ with $c''_1 = c_1$, we see that we just need to find a common ancestor of c' and c''. Since $c'_1 = c''_1$ and $\mathbb{A}_1$ is idempotent, we see that c', c'' have a common ancestor so long as $\mathbb{A}_2 \times \cdots \times \mathbb{A}_n$ is ancestral, which follows from the inductive hypothesis. □

Corollary 3.4.10. *If $\mathbb{A}$ is ancestral and $\mathbb{B} \in HSP_{fin}(\mathbb{A})$, then $\mathbb{B}$ is also ancestral.*

It turns out that ancestral algebras can be defined entirely in terms of collections of partial semilattice operations.

Theorem 3.4.11. *A finite idempotent algebra $\mathbb{A}$ with a fixed partial semilattice operation s is ancestral iff for some $m \ge n \ge 0$ it has a sequence of partial semilattice terms $p_1, p_2, ..., p_m$ such that*

- *$a \rightarrow_s p_i(a,b)$ for all $a, b \in \mathbb{A}$ and all i,*
- *$a \rightarrow_s b$ implies $p_i(a,b) = b$ for all i, and*
- *if we define binary operations f_i recursively by $f_0(x,y) \coloneqq s(x,y)$ and*

$$f_i(x,y) \coloneqq p_i(f_{i-1}(x,y), f_{i-1}(y,x))$$

for $i \ge 1$, then $f_m(x,y) \approx f_n(y,x)$.

Proof. That the existence of such a sequence implies $\mathbb{A}$ is ancestral follows from the fact that for any a, b, each $f_i(a,b)$ is reachable from a and each $f_j(b,a)$ is reachable from b.

For the converse direction, let $\mathbb{F} = \mathcal{F}_{\mathbb{A}}(x,y) \le \mathbb{A}^{\mathbb{A}^2}$ be the free algebra on two generators in the variety generated by $\mathbb{A}$. Since $\mathbb{F} \in SP_{fin}(\mathbb{A})$, $\mathbb{F}$ is ancestral, so there is some sequence of elements $f_0, ..., f_n \in \mathbb{F}$ with $f_0(x,y) = s(x,y)$, such that each $f_{i-1} \rightarrow_s f_i$, each $f_i \in \mathrm{Sg}_{\mathbb{F}}\{f_{i-1}(x,y), f_{i-1}(x,y)\}$, and such that the subset S of elements of the subalgebra $\mathbb{S} = \mathrm{Sg}_{\mathbb{F}}\{f_n(x,y), f_n(y,x)\}$ which are reachable from f_n in $\mathbb{S}$ is minimal given these constraints. Then S must be strongly connected, and for every $g \in S$ we must have $\mathbb{S} = \mathrm{Sg}_{\mathbb{F}}\{g(x,y), g(y,x)\}$. Thus we can extend our sequence $f_0, ..., f_n$ by $f_{n+1}, ..., f_m$ such that each $f_{i-1} \rightarrow_s f_i$, and $f_m(x,y) \approx f_n(y,x)$, and we will automatically have $f_i \in \mathrm{Sg}_{\mathbb{F}}\{f_{i-1}(x,y), f_{i-1}(x,y)\}$ for each i.

Note that $f_{i-1} \rightarrow_s f_i$ and $f_i \in \mathrm{Sg}_{\mathbb{F}}\{f_{i-1}(x,y), f_{i-1}(x,y)\}$ implies the existence of a binary term p_i such that $x \rightarrow_s p_i(x,y)$ and $f_i(x,y) = p_i(f_{i-1}(x,y), f_{i-1}(y,x))$, by the argument of Proposition 3.2.23. Note that the reduct with basic operations s, f_i is ancestral, and has the property that $a \rightarrow_s b$ implies $f_i(a,b) = f_i(b,a) = b$ for all i, so $\{a,b\}$ is a semilattice subalgebra with respect to any nontrivial binary term in $\mathrm{Clo}(f_0, ..., f_m)$. Thus we may assume without loss of generality that $a \rightarrow_s b$ implies $p_i(a,b) = b$ for all i, and then the argument of Proposition 3.2.23 implies that each p_i is a partial semilattice term. □

In fact, we can go further: every ancestral algebra has an ancestral reduct which is prepared. Recall that $\mathbb{A}$ is *prepared* if for all $a, b \in \mathbb{A}$, we have $(b,b) \in \mathrm{Sg}_{\mathbb{A}^2}\{(a,b),(b,a)\}$ iff $\{a,b\}$ is a semilattice subalgebra of $\mathbb{A}$ with $a \rightarrow b$.

Theorem 3.4.12. *Every finite ancestral algebra $\mathbb{A}$ has a reduct which is prepared and ancestral.*

Proof. Let s, f_i be as in Theorem 3.4.11, and assume without loss of generality that these are the basic operations of $\mathbb{A}$. Suppose there is a pair $a, b \in \mathbb{A}$ with $(b,b) \in \mathrm{Sg}_{\mathbb{A}^2}\{(a,b),(b,a)\}$ but $s(a,b) \ne b$. Let s' be a partial semilattice term with $s'(a,b) = b$. Then $c \rightarrow_s d$ implies $c \rightarrow_{s'} d$, and if we define

$$f_0'(x,y) \coloneqq s'(x,y)$$

and

$$f_i'(x,y) \coloneqq f_{i-1}(s'(x,y), s'(y,x))$$

for $i \ge 1$, then the reduct with basic operations s', f_i' is an ancestral algebra (with respect to s') with strictly more semilattice subalgebras than $\mathbb{A}$. □

Due to the structural simplifications we can obtain by passing to reducts, it makes sense to focus on ancestral algebras such that no proper reduct is also ancestral.

Definition 3.4.13. A finite algebra $\mathbb{A}$ is called a *minimal ancestral* algebra if $\mathbb{A}$ is ancestral, and no proper reduct of $\mathbb{A}$ is ancestral.

Since every minimal ancestral algebra is automatically prepared, we don't need to specify a particular choice of partial semilattice operation to define the digraph of semilattice subalgebras.

Proposition 3.4.14. *Every finite ancestral algebra has a reduct which is a minimal ancestral algebra.*

Proof. Whether an algebra is ancestral only depends on the collection of partial semilattice operations in its clone. Since there are only finitely many partial semilattice operations on a given finite set, we don't need to worry about infinite descending chains of smaller and smaller ancestral reducts. □

Proposition 3.4.15. *If $\mathbb{A}$ is a minimal ancestral algebra and $\mathbb{B} \in HSP_{fin}(\mathbb{A})$, then $\mathbb{B}$ is also a minimal ancestral algebra.*

Proof. Let f_i be terms for $\mathbb{A}$ as in Theorem 3.4.11. If we can find a proper reduct of $\mathbb{B}$ which is ancestral, then there is a sequence of terms f_i' of this reduct such that $f_0'(x,y) \approx x$, $f_i'(x,y) \rightarrow f_{i+1}'(x,y)$, and $f_m'(a,b) = f_n'(b,a)$ holds for all $a, b \in \mathbb{B}$. Then if we define additional terms f_{m+i}' by

$$f_{m+i}'(x,y) \coloneqq f_i(f_m'(x,y), f_n'(y,x)),$$

we see that these terms $f_0', ..., f_m', f_{m+1}', ...$ generate the same reduct on $\mathbb{B}$ as $f_0', ..., f_m'$, and generate an ancestral reduct of $\mathbb{A}$. □

Theorem 3.4.16. *If $\mathbb{A}$ is a minimal ancestral algebra, then for any $a, b \in \mathbb{A}$, if S is the maximal strongly connected component of $\mathrm{Sg}_{\mathbb{A}}\{a,b\}$, then we have $\mathrm{Sg}_{\mathbb{A}}\{a,b\} = S \cup \{a,b\}$. If $\{a,b\} \not\subseteq S$, then $\mathrm{Sg}_{\mathbb{A}}\{a,b\}$ has a semilattice quotient with S as a congruence class which acts as the top element.*

Proof. Choose terms f_i as in Theorem 3.4.11. Let $\mathbb{F} = \mathcal{F}_{\mathbb{A}}(x,y)$ be the free algebra on two generators in the variety generated by $\mathbb{A}$. Pick any element $g(x,y)$ in the maximal strongly connected component of $\mathbb{F}$, and note that since $g(x,y)$ is reachable from both x and y in $\mathbb{F}$, every term $t(x_1, ..., x_k) \in \mathrm{Clo}(g)$ which depends on all its inputs has the property that $t(x_1, ..., x_k)$ is reachable from each x_i in $\mathcal{F}_{\mathbb{A}}(x_1, ..., x_k)$.

Applying the semilattice iteration argument, we get a partial semilattice term $s'(x,y) \in \mathrm{Clo}(g)$, which is reachable from each of x, y, and $g(x,y)$ in $\mathbb{F}$. In particular, we see that $s'(x,y)$ is contained in the maximal strongly connected component of $\mathbb{F}$, and if we define terms f_i' by

$$f_0'(x,y) \coloneqq s'(x,y)$$

and

$$f_i'(x,y) \coloneqq f_{i-1}(s'(x,y), s'(y,x))$$

for $i \geq 1$, then the reduct with basic operations f_i' is an ancestral algebra, and each $f_i'(x,y)$ is contained in the maximal strongly connected component of $\mathbb{F}$. Thus the clone generated by the f_i's must be equal to the clone of $\mathbb{A}$, and we see that every element of $\mathbb{F}$ is either equal to one of x, y or is contained in the maximal strongly connected component of $\mathbb{F}$. □

Corollary 3.4.17. *If $\mathbb{A}$ is a minimal ancestral algebra, then the maximal strongly connected component of $\mathbb{A}$ is a strongly absorbing subalgebra of $\mathbb{A}$.*

There is a sense in which even the class of minimal ancestral algebras is unnecessarily large: it contains algebras such as the algebra $\mathbb{A}_4$ from Example 3.4.2 which have proper Taylor reducts with two element majority or affine subalgebras.

Theorem 3.4.18. *Suppose $\mathbb{A}$ is a minimal ancestral algebra which is generated by a and b, is strongly connected, and is simple. Then there are ternary terms $g, g' \in \mathrm{Clo}(\mathbb{A})$ such that $\{a, b\}$ is closed under g and g', $(\{a, b\}, g)$ is a two element majority algebra, and $(\{a, b\}, g')$ is a two element affine algebra.*

Proof. Let $\mathbb{S} = \mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\}$. If $\mathbb{S}$ is linked, then by Theorem 3.3.1(c) we must have $(b, b) \in \mathbb{S}$, so $a \to b$, a contradiction. Otherwise, $\mathbb{S}$ is the graph of an automorphism swapping a and b. In this case, the ternary relation $\mathbb{R} = \mathrm{Sg}_{\mathbb{A}^3}\{(a, a, b), (a, b, a), (b, a, a)\}$ has $(a, a), (a, b), (b, a) \in \pi_{i,j}(\mathbb{R})$ for each i, j, so by Theorem 3.3.1(c) we have $\pi_{i,j}(\mathbb{R}) = \mathbb{A}^2$, and then by Theorem 3.3.5 we have $\mathbb{R} = \mathbb{A}^3$. Thus $(a, a, a) \in \mathbb{R}$ and $(b, b, b) \in \mathbb{R}$, and we can take g, g' to be ternary terms of $\mathbb{A}$ which witness these facts. □

Later we will see that the above result implies that a minimal ancestral algebra which is both strongly connected and generated by two elements has a proper Taylor reduct (and, in fact, has a proper bounded width reduct). For now we will show that minimal ancestral algebras which avoid this situation are actually spirals.

Theorem 3.4.19. *If $\mathbb{A}$ is a minimal ancestral algebra such that for all a, b the subalgebra $\mathrm{Sg}_{\mathbb{A}}\{a, b\}$ has no strongly connected quotient, then $\mathbb{A}$ is term equivalent to a spiral.*

Proof. Let s be a nontrivial partial semilattice operation on $\mathbb{A}$. Define a sequence of terms f_i inductively by $f_0 \coloneqq s$ and

$$f_{i+1}(x) \coloneqq f_i(s(x, y), s(y, x)).$$

We will show by induction on $|\mathbb{A}|$ that for each $a, b \in \mathbb{A}$, there is an n such that $f_n(a, b) = f_n(b, a)$. To see this, note that by Theorem 3.4.16, for any a, b the subalgebra generated by $s(a, b), s(b, a)$ is contained in the maximal strongly connected component S of $\mathrm{Sg}_{\mathbb{A}}\{a, b\}$, so as long as $S \neq \mathbb{A}$ we can apply the induction hypothesis to see that there is some i such that

$$f_i(s(a, b), s(b, a)) = f_i(s(b, a), s(a, b)),$$

and for this i we then have $f_{i+1}(a, b) = f_{i+1}(b, a)$.

Thus there is some n such that $f = f_n$ is commutative (in fact, we can take $n = |\mathbb{A}|$). To finish, we need to show that if $\mathbb{A}$ is generated by two elements a, b with $|\mathbb{A}| > 2$, then the maximal strongly connected component S of $\mathbb{A}$ does not contain either of a, b. To this end, suppose for a contradiction that S contains b. Let $\mathbb{S} = \mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\}$. If S is contained in a linked component of $\mathbb{S}$, then by Theorem 3.3.1(c) we must have $(b, b) \in \mathbb{S}$, so $a \to b$, a contradiction. Otherwise, the linking congruence $\theta \in \mathrm{Con}(\mathbb{A})$ of $\mathbb{S}$ has $|S/\theta| > 1$ and $b/\theta \in S/\theta$, and so we may assume without loss of generality that θ is trivial. But if θ is trivial, then $\mathbb{A}$ has an automorphism which interchanges a and b, so S contains both a and b, so $\mathbb{A}$ is both strongly connected and generated by two elements, a contradiction. □

3.5 Cycle-consistency solves ancestral CSPs

In this section we will prove that any cycle-consistent instance of an ancestral CSP has a solution. This proof is a simple case of Kozik's proof [126] of the fact that cycle-consistency solves CSPs over templates with bounded width: the main purpose of presenting the argument in this special case is to allow the reader to focus on the overall proof strategy before getting into the technical algebraic details.

The ingredients which we will need for the proof are the following facts about ancestral algebras.

- Every ancestral algebra $\mathbb{A}$ has a unique maximal strongly connected component $\mathbb{A}^{\max}$ (Proposition 3.4.5).
- If $\pi : \mathbb{A} \twoheadrightarrow \mathbb{B}$ is a surjective homomorphism, then $\pi(\mathbb{A}^{\max}) = \mathbb{B}^{\max}$ (Corollary 3.3.2 to Theorem 3.3.1(a)).
- If $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$ and $\mathbb{A}^{\max}$ is contained in a linked component of $\mathbb{R}$, then $\mathbb{R}^{\max} = \mathbb{A}^{\max} \times \mathbb{B}^{\max}$ (Theorem 3.3.1(c)).
- In particular, if $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$, $\mathbb{A}$ is generated by $\mathbb{A}^{\max}$, $\mathbb{B}$ is generated by $\mathbb{B}^{\max}$, and $\mathbb{B}$ is simple, then $\mathbb{R}$ is either the graph of a homomorphism $\mathbb{A} \twoheadrightarrow \mathbb{B}$ or $\mathbb{R} = \mathbb{A} \times \mathbb{B}$ (Corollary 3.3.4).
- If $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B} \times \mathbb{C}$ has $\pi_{12}(\mathbb{R}) = \mathbb{A} \times \mathbb{B}$ and $\pi_{13}(\mathbb{R}) = \mathbb{A} \times \mathbb{C}$, then $\mathbb{R}^{\max} = \mathbb{A}^{\max} \times \pi_{23}(\mathbb{R})^{\max}$ (Theorem 3.3.5).
- Applying the above inductively, if $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ has $\pi_{ij}(\mathbb{R}) = \mathbb{A}_i \times \mathbb{A}_j$ for all $i \ne j$, then $\mathbb{R}^{\max} = \mathbb{A}_1^{\max} \times \cdots \times \mathbb{A}_n^{\max}$ (Corollary 3.3.7).
- If $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{A}$ and $\mathbb{R}$ contains the diagonal $\Delta_{\mathbb{A}}$, then $\Delta_{\mathbb{A}^{\max}} \subseteq \mathbb{R}^{\max}$ (Theorem 3.3.9).
- If we start with any cycle-consistent instance of $\mathrm{CSP}(\mathbb{A}_1, ..., \mathbb{A}_n)$ and replace every domain and every relation by the subalgebra generated by its maximal elements, then the resulting instance will still be cycle-consistent (Corollary 3.3.10).

If we assume that our algebras are minimal ancestral (rather than just ancestral), then each $\mathbb{A}^{\max}$ becomes a subalgebra (Corollary 3.4.17), which slightly simplifies the arguments. We won't use this simplification, but the reader should keep it in mind.

The general strategy is to start with a cycle-consistent instance, and to find a way to shrink some of the variable domains and relations to get a strictly smaller cycle-consistent instance. Eventually, we reach a situation where all the variable domains have size 1 and the instance is still cycle-consistent - at this point, there is obviously a solution to the CSP. We have already seen that by shrinking variable domains, we can reach a situation where each variable domain $\mathbb{A}_x$ is generated by $\mathbb{A}_x^{\max}$ (the last bullet point above).

To finish the argument, we need to find another strategy for reducing the variable domains when each $\mathbb{A}_x = \mathrm{Sg}(\mathbb{A}_x^{\max})$. The intuition is that if $\mathbb{A}_x = \mathrm{Sg}(\mathbb{A}_x^{\max})$, then there is some congruence $\theta_x \in \mathrm{Con}(\mathbb{A}_x)$ such that $\mathbb{A}_x/\theta_x$ is simple, and in fact $\mathbb{A}_x/\theta_x$ will be polynomially complete by Corollary 3.3.8. Since polynomially complete algebras should have few interesting subdirect relations, it's plausible that we can replace the domain $\mathbb{A}_x$ with an arbitrary congruence class of θ_x, and always obtain a cycle-consistent instance.

So fix a variable x with $|\mathbb{A}_x| > 1$, a maximal congruence θ_x in $\mathrm{Con}(\mathbb{A}_x)$, and a congruence class $\mathbb{A}_x'$ of θ_x. We now have to restrict the other variable domains in order to, at the very least, get an arc-consistent sub-instance. We will show that a very minimalistic sort of reduction strategy suffices: instead of worrying about all possible issues with ensuring arc-consistency, we will only consider paths from variables y to x through the instance.

Definition 3.5.1. If $\mathbf{X}$ is an instance of a CSP and x, y are variables of $\mathbf{X}$, then a *path* p from x to y is defined as a sequence $x = v_0, (\mathbb{R}_1, i_1, j_1), v_1, ..., v_{n-1}, (\mathbb{R}_n, i_n, j_n), v_n = y$ such that each v_k is a variable, and each $\mathbb{R}_k$ is a relation such that one of the constraints of the instance $\mathbf{X}$ imposes the relation $\mathbb{R}_k$ on a tuple $u = (u_1, ...)$ of variables with $u_{i_k} = v_{k-1}$ and $u_{j_k} = v_k$.

To every path p from x to y, we associate the binary relation $\mathbb{P}_p \le \mathbb{A}_x \times \mathbb{A}_y$ which is given by

$$\mathbb{P}_p \coloneqq \pi_{i_1 j_1}(\mathbb{R}_1) \circ \cdots \circ \pi_{i_n j_n}(\mathbb{R}_n).$$

In other words, $\mathbb{P}_p$ is the set of pairs of values in $\mathbb{A}_x \times \mathbb{A}_y$ which are consistent with the path p.

We define addition and negation of paths in the natural way, so that if p is a path from x to y and q is a path from y to z, then $p + q$ is a path from x to z with $\mathbb{P}_{p+q} = \mathbb{P}_p \circ \mathbb{P}_q$, and $-p$ is a path from y to x with $\mathbb{P}_{-p} = \mathbb{P}_p^-$.

In particular, we see that an instance is arc-consistent iff for all paths p the associated binary relations $\mathbb{P}_p$ are subdirect, and it is cycle-consistent iff we additionally have $\Delta_{\mathbb{A}_v} \subseteq \mathbb{P}_p$ for every path p from a variable v back to itself.

Definition 3.5.2. Suppose that $\mathbf{X}$ is a cycle-consistent instance such that for all variable domains we have $\mathbb{A}_v = \mathrm{Sg}(\mathbb{A}_v^{\max})$, that x is any variable with $|\mathbb{A}_x| > 1$, that θ_x is any maximal congruence on $\mathbb{A}_x$, and that $\mathbb{A}_x'$ is any congruence class of $\mathbb{A}_x/\theta_x$.

For each variable y, we say that y is *proper* if there is a path p from y to x such that $\mathbb{P}_p/\theta_x \le \mathbb{A}_y \times \mathbb{A}_x/\theta_x$ is the graph of a homomorphism $\iota_y : \mathbb{A}_y \twoheadrightarrow \mathbb{A}_x/\theta_x$. In this case, we define the congruence $\theta_y \in \mathrm{Con}(\mathbb{A}_y)$ to be the kernel of ι_y, and we define $\mathbb{A}_y'$ to be the preimage of $\mathbb{A}_x'$ under ι_y. If y is not proper, then we define $\mathbb{A}_y'$ to be $\mathbb{A}_y$.

We define the reduced instance $\mathbf{X}'$ by replacing the domain of each variable v by $\mathbb{A}_v'$, and replacing each constraint relation $\mathbb{R} \le \mathbb{A}_{v_1} \times \cdots \times \mathbb{A}_{v_m}$ of $\mathbf{X}$ by $\mathbb{R}' = \mathbb{R} \cap (\mathbb{A}_{v_1}' \times \cdots \times \mathbb{A}_{v_m}')$.

The reason for the name "proper" is that a variable v is proper iff the reduced domain $\mathbb{A}_v'$ is a proper subalgebra of $\mathbb{A}_v$. First we need to check that the maps ι_y for the proper variables y are well-defined.

Lemma 3.5.3. *If y is a proper variable and p, q are two paths from y to x such that $\mathbb{P}_p/\theta_x, \mathbb{P}_q/\theta_x$ are graphs of homomorphisms $\iota_p, \iota_q : \mathbb{A}_y \twoheadrightarrow \mathbb{A}_x/\theta_x$, then in fact we have $\iota_p = \iota_q$. Thus ι_y, θ_y, and $\mathbb{A}_y'$ are all well-defined.*

Proof. The path $p - q$ connects y to itself, so by cycle-consistency we must have $\Delta_{\mathbb{A}_y} \subseteq \mathbb{P}_{p-q} = \mathbb{P}_p \circ \mathbb{P}_q^-$. Taking the quotient by θ_x, we see that $\Delta_{\mathbb{A}_y} \subseteq (\mathbb{P}_p/\theta_x) \circ (\mathbb{P}_q/\theta_x)^-$, so for every element $a \in \mathbb{A}_y$ we must have $\iota_p(a) = \iota_q(a)$. □

We sometimes abuse notation, and think of ι_y as an isomorphism from $\mathbb{A}_y/\theta_y$ to $\mathbb{A}_x/\theta_x$.

Lemma 3.5.4. *Suppose p is a path from y to a proper variable z. Then one of the following is true:*

- $\mathbb{P}_p/\theta_z = \mathbb{A}_y \times \mathbb{A}_z/\theta_z$, *or*
- *y is also proper, and $\mathbb{P}_p/(\theta_y \times \theta_z)$ is the graph of an isomorphism $\iota_p : \mathbb{A}_y/\theta_y \xrightarrow{\sim} \mathbb{A}_z/\theta_z$ such that $\iota_y = \iota_z \circ \iota_p$.*

Proof. This follows from Corollary 3.3.4 and cycle-consistency (note that $\mathbb{A}_z/\theta_z$ is simple, since it is isomorphic to $\mathbb{A}_x/\theta_x$). □

We have the ingredients necessary to check that the reduced instance $\mathbf{X}'$ is cycle-consistent. We start with arc-consistency.

Lemma 3.5.5. *Suppose $\mathbb{R} \le_{sd} \mathbb{A}_{v_1} \times \cdots \times \mathbb{A}_{v_n}$ is a constraint of $\mathbf{X}$. Then the reduced constraint $\mathbb{R}' = \mathbb{R} \cap (\mathbb{A}'_{v_1} \times \cdots \times \mathbb{A}'_{v_n})$ is subdirect inside $\mathbb{A}'_{v_1} \times \cdots \times \mathbb{A}'_{v_n}$, that is, $\pi_i(\mathbb{R}') = \mathbb{A}'_{v_i}$ for each i.*

Proof. By symmetry, it's enough to prove that $\pi_1(\mathbb{R}') = \mathbb{A}'_{v_1}$. In other words, for each element $a \in \mathbb{A}'_{v_1}$, we want to find a tuple $s \in \mathbb{R}$ such that $s_i \in \mathbb{A}'_{v_i}$ for all i. We may ignore variables v_i such that $i \ne 1$ and v_i is not proper, since for such i the restriction from $\mathbb{A}_{v_i}$ to $\mathbb{A}'_{v_i} = \mathbb{A}_{v_i}$ has no effect. Similarly, for any two proper variables v_i, v_j such that $\pi_{ij}(\mathbb{R})$ induces an isomorphism between $\mathbb{A}_{v_i}/\theta_{v_i}$ and $\mathbb{A}_{v_j}/\theta_{v_j}$, we may ignore one of the two variables v_i, v_j, since any element $s \in \mathbb{R}$ which satisfies $s_i \in \mathbb{A}'_{v_i}$ will automatically also satisfy $s_j \in \mathbb{A}'_{v_j}$.

To formalize the process of ignoring variables, we define an equivalence relation $\sim$ on the set of indices of proper variables of $\mathbb{R}$, with $i \sim j$ when $\pi_{ij}(\mathbb{R})$ induces an isomorphism between $\mathbb{A}_{v_i}/\theta_{v_i}$ and $\mathbb{A}_{v_j}/\theta_{v_j}$ (that $\sim$ is an equivalence relation is easy to check). Then we let $I \subseteq [n]$ be a set of variable indices such that each $\sim$-class has exactly one representative in I, $1 \in I$, and no index of any non-proper variable other than possibly 1 is in I. We then define a relation $\mathbb{S} \le \mathbb{A}_{v_1} \times \prod_{i \in I\setminus\{1\}} \mathbb{A}_{v_i}/\theta_{v_i}$ by

$$\mathbb{S} \coloneqq \pi_I(\mathbb{R}) \Big/ \prod_{i \in I\setminus\{1\}} \theta_{v_i}.$$

We just need to show that for every $a \in \mathbb{A}'_{v_1}$ there is some $s \in \mathbb{S}$ with $s_1 = a$ and $s_i = \mathbb{A}'_{v_i}/\theta_{v_i}$ for each $i \in I \setminus \{1\}$. Note that by Lemma 3.5.4 and the construction of I, for every pair $i, j \in I$ the projection $\pi_{ij}(\mathbb{S})$ is full. Thus by Corollary 3.3.7, we in fact have

$$\mathbb{S}^{\max} = \mathbb{A}_{v_1}^{\max} \times \prod_{i \in I\setminus\{1\}} \mathbb{A}_{v_i}^{\max}/\theta_{v_i},$$

and since each $\mathbb{A}_{v_i}$ is generated by $\mathbb{A}_{v_i}^{\max}$, we have

$$\mathbb{S} = \mathbb{A}_{v_1} \times \prod_{i \in I\setminus\{1\}} \mathbb{A}_{v_i}/\theta_{v_i}.$$ □

Now we can check that cycle-consistency also holds for the reduced instance.

Lemma 3.5.6. *Suppose p is a path from v to v in the instance $\mathbf{X}$, and let p' be the corresponding path in $\mathbf{X}'$. If $\Delta_{\mathbb{A}_{v_1}} \subseteq \mathbb{P}_p$, then $\Delta_{\mathbb{A}'_{v_1}} \subseteq \mathbb{P}_{p'}$.*

Proof. Suppose that p is the path $v = v_0, (\mathbb{R}_1, i_1, j_1), v_1, ..., v_{n-1}, (\mathbb{R}_n, i_n, j_n), v_n = v$. Note that in the corresponding path p', we must replace each $\mathbb{R}_i$ with $\mathbb{R}'_i$, so we must also worry about the proper variables which occur in $\mathbb{R}_i$ but do not lie along the path p. In order to do this cleanly, we consider the relation $\mathbb{R}$ defined by

$$\mathbb{R} := \Big\{(v_0, u^1, ..., u^n, v_n) \in \mathbb{A}_v \times \prod_{i\le n} \mathbb{R}_i \times \mathbb{A}_v \;\Big|\; v_0 = u^1_{i_1}, u^1_{j_1} = u^2_{i_2}, ..., u^{n-1}_{j_{n-1}} = u^n_{i_n}, u^n_{j_n} = v_n\Big\}.$$

If each $\mathbb{R}_i$ has arity m_i, then $\mathbb{R}$ is thought of as a relation of arity $m = 2 + \sum_i m_i$, and the indices of $\mathbb{R}$ might contain several copies of variables of the instance $\mathbf{X}$. Let the ith index of $\mathbb{R}$ correspond to the variable y_i in $\mathbf{X}$, with $y_1 = v_0 = v$ and $y_m = v_n = v$, so

$$\mathbb{R} \le_{sd} \mathbb{A}_{y_1} \times \cdots \times \mathbb{A}_{y_m}.$$

Note that by the arc-consistency of the instance $\mathbf{X}$, for any two indices i, j of the relation $\mathbb{R}$, the projection $\pi_{ij}(\mathbb{R})$ is the same as $\mathbb{P}_q$ for some path q from y_i to y_j formed out of the relations $\mathbb{R}_i$, and that $\pi_{1m}(\mathbb{R}) = \mathbb{P}_p$, so $\pi_{1m}(\mathbb{R}) \supseteq \Delta_{\mathbb{A}_v}$.

As in the argument for arc-consistency, we define an equivalence relation $\sim$ on the proper indices of $\mathbb{R}$ defined by $i \sim j$ when $\pi_{ij}(\mathbb{R})$ induces an isomorphism between $\mathbb{A}_{y_i}/\theta_{y_i}$ and $\mathbb{A}_{y_j}/\theta_{y_j}$. We let $I \subseteq [m]$ to be a set of indices of $\mathbb{R}$ with $1, m \in I$, such that I contains no indices of non-proper variables of $\mathbb{R}$ other than possibly 1 and m, such that $I \setminus \{m\}$ contains one representative from each $\sim$ class of $\{1, ..., m-1\}$, and such that $I \setminus \{1\}$ contains one representative from each $\sim$ class of $\{2, ..., m\}$. As before, we define a relation $\mathbb{S}$ by

$$\mathbb{S} := \pi_I(\mathbb{R}) \Big/ \prod_{i\in I\setminus\{1,m\}} \theta_{y_i}.$$

We just need to show that for every $a \in \mathbb{A}'_v$, there is some $s \in \mathbb{S}$ with $s_1 = s_m = a$ and $s_i = \mathbb{A}'_{v_i}/\theta_{v_i}$ for each $i \in I \setminus \{1, m\}$. By Lemma 3.5.4 and the construction of I, for every pair $i, j \in I$ with $\{i, j\} \ne \{1, m\}$ the projection $\pi_{ij}(\mathbb{S})$ is full. Thus by Corollary 3.3.7, we have

$$\pi_{I\setminus\{m\}}(\mathbb{S}) = \mathbb{A}_{y_1} \times \prod_{i\in I\setminus\{1,m\}} \mathbb{A}_{y_i}/\theta_{y_i}$$

and

$$\pi_{I\setminus\{1\}}(\mathbb{S}) = \mathbb{A}_{y_m} \times \prod_{i\in I\setminus\{1,m\}} \mathbb{A}_{y_i}/\theta_{y_i}.$$

Thus by Theorem 3.3.5, we have

$$\mathbb{S}^{\max} = \pi_{1m}(\mathbb{S})^{\max} \times \prod_{i\in I\setminus\{1,m\}} \mathbb{A}^{\max}_{y_i}/\theta_{y_i},$$

and by Theorem 3.3.9 and the assumption $\pi_{1m}(\mathbb{S}) = \pi_{1m}(\mathbb{R}) \supseteq \Delta_{\mathbb{A}_v}$, we have $\pi_{1m}(\mathbb{S})^{\max} \supseteq \Delta_{\mathbb{A}^{\max}_v}$. Since each $\mathbb{A}_y$ is generated by $\mathbb{A}^{\max}_y$, we have

$$\mathbb{S} \supseteq \Delta_{\mathbb{A}_v} \times \prod_{i\in I\setminus\{1,m\}} \mathbb{A}_{y_i}/\theta_{y_i},$$

so in particular for every $a \in \mathbb{A}'_v$ we have $\{a\} \times \prod_{i\in I\setminus\{1,m\}} \mathbb{A}'_{y_i}/\theta_{y_i} \times \{a\} \subseteq \mathbb{S}$, so $(a, a) \in \pi_{1m}(\mathbb{R}') = \mathbb{P}_{p'}$. □

Thus the reduced instance $\mathbf{X}'$ is cycle-consistent. Since we can iteratively shrink our instance whenever some variable x has $\mathbb{A}_x \neq \mathrm{Sg}(\mathbb{A}_x^{\max})$ or has $\mathbb{A}_x = \mathrm{Sg}(\mathbb{A}_x^{\max})$ but $|\mathbb{A}_x| > 1$, we see that we eventually reach a situation where each $\mathbb{A}_x$ consists of a single element, and then arc-consistency proves that this collection of single elements gives a solution to the original instance. We have proved our main result.

Theorem 3.5.7. *If $\mathbf{X}$ is a cycle-consistent instance of an ancestral CSP, then $\mathbf{X}$ has a solution.*

In fact, for any variable x of $\mathbf{X}$, and for any element $a \in \mathbb{A}_x$ such that there is a sequence of subalgebras $\mathbb{A}_x \supseteq \mathbb{A}_0 \supseteq \cdots \supseteq \mathbb{A}_n = \{a\}$ with $\mathbb{A}_0 = \mathrm{Sg}(\mathbb{A}_x^{\max})$ and such that for each i, there is a maximal congruence $\theta_i \in \mathrm{Con}(\mathbb{A}_i)$ and a congruence class $\mathbb{A}_i'$ of θ_i with $\mathbb{A}_{i+1} = \mathrm{Sg}(\mathbb{A}_i'^{\max})$, there is a solution to the instance $\mathbf{X}$ in which x is assigned the value a.

The simple construction of the reduced instance $\mathbf{X}'$ can be used to show that we can find a solution to any cycle-consistent instance of an ancestral CSP in linear time.

3.6 Cycle-consistency solves majority CSPs

The paper which prompted the study of cycle-consistency was a preliminary investigation by Chen, Dalmau, and Grußien [57], which studied a slightly stronger consistency notion: singleton arc-consistency. Singleton arc-consistency refers to the strategy of fixing a particular value for some variable, and checking if applying arc-consistency to the remaining variables produces a contradiction. Singleton arc-consistency is clearly at least as powerful as cycle-consistency. One of the main results of [57] showed that singleton arc consistency solves majority CSPs, but in fact their proof strategy was to show that cycle-consistent instances of majority CSPs always have solutions.

The argument for majority algebras is simpler than the argument for ancestral algebras, essentially because the analogue of the case where all the variables domains are strongly connected doesn't need to be considered. Instead, we are always in the situation where some variable domain $\mathbb{A}_x$ has a proper absorbing subalgebra (every singleton is an absorbing subalgebra of a majority algebra), although we need to work slightly harder than we did in the absorbing case of ancestral CSPs since the absorption is no longer binary absorption. Rather than working with absorbing subalgebras, [57] used the closely related concept of an *ideal* of a majority algebra.

Definition 3.6.1. If $\mathbb{A} = (A, m)$ is a majority algebra, then $\mathbb{B} \le \mathbb{A}$ is called an *ideal* of $\mathbb{A}$ if $m(\mathbb{B}, \mathbb{A}, \mathbb{B}) \subseteq \mathbb{B}$.

The word "ideal" comes from the theory of median algebras - a subset $\mathbb{B}$ is an ideal of a median algebra $\mathbb{A}$ iff there is a congruence θ of $\mathbb{A}$ such that $\mathbb{B}$ is a congruence class of θ. The corresponding statement is not true of majority algebras in general: every subset of the dual discriminator algebra from Example 1.6.5 is an ideal, but the dual discriminator algebra on n elements is simple (and polynomially complete) for $n \ge 3$.

The next result shows that ideals interact with standard algebraic constructions (products, quotients, intersections) nicely. A similar result holds for absorbing subalgebras, with the same proof.

Proposition 3.6.2. *Suppose that a relation $\mathbb{R}$ is defined by a primitive positive formula Φ involving the relations $\mathbb{R}_1, ..., \mathbb{R}_k$. If we replace each $\mathbb{R}_i$ with an ideal $\mathbb{R}_i'$ of $\mathbb{R}_i$ to make a primitive positive formula Φ', then the relation $\mathbb{R}'$ which is defined by Φ' is an ideal of $\mathbb{R}$.*

Proof. Let $\Phi(x) = \exists y \Psi(x, y)$, with Ψ quantifier-free, and let Ψ' be the corresponding formula with $\mathbb{R}_i$s replaced by $\mathbb{R}_i'$s. Then for any a, b, c with $a, c \in \mathbb{R}'$ and $b \in \mathbb{R}$, there exist d, e, f such that $\Psi'(a, d), \Psi(b, e), \Psi'(c, f)$ hold, so $\Psi'(m(a, b, c), m(d, e, f))$ holds since each $\mathbb{R}_i'$ is an ideal, so $\Phi'(m(a, b, c))$ holds. □

Recall the definition of a path in an instance (Definition 3.5.1). It's notationally convenient to allow paths to act on subsets of the variable domains.

Definition 3.6.3. If p is a path connecting variables x, y of an instance $\mathbf{X}$, and if B is a subset of the variable domain $\mathbb{A}_x$, then we define $B + p$ to be the subset of $\mathbb{A}_y$ given by

$$B + p \coloneqq \{c \in \mathbb{A}_y \mid \exists b \in B \text{ s.t. } (b, c) \in \mathbb{P}_p\} = \pi_2(\mathbb{P}_p \cap (B \times \mathbb{A}_y)).$$

Proposition 3.6.4. *If $\mathbb{B} \le \mathbb{A}_x$ and p is a path from x to y, then $\mathbb{B} + p$ is a subalgebra of $\mathbb{A}_y$. If $\mathbb{B}$ is an ideal of $\mathbb{A}_x$ and the instance is arc-consistent, then $\mathbb{B} + p$ is an ideal of $\mathbb{A}_y$.*

Our overall strategy will be to start with a cycle-consistent instance $\mathbf{X}$, and find a collection of ideals $\mathbb{A}_x'$ of the variable domains $\mathbb{A}_x$ such that reducing each domain to $\mathbb{A}_x'$ produces an arc-consistent instance $\mathbf{X}'$. Then we will show that any such $\mathbf{X}'$ is automatically cycle-consistent.

In order to find an arc-consistent family of ideal subdomains, we consider the set $\mathcal{I}$ of pairs $(x, \mathbb{B})$ where x is a variable and $\mathbb{B}$ is a proper ideal of $\mathbb{A}_x$. Note that $\mathcal{I}$ is nonempty as long as some x has $|\mathbb{A}_x| > 1$, since every singleton is an ideal.

Definition 3.6.5. Let $\mathcal{I}$ be the set of pairs $(x, \mathbb{B})$ where x is a variable and $\mathbb{B}$ is a proper ideal of $\mathbb{A}_x$. We define a quasiorder $\preceq$ on $\mathcal{I}$ by $(x, \mathbb{B}) \preceq (y, \mathbb{B} + p)$ for every path p from x to y with $\mathbb{B} + p \ne \mathbb{A}_y$.

Proposition 3.6.6. *If $\mathbf{X}$ is a cycle-consistent instance, x is a variable, and $(x, \mathbb{B}) \preceq (x, \mathbb{C})$, then $\mathbb{B} \le \mathbb{C}$.*

Proof. Suppose p is a path from x to itself with $\mathbb{B} + p = \mathbb{C}$. By cycle-consistency we must have $\Delta_{\mathbb{A}_x} \subseteq \mathbb{P}_p$, so $\mathbb{B} \subseteq \mathbb{B} + p$. □

Definition 3.6.7. Suppose $\mathbf{X}$ is a cycle-consistent instance of a majority CSP, and assume without loss of generality that each constraint of $\mathbf{X}$ is binary. Fix a maximal element $(x, \mathbb{A}_x')$ of $\mathcal{I}$ under the quasiorder $\preceq$.

Call a variable y *proper* if there is a path p from x to y such that $\mathbb{A}_x' + p \ne \mathbb{A}_y$, and in this case set $\mathbb{A}_y' = \mathbb{A}_x' + p$. If y is not proper, then set $\mathbb{A}_y' = \mathbb{A}_y$.

Define the reduced instance $\mathbf{X}'$ by replacing the domain of each variable v by $\mathbb{A}_v'$, and by replacing each constraint $\mathbb{R} \le \mathbb{A}_u \times \mathbb{A}_v$ with $\mathbb{R}' = \mathbb{R} \cap (\mathbb{A}_u' \times \mathbb{A}_v')$.

First we need to check that the sets $\mathbb{A}_y'$ are well-defined.

Lemma 3.6.8. *If there are paths p, q from x to y such that $\mathbb{A}_x' + p \ne \mathbb{A}_y$ and $\mathbb{A}_x' + q \ne \mathbb{A}_y$, then $\mathbb{A}_x' + p = \mathbb{A}_x' + q$.*

Proof. Since $(x, \mathbb{A}_x')$ is maximal and $(x, \mathbb{A}_x') \preceq (y, \mathbb{A}_x' + p)$, we must have $(y, \mathbb{A}_x' + p) \preceq (x, \mathbb{A}_x') \preceq (y, \mathbb{A}_x' + q)$, so $\mathbb{A}_x' + p \le \mathbb{A}_x' + q$. Similarly we have $\mathbb{A}_x' + q \le \mathbb{A}_x' + p$, so $\mathbb{A}_x' + p = \mathbb{A}_x' + q$. □

Next we check arc-consistency.

Lemma 3.6.9. *If p is a path from y to z and p' is the corresponding path in $\mathbf{X}'$, then $\mathbb{A}'_y + p' = \mathbb{A}'_z$.*

Proof. We just need to check this in the case when p has length 1, corresponding to a binary relation $\mathbb{R} \le_{sd} \mathbb{A}_y \times \mathbb{A}_z$. If $\mathbb{A}'_y + p \neq \mathbb{A}_z$, then y, z must both be proper with $\mathbb{A}'_y + p = \mathbb{A}'_z$. Either way we see that $\mathbb{A}'_y + p \supseteq \mathbb{A}'_z$, and since $\mathbb{R}' = \mathbb{R} \cap (\mathbb{A}'_y \times \mathbb{A}'_z)$ we have $\mathbb{A}'_y + p' = \mathbb{A}'_z$ in the reduced instance. □

Finally, we check that arc-consistency of $\mathbf{X}'$ and cycle-consistency of $\mathbf{X}$ implies cycle-consistency of $\mathbf{X}'$. For this, we note that if p is a path from v back to itself in $\mathbf{X}$, and if p' is the corresponding path in $\mathbf{X}'$, then $\mathbb{P}_{p'}$ is an ideal of $\mathbb{P}_p$. Since $\mathbb{P}_p \supseteq \Delta_{\mathbb{A}'_v}$ we have

$$m(\mathbb{P}_{p'}, \Delta_{\mathbb{A}'_v}, \mathbb{P}_{p'}) \subseteq \mathbb{P}_{p'},$$

so the cycle-consistency of $\mathbf{X}'$ follows from the following result.

Theorem 3.6.10. *Suppose that $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{A}$ is subdirect with $m(\mathbb{R}, \Delta_{\mathbb{A}}, \mathbb{R}) \subseteq \mathbb{R}$, where m is a majority operation. Then $\Delta_{\mathbb{A}} \subseteq \mathbb{R}$.*

In fact, if $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is subdirect and satisfies $m(\mathbb{R}, S, \mathbb{R}) \subseteq \mathbb{R}$, where S is any subset of $\mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, then $S \subseteq \mathbb{R}$.

Proof. First we prove the statement about binary relations, since this is all we will need. Let a be any element of $\mathbb{A}$. Since $\mathbb{R}$ is subdirect, there are $b, c \in \mathbb{A}$ such that $(a, b) \in \mathbb{R}$ and $(c, a) \in \mathbb{R}$. Then since $(a, a) \in \Delta_{\mathbb{A}}$, we have

$$\begin{bmatrix} a \\ a \end{bmatrix} = m\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a \\ a \end{bmatrix}, \begin{bmatrix} c \\ a \end{bmatrix}\right) \in m(\mathbb{R}, \Delta_{\mathbb{A}}, \mathbb{R}) \subseteq \mathbb{R}.$$

For the more general statement, we show by induction on k that $\pi_{[k]}(S) \subseteq \pi_{[k]}(\mathbb{R})$ for each $k \le n$. The base case $k = 1$ follows from the assumption that $\mathbb{R}$ is subdirect, and for the inductive step we may as well assume that we have already proven this for $k = n - 1$, and wish to show it for n. Let $(a_1, ..., a_n)$ be any element of S. Then by the inductive hypothesis there is some b such that $(a_1, ..., a_{n-1}, b) \in \mathbb{R}$, and by the assumption that $\mathbb{R}$ is subdirect there are $c_1, ..., c_{n-1}$ such that $(c_1, ..., c_{n-1}, a_n) \in \mathbb{R}$. Then we have

$$\begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = m\left(\begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \\ b \end{bmatrix}, \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix}, \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \\ a_n \end{bmatrix}\right) \in m(\mathbb{R}, S, \mathbb{R}) \subseteq \mathbb{R}.$$ □

Corollary 3.6.11. *The reduced instance $\mathbf{X}'$ is cycle-consistent.*

We have proved the main result of this section.

Theorem 3.6.12. *Every cycle-consistent instance $\mathbf{X}$ of a majority CSP has a solution.*

In fact, for any variable v of $\mathbf{X}$ and any value $a \in \mathbb{A}_v$, the instance $\mathbf{X}$ has a solution in which the variable v is assigned the value a.

Proof. For the second statement, we note that if $|\mathbb{A}_v| > 1$, then $(v, \{a\}) \in \mathcal{I}$, so there is some maximal element $(x, \mathbb{A}'_x) \in \mathcal{I}$ such that $(v, \{a\}) \preceq (x, \mathbb{A}'_x)$, and we define the reduction $\mathbf{X}'$ in terms of the maximal element $(x, \mathbb{A}'_x)$. If v is proper, then from $(v, \{a\}) \preceq (x, \mathbb{A}'_x) \preceq (v, \mathbb{A}'_v)$ we must have $a \in \mathbb{A}'_v$, and if v is not proper then we have $a \in \mathbb{A}_v = \mathbb{A}'_v$. Either way, we see by induction that the reduced instance $\mathbf{X}'$ has a solution in which the variable v is assigned the value a. □

Corollary 3.6.13. *Suppose $\mathbb{A}$ is an algebra with a partial semilattice term s and a ternary term g such that for any subalgebra $\mathbb{B} \le \mathbb{A}$, the restriction of g to $\mathrm{Sg}(\mathbb{B}^{\max})$ is a majority operation. Then every cycle-consistent instance of $\mathrm{CSP}(\mathbb{A})$ has a solution.*

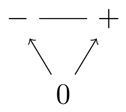
Proof. By Corollary 3.3.10, if we start with a cycle-consistent instance $\mathbf{X}$ and restrict all the variable domains $\mathbb{A}_i$ to $\mathrm{Sg}(\mathbb{A}_i^{\max})$ to create a new instance $\mathbf{X}'$, then $\mathbf{X}'$ will still be cycle-consistent, and by assumption $\mathbf{X}'$ will be preserved by the majority operation g. Then by the previous theorem, $\mathbf{X}'$ will have a solution. □

Example 3.6.1. Consider $\mathbb{A} = (\{-, 0, +\}, g)$, where g is the idempotent cyclic ternary operaton with

$$\begin{aligned} g(0,0,-) = g(0,-,-) &= -, \\ g(0,-,+) = g(-,-,+) &= -, \\ g(0,0,+) = g(0,+,+) &= +, \\ g(0,+,-) = g(-,+,+) &= +. \end{aligned}$$

This can be described more succinctly as follows: the permutation $(-\ +)$ is an automorphism of $\mathbb{A}$, $\{-,+\}$ is a majority subalgebra of $\mathbb{A}$, and $\{0,-\}, \{0,+\}$ are semilattice subalgebras of $\mathbb{A}$ with $0 \to -, +$. The term $s(x,y) := g(x,x,y)$ is a partial semilattice, and s, g satisfy the assumptions of the Corollary above, so every cycle-consistent instance of $\mathrm{CSP}(\mathbb{A})$ has a solution. We give a table for s and draw the graph of two element subalgebras of $\mathbb{A}$ (with undirected edges for majority subalgebras and directed edges for semilattice subalgebras) below.

s	$-$	0	$+$
$-$	$-$	$-$	$-$
0	$-$	0	$+$
$+$	$+$	$+$	$+$



The relational clone $\mathrm{Inv}(g)$ is generated by the unary relation $x \ne 0$, the binary relation $x = -y$, the binary relation $x \le y$, and the ternary relation $x = 0 \implies y = z$.

The clone $\langle g \rangle$ is properly contained in the clone $\langle s_2 \rangle$ from Example 1.6.8, and it does not contain any proper subclone with a Taylor operation. In some sense the algebra considered in this example is the prototypical example of a bounded width algebra: Bulatov [44] has shown that in every minimal bounded width clone, the maximal strongly connected components behave as if there is a majority operation preserving them, and for every pair of maximal strongly connected components there is a two-element majority subalgebra which connects them.

Remark 3.6.1. It's tempting to try to generalize Theorem 3.6.10 to near-unanimity operations. We say that a subalgebra $\mathbb{B}$ *absorbs* $\mathbb{A}$ with respect to a near-unanimity operation t if

$$t(\mathbb{B}, ..., \mathbb{B}, \mathbb{A}, \mathbb{B}, ..., \mathbb{B}) \subseteq \mathbb{B}$$

for each possible location of $\mathbb{A}$. Suppose that $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{A}$ absorbs $\Delta_{\mathbb{A}}$ with respect to t - can we conclude that $\mathbb{R}$ contains the diagonal?

Unfortunately the answer is no: even if $\mathbb{R}$ is subdirect and absorbs $\mathbb{A}^2$ with respect to a near-unanimity term, we might not have $\Delta_{\mathbb{A}} \subseteq \mathbb{R}$. Consider the threshold function t_2^n from Example 1.1.3 defined by

$$t_2^n(x_1, ..., x_n) = \begin{cases} 1 & \sum_i x_i \ge 2, \\ 0 & \sum_i x_i \le 1. \end{cases}$$

For $n \geq 4$, the relation

$$\mathbb{R} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

absorbs $\{0,1\}^2$ with respect to t_2^n, but does not contain the diagonal element $(0,0)$. However, $\mathbb{R}$ *does* intersect the diagonal at $(1,1)$. In the next section we will see that this weaker claim generalizes: if $\mathbb{R} \leq_{sd} \mathbb{A} \times \mathbb{A}$ absorbs $\Delta_\mathbb{A}$, then $\mathbb{R} \cap \Delta_\mathbb{A} \neq \emptyset$.

3.7 Absorption, Jónsson absorption, and connectivity

Absorption is a common generalization of ideals of majority algebras and maximal strongly connected components of minimal ancestral algebras, and a lot of the theory of absorbing subalgebras applies to general (finite, idempotent) algebras, without assuming the existence of a Taylor term. After introducing absorption, we will show that absorbing subalgebras $\mathbb{R}'$ of binary relations $\mathbb{R}$ retain some of the connectivity properties of the original relations $\mathbb{R}$.

Definition 3.7.1. A subalgebra $\mathbb{B} \leq \mathbb{A}$ *absorbs* $\mathbb{A}$ with respect to an idempotent term t if

$$t(\mathbb{B}, ..., \mathbb{B}, \mathbb{A}, \mathbb{B}, ..., \mathbb{B}) \subseteq \mathbb{B}$$

for each possible location of $\mathbb{A}$. We just say that $\mathbb{B}$ absorbs $\mathbb{A}$, written $\mathbb{B} \lhd \mathbb{A}$, if there exists some idempotent term t such that $\mathbb{B}$ absorbs $\mathbb{A}$ with respect to t.

More generally, we sometimes say that a set B absorbs a set A with respect to an idempotent term t if

$$t(B, ..., B, A, B, ..., B) \subseteq B$$

for each possible location of A. Note that if $B \subseteq A$, then B must be closed under t.

The reason we avoid specifying the idempotent term t in the notation $\mathbb{B} \lhd \mathbb{A}$ is that there exists a common term t which witnesses all absorption within any finite collection of pairs $\mathbb{B}_i \lhd \mathbb{A}_i$.

Proposition 3.7.2. *If $\mathbb{B}_1 \lhd \mathbb{A}_1$ with respect to t_1 and $\mathbb{B}_2 \lhd \mathbb{A}_2$ with respect to t_2, then each $\mathbb{B}_i \lhd \mathbb{A}_i$ with respect to the star composition $t_1 * t_2$ (see Definition 1.5.3). If $\mathbb{A}_1 = \mathbb{B}_2$, then $\mathbb{B}_1 \lhd \mathbb{A}_2$ with respect to $t_1 * t_2$.*

Corollary 3.7.3. *A finite algebra $\mathbb{A}$ has a near-unanimity term iff for all $a \in \mathbb{A}$, the singleton $\{a\}$ absorbs $\mathbb{A}$.*

A common strategy in arguments involving absorbing operations t of high arity n is to consider expressions of the form

$$t(x, ..., x, y, z, ..., z),$$

where just a single y occurs, and iteratively march the location of the y one step to the left at a time. We can make such arguments more transparent by phrasing them in terms of the sequence of ternary terms

$$d_i(x, y, z) \coloneqq t(\underbrace{x, ..., x}_{n-i}, y, \underbrace{z, ..., z}_{i-1}),$$

with $d_0(x,y,z) \coloneqq x$ and $d_{n+1}(x,y,z) \coloneqq z$, so that the d_i satisfy the system of identities

$$\begin{aligned} d_0(x,y,z) &\approx x, \\ d_i(x,y,y) &\approx d_{i+1}(x,x,y), \\ d_{n+1}(x,y,z) &\approx z. \end{aligned}$$

If $\mathbb{B}$ absorbs $\mathbb{A}$ with respect to the term t, then we will additionally have

$$d_i(\mathbb{B},\mathbb{A},\mathbb{B}) \subseteq \mathbb{B}$$

for all i.

Definition 3.7.4. A *Jónsson absorption chain* is a sequence of ternary terms $d_1, ..., d_n$ which satisfy the identities

$$\begin{aligned} d_1(x,x,y) &\approx x, \\ d_i(x,y,y) &\approx d_{i+1}(x,x,y), \\ d_n(x,y,y) &\approx y. \end{aligned}$$

We say that $\mathbb{B}$ *Jónsson absorbs* $\mathbb{A}$ with respect to the Jónsson chain $d_1, ..., d_n$ if for each $i \in [n]$ we have

$$d_i(\mathbb{B},\mathbb{A},\mathbb{B}) \subseteq \mathbb{B}.$$

If $\mathbb{B}$ Jónsson absorbs $\mathbb{A}$ with respect to some Jónsson chain, then we write $\mathbb{B} \lhd_J \mathbb{A}$.

Proposition 3.7.5. *If $\mathbb{B} \lhd \mathbb{A}$, then $\mathbb{B} \lhd_J \mathbb{A}$.*

As with absorption, we can witness several instances of Jónsson absorption simultaneously with a single Jónsson absorption chain $d_1, ..., d_n$.

Proposition 3.7.6. *If $\mathbb{B}_1 \lhd_J \mathbb{A}_1$ with respect to $d_1, ..., d_m$ and $\mathbb{B}_2 \lhd_J \mathbb{A}_2$ with respect to $e_1, ..., e_n$, then the sequence of terms $f_1, ..., f_{mn}$ defined by*

$$f_{n(i-1)+j} \coloneqq d_i(x, e_j(x,y,z), z)$$

is a Jónsson absorption chain which witnesses both $\mathbb{B}_1 \lhd_J \mathbb{A}_1$ and $\mathbb{B}_2 \lhd_J \mathbb{A}_2$. If $\mathbb{A}_1 = \mathbb{B}_2$, then $\mathbb{B}_1 \lhd_J \mathbb{A}_2$ with respect to $f_1, ..., f_{mn}$.

Corollary 3.7.7. *A finite algebra $\mathbb{A}$ generates a congruence distributive variety iff for all $a \in \mathbb{A}$, the singleton $\{a\}$ Jónsson absorbs $\mathbb{A}$.*

Proof. A Jónsson absorbing chain which witnesses $\{a\} \lhd_J \mathbb{A}$ for all $a \in \mathbb{A}$ is the same as a sequence of terms $d_1, ..., d_m$ which satisfy the system of identities

$$\begin{aligned} d_1(x,x,y) &\approx x, \\ d_i(x,y,x) &\approx x \text{ for all } i, \\ d_i(x,y,y) &\approx d_{i+1}(x,x,y) \text{ for all } i, \\ d_m(x,y,y) &\approx y, \end{aligned}$$

that is, $d_1, ..., d_m$ are a sequence of directed Jónsson terms. By Theorem A.4.8, a variety is congruence distributive iff it has directed Jónsson terms. □

Example 3.7.1. If $\mathbb{A} = (A, s)$ is a 2-semilattice, then $\mathbb{B} \lhd_J \mathbb{A}$ iff $s(\mathbb{A}, \mathbb{B}) = s(\mathbb{B}, \mathbb{A}) \subseteq \mathbb{B}$, that is, iff $\mathbb{B} \lhd_{str} \mathbb{A}$.

Example 3.7.2. If $\mathbb{A}$ is abelian, then $\mathbb{A}$ has no Jónsson absorbing singleton subalgebras. To see this, note that if $\mathbb{A}$ is abelian, then for any Jónsson chain $d_1, ..., d_n$ witnessing $\{b\} \lhd_J \mathbb{A}$ and any $a \neq b$, we have $d_1(b, b, a) = b$, and then by induction we have

$$d_i(b, \boxed{b}, a) = b = d_i(b, \boxed{b}, b) \implies d_i(b, \boxed{a}, a) = d_i(b, \boxed{a}, b) = b \implies d_{i+1}(b, b, a) = d_i(b, a, a) = b,$$

so $a = d_n(b, a, a) = b$, a contradiction.

In particular, no affine algebra $\mathbb{A}$ has any proper Jónsson absorbing subalgebra $\mathbb{B}$, because we can apply the above argument to the quotient $\mathbb{A}/\theta_{\mathbb{B}}$, where $\theta_{\mathbb{B}}$ is the congruence of $\mathbb{A}$ which has $\mathbb{B}$ as a congruence class.

Example 3.7.3. Suppose $\mathbb{B}$ is an ideal of a majority algebra $\mathbb{A} = (A, m)$. Then $\mathbb{B} \lhd_J \mathbb{A}$ with respect to the Jónsson absorption chain $d_1(x, y, z) = m(x, y, z)$ (of length 1):

$$\begin{aligned} m(x, x, y) &\approx x, \\ m(x, y, y) &\approx y, \\ m(\mathbb{B}, \mathbb{A}, \mathbb{B}) &\subseteq \mathbb{B}. \end{aligned}$$

In fact, the converse holds: if $\mathbb{B} \lhd_J \mathbb{A}$, then there must be a majority term $m' \in \text{Clo}(m)$ such that $m'(\mathbb{B}, \mathbb{A}, \mathbb{B}) \subseteq \mathbb{B}$. This follows from the fact that every ternary term in a majority algebra is either a projection or another majority operation.

If $\mathbb{A}$ generates a locally finite variety, then by applying the construction of Proposition 3.7.6 iteratively to all the majority operations in $\text{Clo}(m)$, we can find a single majority term $\hat{m} \in \text{Clo}(m)$ such that for any $\mathbb{C} \leq \mathbb{B} \in HSP(\mathbb{A})$ we have

$$\mathbb{C} \lhd_J \mathbb{B} \iff \hat{m}(\mathbb{C}, \mathbb{B}, \mathbb{C}) \subseteq \mathbb{C}.$$

As we will see later, for finite majority algebras $\mathbb{B} \lhd_J \mathbb{A}$ implies that $\mathbb{B} \lhd \mathbb{A}$ - possibly with respect to a term of very high arity (for instance, in the case where $\mathbb{A}$ is the dual discriminator algebra from Example 1.6.5 and $|\mathbb{B}| = |\mathbb{A}| - 1$, the minimal arity of a term t which witnesses $\mathbb{B} \lhd \mathbb{A}$ is $|\mathbb{A}| + 1$). So ideals of finite majority algebras are actually the same thing as absorbing subalgebras!

Remark 3.7.1. If we define a concept called *ideal absorption* by $\mathbb{B} \lhd_I \mathbb{A}$ when there is a ternary term d such that $d(x, x, y) \approx x \approx d(y, x, x)$ and $d(\mathbb{B}, \mathbb{A}, \mathbb{B}) \subseteq \mathbb{B}$, then all of the results about ideals of majority algebras generalize. I don't know any applications of this idea outside the context of majority algebras.

Like ideals of majority algebras, absorbing subalgebras play nice with primitive positive formulas.

Proposition 3.7.8. *Suppose that a relation $\mathbb{R}$ is defined by a primitive positive formula Φ involving the relations $\mathbb{R}_1, ..., \mathbb{R}_k$. If we replace each $\mathbb{R}_i$ with an absorbing subalgebra $\mathbb{R}_i' \lhd \mathbb{R}_i$ to make a primitive positive formula Φ', then the relation $\mathbb{R}'$ which is defined by Φ' is an absorbing subalgebra of $\mathbb{R}$. The same is true with "absorbing" replaced by "Jónsson absorbing".*

Proof. Since only finitely many relations $\mathbb{R}_i$ show up in Φ, we can find a single absorbing term (or Jónsson chain) which witnesses all absorptions $\mathbb{R}_i' \lhd \mathbb{R}_i$ (or $\mathbb{R}_i' \lhd_J \mathbb{R}_i$) simultaneously. From here the proof is similar to the proof in the case of ideals of majority algebras. □

Now we will illustrate how Jónsson absorption is used, by proving a few connectivity results. Recall that every binary relation $\mathbb{R} \le \mathbb{A} \times \mathbb{A}$ can be visualized as a graph in two different ways: we can either think of $\mathbb{R}$ as a bipartite graph on the disjoint union $\mathbb{A} \sqcup \mathbb{A}$, or we can think of $\mathbb{R}$ as a directed graph on $\mathbb{A}$. The next result is perhaps the most crucial.

Theorem 3.7.9 (Absorbing directed paths [16]). *If $\mathbb{S}, \mathbb{R} \le \mathbb{A} \times \mathbb{A}$ are binary relations with $\mathbb{S} \lhd_J \mathbb{R}$, and $a, b \in \mathbb{A}$ satisfy*

- *$(a, a), (b, b) \in \mathbb{S}$, and*
- *$(a, b) \in \mathbb{R}$,*

then if we think of $\mathbb{S}$ as a directed graph on $\mathbb{A}$, there is a directed path from a to b in $\mathbb{S}$, that is, $(a, b) \in \mathbb{S}^{\circ n}$ for some n.

Proof. Suppose $\mathbb{S} \lhd_J \mathbb{R}$ with respect to the Jónsson chain $d_1, ..., d_n$. Then for each i we have

$$\begin{bmatrix} d_i(a,a,b) \\ d_{i+1}(a,a,b) \end{bmatrix} = \begin{bmatrix} d_i(a,a,b) \\ d_i(a,b,b) \end{bmatrix} = d_i\left(\begin{bmatrix} a \\ a \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ b \end{bmatrix}\right) \in d_i(\mathbb{S}, \mathbb{R}, \mathbb{S}) \subseteq \mathbb{S}.$$

Stringing these together, we get a directed path from $d_1(a, a, b) = a$ to $d_n(a, b, b) = b$ of length n, so in fact $(a, b) \in \mathbb{S}^{\circ n}$. □

Applying the above to $\mathbb{S}^{\circ m} \lhd_J \mathbb{R}^{\circ m}$ for a sufficiently large m, we get the following stronger-looking corollary.

Corollary 3.7.10. *If $\mathbb{S}, \mathbb{R} \le \mathbb{A} \times \mathbb{A}$ have $\mathbb{S} \lhd_J \mathbb{R}$, and $a, b \in \mathbb{A}$ satisfy*

- *each of a, b is contained in a directed cycle of the digraph $\mathbb{S}$, and*
- *there is a directed path from a to b in the digraph $\mathbb{R}$,*

then there is a directed path from a to b in the digraph $\mathbb{S}$.

For the sake of applying the previous result, it is useful to keep in mind the following basic fact about finite directed graphs.

Proposition 3.7.11. *If (A, R) is a finite directed graph such that each vertex of A has in-degree at least 1 (in other words, such that $\pi_2(R) = A$), then for every vertex $a \in A$ there is some $a' \in A$ and some n such that a' is contained in a directed cycle of length n and such that there is a directed path from a' to a of length n (that is, $(a', a'), (a', a) \in R^{\circ n}$).*

Proof. Define a function $\varphi : A \to A$ such that for each $a \in A$ we have $(\varphi(a), a) \in R$. Then there is some n such that $\varphi^{\circ 2n} = \varphi^{\circ n}$ by the finiteness of A: in fact, we may take $n = \text{lcm}\{1, ..., |A|\}$. □

In the next result, we think of binary relations as bipartite graphs. Recall that the *linked components* of a binary relation $\mathbb{R} \le \mathbb{A} \times \mathbb{B}$ are the connected components of $\mathbb{R}$ considered as a bipartite graph on $\mathbb{A} \sqcup \mathbb{B}$, and that the linked components of size greater than 1 are the same as the congruence classes of the linking congruence $\ker \pi_1 \vee \ker \pi_2$ on $\mathbb{R}$.

Theorem 3.7.12 (Absorbing linked components [16]). *If $\mathbb{S}, \mathbb{R} \le \mathbb{A} \times \mathbb{B}$ are binary relations with $\mathbb{S} \lhd_J \mathbb{R}$, and $a, b \in \pi_1(\mathbb{S})$ are in the same linked component of $\mathbb{R}$, then a, b are in the same linked component of $\mathbb{S}$.*

Proof. If a, b are linked in $\mathbb{R}$, then there is some m such that $(a, b) \in (\mathbb{R} \circ \mathbb{R}^-)^{\circ m}$. Since $(\mathbb{S} \circ \mathbb{S}^-)^{\circ m} \lhd_J (\mathbb{R} \circ \mathbb{R}^-)^{\circ m}$ and $(a, a), (b, b) \in \mathbb{S} \circ \mathbb{S}^-$ by $a, b \in \pi_1(\mathbb{S})$, we can apply Theorem 3.7.9 to see that there is some n such that $(a, b) \in (\mathbb{S} \circ \mathbb{S}^-)^{\circ mn}$. Thus a, b are in the same linked component of $\mathbb{S}$. □

The next result is an analogue of Theorem 3.3.9 and Theorem 3.6.10 for Jónsson absorption.

Theorem 3.7.13 (Loop Lemma, finite absorbing case [14])**.** *If $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{A}$ is subdirect, $\mathbb{A}$ is finite, and $\mathbb{R}$ Jónsson absorbs the diagonal $\Delta_{\mathbb{A}}$, then $\mathbb{R} \cap \Delta_{\mathbb{A}} \ne \emptyset$.*

Proof. We may assume without loss of generality that $\mathbb{A}$ is idempotent. As long as $|\mathbb{A}| > 1$, we will try to find a proper subalgebra $\mathbb{B} \le \mathbb{A}$ with $\mathbb{R} \cap (\mathbb{B} \times \mathbb{B})$ subdirect. Then $\mathbb{R} \cap (\mathbb{B} \times \mathbb{B})$ will Jónsson absorb $\Delta_{\mathbb{B}}$, and we can show by induction that $\mathbb{R} \cap \Delta_{\mathbb{B}} \ne \emptyset$.

Let b be any element of $\mathbb{A}$, and define a sequence of subalgebras $\mathbb{B}_i$ by $\mathbb{B}_0 = \{b\}$, $\mathbb{B}_{i+1} = \mathbb{B}_i + \mathbb{R}$, i.e. $\mathbb{B}_{i+1} = \pi_2(\mathbb{R} \cap (\mathbb{B}_i \times \mathbb{A}))$. If there is any i such that $\mathbb{B}_i \ne \mathbb{A}$ but $\mathbb{B}_{i+1} = \mathbb{A}$, then for every $\mathbb{C} \le \mathbb{A}$ we have $(\mathbb{C} + \mathbb{R}^-) \cap \mathbb{B}_i \ne \emptyset$, so by the finiteness of $\mathbb{B}_i$ we may take

$$\mathbb{B} = \bigcap_{k \ge 0} \mathbb{B}_i + (\mathbb{R}^-)^{\circ k} = \{a \mid \exists a_0, a_1, ... \in \mathbb{B}_i \text{ s.t. } a_0 = a \text{ and } \forall j\ (a_j, a_{j+1}) \in \mathbb{R}\}.$$

Otherwise, each $\mathbb{B}_i \ne \mathbb{A}$, and by the finiteness of $\mathbb{A}$ there must be some m, n such that $\mathbb{B}_m = \mathbb{B}_{m+n}$. We will show that in this case we have $\mathbb{B}_{m+i} = \mathbb{B}_m$ for each i, so we may take $\mathbb{B} = \mathbb{B}_m$.

Consider any directed cycle $a_0, ..., a_{kn} = a_0$ of $\mathbb{R}$ (considered as a digraph) with $a_0 \in \mathbb{B}_m$. We will show that each $a_i \in \mathbb{B}_m$. Note that $(a_0, a_0), (a_i, a_i) \in \mathbb{R}^{\circ kn}$, that $\mathbb{R}^{\circ kn}$ Jónsson absorbs $(\mathbb{R} \cup \Delta_{\mathbb{A}})^{\circ kn}$, and that $(a_0, a_i) \in \mathbb{R}^{\circ i} \subseteq (\mathbb{R} \cup \Delta_{\mathbb{A}})^{\circ kn}$. Thus by Theorem 3.7.9 there is some l such that $(a_0, a_i) \in \mathbb{R}^{\circ ln}$, and since $\mathbb{B}_m + \mathbb{R}^{\circ ln} = \mathbb{B}_{m+ln} = \mathbb{B}_m$, we see that $a_i \in \mathbb{B}_m$.

Since $\mathbb{B}_{m+i} + \mathbb{R}^{\circ n} = \mathbb{B}_{m+i}$ and $\mathbb{B}_{m+i}$ is finite, for each element a of $\mathbb{B}_{m+i}$ there is an a_i contained in a directed cycle of $\mathbb{R}^{\circ n}$ and a directed path of $\mathbb{R}^{\circ n}$ from a_i to a, so in fact we have $a \in \mathbb{B}_m$ as well, and we see that $\mathbb{B}_{m+i} \subseteq \mathbb{B}_m$. Similarly we have $\mathbb{B}_m \subseteq \mathbb{B}_{m+i}$, so $\mathbb{B}_m = \mathbb{B}_{m+i}$. □

Corollary 3.7.14. *If $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{A}$ is subdirect, $\mathbb{A}$ is finite and has no proper absorbing subalgebra, and $\mathbb{R}$ absorbs the diagonal $\Delta_{\mathbb{A}}$, then $\Delta_{\mathbb{A}} \subseteq \mathbb{R}$.*

Proof. Since $\mathbb{R} \cap \Delta_{\mathbb{A}} \ne \emptyset$ is an absorbing subalgebra of $\Delta_{\mathbb{A}}$ and $\Delta_{\mathbb{A}} \cong \mathbb{A}$ has no proper absorbing subalgebra, we must have $\mathbb{R} \cap \Delta_{\mathbb{A}} = \Delta_{\mathbb{A}}$. □

Definition 3.7.15. We say that $\mathbb{B}$ is a *minimal absorbing subalgebra* of $\mathbb{A}$, written $\mathbb{B} \lll \mathbb{A}$, if $\mathbb{B} \lhd \mathbb{A}$ and $\mathbb{B}$ has no proper absorbing subalgebra.

Proposition 3.7.16. *Every finite absorbing subalgebra of $\mathbb{A}$ contains a minimal absorbing subalgebra of $\mathbb{A}$, and any pair of distinct minimal absorbing subalgebras of $\mathbb{A}$ are disjoint.*

Proof. This follows from the fact that $\mathbb{C} \lhd \mathbb{B} \lhd \mathbb{A}$ implies $\mathbb{C} \lhd \mathbb{A}$, and the fact that the intersection of any pair of absorbing subalgebras is an absorbing subalgebra. □

Theorem 3.7.17. *Suppose that $\mathbf{X}$ is an arc-consistent instance of a CSP, and suppose that for each variable domain $\mathbb{A}_v$ there is a minimal absorbing subalgebra $\mathbb{A}'_v \lll \mathbb{A}_v$ such that the reduced instance $\mathbf{X}'$ with variable domains replaced by $\mathbb{A}'_v$ and relations $\mathbb{R} \le_{sd} \mathbb{A}_{v_1} \times \cdots \times \mathbb{A}_{v_n}$ replaced by $\mathbb{R}' = \mathbb{R} \cap (\mathbb{A}'_{v_1} \times \cdots \times \mathbb{A}'_{v_n})$ is arc-consistent.*

Then for any path p in $\mathbf{X}$ from a variable v to itself such that $\mathbb{P}_p \supseteq \Delta_{\mathbb{A}_v}$, the corresponding path p' of $\mathbf{X}'$ has $\mathbb{P}_{p'} \supseteq \Delta_{\mathbb{A}'_v}$. In particular, if $\mathbf{X}$ is cycle-consistent then so is $\mathbf{X}'$.

Proof. Note that $\mathbb{P}_{p'}$ is an absorbing subalgebra of $\mathbb{P}_p$, so $\mathbb{P}_{p'}$ absorbs $\Delta_{\mathbb{A}'_v}$. Since $\mathbb{P}_{p'}$ is subdirect in $\mathbb{A}'_v \times \mathbb{A}'_v$ by the arc-consistency of $\mathbf{X}'$ and $\mathbb{A}'_v$ has no proper absorbing subalgebra, we may apply Corollary 3.7.14 to see that $\mathbb{P}_{p'} \supseteq \Delta_{\mathbb{A}'_v}$. □

Later we will show that any cycle-consistent instance $\mathbf{X}$ has an arc-consistent reduction $\mathbf{X}'$ where all variable domains are replaced by minimal absorbing subalgebras, which will set us up to apply Theorem 3.7.17. The argument strategy will be fairly generic, not using any specific properties of absorbing subalgebras other than Theorem 3.7.9 and the fact that absorption is compatible with primitive positive formulas. Additionally, we will be able to weaken cycle-consistency to a property known as pq-consistency, which says that for any pair of paths p, q from a variable v to itself, there is some $j \ge 0$ such that $\mathbb{P}_{j(p+q)+p} \supseteq \Delta_{\mathbb{A}_v}$.

3.7.1 Local criterion for Jónsson absorption

Since a finite algebra $\mathbb{A}$ has bounded strict width iff every singleton is an absorbing subalgebra of $\mathbb{A}$, we'd like to have a way to test whether a given subalgebra $\mathbb{B} \le \mathbb{A}$ is an absorbing subalgebra. Since the arity of a potential absorbing term is unbounded, we'll start with the easier problem of testing whether $\mathbb{B}$ is a *Jónsson* absorbing subalgebra, since in this case there is an obvious algorithm which will at least eventually halt: list out every possible ternary term of $\mathbb{A}$ by brute force, and make a digraph of possible Jónsson chains.

The idea behind finding a better way to test whether $\mathbb{B} \lhd_J \mathbb{A}$ is to try to find a converse to the fundamental digraph connectivity result characterizing Jónsson absorption (Theorem 3.7.9). In order to formulate the converse, we need to consider generic pairs of digraphs $\mathbb{S} \le \mathbb{R} \le \mathbb{C} \times \mathbb{C}$ such that

$$\mathbb{B} \lhd_J \mathbb{A} \implies \mathbb{S} \lhd_J \mathbb{R}.$$

One natural way to do this is to write $\mathbb{R}$ as the projection to the last two coordinates of a ternary relation $\mathbb{X} \le \mathbb{A} \times \mathbb{C} \times \mathbb{C}$, and to take $\mathbb{S}$ to be the corresponding projection of $\mathbb{X} \cap (\mathbb{B} \times \mathbb{C} \times \mathbb{C})$.

Definition 3.7.18. For $\mathbb{B} \le \mathbb{A}$ and $\mathbb{C} \in \mathcal{V}(\mathbb{A})$ all idempotent, we say that $\mathbb{A}, \mathbb{B}, \mathbb{C}$ satisfy the condition $J(\mathbb{A}, \mathbb{B}; \mathbb{C})$ if for every $a \in \mathbb{A}$, $b, b' \in \mathbb{B}$, and $c, d \in \mathbb{C}$, if we set

$$\mathbb{S} = \pi_{23}\left(\mathrm{Sg}_{\mathbb{A}\times\mathbb{C}\times\mathbb{C}}\left\{\begin{bmatrix} b \\ c \\ c \end{bmatrix}, \begin{bmatrix} a \\ c \\ d \end{bmatrix}, \begin{bmatrix} b' \\ d \\ d \end{bmatrix}\right\} \cap \begin{bmatrix} \mathbb{B} \\ \mathbb{C} \\ \mathbb{C} \end{bmatrix}\right),$$

then there is some n such that $(c, d) \in \mathbb{S}^{\circ n}$.

We will show that the condition $J(\mathbb{A}, \mathbb{B}; \mathbb{A})$ is equivalent to $\mathbb{B} \lhd_J \mathbb{A}$, following the strategy of [16]. Note that Theorem 3.7.9 proves one direction of the equivalence, so we just need to prove that $J(\mathbb{A}, \mathbb{B}; \mathbb{A}) \implies \mathbb{B} \lhd_J \mathbb{A}$. The strategy will be to use induction to show that $J(\mathbb{A}^m, \mathbb{B}^m; \mathbb{A}^n)$ holds for all m, n, and then to take $m = |\mathbb{A}||\mathbb{B}|^2, n = |\mathbb{A}|^2$ to show that a certain directed path exists between binary terms in the free algebra on two generators, which will correspond to a Jónsson absorption chain. Before diving into the details, we will outline how this criterion could be used to test whether $\mathbb{B} \lhd_J \mathbb{A}$.

Note that if $\mathbb{A}$ is given in terms of tables for its basic operations, then the condition $J(\mathbb{A}, \mathbb{B}; \mathbb{A})$ can be tested in time polynomial in $|\mathbb{A}|, |\mathbb{B}|$ (with the degree of the polynomial depending on the arities of the basic operations), since the total number of tuples a, b, b', c, d is $|\mathbb{A}|^3|\mathbb{B}|^2$, computing $\mathbb{S}$

requires us to compute a ternary relation of size at most $|\mathbb{A}|^3$, and we only need to check whether $(c, d) \in \mathbb{S}^{\circ n}$ for $n \le |\mathbb{A}|$.

If $\mathbb{A}$ is instead given in terms of a list of basic relations, then testing the condition $J(\mathbb{A}, \mathbb{B}; \mathbb{A})$ can be reduced to solving polynomially many polynomially large constraint satisfaction problems over the domain $\mathbb{A}$ - so in particular if $\mathrm{CSP}(\mathbb{A})$ can be solved in polynomial time, then we can test $J(\mathbb{A}, \mathbb{B}; \mathbb{A})$ in polynomial time. To see this, note that in order to test whether a given edge (e, f) is an element of $\mathbb{S}$, we just need to test whether $\mathbb{A}$ has a ternary polymorphism f such that

$$\begin{aligned} f(b, a, b') &\in \mathbb{B}, \\ f(c, c, d) &= e, \\ f(c, d, d) &= f, \end{aligned}$$

and the set of ternary polymorphisms $f \in \mathcal{F}_{\mathbb{A}}(x, y, z) \le \mathbb{A}^{\mathbb{A}^3}$ can be described by a primitive positive formula involving only $|\mathbb{A}|^3$ variables.

Lemma 3.7.19. *If $J(\mathbb{A}_1, \mathbb{B}_1; \mathbb{C})$ and $J(\mathbb{A}_2, \mathbb{B}_2; \mathbb{C})$ both hold, then so does $J(\mathbb{A}_1 \times \mathbb{A}_2, \mathbb{B}_1 \times \mathbb{B}_2; \mathbb{C})$.*

Proof. Suppose $a = (a_1, a_2) \in \mathbb{A}_1 \times \mathbb{A}_2$ and $b = (b_1, b_2), b' = (b_1', b_2') \in \mathbb{B}_1 \times \mathbb{B}_2$, $c, d \in \mathbb{C}$. Define $\mathbb{S}, \mathbb{R} \le \mathbb{C} \times \mathbb{C}$ as usual, and define an intermediate digraph $\mathbb{S}_1$, where instead of restricting to $\mathbb{B}_1 \times \mathbb{B}_2$, we restrict to $\mathbb{B}_1 \times \mathbb{A}_2$ instead - so for the purposes of computing $\mathbb{S}_1$, we can ignore the $\mathbb{A}_2$ components. Then by $J(\mathbb{A}_1, \mathbb{B}_1; \mathbb{C})$, from $(c, d) \in \mathbb{R}$ we see that there is a directed path from c to d in $\mathbb{S}_1$.

To finish, we just need to check that for each $(e, f) \in \mathbb{S}_1$, there is a directed path from e to f in $\mathbb{S}$. Note that $(e, f) \in \mathbb{S}_1$ means that there are some $b_1'' \in \mathbb{B}_1, a_2'' \in \mathbb{A}_2$ such that

$$\begin{bmatrix} (b_1'', a_2'') \\ e \\ f \end{bmatrix} \in \mathrm{Sg}\left\{ \begin{bmatrix} (b_1, b_2) \\ c \\ c \end{bmatrix}, \begin{bmatrix} (a_1, a_2) \\ c \\ d \end{bmatrix}, \begin{bmatrix} (b_1', b_2') \\ d \\ d \end{bmatrix} \right\}.$$

Then from $e, f \in \mathrm{Sg}\{c, d\}$, there are some $(b_1''', b_2''') \in \mathbb{B}_1 \times \mathbb{B}_2$ with

$$\begin{bmatrix} (b_1''', b_2''') \\ e \\ e \end{bmatrix} \in \mathrm{Sg}\left\{ \begin{bmatrix} (b_1, b_2) \\ c \\ c \end{bmatrix}, \begin{bmatrix} (b_1', b_2') \\ d \\ d \end{bmatrix} \right\},$$

and similarly for (f, f), so we just need to check that

$$\pi_{23}\left(\mathrm{Sg}\left\{ \begin{bmatrix} (b_1''', b_2''') \\ e \\ e \end{bmatrix}, \begin{bmatrix} (b_1'', a_2'') \\ e \\ f \end{bmatrix}, \begin{bmatrix} (b_1'''', b_2'''') \\ f \\ f \end{bmatrix} \right\} \cap \begin{bmatrix} \mathbb{B}_1 \times \mathbb{B}_2 \\ \mathbb{C} \\ \mathbb{C} \end{bmatrix} \right)$$

contains a directed path from e to f. But now we can ignore the $\mathbb{B}_1$ component, so this follows from $J(\mathbb{A}_2, \mathbb{B}_2; \mathbb{C})$. □

Lemma 3.7.20. *If $J(\mathbb{A}, \mathbb{B}; \mathbb{C}_1)$ and $J(\mathbb{A}, \mathbb{B}; \mathbb{C}_2)$ both hold and $\mathbb{C}_1, \mathbb{C}_2$ are finite and idempotent, then $J(\mathbb{A}, \mathbb{B}; \mathbb{C}_1 \times \mathbb{C}_2)$ holds as well.*

Proof. Suppose not. Choose $c = (c_1, c_2), d = (d_1, d_2) \in \mathbb{C}_1 \times \mathbb{C}_2$ such that $\mathrm{Sg}\{c, d\}$ is minimal among all pairs such that there exist $a \in \mathbb{A}, b, b' \in \mathbb{B}$ so that the associated digraph $\mathbb{S}$ has no directed path from c to d.

Ignoring the $\mathbb{C}_2$ components, we can apply $J(\mathbb{A}, \mathbb{B}; \mathbb{C}_1)$ to find a sequence of edges $(e^i, f^{i+1}) \in \mathbb{S}$ such that $f_1^i = e_1^i$ for each $i \le n$, $c = f^1$, and $e^n = d$. Since we assumed that there is no directed path from c to $e^n = d$, we can consider the first i such that there is no directed path from c to e^i.

Since $e^i \in \mathrm{Sg}\{c, d\}$, we have

$$\begin{bmatrix} c \\ e^i \end{bmatrix} \in \mathrm{Sg}\left\{ \begin{bmatrix} c \\ c \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} d \\ d \end{bmatrix} \right\} = \mathbb{R},$$

and since there is no directed path from c to e^i in $\mathbb{S}$, we see that we must have $\mathrm{Sg}\{c, e^i\} = \mathrm{Sg}\{c, d\}$ by our minimality assumption, so in particular we have $f^i \in \mathrm{Sg}\{c, e^i\}$. Thus we have

$$\begin{bmatrix} f^i \\ e^i \end{bmatrix} \in \mathrm{Sg}\left\{ \begin{bmatrix} c \\ e^i \end{bmatrix}, \begin{bmatrix} e^i \\ e^i \end{bmatrix} \right\} \subseteq \mathbb{R}.$$

By the choice of i there is a path from c to f^i in $\mathbb{S}$ (passing through e^{i-1} if $i > 1$). To get a contradiction, we just need to show that there is a directed path from f^i to e^i in $\mathbb{S}$. Since $(e^i, e^i), (f^i, f^i) \in \mathbb{S}$, there are $a' \in \mathbb{A}, b'', b''' \in \mathbb{B}$ such that

$$\pi_{23}\left(\mathrm{Sg}\left\{ \begin{bmatrix} b'' \\ f^i \\ f^i \end{bmatrix}, \begin{bmatrix} a' \\ f^i \\ e^i \end{bmatrix}, \begin{bmatrix} b''' \\ e^i \\ e^i \end{bmatrix} \right\} \cap \begin{bmatrix} \mathbb{B} \\ \mathbb{C}_1 \times \mathbb{C}_2 \\ \mathbb{C}_1 \times \mathbb{C}_2 \end{bmatrix} \right) \subseteq \mathbb{S}.$$

Since $f_1^i = e_1^i$, we can ignore the $\mathbb{C}_1$ components in the above, so by $J(\mathbb{A}, \mathbb{B}; \mathbb{C}_2)$ there is a directed path from f^i to e^i in $\mathbb{S}$. □

Theorem 3.7.21 (Local criterion for Jónsson absorption [16]). *If $\mathbb{B} \le \mathbb{A}$ are finite and idempotent, then $\mathbb{B} \lhd_J \mathbb{A}$ if and only if $J(\mathbb{A}, \mathbb{B}; \mathbb{A})$ holds.*

Proof. By the previous two lemmas, $J(\mathbb{A}^m, \mathbb{B}^m; \mathbb{A}^n)$ holds for $m = \mathbb{B} \times \mathbb{A} \times \mathbb{B}$ and $n = \mathbb{A} \times \mathbb{A}$. There is a natural map $\Phi : \mathcal{F}_{\mathbb{A}}(x, y, z) \to \mathbb{A}^{\mathbb{B} \times \mathbb{A} \times \mathbb{B}}$ and a pair of natural maps $\Psi_1, \Psi_2 : \mathcal{F}_{\mathbb{A}}(x, y, z) \to \mathbb{A}^{\mathbb{A} \times \mathbb{A}}$: the first takes f to the restriction $f|_{\mathbb{B} \times \mathbb{A} \times \mathbb{B}}$, the other two take f to the functions $f(x, x, y)$, $f(x, y, y)$.

Then we can apply $J(\mathbb{A}^m, \mathbb{B}^m; \mathbb{A}^n)$ with $a = \Phi(\pi_2)$, $b = \Phi(\pi_2), b' = \Phi(\pi_3)$, $c = \Psi_i(\pi_1) = \Psi_1(\pi_2), d = \Psi_i(\pi_3) = \Psi_2(\pi_2)$. If we set

$$\mathbb{S} = \pi_{23}\left(\mathrm{Sg}\left\{ \begin{bmatrix} x|_{x,z \in \mathbb{B}} \\ x \\ x \end{bmatrix}, \begin{bmatrix} y|_{x,z \in \mathbb{B}} \\ x \\ y \end{bmatrix}, \begin{bmatrix} z|_{x,z \in \mathbb{B}} \\ y \\ y \end{bmatrix} \right\} \cap \begin{bmatrix} \mathbb{B}^m \\ \mathbb{A}^n \\ \mathbb{A}^n \end{bmatrix} \right),$$

then the inner ternary subalgebra is exactly $\mathrm{Im}(\Phi, \Psi_1, \Psi_2)$, so $\mathbb{S}$ is exactly the digraph of pairs of binary terms $g(x, y), h(x, y)$ such that there is some ternary term $f(x, y, z)$ satisfying

$$\begin{aligned} f(\mathbb{B}, \mathbb{A}, \mathbb{B}) &\subseteq \mathbb{B}, \\ f(x, x, y) &\approx g(x, y), \\ f(x, y, y) &\approx h(x, y). \end{aligned}$$

The condition $J(\mathbb{A}^m, \mathbb{B}^m; \mathbb{A}^n)$ says that this digraph contains a path from the term x to the term y, which is the same as a Jónsson absorption chain for $\mathbb{B} \lhd_J \mathbb{A}$. □

Note that the same argument shows that it is enough to check $J(\mathbb{A}, \mathbb{B}; \mathbb{C}_i)$ for any collection of algebras $\mathbb{C}_1, ..., \mathbb{C}_n$ generating a variety $\mathcal{V}$ such that $\mathcal{F}_{\mathbb{A}}(x, y) = \mathcal{F}_{\mathcal{V}}(x, y)$. In cases where $\mathcal{F}_{\mathbb{A}}(x, y)$ is small the criterion becomes especially nice.

Corollary 3.7.22. *If $\mathbb{A} = (A, m)$ is a majority algebra, then $\mathbb{B} \lhd_J \mathbb{A}$ iff there do* not *exist $a \in \mathbb{A}$ and $b, c \in \mathbb{B}$ such that*

- *a, b, c are distinct,*
- $\mathrm{Sg}_{\mathbb{A}}\{a, b, c\} \cap \mathbb{B} = \{b, c\}$,
- *the partitions $\{\{b\}, \mathrm{Sg}\{a, b, c\} \setminus \{b\}\}$ and $\{\{c\}, \mathrm{Sg}\{a, b, c\} \setminus \{c\}\}$ of $\mathrm{Sg}\{a, b, c\}$ correspond to congruences θ_b, θ_c on $\mathrm{Sg}\{a, b, c\}$.*

The third bullet point can also be stated in the equivalent form: $\mathrm{Sg}\{a, b, c\}/(\theta_b \wedge \theta_c)$ is isomorphic to the three element median algebra, with median element $a/(\theta_b \wedge \theta_c) = \mathrm{Sg}\{a, b, c\} \setminus \{b, c\}$.

3.8 Absorption and $\mathbb{B}$-essential relations

In this section we'll give a relational description of absorption, as well as a first simplification via Ramsey theory. The relational description is a generalization of the way relations over near-unanimity algebras decompose.

Definition 3.8.1. Suppose $\mathbb{B} \le \mathbb{A}$. We say that a relation $\mathbb{R} \le \mathbb{A}^m$ is $\mathbb{B}$-*essential* if for every $1 \le i \le n$ we have

$$\mathbb{R} \cap (\mathbb{B}^{i-1} \times \mathbb{A} \times \mathbb{B}^{n-i}) \neq \emptyset,$$

but

$$\mathbb{R} \cap \mathbb{B}^n = \emptyset.$$

More generally, if $\mathbb{B}_i \le \mathbb{A}_i$ for all i, then we say that $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_m$ is $(\mathbb{B}_1, ..., \mathbb{B}_m)$-*essential* if

$$\mathbb{R} \cap (\mathbb{B}_1 \times \cdots \times \mathbb{B}_{i-1} \times \mathbb{A}_i \times \mathbb{B}_{i+1} \times \cdots \times \mathbb{B}_m) \neq \emptyset$$

for each i, but

$$\mathbb{R} \cap (\mathbb{B}_1 \times \cdots \times \mathbb{B}_m) = \emptyset.$$

Proposition 3.8.2. *If $\mathbb{R} \le \mathbb{A}^m$ is $\mathbb{B}$-essential, then so is*

$$\pi_{[m-1]}(\mathbb{R} \cap (\mathbb{A}^{m-1} \times \mathbb{B})).$$

In particular, if there is a $\mathbb{B}$-essential relation of some arity, then there are $\mathbb{B}$-essential relations of all smaller arities.

Proposition 3.8.3. *If $\mathbb{B}$ absorbs $\mathbb{A}$ with respect to a term t of arity m, then there are no $\mathbb{B}$-essential relations $\mathbb{R} \le \mathbb{A}^m$ of arity m.*

Proof. Suppose for contradiction that $\mathbb{R} \le \mathbb{A}^m$ is $\mathbb{B}$-essential, and let $b_{ij} \in \mathbb{B}, a_i \in \mathbb{A}$ be such that

$$\begin{bmatrix} a_1 \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}, \begin{bmatrix} b_{12} \\ a_2 \\ \vdots \\ b_{m2} \end{bmatrix}, ..., \begin{bmatrix} b_{1m} \\ b_{2m} \\ \vdots \\ a_m \end{bmatrix} \in \mathbb{R}.$$

Then if we apply t, we have

$$t\left(\begin{bmatrix} a_1 & b_{12} & \cdots & b_{1m} \\ b_{21} & a_2 & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & a_m \end{bmatrix}\right) \in \mathbb{R} \cap \mathbb{B}^m$$

since $\mathbb{B}$ absorbs $\mathbb{A}$ with respect to t, which is a contradiction. □

Corollary 3.8.4. *If $\mathbb{B}$ absorbs $\mathbb{A}$ with respect to a term t of arity m, then for any $n \ge m$ and any relation $\mathbb{R} \le \mathbb{A}^n$ such that*

$$\pi_I(\mathbb{R}) \cap \mathbb{B}^{m-1} \ne \emptyset$$

for all $I \subseteq [n]$ with $|I| = m - 1$, we have

$$\mathbb{R} \cap \mathbb{B}^n \ne \emptyset.$$

Proof. We prove this by induction on $n \ge m$. The base case $n = m$ follows from the fact that there are no $\mathbb{B}$-essential relations of arity m by Proposition 3.8.3. For the inductive step, note that by the inductive hypothesis we have

$$\pi_{[n]\setminus\{i\}}(\mathbb{R}) \cap \mathbb{B}^{n-1} \ne \emptyset$$

for all $i \in [n]$, and since $n \ge m$ we must have $\mathbb{R} \cap \mathbb{B}^n \ne \emptyset$, since there are no $\mathbb{B}$-essential relations of arity n either by Proposition 3.8.2. □

Our main result is the converse to Proposition 3.8.3.

Theorem 3.8.5 (Relational description of absorption [16])**.** *If $\mathbb{A}$ is finite and idempotent, then $\mathbb{B}$ absorbs $\mathbb{A}$ with respect to a term of arity m if and only if there are no $\mathbb{B}$-essential relations of arity m. In particular, we have $\mathbb{B} \lhd \mathbb{A}$ if and only if there is a bound on the arity of $\mathbb{B}$-essential relations.*

The strategy of the proof is to show that if there are no m-ary terms t which absorb $\mathbb{B}$, then the projection of the free algebra $\mathcal{F}_{\mathbb{A}}(x_1, ..., x_m) \le \mathbb{A}^{\mathbb{A}^m}$ onto the coordinates where all but one input x_i are in $\mathbb{B}$ looks like a $\mathbb{B}$-essential relation. The arity of this projection will be much higher than m, but the set of coordinates can be naturally grouped into m parts.

Lemma 3.8.6. *If $n_1, ..., n_m \ge 1$ and $\mathbb{R} \le \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m}$ is $(\mathbb{B}^{n_1}, ..., \mathbb{B}^{n_m})$-essential, then there is a $\mathbb{B}$-essential relation $\mathbb{R}' \le \mathbb{A}^m$ of arity m. In fact, $\mathbb{R}'$ can be chosen to have the form*

$$\mathbb{R}' = \pi_I\Big(\mathbb{R} \cap \Big(\prod_i \mathbb{C}_i\Big)\Big)$$

for some $I \subseteq [n_1 + \cdots + n_m]$ with $|I| = m$ and for some choice of $\mathbb{C}_i \in \{\mathbb{A}, \mathbb{B}\}$ for each i.

Proof. We prove this by induction on $n = n_1 + \cdots + n_m$. If all $n_i = 1$, then $\mathbb{R}$ is an m-ary $\mathbb{B}$-essential relation already. Otherwise, we may assume $n_m > 1$ without loss of generality. First consider the relation

$$\mathbb{R}_1 = \pi_{[n-1]}(\mathbb{R} \cap (\mathbb{A}^{n-1} \times \mathbb{B})) \le \mathbb{A}^{n_1} \times \cdots \times \mathbb{A}^{n_m - 1}.$$

We have

$$\mathbb{R}_1 \cap (\mathbb{B}^{n_1} \times \cdots \times \mathbb{A}^{n_i} \times \cdots \times \mathbb{B}^{n_{m-1}} \times \mathbb{B}^{n_m - 1}) \ne \emptyset$$

for each $i \ne m$, and

$$\mathbb{R}_1 \cap \mathbb{B}^{n-1} = \emptyset,$$

so the only way for $\mathbb{R}_1$ to fail to be $(\mathbb{B}^{n_1}, ..., \mathbb{B}^{n_{m-1}}, \mathbb{B}^{n_m - 1})$-essential is if

$$\pi_{[n-n_m] \cup \{n\}}(\mathbb{R}) \cap \mathbb{B}^{n - n_m + 1} = \emptyset.$$

In this case, we see that

$$\mathbb{R}_2 = \pi_{[n-n_m] \cup \{n\}}(\mathbb{R})$$

is a $(\mathbb{B}^{n_1}, ..., \mathbb{B}^{n_{m-1}}, \mathbb{B})$-essential relation. □

Proof of Theorem 3.8.5. We just need to prove that if there is no m-ary $\mathbb{B}$-essential relation, then $\mathbb{B}$ absorbs $\mathbb{A}$ with respect to some m-ary term t. For each i, let X_i be the set of tuples $(x_1, ..., x_m) \in \mathbb{A}^m$ such that $x_j \in \mathbb{B}$ for $j \ne i$, and $x_i \in \mathbb{A} \setminus \mathbb{B}$. Consider the relation

$$\mathbb{R} = \pi_{X_1 \cup \cdots \cup X_m}(\mathcal{F}_{\mathbb{A}}(x_1, ..., x_m)) \le \mathbb{A}^{X_1} \times \cdots \times \mathbb{A}^{X_m}.$$

Since $\mathcal{F}_{\mathbb{A}}(x_1, ..., x_m)$ contains the projection functions $\pi_i : \mathbb{A}^m \to \mathbb{A}$, by the definition of the sets X_i we have

$$\mathbb{R} \cap (\mathbb{B}^{X_1} \times \cdots \times \mathbb{A}^{X_i} \times \cdots \times \mathbb{B}^{X_m}) \ne \emptyset$$

for all i. Since there is no $\mathbb{B}$-essential relation of arity m, we see that $\mathbb{R}$ can't be $(\mathbb{B}^{X_1}, ..., \mathbb{B}^{X_m})$-essential by Lemma 3.8.6, so we must have

$$\mathbb{R} \cap (\mathbb{B}^{X_1} \times \cdots \times \mathbb{B}^{X_m}) \ne \emptyset$$

as well. Then by the definition of $\mathbb{R}$, we see that there is a term $t \in \mathcal{F}_{\mathbb{A}}(x_1, ..., x_m)$ which absorbs $\mathbb{B}$. □

We can simplify this slightly as follows.

Corollary 3.8.7. *We have $\mathbb{B} \lhd \mathbb{A}$ with respect to an m-ary term t iff for all $b_{ij} \in \mathbb{B}, a_i \in \mathbb{A}$ we have*

$$\mathrm{Sg}_{\mathbb{A}^m}\left\{\begin{bmatrix} a_1 \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix}, \begin{bmatrix} b_{12} \\ a_2 \\ \vdots \\ b_{m2} \end{bmatrix}, ..., \begin{bmatrix} b_{1m} \\ b_{2m} \\ \vdots \\ a_m \end{bmatrix}\right\} \cap \mathbb{B}^m \ne \emptyset.$$

We leave the following generalization as an exercise to the reader.

Theorem 3.8.8. *If $\mathbb{A}_1, ..., \mathbb{A}_k$ are finite and idempotent, and $\mathbb{B}_i \le \mathbb{A}_i$ for each i are such that there is no $(\mathbb{B}_{i_1}, ..., \mathbb{B}_{i_m})$-essential relation $\mathbb{R} \le \mathbb{A}_{i_1} \times \cdots \times \mathbb{A}_{i_m}$ for any choice of $i_1, ..., i_m \in [k]$, then there is an m-ary term t such that each $\mathbb{B}_i$ absorbs $\mathbb{A}_i$ with respect to t.*

Corollary 3.8.9. *A finite idempotent algebra $\mathbb{A}$ has a near-unanimity term of arity m iff for each choice of $a_i, b_i \in \mathbb{A}$, we have*

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathrm{Sg}_{\mathbb{A}^m} \left\{ \begin{bmatrix} a_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \begin{bmatrix} b_1 \\ a_2 \\ \vdots \\ b_m \end{bmatrix}, ..., \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ a_m \end{bmatrix} \right\}.$$

Now we move our focus to finding a simpler characterization of $\mathbb{B} \lhd \mathbb{A}$, without restricting to terms of a particular arity. We'll use the notation $r_k(m)$ for the multicolored Ramsey number $R(m, ..., m)$ (with k copies of m), defined as the least number n such that any edge coloring of K_n with k colors must have a monochromatic copy of K_m.

Theorem 3.8.10. *If $\mathbb{A}$ is finite and idempotent, then $\mathbb{B} \lhd \mathbb{A}$ iff there do* not *exist $a \in \mathbb{A}$ and $b, c \in \mathbb{B}$ such that for every m, we have*

$$\mathrm{Sg}_{\mathbb{A}^m} \left\{ \begin{bmatrix} a & b & b & \cdots & b \\ c & a & b & \cdots & b \\ c & c & a & \cdots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c & c & c & \cdots & a \end{bmatrix} \right\} \cap \mathbb{B}^m = \emptyset.$$

Proof. We just need to show that if $\mathbb{B}$ does not absorb $\mathbb{A}$, then such a, b, c exist for every m. Let $n = |\mathbb{A}|(r_{|\mathbb{B}|^2}(m) - 1) + 1$. Then since $\mathbb{B}$ doesn't absorb $\mathbb{A}$ with respect to any term of arity n, there is some collection of $a_i \in \mathbb{A}, b_{ij} \in \mathbb{B}$ such that

$$\mathrm{Sg}_{\mathbb{A}^n} \left\{ \begin{bmatrix} a_1 & b_{12} & \cdots & b_{1n} \\ b_{21} & a_2 & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & a_n \end{bmatrix} \right\} \cap \mathbb{B}^n = \emptyset.$$

By the pigeonhole principle, there is some a which occurs at least $n' = r_{|\mathbb{B}|^2}(m)$ times among $a_1, ..., a_n$. Suppose without loss of generality that $a_1, ..., a_{n'}$ are all equal to a. If we restrict to the rows and columns with $a_i = a$, we find that

$$\mathrm{Sg}_{\mathbb{A}^{n'}} \left\{ \begin{bmatrix} a & b_{12} & \cdots & b_{1n'} \\ b_{21} & a & \cdots & b_{2n'} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n'1} & b_{n'2} & \cdots & a \end{bmatrix} \right\} \cap \mathbb{B}^{n'} = \emptyset.$$

Now we color the complete graph $K_{n'}$ with $|\mathbb{B}|^2$ colors, coloring the edge $\{i, j\}$ (with $i < j$) with the color corresponding to the ordered pair (b_{ij}, b_{ji}). Then by the definition of the Ramsey number $r_{|\mathbb{B}|^2}(m)$, there is a monochromatic copy of K_m, with all edges colored by the color corresponding to some pair $(b, c) \in \mathbb{B}^2$. By restricting to the rows and columns corresponding to the vertices of this monochromatic K_m, we see that

$$\mathrm{Sg}_{\mathbb{A}^m} \left\{ \begin{bmatrix} a & b & \cdots & b \\ c & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ c & c & \cdots & a \end{bmatrix} \right\} \cap \mathbb{B}^m = \emptyset. \qquad \square$$

Corollary 3.8.11. *If $\mathbb{A} = (A, m)$ is a finite majority algebra, then $\mathbb{B} \lhd \mathbb{A}$ iff there is a majority term $m' \in \mathrm{Clo}(m)$ such that $m'(\mathbb{B}, \mathbb{A}, \mathbb{B}) \subseteq \mathbb{B}$. Equivalently, we have $\mathbb{B} \lhd \mathbb{A} \iff \mathbb{B} \lhd_J \mathbb{A}$.*

More precisely, if $\mathbb{B} \lhd_J \mathbb{A}$, then $\mathbb{B}$ absorbs $\mathbb{A}$ with respect to a term of arity at most $\lceil e \cdot |\mathbb{B}|! \rceil$, where e is Euler's constant $\sum_{n\ge 0} \frac{1}{n!} \approx 2.718$.

Proof. The weaker bound $\lceil e|\mathbb{A}| \cdot |\mathbb{B}|^2! \rceil$ on the arity of an absorbing term follows from the estimate $r_k(3) \le \lceil e \cdot k! \rceil$ and the fact that

$$\begin{bmatrix} b \\ m'(c,a,b) \\ c \end{bmatrix} \in \mathrm{Sg}_{\mathbb{A}^3}\left\{ \begin{bmatrix} a & b & b \\ c & a & b \\ c & c & a \end{bmatrix} \right\}.$$

However, we don't need the exact setup above. It's enough to find n sufficiently large that for every $n \times n$ matrix

$$\begin{bmatrix} a_1 & b_{12} & \cdots & b_{1n} \\ b_{21} & a_2 & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & a_n \end{bmatrix}$$

with off-diagonal entries in $\mathbb{B}$, we can find i, j, k distinct such that $b_{ij} = b_{ik}$ and $b_{ki} = b_{kj}$. If we set $b = b_{ij} = b_{ik}$ and $c = b_{ki} = b_{kj}$, then this will give us a submatrix of the form

$$\begin{bmatrix} a_i & b & b \\ b_{ji} & a_j & b_{jk} \\ c & c & a_k \end{bmatrix},$$

and applying m' will give us an element of $\mathbb{B}^3$.

Taking $n = \lceil e \cdot |\mathbb{B}|! \rceil$ is good enough to find i, j, k with $b_{ij} = b_{ik}$ and $b_{ki} = b_{kj}$. The proof is a minor adaptation of the proof of the upper bound on $r_k(3)$, and is left as an exercise to the reader. □

Remark 3.8.1. It's intriguing that in the case of majority algebras, the bound on the arity of the absorbing operation only depends on the size of $|\mathbb{B}|$.

Problem 3.8.1. Define $m(k)$ to be the least number such that whenever $\mathbb{A}$ is a finite majority algebra, $\mathbb{B} \lhd_J \mathbb{A}$, and $|\mathbb{B}| \le k$, we can always find a term t of arity at most $m(k)$ such that $\mathbb{B}$ absorbs $\mathbb{A}$ with respect to t. How quickly does $m(k)$ grow?

The dual discriminator algebra from Example 1.6.5 shows that we always have $m(k) \ge k + 2$, while the previous result shows that $m(k) \le \lceil e \cdot k! \rceil$. For $k = 1, 2$ we have $m(1) = 3, m(2) = 4$. Could it be that we have $m(k) = k + 2$ for every k?

3.9 Finding an arc-consistent absorbing subinstance

In this section we'll go over Marcin Kozik's proof from [127] (which refined the argument from [126]) of the fact that every cycle-consistent instance has a cycle-consistent subinstance such that every domain is absorption-free. In fact, Kozik proves something stronger, involving a weaker consistency notion known as *pq*-consistency. The technique for the proof can be viewed as a generalization of the argument for the case of majority algebras, but it is much more difficult because we can't assume that all the relations involved are binary. The main idea of the proof was originally developed in [23], for the sake of proving a technical lemma about absorption generalizing Theorem 3.7.13 and Theorem 3.6.10, which was needed to show that near-unanimity CSPs can be solved in NL (nondeterministic logspace).

First we define the weaker consistency notion known as pq-consistency (Kozik names it jpq-consistency in [127]). The basic idea behind this definition is that it is a consistency check which (aside from assuming arc-consistency) only involves pairwise projections of constraints, only computes compositions of these binary relations along cycles, and is strong enough to rule out the existence of a cycle such that each binary relation along it is the graph of a permutation and the composition of all these permutations is not the identity permutation.

Definition 3.9.1. A CSP instance $\mathbf{X}$ with domains $\mathbb{A}_{v_i}$ corresponding to variables v_i is called *pq-consistent* if

- it is arc-consistent, i.e. each relation $\mathbb{R} \le \mathbb{A}_{v_1} \times \cdots \times \mathbb{A}_{v_k}$ imposed on the variables is subdirect, and
- for each variable v and each pair of cycles p, q of $\mathbf{X}$ which begin and end at v, there exists some $j \ge 0$ such that the binary relation $\mathbb{P}_{j(p+q)+p}$ corresponding to the path $j(p+q)+p$ (see Definition 3.5.1) contains the diagonal $\Delta_{\mathbb{A}_v}$, i.e. for each $a \in \mathbb{A}_v$, we have $a \in \{a\}+j(p+q)+p$ (see Definition 3.6.3).

The reader may find it interesting to check that in the proofs that we have already given for the fact that cycle-consistency solves ancestral CSPs and majority CSPs, we may substitute pq-consistency for cycle-consistency everywhere without significantly complicating the arguments. The reason for introducing the slightly more technical notion of pq-consistency is that the "standard semidefinite relaxation" of a CSP naturally produces a pq-consistent instance, but doesn't always produce a cycle-consistent instance - and the semidefinite relaxation is the tool used to "robustly" solve bounded width CSPs in [19].

The main step of the argument is the following technical result.

Theorem 3.9.2 (Kozik [127])**.** *If $\mathbf{X}$ is a pq-consistent instance and $\mathbf{Y}$ is an arc-consistent subinstance of $\mathbf{X}$ defined by restricting each domain and each relation of $\mathbf{X}$ to an absorbing subalgebra, and if any domain of $\mathbf{Y}$ has a proper absorbing subalgebra, then there is a proper arc-consistent subinstance $\mathbf{Z}$ of $\mathbf{Y}$ defined by restricting each domain to an absorbing subalgebra.*

Before diving into the proof of this result, we'll show how it can be used.

Theorem 3.9.3. *If $\mathbf{X}$ is a pq-consistent instance with domains $\mathbb{A}_v$, then there is a pq-consistent subinstance $\mathbf{X}'$ of $\mathbf{X}$ defined by restricting each domain $\mathbb{A}_v$ to a minimal absorbing subalgebra $\mathbb{A}_v'$. If $\mathbf{X}$ is cycle-consistent, then so is $\mathbf{X}'$.*

Proof. By repeatedly applying Theorem 3.9.2, we may find an arc-consistent subinstance $\mathbf{X}'$ of $\mathbf{X}$ such that each domain has no proper absorbing subalgebra. Then by Theorem 3.7.17, for every cycle r from v to v of $\mathbf{X}$ such that $\mathbb{P}_r \supseteq \Delta_{\mathbb{A}_v}$, if r' is the corresponding cycle in $\mathbf{X}'$, then we have $\mathbb{P}_{r'} \supseteq \Delta_{\mathbb{A}_v'}$. If $\mathbf{X}$ is pq-consistent, then for any cycles p, q from v to v there is some j such that $\mathbb{P}_{j(p+q)+p} \supseteq \Delta_{\mathbb{A}_v}$, so on taking $r = j(p+q)+p$ we see that the corresponding cycles p', q' have $\mathbb{P}_{j(p'+q')+p'} \supseteq \Delta_{\mathbb{A}_v'}$. □

The proof of Theorem 3.9.2 will only rely on three properties of absorption. Since there are several absorption-like concepts that have proven useful, and most of them satisfy these properties, we will consider an arbitrary "absorption concept" $\lhd_X$ which applies to certain pairs $\mathbb{B} \le \mathbb{A}$, and which satisfies the following three properties.

- **Compatibility with pp-formulas.** If $\mathbb{S}_i \lhd_X \mathbb{R}_i$ are relations, and if a relation $\mathbb{R}$ is defined by a pp-formula Φ involving the relations $\mathbb{R}_1, ..., \mathbb{R}_k$ (and possibly some other relations), then if we define a relation $\mathbb{S}$ by the pp-formula Φ' defined by replacing each $\mathbb{R}_i$ by $\mathbb{S}_i$ in Φ, we have $\mathbb{S} \lhd_X \mathbb{R}$.

- **Transitive closure.** If $\mathbb{C} \lhd_X \mathbb{B} \lhd_X \mathbb{A}$, then $\mathbb{C} \lhd_X \mathbb{A}$.

- **Connectivity transfers.** If $\mathbb{S} \lhd_X \mathbb{R}$ and $\mathbb{R} \le \mathbb{A} \times \mathbb{A}$, and if $a, b \in \mathbb{A}$ are such that $(a, a), (b, b) \in \mathbb{S}$ and $(a, b) \in \mathbb{R}$, then there is some k such that $(a, b) \in \mathbb{S}^{\circ k}$.

Note that by the local criterion for Jónsson absorption (Theorem 3.7.21), if $\lhd_X$ is compatible with pp-formulas, then the connectivity transfer property of $\lhd_X$ is equivalent to the implication $\mathbb{B} \lhd_X \mathbb{A} \implies \mathbb{B} \lhd_J \mathbb{A}$. Also, a trivial case of compatibility with pp-formulas implies that for all $\mathbb{A}$, we have $\mathbb{A} \lhd_X \mathbb{A}$.

Throughout most of the proof, we will be focusing on the arc-consistent instance $\mathbf{Y}$. Therefore, for each variable v of $\mathbf{Y}$, we let $\mathbb{A}_v$ be the corresponding domain in $\mathbf{Y}$, and let $\mathbb{A}_v^{\mathbf{X}}$ be the corresponding domain in the original pq-consistent instance $\mathbf{X}$, so we have $\mathbb{A}_v \lhd_X \mathbb{A}_v^{\mathbf{X}}$. Similarly, if $\mathbb{R}$ refers to a relation in $\mathbf{Y}$, then we let $\mathbb{R}^{\mathbf{X}}$ refer to the corresponding relation in $\mathbf{X}$, with $\mathbb{R} \lhd_X \mathbb{R}^{\mathbf{X}}$.

The argument strategy generalizes the strategy used for majority algebras. We will consider the set $\mathcal{B}$ of ordered pairs $(x, \mathbb{B})$ such that x is a variable of $\mathbf{Y}$, $\mathbb{B} \lhd_X \mathbb{A}_x$, and $\mathbb{B} \ne \emptyset, \mathbb{A}_x$. We want to define a quasiorder $\preceq$ on $\mathcal{B}$, such that if restricting the domain of the variable x to $\mathbb{B}$ and imposing arc-consistency forces another variable y to have its domain restricted to $\mathbb{C}$, then we have $(x, \mathbb{B}) \preceq (y, \mathbb{C})$. Unfortunately, it is not enough to consider paths alone to define this partial order: general deductions involving arc-consistency involve reasoning about *trees*.

Definition 3.9.4. To every relational structure $\mathbf{A} = (A, R_1, ...)$ we associate the bipartite graph $\mathcal{G}_{\mathbf{A}}$ with vertex sets A and $R_1 \sqcup \cdots$, and edge set consisting of pairs (a, r) for every $a \in A$ and $r \in R_i$ such that some coordinate of r is equal to a (if a occurs as a coordinate of r multiple times, then we make multiple copies of the edge (a, r)).

We say that $\mathbf{A}$ is a *tree* if the associated bipartite graph $\mathcal{G}_{\mathbf{A}}$ is a tree (so in particular, no tuple r in any relation R_i can have any repeated coordinates).

Kozik [127] extends the concepts of paths and addition of paths to trees in order to define the partial order $\preceq$ on $\mathcal{B}$ properly.

Definition 3.9.5. If $\mathbf{Y}$ is a CSP instance, viewed as a relational structure, then we define a *tree pattern* p from x to y to consist of the following information:

- a relational structure $\mathbf{A} = (A, R_1, ...)$ which is a tree, with each relation of $\mathbf{A}$ corresponding to a relation of $\mathbf{Y}$,

- a homomorphism of relational structures $h : \mathbf{A} \to \mathbf{Y}$,

- a subset $I \subseteq A$ of the elements of A which we call the set of *inputs* to the pattern, such that for all $i \in I$ we have $h(i) = x$, and

- an element $o \in A$ which we call the *output* of the pattern, such that $h(o) = y$.

If p is a tree pattern from x to y, then we may view it as a CSP instance via the homomorphism $h : \mathbf{A} \to \mathbf{Y}$. If $\mathbb{B} \le \mathbb{A}_x$, then we define $\mathbb{B} + p$ to be the subalgebra of values $b \in \mathbb{A}_y$ such that the instance $\mathbf{A}$ has a solution with the variables from I assigned to values in $\mathbb{B}$, and with the variable o assigned to the value b.

If p is a tree pattern from x to y, and if q is a tree pattern from y to z, then we define the tree pattern $p + q$ by attaching a copy of p to each input of q, combining the output of each copy of p to the corresponding input of q. This definition is set up to ensure that $\mathbb{B} + (p + q) = (\mathbb{B} + p) + q$ for any $\mathbb{B} \le \mathbb{A}_x$.

Proposition 3.9.6. *If p is a tree pattern from x to y in an arc-consistent instance $\mathbf{Y}$ and $\mathbb{B} \lhd_X \mathbb{A}_x$, then $\mathbb{B} + p \lhd_X \mathbb{A}_y$.*

Proof. This follows from the fact that $\lhd_X$ is compatible with pp-formulas: we have $\mathbb{A}_x + p = \mathbb{A}_y$ if $\mathbf{Y}$ is arc-consistent, and so $\mathbb{B} + p \lhd_X \mathbb{A}_x + p = \mathbb{A}_y$. □

Note that unlike the situation for path patterns, arc-consistency of the instance $\mathbf{Y}$ is no longer enough to ensure that $\mathbb{B} \ne \emptyset \implies \mathbb{B} + p \ne \emptyset$ for all tree patterns p. So we can no longer take as given that the subalgebras we construct will always be nonempty.

Definition 3.9.7. Define the quasiordered set $(\mathcal{B}, \preceq)$ to be the set of ordered pairs $(x, \mathbb{B})$ such that x is a variable of the instance $\mathbf{Y}$, $\mathbb{B} \lhd_X \mathbb{A}_x$, and $\mathbb{B} \ne \emptyset, \mathbb{A}_x$, with the quasiorder defined by $(x, \mathbb{B}) \preceq (y, \mathbb{C})$ if there exists a tree pattern p from x to y with $\mathbb{B} + p = \mathbb{C}$.

As in the argument for majority algebras, we now pick a maximal component $\mathcal{C}$ of the quasiordered set $(\mathcal{B}, \preceq)$ (since $\mathcal{B}$ is nonempty by assumption and is finite, such a maximal component exists). We would like to use $\mathcal{C}$ to define our reduced instance $\mathbf{Z}$, but we no longer have a guarantee that there is at most one set $\mathbb{B}$ with $(x, \mathbb{B}) \in \mathcal{C}$ for a given variable x.

A worst case scenario would be that there exist $\mathbb{B}_1, \mathbb{B}_2$ with $(x, \mathbb{B}_i) \in \mathcal{C}$ such that $\mathbb{B}_1 \cap \mathbb{B}_2 = \emptyset$: in this case, we would have no hope of using $\mathcal{C}$ to define an arc-consistent reduction, because no matter which $(y, \mathbb{C}) \in \mathcal{C}$ we pick, there exist tree patterns p_1, p_2 from y to x with $\mathbb{C} + p_i = \mathbb{B}_i$, so reducing the domain $\mathbb{A}_y$ to $\mathbb{C}$ and imposing arc-consistency would make it impossible to assign any value to x. The main step of the proof is ruling out this scenario.

Lemma 3.9.8. *If $\mathcal{C}$ is a maximal component of $(\mathcal{B}, \preceq)$, and if $(x, \mathbb{B}), (x, \mathbb{C}) \in \mathcal{C}$, then $\mathbb{B} \cap \mathbb{C} \ne \emptyset$.*

Before proving the lemma, we'll show how we can use it to finish the proof of Theorem 3.9.2. This step won't use the fact that the instance $\mathbf{X}$ is *pq*-consistent, or the fact that $\lhd_X$ transfers connectivity: the lemma is where these crucial facts are used.

Proof of Theorem 3.9.2, assuming the lemma. Note that if $(x, \mathbb{B}), (x, \mathbb{C}) \in \mathcal{C}$, then we can splice together tree patterns to show that $(x, \mathbb{B} \cap \mathbb{C}) \in \mathcal{C}$ as well (so long as $\mathbb{B} \cap \mathbb{C} \ne \emptyset$, which follows from the lemma). So for every x, we can define a subalgebra $\mathbb{B}_x \lhd_X \mathbb{A}_x$ by taking $\mathbb{B}_x$ to be the intersection of all $\mathbb{B}$ such that $(x, \mathbb{B}) \in \mathcal{C}$ (or taking $\mathbb{B}_x = \mathbb{A}_x$ if no such $\mathbb{B}$ exist). We define the absorbing subinstance $\mathbf{Z}$ by reducing the domains of $\mathbf{Y}$ from $\mathbb{A}_x$ to $\mathbb{B}_x$. We need to check that $\mathbf{Z}$ is arc-consistent.

Consider a single relation $\mathbb{R} \le_{sd} \mathbb{A}_{x_1} \times \cdots \times \mathbb{A}_{x_k}$ of $\mathbf{Y}$. We wish to show that $\mathbb{R} \cap \prod_i \mathbb{B}_{x_i}$ is subdirect in $\prod_i \mathbb{B}_{x_i}$. We will show by induction on i that

$$\pi_i(\mathbb{R} \cap \prod_{j \le i} \mathbb{B}_{x_j} \times \prod_{l > i} \mathbb{A}_{x_l}) = \mathbb{B}_{x_i}.$$

The base case $i = 1$ follows from the fact that $\mathbf{Y}$ is arc-consistent. For the inductive step, we pick any $(y, \mathbb{C}) \in \mathcal{C}$ and splice together tree patterns p_j from y to x_j with $\mathbb{C} + p_j = \mathbb{B}_{x_j}$ for $j < i$ such that $\mathbb{B}_{x_j} \ne \mathbb{A}_{x_j}$ together with the relation $\mathbb{R}$ to make a tree pattern p from y to x_i with

$$\mathbb{C} + p = \pi_i(\mathbb{R} \cap \prod_{j \le i-1} \mathbb{B}_{x_j} \times \prod_{l > i-1} \mathbb{A}_{x_l}),$$

and note that by the induction hypothesis the right hand side is nonempty. Thus we either have $\mathbb{C} + p = \mathbb{A}_{x_i}$ or $(x_i, \mathbb{C} + p) \in \mathcal{C}$, and in either case we have $\mathbb{B}_{x_i} \subseteq \mathbb{C} + p$ (by the lemma), which completes the proof. □

Now we finally prove the crucial lemma.

Proof of the lemma. Suppose for contradiction that the lemma is not true, and choose $\mathbb{C}$ maximal such that $(x, \mathbb{C}) \in \mathcal{C}$ and such that there exists $(x, \mathbb{B}) \in \mathcal{C}$ with $\mathbb{B} \cap \mathbb{C} = \emptyset$. Let $\mathbb{B}_1, ..., \mathbb{B}_k$ be the set of minimal $\mathbb{B}$s such that $(x, \mathbb{B}) \in \mathcal{C}$ and $\mathbb{B} \cap \mathbb{C} = \emptyset$. Note that since the set of $\mathbb{B}$s with $(x, \mathbb{B}) \in \mathcal{C}$ is closed under nonempty intersection, we must have $\mathbb{B}_i \cap \mathbb{B}_j = \emptyset$ for all $i \ne j$. Additionally, any $\mathbb{B}$ with $(x, \mathbb{B}) \in \mathcal{C}$ and $\mathbb{B} \cap \mathbb{C} = \emptyset$ must contain at least one $\mathbb{B}_i$.

Choose tree patterns p_i, q, r from x to x such that $\mathbb{B}_i + p_i = \mathbb{B}_{i+1}$, $\mathbb{C} + q = \mathbb{B}_1$, $\mathbb{B}_k + r = \mathbb{C}$. Define the tree pattern p by $p = q + p_1 + \cdots + p_{k-1} + r$, and note that $\mathbb{C} + p = \mathbb{C}$. We will mainly work inside the instance $\mathbf{A}$ corresponding to the tree pattern p.

First we prune the inputs of the tree pattern p a little bit to make a new tree pattern p' (with the same instance $\mathbf{A}$), removing variables of $\mathbf{A}$ from the input set one at a time as long as we can remove one while keeping $\mathbb{C} + p' = \mathbb{C}$. Now pick any remaining input variable $s \in \mathbf{A}$ of p' (at least one input variable remains at the end of the pruning process, by the arc-consistency of $\mathbf{Y}$), and let t be the output variable of p' (note that s, t are both mapped to x in $\mathbf{Y}$). Let p'' be p' with s removed from its input set. Consider the binary relation $\mathbb{S} \le \mathbb{A}_x \times \mathbb{A}_x$ consisting of pairs (a, b) such that some solution of the instance $\mathbf{A}$ assigns the value a to s, assigns the value b to t, and assigns all input variables of p'' to values in $\mathbb{C}$.

Since $\mathbb{C} + p' = \mathbb{C}$, we have

$$\mathbb{C} + \mathbb{S} = \mathbb{C} + p' = \mathbb{C},$$

and because of the pruning process we have

$$\pi_2(\mathbb{S}) = \mathbb{C} + p'' \ne \mathbb{C},$$

so by the maximal choice of $\mathbb{C}$ we have $\pi_2(\mathbb{S}) \cap \mathbb{B}_i \ne \emptyset$ for all i. By splicing p'' together with a tree pattern q_i with $\mathbb{C} + q_i = \mathbb{B}_i$ (merging their outputs together), we see that $(x, \pi_2(\mathbb{S}) \cap \mathbb{B}_i) \in \mathcal{C}$, so by the minimality of $\mathbb{B}_i$ we have

$$\pi_2(\mathbb{S}) \supseteq \mathbb{B}_i$$

for all i. Thus the subalgebra

$$\mathbb{B}_i - \mathbb{S} = \pi_1(\mathbb{S} \cap \mathbb{A}_x \times \mathbb{B}_i)$$

is nonempty, has $(\mathbb{B}_i - \mathbb{S}) \cap \mathbb{C} = \emptyset$ since $(\mathbb{C} + \mathbb{S}) \cap \mathbb{B}_i = \emptyset$, and by splicing p'' with the same q_i and changing the output to s, we see that $(x, \mathbb{B}_i - \mathbb{S}) \in \mathcal{C}$. Thus there is some j_i such that $\mathbb{B}_i - \mathbb{S} \supseteq \mathbb{B}_{j_i}$. Then we have

$$(\mathbb{B}_{j_i} + \mathbb{S}) \cap \mathbb{B}_i \neq \emptyset,$$

and by another tree splice (this time splicing q_{j_i} into p'' by merging the output of q_{j_i} with s) we see that either $\mathbb{B}_{j_i} + \mathbb{S} = \mathbb{A}_x$ or $(x, \mathbb{B}_{j_i} + \mathbb{S}) \in \mathcal{C}$, so by the minimality of $\mathbb{B}_i$ we have

$$\mathbb{B}_{j_i} + \mathbb{S} \supseteq \mathbb{B}_i.$$

Thus we have

$$\cup_i \mathbb{B}_i + \mathbb{S} \supseteq \cup_i \mathbb{B}_i,$$

so if we consider $\mathbb{S}$ as a digraph on $\mathbb{A}_x$, we see that there is some directed cycle of $\mathbb{S}$ which is entirely contained in $\cup_i \mathbb{B}_i$. From $\mathbb{C} + \mathbb{S} = \mathbb{C}$, we also see that there is some directed cycle of $\mathbb{S}$ which is entirely contained in $\mathbb{C}$. The plan is to apply Corollary 3.7.10 to produce a directed path in $\mathbb{S}$ from an element of $\mathbb{C}$ to an element of $\cup_i \mathbb{B}_i$, which will give us a contradiction since any directed path in $\mathbb{S}$ which starts in $\mathbb{C}$ must end up in $\mathbb{C}$.

In order to apply Corollary 3.7.10, we need to construct a binary relation $\mathbb{R}$ such that $\mathbb{S} \lhd_X \mathbb{R}$ and such that there is a directed path from $\mathbb{C}$ to $\cup_i \mathbb{B}_i$ in $\mathbb{R}$. This is where we will finally use the assumption that $\mathbf{Y}$ absorbs a bigger instance $\mathbf{X}$ which is pq-consistent. We define an instance $\mathbf{A}^{\mathbf{X}}$ similarly to $\mathbf{A}$, but with each domain replaced with the corresponding domain in $\mathbf{X}$ and similarly for the relations, and define $\mathbb{R}$ to be the projection of the solution set to $\mathbf{A}^{\mathbf{X}}$ onto the variables s, t. Then since $\lhd_X$ is compatible with pp-formulas and since every domain/relation restriction in sight is absorbing, we have $\mathbb{S} \lhd_X \mathbb{R}$.

Now pick any $b \in \cup_i \mathbb{B}_i$ which is contained in a directed cycle of $\mathbb{S}$. Suppose $b \in \mathbb{B}_i$. Consider the path from s to the output variable of $q + p_1 + \cdots + p_{i-1}$ in $\mathbf{A}$, call this path α, and let β be the path from that output variable to t in $\mathbf{A}$. The images of these paths in $\mathbf{X}$ are cycles $\alpha_{\mathbf{X}}, \beta_{\mathbf{X}}$ from x to x, so by the pq-consistency of $\mathbf{X}$ there must exist some $j \geq 0$ such that $b \in \{b\} + j(\beta_{\mathbf{X}} + \alpha_{\mathbf{X}}) + \beta_{\mathbf{X}}$. Note that by the arc-consistency of $\mathbf{X}$, $\mathbb{R}$ is the binary relation corresponding to the cycle $\alpha_{\mathbf{X}} + \beta_{\mathbf{X}}$. Additionally, since

$$\mathbb{C} + q + p_1 + \cdots + p_{i-1} = \mathbb{B}_i,$$

there is some $c \in \mathbb{C}$ such that $b \in \{c\} + \alpha_{\mathbf{X}}$. Thus we have

$$b \in \{c\} + \alpha_{\mathbf{X}} + j(\beta_{\mathbf{X}} + \alpha_{\mathbf{X}}) + \beta_{\mathbf{X}} = \{c\} + (j+1)(\alpha_{\mathbf{X}} + \beta_{\mathbf{X}}) = \{c\} + \mathbb{R}^{\circ(j+1)}.$$

Additionally, by following paths of $\mathbb{S}$ backwards sufficiently many times, we see that c is reachable from a directed cycle of $\mathbb{S}$ which is entirely contained in $\mathbb{C}$. Thus there is some m such that for some $a \in \mathbb{C}$, we have $(a, a), (b, b) \in \mathbb{S}^{\circ m}$ and $(a, b) \in \mathbb{R}^{\circ m}$, and since $\mathbb{S}^{\circ m} \lhd_X \mathbb{R}^{\circ m}$ we may apply the transfer of connectivity property to see that for some n we have $(a, b) \in \mathbb{S}^{\circ n}$, which gives us our contradiction. □

To finish the analysis of bounded width algebras, we just need to understand the case where all the domains are absorption free. For this we need two main ingredients: first is that binary relations are forced to be boring unless some absorption occurs, and second is that if a simple algebra has an exciting ternary relation whose binary projections are boring, then the algebra must be affine and therefore does not have bounded width.

3.9.1 Absorption constants

In this subsection, we'll go over the proof of an interesting generalization of the Theorem 3.7.13 to higher arity relations, from [23], which we mentioned at the beginning of the last section. Since the proof of this result is so similar to the proof of Kozik's result from the last section, this seems like an appropriate place to cover the argument.

We will need some notation for the diagonal of a power $\mathbb{A}^n$. One option is to use the notation $\Delta_{\mathbb{A}}^n$, but this looks very similar to the notation we use in the appendix on commutator theory (Appendix A). Another notation some authors use is $0_{\mathbb{A}}^n$, so that when $n = 2$ we get the least congruence $0_{\mathbb{A}}$, but I am not a big fan of this notation either. Yet another possibility is $=_{\mathbb{A}}^n$. I decided on a fourth option, which allows me to refer to specific elements of the diagonal without too much ugliness, and which emphasizes the fact that the diagonal is isomorphic to $\mathbb{A}$.

Definition 3.9.9. For any n, define the *diagonal subalgebra* of arity n to be the subalgebra $\mathbb{A}^{(n)} \le_{sd} \mathbb{A}^n$, given by

$$\mathbb{A}^{(n)} = \{(a, ..., a) \mid a \in \mathbb{A}\}.$$

Additionally, for each $a \in \mathbb{A}$, we define the corresponding *constant tuple* to be

$$a^{(n)} = (a, ..., a) \in \mathbb{A}^{(n)}.$$

Theorem 3.9.10 (Theorem 6 of [23])**.** *If $\mathbb{A}$ is finite, $\mathbb{R} \le_{sd} \mathbb{A}^n$ is subdirect, and $\mathbb{R}$ Jónsson absorbs the diagonal $\mathbb{A}^{(n)}$, then $\mathbb{R} \cap \mathbb{A}^{(n)} \ne \emptyset$.*

Proof. The proof strategy is similar to the proof of Theorem 3.7.13. We assume without loss of generality that $\mathbb{A}$ is idempotent, and we induct on $|\mathbb{A}|$. It's enough to show that there is some proper subalgebra $\mathbb{B} \le \mathbb{A}$ such that

$$\mathbb{R} \cap \mathbb{B}^n \le_{sd} \mathbb{B}^n,$$

since $\mathbb{R} \cap \mathbb{B}^n$ will automatically Jónsson absorb $\mathbb{B}^{(n)}$.

Similarly to the proof of Theorem 3.9.2, we define the a quasiordered set $(\mathcal{B}, \preceq)$ to be the set of subalgebras $\mathbb{B} \le \mathbb{A}$ with $\mathbb{B} \ne \emptyset, \mathbb{A}$, with the quasiorder defined by $\mathbb{B} \preceq \mathbb{C}$ if there is a tree pattern p built out of copies of the relation $\mathbb{R}$ such that $\mathbb{B} + p = \mathbb{C}$.

Pick some maximal component $\mathcal{C}$ of $(\mathcal{B}, \preceq)$. Note that if $\mathbb{B}, \mathbb{C} \in \mathcal{C}$ have $\mathbb{B} \cap \mathbb{C} \ne \emptyset$, then we can splice together tree patterns to see that $\mathbb{B} \cap \mathbb{C} \in \mathcal{C}$ as well.

First suppose that every pair $\mathbb{B}, \mathbb{C} \in \mathcal{C}$ have $\mathbb{B} \cap \mathbb{C} \ne \emptyset$. Then there is some $\mathbb{B} \in \mathcal{C}$ which is contained in all other elements of $\mathcal{C}$. We claim that in this case, we have

$$\mathbb{R} \cap \mathbb{B}^n \le_{sd} \mathbb{B}^n,$$

which will allow us to complete the proof. To check this, we prove by induction on i that

$$\mathbb{B} \subseteq \pi_i(\mathbb{R} \cap \mathbb{B}^{i-1} \times \mathbb{A}^{n-i+1}).$$

For $i = 1$ this follows from the assumption that $\mathbb{R}$ is subdirect. For the induction step, note that the induction hypothesis implies that

$$\pi_i(\mathbb{R} \cap \mathbb{B}^{i-1} \times \mathbb{A}^{n-i+1}) \ne \emptyset,$$

so

$$\pi_i(\mathbb{R} \cap \mathbb{B}^{i-1} \times \mathbb{A}^{n-i+1}) \in \mathcal{C} \cup \{\mathbb{A}\},$$

and either way it contains $\mathbb{B}$.

Now suppose, for the sake of contradiction, that there are $\mathbb{B}, \mathbb{C} \in \mathcal{C}$ with $\mathbb{B} \cap \mathbb{C} = \emptyset$. As in the proof of Lemma 3.9.8, we take $\mathbb{C} \in \mathcal{C}$ maximal under inclusion such that there exists some $\mathbb{B} \in \mathcal{C}$ with $\mathbb{B} \cap \mathbb{C} = \emptyset$, and we let $\mathbb{B}_1, ..., \mathbb{B}_k$ be the set of minimal (under inclusion) $\mathbb{B}$s such that $\mathbb{B} \in \mathcal{C}$ and $\mathbb{B} \cap \mathbb{C} = \emptyset$.

We continue following the proof of Lemma 3.9.8, defining tree patterns p_i, q, r built out of copies of the relation $\mathbb{R}$ such that $\mathbb{B}_i + p_i = \mathbb{B}_{i+1}$, $\mathbb{C} + q = \mathbb{B}_1$, $\mathbb{B}_k + r = \mathbb{C}$, and defining p by

$$p = q + p_1 + \cdots + p_k + r.$$

Then we prune the inputs of the pattern p to make a pattern p' with as few inputs as possible such that

$$\mathbb{C} + p' = \mathbb{C},$$

and let p'' be the pattern we get from p' by removing one additional input s from the input set, and define

$$\mathbb{S} \leq \mathbb{A} \times \mathbb{A}$$

as the set of possible pairs of values for the pruned input s and the output of the pattern p'' which extend to assignments where every remaining input of p'' is given a value in $\mathbb{C}$, as in the proof of Lemma 3.9.8.

By the exact same argument from Lemma 3.9.8, we have

$$\mathbb{C} + \mathbb{S} = \mathbb{C} + p' = \mathbb{C}$$

and

$$\cup_i \mathbb{B}_i + \mathbb{S} \supseteq \cup_i \mathbb{B}_i,$$

so every element $c \in \mathbb{C}$ is reachable from a directed cycle of $\mathbb{S}$ which is entirely contained in $\mathbb{C}$, and $\cup_i \mathbb{B}_i$ contains a directed cycle of $\mathbb{S}$.

Now we finally deviate slightly from the proof of Lemma 3.9.8. Define a pattern $p_=$ by replacing each occurence of $\mathbb{R}$ by $\mathbb{R} \cup \mathbb{A}^{(n)}$ in the pattern p, and similarly define $p'_=, p''_=$, and define

$$S_= \subseteq \mathbb{A} \times \mathbb{A}$$

as the set of possible pairs of values for the pruned input s and the output of the pattern $p''_=$ which extend to assignments where every remaining input of $p''_=$ is given a value in $\mathbb{C}$. The $\mathbb{S} \subseteq S_=$ and $\mathbb{S}$ Jónsson absorbs $S_=$. Additionally, for each i we have

$$\mathbb{C} + p_= \supseteq \mathbb{C} + q + p_1 + \cdots + p_{i-1} = \mathbb{B}_i,$$

since we can simply feed a bunch of equal copies of an element $b \in \mathbb{B}_i$ to each of the remaining levels of the tree pattern. Thus we have

$$\mathbb{C} + S_= \supseteq \cup_i \mathbb{B}_i.$$

Thus we can find a directed path in $S_=$ from some element $c \in \mathbb{C}$ which is contained in a directed cycle of $\mathbb{S}$ to some element of $\cup_i \mathbb{B}_i$ which is contained in a directed cycle of $\mathbb{S}$. This allows us to apply Corollary 3.7.10 to conclude that there is some directed path from $\mathbb{C}$ to $\cup_i \mathbb{B}_i$ in $\mathbb{S}$, which gives us our contradiction. □

Ross Willard points out the following consequence of this result.

Corollary 3.9.11. *If $\mathbb{A}$ is a finite algebra, then there is at least one element $a \in \mathbb{A}$ such that, for all subdirect relations $\mathbb{R} \le_{sd} \mathbb{A}^n$, we have*

$$\mathbb{R} \lhd_J \mathrm{Sg}_{\mathbb{A}^n}(\mathbb{R} \cup \mathbb{A}^{(n)}) \quad \implies \quad a^{(n)} \in \mathbb{R}.$$

The set of such elements a forms a Jónsson absorbing subalgebra of $\mathbb{A}$.

Definition 3.9.12. Say that a is an *absorption constant* of $\mathbb{A}$ with respect to the absorption concept $\lhd_X$ if

$$\mathbb{R} \lhd_X \mathrm{Sg}_{\mathbb{A}^n}(\mathbb{R} \cup \mathbb{A}^{(n)}) \quad \implies \quad a^{(n)} \in \mathbb{R}$$

for all subdirect relations $\mathbb{R} \le_{sd} \mathbb{A}^n$. Let

$$\mathrm{Abs}_X(\mathbb{A}) \lhd_X \mathbb{A}$$

be the set of absorption constants of $\mathbb{A}$ with respect to $\lhd_X$.

Problem 3.9.1 (Ross Willard)**.** Can we give an independent characterization of the canonical absorbing subalgebra $\mathrm{Abs}(\mathbb{A})$? What can we do with it?

3.10 Zhuk's centers and ternary absorption

In this section we'll go over a very strong technique introduced by Zhuk in his proof of the dichotomy conjecture [190], which produces ternary absorption as soon as we have a certain type of binary relation on a pair of Taylor algebras. This technique allows us to both simplify and strengthen one of the key results needed for the study of general Taylor algebras, known as the "absorption theorem".

First, we'll go over the history of this idea, so the reader can understand where the definition comes from and why it is (somewhat) natural.

The main idea behind Zhuk's approach in [190] is to note that if an algebra is not polynomially complete, then its polynomial clone must be contained in a maximal proper subclone of the clone of all functions (that every proper subclone is contained in a *maximal* proper subclone follows from the fact that the clone of all functions is finitely generated: in fact, it's generated by the set of functions of arity 2). A maximal clone corresponds under the $\mathrm{Inv} - \mathrm{Pol}$ Galois connection to a minimal relational clone, and every minimal relational clone can be generated by a single relation, of one of several special forms. Zhuk is very familiar with the theory of relational clones, so he was aware of Rosenberg's Completeness Theorem [165] (see [154] or chapter II.6 of [134] for alternate expositions), which completely classifies the special relations which correspond to maximal clones into six different types.

Zhuk then considered each of the types of relations from Rosenberg's classification, and investigated which of them might be preserved by the polynomial clone of a Taylor algebra, and what the existence of such a relation implies about the structure of the Taylor algebra. The most interesting case is the case of the relations known as *central relations*.

Definition 3.10.1. A relation $\mathbb{R} \le \mathbb{A}^n$ is *central* if it has the following properties:

- $\mathbb{R}$ is symmetric under permuting its coordinates,

- $\mathbb{R}$ contains every tuple which has any pair of equal coordinates, and
- the set $\mathbb{C} \le \mathbb{A}$ defined by

  $$\mathbb{C} = \{c \in \mathbb{A} \mid \forall a_2, ..., a_n \in \mathbb{A}, (c, a_2, ..., a_n) \in \mathbb{R}\}$$

  is not empty and is not equal to $\mathbb{A}$.

The set $\mathbb{C}$ is known as the *center* of the central relation $\mathbb{R}$.

Since relations of high arity are hard to think about, Zhuk simplifies this to a special type of binary relation on $\mathbb{A} \times \mathbb{B}$, where $\mathbb{B}$ is secretly taken to be $\mathbb{A}^{n-1}$. To see that this step doesn't lose anything essential, we use the following fact about absorbing subalgebras of powers.

Proposition 3.10.2. *Suppose that $\mathbb{A}$ is idempotent and that $\mathbb{A}^k$ has a proper absorbing subalgebra for some k. Then $\mathbb{A}$ has a proper absorbing subalgebra.*

In fact, this holds for any absorption concept $\lhd_X$ which is compatible with pp-formulas.

Proof. We induct on k. Suppose that $\mathbb{B} \lhd_X \mathbb{A}^k$. If $\pi_1(\mathbb{B}) \ne \mathbb{A}$ then $\pi_1(\mathbb{B}) \lhd_X \mathbb{A}$ and we are done, otherwise since $\mathbb{B} \ne \mathbb{A}^k$ there must exist some $a \in \mathbb{A}$ such that $\pi_{[k]\setminus\{1\}}(\mathbb{B} \cap \{a\} \times \mathbb{A}^{k-1}) \ne \mathbb{A}^{k-1}$. Since $\lhd_X$ is compatible with pp-formulas and $\{a\} \le \mathbb{A}$ by the idempotence of $\mathbb{A}$, we have

$$\pi_{[k]\setminus\{1\}}(\mathbb{B} \cap \{a\} \times \mathbb{A}^{k-1}) \lhd_X \mathbb{A}^{k-1},$$

so we can apply the induction hypothesis. □

With this in mind, it's natural to restrict our attention to binary relations $\mathbb{R} \le \mathbb{A} \times \mathbb{B}$ which have a nontrivial proper "left center", and to try to use them to produce an absorbing subalgebra inside either $\mathbb{A}$ or $\mathbb{B}$.

Definition 3.10.3. If $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$ is subdirect and $\mathbb{B}$ is finite and idempotent, then the *left center* of $\mathbb{R}$ is the subalgebra $\mathbb{C} \le \mathbb{A}$ defined by

$$\mathbb{C} = \{c \in \mathbb{A} \mid \forall b \in \mathbb{B}, (c, b) \in \mathbb{R}\}.$$

The *right center* of a subdirect binary relation is defined similarly (so the right center of $\mathbb{R}$ is the left center of $\mathbb{R}^-$, and is a subalgebra of $\mathbb{B}$).

To see that the left center $\mathbb{C}$ is automatically a subalgebra of $\mathbb{A}$, note that it can be defined by the following pp-formula:

$$c \in \mathbb{C} \iff \bigwedge_{b \in \mathbb{B}} \exists x (x \in \{b\} \wedge (c, x) \in \mathbb{R}).$$

In order to do anything useful with such a binary relation, we will need to assume that $\mathbb{B}$ is Taylor. We will attempt to exploit the Taylor term to produce binary absorption on $\mathbb{B}$, using the following lemma.

Lemma 3.10.4. *Suppose $\mathbb{B} \le \mathbb{A}$ and that there is an idempotent term $t \in \mathrm{Clo}_k(\mathbb{A})$ with the following two properties:*

- *t satisfies an identity of the form $t(x, u_2, ..., u_k) \approx t(y, v_2, ..., v_k)$, where each $u_i, v_i \in \{x, y\}$, and*
- *$t(\mathbb{B}, \mathbb{A}, ..., \mathbb{A}) \subseteq \mathbb{B}$.*

Then $\mathbb{B}$ absorbs $\mathbb{A}$ with respect to some idempotent binary operation f.

Proof. To make the notation more clear, we treat each u_i, v_i as a binary function, with $u_i = u_i(x, y)$ and $v_i = v_i(x, y)$. Define $f(x, y)$ by

$$f(x, y) \coloneqq t(x, u_2(x, y), ..., u_k(x, y)) \approx t(y, v_2(x, y), ..., v_k(x, y)).$$

Then for any $a \in \mathbb{A}$ and $b \in \mathbb{B}$, we have

$$f(a, b) = t(b, v_2(a, b), ..., v_k(a, b)) \in t(\mathbb{B}, \mathbb{A}, ..., \mathbb{A}) \subseteq \mathbb{B},$$

and

$$f(b, a) = t(b, u_2(b, a), ..., u_k(b, a)) \in t(\mathbb{B}, \mathbb{A}, ..., \mathbb{A}) \subseteq \mathbb{B}. \qquad \square$$

Theorem 3.10.5 (Zhuk [190]). *Suppose that $\mathbb{A}, \mathbb{B}$ are finite idempotent algebras, and that there is a term t which is Taylor on $\mathbb{B}$. If $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$ is subdirect and has a nontrivial left center $\mathbb{C}$, then either $\mathbb{B}$ has a proper binary absorbing subalgebra, or $\mathbb{C}$ absorbs $\mathbb{A}$ with respect to the term $t * \cdots * t$, with $|\mathbb{B}| - 1$ copies of t.*

Proof. Suppose t has arity k. We will show that if $\mathbb{B}$ has no proper binary absorbing subalgebra, then for any $a \in \mathbb{A} \setminus \mathbb{C}$ and for any $c_1, ..., c_k \in \mathbb{C}$ and any $i \le k$, the value

$$t(c_1, ..., c_{i-1}, a, c_{i+1}, ..., c_k)$$

is "closer" to being in $\mathbb{C}$ than a is. To make this precise, we measure how close an element a is to being in $\mathbb{C}$ by looking at the size of the set

$$a + \mathbb{R} = \pi_2(\mathbb{R} \cap \{a\} \times \mathbb{B}).$$

By the definition of $\mathbb{C}$, we have $|a + \mathbb{R}| = |\mathbb{B}|$ if and only if $a \in \mathbb{C}$.

Since $\mathbb{R}$ is preserved by t, we have

$$t(c_1, ..., a, ..., c_k) + \mathbb{R} \supseteq t(c_1 + \mathbb{R}, ..., a + \mathbb{R}, ..., c_k + \mathbb{R}) = t(\mathbb{B}, ..., a + \mathbb{R}, ..., \mathbb{B}).$$

Since t is idempotent, the right hand side of the above must contain $a+\mathbb{R}$, and if it is equal to $a+\mathbb{R}$ then we can apply the previous lemma (since t is Taylor) to see that $a + \mathbb{R}$ is a binary absorbing subalgebra of $\mathbb{B}$. Thus if $a \notin \mathbb{C}$, then either $a + \mathbb{R}$ is a proper binary absorbing subalgebra of $\mathbb{B}$, or else

$$|t(c_1, ..., a, ..., c_k) + \mathbb{R}| > |a + \mathbb{R}|. \qquad \square$$

Keeping the same setup, the left center $\mathbb{C}$ has an additional nice property, which is much stronger than it looks.

Theorem 3.10.6 (Zhuk [190]). *Suppose $\mathbb{A}, \mathbb{B}$ are finite idempotent algebras. If $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$ has a left center $\mathbb{C}$ and $\mathbb{B}$ has no proper binary absorbing subalgebras, then for any $a \in \mathbb{A}$ we have*

$$a \notin \mathbb{C} \quad \implies \quad \begin{bmatrix} a \\ a \end{bmatrix} \notin \mathrm{Sg}_{\mathbb{A}^2}\left\{\begin{bmatrix} a \\ \mathbb{C} \end{bmatrix}, \begin{bmatrix} \mathbb{C} \\ \mathbb{C} \end{bmatrix}, \begin{bmatrix} \mathbb{C} \\ a \end{bmatrix}\right\}.$$

Proof. Suppose otherwise. Then there are i, j and $c_1, ..., c_i, c_j', ..., c_n' \in \mathbb{C}$ with $j \le i+1$ and a term t of arity n such that

$$\begin{bmatrix} a \\ a \end{bmatrix} = t\left(\begin{bmatrix} a \\ c_1 \end{bmatrix}, \dots, \begin{bmatrix} a \\ c_{j-1} \end{bmatrix}, \begin{bmatrix} c_j' \\ c_j \end{bmatrix}, \dots, \begin{bmatrix} c_i' \\ c_i \end{bmatrix}, \begin{bmatrix} c_{i+1}' \\ a \end{bmatrix}, ..., \begin{bmatrix} c_n' \\ a \end{bmatrix}\right).$$

Looking at the neighbors via $\mathbb{R}$, we have

$$\begin{bmatrix} a+\mathbb{R} \\ a+\mathbb{R} \end{bmatrix} \supseteq t\left(\begin{bmatrix} a+\mathbb{R} & \cdots & a+\mathbb{R} & \mathbb{B} & \cdots & \mathbb{B} & \mathbb{B} & \cdots & \mathbb{B} \\ \mathbb{B} & \cdots & \mathbb{B} & \mathbb{B} & \cdots & \mathbb{B} & a+\mathbb{R} & \cdots & a+\mathbb{R} \end{bmatrix}\right).$$

Thus $a+\mathbb{R}$ absorbs $\mathbb{B}$ with respect to the binary term

$$f(x, y) \coloneqq t(x, ..., x, y, ..., y)$$

as long as the number of xs is between $j-1$ and i. □

We can combine the previous two results about left centers to define a new type of absorption. We won't need the full power of the previous result, and instead will use a slightly weaker property.

Definition 3.10.7. We say that $\mathbb{C}$ *centrally absorbs* $\mathbb{A}$, written $\mathbb{C} \lhd_Z \mathbb{A}$, if the following two properties hold:

- $\mathbb{C} \lhd \mathbb{A}$, and
- for any $a \notin \mathbb{C}$, we have $\begin{bmatrix} a \\ a \end{bmatrix} \notin \mathrm{Sg}_{\mathbb{A}^2}\left\{\begin{bmatrix} a \\ \mathbb{C} \end{bmatrix}, \begin{bmatrix} \mathbb{C} \\ a \end{bmatrix}\right\}$.

Corollary 3.10.8. *Suppose $\mathbb{A}, \mathbb{B}$ are finite and idempotent. If $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$ has left center $\mathbb{C}$ and $\mathbb{B}$ is Taylor and binary absorption free, then $\mathbb{C} \lhd_Z \mathbb{A}$.*

There is an unfortunate naming collision between the centers considered here, and the centers considered in commutator theory. Generally it should be clear from context which sort of center is meant. (I have proposed the alternate name *stable absorption* instead of central absorption, but it seems unlikely to catch on.)

The key fact about central absorption that makes it so much more powerful than ordinary absorption is the following doubling trick due to Zhuk and Kozik.

Lemma 3.10.9 (Essential doubling trick [190])**.** *Suppose that $\mathbb{R} \le \mathbb{A}_0 \times \cdots \times \mathbb{A}_{n+1}$ is $(\mathbb{C}, \mathbb{B}_1, ..., \mathbb{B}_n, \mathbb{C}')$-essential, with $\mathbb{C}' \lhd_Z \mathbb{A}_{n+1}$ and $\mathbb{A}_{n+1}$ finite and idempotent. Then there is a relation*

$$\mathbb{R}' \le \mathbb{A}_0 \times \cdots \times \mathbb{A}_n \times \mathbb{A}_n \times \cdots \times \mathbb{A}_0$$

which is $(\mathbb{C}, \mathbb{B}_1, ..., \mathbb{B}_n, \mathbb{B}_n, ..., \mathbb{B}_1, \mathbb{C})$-essential.

Proof. Suppose $\mathbb{R}$ is chosen such that, subject to satisfying the assumptions of the lemma, the subalgebra $\mathbb{B}' \le \mathbb{A}_{n+1}$ defined by

$$\mathbb{B}' = \pi_{n+1}(\mathbb{R} \cap \mathbb{C} \times \mathbb{B}_1 \times \cdots \times \mathbb{B}_n \times \mathbb{A}_{n+1})$$

is as small as possible. Note that $\mathbb{B}'$ is necessarily nonempty and disjoint from $\mathbb{C}'$ if $\mathbb{R}$ is $(\mathbb{C}, \mathbb{B}_1, ..., \mathbb{B}_n, \mathbb{C}')$-essential.

Since we may shrink $\mathbb{R}$ to the subalgebra generated by any collection of tuples witnessing $\mathbb{R} \cap (\mathbb{C} \times \cdots \times \mathbb{A}_i \times \cdots \times \mathbb{C}') \neq \emptyset$ for all i from 0 to $n+1$, we see that

$$b, b' \in \mathbb{B}' \quad \implies \quad b' \in \mathrm{Sg}_{\mathbb{A}_{n+1}}(\mathbb{C}' \cup \{b\}).$$

In particular, if we pick some $b \in \mathbb{B}'$ and define the symmetric binary relation $\mathbb{S} \leq \mathbb{A}_{n+1} \times \mathbb{A}_{n+1}$ by

$$\mathbb{S} = \mathrm{Sg}_{\mathbb{A}_{n+1}^2} \left\{ \begin{bmatrix} b \\ \mathbb{C}' \end{bmatrix}, \begin{bmatrix} \mathbb{C}' \\ b \end{bmatrix} \right\},$$

then $\pi_1(\mathbb{S}) \supseteq \mathbb{B}'$.

We now define the relation $\mathbb{R}'$ by

$$(x_0, ..., x_n, y_n, ..., y_0) \in \mathbb{R}' \iff \exists x_{n+1}, y_{n+1}\ (x_0, ..., x_{n+1}) \in \mathbb{R} \wedge (x_{n+1}, y_{n+1}) \in \mathbb{S} \wedge (y_0, ..., y_{n+1}) \in \mathbb{R}.$$

To see that

$$\mathbb{R}' \cap \mathbb{C} \times \mathbb{B}_1 \times \cdots \times \mathbb{A}_i \times \cdots \times \mathbb{B}_n \times \mathbb{B}_n \times \cdots \times \mathbb{B}_1 \times \mathbb{C} \neq \emptyset$$

for any $0 \leq i \leq n$, we choose $(x_0, ..., x_{n+1}) \in \mathbb{R} \cap (\mathbb{C} \times \cdots \times \mathbb{A}_i \times \cdots \times \mathbb{C}')$ and choose $(y_0, ..., y_{n+1}) \in \mathbb{R} \cap \mathbb{C} \times \mathbb{B}_1 \times \cdots \times \mathbb{B}_n \times \{b\}$, which is possible since $b \in \mathbb{B}'$ and $\mathbb{C}' \times \{b\} \subseteq \mathbb{S}$. We can check that

$$\mathbb{R}' \cap \mathbb{C} \times \mathbb{B}_1 \times \cdots \times \mathbb{B}_n \times \mathbb{B}_n \times \cdots \times \mathbb{A}_i \times \cdots \times \mathbb{B}_1 \times \mathbb{C} \neq \emptyset$$

for $0 \leq i \leq n$ similarly, by interchanging the roles of the x_is and y_is.

To finish, we just need to check that

$$\mathbb{R}' \cap \mathbb{C} \times \mathbb{B}_1 \times \cdots \times \mathbb{B}_n \times \mathbb{B}_n \times \cdots \times \mathbb{B}_1 \times \mathbb{C} = \emptyset,$$

or equivalently, that

$$\mathbb{S} \cap \mathbb{B}' \times \mathbb{B}' = \emptyset.$$

So suppose for contradiction that there are $b', b'' \in \mathbb{B}'$ with $(b', b'') \in \mathbb{S}$. Since

$$b \in \mathrm{Sg}(\mathbb{C}' \cup \{b'\}) \subseteq \mathbb{B}' - \mathbb{S},$$

we see that there is some $b''' \in \mathbb{B}'$ such that $(b, b''') \in \mathbb{S}$. But then we have

$$\{b\} + \mathbb{S} \supseteq \mathrm{Sg}(\mathbb{C}' \cup \{b'''\}) \supseteq \mathbb{B}',$$

so $(b, b) \in \mathbb{S}$, contradicting our assumption that $\mathbb{C}' \lhd_Z \mathbb{A}_{n+1}$. □

Corollary 3.10.10. *If $\mathbb{C} \lhd_Z \mathbb{A}$ and $\mathbb{A}$ is finite and idempotent, then $\mathbb{C}$ absorbs $\mathbb{A}$ with respect to some ternary term.*

Proof. If $\mathbb{C}$ does not absorb $\mathbb{A}$ with respect to any ternary term, then by Theorem 3.8.5 there is some ternary $\mathbb{C}$-essential relation $\mathbb{R} \leq \mathbb{A}^3$. By repeatedly applying the doubling trick, we see that there exists some $\mathbb{C}$-essential relation of arity $2 + 2^k$ for every $k \geq 0$, so $\mathbb{C}$ can't absorb $\mathbb{A}$ with respect to a term of any arity, contradicting the assumption $\mathbb{C} \lhd_Z \mathbb{A}$. □

Corollary 3.10.11. *If $\mathbb{C}_1 \lhd_Z \mathbb{A}_1, \mathbb{B}_2 \lhd \mathbb{A}_2$, and $\mathbb{C}_3 \lhd_Z \mathbb{A}_3$ with $\mathbb{A}_i$ finite and idempotent, then no $(\mathbb{C}_1, \mathbb{B}_2, \mathbb{C}_3)$-essential relation can exist.*

Proof. If a $(\mathbb{C}_1, \mathbb{B}_2, \mathbb{C}_3)$-essential relation exists, then by repeatedly applying the doubling trick we can find $(\mathbb{C}_1, \mathbb{B}_2, ..., \mathbb{B}_2, \mathbb{C}_1)$-essential relations of arbitrarily high arity. By forcing the first and last coordinates to be in $\mathbb{C}_1$ and existentially projecting, we see that there are $\mathbb{B}_2$-essential relations of arbitrarily high arity, which contradicts the assumption $\mathbb{B}_2 \lhd \mathbb{A}_2$. □

Corollary 3.10.12. *If $\mathbb{A}_i$ are finite and idempotent, $\mathbb{C}_i \lhd \mathbb{A}_i$ for all i and for all but at most one i we have $\mathbb{C}_i \lhd_Z \mathbb{A}_i$, then for any relation $\mathbb{R} \le \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ such that $\pi_{i,j}(\mathbb{R}) \cap \mathbb{C}_i \times \mathbb{C}_j \ne \emptyset$ for all i, j, we have*

$$\mathbb{R} \cap \mathbb{C}_1 \times \cdots \times \mathbb{C}_n \ne \emptyset.$$

Proof. We show by induction on $|I|$ that for all $I \subseteq [n]$ we have

$$\pi_I(\mathbb{R}) \cap \prod_{i \in I} \mathbb{C}_i \ne \emptyset.$$

The base case $|I| \le 2$ is our assumption. For $|I| \ge 3$, pick $i, j, k \in I$ distinct. By the induction hypothesis, there are tuples $x_i, x_i, x_k \in \mathbb{R}$ such that $\pi_{I \setminus \{i\}}(x_i) \in \prod_{i' \in I \setminus \{i\}} \mathbb{C}_{i'}$, and similarly for x_j, x_k.

Now consider the subalgebra of $\mathbb{A}_i \times \mathbb{A}_j \times \mathbb{A}_k$ generated by $\pi_{i,j,k}(x_i), \pi_{i,j,k}(x_j), \pi_{i,j,k}(x_k)$. Since this subalgebra can't be a $(\mathbb{C}_i, \mathbb{C}_j, \mathbb{C}_k)$-essential relation (since at least two of $\mathbb{C}_i, \mathbb{C}_j, \mathbb{C}_k$ are centrally absorbing and the third is absorbing), it must contain an element of $\mathbb{C}_i \times \mathbb{C}_j \times \mathbb{C}_k$. Thus there is some $x \in \mathrm{Sg}\{x_i, x_j, x_k\}$ such that

$$\pi_{i,j,k}(x) \in \mathbb{C}_i \times \mathbb{C}_j \times \mathbb{C}_k,$$

and this x automatically satisfies

$$\pi_{I \setminus \{i,j,k\}}(x) \in \prod_{i' \in I \setminus \{i,j,k\}} \mathbb{C}_{i'}$$

since each of x_i, x_j, x_k do, which completes the inductive step. □

Corollary 3.10.13. *If $\mathbb{A}$ is finite and idempotent, then there is a ternary term $t \in \mathrm{Clo}_3(\mathbb{A})$ such that for all finite $\mathbb{B} \in HSP(\mathbb{A})$ and each $\mathbb{C} \lhd_Z \mathbb{B}$, $\mathbb{C}$ absorbs $\mathbb{B}$ with respect to the term t.*

Proof. For any finite collection of pairs $\mathbb{C}_i \lhd_Z \mathbb{B}_i \in HSP(\mathbb{A})$, we can apply the previous corollary to find a term $t \in \mathrm{Clo}_3(\mathbb{A})$ which simultaneously witnesses all $\mathbb{C}_i \lhd \mathbb{B}_i$. Since there are only finitely many ternary terms t of $\mathbb{A}$, some t must work for all pairs $\mathbb{C} \lhd_Z \mathbb{B} \in HSP(\mathbb{A})$. □

Central absorption turns out to be a good absorption concept (in the sense of the previous section), as long as we restrict ourselves to finite idempotent algebras. Unlike previous absorption concepts, in this case it is not so easy to see that $\lhd_Z$ is compatible with pp-formulas. For this, we need to consider the basic types of pp-formulas separately. The hardest case is the case of projections.

Proposition 3.10.14. *If $\mathbb{C} \lhd_Z \mathbb{A}$ with $\mathbb{A}$ finite and idempotent, and if there is a surjective homomorphism $\pi : \mathbb{A} \twoheadrightarrow \mathbb{B}$, then $\pi(\mathbb{C}) \lhd_Z \mathbb{B}$.*

Proof. Suppose there is some $b \in \mathbb{B} \setminus \pi(\mathbb{C})$ such that $(b,b) \in \mathrm{Sg}(\pi(\mathbb{C}) \times \{b\} \cup \{b\} \times \pi(\mathbb{C}))$. Choose $a \in \pi^{-1}(b)$ such that the subalgebra $\mathrm{Sg}(\mathbb{C} \cup \{a\})$ is as small as possible. Set

$$\mathbb{S} = \mathrm{Sg}_{\mathbb{A}^2}\left\{\begin{bmatrix} a \\ \mathbb{C} \end{bmatrix}, \begin{bmatrix} \mathbb{C} \\ a \end{bmatrix}\right\}.$$

By the choice of b, there exist $a', a'' \in \mathbb{A}$ such that $(a', a'') \in \mathbb{S}$ and $\pi(a') = \pi(a'') = b$. By the choice of a, we have $a \in \mathrm{Sg}(\mathbb{C} \cup \{a''\})$. Thus we have

$$\mathrm{Sg}\{a, a'\} + \mathbb{S} \supseteq \mathrm{Sg}(\mathbb{C} \cup \{a''\}) \supseteq \{a\},$$

so there is some $a''' \in \mathrm{Sg}\{a, a'\}$ with $(a''', a) \in \mathbb{S}$, and by idempotence we have $\pi(a''') = b$, so $a \in \mathrm{Sg}(\mathbb{C} \cup \{a'''\})$. By idempotence we have $\{a\} \le \mathbb{A}$, so

$$\{a\} - \mathbb{S} \supseteq \mathrm{Sg}(\mathbb{C} \cup \{a'''\}) \supseteq \{a\},$$

so $(a,a) \in \mathbb{S}$, which contradicts the assumption $\mathbb{C} \lhd_Z \mathbb{A}$. □

Proposition 3.10.15. *If $\mathbb{C} \lhd_Z \mathbb{B} \lhd_Z \mathbb{A}$, then $\mathbb{C} \lhd_Z \mathbb{A}$. As a consequence, if $\mathbb{C}_i \lhd_Z \mathbb{B}_i \le \mathbb{A}$, then $\mathbb{C}_1 \cap \mathbb{C}_2 \lhd_Z \mathbb{B}_1 \cap \mathbb{B}_2$.*

Proof. Suppose there is some $a \in \mathbb{A}$ such that $(a,a) \in \mathrm{Sg}(\mathbb{C} \times \{a\} \cup \{a\} \times \mathbb{C})$. Since $\mathbb{C} \le \mathbb{B}$ and $\mathbb{B} \lhd_Z \mathbb{A}$, we must have $a \in \mathbb{B}$. Then since $\mathbb{C} \lhd_Z \mathbb{B}$, we must have $a \in \mathbb{C}$. Thus $\mathbb{C} \lhd_Z \mathbb{A}$.

For the second statement, note that $\mathbb{C}_2 \lhd_Z \mathbb{B}_2$ implies $\mathbb{C}_1 \cap \mathbb{C}_2 \lhd_Z \mathbb{C}_1 \cap \mathbb{B}_2$ and $\mathbb{C}_1 \lhd_Z \mathbb{B}_1$ implies $\mathbb{C}_1 \cap \mathbb{B}_2 \lhd_Z \mathbb{B}_1 \cap \mathbb{B}_2$. □

Proposition 3.10.16. *If $\mathbb{C}_1 \lhd_Z \mathbb{A}_1$, then $\mathbb{C}_1 \times \mathbb{A}_2 \lhd_Z \mathbb{A}_1 \times \mathbb{A}_2$.*

Putting these three results together, we see that central absorption is a good absorption concept.

Proposition 3.10.17. *The absorption concept $\lhd_Z$, restricted to finite idempotent algebras, is compatible with pp-formulas, is transitively closed, and transfers connectivity.*

Remark 3.10.1. Annoyingly, binary absorption fails to be transitively closed or compatible with pp-formulas (the intersection of two binary absorbing subalgebras might not be binary absorbing). However, if we restrict ourselves to finite idempotent algebras which are *prepared*, that is, such that $(b,b) \in \mathrm{Sg}\{(a,b),(b,a)\}$ implies that $\{a,b\}$ is a semilattice subalgebra with absorbing element b, then binary absorption becomes compatible with pp-formulas and transitively closed (see Proposition 3.2.22).

In some cases central absorption implies binary absorption. To describe a criterion for when this happens, we will exploit partial semilattice operations.

Proposition 3.10.18. *Suppose that $\mathbb{C} \lhd_Z \mathbb{A}$ and that s is any partial semilattice operation. Then $s(\mathbb{C}, \mathbb{A}) \subseteq \mathbb{C}$.*

Proof. Suppose $c \in \mathbb{C}$ and $a \in \mathbb{A}$, and let $b = s(c,a)$. Then $s(c,b) = s(b,c) = b$ by the defining property of partial semilattice operations, so $(b,b) \in \mathrm{Sg}(\{b\} \times \mathbb{C} \cup \mathbb{C} \times \{b\})$. Thus by the definition of central absorption, we have $b \in \mathbb{C}$, that is, $s(c,a) \in \mathbb{C}$. □

Proposition 3.10.19. *Suppose that $\mathbb{C} \lhd_Z \mathbb{A}$ in a finite idempotent algebra $\mathbb{A}$, or just that $s(\mathbb{C}, \mathbb{A}) \subseteq \mathbb{C}$ for all partial semilattice terms $s \in \mathrm{Clo}(\mathbb{A})$. Then the following are equivalent:*

(a) $\mathbb{C}$ binary absorbs $\mathbb{A}$,

(b) for all $a \in \mathbb{A}\backslash\mathbb{C}$ and all $c \in \mathbb{C}$, the subalgebra $\mathrm{Sg}\{a,c\}$ has a proper binary absorbing subalgebra,

(c) for all $a \in \mathbb{A}$ and all $c \in \mathbb{C}$, there is a sequence of elements $a = a_0, a_1, ..., a_n \in \mathrm{Sg}\{a,c\}$ with $a_n \in \mathbb{C}$ such that $(a_i, a_i) \in \mathrm{Sg}\{(a_{i-1}, a_i), (a_i, a_{i-1})\}$ for all i.

If $\mathbb{A}$ is prepared, then the third condition is equivalent to the assumption that for all a and for all $c \in \mathbb{C}$, the subalgebra $\mathrm{Sg}\{a,c\}$ contains a directed path from a to $\mathbb{C}$.

Proof. To see that (a) implies (b), note that $\mathbb{C} \lhd_{bin} \mathbb{A}$ implies that $\mathbb{C} \cap \mathrm{Sg}\{a,c\} \lhd_{bin} \mathrm{Sg}\{a,c\}$. To see that (b) implies (c), we induct on the size of $\mathrm{Sg}\{a,c\}$. Let $\mathbb{B}$ be a proper binary absorbing subalgebra of $\mathrm{Sg}\{a,c\}$, and let s be a partial semilattice term that witnesses this absorption (such an s exists by Proposition 3.2.17). Then for any $b \in \mathbb{B}$ we have $s(a,b) \in \mathbb{B}$, and if we take $a_1 = s(a,b)$ then $(a_1, a_1) \in \mathrm{Sg}\{(a, a_1), (a_1, a)\}$. Let $c_1 = s(c,b)$, then $c_1 \in \mathbb{B} \cap \mathbb{C}$, and so $\mathrm{Sg}\{a_1, c_1\} \subseteq \mathbb{B} < \mathrm{Sg}\{a,c\}$, so by the inductive hypothesis we can complete this to a sequence $a_1, ..., a_n \in \mathrm{Sg}\{a_1, c_1\}$ as in (c).

Now suppose that (c) holds. For each a, c with $c \in \mathbb{C}$, we will construct a binary function f_{ac} such that $f_{ac}(a,c) \in \mathbb{C}$ and $f_{ac}(\mathbb{C}, \mathbb{A}) \subseteq \mathbb{C}$. Then by cyclically composing the functions f_{ac} together, we can produce a binary term which absorbs $\mathbb{C}$. To construct f_{ac}, we pick a sequence of partial semilattice terms s_i such that $s_i(a_{i-1}, a_i) = a_i$ as well as binary terms t_i such that $t_i(a,c) = a_i$. We set

$$f_{ac}(x,y) \coloneqq s_n(\cdots s_2(s_1(x, t_1(x,y)), t_2(x,y)) \cdots, t_n(x,y)).$$

Then we have

$$f_{ac}(a,c) = s_n(\cdots s_2(s_1(a, a_1), a_2) \cdots, a_n) = a_n \in \mathbb{C}$$

and

$$f_{ac}(\mathbb{C}, \mathbb{A}) \subseteq s_n(\cdots s_2(s_1(\mathbb{C}, \mathbb{A}), \mathbb{A}) \cdots, \mathbb{A}) \subseteq \mathbb{C},$$

as required. □

3.11 Binary relations in Taylor algebras: the Absorption Theorem and the Loop Lemma

In this section we'll go over two of the main results from Barto and Kozik's paper [18] about absorption, known as the "absorption theorem" and the "loop lemma". The first of these results can be used to constrain the possible subdirect binary relations in simple absorption free algebras, while the second result makes no direct mention of absorption, but combines the theory of absorbing subalgebras with an elementary argument in the absorption free case to give a criterion for a subdirect binary relation to intersect the diagonal.

The loop lemma was originally introduced in order to settle a special case of the dichotomy problem, where the template structure $\mathbf{A}$ consists of a set together with a single subdirect binary relation (considered as a directed graph). As a bonus, the loop lemma easily implies the existence of a Taylor term of a special form, known as a *Siggers* operation, named after the first person to notice that such special Taylor terms exist in the finite case [174] (this result was quickly refined, after the initial discovery: see [113] for the paper which introduced the 4-ary operations which are now commonly known as Siggers operations).

Here is a strong form of the absorption theorem, stated in terms of Zhuk's centers.

Theorem 3.11.1 (Absorption Theorem [18]). *If $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$ is a subdirect binary relation and $\mathbb{A}, \mathbb{B}$ are finite idempotent Taylor algebras, and if $\mathbb{R}$ is* linked, *then either*

- $\mathbb{R} = \mathbb{A} \times \mathbb{B}$,
- *$\mathbb{A}$ has a proper binary absorbing or centrally absorbing subalgebra, or*
- *$\mathbb{B}$ has a proper subalgebra which is both binary absorbing and centrally absorbing.*

The absorption theorem can be viewed as a strengthening of Zhuk's results about central relations: as we will see, it actually follows from Zhuk's result by applying a few simple tricks. First we will bootstrap to the case of a subdirect relation $\mathbb{R}$ such that $\mathbb{R} \circ \mathbb{R}^- = \mathbb{A} \times \mathbb{A}$.

Lemma 3.11.2. *If $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$ is a subdirect binary relation and $\mathbb{A}, \mathbb{B}$ are finite idempotent Taylor algebras, and if $\mathbb{R} \circ \mathbb{R}^- = \mathbb{A} \times \mathbb{A}$, then either*

- *there is some $b \in \mathbb{B}$ such that $\mathbb{A} \times \{b\} \subseteq \mathbb{R}$,*
- *$\mathbb{A}$ has a proper binary absorbing subalgebra, or*
- *every element of $\mathbb{A}$ is contained in a proper centrally absorbing subalgebra.*

Proof. Suppose that there is no $b \in \mathbb{B}$ with $\mathbb{A} \times \{b\} \subseteq \mathbb{R}$ and that $\mathbb{A}$ is binary absorption free, and choose any $a \in \mathbb{A}$. Choose a sequence of subalgebras $\{a\} + \mathbb{R} = \mathbb{D}_0 \ge \mathbb{D}_1 \ge \cdots \ge \mathbb{D}_n$ such that each $\mathbb{D}_{i+1}$ is a proper binary absorbing subalgebra of $\mathbb{D}_i$ and such that $\mathbb{D}_n$ has no proper binary absorbing subalgebras. We will first show that $\mathbb{D}_n - \mathbb{R} = \mathbb{A}$, and then we will apply Zhuk's result (Corollary 3.10.8) to the binary relation $\mathbb{R} \cap (\mathbb{A} \times \mathbb{D}_n)$.

We will show that $\mathbb{D}_i - \mathbb{R} = \mathbb{A}$ for each i, by induction on i. Note that $\mathbb{D}_0 - \mathbb{R} = \{a\} + \mathbb{R} - \mathbb{R} = \mathbb{A}$ by the assumption $\mathbb{R} \circ \mathbb{R}^- = \mathbb{A} \times \mathbb{A}$. For the inductive step, note that since $\mathbb{D}_{i+1} \lhd_{bin} \mathbb{D}_i$, we have

$$\mathbb{D}_{i+1} - \mathbb{R} \lhd_{bin} \mathbb{D}_i - \mathbb{R} = \mathbb{A},$$

so we must have $\mathbb{D}_{i+1} - \mathbb{R} = \mathbb{A}$ since $\mathbb{A}$ has no proper binary absorbing subalgebra.

If we set $\mathbb{R}' = \mathbb{R} \cap (\mathbb{A} \times \mathbb{D}_n)$, then we have

$$\{a\} \times \mathbb{D}_n \subseteq \mathbb{R}' \le_{sd} \mathbb{A} \times \mathbb{D}_n.$$

Thus the left center $\mathbb{C}$ of $\mathbb{R}'$ contains a. Since $\mathbb{D}_n$ is binary absorption free, we see that $\mathbb{C}$ centrally absorbs $\mathbb{A}$ by Corollary 3.10.8. If $\mathbb{C} = \mathbb{A}$, then $\mathbb{A} \times \mathbb{D}_n \subseteq \mathbb{R}$, contradicting the assumption that there is no $b \in \mathbb{B}$ with $\mathbb{A} \times \{b\} \subseteq \mathbb{R}$. □

In the case where $\mathbb{A}$ has no proper binary absorbing or centrally absorbing subalgebra and $\mathbb{R}$ has a nontrivial right center, we will use the criterion developed in Proposition 3.10.19 to show that the right center of $\mathbb{R}$ must actually be a binary absorbing subalgebra of $\mathbb{B}$.

Lemma 3.11.3. *If $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$ is a subdirect binary relation and $\mathbb{A}, \mathbb{B}$ are finite idempotent Taylor algebras, and if there is some $b \in \mathbb{B}$ such that $\mathbb{A} \times \{b\} \subseteq \mathbb{R}$, then either*

- $\mathbb{R} = \mathbb{A} \times \mathbb{B}$,
- *$\mathbb{A}$ has a proper binary absorbing or centrally absorbing subalgebra, or*

- *the right center of $\mathbb{R}$ is a proper binary absorbing subalgebra of $\mathbb{B}$.*

Proof. Let $\mathbb{C} \le \mathbb{B}$ be the right center of $\mathbb{R}$. By Corollary 3.10.8, if $\mathbb{A}$ has no proper binary absorbing subalgebra then we have $\mathbb{C} \lhd_Z \mathbb{B}$. If $\mathbb{C}$ is not a binary absorbing subalgebra of $\mathbb{B}$, then by Proposition 3.10.19 there must be some $b \in \mathbb{B} \setminus \mathbb{C}$ and $c \in \mathbb{C}$ such that $\mathrm{Sg}\{b, c\}$ has no proper binary absorbing subalgebra.

Since $\mathbb{R}$ is subdirect, there is some $a \in \mathbb{A}$ such that $(a, b) \in \mathbb{R}$. Since c is in the right center of $\mathbb{R}$, we also have $\mathbb{A} \times \{c\} \subseteq \mathbb{R}$. Thus if we set $\mathbb{R}' = \mathbb{R} \cap (\mathbb{A} \times \mathrm{Sg}\{b, c\})$, then we have

$$\{a\} \times \mathrm{Sg}\{b, c\} \subseteq \mathbb{R}' \le_{sd} \mathbb{A} \times \mathrm{Sg}\{b, c\}$$

Since b is *not* in the right center of $\mathbb{R}$, the left center of $\mathbb{R}'$ is a proper subalgebra of $\mathbb{A}$. Then since $\mathrm{Sg}\{b, c\}$ has no proper binary absorbing subalgebra, Corollary 3.10.8 shows that the left center of $\mathbb{R}'$ is a proper centrally absorbing subalgebra of $\mathbb{A}$. □

Proof of the Absorption Theorem. Let $\mathbb{S} = \mathbb{R} \circ \mathbb{R}^- \le_{sd} \mathbb{A} \times \mathbb{A}$. If $\mathbb{S} = \mathbb{A} \times \mathbb{A}$, we may apply the lemmas to see that either $\mathbb{A}$ has a proper binary absorbing or centrally absorbing subalgebra, or that $\mathbb{B}$ has a proper subalgebra which is both binary absorbing and centrally absorbing. Otherwise, by the fact that $\mathbb{R}$ is linked and the finiteness of $\mathbb{A}$ there must be some minimal $k > 1$ such that $\mathbb{S}^{\circ k} = \mathbb{A} \times \mathbb{A}$. Then we can apply the lemmas to $\mathbb{S}^{\circ(k-1)}$ to see that $\mathbb{A}$ must have either a binary absorbing or centrally absorbing subalgebra. □

Corollary 3.11.4. *Let $\mathbb{A}, \mathbb{B}$ be finite idempotent Taylor algebras with no proper binary or centrally absorbing subalgebras. If $\mathbb{B}$ is simple, then every subdirect binary relation $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$ is either the full relation or the graph of a surjective homomorphism $\mathbb{A} \twoheadrightarrow \mathbb{B}$.*

Proof. Since $\mathbb{B}$ is simple, the linking congruence of $\mathbb{R}$ on $\mathbb{B}$ is either trivial or is full. If the linking congruence of $\mathbb{R}$ on $\mathbb{B}$ is trivial, then $\mathbb{R}$ must be the graph of a surjective homomorphism $\mathbb{A} \twoheadrightarrow \mathbb{B}$. Otherwise, $\mathbb{R}$ is linked, so we can apply the Absorption Theorem 3.11.1 to see that $\mathbb{R} = \mathbb{A} \times \mathbb{B}$. □

Next we switch our focus to subdirect relations $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{A}$. In this case, it is often appropriate to think of $\mathbb{R}$ as a digraph on the vertex set $\mathbb{A}$, and we can ask questions about whether $\mathbb{R}$ (viewed as a digraph) is weakly connected, strongly connected, whether it contains any loops, etc. To be precise, the associated digraph is the relational structure $\mathbf{R} = (A, R)$, where A is the underlying set of $\mathbb{A}$ and $R \subseteq A \times A$ is the underlying set of $\mathbb{R}$ (often I abuse notation and write $\mathbf{R} = (\mathbb{A}, \mathbb{R})$ instead of explicitly replacing $\mathbb{A}, \mathbb{R}$ with their underlying sets).

Remark 3.11.1. Note that if $\mathbb{R} \le \mathbb{A} \times \mathbb{A}$ and $\mathbb{S} \le \mathbb{B} \times \mathbb{B}$ are subpowers of $\mathbb{A}, \mathbb{B}$, then a homomorphism $\mathbb{R} \to \mathbb{S}$ and a homomorphism $\mathbf{R} \to \mathbf{S}$ of the associated digraphs $\mathbf{R} = (\mathbb{A}, \mathbb{R}), \mathbf{S} = (\mathbb{B}, \mathbb{S})$ are completely different things! The first is a homomorphism of algebraic structures, and doesn't depend on how $\mathbb{R}, \mathbb{S}$ are represented as collections of ordered pairs of elements in $\mathbb{A}$ or $\mathbb{B}$ (but does depend on how the algebraic operations behave). The second is a digraph homomorphism, which ignores the algebraic structure, and is completely determined by a map $A \to B$ of the underlying sets of $\mathbb{A}, \mathbb{B}$ which is compatible with the digraph structures $\mathbb{R}, \mathbb{S}$.

In the context of digraphs, the case of a *subdirect* relation $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{A}$ is actually rather special. The assumption $\pi_1(\mathbb{R}) = \mathbb{A}$ means that every vertex of the digraph $\mathbf{R}$ has outdegree at least one, and the assumption $\pi_2(\mathbb{R}) = \mathbb{A}$ means that every vertex of $\mathbb{R}$ has indegree at least one.

Definition 3.11.5. A digraph $\mathbf{D} = (V, E)$ is called *smooth* if every vertex of $\mathbf{D}$ has indegree at least one and outdegree at least one. Note that this is equivalent to the relation $E \subseteq V \times V$ being subdirect.

If a digraph is not smooth, it is often desirable to find a smooth digraph within it. The natural thing to do is to simply prune all of the vertice with indegree 0 or outdegree 0. Unfortunately, after this pruning step we may find ourselves with more vertices that need to be pruned, and so on - possibly ending up with no vertices at all! For instance, this actually occurs if our initial digraph is a finite directed path. Additionally, it may not be clear that these pruning operations are compatible with the algebraic structures which we started with. Luckily, there is a standard way to describe the result of this pruning process via a primitive positive formula, as well as a simple criterion for when the pruned digraph will be nonempty.

Proposition 3.11.6. *If $\mathbf{D} = (V, E)$ is a digraph, then the largest smooth digraph $\mathbf{D}_{sm}$ which is contained in $\mathbf{D}$ is exactly the set of vertices v of $\mathbf{D}$ such that there exists a bi-infinite directed walk through v. If $\mathbf{D}$ is finite, with n vertices, then the vertex set of $\mathbf{D}_{sm}$ may be defined by the pp-formula*

$$v \in \mathbf{D}_{sm} \iff \exists v_{-n}, ..., v_n \ (v_0 = v) \wedge \bigwedge_{-n \le i < n} (v_i, v_{i+1}) \in E.$$

The set $\mathbf{D}_{sm}$ will be nonempty iff $\mathbf{D}$ contains a directed cycle (or a bi-infinite directed path, in the infinite case).

Definition 3.11.7. If $\mathbf{D}$ is a digraph and $\mathbf{D}_{sm}$ is defined as in the previous proposition, then we call $\mathbf{D}_{sm}$ the *smooth part* of the digraph $\mathbf{D}$.

Note that the smooth part of a digraph may contain vertices which are not themselves part of any directed cycles: it may also contain intermediate vertices along directed paths connecting two directed cycles. In fact, the smooth part of a digraph enjoys the following convexity property.

Proposition 3.11.8. *If $\mathbf{D}$ is a digraph and a, b are in the smooth part of $\mathbf{D}$, then every vertex of $\mathbf{D}$ which can be found along any directed path from a to b is also contained in the smooth part of $\mathbf{D}$.*

One reason for introducing this terminology is that it lets us easily state results such as the following one.

Proposition 3.11.9. *If $\mathbb{S} \lhd \mathbb{R}$ and $\mathbb{R}, \mathbb{S} \le \mathbb{A} \times \mathbb{A}$ correspond to digraphs $\mathbf{R}, \mathbf{S}$ with vertex set $\mathbb{A}$, and if $\mathbf{R}$ is smooth, then the smooth part $\mathbf{S}_{sm}$ of the digraph $\mathbf{S}$ has vertex set equal to an absorbing subalgebra of $\mathbb{A}$, which will be nonempty as long as $\mathbf{S}$ contains some directed cycle.*

Of course, we will often abuse notation a little further, and talk about the "smooth part of the digraph $\mathbb{S}$" as long as this does not seem likely to cause confusion. It will be convenient to have the following criterion for the existence of a directed cycle contained in a subalgebra $\mathbb{B} \le \mathbb{A}$.

Proposition 3.11.10. *If $\mathbb{R} \le \mathbb{A} \times \mathbb{A}$, and if $\mathbb{B} \le \mathbb{A}$ is finite and satisfies either $\mathbb{B} \subseteq \mathbb{B} + \mathbb{R}$ or $\mathbb{B} \subseteq \mathbb{B} - \mathbb{R}$, then the restriction $\mathbb{R} \cap (\mathbb{B} \times \mathbb{B})$ of $\mathbb{R}$ to $\mathbb{B}$ has nonempty smooth part.*

Proof. Suppose that $\mathbb{B} \subseteq \mathbb{B} - \mathbb{R}$. Then every vertex in $\mathbb{B}$ has an edge leaving it which lands in $\mathbb{B}$, so we can find an arbitrarily long directed walk of $\mathbb{R}$ which is entirely contained in $\mathbb{B}$. Since $\mathbb{B}$ is finite, this implies that there is some directed cycle which is entirely contained in $\mathbb{B}$. □

As a warmup to the full loop lemma, we will first focus on the special case where the relation $\mathbb{R}$ is linked. This special case is usually enough to handle most applications.

Lemma 3.11.11 (Loop Lemma, linked case)**.** *Suppose that $\mathbb{A}$ is a finite Taylor algebra and that $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{A}$ is a linked subdirect relation. Then $\mathbb{R}$ contains a loop, that is, $\mathbb{R} \cap \Delta_{\mathbb{A}} \ne \emptyset$.*

Proof. We prove this by induction on $|\mathbb{A}|$. We may assume that $\mathbb{A}$ is idempotent without loss of generality. If $\mathbb{R} \ne \mathbb{A} \times \mathbb{A}$, then $\mathbb{A}$ must have some proper absorbing subalgebra $\mathbb{B} \lhd \mathbb{A}$ by the Absorption Theorem 3.11.1. If we define a sequence of absorbing subalgebras $\mathbb{B} = \mathbb{B}_0, \mathbb{B}_1, ...$ of $\mathbb{A}$ by $\mathbb{B}_{i+1} = \mathbb{B}_i + \mathbb{R}$ for i even and $\mathbb{B}_{i+1} = \mathbb{B}_i - \mathbb{R}$ for i odd, then since $\mathbb{R}$ is linked and $\mathbb{A}$ is finite there must be some i such that $\mathbb{B}_{i+1} = \mathbb{A}$ but $\mathbb{B}_i \ne \mathbb{A}$. Since this $\mathbb{B}_i$ satisfies $\mathbb{B}_i \subseteq \mathbb{A} = \mathbb{B}_{i+1}$, we see that either $\mathbb{B}_i \subseteq \mathbb{B}_i + \mathbb{R}$ or $\mathbb{B}_i \subseteq \mathbb{B}_i - \mathbb{R}$, so by the previous proposition the relation $\mathbb{R} \cap (\mathbb{B}_i \times \mathbb{B}_i)$ has a nonempty smooth part $\mathbb{B}_{sm} \lhd \mathbb{B}_i$, with edge set $\mathbb{S} = \mathbb{R} \cap (\mathbb{B}_{sm} \times \mathbb{B}_{sm})$.

Since $\mathbb{B}_{sm} \lhd \mathbb{A}$, we have $\mathbb{S} \lhd \mathbb{R}$. Since $\mathbb{S}$ is smooth, we can transfer the linkedness of $\mathbb{R}$ to $\mathbb{S}$ using Theorem 3.7.12, to see that $\mathbb{S}$ must also be linked. By the inductive hypothesis applied to $\mathbb{B}_{sm}$, we see that $\mathbb{S}$ must contain a loop, and this loop will also be contained in $\mathbb{R}$ since $\mathbb{S} \le \mathbb{R}$. □

To state the full loop lemma, we need another digraph concept.

Definition 3.11.12. The *algebraic length* of a weakly connected digraph $\mathbf{D}$ is the least common multiple of all integers k such that there is a digraph homomorphism from $\mathbf{D}$ to a directed cycle of length k.

Proposition 3.11.13. *The algebraic length of a weakly connected digraph $\mathbf{D} = (V, E)$ is the greatest common divisor of all integers k such that there exist $v \in V$ and $k_1, k_2, ..., k_m \in \mathbb{N}$ such that*

$$v \in \{v\} + k_1E - k_2E + \cdots \pm k_mE$$

and $k = k_1 - k_2 + \cdots \pm k_m$.

Furthermore, there exists a digraph homomorphism from $\mathbf{D}$ to a directed cycle $\mathbf{C}$ iff the algebraic length of $\mathbf{D}$ is a multiple of the length of the cycle $\mathbf{C}$.

Proposition 3.11.14. *If $\mathbf{D} = (V, E)$ is a smooth, weakly connected digraph of algebraic length k, then the digraph $\mathbf{D}^{\circ m} = (V, E^{\circ m})$ has $\gcd(k, m)$ weakly connected components, and each weakly connected component of $\mathbf{D}^{\circ m}$ has algebraic length $\frac{k}{\gcd(k,m)}$.*

Proposition 3.11.15. *If $\mathbf{D} = (V, E)$ is smooth and weakly connected, then $\mathbf{D}$ has algebraic length 1 if and only if there is some $m \ge 0$ such that the relation $E^{\circ m}$ is linked.*

Corollary 3.11.16. *If $\mathbf{D} = (V, E)$ is smooth and has a weakly connected component $C \subseteq V$ of algebraic length 1, and if $v \in C$, then the set C can be defined by a primitive positive formula using the singleton unary relation $\{v\}$ and the binary relation E.*

With these preliminaries out of the way, we can finally state the full version of the loop lemma for finite Taylor algebras.

Theorem 3.11.17 (Loop Lemma [18])**.** *If $\mathbb{A}$ is a finite Taylor algebra and $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{A}$ corresponds to a smooth digraph $\mathbf{R} = (\mathbb{A}, \mathbb{R})$ which has a weakly connected component of algebraic length 1, then $\mathbb{R}$ has a loop, i.e. $\mathbb{R} \cap \Delta_{\mathbb{A}} \ne \emptyset$.*

Proof. We prove this by induction on $|\mathbb{A}|$. We may assume that $\mathbb{A}$ is idempotent without loss of generality. We may also assume that $\mathbf{R}$ is weakly connected by restricting to a weakly connected component of algebraic length 1 (which forms a subalgebra of $\mathbb{A}$ by the results above). Let m be minimal such that $\mathbb{R}^{\circ m}$ is linked. We split into cases based on whether $\mathbb{R}^{\circ m} = \mathbb{A} \times \mathbb{A}$ or not.

If $\mathbb{R}^{\circ m} \neq \mathbb{A} \times \mathbb{A}$, then by the Absorption Theorem 3.11.1 we see that $\mathbb{A}$ must have some proper absorbing subalgebra. By a similar argument to the linked case (Lemma 3.11.11), we see that there is some proper absorbing $\mathbb{B} \lhd \mathbb{A}$ such that $\mathbb{S} = \mathbb{R} \cap (\mathbb{B} \times \mathbb{B})$ is subdirect in $\mathbb{B} \times \mathbb{B}$. Then since $\mathbb{S}^{\circ m} \lhd \mathbb{R}^{\circ m}$, we can apply Theorem 3.7.12 to see that $\mathbb{S}^{\circ m}$ is linked, so the smooth digraph $\mathbf{S} = (\mathbb{A}, \mathbb{S})$ has algebraic length 1 and $\mathbb{S}$ has a loop by the inductive hypothesis.

If $\mathbb{R}^{\circ m} = \mathbb{A} \times \mathbb{A}$, then we let $\mathbb{B}$ be any linked component of $\mathbb{R}^{\circ(m-1)}$ on the first coordinate (note that $\mathbb{R}^{\circ(m-1)}$ is not linked by the choice of m, so $\mathbb{B}$ is a proper subalgebra of $\mathbb{A}$). First we will show that $\mathbb{B} \subseteq \mathbb{B} - \mathbb{R}$. To see this, let $b \in \mathbb{B}$ be arbitrary, pick any $c \in b + \mathbb{R}^{\circ(m-1)}$. Then since $\mathbb{R}^{\circ m} = \mathbb{A} \times \mathbb{A}$, we have

$$c \in b + \mathbb{R}^{\circ m},$$

and if we let d be the first element along a directed path of length m from b to c, then we have

$$d \in (b + \mathbb{R}) \cap (c - \mathbb{R}^{\circ(m-1)}) \subseteq (b + \mathbb{R}) \cap (b + \mathbb{R}^{\circ(m-1)} - \mathbb{R}^{\circ(m-1)}) \subseteq (b + \mathbb{R}) \cap \mathbb{B}.$$

Thus $b \in \mathbb{B} - \mathbb{R}$, and since b was an arbitrary element of $\mathbb{B}$ we see that $\mathbb{B} \subseteq \mathbb{B} - \mathbb{R}$. Thus the smooth part $\mathbb{B}_{sm}$ of $\mathbb{R} \cap (\mathbb{B} \times \mathbb{B})$ is nonempty.

To finish, we just need to check that the smooth digraph corresponding to $\mathbb{S} = \mathbb{R} \cap (\mathbb{B}_{sm} \times \mathbb{B}_{sm})$ has algebraic length 1. For this, we pick any $(b, c) \in \mathbb{S}^{\circ(m-1)}$, and pick any directed path $b = b_1, ..., b_m = c$ of length $m - 1$ with all $b_i \in \mathbb{B}_{sm}$. Since $(b, c) \in \mathbb{R}^{\circ m}$, we may also find directed path $b = c_0, ..., c_m = c$ from b to c of length m in $\mathbf{R}$. We will show that every c_i along this path is actually in $\mathbb{B}_{sm}$. For this, we just note that b_i, c_i are in the same linked component of $\mathbb{R}^{\circ(m-1)}$ for each $i \geq 1$ (since b_i and c_i can both reach c in exactly $m - i$ steps), so each c_i is at least in $\mathbb{B}$, and then since each c_i is along a directed path between two vertices of $\mathbb{B}_{sm}$ we see that each c_i belongs to the smooth part $\mathbb{B}_{sm}$ as well. Thus $b \in b + \mathbb{S}^{\circ(m-1)} - \mathbb{S}^{\circ m}$, so $\mathbb{S}$ has algebraic length 1 and we may apply the inductive hypothesis to see that $\mathbb{S}$ contains a loop. □

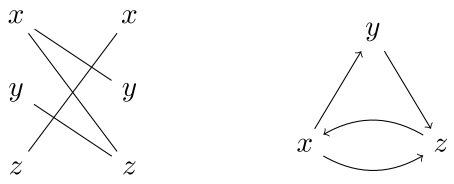
Corollary 3.11.18 (Siggers term [174], [113])**.** *If $\mathbb{A}$ is a finite Taylor algebra, then $\mathbb{A}$ has a 4-ary idempotent term t which satisfies the identity*

$$t(x, x, y, z) \approx t(y, z, z, x).$$

Proof. Assume without loss of generality that $\mathbb{A}$ is idempotent. Let $\mathbb{F} = \mathcal{F}_{\mathbb{A}}(x, y, z)$ be the free algebra on three generators in the variety generated by $\mathbb{A}$. Let $\mathbb{R}$ be the binary relation

$$\mathbb{R} = \mathrm{Sg}_{\mathbb{F}^2}\left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} y \\ z \end{bmatrix}, \begin{bmatrix} z \\ x \end{bmatrix} \right\}.$$

Then $\mathbb{R}$ is clearly subdirect, and the generating set of $\mathbb{R}$ forms the binary relation on $\{x, y, z\}$ pictured below, as both a bipartite graph and as a digraph.



This digraph is smooth, strongly connected (in fact, it has $x + \mathbb{R}^{\circ 3} = \mathbb{F}$ and $\mathbb{R}^{\circ 5} = \mathbb{F} \times \mathbb{F}$), and has algebraic length 1 (since $x \in x + \mathbb{R}^{\circ 2} - \mathbb{R}^{\circ 1}$), so we can apply the Loop Lemma to see that $\mathbb{R}$ contains some loop (f, f) (we are using here the fact that $\mathbb{F} \le \mathbb{A}^{\mathbb{A}^3}$ is finite and Taylor). Then since $(f, f) \in \mathbb{R}$, there must be some 4-ary term t such that

$$t\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} y \\ z \end{bmatrix}, \begin{bmatrix} z \\ x \end{bmatrix}\right) = \begin{bmatrix} f \\ f \end{bmatrix},$$

and this t then satisfies the identity

$$t(x, x, y, z) = f = t(y, z, z, x). \qquad \square$$

Remark 3.11.2. Suppose that t is a Siggers term, i.e. that $t(x, x, y, z) \approx t(y, z, z, x)$. If we substitute $y = z$ into the Siggers identity and rename variables, we see that

$$t(y, y, x, x) \approx t(x, x, x, y),$$

and if we substitute $x = y$ into the Siggers identity and rename variables, then we get

$$t(x, x, x, y) \approx t(x, y, y, x).$$

Thus there is some binary term $f(x, y)$ such that

$$t\left(\begin{bmatrix} y & y & x & x \\ x & y & y & x \\ x & x & x & y \end{bmatrix}\right) \approx \begin{bmatrix} f(x,y) \\ f(x,y) \\ f(x,y) \end{bmatrix}.$$

If we reorder the first and second inputs to t, the left hand side exactly becomes the left hand side of the equation for the 3-edge term. If $f(x, y)$ was equal to x, then t would become a 3-edge term (up to reordering inputs).

If $f(x, y)$ was instead equal to y, then $p(x, y, z) = t(x, x, y, z)$ would become a Mal'cev term, which is even better than a 3-edge term. However, if we allow for the possibility of semilattice subalgebras, then $f(x, y)$ must act as the semilattice operation on any two-element semilattice subalgebra, and of course in this case there couldn't possibly be any cube term of any arity. For this reason, the system of equations satisfied by t above are often summarized by calling such a t a "weak 3-edge term".

The fact that a Siggers term looks suspiciously similar to a 3-edge term is more than a coincidence: later we will see that every finite Taylor algebra either has a 3-edge term or has some pair of elements $a \ne b$ such that $(b, b) \in \mathrm{Sg}\{(a, b), (b, a)\}$.

3.12 Finite abelian Taylor algebras are affine, and Zhuk's four cases

First we recall the definition of an abelian algebra.

Definition 3.12.1. An algebraic structure $\mathbb{A}$ is called *abelian* if there is a congruence Θ on $\mathbb{A} \times \mathbb{A}$ such that the diagonal $\Delta_{\mathbb{A}} = \{(a, a) \mid a \in \mathbb{A}\}$ is one of the congruence classes of Θ.

The reader might be sceptical about how often such a congruence Θ actually shows up. After all, such a congruence is most naturally viewed as a 4-ary relation on $\mathbb{A}$, and for the most part we have only been able to prove interesting structural results about binary relations so far. The next result illustrates the most common situation which leads to the existence of such a congruence.

Proposition 3.12.2. *Suppose that $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{A} \times \mathbb{A}$ has the property that for each $a \in \mathbb{A}$, and for each permutation (i,j,k) of $(1,2,3)$, the binary relation*

$$\pi_{ij}(x \in \mathbb{R} \wedge x_k = a)$$

is the graph of an automorphism of $\mathbb{A}$. Then $\mathbb{A}$ is abelian.

Proof. Note that the assumption on $\mathbb{R}$ can be rephrased as saying that if we fix any pair of coordinates of a tuple in $\mathbb{R}$, then the last coordinate is uniquely determined. Therefore $\mathbb{R}$ can be viewed as the graph of a homomorphism

$$m : \mathbb{A} \times \mathbb{A} \twoheadrightarrow \mathbb{A}$$

such that the preimage $m^{-1}(a)$ is the graph of an automorphism of $\mathbb{A}$ for every $a \in \mathbb{A}$ (equivalently, m is the multiplication of some quasigroup which commutes with the operations of $\mathbb{A}$). In other words, every congruence class of the kernel $\ker m \in \operatorname{Con}(\mathbb{A} \times \mathbb{A})$ is the graph of an automorphism of $\mathbb{A}$. Twisting $\ker m$ by one of these automorphisms yields a congruence $\Theta \in \operatorname{Con}(\mathbb{A} \times \mathbb{A})$ such that one of its congruence classes is the graph of the identity permutation of $\mathbb{A}$. □

The proof we give in this section - following [22] - of the fact that finite abelian Taylor algebras are affine breaks into three steps:

- every finite abelian algebra is (hereditarily) absorption free,
- every finite, idempotent, Taylor, hereditarily absorption free algebra is Mal'cev, and
- every abelian Mal'cev algebra is affine.

We have already completed the third step in Section 1.9, Theorem 1.9.23. We will complete the remaining steps in reverse order as well.

Definition 3.12.3. We say that an algebra $\mathbb{A}$ is *hereditarily absorption free* if every subalgebra of $\mathbb{A}$ is absorption free, that is, if $\mathbb{C} \lhd \mathbb{B} \le \mathbb{A}$ implies that $\mathbb{C} = \mathbb{B}$ or $\mathbb{C} = \emptyset$.

Proposition 3.12.4. *Suppose $\mathbb{A}, \mathbb{B}$ are idempotent and hereditarily absorption free. Then $\mathbb{A} \times \mathbb{B}$ is also hereditarily absorption free.*

Proof. Suppose that $\mathbb{S} \lhd \mathbb{R} \le \mathbb{A} \times \mathbb{B}$, with $\mathbb{S} \ne \emptyset$. Then since $\pi_1(\mathbb{S}) \lhd \pi_1(\mathbb{R}) \le \mathbb{A}$ and $\mathbb{A}$ is hereditarily absorption free, we see that $\pi_1(\mathbb{S}) = \pi_1(\mathbb{R})$. Thus for every $a \in \pi_1(\mathbb{R})$ we have $a + \mathbb{S} \ne \emptyset$, and since $\mathbb{A}$ is idempotent, we have

$$a + \mathbb{S} \lhd a + \mathbb{R} \le \mathbb{B}.$$

Then since $\mathbb{B}$ is hereditarily absorption free, we see that $a + \mathbb{S} = a + \mathbb{R}$. Since a was an arbitrary element of $\pi_1(\mathbb{R})$, we have $\mathbb{S} = \mathbb{R}$. □

Theorem 3.12.5 (HAF implies Mal'cev [22])**.** *If $\mathbb{A}$ is finite, idempotent, Taylor, and hereditarily absorption free, then $\mathbb{A}$ is Mal'cev.*

Proof. By repeatedly applying the previous proposition, we see that the free algebra on two generators $\mathbb{F} = \mathcal{F}_{\mathbb{A}}(x, y) \le \mathbb{A}^{\mathbb{A}^2}$ is absorption free. Consider the binary relation $\mathbb{R} \le_{sd} \mathbb{F} \times \mathbb{F}$ defined by

$$\mathbb{R} = \mathrm{Sg}_{\mathbb{F}^2}\left\{\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix}\right\}.$$

Then $x + \mathbb{R} \supseteq \mathrm{Sg}_{\mathbb{F}}\{x, y\} = \mathbb{F}$, so x is contained in the left center of $\mathbb{R}$. Thus by the Absorption Theorem 3.11.1 (or just Zhuk's result Corollary 3.10.8) we must have $\mathbb{R} = \mathbb{F} \times \mathbb{F}$, and in particular $(y, y) \in \mathbb{R}$. Thus there is some ternary term p such that

$$p\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \end{bmatrix}\right) = \begin{bmatrix} y \\ y \end{bmatrix}.$$

□

To finish the proof that finite abelian Taylor algebras are affine, we just need to check that every abelian algebra is absorption free. Note that every subalgebra of an abelian algebra is also abelian, so this will imply that abelian algebras are *hereditarily* absorption free as well. Additionally, every reduct of an abelian algebra is also abelian (since taking reducts can only increase the congruence lattice), so we see that the idempotent reduct of a finite abelian Taylor algebra will also be hereditarily absorption free, allowing us to apply the previous result to it.

It is not so easy to see how to use abelianness to rule out absorption. As a warmup, we will show that abelian algebras can't have any near-unanimity terms: this will give us the hint about how to show that finite abelian algebras are absorption free.

Proposition 3.12.6. *If an algebra $\mathbb{A}$ is abelian and has at least two elements, then $\mathbb{A}$ does not have a near-unanimity term.*

Proof. Let $\Theta \in \mathrm{Con}(\mathbb{A} \times \mathbb{A})$ be a congruence with the diagonal $\Delta_{\mathbb{A}}$ as a congruence class. Suppose for contradiction that t is a near-unanimity term of minimal arity n, and note that n must be at least 3 since $\mathbb{A}$ has at least two elements. Let a, b be any pair of elements of $\mathbb{A}$. Then we have

$$t\left(\begin{bmatrix} a & b & b & \cdots & b \\ b & b & b & \cdots & b \end{bmatrix}\right) = \begin{bmatrix} b \\ b \end{bmatrix} \in \Delta_{\mathbb{A}}.$$

Since the second column of inputs to t is $(b, b) \in \Delta_{\mathbb{A}}$, we can replace it with any other element of $\Delta_{\mathbb{A}}$ without changing the the result modulo Θ. Thus we have

$$t\left(\begin{bmatrix} a & a & b & \cdots & b \\ b & a & b & \cdots & b \end{bmatrix}\right) \equiv_{\Theta} \begin{bmatrix} b \\ b \end{bmatrix} \in \Delta_{\mathbb{A}}.$$

Since $t(b, a, b, ..., b) = b$, we see that we must have

$$t\left(\begin{bmatrix} a & a & b & \cdots & b \\ b & a & b & \cdots & b \end{bmatrix}\right) = \begin{bmatrix} b \\ b \end{bmatrix}.$$

Since a, b were arbitrary elements of $\mathbb{A}$, we see that

$$t(y, y, x, ..., x) \approx x,$$

so the term $t(x, x, y_2, ..., y_{n-1})$ is a near-unanimity term of arity $n - 1$, contradicting the choice of t. □

In order to mimic this argument to rule out absorption, we will need to assume finiteness of $\mathbb{A}$ and apply an iteration argument.

Theorem 3.12.7 (Abelian implies HAF [22])**.** *If a finite algebra $\mathbb{A}$ is abelian, then it is absorption free.*

Proof. Let $\Theta \in \mathrm{Con}(\mathbb{A} \times \mathbb{A})$ be a congruence with the diagonal $\Delta_{\mathbb{A}}$ as a congruence class. Suppose for contradiction that $\mathbb{B} \lhd \mathbb{A}$ is nonempty and proper, and let t be a term of minimal arity n among those which absorb $\mathbb{B}$. Note that $n \geq 2$ since $\mathbb{B}$ is a proper subalgebra of $\mathbb{A}$. Now iterate t on its first argument, i.e. define a sequence of terms t_i with $t_1 = t$ and

$$t_{i+1}(x, y_1, ..., y_{n-1}) \coloneqq t(t_i(x, y_1, ..., y_{n-1}), y_1, ..., y_{n-1}).$$

By induction on i, each t_i absorbs $\mathbb{B}$. Since $\mathbb{A}$ is finite, there is some i such that $t_i = t_{2i}$, call this t_i t_∞. Then we have

$$t_\infty(t_\infty(x, y_1, ..., y_{n-1}), y_1, ..., y_{n-1}) \approx t_\infty(x, y_1, ..., y_{n-1}),$$

and t_∞ absorbs $\mathbb{B}$.

Now we argue as in the near-unanimity case: let $a \in \mathbb{A}$ and $b_1, b_2, ..., b_{n-1} \in \mathbb{B}$, and set

$$b = t_\infty(a, b_1, b_2, ..., b_{n-1}) \in \mathbb{B}.$$

Then we have

$$t_\infty\left(\begin{bmatrix} a & b_1 & b_2 & \cdots & b_{n-1} \\ b & b_1 & b_2 & \cdots & b_{n-1} \end{bmatrix}\right) = \begin{bmatrix} b \\ b \end{bmatrix} \in \Delta_{\mathbb{A}},$$

so since $(b_1, b_1) \equiv_\Theta (a, a)$, we have

$$t_\infty\left(\begin{bmatrix} a & a & b_2 & \cdots & b_{n-1} \\ b & a & b_2 & \cdots & b_{n-1} \end{bmatrix}\right) \equiv_\Theta \begin{bmatrix} b \\ b \end{bmatrix} \in \Delta_{\mathbb{A}}.$$

Thus since $\Delta_{\mathbb{A}}$ is a congruence class of Θ and $\mathbb{B}$ absorbs $\mathbb{A}$ with respect to t_∞, we have

$$t_\infty(a, a, b_2, ..., b_{n-1}) = t_\infty(b, a, b_2, ..., b_{n-1}) \in \mathbb{B}.$$

Since a was an arbitrary element of $\mathbb{A}$ and $b_2, ..., b_{n-1}$ were arbitrary elements of $\mathbb{B}$, we see that the term

$$t_\infty(x, x, y_2, ..., y_{n-1})$$

absorbs $\mathbb{B}$ and has arity $n - 1$, contradicting the choice of t. □

Now we can put all the pieces together and get our main result.

Theorem 3.12.8 (Fundamental Theorem of Abelian Algebras, finite Taylor case [95], [22], [177], [193])**.** *If $\mathbb{A}$ is a finite abelian Taylor algebra, then $\mathbb{A}$ is affine.*

Proof. Let $\mathbb{A}^{id}$ be the idempotent reduct of $\mathbb{A}$, note that $\mathbb{A}^{id}$ is still abelian and Taylor (since Taylor terms are idempotent by definition). Then every subalgebra of $\mathbb{A}^{id}$ is also abelian, so by Theorem 3.12.7 $\mathbb{A}^{id}$ is hereditarily absorption free. Since $\mathbb{A}^{id}$ is finite, idempotent, Taylor, and hereditarily absorption free it has a Mal'cev term p by Theorem 3.12.5. Then p is also a Mal'cev term of $\mathbb{A}$, so we can apply Theorem 1.9.23 to see that $\mathbb{A}$ is affine. □

Remark 3.12.1. It is not hard to generalize Theorem 3.12.7 to show that if a finite algebra $\mathbb{A}$ is solvable, then $\mathbb{A}$ is hereditarily absorption free (this follows from the fact that every solvable idempotent algebra $\mathbb{A}$ has a congruence θ such that $\mathbb{A}/\theta$ is abelian and every congruence class of θ is solvable). Thus finite solvable Taylor algebras are also Mal'cev by Theorem 3.12.5.

Now we can apply the fundamental theorem of abelian algebras to further constrain relations on absorption free algebras.

Theorem 3.12.9 (Zhuk [190])**.** *Suppose that $\mathbb{A}$ is finite, simple, idempotent, Taylor, absorption free, and* not *affine. Then every subdirect relation $\mathbb{R} \le_{sd} \mathbb{A}^n$ is the intersection of its binary projections, each of which is either a full relation or the graph of an automorphism of $\mathbb{A}$.*

In fact, we can weaken the assumption that $\mathbb{A}$ is absorption free to the assumption that $\mathbb{A}$ has no binary or centrally absorbing subalgebras.

Proof. We call a subdirect relation $\mathbb{R} \le \mathbb{A}^n$ *irredundant* if no $\pi_{ij}(\mathbb{R})$ is the graph of an automorphism of $\mathbb{A}$. We will prove by induction on n that every irredundant subdirect relation on $\mathbb{A}$ is the full relation.

The base cases of the induction are the cases $n = 1, 2, 3$. The case $n = 1$ is trivial (a unary subdirect relation must be full). The case $n = 2$ follows from the Absorption Theorem 3.11.1, since every subdirect binary relation on $\mathbb{A}$ is either the graph of an automorphism of $\mathbb{A}$, or is linked (since $\mathbb{A}$ is simple) and therefore is equal to the full relation (since $\mathbb{A}$ has no binary or centrally absorbing subalgebras). For the case $n = 3$, note by the $n = 2$ case both $\pi_{13}(\mathbb{R})$ and $\pi_{23}(\mathbb{R})$ must be full relations, so for any $a \in \mathbb{A}$ the binary relation

$$\mathbb{R}^a := \pi_{12}(\mathbb{R} \cap (\mathbb{A}^2 \times \{a\}))$$

is subdirect. Then by the $n = 2$ case again, we see that $\mathbb{R}^a$ is either the graph of an automorphism or is equal to $\mathbb{A}^2$. If there is any $a \in \mathbb{A}$ such that $\mathbb{R}^a = \mathbb{A}^2$, then a is contained in the right center of $\mathbb{R}$, considered as a binary relation on $(\mathbb{A}^2) \times \mathbb{A}$, so $\mathbb{R} = \mathbb{A}^3$ by the Absorption Theorem 3.11.1 (or just Corollary 3.10.8). Otherwise every $\mathbb{R}^a$ is the graph of an automorphism, and a similar argument applies if we permute the coordinates of $\mathbb{R}$, so we may apply Proposition 3.12.2 to see that $\mathbb{A}$ is abelian. But then by the fundamental theorem of abelian algebras 3.12.8 we see that $\mathbb{A}$ is affine, which contradicts our assumptions.

For the induction step, assume that $n > 3$. Then for every pair of distinct $i, j \le n - 1$, the ternary relation $\pi_{ijn}(\mathbb{R})$ is full by the $n = 3$ case, so for every $a \in \mathbb{A}$, the binary relation

$$\pi_{ij}(\mathbb{R} \cap (\mathbb{A}^{n-1} \times \{a\}))$$

is the full relation $\mathbb{A}^2$. Thus the relation

$$\mathbb{R}^a := \pi_{[n-1]}(\mathbb{R} \cap (\mathbb{A}^{n-1} \times \{a\}))$$

is irredundant, so by the inductive hypothesis, $\mathbb{R}^a$ is the full relation $\mathbb{A}^{n-1}$ for every $a \in \mathbb{A}$. In other words, $\mathbb{R}$ is the full relation $\mathbb{A}^n$. □

Since the conclusion of Theorem 3.12.9 is actually much stronger than merely being polynomially complete, we will give it a special name.

Definition 3.12.10. We say that an algebra $\mathbb{A}$ is *subdirectly complete* if every subdirect relation $\mathbb{R} \le_{sd} \mathbb{A}^n$ is the intersection of its binary projections, each of which is either a full relation or the graph of an automorphism of $\mathbb{A}$.

Proposition 3.12.11. *Every subdirectly complete finite algebra is polynomially complete.*

Corollary 3.12.12 (Zhuk's four cases [190])**.** *If $\mathbb{A}$ is a nontrivial finite idempotent Taylor algebra, then at least one of the following is true.*

- *$\mathbb{A}$ has a proper binary absorbing subalgebra,*
- *$\mathbb{A}$ has a proper centrally absorbing subalgebra,*
- *$\mathbb{A}$ has a nontrivial affine quotient, or*
- *$\mathbb{A}$ has a nontrivial subdirectly complete quotient.*

Proof. Let $\theta \in \mathrm{Con}(\mathbb{A})$ be a maximal congruence on $\mathbb{A}$, so $\mathbb{A}/\theta$ is simple. If $\mathbb{A}/\theta$ has a proper binary or centrally absorbing subalgebra $\mathbb{B}$, then the preimage of $\mathbb{B}$ under the projection $\mathbb{A} \twoheadrightarrow \mathbb{A}/\theta$ is a proper binary or centrally absorbing subalgebra of $\mathbb{A}$. Otherwise, Theorem 3.12.9 shows that if $\mathbb{A}/\theta$ is not affine, then it is subdirectly complete. □

Remark 3.12.2. For the sake of proving Theorem 3.12.9 and Corollary 3.12.12, we only need to show that if $\mathbb{A}$ is a finite idempotent Taylor algebra with a ternary relation $\mathbb{R} \le_{sd} \mathbb{A}^3$ as in Proposition 3.12.2, then $\mathbb{A}$ is affine. It's possible to give a direct argument for this, as follows.

First, we reinterpret $\mathbb{R}$ as the graph of a quasigroup operation $\cdot : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$. Using this quasigroup operation $\cdot$, we can define a Mal'cev operation $p : \mathbb{A}^3 \to \mathbb{A}$ which is centralized by the clone of $\mathbb{A}$, such that p is invertible in its first and last variables. We then pick any element $0 \in \mathbb{A}$, and define the binary operation $m : \mathbb{A}^2 \to \mathbb{A}$ by $m(x,y) := p(x,0,y)$. Then we have $m(x,0) = p(x,0,0) = x$ and $m(0,x) = p(0,0,x) = x$ for all $x \in \mathbb{A}$, so we can apply the variant of the Eckmann-Hilton principle from Remark 1.5.3 to see that m must be commutative and associative. This m will also be cancellative by construction, so by the finiteness of $\mathbb{A}$ we see that m defines an abelian group structure on $\mathbb{A}$, which shows that $\mathbb{A}$ is quasiaffine. One then needs to check that any finite Taylor algebra which is quasiaffine has a Mal'cev polynomial to finish the argument.

3.13 Bounded width: affine-free CSPs are solved by cycle-consistency

Really, the title of this section should be referring to pq-consistency (see Definition 3.9.1), but I wanted to keep the table of contents understandable. We have already shown in Theorem 3.9.3 that if we have a pq-consistent instance of a CSP, then we can reduce some of the domains to find a pq-consistent instance in which every domain is absorption free. In this section, we will show that if every domain is absorption free and affine free, then we can reduce the instance further while preserving pq-consistency.

Definition 3.13.1. We say that a finite idempotent algebra $\mathbb{A}$ is *affine-free* if no quotient of any subalgebra of $\mathbb{A}$ is affine.

The argument strategy is very similar to the argument in the case of strongly connected algebras. We already have most of the pieces.

- If a binary subdirect relation $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$ is linked and $\mathbb{A}, \mathbb{B}$ are absorption free and Taylor, then $\mathbb{R} = \mathbb{A} \times \mathbb{B}$ by the Absorption Theorem 3.11.1.

- If a binary relation $\mathbb{R} \le \mathbb{A}\times\mathbb{A}$ absorbs the diagonal $\Delta_{\mathbb{A}}$ and $\mathbb{A}$ is absorption free, then $\Delta_{\mathbb{A}} \subseteq \mathbb{R}$ by Theorem 3.7.13.

- If a binary subdirect relation $\mathbb{R} \le_{sd} \mathbb{A}\times\mathbb{A}$ is linked and $\mathbb{A}$ is Taylor, then $\mathbb{R}\cap\Delta_{\mathbb{A}} \ne \emptyset$ by the linked case of the Loop Lemma 3.11.11.

- If $\mathbb{A}$ is simple, idempotent, Taylor, absorption free, and not affine, then $\mathbb{A}$ is subdirectly complete, by Theorem 3.12.9.

The missing ingredient is an analogue of Theorem 3.3.5.

Theorem 3.13.2. *Suppose $\mathbb{R} \le_{sd} \mathbb{A}\times\mathbb{B}\times\mathbb{C}$ is subdirect, $\mathbb{A}$ has no proper binary or centrally absorbing subalgebra and no affine quotient, $\pi_{23}(\mathbb{R})$ has no proper binary absorbing subalgebra, $\pi_{12}(\mathbb{R}) = \mathbb{A}\times\mathbb{B}$, $\pi_{13}(\mathbb{R}) = \mathbb{A}\times\mathbb{C}$, and $\mathbb{A},\mathbb{B},\mathbb{C}$ are finite idempotent Taylor algebras. Then $\mathbb{R} = \mathbb{A}\times\pi_{23}(\mathbb{R})$.*

Note that by the Absorption Theorem 3.11.1, we just need to prove that if we consider $\mathbb{R}$ as a subdirect binary relation $\mathbb{R} \le_{sd} \mathbb{A}\times\pi_{23}(\mathbb{R})$, then $\mathbb{R}$ is linked. If not, then the linking congruence of $\mathbb{R}$ on $\mathbb{A}$ is contained in some maximal congruence $\theta \in \mathrm{Con}(\mathbb{A})$, and if we replace $\mathbb{R}$ by the quotient $\mathbb{R}/\theta \le_{sd} \mathbb{A}/\theta\times\mathbb{B}\times\mathbb{C}$, then we have a smaller counterexample to Theorem 3.13.2 such that $\mathbb{A}$ is simple. So we just need to rule out the case where $\mathbb{A}$ is simple and $\mathbb{R}$ is the graph of a homomorphism $f : \pi_{23}(\mathbb{R}) \twoheadrightarrow \mathbb{A}$. For this, we will use a consequence of the linked case of the Loop Lemma 3.11.11.

Lemma 3.13.3. *If $\mathbb{A},\mathbb{B}$ are finite Taylor algebras, $\mathbb{R},\mathbb{S} \le_{sd} \mathbb{A}\times\mathbb{B}$, and the linked components of $\mathbb{R}$ on $\mathbb{A}\sqcup\mathbb{B}$ contain the corresponding linked components of $\mathbb{S}$, then $\mathbb{R}\cap\mathbb{S}\ne\emptyset$.*

Proof. Let $\mathbb{A}' \le \mathbb{A}, \mathbb{B}' \le \mathbb{B}$ be corresponding linked components of $\mathbb{R}$, with $\mathbb{R}\cap(\mathbb{A}'\times\mathbb{B}') \ne \emptyset$. By replacing $\mathbb{A},\mathbb{B}$ with $\mathbb{A}',\mathbb{B}'$ and shrinking $\mathbb{R},\mathbb{S}$, we may assume without loss of generality that $\mathbb{R}$ is linked. Then $\mathbb{R}\circ\mathbb{S}^- \le_{sd} \mathbb{A}\times\mathbb{A}$ is also linked, so by the linked case of the Loop Lemma 3.11.11, there is some $a \in \mathbb{A}$ such that $(a,a) \in \mathbb{R}\circ\mathbb{S}^-$. By the definition of $\mathbb{R}\circ\mathbb{S}^-$, this means that there is some $b \in \mathbb{B}$ such that $(a,b)\in\mathbb{R}$ and $(b,a)\in\mathbb{S}^-$, so $(a,b)\in\mathbb{R}\cap\mathbb{S}$. □

Proof of Theorem 3.13.2. Write $\mathbb{S} = \pi_{23}(\mathbb{R}) \le_{sd} \mathbb{B}\times\mathbb{C}$. Assume for the sake of contradiction that $\mathbb{A}$ is simple and that $\mathbb{R}$ is the graph of a homomorphism $f : \mathbb{S} \twoheadrightarrow \mathbb{A}$. Note that by the idempotence of $\mathbb{A}$, for each $a \in \mathbb{A}$ the set $f^{-1}(a) \subseteq \mathbb{S}$ is a subalgebra of $\mathbb{S}$, and let $\mathbb{S}_a \coloneqq f^{-1}(a)$. The assumptions $\pi_{12}(\mathbb{R}) = \mathbb{A}\times\mathbb{B}, \pi_{13}(\mathbb{R}) = \mathbb{A}\times\mathbb{C}$ are equivalent to each $\mathbb{S}_a = f^{-1}(a)$ being a subdirect relation on $\mathbb{B}\times\mathbb{C}$.

If we can show that there are $a \ne a' \in \mathbb{A}$ such that $\mathbb{S}_a, \mathbb{S}_{a'} \le_{sd} \mathbb{B}\times\mathbb{C}$ have the same linked components on $\mathbb{B}\sqcup\mathbb{C}$, then we can apply the lemma to see that $f^{-1}(a)\cap f^{-1}(a') = \mathbb{S}_a\cap\mathbb{S}_{a'} \ne \emptyset$, which will give us a contradiction. To accomplish this, we will show that each $\mathbb{S}_a$ has the same linked components on $\mathbb{B}\sqcup\mathbb{C}$ as $\mathbb{S}$. In fact, we will show that for every $a \in \mathbb{A}$, we have $\mathbb{S} \subseteq \mathbb{S}_a\circ\mathbb{S}_a^-\circ\mathbb{S}_a$.

Let (b,c) be any element of $\mathbb{S}$. Define a subalgebra $\mathbb{X}_{bc} \le \mathbb{A}\times\mathbb{A}\times\mathbb{A}$ by

$$\mathbb{X}_{bc} \coloneqq \left\{ \begin{bmatrix} x\\ y\\ z\end{bmatrix} \;\middle|\; \exists b' \in \mathbb{B}, c' \in \mathbb{C} \text{ s.t. } \begin{bmatrix} x\\ b\\ c'\end{bmatrix} \in \mathbb{R} \wedge \begin{bmatrix} y\\ b'\\ c'\end{bmatrix} \in \mathbb{R} \wedge \begin{bmatrix} z\\ b'\\ c\end{bmatrix} \in \mathbb{R} \right\}.$$

Equivalently, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{X}_{bc} \iff \begin{bmatrix} b \\ c \end{bmatrix} \in \mathbb{S}_x \circ \mathbb{S}_y^- \circ \mathbb{S}_z.$$

Since each $\mathbb{S}_a$ is subdirect, we have $(b,b) \in \mathbb{S}_a \circ \mathbb{S}_a^-$ and $(c,c) \in \mathbb{S}_a^- \circ \mathbb{S}_a$. Thus for each $a \in \mathbb{A}$, we have

$$\begin{bmatrix} a \\ a \\ f(b,c) \end{bmatrix}, \begin{bmatrix} f(b,c) \\ a \\ a \end{bmatrix} \in \mathbb{X}_{bc},$$

so $\mathbb{X}_{bc}$ is subdirect in $\mathbb{A}^3$, and for each $i \neq j \le 3$ the projection $\pi_{ij}(\mathbb{X}_{bc})$ is not the graph of an automorphism of $\mathbb{A}$. Thus by Theorem 3.12.9, we see that $\mathbb{X}_{bc} = \mathbb{A}^3$, so in particular we have $(a,a,a) \in \mathbb{X}_{bc}$ for all $a \in \mathbb{A}$. Since this holds for every $(b,c) \in \mathbb{S}$, we see that $\mathbb{S} \subseteq \mathbb{S}_a \circ \mathbb{S}_a^- \circ \mathbb{S}_a$ for all $a \in \mathbb{A}$, so each $\mathbb{S}_a$ has the same linked components on $\mathbb{B} \sqcup \mathbb{C}$ as $\mathbb{S}$, which completes the contradiction. □

Corollary 3.13.4. *If $\mathbb{A}_1, ..., \mathbb{A}_n$ are finite idempotent Taylor algebras with no proper binary or centrally absorbing subalgebras such that all but at most two of the $\mathbb{A}_i$s have no affine quotients, and if $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is a subdirect relation such that each $\pi_{ij}(\mathbb{R})$ is full, then $\mathbb{R} = \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$.*

Corollary 3.13.5. *If $\mathbb{A}_1, ..., \mathbb{A}_n$ are finite idempotent Taylor algebras with no proper binary or centrally absorbing subalgebras and no affine quotients, and if $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n \times \mathbb{A}_1$ is a subdirect relation such that $\Delta_{\mathbb{A}_1} \subseteq \pi_{1,n+1}(\mathbb{R})$ and $\pi_{ij}(\mathbb{R})$ is full for all pairs (i,j) other than $(1,n+1)$, then $\mathbb{R}$ contains every tuple whose first and last coordinates are the same.*

Proof. Suppose first that $\mathbb{R}$ has a proper binary or centrally absorbing subalgebra $\mathbb{R}'$. Note that each $\pi_{ij}(\mathbb{R}')$ with $(i,j) \neq (1,n+1)$ is full since the $\mathbb{A}_i$ have no binary or centrally absorbing subalgebras. Additionally, $\pi_{1,n+1}(\mathbb{R}')$ absorbs $\Delta_{\mathbb{A}_1}$, so by Theorem 3.11.11 we see that $\Delta_{\mathbb{A}_1} \subseteq \pi_{1,n+1}(\mathbb{R}')$ as well. Thus we may replace $\mathbb{R}$ by $\mathbb{R}'$, until we eventually reach a situation where $\mathbb{R}$ has no proper binary or central absorption. In particular, we may assume that $\pi_{1,n+1}(\mathbb{R})$ has no proper binary or centrally absorbing subalgebras.

For any $2 \le i \le n$, we may apply Theorem 3.13.2 to $\pi_{i,1,n+1}(\mathbb{R})$ to see that $\pi_{i,1,n+1}(\mathbb{R}) = \mathbb{A}_i \times \pi_{1,n+1}(\mathbb{R})$. Now consider $\mathbb{R}$ as an n-ary relation

$$\mathbb{R} \le_{sd} \pi_{1,n+1}(\mathbb{R}) \times \mathbb{A}_2 \times \cdots \times \mathbb{A}_n,$$

and apply the previous corollary to see that $\mathbb{R} = \pi_{1,n+1}(\mathbb{R}) \times \mathbb{A}_2 \times \cdots \times \mathbb{A}_n$. In particular, since we have $\Delta_{\mathbb{A}_1} \subseteq \pi_{1,n+1}(\mathbb{R})$, we see that $\mathbb{R}$ contains every tuple whose first and last coordinates are equal. □

Now that we've gathered up all the necessary ingredients, we argue as in the case of strongly connected algebras. We start by picking some variable x with $|\mathbb{A}_x| > 1$, pick a maximal congruence $\theta_x \in \mathrm{Con}(\mathbb{A}_x)$, pick a congruence class $\mathbb{A}'_x \le \mathbb{A}_x$ of θ_x. Then we refer back to Definition 3.5.2 to define the "proper" variables y to be the variables such that there exists a path p from y to x such that

$$\mathbb{P}_p/\theta_x \le_{sd} \mathbb{A}_y \times \mathbb{A}_x/\theta_x$$

is the graph of a homomorphism $\iota_y : \mathbb{A}_y \twoheadrightarrow \mathbb{A}_x/\theta_x$, and define θ_y to be the kernel of ι_y and $\mathbb{A}'_y$ to be $\iota_y^{-1}(\mathbb{A}'_x)$.

As in the case of strongly connected algebras, we need to check that the homomorphism ι_y does not depend on the choice of path p. This time, we will check this using pq-consistency instead of cycle-consistency.

Lemma 3.13.6. *Suppose that the instance* $\mathbf{X}$ *is pq-consistent, and that* x, θ_x *are chosen as above. Suppose that* y *is a proper variable, and that* p, q *are two paths from* y *to* x *such that* $\mathbb{P}_p/\theta_x, \mathbb{P}_q/\theta_x$ *are the graphs of homomorphisms* $\iota_p, \iota_q : \mathbb{A}_y \twoheadrightarrow \mathbb{A}_x/\theta_x$*. Then* $\iota_p = \iota_q$*.*

Proof. Consider the cycles $p - q$ and $q - p$ from y to y, then by the definition of pq-consistency (Definition 3.9.1) we see that there must be some $j \geq 0$ such that for all $a \in \mathbb{A}_y$, we have

$$a \in \{a\} + j(p - q + q - p) + p - q.$$

For any $b \in \mathbb{A}_x$, we have $b/\theta_x - p + p = b/\theta_x$ and $b/\theta_x - q + q = b/\theta_x$ by the assumptions on $\mathbb{P}_p, \mathbb{P}_q$, so we see that

$$\{a\} + j(p - q + q - p) + p - q \subseteq \iota_p(a) - q = \iota_q^{-1}(\iota_p(a)),$$

so we must have $\iota_q(a) = \iota_p(a)$. □

As a consequence, we have the following analogue of Lemma 3.5.4.

Lemma 3.13.7. *Suppose that the instance* $\mathbf{X}$ *is pq-consistent, and that each domain has no proper binary or centrally absorbing subalgebra. Suppose* p *is a path from* y *to a proper variable* z*. Then one of the following is true:*

- $\mathbb{P}_p/\theta_z = \mathbb{A}_y \times \mathbb{A}_z/\theta_z$*, or*
- y *is also proper, and* $\mathbb{P}_p/(\theta_y \times \theta_z)$ *is the graph of an isomorphism* $\iota_p : \mathbb{A}_y/\theta_y \xrightarrow{\sim} \mathbb{A}_z/\theta_z$ *such that* $\iota_y = \iota_z \circ \iota_p$*.*

Proof. Since $\mathbb{A}_z/\theta_z$ is simple, the linking congruence of $\mathbb{P}_p/\theta_z$ must either be trivial or full. If the linking congruence of $\mathbb{P}_p/\theta_z$ is full, then by the Absorption Theorem 3.11.1 we see that $\mathbb{P}_p/\theta_z = \mathbb{A}_y \times \mathbb{A}_z/\theta_z$. Otherwise, $\mathbb{P}_p/\theta_z$ is the graph of a homomorphism from $\mathbb{A}_y$ to $\mathbb{A}_z/\theta_z$, so then by joining the path p with a path from z to x we see that y is proper and $\iota_y = \iota_z \circ \iota_p$. □

To finish, we just need to show that restricting each proper variable's domain $\mathbb{A}_x$ to $\mathbb{A}_x'$ gives us a pq-consistent instance $\mathbf{X}'$. To see that $\mathbf{X}'$ is arc-consistent, we apply Corollary 3.13.4 as in the proof of Lemma 3.5.5. To see that $\mathbf{X}'$ is pq-consistent, we apply Corollary 3.13.5 as in the proof of Lemma 3.5.6. We have proven our main result.

Theorem 3.13.8 (Kozik [127])**.** *If* $\mathbf{X}$ *is a pq-consistent instance of a CSP such that every domain is finite, idempotent, Taylor, and affine-free, then* $\mathbf{X}$ *has a solution.*

As a curiously roundabout consequence, we see that we can't build an affine (or even abelian) algebra out of affine-free algebras.

Corollary 3.13.9 (A special case of Lemma 1.5.9)**.** *If* $\mathbb{A}_1, ..., \mathbb{A}_n$ *are finite, idempotent, Taylor, and affine-free, then the variety* $\mathcal{V}(\mathbb{A}_1, ..., \mathbb{A}_n)$ *which they generate does not contain any nontrivial abelian algebras.*

Proof. Since the variety $\mathcal{V}(\mathbb{A}_1, ..., \mathbb{A}_n)$ is finitely generated, it is locally finite, so any nontrivial abelian algebra in this variety must contain a finite abelian algebra $\mathbb{B}$ with $|\mathbb{B}| > 1$. Since $\mathbb{B}$ is finite, Taylor, and abelian, we see that $\mathbb{B}$ is affine by Theorem 3.12.8. But then $\mathbb{B}$ is a subquotient of some finite product of $\mathbb{A}_i$s, so $\mathrm{CSP}(\prod_i \mathbb{A}_i^k)$ fails to have bounded width for some finite k, which contradicts the fact that $\mathrm{CSP}(\mathbb{A}_1, ..., \mathbb{A}_n)$ is solved by pq-consistency. □

Using commutator theory, we have the following consequence (see Corollary 1.9.34).

Corollary 3.13.10. *If $\mathbb{A}$ is a finite idempotent algebra, then $\mathbb{A}$ is Taylor and affine-free if and only if the variety $\mathcal{V}(\mathbb{A})$ is congruence meet-semidistributive.*

Using the language of pp-constructability (see Definition 1.4.14), we can rephrase Theorem 3.13.8 as follows.

Corollary 3.13.11. *A relational structure* $\mathbf{A}$ *with a finite domain has* $\mathrm{CSP}(\mathbf{A})$ *solved by pq-consistency if and only if* $\mathbf{A}$ *does not pp-construct any of the relational structures* $(\mathbb{Z}/p, \{1\}, x+y = z)$*, p prime.*

Proof. Since $\mathbf{A}$ pp-constructs its rigid core and vice-versa, we may assume without loss of generality that $\mathbf{A}$ is a rigid core. Then the associated algebra $\mathbb{A}$ is idempotent, so $\mathbb{A}$ is Taylor if and only if there is any relational structure which $\mathbf{A}$ does not pp-construct. To finish, we need to check that if $\mathbb{A}$ is not affine-free, then $\mathbf{A}$ pp-constructs some $(\mathbb{Z}/p, \{1\}, x + y = z)$. Since restricting to a subalgebra of $\mathbb{A}$ and taking a quotient can both be accomplished by pp-constructions, we may suppose that $\mathbb{A}$ is affine and nontrivial.

If $\mathbb{A}$ is affine, then by definition $\mathbb{A}$ is polynomially equivalent to some module $\mathbb{M}$. If $\mathbb{A}$ is also idempotent, then the relation $x+y = z$ is preserved by $\mathbb{A}$, as are all singleton unary relations, so $\mathbf{A}$ pp-constructs the relational structure $(\mathbb{M}, x + y = z)^{rig}$ (the superscript is shorthand for throwing in all unary singleton relations). Since $\mathbb{M}$ is finite, some element of $\mathbb{M}$ must have prime order, say order p. Then the set of all elements of $\mathbb{M}$ with order p is pp-definable, so we may suppose without loss of generality that every nonzero element of $\mathbb{M}$ has order exactly p. As an abelian group we then have $\mathbb{M} \cong (\mathbb{Z}/p)^k$ for some k. Letting c be any nonzero element of $\mathbb{M}$, we then see that $(\mathbb{M}, \{c\}, x + y = z)$ is homomorphically equivalent to $(\mathbb{Z}/p, \{1\}, x + y = z)$. □

3.13.1 Weak Prague instances

The original proofs of the bounded width conjecture (i.e., that affine-free CSPs have bounded width) didn't use the concepts of pq-consistency or cycle-consistency. Bulatov's argument [44] used $(2, 3)$-consistency, and leveraged a local structure theory of bounded width algebras in terms of two element semilattice and majority subalgebras. The early arguments due to Barto and Kozik [13], [20] used simpler algebraic ingredients, but used a more complicated consistency condition satisfied by instances called *Prague instances*, which were then simplified to *weak Prague instances*. We won't go over the original Prague instance concept until later, but weak Prague instances have a nice definition.

Definition 3.13.12. An instance $\mathbf{X}$ of a CSP with variable domains $\mathbb{A}_x$ is called a *weak Prague instance* if it satisfies the following three conditions.

(P1) The instance $\mathbf{X}$ is arc-consistent, that is, each constraint relation $\mathbb{R} \le \prod_{x_i} \mathbb{A}_{x_i}$ is subdirect.

(P2) For every variable x, every set $A \subseteq \mathbb{A}_x$, and every cycle p from x to x, we have the implication

$$A + p = A \implies A - p = A.$$

(P3) For every variable x, every set $A \subseteq \mathbb{A}_x$, and every pair of cycles p, q from x to x, we have the implication

$$A + p + q = A \implies A + p = A.$$

We can understand what condition (P2) says about an individual cycle p in terms of the digraph associated to the binary relation $\mathbb{P}_p \le_{sd} \mathbb{A}_x \times \mathbb{A}_x$.

Proposition 3.13.13. *A subdirect binary relation $\mathbb{P} \le_{sd} \mathbb{A} \times \mathbb{A}$ on a finite algebra $\mathbb{A}$ satisfies the implication*

$$A + \mathbb{P} = A \implies A - \mathbb{P} = A$$

for all $A \subseteq \mathbb{A}$ if and only if the digraph $\mathbf{P} = (\mathbb{A}, \mathbb{P})$ satisfies one of the following equivalent conditions:

- *every weakly connected component of $\mathbf{P}$ is strongly connected,*
- *every edge of $\mathbf{P}$ is contained in a directed cycle of $\mathbf{P}$,*
- *there is some $k \ge 0$ such that $\mathbb{P}^- \subseteq \mathbb{P}^{\circ k}$.*

An alternative form of condition (P2) is given in [19].

Proposition 3.13.14 (Barto, Kozik [19])**.** *If an instance satisfies condition (P1), then (P2) is equivalent to the following condition.*

(P2) For all variables x, sets $A \subseteq \mathbb{A}_x$, and cycles p from x to x such that $A + p = A$, if p_1 is the first step of the cycle p, then we have $A + p_1 - p_1 = A$.*

Note that $A + p_1 - p_1 = A$ if and only if A is a union of linked components of p_1.

Proof. It's easy to see that (P1) and (P2) imply (P2*), so we'll focus on proving the more difficult implication: that (P2*) implies (P2). Suppose that $A + p = A$, and write $p = p_1 + p_2 + \cdots + p_k$, where each p_i has length one. By the assumption $A + p = A$, we have

$$(A + p_1 + \cdots + p_i) + (p_{i+1} + \cdots + p_k + p_1 + \cdots + p_i) = (A + p) + p_1 + \cdots + p_i = A + p_1 + \cdots + p_i,$$

so we can apply (P2*) to see that

$$(A + p_1 + \cdots + p_i) + p_{i+1} - p_{i+1} = A + p_1 + \cdots + p_i.$$

Thus we have

$$\begin{aligned} A - p &= (A + p) - p \\ &= A + p_1 + \cdots + p_{k-1} + p_k - p_k - p_{k-1} - \cdots - p_1 \\ &= A + p_1 + \cdots + p_{k-1} - p_{k-1} - \cdots - p_1 \\ &= \cdots \\ &= A + p_1 - p_1 = A. \end{aligned}$$

□

Conditions (P1) and (P2) are closely related to the basic linear relaxation of a CSP, from subsection 1.6.1.

Theorem 3.13.15. *If $\mathbf{X}$ is an instance of a CSP such that the basic linear relaxation of $\mathbf{X}$ has a solution assigning probability vectors p_C to each constraint C of $\mathbf{X}$ and probability vectors p_x to each variable x, then the instance $\mathbf{X}'$ obtained by restricting each constraint relation of $\mathbf{X}$ to the support of the corresponding probability distribution p_C (and similarly for the variable domains) satisfies conditions (P1) and (P2).*

Proof. Assume for simplicity that $\mathbf{X} = \mathbf{X}'$, that is, that all of the probability vectors have full support. The compatibility of the probability vectors p_C with the probability vectors on the variable domains ensures that $\mathbf{X}$ is arc-consistent, so (P1) is satisfied. For (P2), it is easier to check condition (P2*) from Proposition 3.13.14. We attach to each set $A \subseteq \mathbb{A}_x$ a probability $P(A)$, given by

$$P(A) = \sum_{a\in A} p_{x,a}.$$

Now consider any step p_1 from a variable x to an adjacent variable y within a constraint C. Let $\mathbb{P} \subseteq \mathbb{A}_x \times \mathbb{A}_y$ be the binary projection of the corresponding constraint relation onto x and y, and let $p_{\mathbb{P}}$ be the corresponding marginal distribution of p_C. Then we have

$$P(A+\mathbb{P}) = \sum_{b\in A+\mathbb{P}} p_{y,b} \ge \sum_{b\in A+\mathbb{P}} \sum_{a\in A} p_{\mathbb{P},(a,b)} = \sum_{a\in A} p_{x,a} = P(A),$$

with equality when $A + \mathbb{P} - \mathbb{P} = A$. Thus if $A + p = A$, then we have

$$P(A) \le P(A+p_1) \le P(A+p) = P(A),$$

so $P(A+p_1) = P(A)$, and thus we have $A + p_1 - p_1 = A$. □

In fact, Theorem 3.13.15 has a converse when we restrict our attention to a single cycle at a time.

Theorem 3.13.16. *If $\mathbf{X}$ is an instance of a CSP such that the associated hypergraph of variables and relations consists of a single cycle, then $\mathbf{X}$ has properties (P1) and (P2) if and only if the basic linear relaxation of $\mathbf{X}$ has a solution such that for each constraint C of $\mathbf{X}$, the support of the corresponding probability distribution p_C is exactly equal to the relation corresponding to C.*

Proof. Let $v_1, ..., v_n$ be the variables of $\mathbf{X}$ which occur in two constraints, in the order in which they appear around the cycle, and let the constraints $C_1, ..., C_n$ be numbered such that v_i and v_{i+1} are variables of C_i for each i (here we interpret the subscripts $i, i+1$ modulo n, so $v_{n+1} = v_1$).

Consider the following directed graph on the set of pairs (i, a) where $i \in \mathbb{Z}/n\mathbb{Z}$ and $a \in \mathbb{A}_{v_i}$: for every element r of the relation corresponding to constraint C_i, we make a directed edge from $(i, \pi_{v_i}(r))$ to $(i+1, \pi_{v_{i+1}}(r))$. Then conditions (P1) and (P2) guarantee that every edge of this digraph is contained in a directed cycle. Choose some finite set of directed cycles $\mathcal{C}$ of this digraph which covers each edge at least once. Then for each constraint C_i, we let p_{C_i} be the probability distribution defined by first choosing a cycle from $\mathcal{C}$ uniformly at random, and then choosing uniformly among the elements r of the relation corresponding to the constraint C_i such that the edge from $(i, \pi_{v_i}(r))$ to $(i+1, \pi_{v_{i+1}}(r))$ is contained in our chosen cycle. □

The condition (P3) can be rephrased to look slightly more similar to the condition for *pq*-consistency.

Proposition 3.13.17. *An instance* $\mathbf{X}$ *with finite variable domains* $\mathbb{A}_x$ *satisfies condition (P3) if and only if it satisfies the following condition.*

(P3) For all variables* x*, for all pairs of cycles* p, q *from* x *to* x*, and for all* $a \in \mathbb{A}_x$*, there is some* $j \geq 0$ *such that*

$$\{a\} + j(p+q) = \{a\} + j(p+q) + p = \{a\} + j(p+q) + p + q.$$

Proof. First we show that (P3) implies (P3*). For this, note that if we define a sequence of subsets $A_i \subseteq \mathbb{A}_x$ by $A_i = \{a\} + i(p+q)$, then by the finiteness of $\mathbb{A}_x$ there must be some j, k with $k > 0$ such that $A_j = A_{j+k}$. But then (P3) implies that $A_j + p = A_j$ and similarly that $(A_j + p) + q = A_j + p$.

For the reverse direction, let $A \subseteq \mathbb{A}_x$ satisfy $A + p + q = A$. Then by the finiteness of A we can find j sufficiently large such that for each $a \in A$ we have $\{a\} + j(p+q) = \{a\} + j(p+q) + p$. For this choice of j, we then have

$$A = A + j(p+q) = A + j(p+q) + p = A + p. \qquad \square$$

There is also a natural way to certify that a given instance satisfies condition (P3), following a similar philosophy to the method we used to find absorbing reductions of cycle consistent majority CSPs.

Proposition 3.13.18. *An instance* $\mathbf{X}$ *satisfies condition (P3) at a variable* x *if and only if there is a partial order* $\preceq$ *on the power set* $\mathcal{P}(\mathbb{A}_x)$*, such that for every cycle* p *from* x *to* x *and every* $A \subseteq \mathbb{A}_x$*, we have*

$$A \preceq A + p.$$

The instance $\mathbf{X}$ *satisfies (P3) everywhere if and only if there is a quasiorder* $\preceq$ *on the set of ordered pairs* (x, A) *with* $A \subseteq \mathbb{A}_x$*, such that for each binary projection* $\mathbb{R}_{ij} \leq \mathbb{A}_x \times \mathbb{A}_y$ *of any constraint relation of* $\mathbf{X}$ *and for each* $A \subseteq \mathbb{A}_x$*, we have*

$$(x, A) \preceq (y, A + \mathbb{R}_{ij}),$$

and such that for each x*, the restriction of* $\preceq$ *to* $\{x\} \times \mathcal{P}(\mathbb{A}_x)$ *defines a partial order on* $\mathcal{P}(\mathbb{A}_x)$*.*

Weak Prague instances are closely related to *pq*-consistent instances, but they are not quite the same.

Theorem 3.13.19. *Every weak Prague instance is pq-consistent.*

Proof. Suppose $\mathbf{X}$ is a weak Prague instance, that x is a variable of $\mathbf{X}$, that p, q are cycles from x to x, and that $a \in \mathbb{A}_x$. We need to check that there is some $j \geq 0$ such that

$$a \in \{a\} + j(p+q) + p.$$

Since $\mathbb{A}_x$ is finite, there must be some $j > 0$ such that

$$\{a\} + j(p+q) = \{a\} + 2j(p+q).$$

Let $A = \{a\} + j(p+q)$ be the common value of both sides of the above equation (note that if $\mathbb{A}_x$ is idempotent, then A will actually be a subalgebra of $\mathbb{A}_x$). Then by (P2) we have

$$A = A + j(p+q) \implies A = A - j(p+q),$$

so

$$a \in \{a\} + j(p+q) - j(p+q) = A - j(p+q) = A.$$

Additionally, by (P3) we have

$$A = A + p + (q + (j-1)(p+q)) \implies A = A + p,$$

so

$$a \in A = A + p = \{a\} + j(p+q) + p. \qquad \square$$

Example 3.13.1. Here we give an example of a *pq*-consistent instance (in fact, even a singleton arc-consistent instance!) which is not a weak Prague instance. Consider the instance of 2-SAT with just one variable x, domain $\mathbb{A}_x = (\{0,1\}, \text{maj})$, and a binary constraint relation $\mathbb{R} \le_{sd} \mathbb{A}_x \times \mathbb{A}_x$ imposed on (x,x) given by $\mathbb{R} = \{(0,0),(0,1),(1,1)\}$ (that is, $\mathbb{R}$ is the binary relation $\le$).

Since $\Delta_{\{0,1\}} \subseteq \mathbb{R}$, we see that this instance is *pq*-consistent. However, this instance does not satisfy property (P2) of a weak Prague instance: we have

$$\{1\} + \mathbb{R} = \{1\},$$

but

$$\{1\} - \mathbb{R} = \{0,1\} \ne \{1\}.$$

Alternatively, we can check that (P2) is not satisfied by noting that the digraph $(\{0,1\}, \le)$ is weakly connected but not strongly conected.

Although not every *pq*-consistent instance satisfies (P2), we at least have the following implication.

Theorem 3.13.20 (Kozik [127]). *Every pq-consistent instance satisfies conditions (P1) and (P3).*

Proof. Suppose $\mathbf{X}$ is a *pq*-consistent instance, that x is a variable of $\mathbf{X}$, that p, q are cycles from x to x, and that $A \subseteq \mathbb{A}_x$ satisfies

$$A + p + q = A.$$

By *pq*-consistency, there is some $j \ge 0$ such that

$$A \subseteq A + j(p+q) + p = A + p.$$

Similarly, from

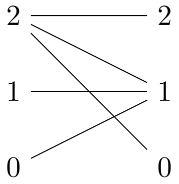
$$(A+p) + q + p = A + p,$$

we see that

$$A + p \subseteq (A+p) + q = A.$$

Thus we have $A = A + p$. $\square$

Example 3.13.2. There is an example of an instance which satisfies (P1) and (P3), but which is not *pq*-consistent. As in the previous example, this instance will have just a single variable x and a single binary constraint $\mathbb{R} \le_{sd} \mathbb{A}_x \times \mathbb{A}_x$. We take the algebra $\mathbb{A}_x$ to be the three-element dual discriminator algebra $(\{0,1,2\}, d(x,y,z))$ from Example 1.6.5. The binary relation $\mathbb{R}$ is the 0/1/all constraint displayed below.



To see that this is not *pq*-consistent, note that there is no j such that $(0,0) \in \mathbb{R}^{\circ j}$. To see that this instance satisfies condition (P3), we use the following total ordering on $\mathcal{P}(\{0,1,2\})$:

$$\emptyset \preceq \{0\} \preceq \{0,1\} \preceq \{1\} \preceq \{0,2\} \preceq \{2\} \preceq \{1,2\} \preceq \{0,1,2\}.$$

We can use weak Prague instances to see that there is a sense in which the linear programming relaxation *almost* solves general CSPs of bounded width.

Definition 3.13.21. We say that a probability distribution μ on a finite set A is *in general position* if we have $\mu(S) \ne \mu(T)$ for every pair of disjoint subsets $S, T \subseteq A$ with $\mu(S), \mu(T) \ne 0$. We say that a solution to the linear relaxation of an instance $\mathbf{X}$ is in general position if the probability distribution which it assigns to each variable domain is in general position.

Proposition 3.13.22. *If there is a solution to the linear programming relaxation of* $\mathbf{X}$ *which is in general position, then the instance we get by restricting each variable domain and relation to the support of this solution is a weak Prague instance (and is therefore pq-consistent as well).*

Proof. We just need to verify condition (P3). Suppose that the solution to the linear relaxation assigns each variable x to the probability distribution μ_x on the variable domain A_x. If $S \subseteq A_x$ is contained in the support of μ_x and p, q are cycles from x to x such that $S + p + q = S$, then we have

$$\mu_x(S) \le \mu_x(S + p) \le \mu_x(S + p + q) = \mu_x(S),$$

so $\mu_x(S) = \mu_x(S + p)$. Thus we have

$$\mu_x(S \setminus (S + p)) = \mu_x((S + p) \setminus S),$$

so the assumption that μ_x is in general position implies that $S = S + p$. □

Libor Barto has raised the following question.

Problem 3.13.1. Is it true that every instance of an affine-free CSP which satisfies conditions (P1) and (P3) has a solution?

We will solve this problem later in these notes.

3.14 Terms for bounded width and the meta-problem

In this section we'll prove the existence of nice ternary terms characterizing bounded width algebras, which were first conjectured to exist by Jovanović [106] and later proved to exist using a Ramsey argument and the fact that bounded width CSPs are solved by $(2,3)$-consistency [107]. Using *pq*-consistency instead of $(2,3)$-consistency, it is possible to prove the existence of these terms directly, as noted by Kozik [127]. These nice ternary terms will allow us to efficiently solve the *meta-problem* for bounded width CSPs: given a core relational structure $\mathbf{A}$ as input, determine whether $\mathrm{CSP}(\mathbf{A})$ has bounded width.

Theorem 3.14.1 (Height 1 identities for bounded width [106], [107], [127])**.** *Suppose* $\mathbf{A}$ *is a relational structure on a finite domain. Then* $\mathrm{CSP}(\mathbf{A})$ *has bounded relational width iff there are ternary polymorphisms* $f, g \in \mathrm{Pol}_3(\mathbf{A})$ *satisfying the height* 1 *identities*

$$g(x,x,y) \approx g(x,y,x) \approx g(y,x,x) \approx f(x,x,y) \approx f(x,y,x) \approx f(x,y,y).$$

In this case, every pq-consistent instance of $\mathrm{CSP}(\mathbf{A})$ *has a solution.*

The identities in the statement of Theorem 3.14.1 may be interpreted as follows. If the common values $c(x,y)$ of $g(x,x,y)$, etc. are all equal to x, then g is a majority function, and f behaves as if it is first projection. If instead we have $c(x,y) = x \vee y$, then f, g both behave as if they are the three-element semilattice operation $x \vee y \vee z$. Finally, if $c(x,y) = y$, then f is a Pixley operation, so $f(x, f(x,y,z), z)$ is a majority operation, and additionally Theorem 3.1.14 applies.

Since having bounded relational width is preserved by homomorphic equivalence, we may reduce proving Theorem 3.14.1 to the special case where $\mathbf{A}$ is a core, and then we can use Theorem 1.4.7 to reduce to the case of a rigid core, so that the associated algebra $\mathbb{A}$ is idempotent. Since any idempotent algebra $\mathbb{A}$ such that $\mathrm{CSP}(\mathbb{A})$ has bounded width must be Taylor and affine-free, we see from Theorem 3.13.8 that $\mathrm{CSP}(\mathbb{A})$ is solved by *pq*-consistency. Furthermore, by Corollary 3.13.9 we see that the free algebra $\mathbb{F} = \mathcal{F}_{\mathbb{A}}(x,y) \le \mathbb{A}^{\mathbb{A}^2}$ is also affine-free, so $\mathrm{CSP}(\mathbb{F})$ is also solved by *pq*-consistency. The plan is to construct a *pq*-consistent instance of $\mathrm{CSP}(\mathbb{F})$ which encodes the existence of such ternary terms f, g, but before we do this we need a basic result about taking closures under algebraic operations.

Definition 3.14.2. Suppose that $\mathbf{X}$ is an instance of a CSP such that every variable domain is contained in $\mathbb{A}$, but possibly the variable domains and the relations of $\mathbf{X}$ are not closed under the operations of $\mathbb{A}$. Define $\mathrm{Sg}_{\mathbb{A}}(\mathbf{X})$ to be the instance of $\mathrm{CSP}(\mathbb{A})$ where every variable domain and every relation of $\mathbf{X}$ is replaced by the subalgebra it generates.

Proposition 3.14.3. *If* $\mathbf{X}$ *is a pq-consistent instance as above, then* $\mathrm{Sg}_{\mathbb{A}}(\mathbf{X})$ *is also pq-consistent.*

Proof. For arc-consistency, let $R \subseteq \mathbb{A}^n$ be any relation, and note that $\mathrm{Sg}_{\mathbb{A}}(\pi_1(R)) = \pi_1(\mathrm{Sg}_{\mathbb{A}}(R))$. For paths, let $R, S \subseteq \mathbb{A} \times \mathbb{A}$ be any binary relations, then we have $\mathrm{Sg}_{\mathbb{A}}(R \circ S) \subseteq \mathrm{Sg}_{\mathbb{A}}(R) \circ \mathrm{Sg}_{\mathbb{A}}(S)$. For cycles interacting well with the diagonal, note that for any $B \subseteq \mathbb{A}$ we have $\mathrm{Sg}_{\mathbb{A}^2}(\Delta_B) = \Delta_{\mathrm{Sg}_{\mathbb{A}}(B)}$. □

We have a similar result for weak Prague instances (Definition 3.13.12), which we won't actually need.

Proposition 3.14.4. *If* $\mathbf{X}$ *is a weak Prague instance as above, then* $\mathrm{Sg}_{\mathbb{A}}(\mathbf{X})$ *is also a weak Prague instance.*

Proof. That $\mathrm{Sg}_{\mathbb{A}}(\mathbf{X})$ satisfies (P1) and (P3) follows from the fact that $\mathbf{X}$ is a pq-consistent instance (Theorem 3.13.19), which implies that $\mathrm{Sg}_{\mathbb{A}}(\mathbf{X})$ is also pq-consistent by the previous proposition, and this in turn implies that $\mathrm{Sg}_{\mathbb{A}}(\mathbf{X})$ satisfies (P1) and (P3) (Theorem 3.13.20). To check that $\mathrm{Sg}_{\mathbb{A}}(\mathbf{X})$ satisfies (P2), we use Proposition 3.13.13: note that if $P \subseteq \mathbb{A} \times \mathbb{A}$ satisfies $P^- \subseteq P^{\circ k}$, then $\mathrm{Sg}_{\mathbb{A}}(P)^- = \mathrm{Sg}_{\mathbb{A}}(P^-) \subseteq \mathrm{Sg}_{\mathbb{A}}(P^{\circ k}) \subseteq \mathrm{Sg}_{\mathbb{A}}(P)^{\circ k}$. □

Lemma 3.14.5. *Suppose* $\mathbf{X}$ *is an instance of a CSP over the two-element domain* $\{x, y\}$ *with no unary relations, such that every binary projection* $\pi_{i,j}(R)$ *of every relation* R *is subdirect in* $\{x, y\}^2$ *and has* $(x, x) \in \pi_{i,j}(R)$*. Then* $\mathbf{X}$ *is pq-consistent.*

Proof. The assumptions on $\mathbf{X}$ directly imply that $\mathbf{X}$ is arc-consistent. Now consider any pair of cycles p, q from a variable v of $\mathbf{X}$ to itself. Note that the collection of binary relations on $\{x, y\}$ which are subdirect and contain (x, x) is closed under composition and reversal, so $\mathbb{P}_p, \mathbb{P}_q$ are both subdirect and contain (x, x). We just need to show that there is some j such that $y \in \{y\} + j(p + q) + p$.

If $(y, y) \in \mathbb{P}_p$, then we may take $j = 0$. Otherwise, we must have $\mathbb{P}_p = \{(x, x), (x, y), (y, x)\}$, and since $(x, x) \in \mathbb{P}_q$ this implies that $\mathbb{P}_p \circ \mathbb{P}_q \circ \mathbb{P}_p = \{x, y\}^2$, so we may take $j = 1$. □

Proof of Theorem 3.14.1. First we prove the existence of such terms in any finite idempotent Taylor affine-free algebra $\mathbb{A}$. Consider the ternary relations $R, S \subseteq \{x, y\}^3$ given by

$$R = \left\{ \begin{bmatrix} x \\ x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ x \end{bmatrix}, \begin{bmatrix} y \\ x \\ x \end{bmatrix} \right\}$$

and

$$S = \left\{ \begin{bmatrix} x \\ x \\ x \end{bmatrix}, \begin{bmatrix} x \\ y \\ y \end{bmatrix}, \begin{bmatrix} y \\ x \\ y \end{bmatrix} \right\}.$$

It's easy to check that each binary projection of R and S is subdirect in $\{x, y\}^2$ and contains (x, x). Now consider the CSP instance $\mathbf{X}$ with just a single variable v, and the apply the constraints R and S to the triple (v, v, v) (if this makes you uncomfortable, you can instead use several different variables and impose equality constraints between them). By the lemma, $\mathbf{X}$ is a pq-consistent instance.

If we let $\mathbb{F} = \mathcal{F}_{\mathbb{A}}(x, y) \le \mathbb{A}^{\mathbb{A}^2}$, then we may consider $\{x, y\}$ to be a subset of $\mathbb{F}$, and apply the proposition to see that $\mathrm{Sg}_{\mathbb{F}}(\mathbf{X})$ is also pq-consistent. Since $\mathbb{F}$ is finite, idempotent, Taylor, and affine-free, we can apply Theorem 3.13.8 to see that $\mathrm{Sg}_{\mathbb{F}}(\mathbf{X})$ has a solution. Suppose that this solution assigns the variable v to the value $c \in \mathbb{F}$. Then we have

$$\begin{bmatrix} c \\ c \\ c \end{bmatrix} \in \mathrm{Sg}_{\mathbb{F}}(R) \cap \mathrm{Sg}_{\mathbb{F}}(S) = \mathrm{Sg}_{\mathbb{F}} \left\{ \begin{bmatrix} x & x & y \\ x & y & x \\ y & x & x \end{bmatrix} \right\} \cap \mathrm{Sg}_{\mathbb{F}} \left\{ \begin{bmatrix} x & x & y \\ x & y & x \\ x & y & y \end{bmatrix} \right\}.$$

Thus there are ternary terms f, g of $\mathbb{A}$ such that

$$g\left(\begin{bmatrix} x & x & y \\ x & y & x \\ y & x & x \end{bmatrix} \right) = \begin{bmatrix} c \\ c \\ c \end{bmatrix} = f\left(\begin{bmatrix} x & x & y \\ x & y & x \\ x & y & y \end{bmatrix} \right),$$

and these f, g satisfy the required identities.

For the converse direction, we will suppose that such terms f, g exist for some idempotent algebra $\mathbb{A}$, and prove that $\mathbb{A}$ is Taylor and affine-free. It's easy to see that $\mathbb{A}$ must be Taylor, since the identities satisfied by g can't be satisfied by any projection. Since any identities which hold in $\mathbb{A}$ also hold in any subquotient of $\mathbb{A}$, we may suppose for contradiction that $\mathbb{A}$ is a nontrivial idempotent affine algebra. Then $\mathbb{A}$ is polynomially equivalent to some module $\mathbb{M}$ over some ring $\mathbb{R}$, and we may write

$$g(x, y, z) \approx \alpha x + \beta y + \gamma z$$

for some $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha + \beta + \gamma = 1$. Plugging in $x = 0$ to the identities

$$g(x, x, y) \approx g(x, y, x) \approx g(y, x, x)$$

gives $\alpha y \approx \beta y \approx \gamma y$, so

$$g(x, y, z) \approx \alpha(x + y + z)$$

and $3\alpha x \approx x$. Then if we plug in $x = 0$ to the identities

$$2\alpha x + \alpha y \approx f(x, x, y) \approx f(x, y, x) \approx f(x, y, y),$$

we see that $\alpha y \approx 2\alpha y$, so $\alpha y \approx 0$. Multiplying by 3, we get $y \approx 0$, so in fact the algebra $\mathbb{A}$ must consist of just the single element 0, a contradiction. □

The proof technique of Theorem 3.14.1 can be used to produce many further terms which mimic the monotone self-dual functions found in the clone of a two-element majority algebra.

Theorem 3.14.6. *Suppose* $\mathrm{CSP}(\mathbf{A})$ *has bounded relational width and* $\mathbf{A}$ *is finite. Then there is a binary polymorphism* $c(x, y)$*, and an infinite family of polymorphisms* $h_n^{\mathcal{F}} \in \mathrm{Pol}_n(\mathbf{A})$ *indexed by the collection of maximal intersecting families* $\mathcal{F}$ *of subsets of* $[n]$*, such that for each set* $S \in \mathcal{F}$ *with* $S \ne [n]$*, if we define* v_i^S *by*

$$v_i^S = \begin{cases} x & i \in S, \\ y & i \notin S, \end{cases}$$

we have the identity

$$h_n^{\mathcal{F}}(v_1^S, ..., v_n^S) \approx c(x, y).$$

Now we show how we can use the ternary terms f, g from Theorem 3.14.1 to solve the meta-problem.

Theorem 3.14.7. *Suppose we are given a finite relational structure* $\mathbf{A} = (A, R_1, ..., R_n)$*, where each relation* R_i *has arity* m_i *and is described by explicitly listing out its tuples, and suppose that we are promised that* $\mathbf{A}$ *is core. Then we can determine whether* $\mathrm{CSP}(\mathbf{A})$ *has bounded width in polynomial time, and in the case where* $\mathrm{CSP}(\mathbf{A})$ *has bounded width, we can explicitly find ternary functions* $f, g \in \mathrm{Pol}_3(\mathbf{A})$ *as in Theorem 3.14.1.*

Proof. We will define an instance $\mathbf{X}$ of $\mathrm{CSP}(\mathbf{A})$ such that every solution to $\mathbf{X}$ corresponds to a pair of terms f, g as in Theorem 3.14.1. The instance $\mathbf{X}$ will have two sets of $|A^3|$ variables, one variable for each value $f(a, b, c)$ for $a, b, c \in A$ and one variable for each value $g(a, b, c)$ for $a, b, c \in A$.

The relations of $\mathbf{X}$ will do two jobs: they will ensure that $f, g \in \mathrm{Pol}_3(\mathbf{A})$, and they will ensure that f, g satisfy the required identities. To ensure that $f \in \mathrm{Pol}_3(\mathbf{A})$, we consider every three tuples $a, b, c \in R_i$ (note that each of a, b, c is an m_i-tuple of values in A), and we impose the constraint

$$\begin{bmatrix} f(a_1, b_1, c_1) \\ f(a_2, b_2, c_2) \\ \vdots \\ f(a_{m_i}, b_{m_i}, c_{m_i}) \end{bmatrix} \in R_i$$

for each such tuple. The number of such constraints we need to impose to ensure that $f \in \mathrm{Pol}_3(\mathbf{A})$ is then

$$\sum_i |R_i|^3,$$

which is at most cubic in the size of the description of $\mathbf{A}$. We ensure that $g \in \mathrm{Pol}_3(\mathbf{A})$ with a similar collection of constraints.

To enforce the required identities between f, g, for every pair $a, b \in A$, we impose the equality constraints

$$g(a, a, b) = g(a, b, a) = g(b, a, a) = f(a, a, b) = f(a, b, a) = f(a, b, b).$$

This requires a total of $5|A|^2$ equality constraints. Thus, the instance $\mathbf{X}$ has overall size at most cubic in the size of the description of $\mathbf{A}$.

In order to solve $\mathbf{X}$, we view it as an instance of CSP($\mathbf{A}^{rig}$), where $\mathbf{A}^{rig}$ is the rigid core obtained from $\mathbf{A}$ by adding a singleton unary relation $\{a\}$ for each element $a \in A$. Note that if $\mathbf{A}$ is a core, then $\mathbf{A}$ has bounded width iff $\mathbf{A}^{rig}$ has bounded width (since each pp-constructs the other). We now attempt to solve the instance $\mathbf{X}$ by using the cycle-consistency algorithm, as follows. For each variable v of $\mathbf{X}$, we go through the values $a \in A$ in order, and temporarily modify $\mathbf{X}$ by adding the extra constraint $v \in \{a\}$ to make an instance $\mathbf{X}_{v=a}$. Then we reduce $\mathbf{X}_{v=a}$ until it either becomes cycle-consistent or until we reach a contradiction. If there is any $a \in A$ such that $\mathbf{X}_{v=a}$ becomes cycle-consistent, then we replace $\mathbf{X}$ by $\mathbf{X}_{v=a}$ and move on to the next variable. If every choice of $a \in A$ leads to $\mathbf{X}_{v=a}$ reaching a contradiction, then we give up and report that CSP($\mathbf{A}$) does not have bounded width.

If the procedure ends without us giving up, then we have found f, g as in Theorem 3.14.1 and these terms prove that CSP($\mathbf{A}$) has bounded width. Conversely, if CSP($\mathbf{A}$) has bounded width, then the original instance $\mathbf{X}$ has a solution, and each time we replace $\mathbf{X}$ by $\mathbf{X}_{v=a}$, the fact that $\mathbf{X}_{v=a}$ can be reduced to a cycle-consistent instance implies that it has a solution, so the whole procedure will end by successfully finding a pair of functions f, g. Of course, if CSP($\mathbf{A}$) does not have bounded width, then we will fail to find a solution to $\mathbf{X}$. □

A simple iteration argument allows us to give a criterion for bounded width involving just one ternary term and a binary term derived from it - however, the identities involved will not have height 1, so these new terms are unsuitable for the application to the meta-problem.

Theorem 3.14.8. *A finite relational structure* $\mathbf{A}$ *has bounded relational width if and only if it has a ternary polymorphism* $g \in \mathrm{Pol}_3(\mathbf{A})$ *such that, if* f *is the binary term* $f(x, y) := g(x, x, y)$*, we have*

$$g(x, x, y) \approx g(x, y, x) \approx g(y, x, x) \approx f(x, y) \approx f(f(x, y), f(y, x)) \approx f(f(x, y), f(x, y)).$$

Proof. Suppose first that $\mathbf{A}$ has bounded relational width, and let $f_3, g_3 \in \mathrm{Pol}_3(\mathbf{A})$ be terms as in Theorem 3.14.1. By an iteration argument applied to the unary operation $x \mapsto g_3(x,x,x)$, we may assume without loss of generality that we have

$$g_3(x,y,z) = h \circ g_3(x,y,z),$$

where $h(x) \coloneqq g_3(x,x,x)$. Define a sequence of terms g^i by $g^1 \coloneqq g_3$ and

$$g^{i+1}(x,y,z) \coloneqq g^i(f_3(x,y,z), f_3(y,z,x), f_3(z,x,y)).$$

Define binary terms f^i by $f^i(x,y) \coloneqq g^i(x,x,y)$. Then we have

$$f^1(x,y) \approx g_3(x,x,y) \approx f_3(x,x,y) \approx f_3(x,y,x) \approx f_3(x,y,y),$$

and for each i we have

$$\begin{aligned} f^{i+1}(x,y) \approx g^{i+1}(x,x,y) &\approx g^i(f_3(x,x,y), f_3(x,y,x), f_3(y,x,x)) \\ &\approx g^i(f^1(x,y), f^1(x,y), f^1(y,x)) \approx f^i(f^1(x,y), f^1(y,x)). \end{aligned}$$

Thus the sequence $f^i(x,y)$ is generated by iterating the map $(x,y) \mapsto (f^1(x,y), f^1(y,x))$. Since $\mathbf{A}$ is finite, there is some N such that $g^N \approx g^{2N}$ and $f^N \approx f^{2N}$. Take $f \coloneqq f^N$ and $g \coloneqq g^N$ to finish the construction.

Now suppose that f, g satisfy the assumed identities. Let e be the unary operation $e(x) \coloneqq f(x,x) = g(x,x,x)$. The identity

$$f(x,y) \approx f(f(x,y), f(x,y)) = e(f(x,y))$$

implies that

$$e(e(x)) \approx e(x),$$

so $\mathbf{A}$ is homomorphically equivalent to $e(\mathbf{A})$, and the restrictions of $f, e \circ g$ to $e(\mathbf{A})$ are idempotent. Let $\mathbb{A}_e$ be the idempotent algebra $(e(\mathbf{A}), f|_{e(\mathbf{A})}, e \circ g|_{e(\mathbf{A})})$. We will show that $\mathbb{A}_e$ is Taylor and affine-free.

That $\mathbb{A}_e$ is Taylor follows from the identity

$$e \circ g(x,x,y) \approx e \circ g(x,y,x) \approx e \circ g(y,x,x).$$

For the sake of contradiction, assume that $\mathbb{B} \in HSP(\mathbb{A}_e)$ is a nontrivial affine algebra. Then we can write

$$e \circ g(x,y,z) \approx \alpha(x+y+z)$$

on $\mathbb{B}$, for some α with $3\alpha x \approx x$. Then we have

$$f(x,y) \approx 2\alpha x + \alpha y,$$

so

$$f(f(x,y), f(y,x)) \approx 2\alpha(2\alpha x + \alpha y) + \alpha(2\alpha y + \alpha x) \approx 5\alpha^2 x + 4\alpha^2 y.$$

Equating these and setting y to 0, we see that $2\alpha x \approx 5\alpha^2 x$. Multiplying by 9 and using $3\alpha x \approx x$, we get $6x \approx 5x$, so $x \approx 0$ on $\mathbb{B}$, a contradiction. □

The identities satisfied by the term g of Theorem 3.14.8 have the following nice consequence.

Proposition 3.14.9. *Suppose that g is a ternary term as in Theorem 3.14.8, and that f is the associated binary term. Then for any a, b, either $f(a,b) = f(b,a)$, or the set $\{f(a,b), f(b,a)\}$ is closed under g, and $(\{f(a,b), f(b,a)\}, g)$ is isomorphic to a two-element majority algebra.*

For small examples of bounded width algebras $\mathbb{A}$ which do not contain large majority subalgebras, most of the structure of a bounded width algebra seems to be controlled by the binary term f from Theorem 3.14.8, with the exact values of the ternary term g only playing an important role on the majority subalgebras. I have also conjectured a very strong refinement of Theorem 3.14.8, which would give a much more explicit structure theory for bounded width algebras.

Conjecture 3.14.1. A finite idempotent algebra $\mathbb{A}$ has bounded relational width if and only if it has a ternary term m and an associated binary term $s(x,y) \coloneqq m(x,x,y)$, which satisfy the identities

$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx s(x,y)$$

and

$$s(x, s(x,y)) \approx s(s(x,y),x) \approx s(x,y).$$

3.15 Stable subalgebras, and even weaker consistency for bounded width

In this section we will introduce a new concept, which is similar to absorption but which is targeted at subdirect relations rather than arbitrary relations. This allows us to unify the treatment of centrally absorbing subalgebras with congruence classes of polynomially complete absorption free quotients, eliminating most of the casework we need to deal with. We will demonstrate the usefulness of this concept by solving the (P1)-(P3) problem (Problem 3.13.1). The approach used in this section is based on Zhuk's theory of "strong subalgebras" [191].

Rather than directly defining stable subalgebras, we will give an axiomatic description of what we want from a concept of "stability".

Definition 3.15.1. Suppose that $\mathcal{V}$ is a pseudovariety of finite algebras. We say that a binary relation $\prec$ on $\mathcal{V}$ is a *stability concept* (or just a *stability*) on $\mathcal{V}$ if $\prec$ satisfies the following axioms.

(Subalgebra) If $\mathbb{B} \prec \mathbb{A}$, then $\mathbb{B} \le \mathbb{A}$.

(Transitivity) If $\mathbb{C} \prec \mathbb{B} \prec \mathbb{A}$, then $\mathbb{C} \prec \mathbb{A}$.

(Intersection) If $\mathbb{B}, \mathbb{C} \prec \mathbb{A}$ and $\mathbb{B} \cap \mathbb{C} \ne \emptyset$, then $\mathbb{B} \cap \mathbb{C} \prec \mathbb{B}$.

(Propagation) If $f : \mathbb{A} \twoheadrightarrow \mathbb{B}$ is a surjective homomorphism, then

(Pushforward) if $\mathbb{C} \prec \mathbb{A}$, then $f(\mathbb{C}) \prec \mathbb{B}$, and

(Pullback) if $\mathbb{D} \prec \mathbb{B}$, then $f^{-1}(\mathbb{D}) \prec \mathbb{A}$.

(Helly) If $\mathbb{B}, \mathbb{C}, \mathbb{D} \prec \mathbb{A}$ are such that $\mathbb{B} \cap \mathbb{C} \ne \emptyset$, $\mathbb{C} \cap \mathbb{D} \ne \emptyset$, and $\mathbb{B} \cap \mathbb{D} \ne \emptyset$, then $\mathbb{B} \cap \mathbb{C} \cap \mathbb{D} \ne \emptyset$.

(Ubiquity) If $\mathbb{A} \in \mathcal{V}$ has $|\mathbb{A}| > 1$, then either

- there is some $\mathbb{B} \prec \mathbb{A}$ such that $\mathbb{B} \neq \mathbb{A}, \emptyset$, or
- there is some proper congruence $\theta \in \mathrm{Con}(\mathbb{A})$ such that $\mathbb{A}/\theta$ is an affine algebra.

Given a stability concept $\prec$ on $\mathcal{V}$, we say that $\mathbb{B}$ is a *stable subalgebra* of $\mathbb{A}$ if $\mathbb{B} \prec \mathbb{A}$. We say that an element $a \in \mathbb{A}$ is a *stable element* if $\{a\} \prec \mathbb{A}$.

The axioms of a stability concept imply apparently stronger versions of themselves.

Proposition 3.15.2. *If $\prec$ is a binary relation on $\mathcal{V}$ which satisfies the propagation axiom from Definition 3.15.1, then for any subdirect relation $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{B}$ in $\mathcal{V}$, we have*

$$\mathbb{C} \prec \mathbb{A} \quad \implies \quad \mathbb{C} + \mathbb{R} \prec \mathbb{B}.$$

Proof. Let π_1, π_2 be the surjective projection maps from $\mathbb{R}$ to $\mathbb{A}$ and $\mathbb{B}$, respectively. Then $\pi_1^{-1}(\mathbb{C}) \prec \mathbb{A}$ by the pullback part of the propagation axiom, so $\mathbb{C} + \mathbb{R} = \pi_2(\pi_1^{-1}(\mathbb{C})) \prec \mathbb{B}$ by the pushforward part of the propagation axiom. □

Proposition 3.15.3. *If $\prec$ satisfies the Helly axiom and the intersection axiom from Definition 3.15.1, then for any n and any $\mathbb{B}_1, ...\mathbb{B}_n \prec \mathbb{A}$ such that $\mathbb{B}_i \cap \mathbb{B}_j \neq \emptyset$ for all $i, j \in [n]$, we have $\bigcap_{i \in [n]} \mathbb{B}_i \neq \emptyset$.*

Proof. We induct on n - the base case $n = 3$ is the Helly axiom. For $n > 3$, set $\mathbb{A}' = \mathbb{B}_n$ and $\mathbb{B}_i' = \mathbb{B}_i \cap \mathbb{B}_n$ for $i < n$, then by the Helly axiom we have

$$\mathbb{B}_i' \cap \mathbb{B}_j' = \mathbb{B}_i \cap \mathbb{B}_j \cap \mathbb{B}_n \neq \emptyset$$

for all $i, j < n$, and by the intersection axiom we have $\mathbb{B}_i' = \mathbb{B}_i \cap \mathbb{B}_n \prec \mathbb{B}_n = \mathbb{A}'$ for all $i < n$, so we can apply the induction hypothesis to $\mathbb{A}'$ to see that

$$\bigcap_{i \in [n]} \mathbb{B}_i = \bigcap_{i < n} \mathbb{B}_i' \neq \emptyset.$$ □

Proposition 3.15.4. *If $\prec$ satisfies the Helly, intersection, and propagation axioms from Definition 3.15.1, then for any subdirect relation $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ in $\mathcal{V}$, if $\mathbb{B}_i \prec \mathbb{A}_i$ for each i and*

$$\pi_{ij}(\mathbb{R}) \cap (\mathbb{B}_i \times \mathbb{B}_j) \neq \emptyset$$

for all $i, j \in [n]$, then we have $\mathbb{R} \cap \prod_{i \in [n]} \mathbb{B}_i \neq \emptyset$.

If $\prec$ additionally satisfies the transitivity axiom, then we also have

$$\mathbb{R} \cap \Big(\prod_{i \in [n]} \mathbb{B}_i \Big) \prec \mathbb{R}.$$

Proof. For each i, the pullback part of the propagation axiom implies that $\pi_i^{-1}(\mathbb{B}_i) \prec \mathbb{R}$, so the previous proposition implies that $\bigcap_{i \in [n]} \pi_i^{-1}(\mathbb{B}_i) \neq \emptyset$. □

Proposition 3.15.5. *If $\prec$ satisfies the transitivity and ubiquity axioms from Definition 3.15.1, then for any idempotent bounded-width algebra $\mathbb{A} \in \mathcal{V}$ there is some $a \in \mathbb{A}$ such that $\{a\} \prec \mathbb{A}$.*

The precise choice of stability concept doesn't matter to us - we can use the following fact as a black box.

Theorem 3.15.6. *If $\mathcal{V}$ is an affine-free pseudovariety of finite idempotent Taylor algebras, then there is at least one stability concept $\prec$ on $\mathcal{V}$.*

Before we prove Theorem 3.15.6, we will apply it to prove that a weaker form of consistency suffices for bounded width CSPs.

Definition 3.15.7. An arc-consistent instance $\mathbf{X}$ of a CSP, with variable domains $\mathbb{A}_x$, is called *weakly consistent* if it satisfies

(W) for all nonempty subsets $A \subseteq \mathbb{A}_x$ and cycles p, q from x to x, we have

$$A + p + q = A \quad \implies \quad A \cap (A + p) \neq \emptyset.$$

Weak consistency is clearly implied by properties (P1) and (P3) from Definition 3.13.12. We can also rephrase weak consistency to make it look more similar to *pq*-consistency.

Proposition 3.15.8. *An arc-consistent instance $\mathbf{X}$ with finite variable domains $\mathbb{A}_x$ is weakly consistent if and only if it satisfies*

(W') for all $a \in \mathbb{A}_x$ and cycles p, q from x to x, there exist $j, k \geq 0$ such that

$$a \in \{a\} + j(p+q) + p - k(p+q).$$

In fact, if $\mathbf{X}$ is weakly consistent then for each x and each pair of cycles p, q we can find some $j \geq 0$ such that $\Delta_{\mathbb{A}_x} \subseteq \mathbb{P}_{j(p+q)+p-j(p+q)}$.

Proof. First we prove that (W) implies (W'). By finiteness we can pick some $j \geq 1$ such that $\{a\} + j(p+q) = \{a\} + 2j(p+q)$ for all $a \in \mathbb{A}_x$. Setting $A = \{a\} + j(p+q)$ and $q' = (j-1)(q+p)+q$, we have

$$A + p + q' = A + j(p+q) = \{a\} + 2j(p+q) = A,$$

so $A \cap A + p \neq \emptyset$, that is,

$$(\{a\} + j(p+q)) \cap (a + j(p+q) + p) \neq \emptyset.$$

Since this is true for all $a \in \mathbb{A}_x$, we have $\Delta_{\mathbb{A}_x} \subseteq \mathbb{P}_{j(p+q)+p-j(p+q)}$.

Next we show that (W') implies (W). Suppose that $A \subseteq \mathbb{A}_x$ satisfies $A + p + q = A$, and pick any element $a \in A$. If $j, k \geq 0$ are such that $a \in \{a\} + j(p+q) + p - k(p+q)$, then we have

$$A \cap (A + p) \supseteq (\{a\} + k(p+q)) \cap (\{a\} + j(p+q) + p) \neq \emptyset. \qquad \square$$

Our argument for showing that weakly consistent instances of bounded-width CSPs have solutions will follow the same general strategy as the argument for *pq*-consistent instances. First we will show that we can find an arc-consistent reduction where the reduced variable domains are all stable subalgebras of the original variable domains, and then we will try to show that any arc-consistent stable reduction is also weakly consistent. Unfortunately, we run into a snag: it is not clear that every arc-consistent stable reduction will really remain weakly consistent. To get around this, we introduce a still weaker condition which will make the strategy work.

Definition 3.15.9. An arc-consistent instance $\mathbf{X}$ of a CSP, with variable domains $\mathbb{A}_x$ contained in a variety $\mathcal{V}$ with a stability concept $\prec$, is called *stably consistent* if it satisfies

(S) for all nonempty stable subalgebras $\mathbb{B} \prec \mathbb{A}_x$ and cycles p, q from x to x, we have

$$\mathbb{B} + p + q = \mathbb{B} \implies \mathbb{B} \cap (\mathbb{B} + p) \neq \emptyset.$$

If all of the variable domains are finite affine-free algebras, then stable consistency is equivalent to the following:

(S') for all stable elements $\{a\} \prec \mathbb{A}_x$ and cycles p, q from x to x, there exist $j, k \geq 0$ such that

$$a \in \{a\} + j(p+q) + p - k(p+q).$$

Lemma 3.15.10. *If* $\mathbf{X}$ *is stably consistent and the variable domains* $\mathbb{A}_x$ *are affine free and are not all singletons, then there is some arc-consistent reduction* $\mathbf{X}'$ *of* $\mathbf{X}$ *such that every variable domain of* $\mathbf{X}'$ *is a stable subalgebra of the corresponding variable domain in* $\mathbf{X}$*, and such that at least one variable domain shrinks.*

Proof. Consider the directed graph with vertices given by pairs $(x, \mathbb{B})$ such that $\mathbb{B} \prec \mathbb{A}_x$ and $\mathbb{B} \neq \mathbb{A}_x, \emptyset$, with an edge from $(x, \mathbb{B})$ to $(y, \mathbb{B} + p)$ for each path p from x to y such that $\mathbb{B} + p \neq \mathbb{A}_y$. Note that by the propagation axiom and the assumption that $\mathbf{X}$ is arc-consistent, we always have

$$\mathbb{B} \prec \mathbb{A}_x \implies \mathbb{B} + p \prec \mathbb{A}_y.$$

Let $\mathcal{S}$ be a maximal strongly connected component of this directed graph.

Claim: For any $(x, \mathbb{B}) \in \mathcal{S}$ and any cycle p from x to x, we have $\mathbb{B} \cap (\mathbb{B} + p) \neq \emptyset$.

Proof of claim: If $(x, \mathbb{B} + p) \not\in \mathcal{S}$, then by maximality of $\mathcal{S}$ we must have $\mathbb{B} + p = \mathbb{A}_x$, so $\mathbb{B} \cap (\mathbb{B} + p) = \mathbb{B} \neq \emptyset$. Otherwise, if $(x, \mathbb{B} + p) \in \mathcal{S}$, then there must be some cycle q from x to x such that $\mathbb{B} + p + q = \mathbb{B}$. Then condition (S) from the definition of stable consistency implies that $\mathbb{B} \cap (\mathbb{B} + p) \neq \emptyset$.

Now define the *universal cover* $\mathbf{T}$ of $\mathbf{X}$ to be the instance whose underlying constraint hypergraph is an infinite tree with a surjective map $\pi : \mathbf{T} \twoheadrightarrow \mathbf{X}$ on the sets of variables, such that for every path p from x to y in $\mathbf{X}$ and every preimage u of x in $\mathbf{T}$ there is a unique lift of the path p to $\mathbf{T}$ which starts at u. The fact that $\mathbf{X}$ is arc-consistent is equivalent to the fact that the solution set to the infinite instance $\mathbf{T}$ is a subdirect relation of infinite arity.

Then for every pair of variables u, v of $\mathbf{T}$ and $\mathbb{B}, \mathbb{C}$ such that $(\pi(u), \mathbb{B}), (\pi(v), \mathbb{C}) \in \mathcal{S}$, there is a unique non-backtracking path p from u to v in $\mathbf{T}$, and by the claim we have $(\mathbb{B}+p) \cap \mathbb{C} \neq \emptyset$. Applying Proposition 3.15.4, we see that the set of solutions to $\mathbf{T}$ such that $u \in \mathbb{B}$ whenever $(\pi(u), \mathbb{B}) \in \mathcal{S}$ is a nonempty, stable subalgebra of the solution set to $\mathbf{T}$ (well, every finite subinstance of $\mathbf{T}$ has this property). Applying the pushforward part of the propagation axiom to this solution set, we obtain an arc-consistent stable reduction $\mathbf{X}'$ of the instance $\mathbf{X}$. □

Lemma 3.15.11. *If* $\mathbf{X}$ *is stably consistent and* $\mathbf{X}'$ *is an arc-consistent stable reduction of* $\mathbf{X}$*, then* $\mathbf{X}'$ *is also stably consistent.*

Proof. Let the variable domains of $\mathbf{X}$ and $\mathbf{X}'$ be $\mathbb{A}_x, \mathbb{A}_x'$, respectively. It's enough to show that if $\mathbb{B} \prec \mathbb{A}_x'$ is a nonempty stable subalgebra, and if p is any cycle from x to x in $\mathbf{X}$ such that $\mathbb{B} \cap (\mathbb{B} + p) \neq \emptyset$, then $\mathbb{B} \cap (\mathbb{B} + p')$ is also nonempty, where p' is the corresponding path in $\mathbf{X}'$. For this, consider the path instance $\mathbf{P}$ we get by unrolling the path p in $\mathbf{X}$, and let

$$\mathbb{R} \leq_{sd} \mathbb{A}_{x_0} \times \cdots \times \mathbb{A}_{x_n}$$

be the solution set to $\mathbf{P}$, with $x_0 = x_n = x$. Now apply Proposition 3.15.4 to $\mathbb{R}$, setting $\mathbb{B}_{x_0} = \mathbb{B}_{x_n} = \mathbb{B}$ and $\mathbb{B}_{x_i} = \mathbb{A}'_{x_i}$ for $i \neq 0, n$. □

Putting these results together, we see that weak consistency implies that a solution exists in bounded width CSPs.

Theorem 3.15.12. *If $\mathbf{X}$ is a weakly consistent instance of* $\mathrm{CSP}(\mathbb{A}_1, ..., \mathbb{A}_n)$*, where the $\mathbb{A}_i$ are finite bounded width algebras, then $\mathbf{X}$ has a solution. In fact, in this case $\mathbf{X}$ has a* stable *solution (i.e. a solution in which each variable x is assigned to a stable element of its variable domain $\mathbb{A}_x$).*

Of course, this all hinged on the existence of a stability concept: we still need to prove Theorem 3.15.6. Our construction of a stability concept won't be particularly elegant, but it will get the job done. First we will show that we can restrict to the case where every binary absorbing algebra is centrally absorbing.

Proposition 3.15.13. *Suppose that $\mathcal{V}$ is a pseudovariety of finite idempotent algebras, and that $\mathcal{V}'$ is an affine-free reduct of $\mathcal{V}$. If $\prec$ is a stability concept on $\mathcal{V}'$, then the restriction of $\prec$ to $\mathcal{V}$ is also a stability concept.*

Proof. The only nontrivial axiom to check is ubiquity. For this, note that since $\mathcal{V}'$ is affine-free, we can apply Proposition 3.15.5 to see that every $\mathbb{A} \in \mathcal{V}'$ has a stable element $\{a\} \prec \mathbb{A}$. Since $\mathcal{V}$ is idempotent, we also have $\{a\} \leq \mathbb{A}$ in $\mathcal{V}$. □

Proposition 3.15.14. *A pseudovariety $\mathcal{V}$ of finite algebras has a stability concept iff every finitely generated subvariety of $\mathcal{V}$ has a stability concept.*

Proof. This is an application of König's Lemma: for each algebra $\mathbb{A}$ in $\mathcal{V}$ with $|\mathbb{A}| > 1$ which has no affine quotients, we need to choose at least one proper subalgebra $\mathbb{B}$ to be a stable subalgebra of $\mathbb{A}$. Since there are only a finite number of choices for $\mathbb{B}$ for each finite $\mathbb{A}$, if we can make a consistent set of choices for every finite collection of algebras $\mathbb{A}_1, ..., \mathbb{A}_n$, then there exists a globally consistent set of choices. □

Definition 3.15.15. We say that a pseudovariety is *strongly prepared* if every binary absorbing algebra $\mathbb{B} \lhd_{bin} \mathbb{A} \in \mathcal{V}$ is also strongly absorbing.

Proposition 3.15.16. *Every locally finite variety of idempotent bounded width algebras has a strongly prepared, bounded width reduct.*

Proof. This follows by repeatedly applying Proposition 3.2.17 and Proposition 3.2.12, and using the fact that the set of two-variable terms can only be shrunk finitely many times if the free algebra on two generators is finite. □

To finish the proof of Theorem 3.15.6, we only need to construct stability concepts on pseudovarieties of idempotent, strongly prepared Taylor algebras.

Definition 3.15.17. If $\mathcal{V}$ is a pseudovariety of finite idempotent strongly prepared Taylor algebras, then we say that $\mathbb{B} \prec \mathbb{A}$ if there is a sequence of subalgebras

$$\mathbb{A} = \mathbb{A}_0 \geq \mathbb{A}_1 \geq \cdots \geq \mathbb{A}_n = \mathbb{B}$$

such that for each i, one of the following is true:

- $\mathbb{A}_{i+1}$ contains a strongly absorbing subalgebra of $\mathbb{A}_i$,
- $\mathbb{A}_{i+1}$ centrally absorbs $\mathbb{A}_i$, or
- there is a congruence θ on $\mathbb{A}_i$ such that $\mathbb{A}_i/\theta$ is polynomially complete, binary absorption-free and central absorption-free, and such that $\mathbb{A}_{i+1}$ is a congruence class of θ.

We say that $\mathbb{B}$ is *stable in one step* if we can take $n = 1$ in the above. Following [191], if the third bullet point above holds for $\mathbb{B}$, then we say that $\mathbb{B}$ is a *PC subalgebra* of $\mathbb{A}$ with *PC congruence* θ.

Algebras which are stable in one step are almost the same thing as what Zhuk calls *strong* subalgebras in [191] - the only difference is in how we handle the case of binary absorption.

Theorem 3.15.18. *If $\mathcal{V}$ is a pseudovariety of finite idempotent strongly prepared Taylor algebras, then the binary relation $\prec$ on $\mathcal{V}$ from Definition 3.15.17 is a stability concept.*

Proof. We just need to verify the axioms for $\prec$. Obviously $\prec$ satisfies the subalgebra axiom, and by Corollary 3.12.12 and the assumption that $\mathcal{V}$ is strongly prepared $\prec$ satisfies ubiquity as well. Transitivity holds for $\prec$ by construction. The remaining axioms are intersection, propagation, and the Helly property.

To verify the intersection axiom, suppose that $\mathbb{B}, \mathbb{C} \prec \mathbb{A}$: we need to check that $\mathbb{B} \cap \mathbb{C} \prec \mathbb{B}$. Inducting on $|\mathbb{A}| + |\mathbb{B}| + |\mathbb{C}|$, we see that it's enough to prove this when $\mathbb{B}$ and $\mathbb{C}$ are both stable in one step. For this, we divide into four cases: either $\mathbb{C} \lhd_Z \mathbb{A}$ (1), $\mathbb{C}$ contains a strongly absorbing subalgebra of $\mathbb{A}$ (2), $\mathbb{B}$ contains a centrally absorbing subalgebra of $\mathbb{A}$ and $\mathbb{C}$ is a PC subalgebra (3), or each of $\mathbb{B}, \mathbb{C}$ is a PC subalgebra (4). Case (1) follows from $\mathbb{B} \cap \mathbb{C} \lhd_Z \mathbb{B}$. For case (2), we need a claim which we will also use elsewhere.

Claim: If $\mathbb{S} \lhd_{str} \mathbb{A}$ and $\mathbb{B} \prec \mathbb{A}$, then $\mathbb{B} \cap \mathbb{S} \neq \emptyset$ and $\mathbb{B} \cap \mathbb{S} \lhd_{str} \mathbb{B}$.

Proof of claim: We just need to check this when $\mathbb{B}$ is stable in one step. If $\mathbb{B}$ contains a strongly absorbing subalgebra of $\mathbb{A}$, we can apply Proposition 3.2.22. If $\mathbb{B} \lhd_Z \mathbb{A}$, then we can apply Proposition 3.2.22 and Proposition 3.10.18. If $\mathbb{B}$ is a PC subalgebra with PC congruence θ, then $\mathbb{S}/\theta \lhd_{str} \mathbb{A}/\theta$ implies that $\mathbb{S}/\theta = \mathbb{A}/\theta$, so $\mathbb{S}$ meets every congruence class of θ, and in particular $\mathbb{B} \cap \mathbb{S} \neq \emptyset$.

Case (2) for intersection axiom: Let $\mathbb{C}'$ be a subalgebra of $\mathbb{C}$ such that $\mathbb{C}' \lhd_{str} \mathbb{A}$. Then $\mathbb{C}' \cap \mathbb{B} \neq \emptyset$ and $\mathbb{B} \cap \mathbb{C}' \lhd_{str} \mathbb{B}$, so $\mathbb{B} \cap \mathbb{C}$ contains a strongly absorbing subalgebra of $\mathbb{B}$.

Case (3) for intersection axiom: Let $\mathbb{B}'$ be a subalgebra of $\mathbb{B}$ such that $\mathbb{B}' \lhd_Z \mathbb{A}$, and let θ be the PC congruence for $\mathbb{C}$. Then $\mathbb{B}'/\theta \lhd_Z \mathbb{A}/\theta$ by Proposition 3.10.14, so since $\mathbb{A}/\theta$ is central absorption-free, we must have $\mathbb{B}'/\theta = \mathbb{A}/\theta$. Since $\mathbb{B}' \le \mathbb{B} \le \mathbb{A}$, we must have $\mathbb{B}/\theta = \mathbb{A}/\theta$ as well, so $\mathbb{C}$ is a PC subalgebra of $\mathbb{B}$ with PC congruence $\theta|_{\mathbb{B}}$.

Case (4) for intersection axiom: Suppose that $\mathbb{B}$ has PC congruence θ and $\mathbb{C}$ has PC congruence ψ. We can consider $\mathbb{A}/(\theta \wedge \psi)$ as a binary subdirect relation:

$$\mathbb{A}/(\theta \wedge \psi) \le_{sd} (\mathbb{A}/\theta) \times (\mathbb{A}/\psi).$$

Since $\mathbb{A}/\theta, \mathbb{A}/\psi$ are both simple (all polynomially complete algebras are simple), this binary relation is either linked or is the graph of an isomorphism. If $\mathbb{A}/(\theta \wedge \psi)$ is the graph of an isomorphism, then we must have $\theta = \psi$, so if $\mathbb{B} \cap \mathbb{C} \neq \emptyset$ then $\mathbb{B} = \mathbb{C}$. Otherwise, if $\mathbb{A}/(\theta \wedge \psi)$ is linked, then by the Absorption Theorem 3.11.1 we must have

$$\mathbb{A}/(\theta \wedge \psi) = (\mathbb{A}/\theta) \times (\mathbb{A}/\psi),$$

since neither $\mathbb{A}/\theta$ nor $\mathbb{A}/\psi$ has a binary or centrally absorbing subalgebras. Thus $\mathbb{B}/\psi = \mathbb{A}/\psi$, so $\mathbb{B} \cap \mathbb{C}$ is a PC subalgebra of $\mathbb{B}$ with PC congruence $\psi|_{\mathbb{B}}$.

That completes the proof of the intersection axiom. For the propagation axiom, the pullback part is almost immediate from the definition. For the pushforward part of the propagation axiom, we can suppose that $\mathbb{A} \twoheadrightarrow \mathbb{A}/\theta$ and that $\mathbb{C}$ is stable in one step in $\mathbb{A}$. If $\mathbb{C} \supseteq \mathbb{C}' \lhd_{str} \mathbb{A}$, then $\mathbb{C}'/\theta \lhd_{str} \mathbb{A}/\theta$. If $\mathbb{C} \lhd_Z \mathbb{A}$, then by Proposition 3.10.14 we have $\mathbb{C}/\theta \lhd_Z \mathbb{A}/\theta$. The tricky case is the case where $\mathbb{C}$ is a PC subalgebra of $\mathbb{A}$ with PC congruence ψ. In this case, we consider $\mathbb{A}/(\theta \wedge \psi)$ as a binary subdirect relation:

$$\mathbb{A}/(\theta \wedge \psi) \le_{sd} (\mathbb{A}/\theta) \times (\mathbb{A}/\psi).$$

Since $\mathbb{A}/\psi$ is simple, this binary relation is either the graph of a homomorphism $(\mathbb{A}/\theta) \twoheadrightarrow (\mathbb{A}/\psi)$ or is linked. If it is the graph of a homomorphism, then we must have $\theta \le \psi$, so $\mathbb{C}/\theta$ is a PC subalgebra of $\mathbb{A}/\theta$ with PC congruence ψ/θ. Now suppose that it is linked, and let $\mathbb{S}$ be a minimal strongly absorbing subalgebra of $\mathbb{A}/\theta$, which exists by Proposition 3.2.22. Then by Theorem 3.7.12, the binary relation

$$\mathbb{A}/(\theta \wedge \psi) \cap (\mathbb{S} \times (\mathbb{A}/\psi)) \le_{sd} \mathbb{S} \times (\mathbb{A}/\psi)$$

is also linked, and then by the Absorption Theorem 3.11.1 it must be the full relation, since $\mathbb{S}$ has no binary absorbing subalgebras and $\mathbb{A}/\psi$ has no binary/centrally absorbing subalgebras. This means that $\mathbb{C}/\theta$ contains $\mathbb{S}$, so $\mathbb{C}/\theta$ is a stable subalgebra of $\mathbb{A}/\theta$.

To finish, we just need to verify the Helly axiom, which states that if $\mathbb{B}, \mathbb{C}, \mathbb{D} \prec \mathbb{A}$ and each pair has a nonempty intersection, then $\mathbb{B} \cap \mathbb{C} \cap \mathbb{D} \ne \emptyset$. We induct on

$$|\mathbb{A}| + |\mathbb{A} \setminus \mathbb{B}| + |\mathbb{A} \setminus \mathbb{C}| + |\mathbb{A} \setminus \mathbb{D}|.$$

Suppose that one of $\mathbb{B}, \mathbb{C}, \mathbb{D}$ is not stable in one step, say $\mathbb{B}$. Then there is some $\mathbb{A}'$ with $\mathbb{B} \prec \mathbb{A}' \prec \mathbb{A}$ such that $|\mathbb{A} \setminus \mathbb{A}'| < |\mathbb{A} \setminus \mathbb{B}|$ and $|\mathbb{A}'| < |\mathbb{A}|$. By the induction hypothesis, we have $\mathbb{A}' \cap \mathbb{C} \cap \mathbb{D} \ne \emptyset$. Set $\mathbb{C}' = \mathbb{A}' \cap \mathbb{C}$, $\mathbb{D}' = \mathbb{A}' \cap \mathbb{D}$, and $\mathbb{B}' = \mathbb{B}$. Then by the intersection property we have $\mathbb{B}', \mathbb{C}', \mathbb{D}' \prec \mathbb{A}'$, and we have

$$\mathbb{C}' \cap \mathbb{D}' = \mathbb{A}' \cap \mathbb{C} \cap \mathbb{D} \ne \emptyset,$$

while

$$\mathbb{B}' \cap \mathbb{C}' = \mathbb{B} \cap \mathbb{C} \ne \emptyset$$

and similarly $\mathbb{B}' \cap \mathbb{D}' \ne \emptyset$. Applying the induction hypothesis again, we see that

$$\mathbb{B} \cap \mathbb{C} \cap \mathbb{D} = \mathbb{B}' \cap \mathbb{C}' \cap \mathbb{D}' \ne \emptyset.$$

So we only need to check the Helly property when each of $\mathbb{B}, \mathbb{C}, \mathbb{D}$ is stable in one step.

Suppose that one of $\mathbb{B}, \mathbb{C}, \mathbb{D}$ contains a strongly absorbing subalgebra $\mathbb{S} \lhd_{str} \mathbb{A}$, say $\mathbb{B} \supseteq \mathbb{S}$. Then $\mathbb{S} \cap \mathbb{C}$ is a nonempty strongly absorbing subalgebra of $\mathbb{C}$ by the earlier claim, and applying that claim again together with the fact that $\mathbb{C} \cap \mathbb{D} \prec \mathbb{C}$, we see that $\mathbb{S} \cap \mathbb{C} \cap \mathbb{D} \ne \emptyset$. So we may assume that each of $\mathbb{B}, \mathbb{C}, \mathbb{D}$ is either centrally absorbing or is a PC subalgebra.

If all three of $\mathbb{B}, \mathbb{C}, \mathbb{D}$ are centrally absorbing, then by Corollary 3.10.13 there is a ternary term t such that each of $\mathbb{B}, \mathbb{C}, \mathbb{D}$ absorbs $\mathbb{A}$ with respect to t. If we pick $x \in \mathbb{B} \cap \mathbb{C}, y \in \mathbb{C} \cap \mathbb{D}, z \in \mathbb{B} \cap \mathbb{D}$, then we must have $t(x, y, z) \in \mathbb{B} \cap \mathbb{C} \cap \mathbb{D}$.

If two of $\mathbb{B}, \mathbb{C}, \mathbb{D}$ are centrally absorbing, suppose that $\mathbb{B}, \mathbb{C}$ are centrally absorbing and $\mathbb{D}$ is a PC subalgebra with PC congruence θ. Then $\mathbb{B} \cap \mathbb{C} \lhd_Z \mathbb{A}$, so we must have $(\mathbb{B} \cap \mathbb{C})/\theta \lhd_Z \mathbb{A}/\theta$, so $(\mathbb{B} \cap \mathbb{C})/\theta = \mathbb{A}/\theta$. Thus $\mathbb{B} \cap \mathbb{C}$ intersects $\mathbb{D}$.

If one of $\mathbb{B},\mathbb{C},\mathbb{D}$ is centrally absorbing, we may suppose that $\mathbb{B} \lhd_Z \mathbb{A}$ and that $\mathbb{C},\mathbb{D}$ are PC subalgebras with PC congruences θ,ψ. As before, we must have $\mathbb{B}/\theta = \mathbb{A}/\theta$ and $\mathbb{B}/\psi = \mathbb{A}/\psi$. Considering $\mathbb{A}/(\theta \wedge \psi)$ as a subdirect binary relation on $(\mathbb{A}/\theta) \times (\mathbb{A}/\psi)$, we see that it must be linked if $\mathbb{C} \ne \mathbb{D}$, since both of $\mathbb{A}/\theta, \mathbb{A}/\psi$ are simple. Then by Theorem 3.7.12, the binary relation

$$\mathbb{B}/(\theta \wedge \psi) \le_{sd} (\mathbb{A}/\theta) \times (\mathbb{A}/\psi)$$

is also linked, so by the Absorption Theorem 3.11.1 it must be the full relation. Thus we have $\mathbb{B} \cap \mathbb{C} \cap \mathbb{D} \ne \emptyset$.

Finally, suppose that all three of $\mathbb{B},\mathbb{C},\mathbb{D}$ are PC subalgebras, with PC congruences θ,ψ,η. Then we can think of $\mathbb{A}/(\theta \wedge \psi \wedge \eta)$ as a ternary subdirect relation:

$$\mathbb{A}/(\theta \wedge \psi \wedge \eta) \le_{sd} (\mathbb{A}/\theta) \times (\mathbb{A}/\psi) \times (\mathbb{A}/\eta).$$

If no two of $\mathbb{B},\mathbb{C},\mathbb{D}$ are equal to each other, then every binary projection of this relation is linked, so every binary projection is full by the Absorption Theorem 3.11.1. If this ternary relation is the full relation, then we have $\mathbb{B} \cap \mathbb{C} \cap \mathbb{D} \ne \emptyset$. Otherwise, pick some $x \in \mathbb{A}/\theta$ such that the binary relation

$$\pi_{23}(\mathbb{A}/(\theta \wedge \psi \wedge \eta) \cap (\{x\} \times (\mathbb{A}/\psi) \times (\mathbb{A}/\eta))) \le_{sd} (\mathbb{A}/\psi) \times (\mathbb{A}/\eta)$$

is not the full relation. Applying the Absorption Theorem 3.11.1 again, we see that the binary relation above must be the graph of an isomorphism between $\mathbb{A}/\psi$ and $\mathbb{A}/\eta$. A similar argument shows that $\mathbb{A}/\theta$ is isomorphic to $\mathbb{A}/\psi$. Thus we can apply Theorem 3.12.9 to show that all three of $\mathbb{A}/\theta, \mathbb{A}/\psi, \mathbb{A}/\eta$ are affine, which contradicts the assumption that they are polynomially complete. □

Problem 3.15.1. Is there a less ad-hoc stability concept?

3.15.1 Ramsey-theoretic upgrade: vague solutions imply solvability

Unsatisfyingly, even weak consistency is too demanding to directly prove the existence of a 4-ary Siggers term satisfying the identity $t(x,x,y,z) \approx t(y,z,z,x)$. Using Ramsey's theorem, we can cure this particular defect. The material in this subsection is based on the theory developed in [36].

In this subsection, we are mainly concerned with the following question: given an instance $\mathbf{X}$ whose variable domains and relations are not assumed to be closed under the basic operations of our bounded width algebra, under what circumstances can we guarantee that $\mathrm{Sg}(\mathbf{X})$ has a solution? If the variable domains of $\mathbf{X}$ consist of generating sets for free algebras, then this question is equivalent to asking which systems of height 1 identities can be solved in every finite bounded width algebra. We will show that if the instance $\mathbf{X}$ has a "vague" solution, then $\mathrm{Sg}(\mathbf{X})$ is guaranteed to have a solution in any finite bounded width algebra which contains the variable domains of $\mathbf{X}$.

Definition 3.15.19. Let $\mathcal{P}_\emptyset(A)$ be the set of non-empty subsets of a set A. A *vague element* of A is defined to be a total quasiorder $\preceq$ on $\mathcal{P}_\emptyset(A)$ such that there is no pair of disjoint subsets $S,T \subset A$ with $S \sim T$, where $S \sim T$ means that $S \preceq T$ and $T \preceq S$.

If $R \subseteq_{sd} A_1 \times \cdots \times A_n$ is subdirect, then we say that a collection of vague elements $\preceq_i$ of the A_is *vaguely satisfies* the relation R if there exists a total quasiorder $\preceq_R$ on the disjoint union

$$\mathcal{P}_\emptyset(A_1) \sqcup \cdots \sqcup \mathcal{P}_\emptyset(A_n)$$

such that the restriction of $\preceq_R$ to $\mathcal{P}_\emptyset(A_i)$ is $\preceq_i$ for each i, and such that for each $i, j \in [n]$ and each nonempty $S \subseteq A_i$, we have

$$S \preceq_R S + \pi_{ij}(R).$$

If $\mathbf{X}$ is an arc-consistent instance with variable domains A_x, then a collection of vague elements $\preceq_x$ of the A_xs is a *vague solution* to $\mathbf{X}$ if it vaguely satisfies every relation of $\mathbf{X}$.

Total quasiorders are also known as *preference relations* - so a vague element of A is a preference relation on the nonempty subsets of A which our hypothetical element might live in, which avoids being caught out as incoherent by requiring that any pair of equally preferable subsets has a nonempty intersection. Vague solutions are closely connected to weak consistency.

Proposition 3.15.20. *Every weakly consistent instance has a vague solution. In fact, there is always a vague solution where each vague element $\preceq_x$ extends the inclusion order $\subseteq$ on $\mathcal{P}_\emptyset(A_x)$ (possibly identifying some subsets with each other as well).*

Proof. Suppose $\mathbf{X}$ is a weakly consistent instance with variable domains A_x, and define a quasiorder $\preceq_0$ on

$$\bigsqcup_x \mathcal{P}_\emptyset(A_x)$$

by

$$S \preceq_0 S + p$$

for every path p in $\mathbf{X}$. Let $\preceq$ be any extension of $\preceq_0$ to a total quasiorder which does not identify any pair of sets which were not already identified by $\preceq_0$. Then if we take $\preceq_x$ to be the restriction of $\preceq$ to $\mathcal{P}_\emptyset(A_x)$ for each variable x, we see that each $\preceq_x$ is a vague element of A_x (since $\mathbf{X}$ is weakly consistent) and that the collection of vague elements $\preceq_x$ is a vague solution to $\mathbf{X}$.

For the second claim, we note that if we add additional tuples to any relation of a weakly consistent instance then it remains weakly consistent by Proposition 3.15.8. If we add extra equality relations from a variable to itself then clearly the instance remains weakly consistent as well, so we see that we may add in the binary relation

$$\Delta_{A_x} \cup \{(a,b)\} \subseteq_{sd} A_x \times A_x$$

for any $a, b \in A_x$ without causing the instance to stop being weakly consistent. If we add all such binary relations in, then we see that the quasiorder $\preceq_0$ has $(x,S) \preceq_0 (x,T)$ for any $S \subseteq T$, so the same will be true in the extension $\preceq$ of $\preceq_0$. □

The converse isn't true - there are instances which have vague solutions, but which are not weakly consistent.

Example 3.15.1. Consider the Siggers instance, which has one variable u with domain $A_u = \{x, y, z\}$, and one binary relation $R \subseteq_{sd} A_u \times A_u$ given by

$$R = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} y \\ z \end{bmatrix}, \begin{bmatrix} z \\ x \end{bmatrix} \right\}.$$

This instance is *not* weakly consistent: we have

$$\{z\} + R = \{x\}, \ \ \{x\} - R = \{z\}$$

and $\{x\} \cap \{z\} = \emptyset$. Nevertheless, the Siggers instance does have a vague solution: take $\preceq_u$ to be the total order

$$\{y\} \prec_u \{x\} \prec_u \{z\} \prec_u \{x, y\} \prec_u \{y, z\} \prec_u \{x, z\} \prec_u \{x, y, z\}.$$

To see that $(\preceq_u, \preceq_u)$ vaguely satisfies R, use the following total quasiorder $\preceq_R$ on $[2] \times \mathcal{P}_\emptyset(A_u)$:

$$\begin{aligned}(1, \{y\}) \prec_R (2, \{y\}) \prec_R (1, \{x\}) \prec_R (2, \{x\}) \sim_R (1, \{z\}) \prec_R (2, \{z\})&\\ \prec_R (2, \{x, y\}) \prec_R (1, \{x, y\}) \sim_R (2, \{y, z\}) \prec_R (1, \{y, z\}) \prec_R (2, \{x, z\}) \prec_R (1, \{x, z\})&\\ \prec_R (1, \{x, y, z\}) \sim_R (2, \{x, y, z\}).&\end{aligned}$$

In order to show that vaguely solvable instances have solutions (after taking the closure by algebraic operations), we will construct a very large weakly consistent instance with many copies of each variable and relation from the original instance. In fact, by König's Lemma we may even allow ourselves to build an infinitely large weakly consistent instance (although we will only truly need a finite portion of it). Roughly speaking, we will take the variables and relations of this instance to be indexed by subsets of $\mathbb{N}$ of certain fixed sizes, which will put us in a position to apply Ramsey's theorem for hypergraphs.

Definition 3.15.21. If $f : S \to \mathbb{N}$ is any function, then we define the *associated total quasiorder* $\preceq_f$ on S by

$$a \preceq_f b \iff f(a) \le f(b).$$

If $\preceq$ is a vague element of A, then we say that a function $f : \mathcal{P}_\emptyset(A) \to \mathbb{N}$ is *compatible* with $\preceq$ if $\preceq_f = \preceq$.

More generally, if $R \subseteq_{sd} A_1 \times \cdots \times A_n$ is a subdirect relation with a vague solution $(\preceq_1, ..., \preceq_n)$, then we say that a function

$$f : \bigsqcup_i \mathcal{P}_\emptyset(A_i) \to \mathbb{N}$$

is *compatible* with R and $(\preceq_1, ..., \preceq_n)$ if $\preceq_f$ can be used as the total quasiorder $\preceq_R$ from the definition of vague satisfaction.

Definition 3.15.22. If $\mathbf{X}$ is an arc-consistent instance with a vague solution given by vague elements $\preceq_x$ of the variable domains A_x, then we define the *associated weakly consistent instance* $\mathbf{X}^*$ as follows:

- for each variable x of $\mathbf{X}$ and each $f : \mathcal{P}_\emptyset(A_x) \to \mathbb{N}$ which is compatible with $\preceq_x$, we have a variable (x, f) of $\mathbf{X}^*$, and
- for each constraint relation $R \subseteq_{sd} A_{x_1} \times \cdots \times A_{x_n}$ of $\mathbf{X}$ and each $f : \bigsqcup_i \mathcal{P}_\emptyset(A_{x_i}) \to \mathbb{N}$ which is compatible with R and $(\preceq_{x_1}, ..., \preceq_{x_n})$, we impose

$$((x_1, f|_{\mathcal{P}_\emptyset(A_{x_1})}), ..., (x_n, f|_{\mathcal{P}_\emptyset(A_{x_n})})) \in R$$

as a constraint in $\mathbf{X}^*$.

Proposition 3.15.23. *If* $\mathbf{X}$ *has a vague solution, then the associated weakly consistent instance* $\mathbf{X}^*$ *is indeed weakly consistent.*

Proof. By the construction of $\mathbf{X}^*$, if there is a path p from (x, f) to (y, g) in $\mathbf{X}^*$ and if $S \subseteq A_x$, then we have

$$f(S) \le g(S + p).$$

Thus if we have $S + p + q = S$ and p, q are cycles from (x, f) to (x, f), then

$$f(S) \le f(S + p) \le f(S + p + q) = f(S)$$

implies that $f(S) = f(S + p)$. Then since f is compatible with $\preceq_x$ we must have $S \sim_x S + p$, so $S \cap (S + p) \ne \emptyset$ since $\preceq_x$ is a vague element. □

Proposition 3.15.24. *If $\mathbf{X}^*$ is weakly consistent and its variable domains are contained in a finite bounded width algebra, then $\mathrm{Sg}(\mathbf{X}^*)$ has a stable solution.*

Proof. By Proposition 3.15.8, an instance is weakly consistent if and only if certain cycles take every element of the variable domain back to themselves, so if $\mathbf{X}^*$ is weakly consistent then so is $\mathrm{Sg}(\mathbf{X}^*)$. By König's Lemma, in order to check that $\mathrm{Sg}(\mathbf{X}^*)$ has a solution we just need to check that every finite subinstance of $\mathrm{Sg}(\mathbf{X}^*)$ has a stable solution, and for this we can apply the main result of the previous section. □

Theorem 3.15.25. *If $\mathbf{X}$ is an arc-consistent instance which has a vague solution, and if the variable domains of $\mathbf{X}$ are contained in a finite bounded width algebra, then $\mathrm{Sg}(\mathbf{X})$ has a stable solution.*

Proof. Let $\mathbf{X}^*$ be the associated weakly consistent instance. By the previous proposition, $\mathrm{Sg}(\mathbf{X}^*)$ has a stable solution. Fix one particular stable solution to $\mathrm{Sg}(\mathbf{X}^*)$.

To finish, we imagine "coloring" the compatible functions $f : \mathcal{P}_\emptyset(A_x) \to \mathbb{N}$ by the values that the variables (x, f) are assigned to in our solution of $\mathrm{Sg}(\mathbf{X}^*)$. Since the variable domains in the instance $\mathrm{Sg}(\mathbf{X}^*)$ are finite, we only have finitely many colors to choose from, so Ramsey's theorem for hypergraphs implies that there is an infinite subset $S \subseteq \mathbb{N}$ such that each compatible function $f : \mathcal{P}_\emptyset(A_x) \to S$ has the same color. Iterating this for each variable x of the instance $\mathbf{X}$, we finally find an infinite subset $U \subseteq \mathbb{N}$ such that for each variable x of $\mathbf{X}$ and each compatible $f : \mathcal{P}_\emptyset(A_x) \to U$, the value assigned to (x, f) in our solution to $\mathrm{Sg}(\mathbf{X}^*)$ only depends on x and does not depend on f.

We claim that assigning the variable x of $\mathbf{X}$ to the value assigned to any such (x, f) (with $f : \mathcal{P}_\emptyset(A_x) \to U$ compatible with $\preceq_x$) in our solution to $\mathrm{Sg}(\mathbf{X}^*)$ solves the instance $\mathrm{Sg}(\mathbf{X})$. To see this, let $R \subseteq A_{x_1} \times \cdots \times A_{x_n}$ be any constraint relation of $\mathbf{X}$. Then since $(\preceq_{x_1}, ..., \preceq_{x_n})$ vaguely satisfies R, there is some total quasiorder $\preceq_R$ as in the definition of vague satisfaction. Since $U \subseteq \mathbb{N}$ is infinite (or just sufficiently large), we can then find a function $f : \bigsqcup_i \mathcal{P}_\emptyset(A_{x_i}) \to U$ such that $\preceq_f = \preceq_R$, and for this f we see that

$$((x_1, f|_{\mathcal{P}_\emptyset(A_{x_1})}), ..., (x_n, f|_{\mathcal{P}_\emptyset(A_{x_n})})) \in \mathrm{Sg}(R)$$

is a constraint of $\mathrm{Sg}(\mathbf{X}^*)$. Therefore, the tuple of values assigned to $(x_1, ..., x_n)$ by this procedure satisfies the constraint $\mathrm{Sg}(R)$ of the instance $\mathrm{Sg}(\mathbf{X})$. □

Remark 3.15.1. The same Ramsey argument can be used to show that if an instance has a vague solution, then it has a vague solution where each vague element $\preceq_x$ extends the inclusion order $\subseteq$ on $\mathcal{P}_\emptyset(A_x)$. A refinement of this argument shows that we can also assume that our vague elements $\preceq_x$ have the following property: whenever $S \preceq_x T$, we also have $A_x \setminus T \preceq_x A_x \setminus S$.

We can use stability to upgrade this result further: we don't need to find a vague solution to the full instance $\mathbf{X}$, it's enough to find a vague solution to just the binary part of $\mathbf{X}$.

Definition 3.15.26. If $\mathbf{X}$ is an instance, then we define the *binary part* of $\mathbf{X}$ to be the instance $\mathbf{X}^{bin}$ given by replacing each k-ary constraint relation $R \subseteq A_{x_1} \times \cdots \times A_{x_k}$ by the collection of binary relations $\pi_{ij}(R) \subseteq A_{x_i} \times A_{x_j}$ for $i, j \in [k]$.

Corollary 3.15.27. *If $\mathbf{X}$ is an arc-consistent instance such that $\mathbf{X}^{bin}$ has a vague solution, and if the variable domains of $\mathbf{X}$ are contained in a finite bounded width algebra, then* $\mathrm{Sg}(\mathbf{X})$ *has a stable solution.*

Proof. By Proposition 3.15.4 and arc-consistency, any stable solution to $\mathrm{Sg}(\mathbf{X}^{bin})$ is also a stable solution to $\mathrm{Sg}(\mathbf{X})$. □

Example 3.15.2. There is an arc-consistent instance $\mathbf{X}$ such that $\mathbf{X}^{bin}$ has a vague solution but $\mathbf{X}$ does not: take the 5-ary relation $R \subseteq_{sd} \{a, b, c, d\}^5$ given by

$$R = \left\{ \begin{bmatrix} a \\ a \\ b \\ c \\ d \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \\ d \\ a \end{bmatrix}, \begin{bmatrix} b \\ c \\ d \\ a \\ a \end{bmatrix}, \begin{bmatrix} c \\ d \\ a \\ a \\ b \end{bmatrix}, \begin{bmatrix} d \\ a \\ a \\ b \\ c \end{bmatrix} \right\},$$

and let $\mathbf{X}$ be the instance with a single variable x which is supposed to satisfy the constraint $(x, x, x, x, x) \in R$. To see that $\mathbf{X}$ has no vague solution, it's enough to consider the relative order of the singleton sets $\{b\}$ and $\{c\}$. On the other hand, $\mathbf{X}^{bin}$ has a vague solution where all five vague elements are given by

$$\{d\} \prec \{c\} \prec \{b\} \prec \{c, d\} \prec \{b, d\} \prec \{a\} \prec \{b, c\} \prec \{a, d\} \sim \cdots \sim \{a, b, c, d\}.$$

The fact that examples like this exist seems to be a hint that vague solutions are nowhere near to being the last word on bounded width CSPs.

Problem 3.15.2. If the sizes of the variable domains in $\mathbf{X}$ are bounded by a constant (perhaps just 3), how hard is it to determine whether $\mathbf{X}^{bin}$ has a vague solution?

We can at least make it a bit simpler for ourselves to check that a particular assignment of vague elements really gives us a vague solution.

Proposition 3.15.28. *If $R \subseteq_{sd} A_x \times A_y$ and $\preceq_x, \preceq_y$ are vague elements of A_x, A_y, respectively, then the pair $(\preceq_x, \preceq_y)$ vaguely satisfies R iff the following implication holds for all $S \subseteq A_x, T \subseteq A_y$:*

$$S + R \preceq_y T \;\;\wedge\;\; T - R \preceq_x S \quad \Longrightarrow \quad S + R \sim_y T \;\;\wedge\;\; T - R \sim_x S.$$

Proof. Consider the digraph on $\mathcal{P}_\emptyset(A_x) \sqcup \mathcal{P}_\emptyset(A_y)$ with an edge from $S \subseteq A_x$ to $T \subseteq A_y$ whenever $S + R \preceq_y T$, and similarly with an edge from $T \subseteq A_y$ to $S \subseteq A_x$ whenever $T - R \preceq_x S$. We just need to check that for every cycle $(S_1, T_1, S_2, T_2, ..., S_k, T_k)$ of this digraph, all S_is are related by $\sim_x$ and all T_js are related by $\sim_y$. For this, suppose that S_i is $\preceq_x$-minimal and that T_j is $\preceq_y$-minimal. Then from

$$S_j + R \preceq_y T_j \preceq_y T_{i-1}$$

and

$$T_{i-1} - R \preceq_x S_i \preceq_x S_j,$$

we see that $S_i \sim_x S_j$ and $T_{i-1} \sim_y T_j$, so S_j is $\preceq_x$-minimal and T_{i-1} is $\preceq_y$-minimal. Applying the same reasoning to S_j, T_{i-1}, we see that S_{i-1} is $\preceq_x$-minimal and T_{j-1} is $\preceq_y$-minimal. Continuing in this fashion, we see that all of the Ss are related by $\sim_x$ and all of the Ts are related by $\sim_y$, which is what we had to check. □

As our first illustration of the theory, we can show the existence of a Siggers term which satisfies a strong collection of additional identities.

Proposition 3.15.29. *A finite algebra $\mathbb{A}$ has bounded relational width if and only if it has a 4-ary term t which satisfies the identities*

$$t(x,x,y,z) \approx t(y,z,z,x) \approx t(z,x,y,x)$$

and

$$t(x,y,x,z) \approx t(x,z,y,x) \approx t(y,z,x,x)$$

simultaneously.

Proof. It is easy to see that the identities satisfied by t imply that the ternary terms f, g defined by

$$g(x,y,z) \coloneqq t(x,x,y,z), \quad f(x,y,z) \coloneqq t(x,y,x,z)$$

satisfy the equations

$$g(x,x,y) \approx g(x,y,x) \approx g(y,x,x) \approx f(x,x,y) \approx f(x,y,x) \approx f(x,y,y).$$

from Theorem 3.14.1, so if such a t exists then $\mathbb{A}$ has bounded relational width.

Now suppose that $\mathbb{A}$ has bounded relational width. Let R be the following 6-ary relation on $\{x,y,z\}$:

$$R = \left\{ \begin{bmatrix} x\\ y\\ z\\ x\\ x\\ y \end{bmatrix}, \begin{bmatrix} x\\ z\\ x\\ y\\ z\\ z \end{bmatrix}, \begin{bmatrix} y\\ z\\ y\\ x\\ y\\ x \end{bmatrix}, \begin{bmatrix} z\\ x\\ x\\ z\\ x\\ x \end{bmatrix} \right\}.$$

Consider the following instance $\mathbf{X}$ on the six variables a,b,c,d,e,f:

$$(a,b,c,d,e,f) \in R \quad \wedge \quad a = b = c \quad \wedge \quad d = e = f.$$

If we consider the domain $\{x,y,z\}$ as a subset of the free algebra $\mathcal{F}_{\mathbb{A}}(x,y,z) \le \mathbb{A}^{\mathbb{A}^3}$ in the natural way, then we just need to show that $\mathrm{Sg}(\mathbf{X})$ has a solution. Since $\mathbf{X}$ is arc-consistent, by Corollary 3.15.27 we just need to find a vague solution to the binary part $\mathbf{X}^{bin}$. We assign the variables a,b,c to the vague element $\preceq_g$ given by

$$\{y\} \prec_g \{z\} \prec_g \{x\} \prec_g \{y,z\} \prec_g \{x,y\} \prec_g \{x,z\} \prec_g \{x,y,z\},$$

and we assign the variables d, e, f to the vague element $\preceq_f$ given by

$$\{y\} \prec_f \{z\} \sim_f \{y, z\} \prec_f \{x\} \sim_f \{x, y\} \prec_f \{x, z\} \prec_f \{x, y, z\}.$$

The reader may check that this assignment vaguely satisfies every binary projection of the relation R. □

As another illustration of the theory, we will show how the Loop Lemma 3.11.17 can be proved for finite bounded width algebras by constructing suitable vague solutions.

Theorem 3.15.30. *If $R \subseteq A_x \times A_x$ is a smooth, weakly connected digraph of algebraic length 1, then the instance $\mathbf{X}$ which consists of only the variable x and the constraint $(x, x) \in R$ has a vague solution. As a consequence, the instance $\mathrm{Sg}(\mathbf{X})$ has a stable solution in any finite bounded width algebra.*

Proof. We will attempt to find a function $f : [2] \times \mathcal{P}_\emptyset(A_x) \to \mathbb{Q}$ such that for each proper nonempty $S \subset A_x$ we have

$$|f(1, S) - f(2, S)| = 1,$$

along with

$$f(1, S) \le f(2, S + R)$$

and

$$f(2, S) \le f(1, S - R).$$

To this end, we define a weighted directed graph $\mathcal{G}$ with vertices corresponding to proper nonempty subsets $S \subset A_x$, and with an edge of weight $+1$ from S to $S + R$ and an edge of weight -1 from S to $S - R$ for each such S (assuming $S + R, S - R \ne A_x$). We will handle each strongly connected component of $\mathcal{G}$ separately.

We call a directed cycle of $\mathcal{G}$ *positive* if the sum of the weights along the cycle is strictly greater than 0, and we define negative cycles similarly. For each positive directed cycle of $\mathcal{G}$ from a vertex $S \subset A_x$ to S, there is a corresponding cycle p of the instance $\mathbf{X}$ which has strictly more $+\mathbb{R}$ steps than $-\mathbb{R}$ steps, with $S + p = S$, and we call such a cycle p "positive" as well.

Claim. No strongly connected component of $\mathcal{G}$ contains both a positive directed cycle and a negative directed cycle.

Proof of claim. Suppose otherwise. Then we can find a vertex $S \subset A_x$ of $\mathcal{G}$, a positive cycle p, and a negative cycle q, such that

$$S = S + p = S + q.$$

We may assume without loss of generality that the total weights of p and q are opposite to each other, so $p + q$ has total weight 0. Then we have

$$S + R^{\circ j} - R^{\circ j} \subseteq S + jp + jq = S$$

for all $j \ge 0$, so S must be a union of linked components of $R^{\circ j}$ for all j. This contradicts Proposition 3.11.15: some $R^{\circ j}$ must be linked if R has algebraic length 1.

Now suppose that $\mathcal{C}$ is a strongly connected component of $\mathcal{G}$ which does not contain any positive directed cycles. We will define the restriction of f to $\mathcal{C}$ such that

$$f(2, S) = f(1, S) - 1$$

for all $S \in \mathcal{C}$. To do this, we pick any $S_0 \in \mathcal{C}$ and any constant $c_{\mathcal{C}}$, and define $f(1,T)$ to be $c_{\mathcal{C}}$ plus the maximum total weight of any directed path from S_0 to T, for all $T \in \mathcal{C}$. That this maximum total weight is well-defined follows from the fact that $\mathcal{C}$ does not contain any positive directed cycles together with the finiteness of $\mathcal{C}$. This definition is easily seen to satisfy

$$f(1,T) \le f(2,T+R) = f(1,T+R) - 1,$$
$$f(1,T) - 1 = f(2,T) \le f(1,T-R),$$

so long as $T+R, T-R$ are in $\mathcal{C}$. Additionally, if $f(1,T) = f(1,U)$ for some $T,U \in \mathcal{C}$, then there is some k such that S_0 has paths of total weight k to each of T and U - in this case, we see that

$$S + kR \subseteq T \cap U,$$

so $T \cap U \ne \emptyset$.

We handle strongly connected components which do not have any negative directed cycles similarly, looking at the negative of the minimum total weight instead of the maximum total weight, and taking $f(2,S) = f(1,S) + 1$ on such components.

To finish, we pick any total order on the strongly connected components of $\mathcal{G}$ which extends the reachability order, and we choose the constants $c_{\mathcal{C}}$ for the various connected components $\mathcal{C}$ according to this order, with sufficient distance between them that there is no interaction between the various strongly connected components of $\mathcal{G}$. □

3.16 Semidefinite Programming robustly solves bounded width CSPs

In this section we finally touch on a difficult topic: trying to maximize the number of satisfied constraints in a CSP instance which has no perfect solution. We consider only a very special case of this problem here: the problem of trying to approximately solve a CSP when we are promised that there exists a way to satisfy all but a tiny fraction of the constraints. This problem was considered by Guruswami and Zhou in [90].

Definition 3.16.1. We say that CSP($\mathbf{A}$) is *robustly solvable* if there is a function $f : [0,1] \to [0,1]$ such that

$$\lim_{\epsilon \to 0} f(\epsilon) = 0,$$

and a polynomial time algorithm that takes as input an instance $\mathbf{X}$ of CSP($\mathbf{A}$), and outputs an assignment to the variables of $\mathbf{X}$ such that if it is possible to satisfy a $1-\epsilon$ fraction of the constraints of $\mathbf{X}$, then the assignment found by the algorithm satisfies at least a $1 - f(\epsilon)$ fraction of the constraints of $\mathbf{X}$.

Before we dive into our main topic, we first give evidence that certain CSPs are *not* robustly solvable. We won't prove the next result here.

Theorem 3.16.2 (Håstad [99]). *Let $\mathbf{A}$ be the affine CSP template with domain $\mathbb{A}$, where $\mathbb{A}$ is the idempotent reduct of any finite abelian group, with relations given by $\mathbb{R}_c = \{(x,y,z) \mid x+y+z = c\} \le_{sd} \mathbb{A}^3$ for every possible $c \in \mathbb{A}$.*

Then for every fixed $\epsilon > 0$, it is NP-hard to solve the following problem: given an instance $\mathbf{X}$ of CSP($\mathbf{A}$) *such that there exists an assignment satisfying at least a $1-\epsilon$ fraction of the constraints, find an assignment which satisfies at least a $\frac{1}{|\mathbb{A}|} + \epsilon$ fraction of the constraints.*

Note that for the affine CSP defined above, randomly guessing values for variables will produce an assignment which satisfies a $\frac{1}{|\mathbb{A}|}$ fraction of the constraints, on average. So Håstad's result tells us that it's NP-hard to find any improvement on randomly guessing, for affine CSPs which are not perfectly solvable.

Corollary 3.16.3. *If* $\mathrm{CSP}(\mathbf{A})$ *is robustly solvable and* $P \neq NP$*, then* $\mathbf{A}$ *must be affine-free (and therefore* $\mathbf{A}$ *has bounded width).*

The best known approach to approximately solving CSPs, based on semidefinite programming, was laid out in Raghavendra's thesis [161] (see [160] for a short overview of the results). Under the Unique Games Conjecture, Raghavendra proved that this approach is actually optimal. The strategy is as follows.

As in the linear programming relaxation of a CSP, we imagine that we are looking for a probability distribution over solutions to the CSP. We do not give a full description of this unknown probability distribution: we only describe the marginal distribution over assignments to tuples of variables belonging to constraints of the CSP, as well as the marginal distribution over assignments to each pair of variables in the CSP. We impose compatibility conditions between the marginal distributions over each tuples of variables $(v_1, ..., v_m)$ belonging to some constraint and the marginal distribution over each pair (v_i, v_j) for $i, j \leq m$.

So far all the conditions given can be described by a system of linear inequalities. The semidefinite aspect comes from the following observation: every covariance matrix of any collection of random variables must be positive semidefinite.

To be more concrete, for each pair of variables x, y and each pair of values $a \in \mathbb{A}_x, b \in \mathbb{A}_y$, we have some variable $p_{(x,a),(y,b)}$ between 0 and 1, describing the probability that x is assigned the value a and y is assigned the value b. We create a matrix M_p with rows and columns indexed by ordered pairs (x, a) with $a \in \mathbb{A}_x$, and fill the $(x, a), (y, b)$ entry with $p_{(x,a),(y,b)}$ (I like to imagine M_p as a block matrix, with each block of rows or columns corresponding to a particular variable x). Then the matrix M_p must be positive semidefinite if these probabilities come from an actual probability distribution.

Before defining everything formally, we give an example.

Example 3.16.1. Consider the following instance of 2-SAT: we have three variables x, y, z, and each pair of variables has a $\neq$ constraint imposed between them. This instance has no perfect solution, but the standard linear programming relaxation is incapable of noticing this. Let's see how the semidefinite relaxation does.

The matrix M_p has six rows and six columns, corresponding to the pairs $(x, 0), (x, 1), (y, 0)$, $(y, 1), (z, 0), (z, 1)$, in that order. If M_p comes from a probability distribution over perfect solutions to this instance of 2-SAT, then it must have the following shape:

$$M_p = \left(\begin{array}{cc|cc|cc} * & 0 & 0 & * & 0 & * \\ 0 & * & * & 0 & * & 0 \\ \hline 0 & * & * & 0 & 0 & * \\ * & 0 & 0 & * & * & 0 \\ \hline 0 & * & 0 & * & * & 0 \\ * & 0 & * & 0 & 0 & * \end{array}\right).$$

Additionally, the entries in each block of M_p must sum to 1 (and be ≥ 0), and for each fixed row or column of M_p, the sum of the entries in the intersection of the row/column with any block must only depend on the row/column. Putting these linear constraints together, we quickly see that every nonzero entry of M_p must actually be equal to $\frac{1}{2}$. So far, this is exactly what the linear programming relaxation will guess.

The matrix M_p found above, with all nonzero entries equal to $\frac{1}{2}$, is *not* positive semidefinite. To see this, note that we have

$$\begin{pmatrix} 1 & -1 & \vert & 1 & -1 & \vert & 1 & -1 \end{pmatrix} \left(\begin{array}{cc|cc|cc} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \hline 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}\right) \left(\begin{array}{c} 1 \\ -1 \\ \hline 1 \\ -1 \\ \hline 1 \\ -1 \end{array}\right) = -3 < 0.$$

So the semidefinite relaxation of the problem can detect that we can't perfectly solve this instance of 2-SAT.

Now suppose that we give up on finding a perfect solution, and instead look for an approximate solution. This means that some of the entries of M_p which were required to be 0 before are instead required to be *small*. One choice of M_p that works is

$$M_p = \frac{1}{8}\left(\begin{array}{cc|cc|cc} 4 & 0 & 1 & 3 & 1 & 3 \\ 0 & 4 & 3 & 1 & 3 & 1 \\ \hline 1 & 3 & 4 & 0 & 1 & 3 \\ 3 & 1 & 0 & 4 & 3 & 1 \\ \hline 1 & 3 & 1 & 3 & 4 & 0 \\ 3 & 1 & 3 & 1 & 0 & 4 \end{array}\right),$$

which the reader may verify is positive semidefinite. This seems to satisfy each particular constraint with a probability of $\frac{3}{4}$. So the semidefinite relaxation thinks it might be possible to satisfy a $\frac{3}{4}$ fraction of the constraints. An easy brute force search reveals that the best we can do in reality is to satisfy a $\frac{2}{3}$ fraction of the constraints.

There is one further step we will take to analyze the semidefinite relaxation, based on a standard fact from linear algebra about positive semidefinite matrices.

Proposition 3.16.4. *If M is an $n \times n$ positive semidefinite matrix, then there is a collection of vectors $x_1, ..., x_n \in \mathbb{R}^n$ such that $M_{ij} = x_i \cdot x_j$ for all $i, j \le n$. Such a collection of vectors $x_1, ..., x_n$ can be computed from M in polynomial time.*

Proof. Perhaps the simplest approach is to compute a Cholesky decomposition of M, writing $M = LL^T$ for some lower triangular matrix L. The columns of L^T can then be used as the vectors $x_1, ..., x_n$. □

Example 3.16.2. The matrix M_p from the end of the previous example is positive semidefinite, so there should exist vectors $x_0, x_1, y_0, y_1, z_0, z_1 \in \mathbb{R}^6$ whose matrix of dot products is equal to M_p. Since M_p has rank 3, we should even be able to find such vectors in $\mathbb{R}^3$. One particularly satisfying choice of vectors that works is

$$x_0 = \frac{1}{\sqrt{24}}\begin{bmatrix} \sqrt{2}-\sqrt{3} \\ \sqrt{2} \\ \sqrt{2}+\sqrt{3} \end{bmatrix}, x_1 = \frac{1}{\sqrt{24}}\begin{bmatrix} \sqrt{2}+\sqrt{3} \\ \sqrt{2} \\ \sqrt{2}-\sqrt{3} \end{bmatrix}, y_0 = \frac{1}{\sqrt{24}}\begin{bmatrix} \sqrt{2} \\ \sqrt{2}+\sqrt{3} \\ \sqrt{2}-\sqrt{3} \end{bmatrix}, y_1 = \frac{1}{\sqrt{24}}\begin{bmatrix} \sqrt{2} \\ \sqrt{2}-\sqrt{3} \\ \sqrt{2}+\sqrt{3} \end{bmatrix},$$

with z_0, z_1 defined similarly by cyclically shifting y_0, y_1.

Now we can give the definition of the basic semidefinite relaxation of a CSP (this is the LC relaxation from [161]).

Definition 3.16.5. Given an instance $\mathbf{X}$ of a CSP, with variable domains $\mathbb{A}_v$ and constraints C imposing relations $\mathbb{R}_C \le \prod_{i \le m_C} \mathbb{A}_{v_{C,i}}$ on the variables $v_{C,1}, ..., v_{C,m}$, the *basic semidefinite relaxation* of $\mathbf{X}$ is the following optimization problem. We wish to find a system of "probabilities" $p_{C,r}$ for $r \in \prod_{i \le m_C} \mathbb{A}_{v_{C,i}}$, such that

$$\sum_r p_{C,r} = 1$$

for each constraint C and

$$p_{C,r} \ge 0$$

for each C, r, and to find vectors

$$x_a \in \mathbb{R}^N$$

for each variable x and value $a \in \mathbb{A}_x$, where $N = \sum_x |\mathbb{A}_x|$ is the number of pairs (x, a), such that for each C and each pair of variables $x = v_{C,i}, y = v_{C,j}$ involved in the constraint C, we satisfy the compatibility condition

$$x_a \cdot y_b = \sum_{r_i = a, r_j = b} p_{C,r}.$$

For each pair of variables x, y of $\mathbf{X}$ which occur together in some constraint C, any solution to the basic semidefinite relaxation will automatically have the following properties:

- For all $a \ne b \in \mathbb{A}_x$, we have $x_a \cdot x_b = 0$.
- We have $\sum_{a \in \mathbb{A}_x} \|x_a\|^2 = \|\sum_{a \in \mathbb{A}_x} x_a\|^2 = 1$.
- For all $a \in \mathbb{A}_x, b \in \mathbb{A}_y$, we have $x_a \cdot y_b \ge 0$.
- We have $\sum_{a \in \mathbb{A}_x, b \in \mathbb{A}_y} x_a \cdot y_b = 1$.

Our goal is to find such a system of probabilities $p_{C,r}$ and vectors x_a such that the quantity

$$\tfrac{1}{\#C} \sum_C \sum_{r \in \mathbb{R}_C} p_{C,r}$$

is maximized. The maximum possible value of that sum is called the *value* of the semidefinite relaxation. If the value of the semidefinite relaxation is equal to 1, then we say that the system of probabilities $p_{C,r}$ and vectors x_a *perfectly solves* the semidefinite relaxation.

Note that the constraints we make on the vectors x_a only involve their dot products, and that they are always linear equalities/inequalities in terms of these dot products. We can deduce from these constraints a result which involves the vectors directly.

Proposition 3.16.6. *Suppose that a system of probabilities $p_{C,r}$ and vectors x_a are as in the basic semidefinite relaxation of a CSP instance $\mathbf{X}$. Then for any variables x, y of $\mathbf{X}$ which occur together in some constraint, we have*

$$\sum_{a \in \mathbb{A}_x} x_a = \sum_{b \in \mathbb{A}_y} y_b.$$

Proof. Let $x_{\mathbb{A}_x} = \sum_{a\in\mathbb{A}_x} x_a$ and similarly let $y_{\mathbb{A}_y} = \sum_{b\in\mathbb{A}_y} y_b$. Then we have $\|x_{\mathbb{A}_x}\|^2 = \|y_{\mathbb{A}_y}\|^2 = x_{\mathbb{A}_x}\cdot y_{\mathbb{A}_y} = 1$, so by the equality case of Cauchy-Schwarz we must have $x_{\mathbb{A}_x} = y_{\mathbb{A}_y}$. □

A useful generalization of the notation for the vectors x_a was used heavily in [19].

Definition 3.16.7. Suppose we are in the setup of the basic semidefinite relaxation of a CSP instance $\mathbf{X}$. If x is a variable of $\mathbf{X}$ and $A \subseteq \mathbb{A}_x$, then we define the vector x_A by

$$x_A = \sum_{a\in A} x_a.$$

Note that since the vectors x_a are pairwise orthogonal for a fixed variable x, we have

$$\|x_A\|^2 = \sum_{a\in A} \|x_a\|^2 = x_A \cdot x_{\mathbb{A}_x}.$$

Additionally, for each pair of variables x, y, we have

$$x_A \cdot y_B = \sum_{a\in A, b\in B} x_a \cdot y_b$$

by the distributive law.

Before explaining how to use the basic semidefinite relaxation to robustly solve affine-free CSPs, we will first show that if an instance $\mathbf{X}$ of an affine-free CSP has a perfect solution to its basic semidefinite relaxation, then in fact $\mathbf{X}$ has a solution. The main idea is to prove that the set of values $a \in \mathbb{A}_x$ such that the vectors x_a are nonzero can be used to restrict the instance to get a weak Prague instance, which will then be pq-consistent by Theorem 3.13.19. The crucial computation is analyzing what happens to the vector x_A when we take a single step along a path.

Lemma 3.16.8. *Suppose we are in the setup of the basic semidefinite relaxation of a CSP instance* $\mathbf{X}$. *Let* x, y *be variables of* $\mathbf{X}$ *which occur together in some constraint, and define a binary relation* $P \subseteq \mathbb{A}_x \times \mathbb{A}_y$ *by*

$$(a, b) \in P \iff x_a \cdot y_b > 0.$$

Then for any set $A \subseteq \mathbb{A}_x$*, we have*

$$\|x_A\|^2 \le \|y_{A+P}\|^2,$$

with equality only when $x_A = y_{A+P}$*. In fact, we have*

$$x_A \cdot (y_{A+P} - x_A) = 0,$$

that is, y_{A+P} *is the sum of* x_A *with a vector which is perpendicular to* x_A*. Furthermore, we have* $x_A = y_{A+P}$ *if and only if* $A + P - P \subseteq A$*.*

Proof. Before diving into the algebraic details of the proof, it may be helpful to note that the dot products $x_a \cdot y_b$ define a probability distribution μ supported on $P \subseteq \mathbb{A}_x \times \mathbb{A}_y$ such that for $A \subseteq \mathbb{A}_x, B \subseteq \mathbb{A}_y$ we have $\mathbb{P}_\mu[A \times B] = x_A \cdot y_B$, and such that the marginal distributions μ_x, μ_y on $\mathbb{A}_x, \mathbb{A}_y$ have probabilities given by $\mathbb{P}_{\mu_x}[A] = \|x_A\|^2$, $\mathbb{P}_{\mu_y}[B] = \|y_B\|^2$. It's possible to argue purely in terms of the probability distribution μ, but the proof we give below won't directly refer to μ at all.

We have $x_A \cdot x_A = x_A \cdot x_{\mathbb{A}_x}$, and by the definition of P we have

$$x_A \cdot y_{A+P} = \sum_{a \in A, b \in A+P} x_a \cdot y_b = \sum_{a \in A, b \in \mathbb{A}_y} x_a \cdot y_b = x_A \cdot y_{\mathbb{A}_y}.$$

Since $x_{\mathbb{A}_x} = y_{\mathbb{A}_y}$, we have

$$x_A \cdot x_A = x_A \cdot x_{\mathbb{A}_x} = x_A \cdot y_{\mathbb{A}_y} = x_A \cdot y_{A+P},$$

so x_A is orthogonal to $y_{A+P} - x_A$.

From the orthogonality of x_A with $y_{A+P} - x_A$, we have

$$\|y_{A+P}\|^2 = \|x_A + (y_{A+P} - x_A)\|^2 = \|x_A\|^2 + \|y_{A+P} - x_A\|^2 \ge \|x_A\|^2,$$

with equality exactly when $y_{A+P} = x_A$.

For the last statement, note that we have $\|x_A\|^2 \le \|y_{A+P}\|^2 \le \|x_{A+P-P}\|^2$, so we just need to check the implication $x_A = y_{A+P} \implies x_{A+P-P} = x_A$. Under the assumption $x_A = y_{A+P}$, we have

$$x_A \cdot y_{A+P} = \|y_{A+P}\|^2 = y_{\mathbb{A}_y} \cdot y_{A+P} = x_{\mathbb{A}_x} \cdot y_{A+P}.$$

Suppose for contradiction that there was some $a \in (A + P - P) \setminus A$. Then by the definition of $A + P - P$ there would be some $b \in A + P$ such that $(a, b) \in P$, that is, such that $x_a \cdot y_b > 0$. But then we would have

$$x_{\mathbb{A}_x} \cdot y_{A+P} \ge x_a \cdot y_b + x_A \cdot y_{A+P} > x_A \cdot y_{A+P},$$

which contradicts the assumption $x_A = y_{A+P}$. □

Theorem 3.16.9. *Suppose* **X** *is an instance of an affine-free CSP, and that there is a system of probabilities $p_{C,r}$ and vectors x_a which perfectly solves the basic semidefinite relaxation of* **X**. *Then* **X** *has a solution.*

Proof. We define a restriction $\mathbf{X}'$ of $\mathbf{X}$ by restricting each relation $\mathbb{R}_C$ to the support R'_C of the marginal distribution $p_{C,r}$, and restricting each variable domain $\mathbb{A}_x$ to the set A'_x of $a \in \mathbb{A}_x$ such that $\|x_a\|^2 \neq 0$. Note that each R'_C will be contained in the original relation $\mathbb{R}_C$ if we have a perfect solution to the basic semidefinite relaxation. Additionally, for each C and each pair of variables $x = v_{C,i}, y = v_{C,j}$, the binary projection $\pi_{i,j}(R'_C)$ will be equal to the set of ordered pairs $(a, b) \in A'_x \times A'_y$ such that $x_a \cdot y_b \neq 0$, by the compatibility between the probabilities $p_{C,r}$ and the vectors x_a.

We will check that $\mathbf{X}'$ is a weak Prague instance (see Definition 3.13.12). Arc-consistency (aka condition (P1)) of $\mathbf{X}'$ follows from the compatibility between the probabilities and the vectors. To check (P2) and (P3), we use Lemma 3.16.8. Let $A \subseteq A'_x$ and let p be a cycle from x to x in the instance $\mathbf{X}'$, with p_1 from x to y the first step of the cycle p. If we have

$$A + p = A,$$

then by Lemma 3.16.8 we have

$$\|x_A\|^2 \le \|y_{A+p_1}\|^2 \le \|x_{A+p}\|^2 = \|x_A\|^2,$$

so by the equality case of Lemma 3.16.8 we must have

$$x_A = y_{A+p_1}.$$

Thus for any $a' \not\in A$, we must have $x_{a'} \cdot y_{A+p_1} = x_{a'} \cdot x_A = 0$, so we have

$$A + p_1 - p_1 = A.$$

Thus by Proposition 3.13.14 we see that $\mathbf{X}'$ satisfies condition (P2). We could have also checked condition (P2) using only the system of probabilities $p_{C,r}$, without mentioning the vectors x_a, by Theorem 3.13.15.

To check (P3), let $A \subseteq A'_x$, and let p, q be cycles from x to x in the instance $\mathbf{X}'$, with

$$A + p + q = A.$$

Then by Lemma 3.16.8 we have

$$\|x_A\|^2 \le \|x_{A+p}\|^2 \le \|x_{A+p+q}\|^2 = \|x_A\|^2,$$

so by the equality case of Lemma 3.16.8 we must have

$$x_A = x_{A+p}.$$

In particular, we must have $A = A + p$, which proves (P3).

To finish, we note that $\mathbf{X}'$ is pq-consistent by Theorem 3.13.19, so $\mathrm{Sg}(\mathbf{X}')$ is also pq-consistent by Proposition 3.14.3, and $\mathrm{Sg}(\mathbf{X}')$ is a restriction of the instance $\mathbf{X}$ since $\mathbf{X} = \mathrm{Sg}(\mathbf{X})$. Thus $\mathrm{Sg}(\mathbf{X}')$ has a solution by Theorem 3.13.8 and its corollaries, which is also a solution to the original instance $\mathbf{X}$. □

In order to extend the previous result to an algorithm for *robustly* solving affine-free CSPs, we need to find some approximate analogue of Lemma 3.16.8. The plan is to start by arguing as in Theorem 1.6.17, using the probabilities $p_{C,r}$ to produce an arc-consistent instance with variable domains A'_x and constraint relations R', such that each tuple $r \in R'$ has probability above some (random) threshold θ. Then we will randomly cut the unit ball of possible values for the vectors x_A into finitely many pieces, so that a version of Lemma 3.16.8 holds when we forget what the exact values of the vectors x_A are, and instead only keep track of which piece of the ball x_A is contained in.

The proof becomes slightly simpler if we use the concept of weak consistency, from the previous section, instead of condition (P3). The simplification we get by aiming for weak consistency instead of (P3) is that when we have

$$A + p + q = A,$$

we only have to ensure that

$$x_A \cdot x_{A+p} > 0 \ \ (\implies A \cap (A + p) \ne \emptyset),$$

rather than needing to ensure that $x_A = x_{A+p}$. This means we only need to make sure that we chop the ball into fine enough pieces to separate each pair x_A, x_B with $x_A \cdot x_B = 0$, instead of needing to separate each pair x_A, x_B with $x_A \ne x_B$.

In order to produce our weakly consistent instance, we will define a quasiorder $\preceq$ (with an associated strict partial order $\prec$ and equivalence relation $\sim$) on the ball. We will first choose a (randomized) sequence of radii $r_1, r_2, ...,$ and a (random) collection of hyperplanes $\mathcal{H}_i$ such that for every variable x, any pair of vectors x_A, x_B which are orthogonal are (with high probability) separated from each other by at least one of the hyperplanes $\mathcal{H}_i$. The plan is to define $\preceq$ by

$$u \prec v \iff \exists i \text{ s.t. } \|u\| < r_i \leq \|v\|$$

and

$$u \sim v \iff (\forall i\ \|u\| < r_i \iff \|v\| < r_i) \wedge (\forall j\ u, v \text{ are on the same side of } \mathcal{H}_j).$$

The plan is to throw away any constraint relation R' which is incompatible with the quasiorder $\preceq$, where we say that R' is incompatible with $\preceq$ if there are variables x, y and some $A \subseteq A'_x$ such that

$$x_A \not\preceq y_{A+\pi_{xy}(R')}.$$

We will choose the radii r_i to guarantee that

$$A + \pi_{xy}(R') - \pi_{xy}(R') \neq A \implies x_A \prec y_{A+\pi_{xy}(R')},$$

by taking $r_{i+1}^2 < r_i^2 + \theta$. On the other hand, if

$$A + \pi_{xy}(R') - \pi_{xy}(R') = A,$$

then x_A will be very close to $y_{A+\pi_{xy}(R')}$. In this case, we will ensure that the r_i are spaced widely enough to make it unlikely that

$$y_{A+\pi_{xy}(R')} \prec x_A.$$

We also need to rule out the possibility that x_A and $y_{A+\pi_{xy}(R')}$ are separated by some hyperplane $\mathcal{H}_i$. The chance that a particular random hyperplane separates x_A and $y_{A+\pi_{xy}(R')}$ is proportional to the angle between x_A and $y_{A+\pi_{xy}(R')}$, which will be low as long as x_A is sufficiently close to $y_{A+\pi_{xy}(R')}$. All we have left to do is to carefully work out the details.

Theorem 3.16.10 (Slight refinement of [19])**.** *If Γ is a finite constraint language and $\mathbf{A} = (A, \Gamma)$ has bounded relational width, then the basic SDP relaxation can be used to* robustly *solve* $\mathrm{CSP}(\mathbf{A})$.

More precisely, if we are given an instance $\mathbf{X}$ such that the basic SDP relaxation is $1-\epsilon$ satisfiable, then we can efficiently find a solution which satisfies a

$$1 - O\Big(\frac{\log(\log(\log(1/\epsilon)))}{\log(1/\epsilon)}\Big)$$

fraction of the constraints.

Proof. Suppose we have a system of probabilities $p_{C,r}$ and vectors x_a which solves the basic SDP relaxation of our instance $\mathbf{X}$ with value at least $1-\epsilon$, that is, such that

$$\tfrac{1}{\#C}\sum_C \sum_{r \in \mathbb{R}_C} p_{C,r} \geq 1 - \epsilon.$$

As an initial simplification to the problem, we will preemptively give up on any constraint C such that

$$\sum_{r\in\mathbb{R}_C} p_{C,r} < 1-\sqrt{\epsilon}.$$

This gives up on at most a $\sqrt{\epsilon} = o(1/\log(1/\epsilon))$ fraction of the constraints of $\mathbf{X}$, and allows us to focus on solving the problem in the special case where every individual constraint C satisfies

$$\sum_{r\in\mathbb{R}_C} p_{C,r} \ge 1-\sqrt{\epsilon}. \tag{3.1}$$

The advantage of this step is that from here on, we can look for a randomized algorithm which has a high probability of satisfying each constraint relation in isolation.

Let $N = \log(1/\sqrt{\epsilon})$. We will make a series of randomized choices in order to produce a weakly consistent instance $\mathbf{X}'$, and after each choice we will give up on some of the constraints of our original instance $\mathbf{X}$. At each step, we just need to confirm that for each constraint C which satisfies (3.1), the chance of giving up on the constraint C is at most

$$O\Big(\frac{\log(\log(N))}{N}\Big).$$

First we will try to produce an arc-consistent instance. Choose a threshold $\theta = \exp(-t) \in [\sqrt{\epsilon}, 1]$ by choosing $t = \log(1/\theta)$ uniformly at random from $[0, N]$. For each constraint C with corresponding constraint relation $\mathbb{R}_C$, we define the reduced relation R'_C by

$$R'_C = \{r \in \mathbb{R}_C \mid p_{C,r} \ge 2\theta\}.$$

For each variable x, we define the reduced variable domain A'_x by

$$A'_x = \{a \in \mathbb{A}_x \mid \|x_a\|^2 \ge \theta\}.$$

Our reduced instance $\mathbf{X}'$ will have variable domains A'_x and constraint relations R'_C for all of the constraints C which we do not choose to give up on by the end of our randomized procedure. In order to ensure arc-consistency of $\mathbf{X}'$, we preemptively give up on any constraint C which does not satisfy

$$\sum_{r\not\in R'_C} p_{C,r} < \theta. \tag{3.2}$$

Claim 1. If a constraint C with $R'_C \subseteq A'_{x_1} \times \cdots \times A'_{x_m}$ satisfies (3.2), then we have

$$\pi_{x_i}(R'_C) = A'_{x_i}$$

for each $i \in \{1, ..., m\}$.

Proof of Claim 1. Let $x = x_i$. First, note that for any $r \in R'_C$, if we set $a = \pi_x(r)$, then we have $\|x_a\|^2 \ge p_{C,r} \ge \theta$, so $a \in A'_x$. Thus we have $\pi_x(R'_C) \subseteq A'_x$.

Conversely, if $a \not\in \pi_x(R'_C)$ then we have

$$\|x_a\|^2 = \sum_{\pi_x(r)=a} p_{C,r} \le \sum_{r\not\in R'_C} p_{C,r} < \theta$$

by (3.2), so $a \notin \pi_x(R'_C) \implies a \notin A'_x$. Thus $\pi_x(R'_C) \supseteq A'_x$, so $\pi_x(R'_C) = A'_x$.

Claim 2. If $\theta = \exp(-t)$ and t is chosen uniformly at random from $[0, N]$, then

$$\mathbb{E}\Big[\frac{1}{\theta}\sum_{r \notin R'_C} p_{C,r}\Big] = O(1/N),$$

and the implied constant only depends on the number of tuples in the constraint $\mathbb{R}_C$. As a consequence, the probability that (3.2) fails to hold for any given constraint C is $O(1/N)$.

Proof of Claim 2. We have

$$\begin{aligned}\mathbb{E}\Big[\frac{1}{\theta}\sum_{r \notin R'_C} p_{C,r}\Big] &= \frac{1}{N}\int_{t=0}^{N} \frac{1}{\theta}\sum_{r \notin R'_C} p_{C,r}\ dt\\ &= \frac{1}{N}\sum_{r \notin \mathbb{R}_C} p_{C,r}\int_{t=0}^{N}\frac{1}{\theta}\ dt + \frac{1}{N}\sum_{r \in \mathbb{R}_C} p_{C,r}\int_{t=0}^{\max(N, \log(2/p_{C,r}))}\frac{1}{\theta}\ dt\\ &\le \frac{1}{N}\int_{t=0}^{N}\frac{\sqrt{\epsilon}}{\theta}\ dt + \frac{1}{N}\sum_{r \in \mathbb{R}_C}\int_{t=0}^{\log(2/p_{C,r})}\frac{p_{C,r}}{\theta}\ dt\\ &< \frac{2|\mathbb{R}_C| + 1}{N}.\end{aligned}$$

Next we slice the ball $\mathcal{B}$ containing the x_As into shells in order to ensure that the consistency condition (P2) is satisfied (note that (P2) is not required in the definition of weak consistency - however, enforcing it makes the rest of the proof simpler). To do this, we choose a second threshold θ_2 uniformly at random from the interval $[0, \theta]$, and define radii r_i by

$$r_i^2 = i\theta + \theta_2.$$

We define the strict partial order $\prec$ on the ball $\mathcal{B}$ by

$$\begin{aligned}u \prec v \quad &\iff \quad \exists i \text{ s.t. } \|u\| < r_i \le \|v\|\\ &\iff \quad \Big\lfloor \frac{\|u\|^2 - \theta_2}{\theta}\Big\rfloor < \Big\lfloor \frac{\|v\|^2 - \theta_2}{\theta}\Big\rfloor.\end{aligned}$$

In order to ensure that (P2) is satisfied, we will preemptively give up on any constraint C such that there is a pair of variables x, y involved in the constraint C which do not satisfy

$$\forall A \subseteq A'_x,\ \ y_{A+\pi_{xy}(R'_C)} \not\prec x_A. \tag{3.3}$$

By the next claim, we will only need to check (3.3) in the special case where $A + \pi_{xy}(R'_C) - \pi_{xy}(R'_C) = A$.

Claim 3. If $A \subseteq A'_x$ has $A + \pi_{xy}(R'_C) - \pi_{xy}(R'_C) \ne A$, and if the constraint C satisfies (3.2), then we automatically have

$$x_A \prec y_{A+\pi_{xy}(R'_C)}.$$

Proof of Claim 3. We just need to prove that

$$\|y_{A+\pi_{xy}(R'_C)}\|^2 \ge \|x_A\|^2 + \theta.$$

If $A + \pi_{xy}(R'_C) - \pi_{xy}(R'_C) \neq A$, then there must be some $a \notin A$ and some $b \in A + \pi_{xy}(R'_C)$ with $(a,b) \in \pi_{xy}(R'_C)$. Picking $r \in R'_C$ with $\pi_{xy}(r) = (a,b)$, we see that

$$x_a \cdot y_b \geq p_{C,r} \geq 2\theta$$

by the definition of R'_C. Since

$$\|y_{A+\pi_{xy}(R'_C)}\|^2 = x_{\mathbb{A}_x} \cdot y_{A+\pi_{xy}(R'_C)} \geq x_A \cdot y_{A+\pi_{xy}(R'_C)} + x_a \cdot y_b,$$

we just need to check that

$$x_A \cdot y_{A+\pi_{xy}(R'_C)} \geq \|x_A\|^2 - \theta.$$

For this, we just note that

$$\begin{aligned}\|x_A\|^2 - x_A \cdot y_{A+\pi_{xy}(R'_C)} &= x_A \cdot y_{\mathbb{A}_y} - x_A \cdot y_{A+\pi_{xy}(R'_C)}\\ &= \sum_{\substack{\pi_x(r)\in A\\ \pi_y(r)\notin A+\pi_{xy}(R'_C)}} p_{C,r}\\ &\leq \sum_{r\notin R'_C} p_{C,r} < \theta.\end{aligned}$$

In order to put a bound on the probability that (3.3) fails to hold when $A+\pi_{xy}(R'_C)-\pi_{xy}(R'_C) = A$, we need to show that the expected value of

$$\frac{\left|\|x_A\|^2 - \|y_{A+\pi_{xy}(R'_C)}\|^2\right|}{\theta}$$

is small when $\theta = \exp(-t)$ and t is chosen uniformly at random from $[0, N]$. In this case we have

$$\begin{aligned}\left|\|x_A\|^2 - \|y_{A+\pi_{xy}(R'_C)}\|^2\right| &\leq \max\left(\|x_A\|^2 - x_A \cdot y_{A+\pi_{xy}(R'_C)},\ \|y_{A+\pi_{xy}(R'_C)}\|^2 - x_A \cdot y_{A+\pi_{xy}(R'_C)}\right)\\ &\leq \sum_{r\notin R'_C} p_{C,r},\end{aligned}$$

so by Claim 2 this is $O(1/N)$ on average.

So far, everything we have done could have been phrased in terms of the linear relaxation rather than the semidefinite relaxation. The next step, where we enforce weak consistency, is the step where we finally use the full power of the semidefinite relaxation. Set

$$M = \lceil \log_2(N) \rceil,$$

and independently pick M uniformly random hyperplanes $\mathcal{H}_1, ..., \mathcal{H}_M$. Now define the equivalence relation $\sim$ on $\mathcal{B}$ by

$$u \sim v \iff (\forall i\ \|u\| < r_i \iff \|v\| < r_i) \wedge (\forall j\ u, v \text{ are on the same side of } \mathcal{H}_j).$$

In order to guarantee weak consistency, we need to preemptively give up on any variable x (along with any constraint involving x) which does not satisfy

$$\forall A, B \subseteq A'_x,\ A, B \neq \emptyset, \quad x_A \sim x_B \implies A \cap B \neq \emptyset \tag{3.4}$$

and we need to preemptively give up on every constraint C such that there is a pair of variables x, y involved in the constraint C which do not satisfy

$$\forall A \subseteq A'_x, \;\; A + \pi_{xy}(R'_C) - \pi_{xy}(R'_C) = A \quad \Longrightarrow \quad y_{A+\pi_{xy}(R'_C)} \sim x_A. \tag{3.5}$$

Claim 4. The chance that any particular variable x fails to satisfy (3.4) is at most $\frac{3^{|A'_x|}}{2^M} = O(1/N)$. As a consequence, the chance that we give up on any particular constraint C due to (3.4) is also $O(1/N)$ by the union bound.

Proof of Claim 4. There are less than $3^{|A'_x|}$ pairs of disjoint, nonempty subsets A, B of A'_x. For any particular pair A, B, the chance that any particular random hyperplane $\mathcal{H}_i$ separates x_A from x_B is exactly $1/2$, since $x_A \cdot x_B = 0$. Thus the chance that none of the M independent random hyperplanes $\mathcal{H}_1, ..., \mathcal{H}_M$ separate x_A from x_B is $1/2^M \le 1/N$.

To finish the argument, we need to find an upper bound on the probability that (3.5) is violated. For a given $A \subseteq A'_x$ with $A + \pi_{xy}(R'_C) - \pi_{xy}(R'_C) = A$, this probability depends mainly on the angle α between x_A and $y_{A+\pi_{xy}(R'_C)}$. From

$$\max\Big(\|x_A\|^2 - x_A \cdot y_{A+\pi_{xy}(R'_C)},\; \|y_{A+\pi_{xy}(R'_C)}\|^2 - x_A \cdot y_{A+\pi_{xy}(R'_C)}\Big) \le \sum_{r \notin R'_C} p_{C,r}$$

we get

$$x_A \cdot y_{A+\pi_{xy}(R'_C)} \ge \|x_A\| \|y_{A+\pi_{xy}(R'_C)}\| - \sum_{r \notin R'_C} p_{C,r}.$$

Using the fact that $\|x_A\|^2, \|y_{A+\pi_{xy}(R'_C)}\|^2 \ge \theta$, we get

$$\cos(\alpha) \ge 1 - \frac{1}{\theta} \sum_{r \notin R'_C} p_{C,r},$$

so

$$\alpha^2 = O\Big(\frac{1}{\theta} \sum_{r \notin R'_C} p_{C,r}\Big).$$

The chance that at least one of the M hyperplanes $\mathcal{H}_i$ separates x_A from $y_{A+\pi_{xy}(R'_C)}$ is then given by

$$\begin{aligned} 1 - \Big(1 - \frac{\alpha}{\pi}\Big)^M &\le \max\Big(1, \frac{M\alpha}{\pi}\Big) \\ &\ll \max\Big(1, M\Big(\frac{1}{\theta} \sum_{r \notin R'_C} p_{C,r}\Big)^{1/2}\Big). \end{aligned}$$

To finish the proof, we just need to verify one final claim.

Claim 5. If $\theta = \exp(-t)$ and t is chosen uniformly at random from $[0, N]$, then

$$\mathbb{E}\Big[\max\Big(1, M\Big(\frac{1}{\theta} \sum_{r \notin R'_C} p_{C,r}\Big)^{1/2}\Big)\Big] = O\Big(\frac{\log(M)}{N}\Big).$$

Proof of Claim 5. The argument is similar to the proof of Claim 2, just slightly more involved. First, we use the inequality $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ to get the bound

$$\max\Big(1, M\Big(\frac{1}{\theta}\sum_{r\notin R'_C} p_{C,r}\Big)^{1/2}\Big) \le \max\Big(1, M\sqrt{\frac{\sqrt{\epsilon}}{\theta}}\Big) + \sum_{\substack{r\in\mathbb{R}_C\\ p_{C,r}<\theta}} \max\Big(1, M\sqrt{\frac{p_{C,r}}{\theta}}\Big).$$

We then bound the contribution of each summand individually. Each max takes the value 1 for a range of values of $t = \log(1/\theta)$ of length $2\log(M)$, and from then on takes exponentially decaying values, so the total expectation ends up being bounded by

$$\frac{2(\log(M)+1)(|\mathbb{R}_C|+1)}{N} = O\Big(\frac{\log(M)}{N}\Big).$$

Thus, after preemptively giving up on at most a

$$O\Big(\frac{\log(M)}{N}\Big) = O\Big(\frac{\log(\log(N))}{N}\Big) = O\Big(\frac{\log(\log(\log(1/\epsilon)))}{\log(1/\epsilon)}\Big)$$

fraction of the constraints, we finally manage to construct a weakly consistent instance $\mathbf{X}'$. Then $\mathrm{Sg}(\mathbf{X}') \subseteq \mathbf{X}$ will also be weakly consistent, so it will have a solution by Theorem 3.15.12. □

The bound given in Theorem 3.16.10 is slightly better than the bound from [19], which only guaranteed that we could find a solution of quality

$$1 - O\Big(\frac{\log(\log(1/\epsilon))}{\log(1/\epsilon)}\Big).$$

We can improve this further, getting rid of the unsightly $\log(\log(\log(1/\epsilon)))$ factor entirely, if we try to construct a vague solution (to the binary part) instead of aiming for weak consistency.

Theorem 3.16.11 (Perfected form of [19])**.** *Suppose Γ is a finite constraint language such that $\mathbf{A} = (A, \Gamma)$ has bounded relational width. If we are given an instance $\mathbf{X}$ such that the basic SDP relaxation is $1-\epsilon$ satisfiable, then we can efficiently find a solution which satisfies a*

$$1 - O\Big(\frac{1}{\log(1/\epsilon)}\Big)$$

fraction of the constraints. In other words, the function f from Definition 3.16.1 satisfies $f(\epsilon) = O(1/\log(1/\epsilon))$.

Proof. We argue as in the proof of Theorem 3.16.10, using the same notation, up to the point where we obtained an arc-consistent instance $\mathbf{X}'$ which satisfied (P2). From here on, we modify the argument: instead of trying to define an equivalence relation $\sim$ on the shell of the ball $\mathcal{B}$ between radius r_i and r_{i+1}, we extend $\prec$ to a (nearly) total order. We do this by picking a uniformly random unit vector U, and setting

$$\forall \|u\|, \|v\| \in (r_i, r_{i+1}], \quad u \prec v \iff U\cdot(u-v) < 0.$$

Now we use the (almost) total order $\prec$ to define a vague element $\prec_x$ for each variable x, by setting

$$A \prec_x B \quad\iff\quad x_A \prec x_B.$$

Note that with probability 1, each $\prec_x$ will be a total order on $\mathcal{P}_\emptyset(A'_x)$, so $\prec_x$ is indeed a vague element of A'_x.

We will now preemptively give up on any constraint relation C if there is any pair of variables x, y such that the vague elements $\prec_x, \prec_y$ fail to vaguely satisfy the binary relation $\pi_{xy}(R'_C)$. We need to find an upper bound on the probability that we will give up on any particular constraint C due to the pair of variables x, y. Set $R' = \pi_{xy}(R'_C)$ to simplify the notation.

We begin trying to construct a quasiorder $\preceq_{R'}$ extending $\prec_x, \prec_y$ on $\mathcal{P}_\emptyset(A'_x) \sqcup \mathcal{P}_\emptyset(A'_y)$ by setting

$$x_A \prec_{R'} y_B \text{ when } \exists i \ \|x_A\| < r_i \le \|y_B\|,$$

and similarly with the roles of x and y reversed. This ensures that

$$A \ne A + R' - R' \implies x_A \prec_{R'} y_{A+R'},$$

and similarly with the roles of x and y reversed. So far we encounter no problems with the construction of $\preceq_{R'}$.

We will also need to require that

$$A = A + R' - R' \implies x_A \sim_{R'} y_{A+R'},$$

and this is the step which could potentially cause an issue. Note that if we haven't already given up on the constraint C while ensuring that (P2) holds, then in this case we have the guarantee that x_A and $y_{A+R'}$ are contained in the same shell of the ball $\mathcal{B}$.

Since the different shells of $\mathcal{B}$ don't interact in any of the orderings $\prec_x, \prec_y, \preceq_{R'}$, we can focus on just one particular shell of $\mathcal{B}$ corresponding to a pair of adjacent radii r_i, r_{i+1}. Within such a shell, all $\preceq_{R'}$ does is identify certain vectors x_A with corresponding vectors $y_{A+R'}$. The only way we could run into trouble is if there was some pair $A, B \subseteq A'_x$ with

$$\begin{aligned} &A \sim_{R'} A + R', \\ &B \sim_{R'} B + R', \\ &A \prec_x B, \end{aligned}$$

but

$$B + R' \prec_y A + R'.$$

In other words, we only need to preemptively give up on the constraint C due to the pair x, y if we can find such a pair of sets $A, B \subseteq A'_x$ with

$$U \cdot (x_A - x_B) < 0 < U \cdot (y_{A+R'} - y_{B+R'}).$$

For a randomly chosen U, the probability of this occuring is proportional to the angle between the vector $x_A - x_B$ and the vector $y_{A+R'} - y_{B+R'}$.

Since A, B are different subsets of A'_x, at least one coordinate of $x_A - x_B$ must have absolute value at least $\sqrt{\theta}$, so we have $\|x_A - x_B\|^2 \ge \theta$, and similarly $\|y_{A+R'} - y_{B+R'}\|^2 \ge \theta$. As in the proof of Theorem 3.16.10, we have

$$\|x_A - y_{A+R'}\|^2, \|x_B - y_{B+R'}\|^2 \le \sum_{r \notin R'_C} p_{C,r},$$

so the angle between $x_A - x_B$ and $y_{A+R'} - y_{B+R'}$ is

$$O\Big(\frac{1}{\sqrt{\theta}}\sqrt{\sum_{r\not\in R'_C} p_{C,r}}\Big).$$

By Claim 5 of Theorem 3.16.10, the average value of this upper bound is at most

$$O(1/N) = O(1/\log(1/\epsilon)),$$

so we only need to give up on $O(1/N)$ of our constraints in order to get an arc-consistent instance $\mathbf{X}'$ whose binary part has a vague solution. Then by Corollary 3.15.27 the instance $\mathrm{Sg}(\mathbf{X}') \subseteq \mathbf{X}$ has a solution, which finishes the proof. □

Remark 3.16.1. The algorithm from Theorem 3.16.11 can be derandomized without too much effort using the method of conditional expectations. After throwing away constraints which are violated by more than a $\sqrt{\epsilon}$ fraction, we pick θ in $[\sqrt{\epsilon}, 1]$ such that

$$\frac{1}{\#C}\sum_C \frac{1}{\theta}\sum_{r\not\in R'_C} p_{C,r} + \frac{1}{\#C}\sum_C \frac{1}{\sqrt{\theta}}\sqrt{\sum_{r\not\in R'_C} p_{C,r}}$$

is minimized - the minimum is guaranteed to be $O(1/N)$, and we only have to examine values of θ which are equal to some $p_{C,r}/2$. Then we use θ to do the first bit of rounding to get to an arc-consistent instance. Next we pick $\theta_2 \in [0, \theta]$ to minimize the number of problems we run into while ensuring that (P2) is satisfied - we only need to try values of θ_2 which are just above or just below a remainder of some $\|x_A\|^2$ modulo θ, so this can be done efficiently.

For the final step, we want to pick a unit vector U which is dual to a hyperplane which separates as few pairs of vectors $x_A - x_B$, $y_{A+R'} - y_{B+R'}$ with $A + R' - R' = A, B + R' - R' = B$ as possible. This step is trickier, but in Appendix C of [19] they cite the paper [108], which shows that you can find a U which is worse by at most $O(1/N)$ compared to what you would get with a uniformly random choice, using a deterministic algorithm which runs in time $\|\mathbf{X}\|^{O(1)}$ times $2^{\log(N)^2} = o(1/\epsilon)$.

We may naturally wonder whether $f(\epsilon) = O(\frac{1}{\log(1/\epsilon)})$ is really the best possible asymptotic one can get in Theorem 3.16.11. The next example shows that at least for HORN-SAT, it is impossible to improve the asymptotic.

Example 3.16.3 (Guruswami and Zhou [90]). For every n, consider the following "simultaneous induction" instance of HORN-SAT, on the $2n+2$ variables $p_0, ..., p_n, q_0, ..., q_n$:

$$\begin{aligned}
&(p_0 = 1) \wedge (q_0 = 1)\\
&\wedge (p_0 \wedge q_0 \implies p_1) \wedge (p_0 \wedge q_0 \implies q_1)\\
&\wedge \ \cdots\\
&\wedge (p_{n-1} \wedge q_{n-1} \implies p_n) \wedge (p_{n-1} \wedge q_{n-1} \implies q_n)\\
&\wedge (p_n = 0) \wedge (q_n = 0).
\end{aligned}$$

This instance has $2n+4$ constraints, and it is possible to satisfy at most $2n+3$ of them. However, the basic SDP relaxation of this instance thinks it can satisfy a $1-1/2^{n/4}$ fraction of the constraints!

Checking that the SDP relaxation has such a solution (for all n) is fairly tricky. One simplification which helps quite a bit is to note that the variables p_i, q_i only interact with $p_{i\pm1}, q_{i\pm1}$, so we just need to analyze the set of solutions to the basic SDP relaxation for the four-variable HORN-SAT instance

$$(x \wedge y \implies z) \wedge (x \wedge y \implies w)$$

in order to understand the possible behavior in the general case where we string together many of these "simultaneous induction" steps.

A second simplification is that since the overall vector

$$x_{\{0,1\}} = y_{\{0,1\}} = z_{\{0,1\}} = w_{\{0,1\}}$$

is a fixed unit vector, and since x_0 and x_1 are perpendicular with

$$x_0 + x_1 = x_{\{0,1\}},$$

etc., we only have to describe the dot products between the four vectors x_0, y_0, z_0, w_0 to determine the whole configuration. (In particular, if our SDP relaxation of this instance has a solution, then it has a solution where all of the vectors live on a 4-dimensional sphere of diameter 1 in $\mathbb{R}^5$.)

The plan is to make the probabilities that z, w are equal to 0 grow to be a constant factor larger than the probabilities that x, y are equal to 0, while keeping the correlation between z and w under control. To this end, we claim that the SDP relaxation of this four-variable instance has a solution which satisfies

$$\begin{aligned} \|x_0\|^2 = \|y_0\|^2 &= 15\epsilon, \\ x_0 \cdot y_0 &= 10\epsilon, \\ \|z_0\|^2 = \|w_0\|^2 &= 18\epsilon, \\ z_0 \cdot w_0 &= 12\epsilon, \\ x_0 \cdot z_0 = \dots = y_0 \cdot w_0 &= 13\epsilon \end{aligned}$$

for any $0 \le \epsilon \le 1/24$. Note that this way we have

$$\frac{\|z_0\|^2}{\|x_0\|^2} = \frac{\|w_0\|^2}{\|y_0\|^2} = \frac{z_0 \cdot w_0}{x_0 \cdot y_0} = \frac{6}{5}$$

and

$$\frac{z_0 \cdot w_0}{\|z_0\|\|w_0\|} = \frac{x_0 \cdot y_0}{\|x_0\|\|y_0\|} = \frac{2}{3},$$

so we can glue the solutions to many units like this together, as long as we can show that each unit is a valid solution to the SDP relaxation on its own. For this, we check that the matrix of dot products between our hypothetical vectors $x_{\{0,1\}}, x_0, y_0, z_0, w_0$ below is positive semidefinite:

$$\begin{bmatrix} 1 & 15\epsilon & 15\epsilon & 18\epsilon & 18\epsilon \\ 15\epsilon & 15\epsilon & 10\epsilon & 13\epsilon & 13\epsilon \\ 15\epsilon & 10\epsilon & 15\epsilon & 13\epsilon & 13\epsilon \\ 18\epsilon & 13\epsilon & 13\epsilon & 18\epsilon & 12\epsilon \\ 18\epsilon & 13\epsilon & 13\epsilon & 12\epsilon & 18\epsilon \end{bmatrix} \overset{?}{\succeq} 0,$$

and we use the probability distribution

$$\begin{aligned}
p_{(1,1,1)} &= 1 - 20\epsilon,\\
p_{(1,0,0)} &= 5\epsilon,\\
p_{(0,1,0)} &= 5\epsilon,\\
p_{(0,0,0)} &= 8\epsilon,\\
p_{(0,0,1)} &= 2\epsilon
\end{aligned}$$

over the set of solutions (x, y, z) to the constraint $x \wedge y \implies z$ (and similarly for the constraint $x \wedge y \implies w$).

In order to check that the 5 by 5 matrix above is always positive semidefinite, the simplest method is to replace the 1 in the upper left corner by $24\epsilon \le 1$ and divide out the ϵs, to get a fixed matrix that doesn't depend on ϵ (which we can then check for positive definiteness by hand, once and for all).

By joining together many copies of the unit described above, we can find a solution to the SDP relaxation where the only constraints which aren't satisfied exactly are the constraints $p_0 = 1, q_0 = 1$ - instead, these will be satisfied with the exponentially small error $0.75/1.2^{n-1}$. We arrange things so that at the second-to-last step, we have

$$\mathbb{P}[p_{n-1} = 0] = \mathbb{P}[q_{n-1} = 0] = 3/4$$

and

$$\mathbb{P}[p_{n-1} = q_{n-1} = 0] = 1/2,$$

at which point we can no longer continue to use the unit described above (since ϵ hits 1/24 at this point). For the last step, we join this with an honest probability distribution over the set of solutions to the 4-variable instance we get by restricting to $p_{n-1}, q_{n-1}, p_n, q_n$:

$$\begin{aligned}
p_{(0,0,0,0)} &= 1/2,\\
p_{(1,0,0,0)} &= 1/4,\\
p_{(0,1,0,0)} &= 1/4.
\end{aligned}$$

As a consequence of this example, we can't hope for improved asymptotics for any relational structure **A** which can pp-construct HORN-SAT. On the other hand, it is certainly possible to get better asymptotics for 2-SAT! Under the Unique Games Conjecture, the best possible function f as in Definition 3.16.1 for 2-SAT is given by $f(\epsilon) \sim \sqrt{\epsilon}$ (see [64] for references and discussion).

Problem 3.16.1 (From [64])**.** Suppose that a finite relational structure **A** has bounded width and does not pp-construct HORN-SAT. Is it necessarily the case that there is some k such that CSP(**A**) can be robustly solved with the function f from Definition 3.16.1 satisfying $f(\epsilon) = O(\epsilon^{1/k})$?

In [63], the authors show that for every $\mathbb{A}$ with a near-unanimity term, there is some k such that CSP(**A**) can be robustly solved (via the basic SDP relaxation) with the function f from Definition 3.16.1 satisfying $f(\epsilon) = O(\epsilon^{1/k})$. The proof involves a consistency condition which doesn't rely on arc-consistency.

Remark 3.16.2. The first part of the proof of Theorem 3.16.10 shows that given a solution to the LP relaxation of an instance $\mathbf{X}$ with value $1 - \epsilon$, we can find a subinstance $\mathbf{X}'$ which satisfies (P1) and (P2) after giving up on an $O(1/\log(1/\epsilon))$ fraction of the constraints. A positive answer to the following problem would then prove that every CSP which is solved by its LP relaxation is also *robustly* solved by its LP relaxation.

Problem 3.16.2 (Conjectured in [37])**.** Suppose that CSP($\mathbb{A}$) is solved by its LP relaxation. Is it necessarily the case that every instance $\mathbf{X}$ of CSP($\mathbb{A}$) which satisfies the consistency conditions (P1) and (P2) has a solution?

Chapter 4

Finite Taylor Algebras

4.1 Cyclic terms

In this section we will prove that every finite Taylor algebra has a cyclic term.

Definition 4.1.1. An m-ary operation c is called *cyclic* if it satisfies the identity

$$c(x_1, x_2, ..., x_m) \approx c(x_2, ..., x_m, x_1).$$

Cyclic terms were first proved to exist for finite congruence modular algebras [21], and most of the basic facts about cyclic terms are developed in that paper. This was extended to finite congruence join-semidistributive algebras in [17], and then finally to all finite Taylor algebras in [18]. We'll start by showing that we only care about cyclic terms of prime arity.

Proposition 4.1.2 (Multiplicative property of cyclic terms [21])**.** *A variety $\mathcal{V}$ has a cyclic term c_{mn} of arity mn if and only if $\mathcal{V}$ has cyclic terms c_m, c_n of arity m and n, respectively.*

Proof. Suppose first that c_{mn} is a cyclic term of arity mn. Then we can define a cyclic term of arity m by plugging in

$$c_{mn}(\underbrace{x_1, ..., x_1}_{n}, \underbrace{x_2, ..., x_2}_{n}, ..., \underbrace{x_m, ..., x_m}_{n}),$$

and we can define a cyclic term of arity n similarly.

Conversely, suppose that c_m, c_n are cyclic terms of arity m and n. We define a cyclic term of arity mn by renumbering the inputs of the star composition $c_n * c_m$:

$$c_n\begin{pmatrix} c_m(x_1, & x_{n+1}, & ..., & x_{(m-1)n+1}), \\ c_m(x_2, & x_{n+2}, & ..., & x_{(m-1)n+2}), \\ \vdots & \vdots & \ddots & \vdots \\ c_m(x_n, & x_{2n}, & ..., & x_{mn}) \end{pmatrix} \approx c_n\begin{pmatrix} c_m(x_2, & x_{n+2}, & ..., & x_{(m-1)n+2}), \\ \vdots & \vdots & \ddots & \vdots \\ c_m(x_n, & x_{2n}, & ..., & x_{mn}), \\ c_m(x_{n+1}, & ..., & x_{(m-1)n+1}, & x_1) \end{pmatrix}. \qquad \square$$

Corollary 4.1.3. *A variety $\mathcal{V}$ has a cyclic term of arity m if and only if $\mathcal{V}$ has a cyclic term of arity p for every prime p which divides m.*

Next we will describe the main obstruction to the existence of a cyclic term of a given arity.

Proposition 4.1.4 (Semantic meaning of cyclic terms [21]). *Suppose that $\mathcal{V}$ is a variety. Then for any $m \in \mathbb{N}$, the following are equivalent.*

(a) $\mathcal{V}$ has no cyclic term of arity m.

(b) There is some $\mathbb{A} \in \mathcal{V}$ and an automorphism $\sigma \in \mathrm{Aut}(\mathbb{A})$ such that $\sigma^m = 1$ and σ has no fixed point.

Proof. We start by showing that (b) implies (a). Suppose that $\mathbb{A}, \sigma$ are as in (b), and suppose for contradiction that $\mathbb{A}$ has some cyclic term c_m of arity m. Let a be any element of $\mathbb{A}$, and define a_i by $a_i = \sigma^i(a)$. Then we have

$$c_m \begin{pmatrix} a_1, & a_2, & ..., & a_m \\ a_2, & a_3, & ..., & a_1 \end{pmatrix} \in \sigma,$$

so $c_m(a_1, ..., a_m)$ is a fixed point of σ, contradicting the assumption in (b).

Now suppose that (a) holds. Let $\mathbb{A} = \mathcal{F}_{\mathcal{V}}(x_1, ..., x_m)$ be the free algebra on m generators, and let σ be the automorphism of $\mathbb{A}$ defined by cyclically permuting the generators $x_1, ..., x_m$. Then a fixed point of σ is precisely the same thing as a cyclic term of $\mathcal{V}$ of arity m, so if $\mathcal{V}$ has no cyclic term of arity m, then σ has no fixed points (and satisfies $\sigma^m = 1$). □

For finite algebras, we can give a local criterion for the existence of a cyclic term.

Proposition 4.1.5 (Local criterion for cyclic terms [21]). *If $\mathbb{A}$ is a finite algebra, then $\mathbb{A}$ has an m-ary cyclic term if and only if it is the case that for all $a_1, ..., a_m \in \mathbb{A}$, there exists some m-ary term t such that*

$$t(a_1, a_2, ..., a_m) = t(a_2, ..., a_m, a_1) = \cdots = t(a_m, a_1, ..., a_{m-1}).$$

Proof. Say that an m-ary term t is cyclic for a tuple $(a_1, ..., a_m)$ if it satisfies the displayed equation from the statement of the proposition. Let c be an m-ary term which is cyclic for a maximal set of tuples (we are using finiteness of $\mathbb{A}$ here). Suppose for contradiction that c is not cyclic, and let $a = (a_1, ..., a_m)$ be any tuple such that c is not cyclic for a.

Define a tuple $a' = (a_1', ..., a_m')$ by

$$a_i' = c(a_i, a_{i+1}, ..., a_{i-1}),$$

with indices taken modulo m. By assumption, there is some m-ary term t which is cyclic for a'. But then the m-ary term

$$t(c(x_1, x_2 ..., x_m), c(x_2, ..., x_m, x_1), ..., c(x_m, x_1, ..., x_{m-1}))$$

is cyclic for a, and is also cyclic for every tuple which c was cyclic for, contradicting the maximality assumption on c. □

For the sake of checking the local condition of Proposition 4.1.5 for a particular tuple $a_1, ..., a_m$, the natural approach is to compute the m-ary relation

$$\mathrm{Sg}_{\mathbb{A}^m} \left\{ \begin{bmatrix} a_1 & a_2 & \cdots & a_m \\ a_2 & a_3 & \cdots & a_1 \\ \vdots & \vdots & \iddots & \vdots \\ a_m & a_1 & \cdots & a_{m-1} \end{bmatrix} \right\},$$

and to check if it contains any constant tuples. This relation is invariant under cyclically permuting its coordinates, which leads us to make the following definition.

Definition 4.1.6. A relation $\mathbb{R} \le \mathbb{A}^m$ is called *cyclic* if $\mathbb{R}$ is invariant under cyclically permuting its coordinates, that is,

$$(a_1, a_2, ..., a_m) \in \mathbb{R} \iff (a_2, ..., a_m, a_1) \in \mathbb{R}.$$

Corollary 4.1.7 (Relational criterion for cyclic terms [21])**.** *If $\mathbb{A}$ is a finite algebra, then $\mathbb{A}$ has a cyclic term of arity m if and only if every m-ary cyclic relation $\mathbb{R} \le \mathbb{A}^m$ contains a constant tuple.*

Now we are finally ready to prove one of the main results of [18], which states that every finite Taylor algebra $\mathbb{A}$ has cyclic terms of every prime arity $p > |\mathbb{A}|$. In fact, we will prove a stronger version of this result due to Zhuk (currently unpublished).

Theorem 4.1.8 (Finite Taylor algebras have cyclic terms [18], refined by Zhuk)**.** *Suppose $\mathbb{A}$ is a finite idempotent Taylor algebra and that p is prime. Then one of the following is true:*

(a) either $\mathbb{A}$ has a cylic term of arity p, or

(b) there is some $\mathbb{B} \in HS(\mathbb{A})$ and some automorphism $\sigma \in \mathrm{Aut}(\mathbb{B})$ such that $\sigma^p = 1$ and σ has no fixed points.

In particular, if $p > |\mathbb{A}|$ then $\mathbb{A}$ has a cyclic term of arity p.

Proof. We prove this by induction on $|\mathbb{A}|$. Suppose that there is no $\mathbb{B} \in HS(\mathbb{A}), \sigma \in \mathrm{Aut}(\mathbb{B})$ as in (b), and let $\mathbb{R} \le \mathbb{A}^p$ be any p-ary cyclic relation. It's enough to show that $\mathbb{R}$ contains a constant tuple.

If $\pi_1(\mathbb{R}) \ne \mathbb{A}$, then since $\mathbb{R}$ is cyclic we have $\mathbb{R} \le \pi_1(\mathbb{R})^p$, so we can apply the induction hypothesis to the algebra $\pi_1(\mathbb{R})$ to see that $\mathbb{R}$ has a constant tuple. Thus we may assume without loss of generality that $\mathbb{R}$ is subdirect in $\mathbb{A}^p$.

If $\mathbb{A}$ has a nontrivial congruence $\theta \in \mathrm{Con}(\mathbb{A})$, then $\mathbb{R}/\theta^p \le (\mathbb{A}/\theta)^p$ is a cyclic relation on $\mathbb{A}/\theta$, so by the induction hypothesis applied to $\mathbb{A}/\theta$ there is some congruence class a/θ such that $\mathbb{R} \cap (a/\theta)^p \ne \emptyset$. Setting $\mathbb{R}' = \mathbb{R} \cap (a/\theta)^p$, we see that $\mathbb{R}'$ is a cyclic relation on a/θ, so by the induction hypothesis applied to a/θ we see that $\mathbb{R}'$ (and therefore also $\mathbb{R}$) has a constant tuple. Thus we may assume that $\mathbb{A}$ is simple.

If any $\pi_{ij}(\mathbb{R})$ is the graph of an automorphism σ of $\mathbb{A}$, then since $\mathbb{R}$ is cyclic, we see that $\pi_{j,2j-i}(\mathbb{R})$ is also the graph of σ, and similarly so is $\pi_{2j-i,3j-2i}(\mathbb{R})$, etc., and we must have $\sigma^p = 1$. Since p is prime, we see that in fact every $\pi_{kl}(\mathbb{R})$ is the graph of some power of the automorphism σ. In this case we see that $\mathbb{R}$ has a constant tuple if and only σ has a fixed point. Thus we may assume without loss of generality that every $\pi_{ij}(\mathbb{R})$ is linked.

By Zhuk's four cases (Corollary 3.12.12), we see that $\mathbb{A}$ is either affine, subdirectly complete, or has a proper ternary absorbing subalgebra.

If $\mathbb{A}$ is affine, with underlying abelian group $(A, +, -, 0)$, then since $x - y + z$ is a term of $\mathbb{A}$ (by the definition of an affine algebra), we see that $k_1x_1 + \cdots + k_mx_m$ is a term of $\mathbb{A}$ for all $k_1, ..., k_m \in \mathbb{Z}$ such that $k_1 + \cdots + k_m \equiv 1 \pmod{|\mathbb{A}|}$. In particular, if $p \nmid |\mathbb{A}|$, then

$$p^{-1}(x_1 + \cdots + x_p)$$

is a p-ary cyclic term of $\mathbb{A}$. On the other hand, if $p \mid |\mathbb{A}|$, then by elementary group theory there must be some element $c \in \mathbb{A}$ of order p, and then by the idempotence of $\mathbb{A}$ the relation

$$\{(x,y) \mid x = y + c\}$$

is a subalgebra of $\mathbb{A}^2$, and it is then the graph of an automorphism σ of $\mathbb{A}$ which has order p and has no fixed points.

If $\mathbb{A}$ is subdirectly complete, then since $\mathbb{R} \le_{sd} \mathbb{A}^p$ is subdirect and every $\pi_{ij}(\mathbb{R})$ is linked, we must have $\mathbb{R} = \mathbb{A}^p$. In this case $\mathbb{R}$ contains *every* constant tuple.

If $\mathbb{A}$ has a proper ternary absorbing subalgebra, then we define a directed graph $\mathbf{D}$ whose vertices are proper ternary absorbing subalgebras $\mathbb{B} \lhd \mathbb{A}$, and with a directed edge $(\mathbb{B}, \mathbb{B} + \pi_{ij}(\mathbb{R}))$ whenever $\mathbb{B} + \pi_{ij}(\mathbb{R}) \ne \mathbb{A}$ and $i \ne j$.

Claim: The digraph $\mathbf{D}$ has no directed cycles.

Proof of claim: Note first that since $\mathbb{R}$ is cyclic we have

$$\pi_{ij}(\mathbb{R})^- \subseteq \pi_{ij}(\mathbb{R})^{\circ(p-1)},$$

so if $\mathbb{B} + \pi_{ij}(\mathbb{R}) = \mathbb{B}$ then we must have

$$\mathbb{B} + \pi_{ij}(\mathbb{R}) - \pi_{ij}(\mathbb{R}) = \mathbb{B},$$

so $\mathbb{B}$ is a union of linked components of $\pi_{ij}(\mathbb{R})$. Since $\pi_{ij}(\mathbb{R})$ is linked and $\mathbb{B}$ is proper, this is impossible. Thus $\mathbf{D}$ has no directed cycles of length 1. Since $\mathbb{R}$ is cyclic, we also have

$$\pi_{ij}(\mathbb{R}) \circ \pi_{kl}(\mathbb{R}) \supseteq \pi_{i+k,j+l}(\mathbb{R}),$$

so if $\mathbf{D}$ has a directed cycle, then $\mathbf{D}$ has a directed cycle of length 2, of the form $\mathbb{B}+\pi_{ij}(\mathbb{R})+\pi_{kl}(\mathbb{R}) = \mathbb{B}$. If $i+k \ne j+l$ this gives us a directed cycle of length 1, while if $i+k = j+l$ then we have $\mathbb{B} + \pi_{ij}(\mathbb{R}) - \pi_{ij}(\mathbb{R}) = \mathbb{B}$, so once again $\mathbb{B}$ must be a union of linked components of $\pi_{ij}(\mathbb{R})$, which is impossible. The claim is proved.

Since the digraph $\mathbf{D}$ is finite, nonempty, and has no directed cycles, there must be a proper ternary absorbing subalgebra $\mathbb{B} \lhd \mathbb{A}$ such that $\mathbb{B} + \pi_{ij}(\mathbb{R}) = \mathbb{A}$ for all $i \ne j$. In particular, we see that $\pi_{ij}(\mathbb{R}) \cap \mathbb{B}^2 \ne \emptyset$ for all i, j. Since $\mathbb{B}$ is ternary absorbing, this implies that in fact $\mathbb{R} \cap \mathbb{B}^p \ne \emptyset$ by Corollary 3.8.4. Setting $\mathbb{R}' = \mathbb{R} \cap \mathbb{B}^p$, we can apply the induction hypothesis to $\mathbb{B}$ to see that $\mathbb{R}'$ contains a constant tuple. Thus $\mathbb{R}$ contains a constant tuple, and we are done. □

Corollary 4.1.9. *If $\mathbb{A}$ is a finite Taylor algebra and m has no prime factors p which are less than or equal to $|\mathbb{A}|$, then $\mathbb{A}$ has an idempotent m-ary cyclic term.*

Example 4.1.1. Let $\mathbb{A}_n$ be the dual discriminator algebra from Example 1.6.5 on a domain of size n. Then every subset of $\mathbb{A}_n$ is a subalgebra with full automorphism group, so $\mathbb{A}_n$ does not have cyclic terms of any arity between 2 and n. By the previous results, we see that $\mathbb{A}_n$ has a cyclic term of arity m if and only if m has no prime factors which are less than or equal to n.

Corollary 4.1.10 (Siggers term from cyclic term)**.** *If $\mathbb{A}$ is a finite Taylor algebra, then $\mathbb{A}$ has an idempotent 4-ary Siggers term t, satisfying the identity $t(x,x,y,z) \approx t(y,z,z,x)$.*

Proof. Let c be an idempotent m-ary cyclic term for some $m > 1$. Then there are numbers $a, b \in \mathbb{N}$ such that $2a + 3b = m$, and we may define t by

$$t(x,y,z,w) \coloneqq c(\underbrace{x,...,x}_{b},\underbrace{y,...,y}_{a},\underbrace{z,...,z}_{b},\underbrace{w,...,w}_{a+b}).$$ □

Corollary 4.1.11 (Daisy chain terms)**.** *If $\mathbb{A}$ is a finite Taylor algebra, then there are idempotent terms $w_i(x,y,z)$ for $i \in \mathbb{Z}$ such that for all i we have*

$$w_i(x,x,y) \approx w_i(y,x,x) \approx w_{i-1}(x,y,x),$$

and the sequence of terms w_i is periodic with some finite period.

Proof. Choose p to be an extremely huge prime, let c be an idempotent p-ary cyclic term, and let $a = \lfloor \frac{p}{3} \rfloor$. Define a long sequence of numbers $a_0, a_1, ...$ by $a_0 = a$ and

$$a_{i+1} = p - 2a_i,$$

stopping as soon as we hit the first a_i with $a_i > \frac{p}{2}$. Define terms w_i' by

$$w_i'(x,y,z) \coloneqq c(\underbrace{x,...,x}_{a_i},\underbrace{y,...,y}_{a_{i+1}},\underbrace{z,...,z}_{a_i}).$$

Since c is cyclic, these w_i's will satisfy the identities

$$w_i'(x,x,y) \approx w_i'(y,x,x) \approx w_{i-1}'(x,y,x).$$

If p is large enough, then there must be some $j < k$ such that $w_j' = w_k'$. Then we define w_i by picking some $i' \in [j,k]$ such that $i \equiv i' \pmod{k-j}$ and setting $w_i = w_{i'}'$. □

Corollary 4.1.12. *A finite algebra $\mathbb{A}$ is Taylor if and only if it has a pair of idempotent ternary terms p,q satisfying the identities*

$$\begin{aligned}&p(x,x,y) \approx p(y,x,x),\\&q(x,x,y) \approx q(y,x,x) \approx p(x,y,x).\end{aligned}$$

Proof. To see that such p,q must exist in a Taylor algebra, we can take p,q to be any pair of consecutive daisy chain terms from the previous corollary. To see that any such p,q define Taylor terms, note that if p is a projection then p must be second projection, but in this case q must be a Mal'cev term. □

4.2 Minimal Taylor clones

Since our main aim in these notes is to understand the most general CSPs which can be solved in polynomial time, it makes sense to study (core) relational structures $\mathbf{A}$ such that CSP($\mathbf{A}$) is in P, but such that adding any additional relations to $\mathbf{A}$ makes the problem NP-complete. According to the CSP dichotomy theorem of Bulatov [48] and Zhuk [190], these maximal relational structures correspond under the Inv − Pol Galois correspondence to *minimal* Taylor clones.

Definition 4.2.1. A clone $\mathcal{O}$ on a finite domain is called a *minimal Taylor clone* if $\mathcal{O}$ is Taylor and every proper subclone of $\mathcal{O}$ is not Taylor. A finite algebra $\mathbb{A}$ is called a *minimal Taylor algebra* if Clo($\mathbb{A}$) is a minimal Taylor clone.

At first it may not be clear that minimal Taylor clones even exist: perhaps every Taylor clone contains a proper Taylor subclone, with the relevant Taylor operations having higher and higher arity. We can rule this out by using the existence of a Siggers term (Corollary 3.11.18).

Proposition 4.2.2. *Every Taylor clone on a finite domain contains a minimal Taylor clone.*

Proof. By Corollary 3.11.18, every Taylor clone contains a 4-ary Siggers operation t satisfying the identity $t(x,x,y,z) \approx t(y,z,z,x)$. Since any such t is Taylor, and since there are only finitely many 4-ary operations on a given finite domain, at least one of the 4-ary Siggers operations $t \in \mathcal{O}$ generates a minimal Taylor clone. □

Since every minimal Taylor clone is generated by a single 4-ary operation, we see that the number of minimal Taylor clones on a domain of size n is at most n^{n^4}. We can get a much better upper bound on the number of minimal Taylor clones by using the daisy chain terms from the previous section.

Proposition 4.2.3. *The number of minimal Taylor clones on a domain of size n is at most n^{2n^3}.*

Proof. By Corollary 4.1.12, every minimal Taylor clone $\mathcal{O}$ contains a pair of ternary idempotent operations p, q satisfying the identities

$$\begin{aligned} &p(x,x,y) \approx p(y,x,x),\\ &q(x,x,y) \approx q(y,x,x) \approx p(x,y,x). \end{aligned}$$

Since $\langle p, q\rangle$ generates a Taylor clone, we must have $\mathcal{O} = \langle p, q\rangle$. Since the number of ordered pairs of ternary operations p, q on a domain of size n is n^{2n^3}, we see that the number of minimal Taylor clones is at most n^{2n^3}. □

Remark 4.2.1. Later we will see that the upper bound n^{2n^3} can be reduced to n^{n^3}, by showing that every minimal Taylor clone is generated by a *single* ternary operation. On a domain of size 2, it is easy to check that every minimal Taylor algebra is term equivalent to either a semilattice, a majority algebra, or to the idempotent reduct of $\mathbb{Z}/2$. On a domain of size 3, there turn out to be a total of 24 minimal Taylor algebras, up to term equivalence and isomorphism.

Unfortunately, the number of minimal Taylor algebras grows quite rapidly as the size of the domain increases: even if we only consider majority algebras, it turns out that the number of minimal majority algebras (up to term-equivalence) such that every three-element subset is a subalgebra is $7^{\binom{n}{3}}$, and identifying isomorphic algebras can only reduce this by a factor of at most $n!$, which makes little difference to the asymptotics.

The key fact that makes the theory of minimal Taylor algebras work is the following result, which essentially says that anything that "looks like" it "could be" a subalgebra or quotient of a minimal Taylor algebra actually *is* a subalgebra or quotient, and is also minimal Taylor as well.

Theorem 4.2.4. *If $\mathbb{A}$ is a minimal Taylor algebra and $\mathbb{B} \in HSP_{fin}(\mathbb{A})$, then $\mathbb{B}$ is also a minimal Taylor algebra.*

In fact, if $S \subseteq \mathbb{B}$ is a subset of $\mathbb{B}$ (not assumed to be a subalgebra), $t \in \mathrm{Clo}(\mathbb{A})$ is any term of $\mathbb{A}$, and θ is an equivalence relation on S such that

- *the set S is closed under t,*
- *every equivalence class of θ is a subalgebra of $\mathbb{B}$,*
- *the equivalence relation θ is a congruence of the algebraic structure (S, t), and*

- *the quotient $(S, t)/\theta$ is a Taylor algebra,*

then in fact the following must all be true:

- *the set S is actually the underlying set of a subalgebra $\mathbb{S}$ of $\mathbb{B}$,*
- *the equivalence relation θ is actually a congruence on the subalgebra $\mathbb{S}$, and*
- *the restriction of every term of $\mathbb{A}$ to the quotient $\mathbb{S}/\theta$ is in the clone generated by the restriction of t to $(S, t)/\theta$.*

Note that taking θ to be the trivial equivalence relation 0_S is always allowed, since every minimal Taylor algebra is automatically idempotent.

Proof. Let p be any prime such that $p > |\mathbb{A}|$ and $p > |(S, t)/\theta|$. By Theorem 4.1.8, there is a p-ary cyclic term $c \in \mathrm{Clo}(\mathbb{A})$, as well as a p-ary term $u \in \mathrm{Clo}(t)$ such that the restriction of u to $(S, t)/\theta$ is cyclic. Define a p-ary term c' by

$$c'(x_1, ..., x_p) \coloneqq c(u(x_1, ..., x_p), u(x_2, ..., x_p, x_1), ..., u(x_p, x_1, ..., x_{p-1})).$$

Then since c is cyclic, c' will automatically be cyclic as well. Since $\mathbb{A}$ is assumed to be minimal Taylor, we must have $\mathrm{Clo}(\mathbb{A}) = \langle c' \rangle$.

Suppose that $x_1, ..., x_p \in S$. Then since $u \in \mathrm{Clo}(t)$ preserves S and acts cyclically on $(S, t)/\theta$, we must have

$$u(x_1, ..., x_p) \equiv_\theta u(x_2, ..., x_p, x_1) \equiv_\theta \cdots \equiv_\theta u(x_p, x_1, ..., x_{p-1}) \in S,$$

and since equivalence classes of θ were assumed to be subalgebras of $\mathbb{B}$, we have

$$c'(x_1, ..., x_p) \equiv_\theta u(x_1, ..., x_p) \in S.$$

Thus c' preserves S as well as the equivalence relation θ, and the restriction of c' to $(S, t)/\theta$ is the same as the restriction of u to $(S, t)/\theta$. Since c' generates $\mathrm{Clo}(\mathbb{A})$, this finishes the proof. □

An immediate consequence of Theorem 4.2.4 is that minimal Taylor algebras are *prepared* in the sense of Definition 3.2.19.

Proposition 4.2.5. *If $\mathbb{A}$ is a minimal Taylor algebra, then $a, b \in \mathbb{A}$ have*

$$\begin{bmatrix} b \\ b \end{bmatrix} \in \mathrm{Sg}_{\mathbb{A}^2}\left\{\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix}\right\}$$

if and only if $\{a, b\}$ is a semilattice subalgebra of $\mathbb{A}$ with absorbing element b.

Proof. If $(b, b) \in \mathrm{Sg}\{(a, b), (b, a)\}$, then there must be some binary term t such that $t(a, b) = t(b, a) = b$. By idempotence, we automatically have $t(a, a) = a$ and $t(b, b) = b$, so the set $S = \{a, b\}$ is closed under t and (S, t) is a two-element semilattice. Thus we can apply Theorem 4.2.4 to see that $\{a, b\}$ must be a subalgebra of $\mathbb{A}$, and that the restriction of every term of $\mathbb{A}$ to $\{a, b\}$ is in the clone generated by the restriction of t to $\{a, b\}$. □

Similarly, we can recognize two-element majority subalgebras and $\mathbb{Z}/2^{\mathrm{aff}}$ subalgebras. To simplify the statements of these results, it is convenient to assume the existence of an order two automorphism.

Proposition 4.2.6. *If $\mathbb{A}$ is a minimal Taylor algebra and $a, b \in \mathbb{A}$ are such that $\mathrm{Sg}_{\mathbb{A}^2}\left\{\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix}\right\}$ is the graph of an automorphism of order two, then*

- *we have* $\begin{bmatrix} a \\ a \\ a \end{bmatrix} \in \mathrm{Sg}_{\mathbb{A}^3}\left\{\begin{bmatrix} a \\ a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \end{bmatrix}\right\}$ *iff $\{a, b\}$ is a majority subalgebra of $\mathbb{A}$, and*
- *we have* $\begin{bmatrix} b \\ b \\ b \end{bmatrix} \in \mathrm{Sg}_{\mathbb{A}^3}\left\{\begin{bmatrix} a \\ a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \end{bmatrix}\right\}$ *iff $\{a, b\}$ is a $\mathbb{Z}/2^{\mathrm{aff}}$ subalgebra of $\mathbb{A}$.*

Corollary 4.2.7. *If a minimal Taylor algebra $\mathbb{A}$ is generated by two elements a, b, then $\mathbb{A}$ is* not *subdirectly complete. As a consequence, either $\mathbb{A}$ has an affine quotient or $\mathbb{A}$ has a proper ternary absorbing subalgebra.*

Proof. Suppose for contradiction that $\mathbb{A}$ is subdirectly complete. Define a subdirect binary relation $\mathbb{S} \le_{sd} \mathbb{A}^2$ by

$$\mathbb{S} = \mathrm{Sg}_{\mathbb{A}^2}\left\{\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix}\right\}.$$

If (a, a) or (b, b) is in $\mathbb{S}$, then $\{a, b\}$ must be a two-element semilattice, which is not subdirectly complete. Otherwise, $\mathbb{S}$ must be the graph of an automorphism of order two by our assumption that $\mathbb{A}$ is subdirectly complete. Now define a subdirect ternary relation $\mathbb{R} \le_{sd} \mathbb{A}^3$ by

$$\mathbb{R} = \mathrm{Sg}_{\mathbb{A}^3}\left\{\begin{bmatrix} a \\ a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \end{bmatrix}\right\}.$$

Since no $\pi_{ij}(\mathbb{R})$ can be the graph of an automorphism, we see that we must have $\mathbb{R} = \mathbb{A}^3$ by our assumption that $\mathbb{A}$ is subdirectly complete. Thus we have $(a, a, a) \in \mathbb{R}$, so $\{a, b\}$ must be a two-element majority algebra, which is not subdirectly complete. This contradiction proves that $\mathbb{A}$ must not be subdirectly complete.

For the last claim, we recall Zhuk's four cases (Corollary 3.12.12), and note that both binary absorption and central absorption imply ternary absorption. □

Problem 4.2.1. Suppose that a minimal Taylor algebra $\mathbb{A}$ is generated by two elements. Is it possible for $\mathbb{A}$ to be polynomially complete?

The general recognition theorem for two-element majority subalgebras is as follows.

Proposition 4.2.8. *If $\mathbb{A}$ is a minimal Taylor algebra, then $a, b \in \mathbb{A}$ have*

$$\begin{bmatrix} a & b \\ a & b \\ a & b \end{bmatrix} \in \mathrm{Sg}_{\mathbb{A}^{3\times 2}}\left\{\begin{bmatrix} a & b \\ a & b \\ b & a \end{bmatrix}, \begin{bmatrix} a & b \\ b & a \\ a & b \end{bmatrix}, \begin{bmatrix} b & a \\ a & b \\ a & b \end{bmatrix}\right\}$$

if and only if $\{a, b\}$ is a majority subalgebra of $\mathbb{A}$.

It is also easy to recognize copies of the free semilattice on two generators.

Proposition 4.2.9. *If $\mathbb{A}$ is a minimal Taylor algebra and $a, b, c \in \mathbb{A}$ satisfy $a \rightarrow c$, $b \rightarrow c$ (i.e. $\{a, c\}$ and $\{b, c\}$ are semilattice subalgebras of $\mathbb{A}$ with absorbing element c), then we have*

$$\begin{bmatrix} c \\ c \end{bmatrix} \in \mathrm{Sg}_{\mathbb{A}^2}\left\{\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix}\right\}$$

if and only if $\{a, b, c\}$ is isomorphic to the free semilattice on two generators.

We can also characterize binary absorbing subalgebras of minimal Taylor algebras, and show that they are always automatically strongly absorbing (and therefore are automatically centrally absorbing as well).

Proposition 4.2.10. *Suppose that $\mathbb{A}$ is a minimal Taylor algebra, and that $\mathbb{B} \lhd_{bin} \mathbb{A}$ is a binary absorbing subalgebra of $\mathbb{A}$. Then the following must hold.*

(a) $\mathbb{B}$ is a strongly absorbing subalgebra of $\mathbb{A}$, that is, any term $f \in \mathrm{Clo}(\mathbb{A})$ which depends on its first input satisfies $f(\mathbb{B}, \mathbb{A}, ..., \mathbb{A}) \subseteq \mathbb{B}$.

(b) There is an equivalence relation $\theta_\mathbb{B} \in \mathrm{Con}(\mathbb{A})$ such that $\mathbb{B}$ is a congruence class of $\theta_\mathbb{B}$, and all other congruence classes of $\theta_\mathbb{B}$ are singletons.

(c) For every $a \not\in \mathbb{B}$, $\mathbb{B} \cup \{a\}$ is a subalgebra of $\mathbb{A}$, and $(\mathbb{B} \cup \{a\})/\theta_\mathbb{B}$ is a two-element semilattice with absorbing element $\mathbb{B}/\theta_\mathbb{B}$.

(d) For every $a \not\in \mathbb{B}$, there is some $b \in \mathbb{B}$ such that $\{a, b\}$ is a two-element semilattice with absorbing element b.

(e) For every $a, b \not\in \mathbb{B}$ such that $\mathrm{Sg}_\mathbb{A}\{a, b\} \cap \mathbb{B} \neq \emptyset$, $\mathrm{Sg}_\mathbb{A}\{a, b\}/\theta_\mathbb{B}$ is isomorphic to the free semilattice on two generators.

(f) For every $a_1, ..., a_k \not\in \mathbb{B}$ such that $\mathrm{Sg}_\mathbb{A}\{a_i, a_j\} \cap \mathbb{B} \neq \emptyset$ for all $i \neq j$, $\mathrm{Sg}_\mathbb{A}\{a_1, ..., a_k\}/\theta_\mathbb{B}$ is isomorphic to a semilattice of size $k + 1$.

In particular, if $\mathbb{A}$ is generated by two elements and $\mathbb{B}$ is a proper binary absorbing subalgebra, then $\mathbb{A}/\theta_\mathbb{B}$ is either a two-element semilattice, or is isomorphic to the free semilattice on two generators.

For the sake of concretely writing down minimal Taylor algebras, we should pick convenient terms. My preference is to write them down in terms of the daisy chain terms from Corollary 4.1.11.

Definition 4.2.11. We say that a sequence of idempotent ternary terms w_i, defined for all $i \in \mathbb{Z}$, is a sequence of *daisy chain terms* if it satisfies the following properties:

- the sequence w_i is purely periodic in i with some finite period, and
- for all $i \in \mathbb{Z}$, we have $w_i(x, x, y) \approx w_i(y, x, x) \approx w_{i-1}(x, y, x)$.

It is useful to work out all possible sequences of daisy chain terms in our three basic examples of minimal Taylor algebras: semilattices, majority algebras, and affine algebras.

Proposition 4.2.12. *If $\mathbb{A} = (A, \vee)$ is a semilattice, then any sequence of daisy chain terms of $\mathbb{A}$ must have*

$$w_i(x, y, z) \approx x \vee y \vee z$$

for all $i \in \mathbb{Z}$.

Proof. It's enough to show that $w_i(x, x, y) \approx w_i(x, y, x) \approx w_i(y, x, x) \approx x \vee y$ for all i. Note that since $w_i(x, x, y) \approx w_i(y, x, x)$, we can't have $w_i(x, x, y) = y$, since semilattices have no Mal'cev terms. Additionally, if we had $w_i(x, x, y) = w_i(y, x, x) = x$, then w_i could not depend on its first or last coordinates, so we would have $w_{i+1}(x, x, y) \approx w_{i+1}(y, x, x) \approx w_i(x, y, x) = y$, which again contradicts the fact that semilattices have no Mal'cev terms.

Since the only binary terms of a semilattice are x, y, and $x \vee y$, we see by process of elimination that we must have $w_i(x, x, y) \approx w_i(y, x, x) \approx x \vee y$, and similar reasoning shows that $w_i(x, y, x) \approx w_{i+1}(x, x, y) \approx x \vee y$, so we are done. □

Proposition 4.2.13. *If $\mathbb{A} = (A, m)$ is a majority algebra, then in any sequence of daisy chain terms of $\mathbb{A}$, each w_i must be a majority term, that is, we have*

$$w_i(x, x, y) \approx w_i(x, y, x) \approx w_i(y, x, x) \approx x$$

for all $i \in \mathbb{Z}$.

Proof. Note that every ternary term of a majority algebra is either a projection or a majority term (as is easily checked by induction on the construction of the term in terms of the majority operation m). If some w_i is a projection, then the identity $w_i(x, x, y) \approx w_i(y, x, x)$ implies that it must be second projection, but then the identity $w_{i+1}(x, x, y) \approx w_{i+1}(y, x, x) \approx w_i(x, y, x) = y$ implies that w_{i+1} is a Mal'cev term, which is impossible. Thus each w_i must be a majority term. □

For the affine case, we have the following simplification in the setting of minimal Taylor algebras.

Proposition 4.2.14. *If $\mathbb{A}$ is minimal Taylor and affine, then there is an abelian group structure on the underlying set A such that $\mathbb{A}$ is term equivalent to $(A, x - y + z)$.*

Proof. Every affine algebra has the ternary function $x - y + z$ as a term, by Proposition 1.9.6. Since the ternary operation $x - y + z$ is Mal'cev, it generates a Taylor clone, so a minimal Taylor algebra is affine if and only if its clone is generated by $x - y + z$. □

Because of this result, we don't need to think about the general case of a module over a (possibly noncommutative) ring if we are only interested in minimal Taylor algebras: we only need to think about algebras of the form $(A, x - y + z)$ for A an abelian group. By the classification of finite abelian groups, we can write such an algebra as a product of cyclic factors of prime power order. Recall that the *exponent* of a group is the least number n such that every cyclic subgroup has order dividing n.

Proposition 4.2.15. *If $\mathbb{A} = (A, x - y + z)$ is an affine algebra such that the abelian group A has exponent n, then for any sequence of daisy chain terms w_i of $\mathbb{A}$, there is a sequence of elements $a_i \in \mathbb{Z}/n$ such that*

$$w_i(x, y, z) \approx a_i x + (1 - 2a_i)y + a_i z$$

and

$$a_{i+1} \equiv 1 - 2a_i \pmod{n}$$

for all $i \in \mathbb{Z}$.

Proof. Every m-ary term $t \in \mathrm{Clo}(x - y + z)$ can be written in the form

$$t(x_1, ..., x_m) \approx k_1x_1 + \cdots + k_mx_m,$$

for some $k_i \in \mathbb{Z}$ satisfying

$$k_1 + \cdots + k_m = 1.$$

Of course, only the congruence classes of the values of the coefficients k_i modulo n matter, and the set of m-ary terms $t \in \mathrm{Clo}(\mathbb{A})$ is in bijection with the set of tuples of $k_i \in \mathbb{Z}/n$ such that $k_1 + \cdots + k_m \equiv 1 \pmod{n}$.

Thus we can write

$$w_i(x, y, z) \approx a_ix + b_iy + c_iz$$

for some $a_i, b_i, c_i \in \mathbb{Z}/n$ such that $a_i + b_i + c_i \equiv 1 \pmod{n}$. The identity $w_i(x, x, y) \approx w(y, x, x)$ then implies that $a_i \equiv c_i$, so $b_i \equiv 1 - 2a_i$, while the identity $w_{i+1}(y, x, x) \approx w_i(x, y, x)$ implies that $a_{i+1} \equiv b_i \equiv 1 - 2a_i$. □

Proposition 4.2.16. *If $\mathbb{A} = (A, x - y + z)$ is an affine algebra such that $|A|$ is a power of 2, then any sequence of daisy chain terms of $\mathbb{A}$ must have*

$$w_i(x, y, z) \approx \frac{x + y + z}{3}$$

for all $i \in \mathbb{Z}$. In particular, if the abelian group A has exponent 2, then each w_i is the Mal'cev operation $x - y + z \approx x + y + z$.

Proof. Suppose the exponent of A is 2^k. Then if we let $a_i \in \mathbb{Z}/2^k$ be the sequence from Proposition 4.2.15, we see from $a_{i+1} \equiv 1 - 2a_i \pmod{2^k}$ that we have

$$a_{i+1} - 1/3 \equiv -2(a_i - 1/3) \pmod{2^k}$$

for all $i \in \mathbb{Z}$, so in fact we must have

$$a_i - 1/3 \equiv 0 \pmod{2^k}$$

for all $i \in \mathbb{Z}$. □

Proposition 4.2.17. *If $\mathbb{A} = (A, x - y + z)$ is an affine algebra such that the abelian group A has exponent 3^k, then any sequence of daisy chain terms of $\mathbb{A}$ must have period equal to 3^k, and there must be some $i \in \mathbb{Z}$ such that*

$$\begin{aligned} w_{i-1}(x, y, z) &\approx \frac{x + z}{2}, \\ w_i(x, y, z) &\approx y, \\ w_{i+1}(x, y, z) &\approx x - y + z. \end{aligned}$$

Proof. We just need to show that the map $a \mapsto 1 - 2a \pmod{3^k}$ defines a cyclic permutation of $\mathbb{Z}/3^k$ for all $k \geq 0$. To see this, it's enough to check that the cycle containing 0 has length exactly 3^k.

Lifting to the integers, if we define a sequence $a_i \in \mathbb{Z}$ by $a_0 = 0$ and $a_{i+1} = 1 - 2a_i$, then we can solve the recurrence to obtain

$$a_i = \frac{1-(-2)^i}{3}.$$

Then we see that $a_i \equiv 0 \pmod{3^k}$ if and only if $3^{k+1} \mid 1-(-2)^i$. By induction on k we may assume that 3^{k-1} divides i, and by the binomial theorem, we have

$$1-(-2)^i = 1-(1-3)^i = 3i - 9\binom{i}{2} + 27\binom{i}{3} - + \cdots \equiv 3i - 0 + 0 - + \cdots \pmod{3^{k+1}},$$

so $3^{k+1} \mid 1-(-2)^i$ if and only if 3^k divides i. □

By going back to the original construction of the daisy chain terms from a huge cyclic term, we can simplify the situation slightly for affine algebras of odd order.

Proposition 4.2.18. *If $\mathbb{A}$ is a minimal Taylor algebra, then it is possible to choose a sequence of daisy chain terms w_i of $\mathbb{A}$ such that for every affine $\mathbb{B} \in HSP_{fin}(\mathbb{A})$ of odd order, the restriction of w_1 to $\mathbb{B}$ is the Mal'cev operation $x - y + z$.*

Proof. Since there are only finitely many ternary terms $w_1 \in \mathrm{Clo}(\mathbb{A})$, it's enough to prove that for every finite k we can find a w_1 that is part of a sequence of daisy chain terms of $\mathbb{A}$, such that w_1 restricts to the Mal'cev operation $x - y + z$ on every affine $\mathbb{B} \in HSP(\mathbb{A})$ such that $|\mathbb{B}|$ is odd and $|\mathbb{B}| \le k$.

Note that for any large prime p, the restriction of a p-ary cyclic term c to $\mathbb{B}$ must be given by

$$c(x_1, ..., x_p) = \frac{x_1 + \cdots + x_p}{p}.$$

Thus, in the construction of the terms w_i' from Corollary 4.1.11 where we plugged in

$$w_i'(x,y,z) \coloneqq c(\underbrace{x,...,x}_{a_i}, \underbrace{y,...,y}_{p-2a_i}, \underbrace{z,...,z}_{a_i}),$$

we will have

$$w_i'(x,y,z) = \frac{a_i x + (p-2a_i)y + a_i z}{p}$$

on $\mathbb{B}$. So as long as we choose a_1 such that $a_1 \equiv p \pmod{k!}$ and $a_1 \approx \frac{p}{3}$ (which is possible as long as we take p much larger than $k!$), we will have $w_1'(x,y,z) = x - y + z$ on every affine algebra $\mathbb{B} \in HSP(\mathbb{A})$ of size at most k. For $|\mathbb{B}|$ odd, the restriction of the sequence of terms w_i' to $\mathbb{B}$ will be purely periodic, so the final sequence of daisy chain terms constructed will have $w_1 = w_1'$ on such $\mathbb{B}$. □

Remark 4.2.2. A similar argument shows that we can instead choose daisy chain terms w_i such that $w_i(x,y,z) \approx \frac{x+y+z}{3}$ for all i on every affine algebra $\mathbb{B} \in HSP(\mathbb{A})$ such that $|\mathbb{B}|$ is not a multiple of 3. In fact, for any profinite integer $a \in \hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n$ such that $a \equiv \frac{1}{3} \pmod{2^k}$ for all k, we can choose daisy chain terms such that $w_0(x,y,z) \approx ax + (1-2a)y + az$ on every affine algebra $\mathbb{B} \in HSP(\mathbb{A})$.

We can also limit the collection of affine algebras which may show up in $HSP(\mathbb{A})$.

Proposition 4.2.19. *If $\mathbb{A}$ is a minimal Taylor algebra and $\mathbb{B} \in HSP(\mathbb{A})$ is affine, then the exponent n of $\mathbb{B}$ is finite, with $n \le |\mathbb{A}|^{|\mathbb{A}|^2}$, and every prime p which divides n is bounded by $|\mathbb{A}|$.*

Proof. First we show that every n such that $\mathbb{Z}/n^{\text{aff}} \in HSP(\mathbb{A})$ has $n \le |\mathbb{A}|^{|\mathbb{A}|^2}$. To see this, note that $\mathbb{Z}/n^{\text{aff}}$ is generated by two elements (to be more specific, it is generated by 0 and 1), so if $\mathbb{Z}/n^{\text{aff}} \in HSP(\mathbb{A})$ then $\mathbb{Z}/n^{\text{aff}}$ must be a quotient of the free algebra on two generators $\mathcal{F}_{\mathbb{A}}(x,y) \le \mathbb{A}^{\mathbb{A}^2}$.

Next, note that if p is prime and $\mathbb{Z}/p^{\text{aff}} \in HSP(\mathbb{A})$, then $\mathbb{A}$ can't have any cyclic term of arity p, since $\mathbb{Z}/p^{\text{aff}}$ has an automorphism of order p with no fixed points. Thus by Theorem 4.1.8 there is no prime $p > |\mathbb{A}|$ such that $\mathbb{Z}/p^{\text{aff}} \in HSP(\mathbb{A})$. □

We will end this section by characterizing Zhuk's centrally absorbing subalgebras in the case of minimal Taylor algebras, and using them to naturally produce majority subquotients of minimal Taylor algebras.

Theorem 4.2.20. *If $\mathbb{A}$ is minimal Taylor, $\mathbb{C} \le \mathbb{A}$, and $\mathbb{M} \in HSP(\mathbb{A})$ is the two element majority algebra on the domain $\{0,1\}$, then the following are equivalent:*

(a) $\mathbb{C}$ is a ternary absorbing subalgebra of $\mathbb{A}$,

(b) for every prime $p > |\mathbb{A}|$ there is a p-ary cyclic term c of $\mathbb{A}$ such that whenever $\#\{i \mid x_i \in \mathbb{C}\} > \frac{p}{2}$, we have

$$c(x_1, ..., x_p) \in \mathbb{C},$$

and furthermore the restriction of c to $\mathbb{M}$ is the p-ary majority operation,

(c) the binary relation $\mathbb{R} \subseteq \mathbb{A} \times \mathbb{M}$ given by

$$\mathbb{R} = (\mathbb{A} \times \{0\}) \cup (\mathbb{C} \times \{0,1\})$$

is a subalgebra of $\mathbb{A} \times \mathbb{M}$,

(d) $\mathbb{C}$ centrally absorbs $\mathbb{A}$,

(e) every daisy chain term $w_i(x,y,z)$ witnesses the fact that $\mathbb{C}$ ternary absorbs $\mathbb{A}$.

Proof. For (a) implies (b): let t be a ternary term which witnesses $\mathbb{C} \lhd \mathbb{A}$. If m is a ternary term of $\mathbb{A}$ which acts as majority on $\mathbb{M}$, then the ternary term

$$t'(x,y,z) \coloneqq m(t(x,y,z), t(y,z,x), t(z,x,y))$$

also witnesses the absorption $\mathbb{C} \lhd \mathbb{A}$, and the restriction of t' to $\mathbb{M}$ is the majority operation. Now let $p > |\mathbb{A}|$ be prime, and let $u \in \text{Clo}(t')$ be any term such that the restriction of u to $\mathbb{M}$ is a p-ary majority operation. Any such u must have the property that whenever $\#\{i \mid x_i \in \mathbb{C}\} > \frac{p}{2}$, we have

$$u(x_1, ..., x_p) \in \mathbb{C}.$$

Now let c' be any p-ary cyclic term of $\mathbb{A}$, and define c by

$$c(x_1, ..., x_p) \coloneqq c'(u(x_1, ..., x_p), u(x_2, ..., x_p, x_1), ..., u(x_p, x_1, ..., x_{p-1})).$$

For (b) implies (c), note that the cyclic term c must generate the clone of $\mathbb{A}$, so it's enough to check that the relation $\mathbb{R}$ is preserved by c, which is easy to prove directly.

That (c) implies (d) follows from Zhuk's Theorem 3.10.8, since the left center of $\mathbb{R}$ is $\mathbb{C}$ and the majority algebra $\mathbb{M}$ is binary absorption free.

That (c) implies (e) follows from a direct computation: we have

$$\begin{bmatrix} w_i(\mathbb{C},\mathbb{A},\mathbb{C}) \\ 1 \end{bmatrix} = w_i\left(\begin{bmatrix}\mathbb{C}\\1\end{bmatrix},\begin{bmatrix}\mathbb{A}\\0\end{bmatrix},\begin{bmatrix}\mathbb{C}\\1\end{bmatrix}\right) \subseteq \mathbb{R},$$

so $w_i(\mathbb{C},\mathbb{A},\mathbb{C}) \subseteq \mathbb{C}$, and similarly $w_i(\mathbb{A},\mathbb{C},\mathbb{C}), w_i(\mathbb{C},\mathbb{C},\mathbb{A}) \subseteq \mathbb{C}$.

That (d) implies (a) follows from Zhuk's Corollary 3.10.10, while (e) implies (a) is immediate. □

Corollary 4.2.21. *Suppose that $\mathbb{A}$ is minimal Taylor and that $\mathbb{C},\mathbb{D} \lhd_Z \mathbb{A}$ are two ternary absorbing subalgebras of $\mathbb{A}$. Then $\mathbb{C}\cup\mathbb{D}$ is a subalgebra of $\mathbb{A}$.*

If $\mathbb{C}\cap\mathbb{D} = \emptyset$, then the equivalence relation θ on $\mathbb{C}\cup\mathbb{D}$ with parts $\mathbb{C}$ and $\mathbb{D}$ is a congruence on $\mathbb{C}\cup\mathbb{D}$, and $(\mathbb{C}\cup\mathbb{D})/\theta$ is isomorphic to the two element majority algebra.

Proof. Note that since $\mathbb{A}$ is minimal Taylor, the clone of $\mathbb{A}$ is generated by any pair of consecutive daisy chain terms, so we just need to check that $\mathbb{C}\cup\mathbb{D}$ is closed under each daisy chain term w_i. For any $a,b,c \in \mathbb{C}\cup\mathbb{D}$, we either have at least two of a,b,c in $\mathbb{C}$ or at least two of a,b,c in $\mathbb{D}$, so the fact that w_i witnesses both $\mathbb{C} \lhd \mathbb{A}$ and $\mathbb{D} \lhd \mathbb{A}$ implies that $w_i(a,b,c) \in \mathbb{C}\cup\mathbb{D}$.

If $\mathbb{C}\cap\mathbb{D} = \emptyset$, then the fact that w_i witnesses both $\mathbb{C} \lhd \mathbb{A}$ and $\mathbb{D} \lhd \mathbb{A}$ implies that w_i is compatible with θ, and that the restriction of w_i to the two-element algebra $(\mathbb{C}\cup\mathbb{D})/\theta$ is the majority operation. □

The following conjecture, if true, would wrap everything up quite neatly.

Conjecture 4.2.1. If $\mathbb{A}$ is a minimal Taylor algebra which is generated by two elements $a, b \in \mathbb{A}$, then at least one of the following is true:

- there is a congruence $\theta \in \mathrm{Con}(\mathbb{A})$ such that $\mathbb{A}/\theta$ is an affine algebra of prime order,
- there is a congruence $\theta \in \mathrm{Con}(\mathbb{A})$ such that $\mathbb{A}/\theta$ is a two element semilattice,
- there is a congruence $\theta \in \mathrm{Con}(\mathbb{A})$ such that $\mathbb{A}/\theta$ is a two element majority algebra, or
- there are proper ternary absorbing subalgebras $\mathbb{C},\mathbb{D} \lhd_Z \mathbb{A}$ such that $a \in \mathbb{C}, b \in \mathbb{D}$, $\mathbb{C}\cup\mathbb{D} = \mathbb{A}$, and $\mathbb{C}\cap\mathbb{D} \neq \emptyset$.

Embarassingly, we don't know the answer to the following basic question.

Problem 4.2.2. Is there any minimal Taylor algebra which is simple, is generated by two elements, has size at least 3, and is not affine?

4.3 Bulatov's colored graph

In Bulatov's approach to the CSP dichotomy conjecture [48], the theory of absorbing subalgebras isn't used. Instead, Bulatov introduces a colored graph in [39] and [47], and uses connectivity properties of this graph to analyze finite Taylor algebras.

Definition 4.3.1. Suppose $\mathbb{A}$ is a finite idempotent algebra, and a, b are any pair of distinct elements of $\mathbb{A}$.

- We say that (a, b) is a *semilattice edge* if there is a binary term t such that $t(a, b) = t(b, a) = b$.
- We say that $\{a, b\}$ is a *weak majority edge* if there is a congruence θ on $\mathrm{Sg}\{a, b\}$ and a ternary term m such that $\{a/\theta, b/\theta\}$ is closed under m and $(\{a/\theta, b/\theta\}, m)$ is a two-element majority algebra.
- We say that $\{a, b\}$ is a *weak affine edge* if there is a congruence θ on $\mathrm{Sg}\{a, b\}$ and a term p such that $(\mathrm{Sg}\{a, b\}/\theta, p)$ is an affine algebra.

We drop the modifier "weak" on an edge if θ is a maximal congruence on $\mathrm{Sg}\{a, b\}$, and for any $a', b' \in \mathrm{Sg}\{a, b\}$ such that $a' \equiv_\theta a$ and $b' \equiv_\theta b$, we have $\mathrm{Sg}\{a, b\} = \mathrm{Sg}\{a', b'\}$. Note that semilattice edges are directed, while majority and affine edges are undirected.

Note that a semilattice edge might not be a subalgebra, and similarly the set $a/\theta \cup b/\theta$ might not be a subalgebra if $\{a, b\}$ is a majority edge. If $\mathbb{A}$ is a *minimal* Taylor algebra, however, then Theorem 4.2.4 shows that $a/\theta \cup b/\theta$ is a subalgebra if (a, b) is a weak majority edge, and similary for semilattice edges. In [48], Bulatov calls an algebra *sm-smooth* if this special case of Theorem 4.2.4 applies to it.

We could have also defined "weak semilattice edges" in a similar way to the way we defined weak majority edges, but this is unnecessary.

Proposition 4.3.2. *If there is a congruence θ on $\mathrm{Sg}\{a, b\}$ and an idempotent binary term t such that $t(a, b) \equiv_\theta t(b, a) \equiv_\theta b$, then there is some $b' \in \mathrm{Sg}\{a, b\}$ such that $b' \equiv_\theta b$ and a partial semilattice term $s \in \mathrm{Clo}(t)$ such that $s(a, b') = s(b', a) = b'$.*

Bulatov defines his colored graph by coloring the semilattice edges red, coloring the majority edges yellow, and coloring the affine edges blue (I don't know why these particular colors were chosen).

Definition 4.3.3. We say that a finite idempotent algebra $\mathbb{A}$ has a *hereditarily connected colored graph* if for all $\mathbb{B} \le \mathbb{A}$, the colored graph of $\mathbb{B}$ is connected (ignoring the directions on the semilattice edges).

For the purposes of checking if an algebra is hereditarily connected, weak edges are interchangeable with edges by the following result.

Proposition 4.3.4. *Let $\mathbb{A}$ be a finite idempotent algebra. If the colored graph of weak edges of $\mathbb{A}$ is hereditarily connected, then the colored graph of edges of $\mathbb{A}$ is also hereditarily connected.*

Proof. Suppose that $\{a,b\}$ is a weak edge of $\mathbb{A}$, with corresponding congruence θ. We will prove by induction on $|\operatorname{Sg}\{a,b\}|$ that a and b are connected in the colored graph of edges of $\operatorname{Sg}\{a,b\}$. We may enlarge θ to a maximal congruence on $\operatorname{Sg}\{a,b\}$ without loss of generality, since any congruence of $\operatorname{Sg}\{a,b\}$ which identifies a and b is the full congruence. If (a,b) is not an edge, then we may pick $a' \in a/\theta$ and $b' \in b/\theta$ such that $\operatorname{Sg}\{a',b'\}$ is strictly smaller than $\operatorname{Sg}\{a,b\}$, and by the inductive hypothesis we see that a', b' are connected in the colored graph of edges of $\operatorname{Sg}\{a,b\}$. Since $\operatorname{Sg}\{a,a'\} \subseteq a/\theta$ and $\operatorname{Sg}\{b,b'\} \subseteq b/\theta$, we also see by the inductive hypothesis that a is connected to a' and b is connected to b' in the colored graph of edges of $\operatorname{Sg}\{a,b\}$. □

Algebras with hereditarily connected colored graphs are closed under the usual algebraic operations.

Proposition 4.3.5. *If $\mathbb{A}, \mathbb{B}$ are finite idempotent algebras (of the same signature) with hereditarily connected colored graphs, then so is $\mathbb{A} \times \mathbb{B}$. More generally, if $\mathbb{A}$ is a finite idempotent algebra and $\theta \in \operatorname{Con}(\mathbb{A})$ is such that $\mathbb{A}/\theta$ is hereditarily connected and every congruence class of θ is also hereditarily connected, then $\mathbb{A}$ is hereditarily connected.*

Proof. We prove the more general statement. Let $a, b \in \mathbb{A}$ be any pair of elements. We will show that a and b are connected by weak edges in $\operatorname{Sg}\{a,b\}$. If $a/\theta = b/\theta$, then $\operatorname{Sg}\{a,b\}$ is contained in a congruence class of θ, so a, b are connected by edges of $\operatorname{Sg}\{a,b\}$. Otherwise, since $a/\theta, b/\theta$ are connected by edges in $\operatorname{Sg}\{a,b\}/\theta$, we can find a sequence of elements $a = a_0, a_1, ..., a_n = b$ such that $(a_i/\theta, a_{i+1}/\theta)$ is an edge of $\mathbb{A}/\theta$ for all i. Then each (a_i, a_{i+1}) will be a weak edge of $\operatorname{Sg}\{a,b\}$, with the corresponding congruence containing θ. □

Proposition 4.3.6. *If $\mathbb{A}$ is a finite idempotent algebra with a hereditarily connected colored graph and $\theta \in \operatorname{Con}(\mathbb{A})$, then $\mathbb{A}/\theta$ also has a hereditarily connected colored graph.*

Proof. We just need to show that if (a,b) is an edge of $\mathbb{A}$ with $a/\theta \ne b/\theta$, then a/θ is connected to b/θ within the subalgebra they generate. We will induct on the size of $|\operatorname{Sg}\{a,b\}|$. If (a,b) is a semilattice edge, then $(a/\theta, b/\theta)$ will automatically be a semilattice edge as well. Otherwise, let η be the maximal congruence on $\operatorname{Sg}\{a,b\}$ corresponding to the edge (a,b).

Since every congruence class of η is a proper subalgebra of $\operatorname{Sg}\{a,b\}$, we see by induction that if $c \equiv_\eta d$, then c/θ and d/θ are connected in the subalgebra they generate, which is contained in $\operatorname{Sg}\{a/\theta, b/\theta\}$. Thus if the restriction of θ to $\operatorname{Sg}\{a,b\}$ is not contained in the maximal congruence η, then a/θ must be connected to b/θ in the subalgebra $\operatorname{Sg}\{a/\theta, b/\theta\}$. Otherwise, if the restriction of θ to $\operatorname{Sg}\{a,b\}$ *is* contained in η, then $(a/\theta, b/\theta)$ is an edge, with witnessing congruence η/θ. □

Corollary 4.3.7. *If $\mathbb{A}$ is a finite idempotent algebra with a hereditarily connected colored graph, then $\mathbb{A}$ is Taylor.*

Bulatov's main result in [39] and [47] is that the converse to the above corollary holds. Since Bulatov didn't have the theory of absorbing subalgebras available to him, he proved this by using tame congruence theory. We will give a different proof, using a pair of consecutive daisy chain terms (whose existence followed from the existence of a cyclic term), and the fact that abelian Taylor algebras are affine.

Theorem 4.3.8 (Bulatov [39], [47]). *A finite idempotent algebra $\mathbb{A}$ is Taylor if and only if it has a hereditarily connected colored graph.*

We will prove Theorem 4.3.8 by induction on $|\mathbb{A}|$. A minimal counterexample $\mathbb{A}$ must be simple by Proposition 4.3.5, and every proper subalgebra of a minimal counterexample must have a hereditarily connected colored graph.

Definition 4.3.9. If $\mathbb{A}$ is any algebra, we define the *hypergraph of proper subalgebras* of $\mathbb{A}$ to be the hypergraph with vertex set equal to the underlying set of $\mathbb{A}$, and with a hyperedge $\mathbb{B}$ for every proper subalgebra $\mathbb{B} \le \mathbb{A}$. We say that $\mathbb{A}$ is *disconnected* if the hypergraph of proper subalgebras of $\mathbb{A}$ is not connected.

We define the connected component equivalence relation $\sim_{\mathbb{A}}$ (or just $\sim$ if $\mathbb{A}$ is clear from context) on $\mathbb{A}$ by $a \sim_{\mathbb{A}} b$ if a is connected to b by a sequence of proper subalgebras of $\mathbb{A}$ (note that in general, $\sim_{\mathbb{A}}$ will *not* be a congruence).

Proposition 4.3.10. *If $\mathbb{A}$ is a disconnected algebra, then for any $a \not\sim_{\mathbb{A}} b$ we have $\mathrm{Sg}\{a, b\} = \mathbb{A}$.*

Proposition 4.3.11. *Suppose that $\mathbb{A}$ is finite, idempotent, simple, and disconnected. For any binary relation $\mathbb{R} \le \mathbb{A} \times \mathbb{A}$ with $\pi_2(\mathbb{R}) = \mathbb{A}$, either $\mathbb{R}$ is the graph of an automorphism of $\mathbb{A}$ or there is some $a \in \mathbb{A}$ such that $\{a\} \times \mathbb{A} \subseteq \mathbb{R}$.*

Proof. If $\mathbb{R}$ is not the graph of an automorphism, then the linking congruence must be nontrivial, hence full (since $\mathbb{A}$ is simple). Thus there is some $a \in \mathbb{A}$ such that $(a, b), (a, c) \in \mathbb{A}$ for some pair of elements b, c with $b \not\sim c$, and from $\mathrm{Sg}_{\mathbb{A}}\{b, c\} = \mathbb{A}$ and idempotence, we see that $\{a\} \times \mathbb{A} \subseteq \mathbb{R}$. □

Proposition 4.3.12. *Suppose that $\mathbb{A}$ is finite, idempotent, simple, and disconnected, and that $a \not\sim_{\mathbb{A}} b$ are such that neither (a, b) nor (b, a) are semilattice edges. Then the binary relation*

$$\mathbb{S}_{ab} \coloneqq \mathrm{Sg}_{\mathbb{A}^2}\left\{\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix}\right\}$$

is the graph of an automorphism of order two which interchanges a and b.

Proof. Assume not. Then by Proposition 4.3.11, there is some $c \in \mathbb{A}$ with $\{c\} \times \mathbb{A} \subseteq \mathbb{S}_{ab}$. Since a, b are in different connected components of $\mathbb{A}$, at least one of them is in a different component than c, so we may suppose that a and c are in different connected components of $\mathbb{A}$ without loss of generality. Then since $(b, a), (b, c) \in \mathbb{S}_{ab}$ and $b \in \mathrm{Sg}_{\mathbb{A}}\{a, c\} = \mathbb{A}$, we have $(b, b) \in \mathbb{S}_{ab}$, so (a, b) is a semilattice edge. □

Definition 4.3.13. Define the equivalence relation $\sim_{\mathbb{A}}^s$ by $a \sim_{\mathbb{A}}^s b$ if a can be connected to b by a chain of proper subalgebras and semilattice edges.

Corollary 4.3.14. *If $\mathbb{A}$ is finite, idempotent, and simple, and if $\sim_{\mathbb{A}}^s$ is not the full equivalence relation $\mathbb{A} \times \mathbb{A}$, then $\mathrm{Aut}(\mathbb{A})$ acts transitively on $\mathbb{A}$.*

Proof. For any pair $a, b \in \mathbb{A}$, either $a \not\sim_{\mathbb{A}}^s b$, or there is some $c \in \mathbb{A}$ such that $a \not\sim_{\mathbb{A}}^s c$ and $c \not\sim_{\mathbb{A}}^s b$. □

Proposition 4.3.15. *Suppose that $\mathbb{A}$ is finite, idempotent, and simple. For any $a \not\sim_{\mathbb{A}}^s b$, if the ternary relation*

$$\mathbb{R}_{ab} \coloneqq \mathrm{Sg}_{\mathbb{A}^3}\left\{\begin{bmatrix} b \\ a \\ a \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} a \\ a \\ b \end{bmatrix}\right\}$$

contains (a, a, a) or (b, a, b), then $\{a, b\}$ is a majority edge.

Proof. Suppose that there is some ternary term t witnessing the presence of one of those tuples in $\mathbb{R}_{ab}$. By Proposition 4.3.12, we see that $\{a, b\}$ is closed under t, and the restriction of t to $\{a, b\}$ is either a majority operation or a Pixley operation. Either way, the ternary term $t(x, t(x, y, z), z)$ acts like a majority operation on $\{a, b\}$. □

Proof of Theorem 4.3.8. We only need to show that if $\mathbb{A}$ is a finite idempotent Taylor algebra, then $\mathbb{A}$ has a connected colored graph. Suppose that $\mathbb{A}$ is a counterexample of minimal size, and note that $\mathbb{A}$ must be simple by Proposition 4.3.5.

Our aim is to show that for any $a \not\sim^s_{\mathbb{A}} b$ such that $\{a, b\}$ is not a majority edge, the ternary relation

$$\mathbb{R}_{ab} \coloneqq \mathrm{Sg}_{\mathbb{A}^3}\left\{\begin{bmatrix} b\\ a\\ a\end{bmatrix}, \begin{bmatrix} a\\ b\\ a\end{bmatrix}, \begin{bmatrix} a\\ a\\ b\end{bmatrix}\right\}$$

must satisfy the conditions of Proposition 3.12.2, which will imply that $\mathbb{A}$ is affine. Note that $a \not\sim_{\mathbb{A}} b$ implies that $\mathbb{R}_{ab}$ is subdirect.

The first step to proving that $\mathbb{R}_{ab}$ satisfies the conditions of Proposition 3.12.2 is showing that $\pi_{12}(\mathbb{R}_{ab}) = \mathbb{A} \times \mathbb{A}$. Since $\pi_{12}(\mathbb{R}_{ab})$ contains $(a, a), (a, b)$, and (b, a), we need to check that $(b, b) \in \pi_{12}(\mathbb{R}_{ab})$. So it is natural to study the set of tuples in $\mathbb{R}_{ab}$ such that two of the coordinates are equal.

The next claim is the main place where we will use the fact that $\mathbb{A}$ is Taylor.

Claim 1: If we define $\mathbb{D}_{ab} \le \mathbb{A} \times \mathbb{A}$ to be the set of pairs (c, d) such that $(c, d, c) \in \mathbb{R}_{ab}$, then $\pi_1(\mathbb{D}_{ab}) \cap \pi_2(\mathbb{D}_{ab}) \neq \emptyset$.

Proof of Claim 1: Let p, q be consecutive daisy chain terms, i.e. p and q are ternary terms satisfying the identities

$$\begin{aligned} p(x, x, y) &\approx p(y, x, x),\\ q(x, x, y) &\approx q(y, x, x) \approx p(x, y, x). \end{aligned}$$

If we set $c = p(a, a, b), d = p(a, b, a) = q(a, a, b)$, and $e = q(a, b, a)$, then we have

$$p\left(\begin{bmatrix} b & a & a\\ a & b & a\\ a & a & b\end{bmatrix}\right) = \begin{bmatrix} c\\ d\\ c\end{bmatrix}$$

and

$$q\left(\begin{bmatrix} b & a & a\\ a & b & a\\ a & a & b\end{bmatrix}\right) = \begin{bmatrix} d\\ e\\ d\end{bmatrix},$$

so $(c, d), (d, e) \in \mathbb{D}_{ab}$, and $d \in \pi_1(\mathbb{D}_{ab}) \cap \pi_2(\mathbb{D}_{ab})$.

Claim 2: The binary relation $\mathbb{D}_{ab}$ from Claim 1 has $\pi_1(\mathbb{D}_{ab}) = \mathbb{A}$.

Proof of Claim 2: Suppose not. First consider the case where neither $\pi_1(\mathbb{D}_{ab}), \pi_2(\mathbb{D}_{ab})$ are equal to $\mathbb{A}$. Then if $c \in \pi_1(\mathbb{D}_{ab}) \cap \pi_2(\mathbb{D}_{ab})$, we see that both $\{a, c\}$ and $\{b, c\}$ are contained in proper subalgebras of $\mathbb{A}$, so $a \sim c \sim b$, contradicting the assumption $a \not\sim b$.

Suppose now that $\pi_1(\mathbb{D}_{ab}) \neq \mathbb{A}$ but $\pi_2(\mathbb{D}_{ab}) = \mathbb{A}$. Then by Proposition 4.3.11, there is some $c \in \mathbb{A}$ such that $\{c\} \times \mathbb{A} \subseteq \mathbb{D}_{ab}$. By Corollary 4.3.14, there is an automorphism $\sigma \in \mathrm{Aut}(\mathbb{A})$ with $\sigma(a) = c$, and from $(\sigma(a), \sigma(b)) \in \{c\} \times \mathbb{A} \subseteq \mathbb{D}_{ab}$, we see that $(\sigma(a), \sigma(b), \sigma(a)) \in \mathbb{R}_{ab}$, so in fact

$\sigma(\mathbb{R}_{ab}) \subseteq \mathbb{R}_{ab}$, and so $\sigma(\mathbb{D}_{ab}) = \mathbb{D}_{ab}$. Thus $(a,a) = \sigma^{-1}(c,c) \in \mathbb{D}_{ab}$, so by Proposition 4.3.15 this contradicts the assumption that $\{a,b\}$ is not a majority edge.

Claim 3: We have $\pi_{1,2}(\mathbb{R}_{ab}) = \mathbb{A} \times \mathbb{A}$.

Proof of Claim 3: By Claim 2, there is some c such that $(b,c) \in \mathbb{D}_{ab}$. Thus $(a,a),(a,b),(b,a),(b,b) \in \pi_{1,2}(\mathbb{R}_{ab})$, and these four elements generate $\mathbb{A}^2$.

Claim 4: For any c, the binary relation $\mathbb{R}_{ab}^c \le \mathbb{A}^2$ defined as the set of pairs (d,e) such that $(c,d,e) \in \mathbb{R}_{ab}$ is the graph of an automorphism of order two.

Proof of Claim 4: Suppose not. Note that by Claim 3, the relation $\mathbb{R}_{ab}^c$ is subdirect. Thus if $\mathbb{R}_{ab}^c$ is not the graph of an automorphism, then by Proposition 4.3.11 there is some d such that $\{d\} \times \mathbb{A} \subseteq \mathbb{R}_{ab}^c$.

First suppose that $c = a$, so $(a,b) \in \mathbb{R}_{ab}^a$. Then either $a \not\sim d$ or $b \not\sim d$. If $a \not\sim d$, then from $(a,b),(d,b) \in \mathbb{R}_{ab}^a$ we see that $(b,b) \in \mathbb{R}_{ab}^a$, contradicting Proposition 4.3.15. Similarly if $b \not\sim d$, then from $(a,b),(a,d) \in \mathbb{R}_{ab}^a$ we see that $(a,a) \in \mathbb{R}_{ab}^a$, contradicting Proposition 4.3.15.

Now suppose that $c \ne a$. There is some $e \ne a$ such that $d \not\sim e$ (if not, then $\sim$ has just two equivalence classes, which are interchanged by the automorphism $\mathbb{S}_{ab}$, and which both have size 1, reducing us to the case $|\mathbb{A}| = 2$). Then $\mathbb{S}_{de}$ is the graph of an automorphism of order two which interchanges d and e, and from $(d,e),(e,d) \in \mathbb{R}_{ab}^c$ we see $\mathbb{S}_{de} \subseteq \mathbb{R}_{ab}^c$. Then since $e \ne a$ there is some $f \ne d$ such that $(a,f) \in \mathbb{S}_{de} \subseteq \mathbb{R}_{ab}^c$. Then from $(a,d),(a,f) \in \mathbb{R}_{ab}^c$ we see that $(c,d),(c,f) \in \mathbb{R}_{ab}^a$, contradicting the fact that $\mathbb{R}_{ab}^a$ is the graph of an automorphism of order two.

To finish the proof, note that Claim 4 shows that the relation $\mathbb{R}_{ab}$ satisfies the assumptions of Proposition 3.12.2, so $\mathbb{A}$ must be abelian. Thus we can apply the Theorem 3.12.8 to see that $\mathbb{A}$ must be affine. □

4.4 Conservative Taylor algebras

Bulatov's colored graph was originally inspired by the study of conservative Taylor algebras. These algebras are easy to classify, and they are a great toy case for testing conjectures about general Taylor algebras.

Definition 4.4.1. A k-ary operation $f : A^k \to A$ is *conservative* if for all $a_1, ..., a_k \in A$ we have

$$f(a_1, ..., a_k) \in \{a_1, ..., a_k\}.$$

An algebraic structure $\mathbb{A}$ is called *conservative* if every basic operation of $\mathbb{A}$ is conservative.

Note that conservative algebras are automatically idempotent.

Proposition 4.4.2. *An algebraic structure $\mathbb{A}$ is conservative if and only if every subset $S \subseteq \mathbb{A}$ is actually a subalgebra of $\mathbb{A}$.*

On the relational side, we define conservative relational structures as follows.

Definition 4.4.3. A relational clone Γ on a domain A is called *conservative* if every unary relation $U \subseteq A$ is an element of Γ, i.e. $\mathcal{P}(A) \subseteq \Gamma$. A relational structure $\mathbf{A}$ is called *conservative* if every unary relation U can be primitively positively defined using the basic relations of $\mathbf{A}$.

Proposition 4.4.4. *If a relational structure $\mathbf{A}$ and an algebraic structure $\mathbb{A}$ are related by the* $\mathrm{Inv}-\mathrm{Pol}$ *Galois correspondence, then $\mathbf{A}$ is conservative if and only if $\mathbb{A}$ is conservative.*

If we are handed a relational structure, then the next result can be useful to decrease the amount of work needed to verify that it is conservative.

Proposition 4.4.5. *A relational structure* $\mathbf{A}$ *with finite underlying set* A *is conservative if and only if, for every* $a \in A$*, the unary relation* $A \setminus \{a\}$ *is primitively positively definable from the basic relations of* $\mathbf{A}$*.*

Example 4.4.1. A natural example of a conservative CSP template (on an infinite domain) is the *list-coloring* problem for graphs: the domain A is an infinite set, and the relations consist of the binary $\neq$ relation and the collection of all possible subsets $U \subseteq A$ as unary relations.

Example 4.4.2. A conservative 2-semilattice is called a *tournament*. The rock-paper-scissors algebra is probably the most famous example of a tournament which is not totally ordered.

Example 4.4.3. If an affine CSP is conservative, then the domain must have size two: the only conservative affine algebra is $\mathbb{Z}/2^{\text{aff}}$, up to term equivalence.

Sometimes we will want to use the following refinement of the concept of conservative algebras.

Definition 4.4.6. We say that a relational clone Γ is *k-conservative* if every unary relation $U \subseteq A$ with size $|U| \le k$ is an element of Γ, and we define k-conservative clones, algebras, and relational structures similarly.

Example 4.4.4. An algebra is idempotent iff it is 1-conservative.

Example 4.4.5. The *k-list-coloring* problem for graphs corresponds to the relational structure with infinite domain A, and relations consisting of the binary $\neq$ relation and the collection of all possible subsets $U \subseteq A$ with $|U| \le k$. This problem is equivalent to 2SAT for $k = 2$, and is NP-hard for $k \ge 3$.

Example 4.4.6. The only 2-conservative affine algebras are $(\mathbb{Z}/2^{\text{aff}})^k$, up to term equivalence and isomorphism.

For the sake of understanding CSPs, we would like to focus on minimal Taylor algebras. The next result shows that we can reduce the study of conservative Taylor algebras to conservative minimal Taylor algebras without losing anything essential.

Proposition 4.4.7. *Every reduct of a conservative algebra is also conservative. In particular, every conservative Taylor clone contains a minimal Taylor clone which is also conservative.*

Since every minimal Taylor clone can be generated by a pair of ternary terms (for instance, we can take a pair of consecutive daisy chain terms), we only have to focus on understanding conservative Taylor algebras of size 3.

Proposition 4.4.8. *A minimal Taylor algebra is conservative if and only if it is* 3*-conservative.*

In fact, looking carefully at how a daisy chain term must act on a conservative Taylor algebra, we have the following simplification.

Proposition 4.4.9. *If* $\mathbb{A}$ *is a* 2*-conservative minimal Taylor algebra and* w_i *is any daisy chain term for* $\mathbb{A}$*, then we have*

$$w_i(x, x, y) \approx w_i(x, y, x) \approx w_i(y, x, x),$$

so in fact every w_i is a ternary weak near-unanimity operation. The binary function

$$f(x, y) \coloneqq w_i(x, x, y)$$

is independent of i, and completely determines the colored graph of $\mathbb{A}$. In addition, the binary function

$$s(x, y) \coloneqq f(x, f(y, x))$$

is a partial semilattice term of $\mathbb{A}$.

Proof. Note that every pair of distinct elements $a, b \in \mathbb{A}$ must form an edge of the colored graph of $\mathbb{A}$ if $\mathbb{A}$ is a 2-conservative Taylor algebra. By our analysis of daisy chain terms on the basic two-element minimal Taylor algebras, we see that:

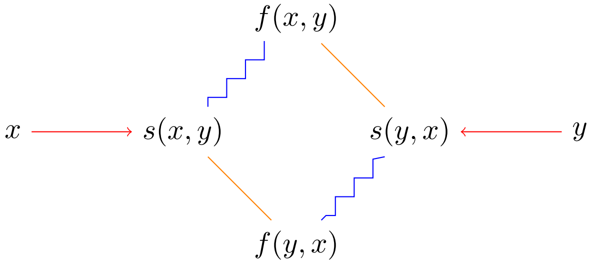
- if (a, b) is a semilattice edge, then $w_i(x, x, y) = w_i(x, y, x) = w_i(y, x, x) = x \vee y$ for $x, y \in \{a, b\}$,
- if $\{a, b\}$ is a majority edge, then $w_i(x, x, y) = w_i(x, y, x) = w_i(y, x, x) = x$ for $x, y \in \{a, b\}$,
- if $\{a, b\}$ is a $\mathbb{Z}/2^{\text{aff}}$ edge, then $w_i(x, x, y) = w_i(x, y, x) = w_i(y, x, x) = y$ for $x, y \in \{a, b\}$.

Thus we can tell what sort of edge $\{a, b\}$ is (as well as how it is directed, in case it is a semilattice edge) by examining the restriction of $f(x, y)$ to the set $\{a, b\}$. The claim about $s(x, y)$ follows easily by considering each of the three possible types of edge individually. □

Thus, from now on we imagine that all conservative minimal Taylor algebras live in a variety $\mathcal{V}$ having just one ternary basic operation w, which satisfies the weak near-unanimity identity

$$w(x, x, y) \approx w(x, y, x) \approx w(y, x, x).$$

Proposition 4.4.10. *The free algebra $\mathcal{F}_{\mathcal{V}}(x, y)$ on two generators in the variety $\mathcal{V}$ generated by conservative minimal Taylor algebras has size 6: its elements are $x, y, f(x, y), f(y, x), s(x, y), s(y, x)$ (defined as in the previous proposition). The colored graph of the algebra $\mathcal{F}_{\mathcal{V}}(x, y)$ is as follows.*



Here the semilattice edges are directed, the majority edges are straight and undirected, and the $\mathbb{Z}/2^{\text{aff}}$ edges are drawn as zigzags. This algebra has $\{f(x, y), f(y, x), s(x, y), s(y, x)\}$ as a binary absorbing subalgebra, corresponding to a semilattice quotient, and has $\{x, f(x, y), s(x, y)\}$ and $\{y, f(y, x), s(y, x)\}$ as ternary absorbing subalgebras, corresponding to a majority quotient.

In order to understand conservative minimal Taylor algebras, Proposition 4.4.8 implies that it's most important to understand the conservative algebras of size 3. Additionally, Proposition 4.4.9 implies that we just need to figure out which ternary weak near-unanimity operations on a three element set generate *minimal* Taylor clones. We get a further simplification by dividing into cases based on whether or not there is a ternary cyclic term. In the case where there is no cyclic term, the following result is useful.

Theorem 4.4.11. *If w is a ternary weak near-unanimity term of a finite algebra $\mathbb{A}$, then there is a ternary weak near-unanimity term $g \in \mathrm{Clo}(w)$ which also satisfies the identity*

$$g(g(x,y,z), g(y,z,x), g(z,x,y)) \approx g(x,y,z).$$

If $|\mathbb{A}| = 3$ and $\mathbb{A}$ has no ternary cyclic term, then any such g satisfies $g(x,y,z) = x$ whenever x, y, z are all different.

Proof. Let $\gamma : \mathbb{A}^3 \to \mathbb{A}^3$ be the map given by

$$\gamma\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \coloneqq \begin{bmatrix} w(x,y,z) \\ w(y,z,x) \\ w(z,x,y) \end{bmatrix}.$$

Then since $\mathbb{A}^3$ is finite, there is some k such that $\gamma^{\circ 2k} = \gamma^{\circ k}$. If we define g by

$$\gamma^{\circ k}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} g(x,y,z) \\ g(y,z,x) \\ g(z,x,y) \end{bmatrix},$$

then $\gamma^{\circ k} \circ \gamma^{\circ k} = \gamma^{\circ k}$ implies that g satisfies the identity

$$g(g(x,y,z), g(y,z,x), g(z,x,y)) \approx g(x,y,z).$$

Note that since w is a weak near-unanimity term, if any two of x, y, z are equal, then $\gamma(x,y,z)$ is a constant tuple, and then by idempotence so is $\gamma^{\circ k}(x,y,z)$. Therefore g is also a weak near-unanimity operation, and $\gamma^{\circ k}(x,y,z)$ can only avoid being a constant tuple if x, y, z are all different.

If $\mathbb{A}$ has no ternary cyclic term and has underlying set $\{a,b,c\}$, then $\gamma^{\circ k}(a,b,c)$ can't be a constant tuple by Proposition 4.1.5, and since $\gamma^{\circ 2k} = \gamma^{\circ k}$, $\gamma^{\circ k}(a,b,c)$ must not have any pair of coordinates equal, so $\gamma^{\circ k}(a,b,c)$ must be a permutation of (a,b,c). By Theorem 4.1.8, if $\mathbb{A}$ has no ternary cyclic term then it must have an automorphism of order three, so $\gamma(a,b,c)$ must be one of $(a,b,c), (b,c,a)$, or (c,a,b), and in each case we have $\gamma^{\circ k}(a,b,c) = (a,b,c)$. Similarly, we must also have $\gamma^{\circ k}(a,c,b) = (a,c,b)$, so we have $g(x,y,z) = x$ whenever x, y, z are all different. □

Theorem 4.4.12. *If a minimal Taylor algebra has size 3 and has no ternary cyclic term, then (after renaming elements) it is term equivalent to one of the following four algebras:*

- *the affine algebra $\mathbb{Z}/3^{\mathrm{aff}}$,*
- *the rock-paper-scissors algebra $\mathbb{A}_{rps}$ from Section 3.1,*
- *the three element dual discriminator algebra from Example 1.6.5, or*
- *the three element simple nonabelian Mal'cev algebra from Example 1.7.2.*

All but the first are conservative, all but the second have a full automorphism group, and the first two have binary cyclic terms.

Proof. By Theorem 4.1.8, if a minimal Taylor algebra $\mathbb{A}$ with underlying set $\{a, b, c\}$ has size 3 and has no ternary cyclic term, then $\mathbb{A}$ must have an automorphism of order three with no fixed points, so the permutation $(a\ b\ c)$ is in $\mathrm{Aut}(\mathbb{A})$. By Theorem 4.3.8, either $\mathbb{A}$ is affine - in which case it must be term-equivalent to $\mathbb{Z}/3^{\mathrm{aff}}$ - or $\mathbb{A}$ has some proper subalgebra of size 2 (since $\mathbb{A}$ has an edge (a, b), and either $\mathrm{Sg}\{a, b\}$ has size 2, or $\mathrm{Sg}\{a, b\} = \mathbb{A}$ has a proper quotient, and one of the congruence classes is a subalgebra of size 2). Since $(a\ b\ c) \in \mathrm{Aut}(\mathbb{A})$, if any 2-element subset of $\mathbb{A}$ is a subalgebra, then *every* 2-element subset of $\mathbb{A}$ is a subalgebra, and all three 2-element subalgebras of $\mathbb{A}$ are isomorphic to $\{a, b\}$.

If $\{a, b\}$ is a semilattice, then any binary operation s that acts like the semilattice term on $\{a, b\}$ has $(\{a, b, c\}, s)$ isomorphic to the rock-paper-scissors algebra. If $\{a, b\}$ is a majority algebra and g is a ternary weak near-unanimity operation as in the previous theorem, then g is a majority operation which acts as first projection whenever all three of its inputs are distinct, so $(\{a, b, c\}, g)$ is isomorphic to the three-element dual discriminator algebra. If $\{a, b\}$ is an affine algebra and g is a ternary weak near-unanimity operation as in the previous theorem, then g is a Mal'cev operation which acts as a minority operation whenever two of its inputs are equal, and which acts as first projection whenever all three of its inputs are distinct, so $(\{a, b, c\}, g)$ is isomorphic to the three element simple nonabelian Mal'cev algebra from Example 1.7.2. □

In most of the remaining cases, the colored graph already does not have any automorphisms of order 3. In these cases, it turns out to be relatively easy to pick out a specific ternary cyclic operation which is determined by the colored graph alone. In fact, we have the following slightly stronger statement.

Theorem 4.4.13. *Suppose that a minimal Taylor algebra $\mathbb{A}$ has the following properties:*

- *$\mathbb{A}$ is 2-conservative, that is, for all $a, b \in \mathbb{A}$ the subset $\{a, b\}$ is a subalgebra of $\mathbb{A}$,*
- *the colored graph of $\mathbb{A}$ does not contain any majority triangles, and*
- *the colored graph of $\mathbb{A}$ does not contain any affine triangles.*

Then $\mathbb{A}$ is conservative, and $\mathrm{Clo}(\mathbb{A})$ is determined by the colored graph of $\mathbb{A}$.

Proof. Let w be any daisy chain term for $\mathbb{A}$. Define a map $\gamma : \mathbb{A}^3 \to \mathbb{A}^3$ as in Theorem 4.4.11. We will make sure to only apply γ to triples where some pair of coordinates are equal, since the values γ takes on such triples is completely determined by the colored graph of $\mathbb{A}$ by Proposition 4.4.9. Define binary terms f, s as in Proposition 4.4.9, and note that f and s are uniquely determined by the colored graph of $\mathbb{A}$. Define maps $\alpha_f, \beta_f : \mathbb{A}^3 \to \mathbb{A}^3$ by

$$\alpha_f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \coloneqq \begin{bmatrix} f(x, y) \\ f(y, z) \\ f(z, x) \end{bmatrix}$$

and

$$\beta_f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \coloneqq \begin{bmatrix} f(x, z) \\ f(y, x) \\ f(z, y) \end{bmatrix},$$

and define maps $\alpha_s, \beta_s : \mathbb{A}^3 \to \mathbb{A}^3$ similarly, with f replaced by s. Note that α_f, β_f, etc. each have the property that if the input has two coordinates the same, then so does the output. As long as

a, b, c do not form a majority triangle, an affine triangle, or a rock-paper-scissors subalgebra, then the triple

$$\alpha_f \circ \beta_f \circ \alpha_s \circ \beta_s \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$$

has two of its three coordinates equal (to check this, consider the case where $\{a, b, c\}$ contains at least one semilattice edge separately from the case where it contains only majority and affine edges). Thus the ternary term

$$t \coloneqq \pi_1 \circ \gamma \circ \alpha_f \circ \beta_f \circ \alpha_s \circ \beta_s : \mathbb{A}^3 \to \mathbb{A}$$

is cyclic on every such triple. Since we assumed that $\mathbb{A}$ has no majority triangles or affine triangles, the only possible triples of $\mathbb{A}$ such that the value of t is not uniquely determined by the colored graph of $\mathbb{A}$ are the rock-paper-scissors subsets of $\mathbb{A}$, which are necessarily subalgebras of $\mathbb{A}$ by Theorem 4.2.4. If we iterate t as in Theorem 4.4.11, then the resulting ternary function g has its values on rock-paper-scissors subalgebras fixed as well, so all of the values of g are determined purely by the colored graph of $\mathbb{A}$. Furthermore, this g is conservative and generates a Taylor clone, so $\mathrm{Clo}(\mathbb{A}) = \mathrm{Clo}(g)$ and $\mathbb{A}$ is conservative. □

Finally, we need to understand the case of a majority triangle or affine triangle $\{a, b, c\}$ with a cyclic term. In these cases, it is helpful to keep track of the subalgebra

$$\pi_1 \left(\mathrm{Sg}_{\mathbb{A}^2} \left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ c \end{bmatrix}, \begin{bmatrix} c \\ a \end{bmatrix} \right\} \cap \Delta_{\mathbb{A}} \right) \le \mathbb{A},$$

since the set of possible outputs of a cyclic term applied to (a, b, c) must be contained in this subalgebra. This subalgebra is an invariant of $\mathrm{Clo}(\mathbb{A})$, and it shrinks when $\mathrm{Clo}(\mathbb{A})$ shrinks.

Definition 4.4.14. If $\mathbb{A}$ is a three element minimal Taylor algebra with underlying set $\{a, b, c\}$, then we will say that an element x of $\mathbb{A}$ is *circled* if $(x, x) \in \mathrm{Sg}\{(a, b), (b, c), (c, a)\}$. Note that the set of circled elements of $\mathbb{A}$ does not depend on the ordering of a, b, c.

Theorem 4.4.15. *Suppose that $\mathbb{A}$ is a conservative three element minimal Taylor algebra with a ternary cyclic term g, such that either all three of the edges of $\mathbb{A}$ are majority or all three are affine. Then (after renaming elements) $\mathbb{A}$ is term equivalent to one of the following three algebras:*

- *the three element solvable nonabelian Mal'cev algebra from Example 1.7.3, with $*$ as the unique circled element,*
- *the three element median algebra $\{0, 1, 2\}$, with the median element 1 as the unique circled element, or*
- *the three element minimal majority algebra $(\{a, b, c\}, m)$, with m a cyclic majority operation such that $m(a, b, c) = b$ and $m(a, c, b) = c$, with $\{b, c\}$ as the set of circled elements.*

In particular, every conservative three element minimal Taylor algebra is determined up to term equivalence by its colored graph and set of circled elements.

Furthermore, in any conservative minimal Taylor algebra, we can choose a ternary operation g as in Theorem 4.4.11 such that if we take g as the basic operation, then every three element majority subalgebra with two circled elements is isomorphic to the third algebra listed above (not just term-equivalent).

Proof. Let g be a ternary cyclic term for $\mathbb{A}$, and suppose that $\mathbb{A}$ has underlying set $\{a, b, c\}$. Once we know the types of the edges of $\mathbb{A}$, we only need to know the values of $g(a, b, c)$ and $g(a, c, b)$ to completely determine g. For each choice of edges, we have two cases: either $g(a, b, c) = g(a, c, b)$, or $g(a, b, c) \neq g(a, c, b)$. This gives us four cases total.

First consider the case where all three edges of $\mathbb{A}$ are affine (so g is Mal'cev), and $g(a, b, c) \neq g(a, c, b)$. Without loss of generality, we may assume that $g(a, b, c) = b$ and $g(a, c, b) = c$. We will show that this case does not occur, by constructing a term w which generates a strictly smaller Taylor clone. Note that the order two permutation which swaps b and c is an automorphism of $(\{a, b, c\}, g)$. Then if we define the ternary operation t by

$$t(x, y, z) \coloneqq g(x, g(x, y, z), g(x, g(x, y, z), g(x, z, y))),$$

then t is also Mal'cev and satisfies

$$t(a, b, c) = a, t(b, a, c) = c, t(c, b, a) = c,$$

so if we define the ternary operation w by

$$w(x, y, z) \coloneqq g(t(x, y, z), t(y, z, x), t(z, x, y)),$$

then w is a symmetric Mal'cev operation, with $w(a, b, c) = w(a, c, b) = a$. Then w generates a strictly smaller Taylor clone, since w preserves the equivalence relation with equivalence classes $\{a\}$ and $\{b, c\}$, while g does not. Thus this case does not occur.

In the remaining three cases, we get the three algebras described in the theorem statement. We need to check that these three algebras are really *minimal* Taylor. Note that in each case, there is a nontrivial congruence on $\mathbb{A}$ with quotient of size two and congruence classes of size at most two, so every Taylor reduct of $\mathbb{A}$ is forced to have a cyclic ternary term. We will show that the clone of each of these algebras contains only one or two ternary cyclic operations w. Note that the only values of $w(x, y, z)$ which are not determined by the types of the edges are the ones where x, y, z are all distinct.

In the case of the solvable Mal'cev algebra from Example 1.7.3 with underlying set $\{0, 1, *\}$, the congruence with congruence classes $\{*\}, \{0, 1\}$ forces the value of $w(0, 1, *)$ to be $*$, and similarly for other permutations of the inputs. Thus there is only one ternary cyclic operation w in the clone.

In the case of the three element median algebra $\{0, 1, 2\}$, the congruences corresponding to the partitions $\{0, 1\}, \{2\}$ and $\{0\}, \{1, 2\}$ force the value of $w(0, 1, 2)$ to be in $\{0, 1\} \cap \{1, 2\} = \{1\}$, and similarly for other permutations of the inputs. Thus there is only one ternary cyclic operation w in the clone.

In the last case, the congruence corresponding to the partition $\{a\}, \{b, c\}$ forces the value of $w(a, b, c)$ to be either b or c. Additionally, the order two automorphism which interchanges b and c forces us to have

$$w(a, b, c) = b \iff w(a, c, b) = c.$$

Thus we either have $w(x, y, z) \approx m(x, y, z)$, or $w(x, y, z) \approx m(x, z, y)$, so there are exactly two ternary cyclic operations w in the clone.

For the last statement, suppose that we have a minimal conservative algebra $\mathbb{A}$, with several majority subalgebras with two circled elements. Let g be any ternary operation as in Theorem 4.4.11. By the last case above, the restriction of g to any of these majority subalgebras either acts like $m(x, y, z)$ or like $m(x, z, y)$. Suppose for contradiction that two of these subalgebras are not isomorphic, i.e., that g acts as $m(x,y,z)$ on one and acts as $m(x,z,y)$ on the other. We will produce a ternary weak near-unanimity term w which acts like $m(x,y,z)$ on both, which will generate a proper Taylor subclone. To this end, we define a ternary term t by

$$t(x,y,z) \coloneqq g(x, g(x,y,z), g(x,z,y)),$$

and define w by

$$w(x,y,z) \coloneqq g(t(x,y,z), t(y,z,x), t(z,x,y)).$$

Then w is cyclic on any subalgebra of $\mathbb{A}$ where g is cyclic, so in particular w is a weak near-unanimity operation. Note that if $\{a,b,c\}$ is a majority subalgebra of $\mathbb{A}$ with $\{b,c\}$ as the set of circled elements, then regardless of whether the restriction of g to $\{a,b,c\}$ is $m(x,y,z)$ or $m(x,z,y)$, we always have

$$t(a,b,c) = b, \quad t(b,c,a) = b, \quad t(c,a,b) = c,$$

so $w(a,b,c) = b$, and so the restriction of w to $\{a,b,c\}$ is $m(x,y,z)$. □

Putting the proofs of the above theorems together, we get a procedure which puts the basic ternary weak near-unanimity operation g of any minimal conservative Taylor algebra into a standard form, such that the restriction of g to any three element subalgebra is completely determined by the edge types and the set of circled elements. In particular, we can exactly count the number of conservative minimal Taylor clones of a given size.

Corollary 4.4.16. *The number of conservative minimal Taylor clones on a set of size n is exactly*

$$\sum_{\text{3-edge-colorings of } K_n} 2^{\#(\mathit{semilattice})} 4^{\Delta(\mathit{affine})} 7^{\Delta(\mathit{majority})} = (1+o(1)) \cdot 7^{\binom{n}{3}},$$

where $\Delta(c)$ is the number of monochromatic triangles of color c. In particular, for large n almost all conservative minimal Taylor clones are clones of majority algebras.

If we only want to know the number of conservative minimal Taylor algebras of a given size up to term equivalence and *isomorphism*, then we can use the Burnside counting theorem, together with the fact that the automorphism group of a conservative minimal Taylor algebra is determined by its colored graph and the choices of circled vertices on its three-element majority and Mal'cev subalgebras in the obvious way. The number of conservative minimal Taylor clones on domains of sizes $2,3,4,5$ is listed below (note: these were computed by hand, so there might be some mistakes).

Domain size	# up to term equiv.	# up to term equiv. and iso.
2	4	3
3	73	19
4	9829	520
5	320668024	2686891

4.4.1 Classification of three-element minimal Taylor algebras

As it turns out, up to term equivalence and isomorphism there are just 24 minimal Taylor algebras on a set of size 3. Of these, 19 are conservative, and the remaining 5 are easy to describe. One of the most obvious non-conservative minimal Taylor algebras of size 3 is the affine algebra $\mathbb{Z}/3^{\text{aff}}$.

Three more are subdirect products of two-element minimal Taylor algebras: specifically, the free semilattice on two generators (which is a subdirect product of two two-element semilattices), the subdirect product of a two-element semilattice and a two-element majority algebra, and the subdirect product of a two-element semilattice and $\mathbb{Z}/2^{\text{aff}}$ (all three of these are quotients of the free algebra from Proposition 4.4.10). There is no three-element subdirect product of a two-element majority algebra and $\mathbb{Z}/2^{\text{aff}}$, but the final example is nearly this: it is the algebra from Example 2.2.1, which has a 3-edge term, a $\mathbb{Z}/2^{\text{aff}}$ quotient, and a two-element centrally absorbing algebra. In this subsection we will prove that this is the complete list of minimal Taylor algebras on a three-element set.

By Theorem 4.4.12, we only have to classify the minimal Taylor algebras of size 3 which have a ternary cyclic term g, and since we have already classified the conservative ones, we just need to classify those which are generated by two elements. By Bulatov's Theorem 4.3.8, each of these algebras has a connected colored graph. Our first step will be to show that for minimal Taylor algebras of size 3, every edge of Bulatov's colored graph is a two-element subalgebra.

Proposition 4.4.17. *If $\mathbb{A}$ is a minimal Taylor algebra of size 3 other than $\mathbb{Z}/3^{\text{aff}}$, then the graph on $\mathbb{A}$ with edges given by the two-element subalgebras of $\mathbb{A}$ is connected - in other words, $\mathbb{A}$ has at least two subalgebras of size two.*

Proof. Suppose for contradiction that the graph of two-element subalgebras of $\mathbb{A}$ is disconnected, and suppose the underlying set of $\mathbb{A}$ is $\{a, b, c\}$. By Theorem 4.3.8, Bulatov's colored graph of $\mathbb{A}$ must be connected, so if $\mathbb{A}$ is not $\mathbb{Z}/3^{\text{aff}}$ then there must be some nontrivial congruence $\theta \in \text{Con}(\mathbb{A})$ such that $\mathbb{A}/\theta$ is either $\mathbb{Z}/2^{\text{aff}}$ or the two-element majority algebra (it can't be the two-element semilattice by Proposition 4.3.2). Suppose without loss of generality that $\{b, c\}$ is the congruence class of θ which has size 2, so that $\{b, c\}$ is a two-element subalgebra of $\mathbb{A}$. By our assumption that the graph of two-element subalgebras is disconnected, neither $\{a, b\}$ nor $\{a, c\}$ can be a subalgebra of $\mathbb{A}$. Let g be a ternary cyclic operation on $\mathbb{A}$, which exists by Theorem 4.4.12 (or directly from Theorem 4.1.8).

Suppose first that $\mathbb{A}/\theta$ is $\mathbb{Z}/2^{\text{aff}}$. Then since any ternary cyclic term of $\mathbb{Z}/2^{\text{aff}}$ acts as the minority operation, we must have

$$g(a, b, b) = g(a, c, c) = a.$$

Since $\{a, b\}$ and $\{a, c\}$ are not closed under g, we must then have

$$g(a, a, b) = c, \quad g(a, a, c) = b.$$

Define a ternary term t by

$$t(x, y, z) = g(g(x, x, y), g(y, y, z), g(z, z, x)).$$

Then t is also cyclic, and we have

$$t(a, a, b) = g(a, c, a) = b,$$

so $\{a, b\}$ is closed under t, and similarly $\{a, c\}$ is also closed under t. Thus t generates a strictly smaller Taylor clone, contradicting the assumption that $\mathbb{A}$ is minimal Taylor.

Now suppose that $\mathbb{A}/\theta$ is the two-element majority algebra. Then since any ternary cyclic term acts on a majority algebra as a majority operation, we must have

$$g(a,a,b) = g(a,a,c) = a,$$

and

$$g(a,b,c), g(a,c,b) \in \{b,c\}.$$

Since $\{a,b\}$ and $\{a,c\}$ are not closed under g, we must have

$$g(a,b,b) = c, \quad g(a,c,c) = b.$$

Assume without loss of generality that $g(a,b,c) = b$, otherwise swap b,c and reorder the last two arguments to g. Define a ternary term t by

$$t(x,y,z) = g(g(x,x,y), g(y,y,z), g(z,z,x)).$$

Then t is also cyclic, and we have

$$t(a,b,b) = g(a,b,c) = b,$$

so $\{a,b\}$ is closed under t. As before, this contradicts the assumption that $\mathbb{A}$ is minimal Taylor. □

From here on, we just need to classify the minimal Taylor algebras on the set $\{a,b,c\}$ such that a and b generate the algebra, while $\{a,c\}$ and $\{b,c\}$ are two-element subalgebras. First we handle the case where one of these subalgebras is a two-element semilattice.

Proposition 4.4.18. *If $\mathbb{A}$ is a minimal Taylor algebra with underlying set $\{a,b,c\}$ such that $\mathbb{A}$ is generated by a and b, then $\mathbb{A}$ does not have a semilattice subalgebra of the form $c \to a$.*

Proof. Suppose for contradiction that $\mathbb{A}$ is generated by a and b, but that $c \to a$. Let g be a ternary cyclic term of $\mathbb{A}$, which exists by Theorem 4.4.12.

First suppose that we do not also have $c \to b$. We will initially attempt to prove that $(a,a) \in \mathrm{Sg}_{\mathbb{A}^2}\{(a,b),(b,a)\}$, so that by Proposition 4.2.5 $\{a,b\}$ will be a two-element semilattice subalgebra with $b \to a$, which will contradict the assumption that a and b generate $\mathbb{A}$. Let $s \in \mathrm{Clo}(g)$ be a partial semilattice term of $\mathbb{A}$ with $s(a,c) = s(c,a) = a$. Since by assumption we do not have $a \to b$, we have $s(a,b) = a$ as well, and since we assumed that $\{b,c\}$ is a subalgebra and that we do not have $c \to b$, we have $s(c,b) = c$. Define $\mathbb{S}$ by

$$\mathbb{S} = \mathrm{Sg}_{\mathbb{A}^2}\left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix} \right\}.$$

If we have $(a,c) \in \mathbb{S}$, then by symmetry we also have $(c,a) \in \mathbb{S}$, so

$$s\left(\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} c \\ a \end{bmatrix}\right) = \begin{bmatrix} a \\ a \end{bmatrix}$$

is in $\mathbb{S}$ as well. If we have $(c,c) \in \mathbb{S}$, then

$$s\left(\begin{bmatrix} c \\ c \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ c \end{bmatrix},$$

so $(a, c) \in \mathbb{S}$, and then we have $(a, a) \in \mathbb{S}$ as before. Since $c \in \mathrm{Sg}_{\mathbb{A}}\{a, b\} = \pi_2(\mathbb{S})$, if neither (a, c) nor (c, c) are in $\mathbb{S}$, then we must have $(b, c) \in \mathbb{S}$. So the only way to avoid a contradiction in this case is for $\mathbb{S}$ to be given by

$$\mathbb{S} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix}, \begin{bmatrix} b \\ c \end{bmatrix}, \begin{bmatrix} c \\ b \end{bmatrix} \right\}.$$

Then the linking congruence θ of $\mathbb{S}$ corresponds to the partition $\{a, c\}, \{b\}$, and $\mathbb{A}/\theta$ is isomorphic to the two-element subalgebra $\{b, c\}$. Since $(b, b), (c, c) \not\in \mathbb{S}$, $\mathbb{A}/\theta \cong \{b, c\}$ is either a majority algebra or is $\mathbb{Z}/2^{\mathrm{aff}}$. Define a ternary term t by

$$t(x, y, z) = g(s(g(x, y, z), x), s(g(x, y, z), y), s(g(x, y, z), z)).$$

Then t is also cyclic, and we will show that $\{a, b\}$ is closed under t, contradicting the assumption that $\mathbb{A}$ is minimal Taylor. If $\mathbb{A}/\theta$ is $\mathbb{Z}/2^{\mathrm{aff}}$, then we have

$$t(a, a, b) = g(a, a, b) = b,$$

and since $\{a, b\}$ is not closed under g we have

$$g(a, b, b) = c,$$

so

$$t(a, b, b) = g(s(c, a), s(c, b), s(c, b)) = g(a, c, c) = a.$$

If $\mathbb{A}/\theta$ is a majority algebra, then we have

$$t(a, b, b) = g(a, b, b) = b,$$

and since $\{a, b\}$ is not closed under g we have

$$g(a, a, b) = c,$$

so

$$t(a, a, b) = g(s(c, a), s(c, a), s(c, b)) = g(a, a, c) = a.$$

Either way, $\{a, b\}$ is closed under t, which gives us a contradiction.

Now suppose that we have both $c \to a$ and $c \to b$. Consider the binary relation $\mathbb{S} = \mathrm{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\}$ as we did before - if either (a, c) or (b, c) are in $\mathbb{S}$, then we easily see that one of $(a, a), (b, b)$ is in $\mathbb{S}$, which together with Proposition 4.2.5 would contradict the assumption that $\mathbb{A}$ is generated by a and b. Thus the only way to immediately avoid a contradiction is for $\mathbb{S}$ to be given by

$$\mathbb{S} = \left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix}, \begin{bmatrix} c \\ c \end{bmatrix} \right\}.$$

In particular, there must be some binary term $f \in \mathrm{Clo}(g)$ such that $f(a, b) = f(b, a) = c$, and since $\mathbb{A}$ is prepared by Proposition 4.2.5, this f must be the commutative binary operation described in the following table.

f	a	b	c
a	a	c	a
b	c	b	b
c	a	b	c

Then $(\{a,b,c\},f)$ is isomorphic to the algebra $(\{-,0,+\},s_2)$ from Example 1.6.8, with the isomorphism given by $a\mapsto +, b\mapsto -, c\mapsto 0$. This algebra is not minimal Taylor: $\mathrm{Clo}(s_2)$ properly contains the clone of the conservative bounded width algebra $(\{-,0,+\},g)$ described in Example 3.6.1. Explicitly, consider the ternary term t on $\mathbb{A}$ given by

$$t(x,y,z) = f(f(f(x,y),f(y,z)),f(x,z)).$$

It is easy to check that this t is symmetric, and that $\{a,b\}$ is closed under t, contradicting the assumption that $\mathbb{A}$ is minimal Taylor. In fact, even this operation t does not generate a minimal Taylor clone (this claim is left as an exercise). □

Proposition 4.4.19. *If $\mathbb{A}$ is a minimal Taylor algebra with underlying set $\{a,b,c\}$ such that $\mathbb{A}$ is generated by a and b, and if $\mathbb{A}$ has a semilattice subalgebra of the form $a\to c$, then $\mathbb{A}$ is isomorphic to a subdirect product of its two-element subalgebras $\{a,c\}$ and $\{b,c\}$.*

Proof. By assumption, we do not have $b\to a$ or $c\to a$, so $s(\{b,c\},\mathbb{A})\subseteq\{b,c\}$ for every partial semilattice term $s\in\mathrm{Clo}(\mathbb{A})$. Since $a\to c$ and $c\in\mathrm{Sg}_{\mathbb{A}}\{a,b\}$, we can apply Proposition 3.10.19 to see that $\{b,c\}$ is a binary absorbing subalgebra of $\mathbb{A}$. Then by Proposition 4.2.10, we see that there is a congruence θ on $\mathbb{A}$ corresponding to the partition $\{a\},\{b,c\}$, such that $\mathbb{A}/\theta\cong\{a,c\}$ is a two-element semilattice. To finish, we just need to show that the equivalence relation ψ on $\mathbb{A}$ corresponding to the partition $\{a,c\},\{b\}$ is also a congruence of $\mathbb{A}$.

If $b\to c$, then the same argument as in the last paragraph shows that $\{a,c\}$ is a binary absorbing subalgebra of $\mathbb{A}$ and that ψ is a congruence of $\mathbb{A}$ - in this case, $\mathbb{A}$ is the free semilattice on two generators. Additionally, the previous proposition shows that we can't have $c\to b$. Thus the only remaining cases are the case where $\{b,c\}$ is a majority algebra and the case where $\{b,c\}$ is $\mathbb{Z}/2^{\mathrm{aff}}$, and we assume from here on that we are in one of these two cases.

Consider the binary relation $\mathbb{S}=\mathrm{Sg}_{\mathbb{A}^2}\{(a,b),(b,a)\}$. Letting s be a partial semilattice operation of $\mathbb{A}$ with $s(a,c)=c$, we must have $s(a,b)\equiv s(a,c)=c \pmod{\theta}$, so since we do not have $a\to b$ we must have $s(a,b)=c$. Then by our assumption that we do not have $b\to c$, we have

$$s\left(\begin{bmatrix}a\\b\end{bmatrix},\begin{bmatrix}b\\a\end{bmatrix}\right)=\begin{bmatrix}c\\b\end{bmatrix},$$

so

$$\mathbb{S}\supseteq\left\{\begin{bmatrix}a\\b\end{bmatrix},\begin{bmatrix}b\\a\end{bmatrix},\begin{bmatrix}b\\c\end{bmatrix},\begin{bmatrix}c\\b\end{bmatrix}\right\}.$$

If this containment is an equality, then the linking congruence of $\mathbb{S}$ is ψ, which would prove that ψ is a congruence of $\mathbb{A}$ and finish the proof. Otherwise, since $(a,a),(b,b)$ are not contained in $\mathbb{S}$ by Proposition 4.2.5, at least one of $(a,c),(c,a)$ or (c,c) must be an element of $\mathbb{S}$. Since $a\to c$ and $\mathbb{S}$ is symmetric, in each case we see that $(c,c)\in\mathbb{S}$. From here on we assume that $(c,c)\in\mathbb{S}$.

From $(b,c),(c,c),(c,b)\in\mathbb{S}$ but $(b,b)\not\in\mathbb{S}$, we see that $\{b,c\}$ can't be $\mathbb{Z}/2^{\mathrm{aff}}$ (since the parallelogram property fails for a binary relation on $\{b,c\}$). We are left with the case where $\{b,c\}$ is a majority algebra. Let g be a cyclic ternary term on $\mathbb{A}$, which exists by Theorem 4.4.12, and let f be a binary term with $f(a,b)=f(b,a)=c$, which must exist if $(c,c)\in\mathbb{S}$. Define a ternary term t by

$$t(x,y,z)=g(f(x,y),f(y,z),f(z,x)).$$

Then t is also cyclic, and we have

$$t(a,a,b) = g(a,c,c) = c, \quad t(a,b,b) = g(c,b,c) = c, \quad t(a,b,c) \in g(c,\{b,c\},c) = \{c\}.$$

This completely determines t, and shows that $\{a,c\}$ is a ternary absorbing subalgebra of $\mathbb{A}$ with absorbing operation t. Therefore $\{a,c\}$ is a centrally absorbing subalgebra of $\mathbb{A}$ by Theorem 4.2.20. If we then define the cyclic ternary term u by

$$u(x,y,z) = t(s(x,y), s(y,z), s(z,x)),$$

then it is easy to check that $\{b\}$ is a ternary absorbing subalgebra of $\mathbb{A}$ with absorbing operation u, so by Theorem 4.2.20 and Corollary 4.2.21 the equivalence relation ψ is a congruence of $\mathbb{A}$ (note that this actually contradicts the assumption $(c,c) \in \mathbb{S}$). □

So far we have classified every minimal Taylor algebra of size 3 which contains at least one semilattice subalgebra. The remaining cases are the cases where our algebra $\mathbb{A}$ has two two-element subalgebras, each of which is either a majority algebra or a copy of $\mathbb{Z}/2^{\text{aff}}$.

Proposition 4.4.20. *If $\mathbb{A}$ is a minimal Taylor algebra with underlying set $\{a,b,c\}$ which is generated by a and b, then at least one of $\{a,c\}, \{b,c\}$ is not a $\mathbb{Z}/2^{\text{aff}}$-subalgebra of $\mathbb{A}$.*

Proof. Suppose for contradiction that $\{a,c\}, \{b,c\}$ are both $\mathbb{Z}/2^{\text{aff}}$-subalgebras of $\mathbb{A}$. By Corollary 4.2.7, $\mathbb{A}$ either has a proper absorbing subalgebra or an affine quotient. If $\mathbb{A}$ has a proper absorbing subalgebra, then one of $\{a,c\}, \{b,c\}$ has a proper absorbing subalgebra, which is impossible if they are both copies of $\mathbb{Z}/2^{\text{aff}}$. Therefore there is a congruence θ of $\mathbb{A}$ such that $\mathbb{A}/\theta$ is affine, and we can assume without loss of generality that θ corresponds to the partition $\{a\}, \{b,c\}$ of $\mathbb{A}$.

Let g be a cyclic ternary term on $\mathbb{A}$, which exists by Theorem 4.4.12. Then since $\mathbb{A}/\theta \cong \{a,c\}$ is a copy of $\mathbb{Z}/2^{\text{aff}}$ we must have

$$g(a,\{b,c\},\{b,c\}) \in \{a\}, \quad g(a,a,c) = c,$$

and since $\{a,b\}$ is not closed under g we must have

$$g(a,a,b) = c$$

as well. This, together with the fact that $\{b,c\}$ is also a copy of $\mathbb{Z}/2^{\text{aff}}$, completely determines g. Define a ternary term t by

$$t(x,y,z) = g(x, g(x,y,y), g(x,y,z)),$$

and note that

$$t\left(\begin{bmatrix} a\\ a\\ b\end{bmatrix}, \begin{bmatrix} a\\ b\\ a\end{bmatrix}, \begin{bmatrix} b\\ a\\ a\end{bmatrix}\right) = g\left(\begin{bmatrix} a\\ a\\ b\end{bmatrix}, \begin{bmatrix} a\\ a\\ c\end{bmatrix}, \begin{bmatrix} c\\ c\\ c\end{bmatrix}\right) = \begin{bmatrix} c\\ c\\ b\end{bmatrix}.$$

Thus the ternary term u defined by

$$u(x,y,z) = g(t(x,y,z), t(y,z,x), t(z,x,y))$$

is a cyclic term with

$$u(a,a,b) = g(c,c,b) = b,$$

so $\{a,b\}$ is closed under u, which contradicts the assumption that $\mathbb{A}$ is a minimal Taylor algebra. □

Proposition 4.4.21. *If $\mathbb{A}$ is a minimal Taylor algebra with underlying set $\{a,b,c\}$ which is generated by a and b, then at least one of $\{a,c\},\{b,c\}$ is not a majority subalgebra of $\mathbb{A}$.*

Proof. Suppose for contradiction that $\{a,c\},\{b,c\}$ are both majority subalgebras of $\mathbb{A}$. Then no subquotient of $\mathbb{A}$ can be affine, so by Theorem 3.13.8 CSP($\mathbb{A}$) has bounded width, and so by Theorem 3.14.8 $\mathbb{A}$ has a binary term f and a ternary term g satisfying the identities

$$g(x,x,y) \approx g(x,y,x) \approx g(y,x,x) \approx f(x,y) \approx f(f(x,y),f(y,x)).$$

We may assume without loss of generality that this g is also cyclic, by Theorem 4.4.12 and the cyclic composition trick. Since $\{a,b\}$ is not closed under g, we may assume without loss of generality that $f(b,a) = c$.

First we show that $f(a,b) \neq b$. If $f(a,b) = b$, then we have $f(a,f(a,b)) = f(a,b) = b$ and

$$f(b,f(b,a)) = f(f(a,b),f(b,a)) = f(a,b) = b,$$

so $(b,b) \in \mathrm{Sg}_{\mathbb{A}^2}\{(a,b),(b,a)\}$, and then by Proposition 4.2.5 we have $a \to b$, which contradicts the assumption that $\mathbb{A}$ is generated by a and b.

Now suppose that $f(a,b) = f(b,a) = c$. Let t be the ternary term defined by

$$t(x,y,z) = g(f(x,f(y,z)),f(y,f(z,x)),f(z,f(x,y))).$$

Then t is cyclic, and we have

$$t(a,a,b) = g(f(a,c),f(a,c),f(b,a)) = g(a,a,c) = a$$

and similarly $t(a,b,b) = b$, so $\{a,b\}$ is closed under t, contradicting the assumption that $\mathbb{A}$ is a minimal Taylor algebra.

Finally, suppose that $f(a,b) = a, f(b,a) = c$. Let h be the ternary term defined by

$$h(x,y,z) = g(f(x,y),f(y,x),g(x,y,z)),$$

let i be the ternary term defined by

$$i(x,y,z) = g(x,h(x,y,z),h(x,z,y)),$$

and let t be the cyclic ternary term defined by

$$t(x,y,z) = g(i(x,y,z),i(y,z,x),i(z,x,y)).$$

Then we have $h(a,a,b) = h(a,b,a) = h(b,a,a) = a$, so

$$t(a,a,b) = g(g(a,a,a),g(a,a,a),g(b,a,a)) = g(a,a,a) = a,$$

and $h(a,b,b) = h(b,a,b) = c, h(b,b,a) = b$, so

$$t(a,b,b) = g(g(a,c,c),g(b,b,c),g(b,c,b)) = g(c,b,b) = b.$$

Thus $\{a,b\}$ is closed under t, contradicting the assumption that $\mathbb{A}$ is a minimal Taylor algebra. □

Proposition 4.4.22. *Suppose that $\mathbb{A}$ is a minimal Taylor algebra with underlying set $\{a,b,c\}$ which is generated by a and b, that $\{a,c\}$ is a $\mathbb{Z}/2^{\text{aff}}$-subalgebra, and that $\{b,c\}$ is a majority subalgebra of $\mathbb{A}$. Then $\mathbb{A}$ is the same as the algebra described in Example 2.2.1, up to isomorphism and term equivalence. In particular, $\{a,c\}$ is a centrally absorbing subalgebra of $\mathbb{A}$, and $\mathbb{A}$ has a congruence θ corresponding to the partition $\{a\},\{b,c\}$ such that $\mathbb{A}/\theta \cong \{a,c\}$ is $\mathbb{Z}/2^{\text{aff}}$.*

Proof. Let $\mathbb{S} = \mathrm{Sg}_{\mathbb{A}^2}\{(a,b),(b,a)\}$, and let g be a ternary cyclic term of $\mathbb{A}$, which exists by Theorem 4.4.12. By Proposition 4.2.5, we have $(a,a),(b,b) \not\in \mathbb{S}$. If $(c,c) \in \mathbb{S}$, then we have

$$\begin{bmatrix} a \\ c \end{bmatrix} = g\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ c \end{bmatrix}\right) \in \mathbb{S},$$

and then that

$$\begin{bmatrix} a \\ a \end{bmatrix} = g\left(\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} c \\ a \end{bmatrix}, \begin{bmatrix} c \\ c \end{bmatrix}\right) \in \mathbb{S},$$

which is a contradiction. Thus we have $(c,c) \not\in \mathbb{S}$, so $\mathbb{S} \cap \Delta_{\mathbb{A}} = \emptyset$. If both $(a,c),(b,c) \in \mathbb{S}$, then we have

$$g\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ c \end{bmatrix}, \begin{bmatrix} c \\ a \end{bmatrix}\right) \in \mathbb{S} \cap \Delta_{\mathbb{A}},$$

which we just showed is impossible. Since $\pi_2(\mathbb{S}) = \mathrm{Sg}_{\mathbb{A}}\{a,b\} = \mathbb{A}$, exactly one of $(a,c),(b,c)$ is in $\mathbb{S}$.

Suppose first that $(b,c) \in \mathbb{S}$. Then the linking congruence θ of $\mathbb{S}$ corresponds to the partition $\{a,c\},\{b\}$ of $\mathbb{A}$, and $\mathbb{A}/\theta \cong \{b,c\}$ is a majority algebra. This implies that

$$g(a,b,b) = g(b,b,c) = b, g(b,c,c) = c, g(a,c,c) = a.$$

Since $\{a,b\}$ isn't closed under g, we must have

$$g(a,a,b) = c.$$

Let $f(x,y) = g(x,x,y)$, and define a ternary term t by

$$t(x,y,z) = g(f(g(x,y,z),f(x,y)),f(g(x,y,z),f(y,z)),f(g(x,y,z),f(z,x))).$$

Then t is cyclic,

$$t(a,a,b) = g(f(c,a),f(c,c),f(c,b)) = g(a,c,c) = a,$$

and

$$t(a,b,b) = g(f(b,c),f(b,b),f(b,b)) = g(b,b,b) = b.$$

Thus $\{a,b\}$ is closed under t, contradicting the assumption that $\mathbb{A}$ is a minimal Taylor algebra.

Now suppose that $(a,c) \in \mathbb{S}$. Then the linking congruence θ of $\mathbb{S}$ corresponds to the partition $\{a\},\{b,c\}$ of $\mathbb{A}$, and $\mathbb{A}/\theta \cong \{a,c\}$ is $\mathbb{Z}/2^{\text{aff}}$. This implies that

$$g(a,\{b,c\},\{b,c\}) = \{a\}, \quad g(a,a,c) = c, g(b,b,c) = b, g(b,c,c) = c.$$

Since $\{a,b\}$ is not closed under g, we must have

$$g(a,a,b) = c.$$

This completely determines g, and we see that $\{a, c\}$ is a ternary absorbing subalgebra of $\mathbb{A}$ with absorbing operation g, so by Theorem 4.2.20 $\{a, c\}$ is a centrally absorbing subalgebra of $\mathbb{A}$. After swapping a and c, g is exactly the same operation as the one described in Example 2.2.1. To see that $\mathbb{A}$ is really minimal Taylor in this case, note that any ternary cyclic $g' \in \mathrm{Clo}(g)$ must also satisfy all of the above identities, including $g'(a, a, b) = c$ since $\{a, c\}$ is centrally absorbing. □

Putting all of the pieces together, we have completed the classification of minimal Taylor algebras on a three-element domain.

Theorem 4.4.23. *If $\mathbb{A}$ is a minimal Taylor algebra on a set of size 3, then up to term equivalence and isomorphism $\mathbb{A}$ is one of the following 24 algebras:*

- *one of the 19 conservative minimal Taylor algebras classified in the previous section,*
- *the affine algebra $\mathbb{Z}/3^{\mathrm{aff}} = (\mathbb{Z}/3, x - y + z)$,*
- *the free semilattice on two generators,*
- *the three-element subdirect product of $(\{0, 1\}, x \vee y \vee z)$ with $(\{0, 1\}, \mathrm{maj}(x, y, z))$,*
- *the three-element subdirect product of $(\{0, 1\}, x \vee y \vee x)$ with $\mathbb{Z}/2^{\mathrm{aff}} = (\mathbb{Z}/2, x + y + z)$,*
- *the three-element algebra from Example 2.2.1, which has a 3-edge term, a two-element centrally absorbing subalgebra, and a $\mathbb{Z}/2^{\mathrm{aff}}$ quotient.*

Problem 4.4.1. For each one of the 24 minimal Taylor algebras on a set of size 3, find a generating set of relations for the corresponding relational clone. Are they all finitely related?

4.5 The strands of an unlinked CSP instance, and a safe recursive strategy

Generally speaking, in order to guarantee a polynomial running time for solving CSPs we attempt to avoid recursion. There is a form of recursion which can be safely applied, however: we can recursively solve polynomially many subproblems as long as the size of *every* variable's domain is strictly reduced in each subproblem. The resulting algorithms will then have the property that the exponent in the running time will depend on the size of the largest domain of any variable. This approach seems to have been introduced with the solutions to the CSP dichotomy conjecture for conservative algebras by Bulatov [45], [46] and Barto [9], as well as Miklós Maróti's "Tree on top of Malcev" algorithm [140] (which used this sort of recursive strategy in a *very* different way from what we will consider in this section).

The challenge now is to find situations where we can usefully reduce to subproblems in which every single variable domain is reduced. The prototypical example of how this may occur is when a CSP instance is *unlinked*.

Definition 4.5.1. Let $\mathbf{X}$ be an instance of a CSP, with variable domains $\mathbb{A}_x$ for each variable x. We say that $\mathbf{X}$ is *unlinked* at a variable x of $\mathbf{X}$ if there are some $a, b \in \mathbb{A}_x$ such that there are *no* cycles p of $\mathbf{X}$ from x to x with $b \in \{a\} + p$. We say that $\mathbf{X}$ is *unlinked* if it is unlinked at every variable.

If an instance $\mathbf{X}$ is unlinked at a variable x, then we define the *linking relation* $\theta_{\mathbf{X}}$ of $\mathbf{X}$ at x as the equivalence relation on $\mathbb{A}_x$ defined by $(a,b) \in \theta_{\mathbf{X}}$ iff there is some cycle p from x to x such that $b \in \{a\} + p$.

Proposition 4.5.2. *If $\mathbf{X}$ is cycle-consistent and unlinked at x, then the linking relation $\theta_{\mathbf{X}}$ is a congruence of $\mathbb{A}_x$.*

Proof. We just need to check that $\theta_{\mathbf{X}}$ is closed under unary polynomial operations of $\mathbb{A}_x$. So suppose that $(a,b) \in \theta_{\mathbf{X}}$, $c_1, ..., c_n \in \mathbb{A}_x$, and f is some $n+1$-ary polymorphism of $\mathbb{A}_x$. Since $(a,b) \in \theta_{\mathbf{X}}$, there is some cycle p from x to x such that $b \in \{a\} + p$. By cycle-consistency, we also have $c_i \in \{c_i\} + p$ for each i, so if $\mathbb{P}_p \le \mathbb{A}_x^2$ is the binary relation corresponding to the cycle p, then we have

$$f\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c_1 \\ c_1 \end{bmatrix}, \cdots, \begin{bmatrix} c_n \\ c_n \end{bmatrix}\right) \in \mathbb{P}_p.$$

Thus we have $(f(a, c_1, ..., c_n), f(b, c_1, ..., c_n)) \in \theta_{\mathbf{X}}$, which completes the proof. □

As a consequence, one way to discover unlinked subinstances of a cycle-consistent instance $\mathbf{X}$ is to go through each maximal congruence θ on some variable domain $\mathbb{A}_x$, and to greedily build up a subinstance $\mathbf{X}'$ which is as large as possible subject to the condition $\theta_{\mathbf{X}'} \le \theta$. The resulting subinstance $\mathbf{X}'$ only depends on θ and not on the choices we make during the greedy construction of $\mathbf{X}'$, so long as $\mathbf{X}$ is cycle-consistent: if there are two paths p, q from x to y such that $p - p$ and $q - q$ each have linking congruences contained in θ, then cycle-consistency applied to $-q + p$ shows that the linking congruence of $p - q$ is also contained in θ. This is the approach Zhuk used in his proof of the general CSP dichotomy conjecture [190].

Proposition 4.5.3. *Suppose that we have a multisorted CSP template* $\mathrm{CSP}(\mathbb{A}_1, ..., \mathbb{A}_n)$ *such that the CSP template* $\mathrm{CSP}(\mathbb{B}_1, ..., \mathbb{B}_m)$ *can be solved in polynomial time, where $\mathbb{B}_1, ..., \mathbb{B}_m$ is the collection of all algebras $\mathbb{B}$ which are isomorphic to a proper subalgebra of some $\mathbb{A}_i$ (note that* $\max|\mathbb{B}_i| < \max|\mathbb{A}_j|$*). Then we can solve any unlinked instance $\mathbf{X}$ of* $\mathrm{CSP}(\mathbb{A}_1, ..., \mathbb{A}_n)$ *in polynomial time.*

Proof. We assume that every pair of variables of $\mathbf{X}$ can be connected by some path without loss of generality, and we shrink $\mathbf{X}$ by enforcing cycle-consistency (if the shrunk instance is no longer unlinked, then every single variable domain has been reduced, and we can solve the instance). Pick any variable x of $\mathbf{X}$, and let $\theta_{\mathbf{X}}$ be the linking congruence on $\mathbb{A}_x$. Then for each congruence class $a/\theta_{\mathbf{X}} \in \mathbb{A}_x/\theta_{\mathbf{X}}$, if we restrict the possible values of x to a/θ and enforce arc-consistency, then every other variable y of $\mathbf{X}$ has its domain restricted to some subset of a congruence class of $\mathbb{A}_y/\theta_{\mathbf{X}}$ (where here we interpret $\theta_{\mathbf{X}}$ as the linking congruence of $\mathbf{X}$ on $\mathbb{A}_y$). Since the value of x must be in *some* congruence class of $\mathbb{A}_x/\theta_{\mathbf{X}}$, and since there are only a constant number of such congruence classes to check, we can solve $\mathbf{X}$ by solving a constant number of instances of CSPs with templates of the form $\mathrm{CSP}(\mathbb{B}_1, ..., \mathbb{B}_m)$, where for each variable y the algebra $\mathbb{B}_i$ is contained in an congruence class of $\mathbb{A}_y/\theta_{\mathbf{X}}$. □

Of course, an entire instance being unlinked is fairly rare. Additionally, the assumption of cycle-consistency is probably unnecessarily strong. The approach used in the algorithms for the conservative CSP dichotomy is a variant of the strategy of looking for unlinked subinstances, starting from the idea of looking for *any* useful way of properly restricting the domains of some subset of the variables. This will naturally lead to different consistency principles (which are likely to also be unecessarily strong).

The most general thing we could do along these lines is the following. For each variable x, and for each element $a \in \mathbb{A}_x$, restrict the domain of x to the singleton $\{a\}$, and run arc-consistency (or cycle-consistency, etc.). Restrict our attention to the set Y of variables y whose domain has shrunk as a result of this restriction (replacing every relation which involves variables outside of Y by its projection onto the variables contained in Y), and solve the resulting instance recursively to see if we can rule out the element $a \in \mathbb{A}_x$ as a possible value for x.

Although this scheme is readily implemented, it is hard to algebraically control what happens as a result. Instead, we will consider the digraph of implications between restrictions on the individual domains, and ask under which conditions this has a nice structure.

Definition 4.5.4. If $\mathbf{X}$ is an instance of a multisorted CSP with variable domains $\mathbb{A}_x$, then we define the *implication digraph* to be the directed graph on $\mathbf{X}$ where the vertices are pairs $(x, \mathbb{B})$ such that x is a variable and $\mathbb{B} \le \mathbb{A}_x$, and where we have a directed edge from $(x, \mathbb{B})$ to $(y, \mathbb{C})$ if there is a path p of length 1 from x to y in $\mathbf{X}$ such that $\mathbb{B} + p = \mathbb{C}$.

We write $(x, \mathbb{B}) \preceq (y, \mathbb{C})$ if there is a path p of any length from x to y such that $\mathbb{B} + p = \mathbb{C}$. The resulting quasiorder is the *implication qoset*.

Let $\mathcal{E}$ be a subdigraph of the implication digraph, and consider $\mathcal{E}$ as a subqoset of the implication qoset by taking its transitive closure. A *strand* of $\mathcal{E}$ is just an equivalence class of $\mathcal{E}$. Often we might take $\mathcal{E}$ to be the subdigraph of pairs $(x, \mathbb{B})$ such that $\mathbb{B}$ is a proper subalgebra of $\mathbb{A}_x$ (or a proper absorbing subalgebra, etc.). We say that $\mathcal{S}$ is a *maximal* strand of $\mathcal{E}$ if it is a maximal equivalence class of the qoset $\mathcal{E}$. If $\mathcal{E}$ is not specified, then a strand can be any subset of any equivalence class of the implication qoset.

A strand is called *absorbing* if for all $(x, \mathbb{B}) \in \mathcal{S}$ we have $\mathbb{B} \lhd \mathbb{A}_x$. Note that as long as $\mathbf{X}$ is arc-consistent, if this occurs for *some* $(x, \mathbb{B}) \in \mathcal{S}$ then it occurs for *all* $(x, \mathbb{B}) \in \mathcal{S}$.

We define the *partial restriction* of $\mathbf{X}$ to the strand $\mathcal{S}$ by reducing the domain of each variable x to $\bigcap_{(x,\mathbb{B}) \in \mathcal{S}} \mathbb{B}$ if some $(x, \mathbb{B}) \in \mathcal{S}$, and leaving the domain of x unchanged otherwise.

Proposition 4.5.5. *If $\mathbf{X}$ is cycle-consistent, $\mathcal{E}$ is a subdigraph of the implication digraph, and we are given some $(x, \mathbb{B}) \in \mathcal{E}$, then we can find a maximal strand of $\mathcal{E}$ in polynomial time.*

Proof. We start from any element of $\mathcal{E}$ and keep following single-step paths until we stabilize at a maximal strand. To see that this takes only polynomially many steps, note that if $(x, \mathbb{B}) \preceq (x, \mathbb{C})$, then cycle-consistency implies that $\mathbb{B} \le \mathbb{C}$, so if $\mathbb{B} \ne \mathbb{C}$ then $|\mathbb{C}| \ge |\mathbb{B}| + 1$. □

In order to restrict an instance $\mathbf{X}$ to a strand $\mathcal{S}$ in a way that guarantees that the restricted instance is arc-consistent, it seems like we should require that the strand $\mathcal{S}$ includes at most one pair $(x, \mathbb{B})$ for each variable $x \in \mathbf{X}$. This will be guaranteed as long as the instance $\mathbf{X}$ is sufficiently consistent.

Proposition 4.5.6. *If $\mathbf{X}$ is an arc-consistent instance, then each strand $\mathcal{S}$ of $\mathbf{X}$ includes at most one pair $(x, \mathbb{B})$ for each variable x as long as $\mathbf{X}$ satisfies property (P3) from Definiion 3.13.12:*

$$\mathbb{B} + p + q = \mathbb{B} \implies \mathbb{B} + p = \mathbb{B}$$

for all pairs of cycles p, q from x to x and all $\mathbb{B} \le \mathbb{A}_x$. In particular, this always occurs if $\mathbf{X}$ is pq-consistent.

Definition 4.5.7. Suppose that $\mathbf{X}$ is an instance, and let $\mathcal{S}$ be any strand of $\mathbf{X}$ such that for each variable x, there is at most one $\mathbb{B}_x$ such that $(x, \mathbb{B}_x) \in \mathcal{S}$. We define the *full restriction* of $\mathbf{X}$ to $\mathcal{S}$ to be the instance $\mathbf{X}'$ whose variable set is the set of variables x such that some $(x, \mathbb{B}_x) \in \mathcal{S}$, with each variable domain reduced to $\mathbb{A}_x$, where for each relation of $\mathbf{X}$ we project it onto the set of variables in $\mathbf{X}'$ and restrict each of its coordinates to $\mathbb{B}_x$.

In order to mimic the situation of an unlinked instance as well as possible, we might also like to have the property that for each variable x such that some $(x, \mathbb{B})$ is in the strand $\mathcal{S}$, we have parallel strands $\mathcal{S}'$ which partition the domain $\mathbb{A}_x$ into disjoint subalgebras $\mathbb{B}'$ with each $(x, \mathbb{B}') \in \mathcal{S}'$.

Proposition 4.5.8. *If $\mathbf{X}$ is an arc-consistent instance, then for each strand $\mathcal{S}$, each $(x, \mathbb{B})$, and each $a \not\in \mathbb{B}$, there is a strand $\mathcal{S}'$, involving the same set of variables as $\mathcal{S}$, such that $(x, \mathbb{B}') \in \mathcal{S}'$ for some $\mathbb{B}' \le \mathbb{A}_x$ with $a \in \mathbb{B}'$ and $\mathbb{B} \cap \mathbb{B}' = \emptyset$, as long as $\mathbf{X}$ also satisfies property (P2) from Definition 3.13.12:*

$$\mathbb{B} + p = \mathbb{B} \quad \implies \quad \mathbb{B} - p = \mathbb{B}$$

for all cycles p from x to x and all $\mathbb{B} \le \mathbb{A}_x$.

In this case, a more precise statement is true: for each $(x, \mathbb{B}) \in \mathcal{S}$ there is a congruence $\theta_{\mathcal{S}}$ on $\mathbb{A}_x$ such that $\mathbb{B}$ is a union of congruence classes of $\theta_{\mathcal{S}}$, and each congruence class $\mathbb{B}'$ of $\theta_{\mathcal{S}}$ is contained in a parallel strand $\mathcal{S}'$.

Proof. Consider the set $\mathcal{C}$ of all cycles p from x to x such that $\mathbb{B} + p = \mathbb{B}$. Property (P2) is the guarantee that for each $p \in \mathcal{C}$, $\mathbb{B}$ will be a union of congruence classes of the linking congruence of $\mathbb{P}_p$. Define $\theta_{\mathcal{S}}$ to be the join of the collection of all linking congruences of $\mathbb{P}_p$ for $p \in \mathcal{C}$. Then $\mathbb{B}$ is still a union of congruence classes of $\theta_{\mathcal{S}}$.

For any congruence class $\mathbb{B}'$ of $\theta_{\mathcal{S}}$, we define the parallel strand $\mathcal{S}'$ as follows: for every way of splitting some $p \in \mathcal{C}$ as $p = q + r$, where q is a path from x to a variable y, we include $(y, \mathbb{B}' + q)$ in $\mathcal{S}'$. Since $\mathbb{B}'$ is a congruence class of $\theta_{\mathcal{S}}$, $\mathbb{B}'$ will be a union of congruence classes of the linking congruence of $\mathbb{P}_p$, so $\mathbb{B}' + p - p = \mathbb{B}'$, so $(\mathbb{B}' + q) + r - p = \mathbb{B}'$, and we see that $(x, \mathbb{B}')$ and $(y, \mathbb{B}' + q)$ are indeed part of the same equivalence class of the implication qoset of $\mathbf{X}$. □

To guarantee that the restriction of our instance $\mathbf{X}$ to any strand is arc-consistent, we need a still stronger consistency principle. This consistency principle was used to give one of the original proofs of the fact that $(2,3)$-consistency was strong enough to ensure satisfiability of any CSP with bounded width [13]. I will define it a bit differently here, but will show that the definition given here is equivalent to the original definition by following an argument from [9].

Definition 4.5.9. An arc-consistent instance $\mathbf{X}$ is called a *Prague instance* if for every $(x, \mathbb{B}), (y, \mathbb{C})$ in the same equivalence class of the implication qoset, and for every path p of length 1 from x to y, we have $\mathbb{B} + p = \mathbb{C}$.

Proposition 4.5.10. *If $\mathbf{X}$ is a Prague instance, then $\mathbf{X}$ satisfies conditions (P2) and (P3) of Definition 3.13.12, so $\mathbf{X}$ is a weak Prague instance.*

Proposition 4.5.11 (Barto and Kozik [11], [9])**.** *If $\mathbf{X}$ is arc-consistent, then the following are equivalent:*

(a) for every $(x, \mathbb{B}), (y, \mathbb{C})$ in the same equivalence class of the implication qoset, and for every path p of length 1 from x to y, we have $\mathbb{B} + p \subseteq \mathbb{C}$.

(b) $\mathbf{X}$ *is a Prague instance,*

(c) for every variable x and every pair of paths p, q from x to x such that the variables involved in p are a subset of the variables involved in q, if $b \in \{a\} + p$ then there is some $j \ge 0$ such that $b \in \{a\} + jq$,

(d) for every variable x and every pair of paths p, q from x to x such that the variables involved in p are a subset of the variables involved in q, then for every sufficiently large j we have $\{a\} + p \subseteq \{a\} + jq$.

Proof. For the implication from (a) to (b), note that from $\mathbb{B} + p \subseteq \mathbb{C}$ and $\mathbb{C} - p \subseteq \mathbb{B}$ we can immediately conclude that $\mathbb{B} + p = \mathbb{C}$.

For (b) $\implies$ (c), let p, q be as in (c). Find a $j \ge 1$ such that $\{a\} + jq = \{a\} + 2jq$, write $\mathbb{B} = \{a\} + jq$, and let $\mathcal{S}$ be the strand of the implication digraph containing $(x, \mathbb{B})$. Since every Prague instance satisfies property (P3), for each variable y which shows up in the cycle q there is a unique $\mathbb{C}$ such that $(y, \mathbb{C}) \in \mathcal{S}$. Since every step of the cycle p goes between variables involved in $\mathcal{S}$ and $\mathbf{X}$ is a Prague instance, we have $\mathbb{B} + p = \mathbb{B}$, and for the same reason we have $\mathbb{B} - jq = \mathbb{B}$. From

$$a \in \{a\} + jq - jq = \mathbb{B} - jq = \mathbb{B}$$

and $b \in \{a\} + p$, we get

$$b \in \{a\} + p \subseteq \mathbb{B} + p = \mathbb{B} = \{a\} + jq.$$

For (c) $\implies$ (d), it's enough to prove that (c) implies $a \in \{a\} + jq$ for every sufficiently large j. Since $\mathbf{X}$ is arc-consistent, there is some b such that $b \in \{a\} + q$, and since $a \in \{b\} - q$ (c) implies there is some $j \ge 0$ such that $a \in \{b\} + jq$. Since $b \in \{a\} + q$, we have $a \in \{a\} + (j+1)q$. Now set $r = (j+1)q$, and by applying (c) again we see that there is some $k \ge 0$ such that

$$a \in \{b\} + kr \subseteq \{a\} + q + k(j+1)q = \{a\} + (k(j+1)+1)q.$$

Since $j+1$ and $k(j+1)+1$ are relatively prime positive integers, every sufficiently large number can be written as a positive combination of $j+1$ and $k(j+1)+1$, which proves (d).

For (d) $\implies$ (a), let q be a path from y to x such that $\mathbb{C} + q = \mathbb{B}$, and let r be a path from x to y such that $\mathbb{B} + r = \mathbb{C}$. Then for every j we have $\mathbb{C} + j(q+r) = \mathbb{C}$. Since the cycle $q+p$ from y to y only involves a subset of the variables involved in $q+r$, by part (d) we have $\mathbb{C} + (q+p) \subseteq \mathbb{C} + j(q+r)$ for every sufficiently large j, so

$$\mathbb{B} + p = \mathbb{C} + q + p \subseteq \mathbb{C} + j(q+r) = \mathbb{C}. \qquad \square$$

Proposition 4.5.12. *If $\mathbf{X}$ is a Prague instance, and if $\mathcal{S}$ is a strand of $\mathbf{X}$, then the full restriction of $\mathbf{X}$ to $\mathcal{S}$ (which restricts both the set of variables and the domains) will be arc-consistent - in fact, it will also be a Prague instance.*

If $\mathcal{S}$ is also a maximal *strand in the qoset of pairs $(x, \mathbb{B})$ such that $\mathbb{B} < \mathbb{A}_x$, then the partial restriction of $\mathbf{X}$ to $\mathcal{S}$ (which restricts the domains but not the set of variables) is also arc-consistent. If $\mathcal{S}$ is additionally absorbing, then the partial restriction of $\mathbf{X}$ to $\mathcal{S}$ will even be a Prague instance.*

Proof. The statement about the full restriction follows from the fact that for any tuple r of any constraint relation $\mathbb{R} \le_{sd} \mathbb{A}_{x_1} \times \cdots \times \mathbb{A}_{x_n}$, if $\pi_{x_i}(r) \in \mathbb{B}_i$ and $(x_i, \mathbb{B}_i) \in \mathcal{S}$, then we necessarily have $\pi_{x_j}(r) \in \mathbb{B}_j$ for any $(x_j, \mathbb{B}_j) \in \mathcal{S}$, by the definition of a Prague instance. For the statement about the partial restriction, suppose that some variable x_i does not not occur in the strand $\mathcal{S}$, and that the relation $\mathbb{R}$ involves a variable x_j with $(x_j, \mathbb{B}_j) \in \mathcal{S}$. Then by the maximality of $\mathcal{S}$, we must have $\mathbb{B}_j + \pi_{x_j x_i}(\mathbb{R}) = \mathbb{A}_{x_i}$, so for each $a \in \mathbb{A}_{x_i}$, there is a tuple $r \in \mathbb{R}$ with $\pi_{x_j}(r) \in \mathbb{B}_j$, and then by the first part we have $\pi_{x_k}(r) \in \mathbb{B}_k$ for each $(x_k, \mathbb{B}_k) \in \mathcal{S}$.

Now suppose that $\mathcal{S}$ is absorbing. By the equivalence between parts (b) and (c) of Proposition 4.5.11, it's enough to show that for every cycle p from x to x in $\mathbf{X}$, if we let p' be the corresponding path in the partially restricted instance, then for any $a, b \in \mathbb{B}_x$ with $(x, \mathbb{B}_x) \in \mathcal{S}$ and $b \in \{a\} + p$, there is some $j \geq 1$ such that $b \in \{a\} + jp'$. For this, we let $\mathbb{R} = \mathbb{P}_p \leq \mathbb{A}_x \times \mathbb{A}_x$ and $\mathbb{S} = \mathbb{P}_{p'} \leq \mathbb{B}_x \times \mathbb{B}_x$, and we aim to apply Corollary 3.7.10.

Since $\mathcal{S}$ is absorbing, we have $\mathbb{S} \lhd \mathbb{R}$. Since the restriction of $\mathbf{X}$ to $\mathcal{S}$ is arc-consistent, $\mathbb{S}$ is subdirect in $\mathbb{B}_x \times \mathbb{B}_x$. Thus, there is some directed cycle $\mathcal{C}_a$ of $\mathbb{S}$ which can be reached from a in $\mathbb{S}$, and there is some directed cycle $\mathcal{C}_b$ of $\mathbb{S}$ with a directed path from $\mathcal{C}_b$ to b in $\mathbb{S}$. Since $\mathbf{X}$ is a Prague instance and $\mathcal{C}_a, \mathcal{C}_b$ are contained in the same weakly connected component of $\mathbb{R}$, there is some directed path from $\mathcal{C}_a$ to $\mathcal{C}_b$ in $\mathbb{R}$ by Proposition 3.13.13. Now we can apply Corollary 3.7.10 to see that there is a directed path from $\mathcal{C}_a$ to $\mathcal{C}_b$ in $\mathbb{S}$, so there is a directed path from a to b in $\mathbb{S}$, and we are done. □

In practice, the local consistency algorithm used to reduce general instances to Prague instances or to cycle-consistent instances actually produces an instance with an even greater level of consistency.

Definition 4.5.13. An instance $\mathbf{X}$ is (l, k)-*minimal*, for $k \geq l$, if

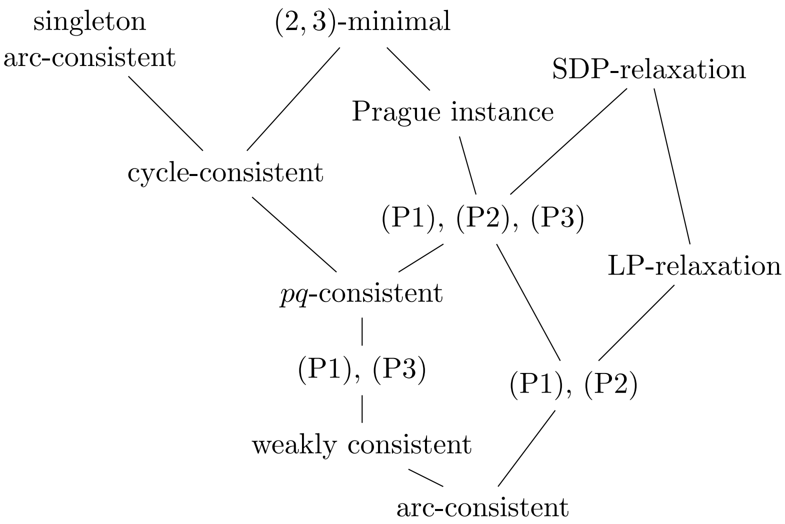
- every set of at most k variables of $\mathbf{X}$ is in the scope of some constraint, and
- for any set S of at most l variables and any pair of constraints C_1, C_2 of $\mathbf{X}$ whose scopes contain S, the existential projections of C_1 and C_2 to the variables in S are the same.

Proposition 4.5.14. *If* $\mathbf{X}$ *is* $(2,3)$*-minimal, then* $\mathbf{X}$ *is a Prague instance, and* $\mathbf{X}$ *is cycle-consistent.*

Proof. We leave cycle-consistency to the reader, and will prove that $\mathbf{X}$ is a Prague instance. By the equivalence of (a) and (b) in Proposition 4.5.11, it's enough to show that for $(x, \mathbb{B}) \preceq (y, \mathbb{C})$ and p a 1 step path from x to y, we always have $\mathbb{B} + p \subseteq \mathbb{C}$. Let q be the path from x to y with $\mathbb{B} + q = \mathbb{C}$, and suppose that the variables which occur along the path q are $x = x_0, x_1, ..., x_n = y$, and let $q = q_1 + \cdots + q_n$ be the decomposition of q into single step paths. For each i, let $\mathbb{B}_i = \mathbb{B} + q_1 + \cdots + q_i$, and let p_i be a single step path from x_i to y (which exists by $(2,3)$-minimality of $\mathbf{X}$).

We prove by induction on i that $\mathbb{B} + p \subseteq \mathbb{B}_i + p_i$. For the inductive step, we need to show that $\mathbb{B}_i + p_i \subseteq \mathbb{B}_{i+1} + p_{i+1}$, that is, $\mathbb{B}_i + p_i \subseteq \mathbb{B}_i + q_{i+1} + p_{i+1}$. This follows from the fact that there is some constraint whose scope contains the three variables x_i, x_{i+1}, and y, together with the fact that the two-variable projections of this constraint onto pairs of variables from $\{x_i, x_{i+1}, y\}$ agree with the binary relations $\mathbb{P}_{p_i}, \mathbb{P}_{q_{i+1}}, \mathbb{P}_{p_{i+1}}$, by $(2,3)$-minimality. □

The relationships between the various types of consistency introduced so far are summarized in the Hasse diagram below (to see that singleton arc-consistency is not implied by $(2,3)$-minimality, consider the four-variable instance of 1-IN-3 SAT where every group of three variables is required to satisfy the 1-IN-3 constraint).



When our instance consists of just a single relation $\mathbb{R}$ (with no repeated variables), all of these consistency conditions become equivalent to subdirectness of the relation $\mathbb{R}$. Studying this very special case is what leads to the main algebraic ingredient we will need for the conservative CSP dichotomy.

Definition 4.5.15. If $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, and if $\mathbb{B}_i \le \mathbb{A}_i$ for all i, then an $(\mathbb{R}, \mathbb{B})$-*strand* is an equivalence class of the quasiorder $\preceq$ on $[n]$ which is the transitive closure of

$$\mathbb{B}_i + \pi_{ij}(\mathbb{R}) = \mathbb{B}_j \quad \implies \quad i \preceq j.$$

The equivalence classes of this quasiorder are the same as the equivalence classes of the more permissive quasiorder

$$i \preceq' j \quad \iff \quad \mathbb{B}_i + \pi_{ij}(\mathbb{R}) \subseteq \mathbb{B}_j,$$

and these quasiorders are the same in the special case where $\mathbb{B}_i$ is a minimal absorbing subalgebra of $\mathbb{A}_i$ for each i.

Using either the fact that the instance consisting of just $\mathbb{R}$ is cycle-consistent, or the fact that it satisfies (P1) and (P2), we have the following result.

Proposition 4.5.16. *If $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$, and if $\mathbb{B}_i \le \mathbb{A}_i$ for all i, then for each $(\mathbb{R}, \mathbb{B})$-strand $S \subseteq [n]$ there is a congruence $\theta_i \in \mathrm{Con}(\mathbb{A}_i)$ for each $i \in S$ such that each $\mathbb{B}_i$ is a union of congruence classes of θ_i, and for any $i, j \in S$, the binary relation*

$$\pi_{ij}(\mathbb{R})/(\theta_i \times \theta_j) \le_{sd} \mathbb{A}_i/\theta_i \times \mathbb{A}_j/\theta_j$$

is the graph of an isomorphism.

The algebraic input needed for conservative CSPs is to show that if we take the $\mathbb{B}_i$s to be minimal absorbing subalgebras of the $\mathbb{A}_i$s such that $\mathbb{R} \cap (\mathbb{B}_1 \times \cdots \times \mathbb{B}_n) \ne \emptyset$, then the $(\mathbb{R}, \mathbb{B})$-strands do not interact with each other. This algebraic miracle is a special property of conservative Taylor algebras - it doesn't seem to hold in general.

4.6 The rectangularity theorem for conservative Taylor algebras

There are two versions of the Rectangularity Theorem for conservative algebras: one uses the theory of absorbing subalgebras and is proved in [9], while the other uses Bulatov's theory of affine-semilattice components (shortened to *as-components*) of the colored graph and is proved in [46]. Generally speaking, the affine-semilattice components of conservative Taylor algebras (and of subdirect products of conservative Taylor algebras) tend to behave like minimal absorbing subalgebras. We will mainly focus on the absorbing subalgebra approach.

First we will state both versions of the Rectangularity Theorem, before diving into the proofs.

Theorem 4.6.1 (Rectangularity Theorem for conservative Taylor algebras, absorbing version [9])**.** *If $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is a subdirect product of conservative Taylor algebras $\mathbb{A}_i$, and if a system of minimal absorbing subalgebras $\mathbb{B}_i \lll \mathbb{A}_i$ for each i satisfies*

$$\mathbb{R} \cap (\mathbb{B}_1 \times \cdots \times \mathbb{B}_n) \neq \emptyset,$$

then

$$\mathbb{R} \cap (\mathbb{B}_1 \times \cdots \times \mathbb{B}_n) = \prod_{S \text{ an } (\mathbb{R},\mathbb{B})\text{-strand}} \Big(\pi_S(\mathbb{R}) \cap \prod_{i\in S} \mathbb{B}_i\Big).$$

The choice of absorption concept used in the Rectangularity Theorem is fairly arbitrary - we could use Jónsson absorption or central absorption (as long as the algebras in question are minimal Taylor) instead, and it would still be true. We will use $\lhd_X$ in this section to refer to any choice of an absorption concept as in Section 3.9 for which the Absorption Theorem 3.11.1 applies (so if we take $\lhd_X$ to be central absorption, then we need to assume that we are in a context where binary absorption implies central absorption, such as the context of minimal Taylor algebras).

Definition 4.6.2. If $\mathbb{A}$ is a subdirect product of conservative minimal Taylor algebras, then we define a quasiorder on the elements of $\mathbb{A}$ by $a \preceq_{as} b$ if there is a sequence $a = a_0, a_1, ..., a_n = b$ such that for each i either $\{a_i, a_{i+1}\}$ is a $\mathbb{Z}/2^{\text{aff}}$-subalgebra or $a_i \to a_{i+1}$.

We say that a subset $B \subseteq \mathbb{A}$ is an *as-component* of $\mathbb{A}$ if B is a maximal equivalence class of the quasiorder $\preceq_{as}$. We say that $\mathbb{A}$ is *strongly as-connected* if $\mathbb{A}$ consists of a single equivalence class of the quasiorder $\preceq_{as}$.

Theorem 4.6.3 (Rectangularity Theorem for conservative Taylor algebras, as-component version [46])**.** *If $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is a subdirect product of conservative minimal Taylor algebras $\mathbb{A}_i$, and if a system of as-components $\mathbb{B}_i \subseteq \mathbb{A}_i$ for each i satisfies*

$$\mathbb{R} \cap (\mathbb{B}_1 \times \cdots \times \mathbb{B}_n) \neq \emptyset,$$

then

$$\mathbb{R} \cap (\mathbb{B}_1 \times \cdots \times \mathbb{B}_n) = \prod_{S \text{ an } (\mathbb{R},\mathbb{B})\text{-strand}} \Big(\pi_S(\mathbb{R}) \cap \prod_{i\in S} \mathbb{B}_i\Big).$$

We will frequently use the following consequence of the Absorption Theorem 3.11.1 throughout the proof. Note that every strongly as-connected algebra is certainly absorption-free, so it applies in that context as well (it's a good exercise to give a direct proof of the analogue for strongly as-connected algebras, along the lines of Theorem 3.3.1 - if you can't solve it, see [46]).

Proposition 4.6.4. *Let $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \mathbb{A}_2$ be a subdirect product of absorption-free idempotent algebras, and let θ_1 be a maximal congruence on $\mathbb{A}_1$. Then either θ_1 contains the linking congruence of $\mathbb{R}$ on $\mathbb{A}_1$, in which case there is a maximal congruence θ_2 on $\mathbb{A}_2$ such that*

$$\mathbb{R}/(\theta_1 \times \theta_2) \le \mathbb{A}_1/\theta_1 \times \mathbb{A}_2/\theta_2$$

is the graph of an isomorphism from $\mathbb{A}_1/\theta_1 \xrightarrow{\sim} \mathbb{A}_2/\theta_2$, or else we have

$$(a/\theta_1) + \mathbb{R} = \mathbb{A}_2$$

for each congruence class a/θ_1 of θ_1.

Proof. This is a restatement of Corollary 3.11.4. □

The next lemma is one of the key places where conservativity is really used in the argument. Its analogue for as-components of conservative minimal Taylor algebras (replacing "absorption-free" with "strongly as-connected") is an easy exercise.

Lemma 4.6.5 (Barto [9])**.** *If $\mathbb{A}$ is an absorption-free conservative algebra, θ is a proper congruence on $\mathbb{A}$, and $\mathbb{B} \le \mathbb{A}$ is any subalgebra such that $\mathbb{B}$ has at least one element from each congruence class of θ, then $\mathbb{B}$ is also absorption-free.*

Proof. Suppose for contradiction that $\mathbb{C} \lhd_X \mathbb{B}$, with $\mathbb{C} \ne \mathbb{B}$. Pick some $b \in \mathbb{B} \setminus \mathbb{C}$ such that $\mathbb{C} \not\subseteq b/\theta$ (this is possible as long as θ has at least two congruence classes), and let $\mathbb{B}'$ be $(\mathbb{B} \setminus (b/\theta)) \cup \{b\}$, that is, $\mathbb{B}'$ is the subalgebra of $\mathbb{B}$ formed by removing every element which is congruent to b other than b itself (that $\mathbb{B}'$ is a subalgebra follows from the fact that $\mathbb{B}$ is conservative). Then since $\lhd_X$ is compatible with pp-formulas, if we set $\mathbb{C}' = \mathbb{C} \cap \mathbb{B}'$, we have $\mathbb{C}' \lhd_X \mathbb{B}'$. Applying compatibility with pp-formulas again, we have

$$\mathbb{C}'/\theta \lhd_X \mathbb{B}'/\theta = \mathbb{A}/\theta,$$

and by the construction of $\mathbb{C}'$ we see that $\mathbb{C}'/\theta \ne \mathbb{A}/\theta$, since $b/\theta \notin \mathbb{C}'/\theta$. Therefore $\mathbb{A}$ has a proper absorbing subalgebra (i.e. the preimage of $\mathbb{C}'/\theta$ under the quotient homomorphism $\mathbb{A} \twoheadrightarrow \mathbb{A}/\theta$), which is a contradiction. □

The next result is also specific to conservative algebras (see Example 3.3.1 for a counterexample in the 2-semilattice case). An analogue for strongly as-connected algebras can be proved using the same argument (see [46]).

Theorem 4.6.6 (Barto [9])**.** *If $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n$ is a subdirect product of absorption-free conservative algebras, then $\mathbb{R}$ is also absorption-free.*

Proof. We induct on n and on the sizes of the $\mathbb{A}_i$s. Suppose that $\mathbb{S} \lhd_X \mathbb{R}$ - we aim to prove that $\mathbb{S} = \mathbb{R}$. By compatibility with pp-formulas we have $\pi_i(\mathbb{S}) \lhd_X \pi_i(\mathbb{R}) = \mathbb{A}_i$ for each i, so since the $\mathbb{A}_i$s are absorption-free $\mathbb{S}$ is also a subdirect product of the $\mathbb{A}_i$s.

Let θ_1 be a maximal congruence on $\mathbb{A}_1$. By Proposition 4.6.4, for each i either

- there is a maximal congruence $\theta_i \in \mathrm{Con}(\mathbb{A}_i)$ such that $\pi_{1i}(\mathbb{R})/\theta_1 \times \theta_i$ is the graph of an isomorphism between $\mathbb{A}_1/\theta_1$ and $\mathbb{A}_i/\theta_i$, or
- we have $a/\theta_1 + \pi_{1i}(\mathbb{R}) = \mathbb{A}_i$ for all congruence classes a/θ_1 of θ_1.

Rearrange the coordinates so that $\mathbb{A}_1, ..., \mathbb{A}_k$ are in the first case, with corresponding maximal congruences θ_i, while $\mathbb{A}_{k+1}, ..., \mathbb{A}_n$ are in the second case. Define a congruence θ on the product by

$$\theta = \theta_1 \times \cdots \times \theta_k \times 0_{\mathbb{A}_{k+1}} \times \cdots \times 0_{\mathbb{A}_n} \in \mathrm{Con}(\mathbb{A}_1 \times \cdots \times \mathbb{A}_n).$$

First suppose that θ is the trivial congruence (i.e. each $\theta_i = 0_{\mathbb{A}_i}$). In this case each $\mathbb{A}_i$ with $1 < i \le k$ is a redundant coordinate (since $\pi_{1i}(\mathbb{R})$ is the graph of an isomorphism between $\mathbb{A}_1$ and $\mathbb{A}_i$), so we can assume without loss of generality that $k = 1$. Let a be any element of $\mathbb{A}$, and consider the relations $\mathbb{S}_a, \mathbb{R}_a$ given by

$$\mathbb{R}_a = \{(x_2, ..., x_n) \mid (a, x_2, ..., x_n) \in \mathbb{R}\} \le \mathbb{A}_2 \times \cdots \times \mathbb{A}_n,$$

with $\mathbb{S}_a$ defined similarly. By compatibility with pp-formulas, we have $\mathbb{S}_a \lhd_X \mathbb{R}_a$, and since $a + \pi_{1i}(\mathbb{R}) = \mathbb{A}_i$ for each $i \ge 2$, $\mathbb{R}_a$ is a subdirect product of $\mathbb{A}_2, ..., \mathbb{A}_n$. By the induction hypothesis, we then have $\mathbb{S}_a = \mathbb{R}_a$, and since this is true for every $a \in \mathbb{A}$, we have $\mathbb{S} = \mathbb{R}$.

Now suppose that θ is nontrivial - suppose without loss of generality that θ_1 is a nontrivial congruence of $\mathbb{A}_1$, with $(a, a') \in \theta_1$ for some $a \ne a'$. Let $\mathbb{B} = \mathbb{A} \setminus \{a\}$, and let $\mathbb{B}' = \mathbb{A} \setminus \{a'\}$. By Lemma 4.6.5, each of $\mathbb{B}, \mathbb{B}'$ is absorption-free. Define $\mathbb{R}_{\mathbb{B}}$ by

$$\mathbb{R}_{\mathbb{B}} = \{(x_2, ..., x_n) \mid \exists b \in \mathbb{B} \ \ (b, x_2, ..., x_n) \in \mathbb{R}\} \le \mathbb{A}_2 \times \cdots \times \mathbb{A}_n,$$

and similarly define $\mathbb{S}_{\mathbb{B}}$, $\mathbb{R}_{\mathbb{B}'}, \mathbb{S}_{\mathbb{B}'}$. We have $\mathbb{S}_{\mathbb{B}} \lhd_X \mathbb{R}_{\mathbb{B}}$ and $\mathbb{S}_{\mathbb{B}'} \lhd_X \mathbb{R}_{\mathbb{B}'}$ by compatibility with pp-formulas. For each $i \le k$, we have

$$\pi_i(\mathbb{R}_{\mathbb{B}})/\theta_i = \pi_i(\mathbb{R}_{\mathbb{B}}/\theta) = \mathbb{B}/\theta_1 + \pi_{1i}(\mathbb{R}/\theta) = \mathbb{A}_1/\theta_1 + \pi_{1i}(\mathbb{R}/\theta) = \mathbb{A}_i/\theta_i,$$

so by Lemma 4.6.5, $\pi_i(\mathbb{R}_{\mathbb{B}})$ is absorption-free for each $i \le k$. For $i > k$, we have

$$\pi_i(\mathbb{R}_{\mathbb{B}}) = \mathbb{B} + \pi_{1i}(\mathbb{R}) = \mathbb{A}_i,$$

since $\mathbb{B}$ contains at least one full congruence class of θ_1. Thus for each $i \in [n]$, $\pi_i(\mathbb{R}_{\mathbb{B}})$ is absorption-free, so since $|\mathbb{B}| < |\mathbb{A}_1|$ we can apply the induction hypothesis to see that $\mathbb{S}_{\mathbb{B}} = \mathbb{R}_{\mathbb{B}}$. A similar argument shows that $\mathbb{S}_{\mathbb{B}'} = \mathbb{R}_{\mathbb{B}'}$, and since $\mathbb{B} \cup \mathbb{B}' = \mathbb{A}_1$ we see that $\mathbb{S} = \mathbb{R}$. □

In order to get a foothold into the Rectangularity Theorem, we start with the case of binary relations. We will only assume that one of the algebras involved is conservative - this way we can later apply the result with the other algebra equal to a larger subdirect product of conservative algebras. Once again, an analogous argument works for as-components (we need a version of Theorem 3.3.1 for as-components to push the argument through, see [46]).

Lemma 4.6.7 (Barto [9]). *Suppose $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \mathbb{A}_2$, where $\mathbb{A}_1$ is conservative, $\mathbb{A}_2$ is idempotent, and both are finite Taylor algebras. Suppose further that $\mathbb{B}_i \lll_X \mathbb{A}_i$ for $i = 1, 2$, that $\mathbb{R} \cap (\mathbb{B}_1 \times \mathbb{B}_2) \ne \emptyset$, and that there is some $(a, b) \in \mathbb{R}$ with $a \in \mathbb{A}_1 \setminus \mathbb{B}_1$ and $b \in \mathbb{B}_2$. Then $\mathbb{B}_1 \times \mathbb{B}_2 \subseteq \mathbb{R}$.*

Proof. Since $(\mathbb{B}_1 + \mathbb{R}) \cap \mathbb{B}_2$ is a nonempty absorbing subalgebra of $\mathbb{B}_2$ and $\mathbb{B}_2$ is absorption-free, we have $\mathbb{B}_1 + \mathbb{R} \supseteq \mathbb{B}_2$, and similarly $\mathbb{B}_1 \subseteq \mathbb{B}_2 - \mathbb{R}$. Thus $\mathbb{R} \cap (\mathbb{B}_1 \times \mathbb{B}_2)$ is subdirect in $\mathbb{B}_1 \times \mathbb{B}_2$. Additionally, $\mathbb{B}_1 \cup \{a\}$ is a subalgebra of $\mathbb{A}_1$ since $\mathbb{A}_1$ is conservative, so by the assumption $a \in \mathbb{B}_2 - \mathbb{R}$ we can assume without loss of generality that $\mathbb{A}_1 = \mathbb{B}_1 \cup \{a\}$ and $\mathbb{A}_2 = \mathbb{B}_2$.

If $\mathbb{R}$ is linked, then by Theorem 3.7.12 so is $\mathbb{R} \cap (\mathbb{B}_1 \times \mathbb{B}_2)$, and then by the Absorption Theorem 3.11.1 we have $\mathbb{B}_1 \times \mathbb{B}_2 \subseteq \mathbb{R}$. Otherwise, the linking congruence of $\mathbb{R}$ is a proper congruence θ_1 on $\mathbb{A}_1$. If $c \in (\{b\} - \mathbb{R}) \cap \mathbb{B}_1$, then we have $(a, c) \in \theta_1$, so $\mathbb{A}_1/\theta_1 = \mathbb{B}_1/\theta_1$. Let

$$\mathbb{A}_1' = (\mathbb{A}_1 \setminus (a/\theta_1)) \cup \{a\},$$

then by compatibility with pp-formulas we have

$$\mathbb{B}_1 \setminus (a/\theta_1) = \mathbb{B}_1 \cap \mathbb{A}_1' \lhd_X \mathbb{A}_1',$$

so

$$(\mathbb{B}_1/\theta_1) \setminus (a/\theta_1) \lhd_X \mathbb{A}_1'/\theta_1 = \mathbb{B}_1/\theta_1,$$

which implies that $\mathbb{B}_1 \setminus (a/\theta_1) \lhd_X \mathbb{B}_1$, which is a contradiction. □

Now we bootstrap our way up.

Lemma 4.6.8 (Barto [9])**.** *Suppose $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_n \times \mathbb{A}_{n+1}$, where $\mathbb{A}_1, ..., \mathbb{A}_n$ are conservative, $\mathbb{A}_{n+1}$ is idempotent, and each $\mathbb{A}_i$ is a finite Taylor algebra. Suppose that we have $\mathbb{B}_i \lll_X \mathbb{A}_i$ for all $i \in [n+1]$, that*

$$\mathbb{R} \cap (\mathbb{B}_1 \times \cdots \times \mathbb{B}_n \times \mathbb{B}_{n+1}) \neq \emptyset,$$

that $[n]$ is an $(\mathbb{R}, \mathbb{B})$-strand, and that there is some $(a_1, ..., a_n, b_{n+1}) \in \mathbb{R}$ such that $a_i \in \mathbb{A}_i \setminus \mathbb{B}_i$ for $i \in [n]$ while $b_{n+1} \in \mathbb{B}_{n+1}$. Then we have

$$\mathbb{R} \cap (\mathbb{B}_1 \times \cdots \times \mathbb{B}_n \times \mathbb{B}_{n+1}) = (\pi_{[n]}(\mathbb{R}) \cap (\mathbb{B}_1 \times \cdots \times \mathbb{B}_n)) \times \mathbb{B}_{n+1}.$$

Proof. We induct on n and on the sizes of the $\mathbb{A}_i$s, and note that we have already proved the case $n = 1$ in the previous lemma. By Proposition 4.5.16 we can find maximal congruences $\theta_i \in \mathrm{Con}(\mathbb{A}_i)$ for $i \in [n]$ such that

$$\pi_{ij}(\mathbb{R})/(\theta_i \times \theta_j) \le \mathbb{A}_i/\theta_i \times \mathbb{A}_j/\theta_j$$

is the graph of an isomorphism for all $i, j \in [n]$. If any θ_i is trivial (i.e. if $\mathbb{A}_i$ is simple for some $i \in [n]$) then the ith coordinate of $\mathbb{R}$ is redundant, so we can apply the induction hypothesis. Otherwise, we have $\theta_1 \neq 0_{\mathbb{A}_1}$.

Let b_1 be any element of $\mathbb{B}_1$. Let $\mathbb{A}_1'$ be any proper subalgebra of $\mathbb{A}_1$ such that $b_1, a_1 \in \mathbb{A}_1'$ and such that $\mathbb{A}_1'/\theta_1 = \mathbb{A}_1/\theta_1$, and let $\mathbb{B}_1' = \mathbb{B}_1 \cap \mathbb{A}_1'$. For each $i \in [n+1]$ define $\mathbb{A}_i', \mathbb{B}_i'$ by

$$\mathbb{A}_i' = \mathbb{A}_1' + \pi_{1i}(\mathbb{R}), \quad \mathbb{B}_i' = \mathbb{B}_i \cap \mathbb{A}_i'.$$

Then since $\pi_{1i}(\mathbb{R})/(\theta_1 \times \theta_i)$ is the graph of an isomorphism for each $i \in [n]$, we have $\mathbb{A}_i'/\theta_i = \mathbb{A}_i/\theta_i$ and $\mathbb{B}_i' = \mathbb{B}_1' + \pi_{1i}(\mathbb{R})$ for all $i \in [n]$. Then since each $\mathbb{B}_i'/\theta_i = \mathbb{B}_i/\theta_i$, we can apply Lemma 4.6.5 to see that $\mathbb{B}_i'$ is absorption-free for each $i \in [n]$. Additionally, by the previous lemma (i.e., the $n = 1$ case) we have

$$\mathbb{B}_1 \times \mathbb{B}_{n+1} \subseteq \pi_{1,n+1}(\mathbb{R}),$$

so $\mathbb{B}_{n+1}' = \mathbb{B}_{n+1}$. Thus if we set

$$\mathbb{R}' = \mathbb{R} \cap (\mathbb{A}_1' \times \cdots \times \mathbb{A}_n' \times \mathbb{A}_{n+1}'),$$

then $(a_1, ..., a_n, b_{n+1}) \in \mathbb{R}'$ and we can apply the induction hypothesis to $\mathbb{R}'$ to see that

$$\mathbb{R}' \cap (\mathbb{B}_1' \times \cdots \times \mathbb{B}_n' \times \mathbb{B}_{n+1}) = (\pi_{[n]}(\mathbb{R}') \cap (\mathbb{B}_1' \times \cdots \times \mathbb{B}_n')) \times \mathbb{B}_{n+1}.$$

In particular, any tuple in $(\pi_{[n]}(\mathbb{R}) \cap \prod_{i \in [n]} \mathbb{B}_i) \times \mathbb{B}_{n+1}$ such that the first coordinate is b_1 is contained in $\mathbb{R}'$, and therefore is also contained in $\mathbb{R}$. Since b_1 was an arbitrary element of $\mathbb{B}_1$, we are done. □

Proof of the Rectangularity Theorem 4.6.1, following [9]. We induct on n, the number of algebras occuring in the product. Suppose for the sake of contradiction that $\mathbb{R}$ is a counterexample, i.e. that there is some

$$b = (b_1, ..., b_n) \in \prod_{S \text{ an } (\mathbb{R},\mathbb{B})\text{-strand}} \Big(\pi_S(\mathbb{R}) \cap \prod_{i \in S} \mathbb{B}_i\Big)$$

such that $b \not\in \mathbb{R}$. By the induction hypothesis, we have $\pi_{[n]\setminus\{i\}}(b) \in \pi_{[n]\setminus\{i\}}(\mathbb{R})$ for each $i \in [n]$.

Consider the quasiorder $\preceq$ on $[n]$ from Definition 4.5.15 defined by

$$i \preceq j \iff \mathbb{B}_i + \pi_{ij}(\mathbb{R}) = \mathbb{B}_j,$$

and suppose without loss of generality that $[k]$ is a $\preceq$-minimal $(\mathbb{R}, \mathbb{B})$-strand for some $k \le n$. If there is any tuple

$$(a_1, ..., a_k, b'_{k+1}, ..., b'_n) \in \mathbb{R}$$

such that $a_i \in \mathbb{A}_i \setminus \mathbb{B}_i$ for $i \in [k]$ and $b'_j \in \mathbb{B}_j$ for $j > k$, then we can apply the previous lemma to the situation

$$\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_k \times \pi_{[k+1,n]}(\mathbb{R})$$

with

$$\pi_{[k+1,n]}(\mathbb{R}) \cap \prod_{j>k} \mathbb{B}_j \lll_X \pi_{[k+1,n]}(\mathbb{R}),$$

by compatibility with pp-formulas and Theorem 4.6.6, to finish the proof. To arrange for this situation, we consider the relation

$$\begin{aligned}\mathbb{R}' &= \mathbb{R} \cap \Big(\Big(\prod_{i\in[k]} (\mathbb{A}_i \setminus \mathbb{B}_i) \cup \{b_i\}\Big) \times \prod_{j>k} \mathbb{A}_j\Big),\\ &= \mathbb{R} \cap \Big(\Big(\{(b_1, ..., b_k)\} \cup \prod_{i\in[k]} (\mathbb{A}_i \setminus \mathbb{B}_i)\Big) \times \prod_{j>k} \mathbb{A}_j\Big),\end{aligned}$$

set $\mathbb{A}'_i = \pi_i(\mathbb{R}')$, and set $\mathbb{B}'_i = \mathbb{A}'_i \cap \mathbb{B}_i$ for each i. By the induction hypothesis applied to $\pi_{[k]\cup\{j\}}(\mathbb{R})$, we see that $\mathbb{B}'_j = \mathbb{B}_j$ for each $j > k$, so we have $\mathbb{B}'_i \lll_X \mathbb{A}'_i$ for all $i \le n$. We will apply the induction hypothesis to $\pi_{[k+1,n]}(\mathbb{R}')$, but first we need to check that $\pi_{[k+1,n]}(\mathbb{R}')$ has more than one strand.

Let $S \subset [n]$ be some $(\mathbb{R}, \mathbb{B})$-strand which is disjoint from $[k]$. By the assumption that $[k]$ was a $\preceq$-minimal $(\mathbb{R}, \mathbb{B})$-strand, there must be some $(c_1, ..., c_n) \in \mathbb{R}$ such that $c_i \not\in \mathbb{B}_i$ for $i \in [k]$ and $c_j \in \mathbb{B}_j$ for $j \in S$. Suppose without loss of generality that there is some $m \ge k$ such that $c_j \in \mathbb{B}_j$ iff $j \in [m + 1, n]$. If $m = k$, then we take $(a_1, ..., a_k, b'_{k+1}, ..., b'_n) = c$ to finish. Otherwise, since c is an element of $\mathbb{R}'$, we see that every strand of $\pi_{[k+1,n]}(\mathbb{R}')$ is either contained in $[k + 1, m]$ or contained in $[m + 1, n]$. Thus, by the induction hypothesis we have

$$\pi_{[k+1,n]}(\mathbb{R}') \cap \prod_{j>k} \mathbb{B}_j = \Big(\pi_{[k+1,m]}(\mathbb{R}') \cap \prod_{j\in[k+1,m]} \mathbb{B}_j\Big) \times \Big(\pi_{[m+1,n]}(\mathbb{R}') \cap \prod_{j\in[m+1,n]} \mathbb{B}_j\Big).$$

Since we have $\pi_{[k+1,m]}(b) \in \pi_{[k+1,m]}(\mathbb{R}')$ and $\pi_{[m+1,n]}(b) \in \pi_{[m+1,n]}(\mathbb{R}')$ by the induction hypothesis applied to $\pi_{[1,m]}(\mathbb{R})$ and $\pi_{[k]\cup[m+1,n]}(\mathbb{R})$, we see that $\pi_{[k+1,n]}(b) \in \pi_{[k+1,n]}(\mathbb{R}')$. Thus either $b \in \mathbb{R}$, or there are some $a_i \in \mathbb{A}_i \setminus \mathbb{B}_i$ for $i \in [k]$ such that

$$(a_1, ..., a_k, b_{k+1}, ..., b_n) \in \mathbb{R},$$

which allows us to apply the previous lemma to finish the proof. □

4.7 The algorithm for conservative CSPs

In this section we present Barto's simple algorithm from [9]. Bulatov's algorithm from [46] is similar in spirit, but it relies on ideas from Maróti's "Tree on top of Malcev" algorithm [140] which we haven't covered yet.

The main idea of Barto's algorithm for conservative CSPs is to try to reduce to the case where all edges of the colored graphs occuring in each of the variable domains $\mathbb{A}_x$ are affine. In this case, any daisy chain term will be a Mal'cev term for each variable domain, and we can solve the problem by using the algorithm for CSPs with a Mal'cev polymorphism. In Bulatov's algorithm from [46], the main idea is to reduce to the case where there are no semilattice edges instead, in which case any daisy chain term will be a ternary generalized majority-minority operation (that a ternary generalized majority-minority operation exists in this case also follows from Theorem 2.1.5).

In order to accomplish this, we aim to show that if any semilattice or majority edge occurs in any variable domain of an instance $\mathbf{X}$, then we can reduce some variable domain by solving an instance where every variable domain has been strictly decreased. We assume that our instance is $(2,3)$-minimal (or perhaps just that it is a Prague instance), and we consider the subdigraph $\mathcal{E}$ of the implication digraph consisting of pairs $(x,\mathbb{B})$ such that that $\mathbb{B} \le \mathbb{A}_x$ is an algebra with at least one proper absorbing subalgebra. The digraph $\mathcal{E}$ will be nonempty as long as any algebra $\mathbb{A}_x$ has any non-affine edge. We pick any maximal strand $\mathcal{S}$ of the digraph $\mathcal{E}$, and we note that the full restriction of our instance $\mathbf{X}$ to the strand $\mathcal{S}$ is then a Prague instance in which every single domain has a proper absorbing subalgebra by Proposition 4.5.12. We can then repeatedly apply Proposition 4.5.12 (or, alternatively, we could apply Kozik's [127] result from Section 3.9, using the fact that every Prague instance is pq-consistent), to find an arc-consistent absorbing reduction $\mathbf{X}'$ of the full restriction of $\mathbf{X}$ to the strand $\mathcal{S}$, such that every variable domain in $\mathbf{X}'$ is absorption free - and as a consequence, such that each variable domain in $\mathbf{X}'$ is a proper subagebra of the corresponding variable domain in the original instance $\mathbf{X}$. We can then apply the following consequence of the Rectangularity Theorem 4.6.1.

Theorem 4.7.1 (Barto [9]). *Suppose that $\mathbf{X}$ is a Prague instance such that each variable domain $\mathbb{A}_x$ is a conservative Taylor algebra. Let $\mathcal{E}$ be the subdigraph of the implication digraph which consists of pairs $(x,\mathbb{B})$ such that $\mathbb{B} \le \mathbb{A}_x$ and $\mathbb{B}$ has a proper absorbing subalgebra, and let $\mathcal{S}$ be any maximal strand of $\mathcal{E}$.*

Suppose that for each $(x,\mathbb{B}_x) \in \mathcal{S}$ we choose a minimal absorbing subalgebra $\mathbb{C}_x \lll \mathbb{B}_x$, such that the system of variable domains $\mathbb{C}_x$ defines an arc-consistent reduction of the full restriction of $\mathbf{X}$ to the strand $\mathcal{S}$. Then one of the following is true:

- *the instance $\mathbf{X}$ has no solutions with any variable x assigned to any value in $\mathbb{B}_x$, for any $(x,\mathbb{B}_x) \in \mathcal{S}$,*

- *the instance $\mathbf{X}'$ which we get by restricting to the variables in $\mathcal{S}$ and by restricting each variable domain to the corresponding $\mathbb{C}_x$ has a solution, and every solution of $\mathbf{X}'$ extends to a solution of $\mathbf{X}$, or*

- *for some strand $\mathcal{T}$ of the subqoset $\mathcal{C}$ of $\mathcal{E}$ consisting of the pairs $(x,\mathbb{C}_x)$ for x occuring in $\mathcal{S}$, the full restriction of $\mathbf{X}$ to $\mathcal{T}$ has no solutions.*

Proof. Suppose that $a \in \prod_x \mathbb{A}_x$ is a solution to the instance $\mathbf{X}$ such that $a_x \in \mathbb{B}_x$ for some $(x,\mathbb{B}_x) \in \mathcal{S}$, and suppose that for each strand $\mathcal{T}$ of the subqoset $\mathcal{C}$ there is a solution $c^{\mathcal{T}} \in \prod_{(x,\mathbb{C}_x)\in\mathcal{T}} \mathbb{C}_x$

of the full restriction of $\mathbf{X}$ to the strand $\mathcal{T}$. We construct a tuple $b \in \prod_x \mathbb{A}_x$ by stitching these solutions together:

$$b_x = \begin{cases} a_x & x \text{ does not occur in the strand } \mathcal{S}, \\ c_x^{\mathcal{T}} & x \text{ occurs in the strand } \mathcal{T} \text{ of the subqoset } \mathcal{C}. \end{cases}$$

We claim that the tuple b is also a solution to the instance $\mathbf{X}$. For this, we can focus our attention on any particular constraint relation

$$\mathbb{R} \le_{sd} \mathbb{A}_{x_1} \times \cdots \times \mathbb{A}_{x_n}$$

of the instance $\mathbf{X}$. Suppose without loss of generality that the variables $x_1, ..., x_k$ occur in the strand $\mathcal{S}$ and that the variables $x_{k+1}, ..., x_n$ do not, and suppose that $k \ge 1$. For each $i \in [n]$, define $\mathbb{B}_i$ by

$$\mathbb{B}_i = \mathbb{B}_{x_1} + \pi_{1i}(\mathbb{R}).$$

Since $\mathbf{X}$ is a Prague instance, we have $\mathbb{B}_i = \mathbb{B}_{x_i}$ for all $i \le k$, and since $\mathcal{S}$ is a maximal strand of $\mathcal{E}$ each $\mathbb{B}_j$ with $j \ge k+1$ is absorption-free. Define the relation $\mathbb{R}_{\mathbb{B}} \le \mathbb{R}$ by

$$\mathbb{R}_{\mathbb{B}} = \mathbb{R} \cap \left(\prod_i \mathbb{B}_i \right) \le_{sd} \mathbb{B}_1 \times \cdots \times \mathbb{B}_n,$$

where subdirectness follows directly from the definition of the $\mathbb{B}_i$s. Set $\mathbb{C}_i = \mathbb{C}_{x_i}$ for $i \le k$, and set $\mathbb{C}_j = \mathbb{B}_j$ for $j \ge k+1$, so we have $\mathbb{C}_i <\!\!\lhd \mathbb{B}_i$ for all $i \in [n]$. Then since $\mathbf{X}$ is a Prague instance, the $(\mathbb{R}_{\mathbb{B}}, \mathbb{C})$-strands are given by $[k+1, n]$ and by the intersections of the strands of $\mathcal{C}$ to $\{x_1, ..., x_k\}$, and we have

$$\mathbb{R}_{\mathbb{B}} \cap \prod_i \mathbb{C}_i \ne \emptyset$$

since the system of variable domains $\mathbb{C}_x$ defines an arc-consistent reduction of the full restriction of $\mathbf{X}$ to the strand $\mathcal{S}$. Then the Rectangularity Theorem 4.6.1 says that

$$\mathbb{R} \supseteq \prod_{T \text{ an } (\mathbb{R}_{\mathbb{B}}, \mathbb{C})\text{-strand}} \left(\pi_T(\mathbb{R}_{\mathbb{B}}) \cap \prod_{i \in T} \mathbb{C}_i \right),$$

so the tuple b satisfies the constraint relation $\mathbb{R}$. □

Using this result, we get the following algorithm for solving conservative CSPs.

Theorem 4.7.2. *Algorithm 12 correctly solves every instance $\mathbf{X}$ of any multisorted CSP where each variable domain is a conservative Taylor algebra. If each variable domain has size at most k, then Algorithm 12 runs in time $\|\mathbf{X}\|^{O(k)}$, where $\|\mathbf{X}\|$ is the number of bits needed to describe the instance $\mathbf{X}$.*

Corollary 4.7.3. *The CSP dichotomy conjecture is true for all CSP templates on a domain of size at most 3.*

Proof. By the classification of minimal Taylor algebras of size 3 from Subsection 4.4.1, every minimal Taylor algebra of size at most 3 is either a subdirect product of conservative Taylor algebras, or has a 3-edge term. □

Algorithm 12 Algorithm for solving an instance $\mathbf{X}$ of a CSP with conservative Taylor variable domains $\mathbb{A}_x$, from [9].

```
 1: Run a local consistency algorithm until $\mathbf{X}$ is a cycle-consistent Prague instance.
 2: Let $\mathcal{E}$ be the subdigraph of the implication digraph consisting of pairs $(x, \mathbb{B})$ such that $\mathbb{B} \leq \mathbb{A}_x$ and $\mathbb{B}$ has a proper absorbing subalgebra.
 3: if $\mathcal{E}$ is non-empty then
 4:     Let $\mathcal{S}$ be any maximal strand of $\mathcal{E}$.    ▷ Proposition 4.5.5.
 5:     Define $\mathbb{C}_x = \mathbb{B}_x$ for each $(x, \mathbb{B}_x) \in \mathcal{S}$.
 6:     while some $\mathbb{C}_x$ is not absorption-free do
 7:         Let $\mathbf{X}_{\mathbb{C}}$ be the Prague instance we get by restricting to variables in $\mathcal{S}$ and restricting each variable domain to $\mathbb{C}_x$.    ▷ Proposition 4.5.12
 8:         Pick any maximal strand $\mathcal{T}$ of the subqoset of the implication qoset of $\mathbf{X}_{\mathbb{C}}$ consisting of $(x, \mathbb{C}')$ such that $\mathbb{C}' \lhd \mathbb{C}_x$ and $\mathbb{C}' \neq \mathbb{C}_x$.
 9:         Set $\mathbb{C}_x \leftarrow \mathbb{C}'$ for each $(x, \mathbb{C}') \in \mathcal{T}$.
10:     Let $\mathcal{C}$ be the subqoset consisting of $(x, \mathbb{C}_x)$ for $x$ occuring in the strand $\mathcal{S}$.
11:     for all strands $\mathcal{T}$ of $\mathcal{C}$ do
12:         Let $\mathbf{X}_{\mathcal{T}}$ be the full restriction of $\mathbf{X}$ to the strand $\mathcal{T}$.
13:         Solve the instance $\mathbf{X}_{\mathcal{T}}$ recursively.    ▷ $\mathbb{C}_x < \mathbb{B}_x$ for all $(x, \mathbb{C}_x) \in \mathcal{T}$.
14:         if $\mathbf{X}_{\mathcal{T}}$ has no solutions then
15:             Set $\mathbb{A}_x \leftarrow \mathbb{A}_x \setminus \mathbb{C}_x$ for each $(x, \mathbb{C}_x) \in \mathcal{T}$.
16:             go to Step 1.
17:         else
18:             Let $c^{\mathcal{T}}$ be a solution to the instance $\mathbf{X}_{\mathcal{T}}$.
19:     Set $\mathbb{A}_x \leftarrow (\mathbb{A}_x \setminus \mathbb{B}_x) \cup \{c_x^{\mathcal{T}}\}$ for all $(x, \mathbb{B}_x) \in \mathcal{S}$, where $\mathcal{T}$ is the strand of $\mathcal{C}$ which contains $(x, \mathbb{C}_x)$.    ▷ Theorem 4.7.1
20:     go to Step 1.
21: Solve $\mathbf{X}$ by using the algorithm for CSPs with a Mal'cev polymorphism.    ▷ Section 1.8
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It seems plausible that a much more careful analysis of the algorithm for conservative CSPs might show that it runs in time $\|\mathbf{X}\|^{O(1)}$, regardless of the sizes of the variable domains.

Problem 4.7.1. Given as input an instance $\mathbf{X}$ of any CSP together with a conservative ternary weak near-unanimity polymorphism which preserves the relations of $\mathbf{X}$, can we solve the instance $\mathbf{X}$ in time polynomial in $\|\mathbf{X}\|$?

The method we have been using to encode relations up to this point has been to explicitly list out the tuples contained in the relation. An alternate way of describing constraint relations on the domain $\{0, 1\}$ via "extension oracles" was introduced in [132], and this way of describing constraint relations seems to generalize naturally to conservative CSPs. I will use the phrase "restriction oracle" instead of "extension oracle" for the generalization I have in mind.

Definition 4.7.4. A *restriction oracle* $\mathcal{O}_R$ for a relation $R \subseteq A_1 \times \cdots \times A_n$ is defined as a black-box function which takes as input a tuple of subsets $B_i \subseteq A_i$, and returns "true" if and only if we have

$$R \cap (B_1 \times \cdots \times B_n) \neq \emptyset.$$

The idea behind a restriction oracle is that it is the bare minimum which is needed to be able to run the (generalized) arc-consistency algorithm. It's easy to see how we could use restriction oracle descriptions of constraint relations to establish cycle-consistency (or even singleton arc-consistency), but it is not clear if it is possible to use restriction oracles to establish $(2, 3)$-minimality, or even to reduce to a subinstance which satisfies condition (P2).

Example 4.7.1. A concrete example of a high-arity relation which has an efficient restriction oracle is the *all-different* relation $\bigwedge_{i \neq j} x_i \neq x_j$. This relation occurs naturally in Sudoku and its generalizations. For any sets $B_1, ..., B_n$, we can determine whether or not

$$\{x \mid \forall i \neq j, x_i \neq x_j\} \cap (B_1 \times \cdots \times B_n) \neq \emptyset$$

as follows. We start by drawing a bipartite graph with parts $A = \{1, ..., n\}$ and $B = \bigcup_i B_i$, with an edge from $i \in A$ to $b \in B$ exactly when $b \in B_i$. Then we use the standard augmenting path algorithm to find a maximum matching in this graph - if there is a matching of size n, then the edges of this matching can be viewed as an assignment from variables x_i to values in B_i which are all different.

Problem 4.7.2. Consider the problem where we are given an instance $\mathbf{X}$ of a CSP together with a conservative ternary weak near-unanimity polymorphism which is promised to preserve the relations of $\mathbf{X}$, but instead of having explicit descriptions of the constraint relations, the constraint relations are given to us implicitly in terms of restriction oracles. Is there an algorithm which determines whether $\mathbf{X}$ has a solution and makes only polynomially many calls to the restriction oracles which describe the constraint relations?

When we leave the context of conservative CSPs, restriction oracles become a less natural concept. The trouble is that it's only natural to call the restriction oracle when the sets B_i are subalgebras of the variable domains. The next example shows how this can become an issue for the algebra $\mathbb{Z}/3^{\text{aff}}$.

Example 4.7.2. We can efficiently describe high-arity relations R on $\mathbb{Z}/3^{\text{aff}}$ by writing down systems of linear equations. If we could convert a description of R as the solution set of a system of linear equations into an efficient restriction oracle $\mathcal{O}_R$, however, then we would be able to solve 1-IN-3 SAT. To see this, note that for $x, y, z \in \mathbb{Z}/3^{\text{aff}}$ we have

$$(x, y, z) \in \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\} \iff x, y, z \in \{0, 1\} \wedge x + y + z \equiv 1 \pmod 3.$$

4.8 The meta-problem for conservative CSP templates

In this section we will go over Carbonnel's solution to the meta-problem for conservative CSPs from their thesis [50]. Recall that in the meta-problem, we are given a CSP template as a relational structure $\mathbf{A} = (A, \Gamma)$ (which we usually assume to be a core), and we wish to either prove that $\mathrm{CSP}(\mathbf{A})$ is NP-complete or to find a Taylor polymorphism of $\mathbf{A}$, in time polynomial in the total size $\|\mathbf{A}\|$ of the description of $\mathbf{A}$, which we define as

$$\|\mathbf{A}\| \coloneqq \sum_{R \in \Gamma} |R|.$$

In the meta-problem for conservative CSPs, we restrict our attention to CSP templates where Γ contains the unary relation $A \setminus \{a\}$ for each $a \in A$ (in particular, any such $\mathbf{A}$ is automatically a rigid core). Note that by our classification of conservative minimal Taylor algebras, if a conservative CSP template $\mathbf{A}$ has a Taylor polymorphism, then it has a ternary weak near-unanimity polymorphism, i.e. an idempotent ternary polymorphism w satisfying the identities

$$w(x, x, y) \approx w(x, y, x) \approx w(y, x, x).$$

Furthermore, we can consider the case of 3-conservative CSP templates without any additional difficulty, since any 3-conservative Taylor algebra has a conservative Taylor reduct (by a 3-conservative template, we mean a CSP template such that Γ contains every unary relation of size at most 3). These facts were not known at the time that Carbonnel wrote their thesis, and by using them we can make Carbonnel's algorithm more concrete.

In Carbonnel's thesis [50], the strategy for solving the meta-problem was described as being similar to a treasure hunt (or perhaps a puzzle hunt): we have a sequence of locked boxes, and a single key which opens the first box, such that each box contains the key to opening the next box. Here, the key is a metaphor for the Taylor polymorphism - once we know a Taylor polymorphism, we can use it to solve instances of our CSP. More specifically, the key is a metaphor for a *partial description* of a Taylor polymorphism. In order to see how a partial description of a Taylor polymorphism can make sense, we first describe a useful rephrasing of the meta-problem for a 3-conservative template in terms of a meta-problem for a multisorted CSP template.

Definition 4.8.1. Suppose that $\mathbf{A} = (A, \Gamma)$ is a 3-conservative CSP template. We define the *associated multisorted template* $\mathbf{A}_3$ to have a sort for each subset of A of size at most 3, with two types of relations:

- for each relation $R \in \Gamma$ of arity m, and for every triple of elements $u, v, w \in \mathbb{R}$ (not necessarily distinct), we have a multisorted relation

$$R \cap (\{u_1, v_1, w_1\} \times \cdots \times \{u_n, v_n, w_n\}) \subseteq \{u_1, v_1, w_1\} \times \cdots \times \{u_n, v_n, w_n\},$$

- for every $a, b, c \in A$ (not necessarily distinct), we have the binary inclusion relation

$$\{(a, a), (b, b)\} \subseteq \{a, b\} \times \{a, b, c\}.$$

Note that the size of the associated multisorted template $\mathbf{A}_3$ is bounded by

$$\|\mathbf{A}_3\| \leq \sum_{R \in \Gamma} \sum_{u,v,w \in R} |R| + \sum_{a,b,c \in A} |\{a, b\}| \leq \|\mathbf{A}\|^4 + 2|A|^3.$$

A polymorphism of a multisorted relational structure is defined to be an operation with a different interpretation on each sort of the structure, such that applying the operation componentwise preserves each multisorted relation.

Proposition 4.8.2. *There is a bijection between conservative ternary polymorphisms of* $\mathbf{A}$ *and ternary polymorphisms of the associated multisorted structure* $\mathbf{A}_3$ *which preserves height* 1 *identities.*

Proof. There is an obvious way to convert any conservative ternary polymorphism of $\mathbf{A}$ into a ternary polymorphism of $\mathbf{A}_3$. Conversely, any ternary polymorphism f of $\mathbf{A}_3$ can be stitched together into a (necessarily conservative) ternary polymorphism $\tilde{f}$ of $\mathbf{A}$: the fact that f preserves the binary inclusion relations guarantees that the values of $\tilde{f}$ are well-defined on two-element sets, and for every $u, v, w \in R \in \Gamma$ the fact that f preserves the multisorted relation

$$R \cap (\{u_1, v_1, w_1\} \times \cdots \times \{u_n, v_n, w_n\})$$

guarantees that $\tilde{f}(u, v, w) \in R$. □

So we have reduced the problem of determining whether a given relational structure $\mathbf{A}$ has a conservative Taylor polymorphism to the problem of determining whether the associated multisorted relational structure $\mathbf{A}_3$ has a ternary weak near-unanimity polymorphism. Now we have a way to understand what a partial description of a ternary weak near-unanimity polymorphism on $\mathbf{A}_3$ should be.

Definition 4.8.3. Suppose that $\mathbf{A} = (A, \Gamma)$ is a 3-conservative CSP template, and let $\mathcal{U} \subseteq \mathcal{P}(A)$ be a collection nonempty subsets of A each of size at most 3 which is closed under taking nonempty subsets. We define the multisorted template $\mathbf{A}_{\mathcal{U}}$ to be the template whose sorts are exactly the elements of $\mathcal{U}$, such that for each m-ary relation R of $\mathbf{A}_3$ with at least one coordinate of a sort in $\mathcal{U}$, if $S \subseteq [m]$ is the set of coordinates of R which have a sort in $\mathcal{U}$, then the relation $\pi_S(R)$ is a relation of $\mathbf{A}_{\mathcal{U}}$.

Later we will want to fix certain coordinates of various relations of $\mathbf{A}_3$ to have certain values, before projecting to the coordinates with sorts in $\mathcal{U}$. To reassure ourselves that this will not cause unexpected problems, we have the following result.

Proposition 4.8.4. *Suppose that* $\mathbf{A} = (A, \Gamma)$ *is a* 3*-conservative CSP template. Let* $\mathcal{U} \subseteq \mathcal{V} \subseteq \mathcal{P}(A)$ *be collections of nonempty subsets of* A *each of size at most* 3 *which are closed under taking nonempty subsets. Suppose that* f *is a ternary polymorphism of the multisorted structure* $\mathbf{A}_{\mathcal{U}}$*, and that* R *is an* m*-ary relation of* $\mathbf{A}_{\mathcal{V}}$*. Let* $S \subseteq [m]$ *be the set of coordinates of the relation* R *with sorts from* $\mathcal{U}$*. Then for any* $y \in \pi_{[m]\setminus S}(R)$*, the relation*

$$R_y \coloneqq \{x \mid \exists r \in R \text{ s.t. } \pi_S(r) = x, \pi_{[m]\setminus S}(r) = y\} \subseteq \pi_S(R)$$

is preserved by the polymorphism f*.*

Proof. We just need to check that for every three tuples $u, v, w \in R_y$, we have $f(u, v, w) \in R_y$. If R is one of the binary inclusion relations, then this follows from the fact that f is conservative on the sorts in $\mathcal{U}$ (which folows from the fact that $\mathcal{U}$ is closed under taking nonempty subsets). Otherwise, R originally came from some relation $\tilde{R} \in \Gamma$. Then there are lifts $\tilde{u}, \tilde{v}, \tilde{w} \in \tilde{R}$ such that projecting to the coordinates of $\tilde{R}$ corresponding to S gives us u, v, w, and such that projecting to the coordinates of $\tilde{R}$ corresponding to $[m] \setminus S$ gives us y in each case. Then there is a relation R' of $\mathbf{A}_{\mathcal{U}}$ given by restricting the ith coordinate of $\tilde{R}$ to the set $\{\tilde{u}_i, \tilde{v}_i, \tilde{w}_i\}$ for all i and projecting to the coordinates corresponding to S, and we have

$$f(u, v, w) \in R' \subseteq R_y$$

since f preserves R'. □

In order to find ternary weak near-unanimity polymorphisms of the structure $\mathbf{A}_{\mathcal{U}}$, we use the idea of solving an *indicator instance* (we used this idea once already to solve the meta-problem for bounded width templates - this idea appears to have shown up for the first time in [103]).

Proposition 4.8.5. *If* $\mathbf{B} = (\mathcal{U}, \Gamma)$ *is a multisorted relational structure with sorts* $\mathcal{U}$*, then* $\mathbf{B}$ *has a ternary weak near-unanimity polymorphism* f *iff the following instance of* $\mathrm{CSP}(\mathbf{B})$ *has a solution:*

$$\exists f = \{f_U \in U^{U^3}\}_{U \in \mathcal{U}} \text{ s.t. } \bigwedge_{U \in \mathcal{U}} \Big(\bigwedge_{a \in U} f_U(a,a,a) \in \{a\} \wedge \bigwedge_{a,b \in U} f_U(a,a,b) = f_U(a,b,a) = f_U(b,a,a) \Big)$$
$$\wedge \bigwedge_{R \in \Gamma} \bigwedge_{u,v,w \in R} f(u,v,w) \in R.$$

Now we can describe the treasure hunt algorithm: we iteratively build ternary weak near-unanimity polymorphisms $f_{\mathcal{U}}$ of the multisorted structures $\mathbf{A}_{\mathcal{U}}$ for progressively larger collections $\mathcal{U}$ of subsets of $\mathbb{A}$ of size at most 3. In each step, we add a single new subset V of A to $\mathcal{U}$ to produce a larger collection $\mathcal{V}$.

In order to solve the indicator instance for $\mathbf{A}_{\mathcal{V}}$, we brute force over all possible ternary weak near-unanimity operations on V, and for each one, we check if it extends to a solution to the indicator instance, using the fact that all of the remaining variables have sorts in $\mathcal{U}$. In fact, we don't need to consider *all* weak near-unanimity operations on V - we only need to check the 73 specific operations from the classification of conservative minimal Taylor algebras of size 3 (if V has size 2, we only need to consider 4 possible operations).

Theorem 4.8.6. *If* $\mathbf{A}$ *is a* 3*-conservative relational structure, then Algorithm 13 runs in time polynomial in* $\|\mathbf{A}\|$*, and either correctly determines that* $\mathrm{CSP}(\mathbf{A})$ *is NP-complete or produces a ternary weak near-unanimity polymorphism of* $\mathbf{A}$*.*

Proof. Given what we have already proved, the only thing left to check is that the algorithm runs in polynomial time. The only step which looks dangerous is the step where we apply Algorithm 12, since the degree of the polynomial in the running time of that algorithm depends on the size of the largest sort which shows up as a variable domain. However, we only ever apply Algorithm 12 to multisorted structures where every sort has size at most 3. □

Remark 4.8.1. Algorithm 13 easily generalizes to multisorted CSP templates. Since an arbitrary instance of an unstructured CSP can be thought of as an instance of a multisorted CSP where each variable has a different sort, with only those relations which actually show up in the instance, we can efficiently check whether there is any possible way to interpret each variable domain as a conservative Taylor algebra such that the relations are compatible with the algebraic structure. If we can impose such an algebraic structure, we can apply Algorithm 12 as long as the variable domains are not too large. If we can solve Problem 4.7.1, then we may not even need the restriction on the sizes of the variable domains!

Algorithm 13 Treasure hunt algorithm for solving the meta-problem for a 3-conservative relational structure $\mathbf{A} = (A, \Gamma)$, from [50].

```
Set $n \leftarrow \binom{|A|}{3} + \binom{|A|}{2}$.
Pick a sequence $\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \cdots \subseteq \mathcal{U}_n = \{U \subseteq A \mid |U| \in [3]\}$ such that $\mathcal{U}_0$ is the set of singleton subsets of $A$, each $\mathcal{U}_{i+1}$ contains exactly one more subset of $A$ then $\mathcal{U}_i$, and each $\mathcal{U}_i$ is closed under taking nonempty subsets.
Let $f_{\mathcal{U}_0}$ be the unique ternary polymorphism of $\mathbf{A}_{\mathcal{U}_0}$, given by $f_{\{a\}}(a, a, a) = a$ for all $a \in A$.
for $i \in [n]$ do
    Let $\mathbf{X}$ be the indicator instance from Proposition 4.8.5 for the multisorted structure $\mathbf{A}_{\mathcal{U}_i}$.
    Let $V$ be the new set in $\mathcal{U}_i \setminus \mathcal{U}_{i-1}$.
    for all ternary weak near-unanimity operations $g_V$ on $V$ do
        Let $\mathbf{X}'$ be the instance we get from $\mathbf{X}$ by replacing each variable $f_V(a, b, c)$ of $\mathbf{X}$ with sort $V$ by the constant $g_V(a, b, c)$.
        Solve the instance $\mathbf{X}'$ by using Algorithm 12 with the Taylor operation $f_{\mathcal{U}_{i-1}}$.
        if $\mathbf{X}'$ has a solution then
            Let $f_{\mathcal{U}_i}$ be any solution to $\mathbf{X}'$.
    if $f_{\mathcal{U}_i}$ hasn't been defined then
        return "NP-complete".
Stitch $f_{\mathcal{U}_n}$ into a ternary polymorphism $f$ of $\mathbf{A}$ using Proposition 4.8.2.
return $f$.
```

Appendix A

Commutator theory in congruence modular varieties

Before diving into commutator theory, we'll review of some of the theory of modular lattices. The theory really begins with the observation that in any module, the lattice of submodules is always *ranked* (so long as there are no infinite chains of submodules). In fact, not only is this lattice ranked, but also every (finite) *sublattice* of the lattice of submodules is ranked as well. So it is natural to study lattices which have this property.

Definition A.0.1. The *length* of a finite chain is the number of elements in the chain minus 1. The *length* of a poset is the supremum of the lengths of all of its chains.

Definition A.0.2. A poset satisfies the *Jordan-Dedekind chain condition* if for any $a \le b$, any two maximal chains from a to b have equal length.

The simplest situation to consider is the situation where some element a has two distinct covers b, c. Then $a = b \wedge c$, and we may start by considering sublattices of the interval $[\![a, b \vee c]\!]$. The claim is that in this scenario, if we want every sublattice of the interval $[\![a, b \vee c]\!]$ to be ranked, then we need $b \vee c$ to cover both b and c (so the interval $[\![a, b \vee c]\!]$ must have length two). If $b \vee c$ does *not* cover c, say $c < d < b \vee c$ for some d, then we have a problem: the sublattice generated by b, c, d is a copy of the pentagon lattice $\mathcal{N}_5$, which is not ranked. The only hard part of verifying this is checking that $b \wedge d = a$, but this follows from $a \le b \wedge d \le b$ and $b \not\le d$.

Definition A.0.3. A poset is called *upper semimodular* if whenever an element a has two distinct covers b, c, there is some element d which covers both b and c.

Surprisingly, it turns out that any upper semimodular poset which has no infinite chains satisfies the Jordan-Dedekind chain condition. Note that every chain is contained in a maximal chain (by Zorn's Lemma).

Proposition A.0.4. *If a is any element of an upper semimodular poset which has no infinite chains, then any two maximal chains starting at a (going upwards) have the same length.*

Proof. Let $a < a_1 < \cdots$ and $a < a_1' < \cdots$ be two maximal chains starting from a of lengths m, n, and induct on $\min(m, n)$. We may assume without loss of generality that $m \le n$. By upper semimodularity, there is some element a_2'' which covers both a_1 and a_1'. Pick some maximal chain

$a_2'' < a_3'' < \cdots$ starting from a_2''. Then the maximal chains $a_1 < a_2 < \cdots$ and $a_1 < a_2'' < \cdots$ must both have length $m-1$ by the induction hypothesis. Since the maximal chain $a_1' < a_2'' < \cdots$ then also has length $m-1$, we can apply the induction hypothesis to see that the maximal chain $a_1' < a_2' < \cdots$ has length $m-1$ as well, so $m = n$. □

Corollary A.0.5 (Birkhoff [28]). *An upper semimodular poset which has no infinite chains satisfies the Jordan-Dedekind chain condition.*

Proof. If $a \le b$, then we can pick some fixed maximal chain $b < b_1 < \cdots$ starting from b. By appending it to any two maximal chains from a to b of different lengths, we obtain two maximal chains starting from a which have different lengths, contradicting the previous proposition. □

On any poset of finite length which satisfies the Jordan-Dedekind chain condition and has upper or lower bounds, we can define a *height function* h such that whenever a is covered by b, we have $h(b) = h(a) + 1$.

Proposition A.0.6 (Birkhoff [28]). *A ranked lattice of finite length is upper semimodular if and only if its height function satisfies the inequality*

$$h(x) + h(y) \ge h(x \vee y) + h(x \wedge y).$$

Proof. The inequality clearly implies upper semimodularity. Now suppose our lattice is upper semimodular, and pick maximal chains

$$x \wedge y = x_0 < x_1 < \cdots < x_m = x,$$
$$x \wedge y = y_0 < y_1 < \cdots < y_n = y.$$

We claim that for each i, j, $x_i \vee y_j$ is either covered by or equal to $x_{i+1} \vee y_j$ and $x_i \vee y_{j+1}$. We can prove this by induction on i, j: if it's true for i, j, then by upper semimodularity $x_{i+1} \vee y_{j+1}$ will either cover or be equal to both of $x_{i+1} \vee y_j$ and $x_i \vee y_{j+1}$.

Thus, the sequence

$$x = x \vee y_0 \le x \vee y_1 \le \cdots \le x \vee y_n = x \vee y$$

has every adjacent pair either equal or a cover, so

$$h(x \vee y) - h(x) \le h(y) - h(x \wedge y).$$ □

There is also a corresponding notion of lower semimodularity, and a dual version of the above result. Putting them together, we get the following.

Theorem A.0.7 (Birkhoff [28]). *A lattice of finite length is modular iff it satisfies the Jordan-Dedekind chain condition and its height function satisfies*

$$h(x) + h(y) = h(x \vee y) + h(x \wedge y).$$

Proof. Since modular implies both upper and lower semimodular, it implies the chain condition and the condition on the height function. For the other direction, suppose that we have a ranked lattice whose height function satisfies the given condition.

Suppose for contradiction that there is a sublattice isomorphic to the pentagon $\mathcal{N}_5$ (recall from the discussion around Definition 1.7.7 that a lattice is modular iff it doesn't have $\mathcal{N}_5$ as a sublattice).

Suppose this sublattice is generated by a, b, c, with $b < c$ and $a \wedge b = a \wedge c$, $a \vee b = a \vee c$. Then we have

$$h(a) + h(b) = h(a \vee b) + h(a \wedge b) = h(a \vee c) + h(a \wedge c) = h(a) + h(c),$$

so $h(b) = h(c)$, contradicting $b < c$. □

The next result can be viewed as a strengthening of the fact that a modular lattice is both upper and lower semimodular.

Theorem A.0.8 (Diamond Isomorphism Theorem). *If a, b are elements of a modular lattice, then the maps $\phi : [\![a, a \vee b]\!] \to [\![a \wedge b, b]\!]$ and $\varphi : [\![a \wedge b, b]\!] \to [\![a, a \vee b]\!]$ given by*

$$\phi : x \mapsto x \wedge b \textit{ and } \varphi : y \mapsto y \vee a$$

are lattice isomorphisms.

Proof. First we check that ϕ, φ are inverse to each other. By the modular law, for $x \in [\![a, a \vee b]\!]$ we have

$$\varphi(\phi(x)) = (x \wedge b) \vee a = x \wedge (b \vee a) = x,$$

and for $y \in [\![a \wedge b, b]\!]$ we have

$$\phi(\varphi(y)) = (y \vee a) \wedge b = y \vee (a \wedge b) = y.$$

It is clear that ϕ respects meets and that φ respects joins, so from the fact that they are inverse to each other we see that they are both lattice isomorphisms. □

Definition A.0.9. If a, b are elements of a lattice, then we say that the intervals $[\![a, a \vee b]\!]$ and $[\![a \wedge b, b]\!]$ are *perspective* to each other, and we abbreviate this with either the notation

$$[\![a, a \vee b]\!] \searrow [\![a \wedge b, b]\!]$$

or the notation

$$[\![a \wedge b, b]\!] \nearrow [\![a, a \vee b]\!].$$

If two intervals in a lattice can be connected by a chain of perspectivities, then we say that they are *projective* to each other.

The fact that all maximal chains in a finite length semimodular lattice have the same length can be strengthened to a lattice version of the Jordan-Hölder Theorem.

Theorem A.0.10 (Jordan-Hölder for semimodular lattices [87]). *Suppose we have two maximal chains*

$$0 = a_0 < a_1 < \cdots < a_n = 1,$$
$$0 = b_0 < b_1 < \cdots < b_n = 1$$

in an upper semimodular lattice of finite length. Then there is a permutation $\sigma \in S_n$ such that each $[\![a_{i-1}, a_i]\!]$ is projective in two steps (going $\nearrow$, $\searrow$) to $[\![b_{\sigma(i)-1}, b_{\sigma(i)}]\!]$.

Proof. We induct on the length n. If $a_1 = b_1$ then we can apply the inductive hypothesis. Otherwise, for each i, let $c_i = a_1 \vee b_i$. If k is maximal such that $a_1 \not\leq b_k$, then

$$a_1 = c_0 < c_1 < \cdots < c_k = c_{k+1} < \cdots < c_n = 1$$

where the strict inequalities up to c_k follow from upper semimodularity, and in the portion after c_{k+1} we have $c_j = b_j$.

Applying the induction hypothesis, we get a bijection $\sigma' : [n]\setminus\{1\} \to [n]\setminus\{k+1\}$ such that each $[\![a_{i-1}, a_i]\!]$ is projective going $\nearrow, \searrow$ to $[\![c_{\sigma'(i)-1}, c_{\sigma'(i)}]\!]$. Since $[\![c_{\sigma'(i)-1}, c_{\sigma'(i)}]\!] \searrow [\![b_{\sigma'(i)-1}, b_{\sigma'(i)}]\!]$, and since $[\![a_0, a_1]\!] \nearrow [\![b_k, b_{k+1}]\!]$, we can take σ to be the extension of σ' given by setting $\sigma(1) = k+1$. □

To relate this to the usual Jordan-Hölder Theorem, we have to consider the lattice of *subnormal subgroups* of a group. A subgroup $\mathbb{M} \leq \mathbb{G}$ is called subnormal if there is a finite chain of subgroups connecting it to $\mathbb{G}$, such that each is a normal subgroup of the next.

Proposition A.0.11. *A subgroup $\mathbb{M} \leq \mathbb{G}$ is subnormal iff the sequence of groups $\mathbb{G} = \mathbb{G}_0 \rhd \mathbb{G}_1 \rhd \cdots$ defined by taking $\mathbb{G}_{i+1}$ to be the normal closure of $\mathbb{M}$ inside $\mathbb{G}_i$ eventually reaches $\mathbb{M}$. As a consequence, the intersection of two subnormal subgroups is also subnormal.*

Proposition A.0.12. *If $\mathbb{G}$ is a group of finite composition length, then the collection of subnormal subgroups of $\mathbb{G}$ forms a lower semimodular lattice. If $[\![\mathbb{N}_1, \mathbb{M}_1]\!]$, $[\![\mathbb{N}_2, \mathbb{M}_2]\!]$ are $\searrow, \nearrow$ projective covers in this lattice, then $\mathbb{M}_1/\mathbb{N}_1 \cong \mathbb{M}_2/\mathbb{N}_2$.*

Note that the modular law is equivalent to the following identity, which recovers the usual modular law in the case $a \leq b$ by replacing $a \wedge b$ with a:

$$(a \wedge b) \vee (c \wedge b) \approx ((a \wedge b) \vee c) \wedge b.$$

Thus modular lattices form a variety of lattices. We finish our review of modular lattices by mentioning a famous result of Dedekind.

Proposition A.0.13 (Dedekind [68])**.** *The free modular lattice on 3 generators is finite, with exactly 28 elements and length 8. It is isomorphic to a subdirect product of six copies of the two-element lattice and a single copy of the diamond lattice $\mathcal{M}_3$.*

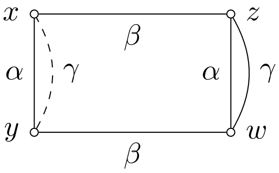
In particular, one can test whether a given 3-variable lattice identity is a consequence of modularity in finite time, by testing whether it holds on $\mathcal{M}_3$.

A corresponding result for 4 generators does not exist: the free modular lattice on 4 generators is infinite. To see this, note that if you start with four generic points on the projective plane and repeatedly generate new points and lines, the resulting set of points and lines you obtain is infinite. Determining whether a 4-variable lattice identity follows from the modular law is undecidable in general [93].

A.1 The Shifting Lemma and the Day terms

We will follow Freese and McKenzie [80], with some arguments taken from Gumm [89] and some from [149]. The starting point for proving things in congruence modular varieties is the Shifting Lemma (this is the main place in the theory where the modular law is actually used).

Lemma A.1.1 (Shifting Lemma)**.** *If $\mathbb{A}$ is congruence modular, $x, y, z, w \in \mathbb{A}$ and $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbb{A})$ with $\alpha \wedge \beta \le \gamma$ and $x \equiv_\alpha y, z \equiv_\alpha w, x \equiv_\beta z, y \equiv_\beta w$, then $z \equiv_\gamma w \implies x \equiv_\gamma y$.*



Proof. We have $(x, y) \in \alpha \wedge (\beta \circ (\alpha \wedge \gamma) \circ \beta) \subseteq \alpha \wedge (\beta \vee (\alpha \wedge \gamma))$. Since $\alpha \wedge \gamma \le \alpha$, we can apply the modular law to get $\alpha \wedge (\beta \vee (\alpha \wedge \gamma)) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$, and this is contained in γ by the assumption $\alpha \wedge \beta \le \gamma$, so $(x, y) \in \gamma$. □

Corollary A.1.2 (Day terms)**.** *In any congruence modular variety $\mathcal{V}$, if $\mathcal{F}_\mathcal{V}(x, y, z, w)$ is the free algebra on four generators, and if we let $\theta_{a,b}$ be the congruence generated by identifying a, b, then there are quaternary terms $m_0, ..., m_n \in \mathcal{F}_\mathcal{V}(x, y, z, w)$ such that*

$$\begin{aligned}
&m_0 = x,\\
&m_i\ (\theta_{x,y} \vee \theta_{z,w}) \wedge (\theta_{x,z} \vee \theta_{y,w})\ m_{i+1} \text{ for } i \text{ even},\\
&m_i\ \theta_{z,w}\ m_{i+1} \text{ for } i \text{ odd},\\
&m_n = y.
\end{aligned}$$

In other words, the m_i satisfy the following system of identities:

$$\begin{aligned}
m_0(x, y, z, w) &\approx x,\\
m_i(x, x, z, z) &\approx x \text{ for all } i,\\
m_i(x, y, x, y) &\approx m_{i+1}(x, y, x, y) \text{ for } i \text{ even},\\
m_i(x, y, z, z) &\approx m_{i+1}(x, y, z, z) \text{ for } i \text{ odd},\\
m_n(x, y, z, w) &\approx y.
\end{aligned}$$

Proof. Apply the Shifting Lemma with $\alpha = \theta_{x,y} \vee \theta_{z,w}, \beta = \theta_{x,z} \vee \theta_{y,w}$, and $\gamma = (\alpha \wedge \beta) \vee \theta_{z,w}$ to see that $(x, y) \in (\alpha \wedge \beta) \vee \theta_{z,w} = \bigcup_n((\alpha \wedge \beta) \circ \theta_{z,w})^{\circ n}$. □

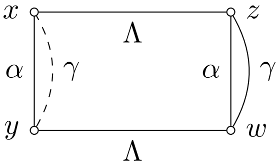
Lemma A.1.3. *Let $\mathbb{A}$ be an algebra with Day terms $m_0, ..., m_n$, $\theta \in \mathrm{Con}(\mathbb{A})$, and $a, b, c, d \in \mathbb{A}$ with $(c, d) \in \theta$. Then $(a, b) \in \theta$ iff for all $i \le n$ we have $m_i(a, b, a, b) \equiv_\theta m_i(a, b, c, d)$.*

Proof. If $(a, b) \in \theta$, then for each i we have $m_i(a, b, a, b) \equiv_\theta m_i(a, a, a, a) = a$ and $m_i(a, b, c, d) \equiv_\theta m_i(a, a, c, c) = a$. For the converse direction, we will show that if $c \equiv_\theta d$ and $m_i(a, b, a, b) \equiv_\theta m_i(a, b, c, d)$ for all i, then $m_i(a, b, c, d) \equiv_\theta m_{i+1}(a, b, c, d)$ for all i, and then we can conclude $a = m_0(a, b, c, d) \equiv_\theta m_n(a, b, c, d) = b$.

For i even, we use $m_i(a, b, a, b) = m_{i+1}(a, b, a, b)$ together with the assumed congruences relating $m_i(a, b, a, b)$ to $m_i(a, b, c, d)$, while for i odd we use $m_i(a, b, c, c) = m_{i+1}(a, b, c, c)$ together with $c \equiv_\theta d$. □

The existence of Day terms implies a result slightly stronger than the Shifting Lemma, called the Shifting Principle.

Lemma A.1.4 (The Shifting Principle)**.** *If $\mathbb{A}$ has Day terms $m_0, ..., m_n$, then $\mathbb{A}$ satisfies the Shifting Principle: if $x, y, z, w \in \mathbb{A}$ and $\alpha, \gamma \in \mathrm{Con}(\mathbb{A})$ and $\Lambda \leq \mathbb{A}^2$ is a reflexive relation preserved by the m_i with $\alpha \cap \Lambda \subseteq \gamma$ and $x \equiv_\alpha y, z \equiv_\alpha w, (x, z) \in \Lambda, (y, w) \in \Lambda$, then $z \equiv_\gamma w \implies x \equiv_\gamma y$.*



Proof. By Lemma A.1.3, it's enough to show that $m_i(x, y, x, y) \equiv_\gamma m_i(x, y, z, w)$ for each i. Since Λ is preserved by the m_i and is reflexive, we have

$$\begin{bmatrix} m_i(x, y, x, y) \\ m_i(x, y, z, w) \end{bmatrix} = m_i\left(\begin{bmatrix} x \\ x \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix}, \begin{bmatrix} x \\ z \end{bmatrix}, \begin{bmatrix} y \\ w \end{bmatrix}\right) \in \Lambda,$$

while $m_i(x, y, x, y) \equiv_\alpha m_i(x, y, z, w)$ by Lemma A.1.3, so $(m_i(x, y, x, y), m_i(x, y, z, w)) \in \alpha \cap \Lambda \subseteq \gamma$. □

Lemma A.1.5. *If the Shifting Principle holds for an algebra $\mathbb{A}$ in the special case where $\alpha \geq \gamma$, then $\mathbb{A}$ is congruence modular.*

Proof. Suppose that $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbb{A})$ with $\alpha \geq \gamma \geq \alpha \wedge \beta$, then to verify congruence modularity we just need to check that $\alpha \wedge (\beta \vee \gamma) \leq \gamma$, as this rules out the existence of a sublattice of $\mathrm{Con}(\mathbb{A})$ isomorphic to the pentagon $\mathcal{N}_5$. Defining reflexive, symmetric relations Λ_i by $\Lambda_i = \beta \circ (\gamma \circ \beta)^{\circ i}$, we see that we just need to prove that $\alpha \cap \Lambda_i \subseteq \gamma$ for each i.

We will prove this by induction on i: note that the base case $i = 0$ is trivial, since $\Lambda_0 = \beta$. For the inductive step, we apply the Shifting Principle to α, Λ_i, and γ see that if $\alpha \cap \Lambda_i \subseteq \gamma$, then

$$\alpha \cap \Lambda_{2i+1} = \alpha \cap (\Lambda_i \circ \gamma \circ \Lambda_i) = \alpha \cap (\Lambda_i \circ (\alpha \wedge \gamma) \circ \Lambda_i) \subseteq \gamma.$$ □

Corollary A.1.6. *A variety is congruence modular iff it has Day terms.*

Example A.1.1. If $p(x, y, z)$ is a Mal'cev term, then we can take

$$\begin{aligned} m_0(x, y, z, w) &= x, \\ m_1(x, y, z, w) &= p(z, w, y), \\ m_2(x, y, z, w) &= y \end{aligned}$$

as a sequence of Day terms. Rather than laboriously checking the Day identities, it is easier to verify that this sequence of terms can be used in the Shifting Lemma setup to show that $x \equiv_\gamma y$. We have $x\ (\alpha \wedge \beta)\ p(z, w, y)\ \gamma\ y$, so from $\alpha \wedge \beta \leq \gamma$ we get $x\ \gamma\ y$.

Example A.1.2. If $g(x, y, z)$ is a majority term, then we can take

$$\begin{aligned} m_0(x, y, z, w) &= x, \\ m_1(x, y, z, w) &= g(x, y, z), \\ m_2(x, y, z, w) &= g(x, y, w), \\ m_3(x, y, z, w) &= y \end{aligned}$$

as a sequence of Day terms. Again, in the Shifting Lemma setup, we have $x\ (\alpha \wedge \beta)\ g(x, y, z)\ \gamma\ g(x, y, w)\ (\alpha \wedge \beta)\ y$, so $x\ \gamma\ y$.

The next corollary gives us a large class of examples of congruence modular varieties, generalizing groups and rings.

Definition A.1.7. An algebra is *congruence regular* if every congruence on $\mathbb{A}$ is uniquely determined by any of its congruence classes.

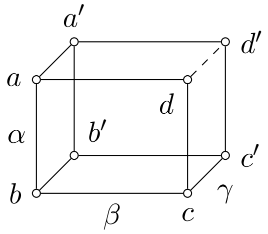
Corollary A.1.8 (Gumm [89])**.** *If every subalgebra of $\mathbb{A}^2$ is congruence regular, then $\mathbb{A}$ is congruence modular.*

Proof. We just need to verify the Shifting Principle for $\mathbb{A}$. Let $\Lambda \le \mathbb{A}^2$ be reflexive, let $\alpha \ge \gamma$ be congruences on $\mathbb{A}$ with $\alpha \cap \Lambda \subseteq \gamma$, and consider the congruences $\alpha \times \gamma$ and $\gamma \times \gamma$ restricted to Λ. We will show that for any $a \in \mathbb{A}$, the congruence classes containing (a, a) in these restrictions are equal, so congruence regularity will imply that $\alpha \times \gamma|_\Lambda = \gamma \times \gamma|_\Lambda$, which is the Shifting Principle for α, Λ, γ.

So suppose that $(a, a) \equiv_{\alpha\times\gamma} (b, c) \in \Lambda$. Then $(b, c) \in \alpha \circ \gamma = \alpha$, so $(b, c) \in \alpha \cap \Lambda \subseteq \gamma$, and this implies that $(a, b) \in \gamma \circ \gamma = \gamma$. Thus $(a, a) \equiv_{\gamma\times\gamma} (b, c)$ as well, and we are done. □

To finish this section, we will prove one of Gumm's "geometric" results on congruence modular varieties, which generalizes the result used to prove associativity of the loop operation in the case of abelian Mal'cev algebras.

Lemma A.1.9 (The Cube Lemma [89])**.** *Suppose every subalgebra of $\mathbb{A}^2$ satisfies the Shifting Lemma. If $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbb{A})$ with $\gamma \ge \alpha \wedge \beta$, and if $a, b, c, d, a', b', c', d' \in \mathbb{A}$ with $(a, b), (c, d), (a', b'), (c', d') \in \alpha$, $(a, d), (b, c), (a', d'), (b', c') \in \beta$, and $(a, a'), (b, b'), (c, c') \in \gamma$, then $(d, d') \in \gamma$.*



Proof. We apply the Shifting Lemma to the algebra $\beta \le \mathbb{A}^2$, and the congruences $\gamma \times 1_\mathbb{A}|_\beta, \alpha \times \alpha|_\beta$, and $\gamma \times \gamma|_\beta$. By the Shifting Lemma applied to α, β, γ, we have $(\alpha \times \alpha|_\beta) \wedge (\gamma \times 1_\mathbb{A}|_\beta) \le \gamma \times \gamma|_\beta$, so the Shifting Lemma applies to $\gamma \times 1_\mathbb{A}|_\beta, \alpha \times \alpha|_\beta, \gamma \times \gamma|_\beta$.

Thus, from $((b, c), (b', c')) \in \gamma\times\gamma|_\beta$, $((a, d), (a', d')) \in \gamma\times 1_\mathbb{A}|_\beta$, and $((b, c), (a, d)), ((b', c'), (a', d')) \in \alpha \times \alpha|_\beta$, the Shifting Lemma allows us to conclude that $((a, d), (a', d')) \in \gamma \times \gamma|_\beta$, so $(d, d') \in \gamma$. □

A.2 The modular commutator

First we go over a slick proof of the main properties of the commutator, using the Day terms to construct an explicit set of generators $X(\alpha, \beta)$ for the congruence $[\alpha, \beta]$ - however, as the approach feels somewhat ad-hoc, we will also prove these properties via a different approach based on the Shifting Lemma applied to congruences (as in the proof of the Cube Lemma). The definition of $X(\alpha, \beta)$ is based on the algebra of matrices $\mathbb{M}(\alpha, \beta)$ used to visualize the term condition (Definition 1.9.28) and Lemma A.1.3.

Definition A.2.1. Suppose $\mathbb{A}$ has Day terms $m_0, ..., m_n$. For $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$, we define $X(\alpha, \beta)$ to be the set of pairs $(m_i(a,b,a,b), m_i(a,b,c,d))$ for $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathbb{M}(\alpha, \beta)$ and $i \le n$.

Example A.2.1. If we have a Mal'cev term $p(x,y,z)$ and take $m_0 = x, m_1 = p(z,w,y), m_2 = y$ as our sequence of Day terms, then $X(\alpha, \beta)$ is the set of pairs $(a, p(c,d,b))$ for $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathbb{M}(\alpha, \beta)$.

Example A.2.2. If we have a majority term $g(x,y,z)$ and take $m_0 = x, m_1 = g(x,y,z), m_2 = g(x,y,w), m_3 = y$ as our sequence of Day terms, then $X(\alpha, \beta)$ is the set of pairs $(a, g(a,b,c))$ for $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathbb{M}(\alpha, \beta)$. For $(a,b) \in \alpha \wedge \beta$, we have

$$g\left(\begin{bmatrix} a & a \\ b & b \end{bmatrix}, \begin{bmatrix} a & b \\ a & b \end{bmatrix}, \begin{bmatrix} b & b \\ b & b \end{bmatrix}\right) = \begin{bmatrix} a & b \\ b & b \end{bmatrix} \in \mathbb{M}(\alpha, \beta),$$

so $(a, g(a,b,b)) = (a,b) \in X(\alpha, \beta)$. Thus $X(\alpha, \beta) = \alpha \wedge \beta$ for majority algebras.

Theorem A.2.2 (Commutator via Day terms)**.** *If $\mathbb{A}$ has Day terms $m_0, ..., m_n$ and $\alpha, \beta, \delta \in \mathrm{Con}(\mathbb{A})$, then the following are equivalent.*

(i) $X(\alpha, \beta) \subseteq \delta$,

(ii) $X(\beta, \alpha) \subseteq \delta$,

(iii) $C(\alpha, \beta; \delta)$ *holds,*

(iv) $C(\beta, \alpha; \delta)$ *holds,*

(v) $[\alpha, \beta] \le \delta$.

Proof. It's enough to show (iii) $\implies$ (i) $\implies$ (iv). For (iii) $\implies$ (i), suppose that $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathbb{M}(\alpha, \beta)$, then

$$m_i\left(\begin{bmatrix} a & a \\ a & a \end{bmatrix}, \begin{bmatrix} a & a \\ b & b \end{bmatrix}, \begin{bmatrix} a & c \\ a & c \end{bmatrix}, \begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = \begin{bmatrix} a & a \\ m_i(a,b,a,b) & m_i(a,b,c,d) \end{bmatrix} \in \mathbb{M}(\alpha, \beta),$$

so $C(\alpha, \beta; \delta)$ implies that we have $(m_i(a,b,a,b), m_i(a,b,c,d)) \in \delta$.

For (i) $\implies$ (iv), we apply Lemma A.1.3 to see that if $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathbb{M}(\alpha, \beta)$, $(c,d) \in \delta$, and $X(\alpha, \beta) \subseteq \delta$, then we must have $(a,b) \in \delta$ as well, so $C(\beta, \alpha; \delta)$ holds. □

Now we can finally prove some useful properties of commutators.

Proposition A.2.3. *If $\mathbb{A}$ is contained in a congruence modular variety, then for congruences on $\mathbb{A}$ we have*

(a) $[\alpha, \beta] = [\beta, \alpha]$,

(b) $[\alpha \wedge \gamma, \beta] \le [\alpha, \beta] \wedge \gamma$,

(c) $[\bigvee_i \alpha_i, \beta] = \bigvee_i [\alpha_i, \beta]$,

(d) *if* $f : \mathbb{A} \twoheadrightarrow \mathbb{B}$ *is surjective, then* $f([\alpha, \beta] \vee \ker f) = [f(\alpha \vee \ker f), f(\beta \vee \ker f)]$,

(e) *if* $\mathbb{B} \le \mathbb{A}$, *then* $[\alpha|_{\mathbb{B}}, \beta|_{\mathbb{B}}] \le [\alpha, \beta]|_{\mathbb{B}}$,

(f) *if* $\mathbb{A} = \prod_{i \in I} \mathbb{A}_i$, *then* $[\bigoplus_i \alpha_i, \bigoplus_i \beta_i] = \bigoplus_i [\alpha_i, \beta_i]$, *where* $\bigoplus_i \alpha_i$ *is the set of pairs* (a, b) *in* $\prod_i \alpha_i$ *such that for all but finitely many* i *we have* $a_i = b_i$,

(g) *if* $\mathbb{A} = \prod_{i \in I} \mathbb{A}_i$, *then* $[\prod_i \alpha_i, \prod_i \beta_i] \le \prod_i [\alpha_i, \beta_i]$.

Proof. Part (a) follows from Theorem A.2.2, part (b) follows from Proposition 1.9.30(d), and part (e) is Proposition 1.9.30(g). For part (c), Theorem A.2.2 shows that $C(\alpha_j, \beta; \bigvee_i [\alpha_i, \beta])$ holds for each j, so we can use Proposition 1.9.30(e) to see that $[\bigvee_i \alpha_i, \beta] \le \bigvee_i [\alpha_i, \beta]$, while the other inequality follows from monotonicity of the commutator.

For part (d), note that part (c) implies $[\alpha, \beta] \vee \ker f = [\alpha \vee \ker f, \beta \vee \ker f] \vee \ker f$, so we may assume that $\alpha, \beta \ge \ker f$ without loss of generality. By Theorem A.2.2, $[\alpha, \beta] \vee \ker f$ is the congruence generated by $X(\alpha, \beta) \cup \ker f$, and $[f(\alpha), f(\beta)]$ is the congruence generated by $X(f(\alpha), f(\beta)) = f(X(\alpha, \beta))$, so $f([\alpha, \beta] \vee \ker f) = [f(\alpha), f(\beta)]$.

Parts (f) and (g) follow directly from Theorem A.2.2, but they can also be proved using only parts (a) - (d) (left as an exercise to the reader). □

Proposition A.2.3(d) tells us that we can compute commutators on quotients of $\mathbb{A}$ directly in $\mathbb{A}$. Since $\mathrm{Con}(\mathbb{A}/\pi)$ is naturally isomorphic to the interval $[\![\pi, 1_{\mathbb{A}}]\!]$ in $\mathrm{Con}(\mathbb{A})$, computing commutators on $\mathbb{A}/\pi$ is equivalent to computing *relative commutators* on $\mathbb{A}$. Recall that if $\alpha, \beta \ge \pi$, then their *relative commutator* $[\alpha, \beta]_\pi$ is defined to be the least $\delta \ge \pi$ which satisfies the term condition $C(\alpha, \beta; \delta)$.

Corollary A.2.4. *If* $\alpha, \beta \ge \pi$ *are congruences in a congruence modular variety, then their relative commutator is given by the formula* $[\alpha, \beta]_\pi = [\alpha, \beta] \vee \pi$.

Theorem A.2.5 (Diamond Isomorphism Theorem for relative commutators)**.** *If* $\mathbb{A}$ *is in a congruence modular variety and* $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$, *then the maps* $\phi : [\![\alpha, \alpha \vee \beta]\!] \to [\![\alpha \wedge \beta, \beta]\!]$ *and* $\varphi : [\![\alpha \wedge \beta, \beta]\!] \to [\![\alpha, \alpha \vee \beta]\!]$ *given by*

$$\phi : x \mapsto x \wedge \beta \quad \textit{and} \quad \varphi : y \mapsto y \vee \alpha$$

are lattice isomorphisms which respect the relative commutators $[\cdot, \cdot]_\alpha, [\cdot, \cdot]_{\alpha \wedge \beta}$.

Furthermore, in this case we have the equality of relative centralizers $(\alpha : \alpha \vee \beta) = (\alpha \wedge \beta : \beta)$.

Proof. By Theorem A.0.8, ϕ and φ are lattice isomorphisms. If $\gamma, \delta \ge \alpha \wedge \beta$, then from $[\gamma \vee \alpha, \delta \vee \alpha] \le [\gamma, \delta] \vee \alpha$ we have

$$\varphi([\gamma, \delta]_{\alpha \wedge \beta}) = [\gamma, \delta]_{\alpha \wedge \beta} \vee \alpha = [\gamma, \delta] \vee \alpha = [\gamma \vee \alpha, \delta \vee \alpha] \vee \alpha = [\varphi(\gamma), \varphi(\delta)]_\alpha.$$

If $\gamma, \delta \in [\![\alpha, \alpha \vee \beta]\!]$, then from $\gamma = \phi(\gamma) \vee \alpha, \delta = \phi(\delta) \vee \alpha$ and $[\phi(\gamma), \phi(\delta)] \le \beta$ we have

$$\begin{aligned} \phi([\gamma, \delta]_\alpha) &= [\gamma, \delta]_\alpha \wedge \beta = [\phi(\gamma) \vee \alpha, \phi(\delta) \vee \alpha]_\alpha \wedge \beta \\ &= ([\phi(\gamma), \phi(\delta)] \vee \alpha) \wedge \beta = [\phi(\gamma), \phi(\delta)] \vee (\alpha \wedge \beta) = [\phi(\gamma), \phi(\delta)]_{\alpha \wedge \beta}. \end{aligned}$$

For the last statement, note that

$$[\delta, \alpha \vee \beta] \le \alpha \iff [\delta, \alpha] \vee [\delta, \beta] \le \alpha \iff [\delta, \beta] \le \alpha \iff [\delta, \beta] \le \alpha \wedge \beta,$$

so $\delta \le (\alpha : \alpha \vee \beta) \iff \delta \le (\alpha \wedge \beta : \beta)$. □

For the second approach to the commutator, we will follow Gumm [89] and take the transitive closure of $\mathbb{M}(\alpha, \beta)$ to produce a congruence on α, considered as a subalgebra of $\mathbb{A}^2$.

Definition A.2.6. For $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$, if we consider $\alpha \le \mathbb{A}^2$ as an algebra of column vectors then we can treat $\mathbb{M}(\alpha, \beta)$ (from Definition 1.9.28) as a binary relation on α, so $\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) \in \mathbb{M}(\alpha, \beta)$ means that $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathbb{M}(\alpha, \beta)$. We define Δ_α^β to be the transitive closure of this binary relation on α.

Note that Δ_α^β is the least congruence on α which contains the binary relation $\beta \times \beta|_{\Delta_{\mathbb{A}}}$. When $\mathbb{A}$ has a Mal'cev term, Δ_α^β simplifies to $\mathbb{M}(\alpha, \beta)$.

Proposition A.2.7. *If $\mathbb{A}$ has a Mal'cev polynomial p and $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$, then $\Delta_\alpha^\beta = \mathbb{M}(\alpha, \beta)$.*

Proof. We just need to check that $\mathbb{M}(\alpha, \beta)$ is transitively closed, so supposed that $\begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} c & e \\ d & f \end{bmatrix} \in \mathbb{M}(\alpha, \beta)$. Then we have

$$\begin{bmatrix} a & e \\ b & f \end{bmatrix} = p\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} c & c \\ d & d \end{bmatrix}, \begin{bmatrix} c & e \\ d & f \end{bmatrix}\right) \in \mathbb{M}(\alpha, \beta).$$ □

Theorem A.2.8. *Suppose that the Shifting Lemma holds for every subalgebra of $\mathbb{A}^2$. Then for $x, y \in \mathbb{A}$ and $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$, the following are equivalent:*

(a) $(x, y) \in [\beta, \alpha]$,

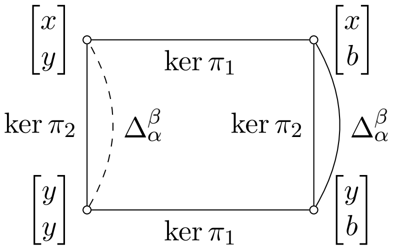
(b) $\begin{bmatrix} x & y \\ y & y \end{bmatrix} \in \Delta_\alpha^\beta$,

(c) there exists $a \in \mathbb{A}$ such that $\begin{bmatrix} x & a \\ y & a \end{bmatrix} \in \Delta_\alpha^\beta$,

(d) there exists $b \in \mathbb{A}$ such that $\begin{bmatrix} x & y \\ b & b \end{bmatrix} \in \Delta_\alpha^\beta$.

Proof. That (b) implies (c), (d) are clear, and (c) implies (a) directly from the term condition $C(\beta, \alpha; [\beta, \alpha])$ and the definition of Δ_α^β. That (c) implies (b) follows from the fact that $(a, y) \in \beta \implies \begin{bmatrix} a & y \\ a & y \end{bmatrix} \in \mathbb{M}(\alpha, \beta)$, and since Δ_α^β is the transitive closure of $\mathbb{M}(\alpha, \beta)$ we have $\begin{bmatrix} x & y \\ y & y \end{bmatrix} \in \Delta_\alpha^\beta \circ \mathbb{M}(\alpha, \beta) = \Delta_\alpha^\beta$.

For (d) $\implies$ (b) we apply the Shifting Lemma to the algebra $\alpha \le_{sd} \mathbb{A} \times \mathbb{A}$, the congruences $\ker \pi_1, \ker \pi_2, \Delta_\alpha^\beta \in \mathrm{Con}(\alpha)$, and the elements $\begin{bmatrix} x \\ b \end{bmatrix}, \begin{bmatrix} y \\ b \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} y \\ y \end{bmatrix} \in \alpha$ (that $x \equiv_\alpha y$ follows from $x \equiv_\alpha b \equiv_\alpha y$).


$\begin{bmatrix}x\\y\end{bmatrix}$
$\begin{bmatrix}x\\b\end{bmatrix}$
$\ker\pi_1$
$\ker\pi_2$
Δ_α^β
$\ker\pi_2$
Δ_α^β
$\begin{bmatrix}y\\y\end{bmatrix}$
$\begin{bmatrix}y\\b\end{bmatrix}$
$\ker\pi_1$

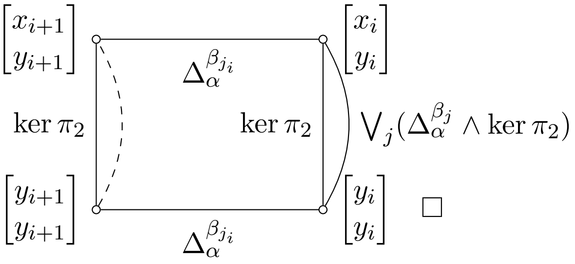
For (a) $\implies$ (b), we will show that the relation Θ defined by $(x, y) \in \Theta \iff \begin{bmatrix}x & y\\ y & y\end{bmatrix} \in \Delta_\alpha^\beta$ is a congruence which satisfies $C(\beta, \alpha; \Theta)$, which will show that $[\beta, \alpha] \le \Theta$. That Θ is reflexive is obvious, that it is symmetric follows from the equivalence of (b) with (c) or (d). If $(x, y), (y, z) \in \Theta$, then from $\begin{bmatrix}x & y\\ y & y\end{bmatrix}, \begin{bmatrix}y & z\\ y & y\end{bmatrix} \in \Delta_\alpha^\beta$ and the fact that Δ_α^β is transitively closed, we get $\begin{bmatrix}x & z\\ y & y\end{bmatrix} \in \Delta_\alpha^\beta$, so $(x, z) \in \Theta$ by the equivalence of (b) and (d).

To finish, we just need to show that Θ satisfies $C(\beta, \alpha; \Theta)$, that is, if $\begin{bmatrix}a & c\\ b & d\end{bmatrix} \in \mathbb{M}(\alpha, \beta)$ with $(c, d) \in \Theta$, then $(a, b) \in \Theta$. But if $(c, d) \in \Theta$, then $\begin{bmatrix}c & d\\ d & d\end{bmatrix} \in \Delta_\alpha^\beta$, so since Δ_α^β is the transitive closure of $\mathbb{M}(\alpha, \beta)$, we see that $\begin{bmatrix}a & d\\ b & d\end{bmatrix} \in \Delta_\alpha^\beta$, so by the equivalence of (b) with (c) we have $(a, b) \in \Theta$. □

Corollary A.2.9. *If every subalgebra of $\mathbb{A}^2$ satisfies the Shifting Lemma, then for $\alpha, \beta_i \in \mathrm{Con}(\mathbb{A})$ we have $[\bigvee_i \beta_i, \alpha] = \bigvee_i [\beta_i, \alpha]$.*

Proof. We have $\Delta_\alpha^{\bigvee_i \beta_i} = \bigvee_i \Delta_\alpha^{\beta_i}$, so $(x, y) \in [\bigvee_i \beta_i, \alpha]$ iff $\begin{bmatrix}x & z\\ y & z\end{bmatrix} \in \bigvee_i \Delta_\alpha^{\beta_i}$ for some z. So there must be a sequence $(x_i, y_i) \in \alpha$, j_i, with $\begin{bmatrix}x_i & x_{i+1}\\ y_i & y_{i+1}\end{bmatrix} \in \Delta_\alpha^{\beta_{j_i}}$ and $(x, y) = (x_n, y_n)$, $x_0 = y_0$.

We show by induction on i that $\begin{bmatrix}x_i & y_i\\ y_i & y_i\end{bmatrix} \in \bigvee_j (\Delta_\alpha^{\beta_j} \wedge \ker \pi_2)$, this will show that $(x_i, y_i) \in \bigvee_j [\beta_j, \alpha]$. For the inductive step, we apply the Shifting Lemma to $\alpha \le \mathbb{A}^2$ with the congruences $\ker \pi_2, \Delta_\alpha^{\beta_{j_i}}, \bigvee_j (\Delta_\alpha^{\beta_j} \wedge \ker \pi_2)$.


$\begin{bmatrix}x_{i+1}\\y_{i+1}\end{bmatrix}$
$\begin{bmatrix}x_i\\y_i\end{bmatrix}$
$\Delta_\alpha^{\beta_{j_i}}$
$\ker\pi_2$
$\ker\pi_2$
$\bigvee_j(\Delta_\alpha^{\beta_j} \wedge \ker\pi_2)$
$\begin{bmatrix}y_{i+1}\\y_{i+1}\end{bmatrix}$
$\begin{bmatrix}y_i\\y_i\end{bmatrix}$
$\Delta_\alpha^{\beta_{j_i}}$

□

Corollary A.2.10. *If every subalgebra of $\mathbb{A}^2$ satisfies the Shifting Lemma, then for $f : \mathbb{A} \twoheadrightarrow \mathbb{B}$ surjective and $\alpha, \beta \ge \ker f$, we have $f([\beta, \alpha] \vee \ker f) = [f(\beta), f(\alpha)]$.*

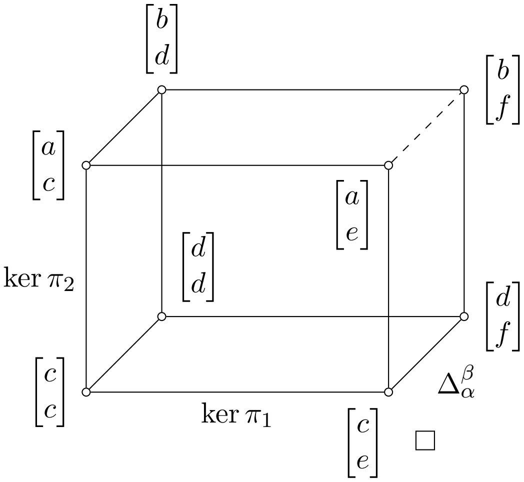
Proof. The hard direction is to check that if $(f(x), f(y)) \in [f(\beta), f(\alpha)]$, then $(x, y) \in [\beta, \alpha] \vee \ker f$. In this case we have $\begin{bmatrix} x & y \\ y & y \end{bmatrix} \in \Delta_\alpha^\beta \vee (\ker f \times \ker f|_\alpha)$. Using a similar argument to the previous corollary, we can show that this implies $\begin{bmatrix} x & y \\ y & y \end{bmatrix} \in (\Delta_\alpha^\beta \wedge \ker \pi_2) \vee (\ker f \times \ker f|_\alpha)$ by repeatedly applying the Shifting Lemma on α. Thus we have $(x, y) \in [\beta, \alpha] \vee \ker f$ by Theorem A.2.8. □

To prove the symmetry of the commutator, we will actually prove a stronger statement: Δ_α^β is in fact the *transpose* of Δ_β^α. In particular, if we view Δ_α^β as a binary relation on *row* vectors in β, then Δ_α^β will be transitively closed (which is far from obvious from the definition!).

Theorem A.2.11. *Suppose that the Shifting Lemma holds for every subalgebra of $\mathbb{A}^4$. If $\overline{\Delta}_\alpha^\beta$ denotes the set of transposes of matrices from Δ_α^β, then $\overline{\Delta}_\alpha^\beta$ is transitively closed as a binary relation on β and we have $\overline{\Delta}_\alpha^\beta = \Delta_\beta^\alpha$. In particular, we have $[\alpha, \beta] = [\beta, \alpha]$.*

Proof. It's enough to prove that $\overline{\Delta}_\alpha^\beta$ is transitively closed as a binary relation on β, as we will then have $\Delta_\beta^\alpha = \bigcup_n \mathbb{M}(\beta, \alpha)^{\circ n} \subseteq \overline{\Delta}_\alpha^\beta$, and a symmetric argument with α, β swapped will show that $\Delta_\alpha^\beta \subseteq \overline{\Delta}_\beta^\alpha$, so $\Delta_\beta^\alpha \subseteq \overline{\Delta}_\alpha^\beta \subseteq \Delta_\beta^\alpha$.

Suppose that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} c & d \\ e & f \end{bmatrix} \in \Delta_\alpha^\beta$. To finish, we just need to show that $\begin{bmatrix} a & b \\ e & f \end{bmatrix} \in \Delta_\alpha^\beta$. This follows from the following application of the Cube Lemma (Lemma A.1.9) applied to the congruences $\ker \pi_1, \ker \pi_2, \Delta_\alpha^\beta$ on α.



$\begin{bmatrix} b \\ d \end{bmatrix}$
$\begin{bmatrix} b \\ f \end{bmatrix}$
$\begin{bmatrix} a \\ c \end{bmatrix}$
$\begin{bmatrix} a \\ e \end{bmatrix}$
$\begin{bmatrix} d \\ d \end{bmatrix}$
$\ker \pi_2$
$\begin{bmatrix} d \\ f \end{bmatrix}$
$\begin{bmatrix} c \\ c \end{bmatrix}$
Δ_α^β
$\ker \pi_1$
$\begin{bmatrix} c \\ e \end{bmatrix}$

□

Theorem A.2.12. *In a congruence modular variety, any alternative commutator $[\cdot, \cdot]'$ which satisfies $[\alpha, \beta]' \le \alpha \wedge \beta$ and $f([\alpha, \beta]' \vee \ker f) = [f(\alpha), f(\beta)]'$ for f surjective and $\alpha, \beta \ge \ker f$ has $[\alpha, \beta]' \le [\alpha, \beta]$ for all α, β.*

Proof. Consider congruences on $\alpha \le_{sd} \mathbb{A} \times \mathbb{A}$. We have $[\Delta_\alpha^\beta, \ker \pi_2]' \le \Delta_\alpha^\beta \wedge \ker \pi_2 \le \pi_1^{-1}[\beta, \alpha]$ by Theorem A.2.8. Also, $\alpha = \pi_1(\ker \pi_2 \vee \ker \pi_1), \beta = \pi_1(\Delta_\alpha^\beta \vee \ker \pi_1)$, so

$$[\beta, \alpha]' = [\pi_1(\Delta_\alpha^\beta \vee \ker \pi_1), \pi_1(\ker \pi_2 \vee \ker \pi_1)]' = \pi_1([\Delta_\alpha^\beta, \ker \pi_2]' \vee \ker \pi_1) \le [\beta, \alpha].$$ □

A.3 The Gumm difference term

In this section we prove that congruence modular varieties have a ternary term p, called a *Gumm difference term*, which acts like a Mal'cev operation on all abelian algebras. This will imply that abelian algebras in congruence modular varieties are affine.

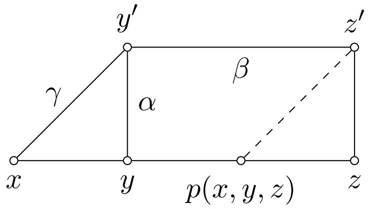
Theorem A.3.1 (Gumm difference term)**.** *For any variety with Day terms $m_0, ..., m_n$, there is a ternary term p satisfying the following two properties:*

(i) p satisfies the identity $p(y, y, x) \approx x$, and

(ii) for any $(x, y) \in \theta$, θ any congruence, we have $p(x, y, y)$ $[\theta, \theta]$ x.

Furthermore, in a congruence modular variety, a ternary term p satisfies (i) and (ii) iff it satisfies the following property:

(iii) for any congruences α, β, γ with $\alpha \wedge \beta \leq \gamma$, the implication in the following picture holds.



Finally, if a variety has a term p which satisfies (iii), then it is congruence modular.

Proof. Recall the identities satisfied by Day terms:

$$\begin{aligned} m_0(x, y, z, w) &\approx x, \\ m_i(x, x, z, z) &\approx x \text{ for all } i, \\ m_i(x, y, x, y) &\approx m_{i+1}(x, y, x, y) \text{ for } i \text{ even}, \\ m_i(x, y, z, z) &\approx m_{i+1}(x, y, z, z) \text{ for } i \text{ odd}, \\ m_n(x, y, z, w) &\approx y. \end{aligned}$$

We inductively define a sequence of ternary terms $q_i(x, y, z)$ by $q_0(x, y, z) = z$, and

$$q_{i+1}(x, y, z) = \begin{cases} m_{i+1}(q_i(x, y, z), q_i(x, y, z), y, x) & i \text{ odd}, \\ m_{i+1}(q_i(x, y, z), q_i(x, y, z), x, y) & i \text{ even}, \end{cases}$$

and we set $p(x, y, z) = q_n(x, y, z)$.

To see that (i) holds, we just check inductively that $q_i(y, y, x) \approx x$:

$$q_{i+1}(y, y, x) = m_{i+1}(q_i(y, y, x), q_i(y, y, x), y, y) \approx m_{i+1}(x, x, y, y) \approx x.$$

For (ii), we will inductively check that

$$q_i(y, x, x)\ [\theta, \theta]\ \begin{cases} m_i(x, y, x, y) & i \text{ even}, \\ m_i(x, y, x, x) & i \text{ odd}. \end{cases}$$

Taking $i = n$, this will give us $p(y,x,x)\ [\theta,\theta]\ m_n(x,y,x,?) = y$.

The base case is easy: $q_0(y,x,x) = x = m_0(x,y,x,y)$. For the inductive step, we divide into cases based on whether i is even or odd.

If i is even, then the induction hypothesis gives

$$q_{i+1}(y,x,x) = m_{i+1}(q_i(y,x,x), q_i(y,x,x), y, x)\ [\theta,\theta]\ m_{i+1}(m_i(x,y,x,y), m_i(x,y,x,y), y, x).$$

Using the term condition $C(\theta,\theta;[\theta,\theta])$, from

$$\begin{aligned} m_{i+1}(m_i(x,y,x,y), m_i(x,y,x,y), y, \boxed{y}) &= m_i(x,y,x,y) = m_{i+1}(x,y,x,y) \\ &= m_{i+1}(m_i(x,x,x,x), m_i(y,y,y,y), x, \boxed{y}), \end{aligned}$$

we conclude

$$\begin{aligned} m_{i+1}(m_i(x,y,x,y), m_i(x,y,x,y), y, \boxed{x})\ [\theta,\theta]\ & m_{i+1}(m_i(x,x,x,x), m_i(y,y,y,y), x, \boxed{x}) \\ &= m_{i+1}(x,y,x,x), \end{aligned}$$

so $q_{i+1}(y,x,x)\ [\theta,\theta]\ m_{i+1}(x,y,x,x)$.

When i is odd, the proof is very similar. Inductively, we have

$$q_{i+1}(y,x,x) = m_{i+1}(q_i(y,x,x), q_i(y,x,x), x, y)\ [\theta,\theta]\ m_{i+1}(m_i(x,y,x,x), m_i(x,y,x,x), x, y).$$

Using the term condition $C(\theta,\theta;[\theta,\theta])$, from

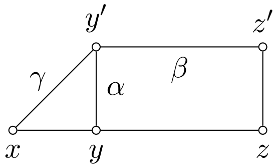
$$\begin{aligned} m_{i+1}(m_i(x,y,x,x), m_i(x,y,x,x), x, \boxed{x}) &= m_i(x,y,x,x) = m_{i+1}(x,y,x,x) \\ &= m_{i+1}(m_i(x,x,x,x), m_i(y,y,y,y), x, \boxed{x}), \end{aligned}$$

we conclude

$$\begin{aligned} m_{i+1}(m_i(x,y,x,x), m_i(x,y,x,x), x, \boxed{y})\ [\theta,\theta]\ & m_{i+1}(m_i(x,x,x,x), m_i(y,y,y,y), x, \boxed{y}) \\ &= m_{i+1}(x,y,x,y), \end{aligned}$$

so $q_{i+1}(y,x,x)\ [\theta,\theta]\ m_{i+1}(x,y,x,y)$. This conclude the proof of (ii).

Now we show that (i) and (ii) imply (iii). Suppose we have the configuration



with $\gamma \ge \alpha \wedge \beta$. From $x \equiv_\beta y \equiv_\beta z$, we have $p(x,y,z) \equiv_\beta z$. Additionally, we have $p(x,y,z) \equiv_\gamma p(y',y,z)$, so we just need to prove that $p(y',y,z) \equiv_\gamma z'$ to finish.

We have $p(y',y,z) \equiv_\alpha p(y',y',z') = z'$, and $p(y',y,z) \equiv_\beta p(z',z,z)$. From $(z,z') \in \alpha \wedge (\beta \vee \gamma)$, we have

$$p(z',z,z)\ [\alpha \wedge (\beta \vee \gamma), \alpha \wedge (\beta \vee \gamma)]\ z'.$$

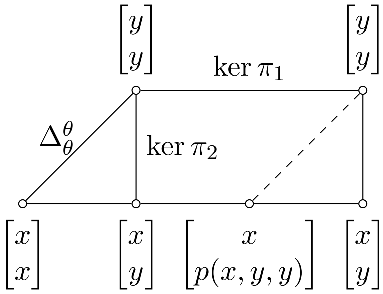
The commutator above is bounded by

$$[\alpha \wedge (\beta \vee \gamma), \alpha \wedge (\beta \vee \gamma)] \le [\alpha, \beta \vee \gamma] = [\alpha,\beta] \vee [\alpha,\gamma] \le (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) = \alpha \wedge \gamma.$$

Thus, we have $(p(y', y, z), z') \in \alpha \wedge (\beta \vee (\alpha \wedge \gamma))$, and by the modular law this is $(\alpha \wedge \beta) \vee (\alpha \wedge \gamma) = \alpha \wedge \gamma \le \gamma$.

Finally, assume that p is a term which satisfies (iii). Taking $x = y = y', z = z', \alpha = \gamma = 0_{\mathbb{A}}, \beta = 1_{\mathbb{A}}$, we get $p(y, y, z) = z$, which is (i). Taking $x = y$ and using $p(y, y, z) = z$, we see that (iii) implies the Shifting Lemma in every algebra, so our variety is congruence modular.

To prove that (iii) implies (ii), suppose $(x, y) \in \theta$, and consider the congruences $\ker \pi_1, \ker \pi_2, \Delta_\theta^\theta$ on θ. Applying (iii) in the picture


$\begin{bmatrix} y \\ y \end{bmatrix}$
$\begin{bmatrix} y \\ y \end{bmatrix}$
$\ker \pi_1$
Δ_θ^θ
$\ker \pi_2$
$\begin{bmatrix} x \\ x \end{bmatrix}$
$\begin{bmatrix} x \\ y \end{bmatrix}$
$\begin{bmatrix} x \\ p(x,y,y) \end{bmatrix}$
$\begin{bmatrix} x \\ y \end{bmatrix}$

we see that $\begin{bmatrix} x & y \\ p(x, y, y) & y \end{bmatrix} \in \Delta_\theta^\theta$, so by Theorem A.2.8 we have $p(x, y, y)\ [\theta, \theta]\ x$. □

Corollary A.3.2 (Factor Permutability). *If $\mathbb{A} = \mathbb{A}_1 \times \mathbb{A}_2$ is contained in a congruence modular variety, then the factor congruences $\ker \pi_1, \ker \pi_2$ permute with every congruence $\gamma \in \mathrm{Con}(\mathbb{A})$.*

Proof. A pair of congruences $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$ correspond to a pair of factor congruences iff they satisfy $\alpha \wedge \beta = 0_{\mathbb{A}}$ and $\alpha \circ \beta = 1_{\mathbb{A}}$. Thus, if $x\ \gamma\ y'\ \alpha\ z'$, then by $\alpha \circ \beta = 1_{\mathbb{A}}$ we can find $y, z \equiv_\alpha x$ with $(y, y'), (z, z') \in \beta$. Then from $\gamma \ge 0_{\mathbb{A}} = \alpha \wedge \beta$ we can use property (iii) of a difference term to see that $x\ \alpha\ p(x, y, z)\ \gamma\ z'$, so $(x, z') \in \gamma \circ \alpha \implies (x, z') \in \alpha \circ \gamma$. □

Corollary A.3.3. *Any abelian algebra which is contained in a congruence modular variety is affine.*

Corollary A.3.4. *A nontrivial algebra $\mathbb{A}$ in a congruence modular variety is abelian iff there is some $\mathbb{B} \le_{sd} \mathbb{A} \times \mathbb{A}$ such that $\mathcal{M}_3$ is a $0, 1$-sublattice of $\mathrm{Con}(\mathbb{B})$.*

Proof. If $\mathbb{A}$ is abelian, then it is affine and we can take $\mathbb{B} = \mathbb{A} \times \mathbb{A}$. For the other direction, it suffices to prove that $\mathbb{B}$ is abelian if $\mathcal{M}_3$ is a $0, 1$-sublattice of $\mathrm{Con}(\mathbb{B})$, since then $\mathbb{B}$ is affine and $\mathbb{A}$ is a quotient of $\mathbb{B}$, so $\mathbb{A}$ is also affine.

Let $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbb{B})$ generate a copy of $\mathcal{M}_3$ which is a $0, 1$-sublattice. Then

$$[1, 1] = [\alpha \vee \beta, \alpha \vee \gamma] = [\alpha, \alpha] \vee [\alpha, \gamma] \vee [\beta, \alpha] \vee [\beta, \gamma] \le \alpha \vee (\beta \wedge \gamma) = \alpha.$$

Similarly we have $[1, 1] \le \beta$, so $[1, 1] \le \alpha \wedge \beta = 0$. □

By plugging a difference term into itself, we can strengthen property (ii) of a Gumm difference term, to get terms which act as Mal'cev operations on solvable algebras.

Definition A.3.5. For any congruence α, define $[\alpha]^n$ inductively by $[\alpha]^0 = \alpha, [\alpha]^{n+1} = [[\alpha]^n, [\alpha]^n]$.

Proposition A.3.6. *If p is a Gumm difference term, and if we define terms p_n inductively by $p_0 = p$ and*

$$p_{n+1}(x, y, z) = p_n(x, p_n(x, y, y), p_n(x, y, z)),$$

then each p_n is also a Gumm difference term, and for any $(x, y) \in \theta$ we have $p_n(x, y, y)\ [\theta]^{2^n}\ x$.

Proof. Inductively, we have

$$p_{n+1}(y,y,x) = p_n(y, p_n(y,y,y), p_n(y,y,x)) = p_n(y,y,x) = x,$$

and from $(x, p_n(x,y,y)) \in [\theta]^{2^n}$, we have

$$p_{n+1}(x,y,y) = p_n(x, p_n(x,y,y), p_n(x,y,y))\ [[\theta]^{2^n}]^{2^n}\ x. \qquad \square$$

Corollary A.3.7. *Any solvable algebra in a congruence modular variety is Mal'cev.*

The last result of this section is useful for understanding the center of an algebra in terms of the difference term.

Theorem A.3.8. *Suppose p is a Gumm difference term for a congruence modular variety and $\alpha \ge \beta$. Then Δ_β^α is given by*

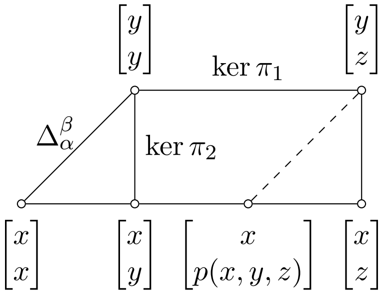
$$\begin{bmatrix} x & w \\ y & z \end{bmatrix} \in \Delta_\beta^\alpha \iff (p(x,y,z)\ [\alpha,\beta]\ w) \wedge (x\ \beta\ y\ \alpha\ z).$$

Proof. If $\begin{bmatrix} x & w \\ y & z \end{bmatrix} \in \Delta_\beta^\alpha$ then clearly $(x\ \beta\ y\ \alpha\ z)$, and from

$$p\left(\begin{bmatrix} x & x \\ x & x \end{bmatrix}, \begin{bmatrix} x & x \\ y & y \end{bmatrix}, \begin{bmatrix} x & w \\ y & z \end{bmatrix}\right) = \begin{bmatrix} x & w \\ p(x,y,y) & p(x,y,z) \end{bmatrix} \in \Delta_\beta^\alpha,$$

we see that from $p(x,y,y)\ [\beta,\beta]\ x$, $[\beta,\beta] \le [\alpha,\beta]$, and the term condition for $[\alpha,\beta]$ we have $w\ [\alpha,\beta]\ p(x,y,z)$.

For the other direction, if $x\ \beta\ y\ \alpha\ z$ then from $\alpha \ge \beta$ we have $(x,y),(x,z) \in \alpha$, so we can apply the defining property (iii) of the difference term to congruences on α to see the implication in the following picture.


$\begin{bmatrix} y \\ y \end{bmatrix}$
$\begin{bmatrix} y \\ z \end{bmatrix}$
$\ker \pi_1$
Δ_α^β
$\ker \pi_2$
$\begin{bmatrix} x \\ x \end{bmatrix}$
$\begin{bmatrix} x \\ y \end{bmatrix}$
$\begin{bmatrix} x \\ p(x,y,z) \end{bmatrix}$
$\begin{bmatrix} x \\ z \end{bmatrix}$

Taking transposes, we have $\begin{bmatrix} x & p(x,y,z) \\ y & z \end{bmatrix} \in \Delta_\beta^\alpha$ by Theorem A.2.11. By Theorems A.2.8 and A.2.11, if $p(x,y,z)\ [\alpha,\beta]\ w$ then $\begin{bmatrix} p(x,y,z) & w \\ z & z \end{bmatrix} \in \Delta_\beta^\alpha$, so $\begin{bmatrix} x & w \\ y & z \end{bmatrix} \in \Delta_\beta^\alpha$ by the fact that Δ_β^α is transitively closed. $\square$

Corollary A.3.9. *If $\alpha \ge \beta$ and $[\alpha,\beta] = 0$, then the restriction of the graph of $p(x,y,z)$ to triples with $x\ \beta\ y\ \alpha\ z$ is preserved by all polynomial operations of $\mathbb{A}$.*

Using this, it's possible to show that if $\mathbb{A}$ has center ζ, then we can write $\mathbb{A}$ as an extension of the quotient $\mathbb{A}/\zeta$ by the abelian algebra ζ/Δ_ζ^1 after making a choice of a section $s : \mathbb{A}/\zeta \to \mathbb{A}$, with each n-ary basic operation f inducing a map $t : (\mathbb{A}/\zeta)^n \to \zeta/\Delta_\zeta^1$ so that the action of f on $\mathbb{A}$ can be decomposed as $(x, y) \mapsto (f^{\zeta/\Delta_\zeta^1}(x) + t(y), f^{\mathbb{A}/\zeta}(y))$. If $\mathbb{A}$ is idempotent, then we can simplify this description slightly by noting that in this case, ζ/Δ_ζ^1 is isomorphic to any congruence class of ζ.

As a consequence of the decomposition of an algebra via its center, nilpotent algebras in congruence modular varieties turn out to be very well-behaved (e.g. they are always Mal'cev and they have regular congruences), and after selecting an element to serve as the identity, one can define an associated nilpotent loop. See Chapter 7 of Freese and McKenzie [80] for details.

A.4 (Directed) Jónsson and Gumm terms

First we give Jónsson's [105] characterization of congruence distributive varieties.

Definition A.4.1. A variety $\mathcal{V}$ is congruence distributive if for every $\mathbb{A} \in \mathcal{V}$, $\mathrm{Con}(\mathbb{A})$ is a distributive lattice, that is, if the inequality

$$\alpha \wedge (\beta \vee \gamma) \le (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$$

holds for all $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbb{A})$.

The prototypical modular lattice which is *not* distributive is the lattice $\mathcal{M}_3$, as the next proposition shows.

Proposition A.4.2 (Birkhoff [28])**.** *In any modular lattice, if a, b, c do not satisfy the distributive law, and if we define elements d, e, f by*

$$d = (b \wedge c) \vee (a \wedge (b \vee c)) = ((b \wedge c) \vee a) \wedge (b \vee c),$$

with e, f defined by cyclic permutations of the variables a, b, c in the above formula, then d, e, f generate a sublattice isomorphic to the diamond lattice $\mathcal{M}_3$.

Proof. Using the modular law, we can check the formulas

$$d \wedge e = e \wedge f = f \wedge d = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$$

and

$$d \vee e = e \vee f = f \vee d = (a \vee b) \wedge (b \vee c) \wedge (c \vee a).$$

If any two of d, e, f are equal, then so are the two displayed expressions, and if we take the wedge of both with a we get

$$a \wedge ((a \vee b) \wedge (b \vee c) \wedge (c \vee a)) = a \wedge (b \vee c)$$

and (using the modular law again)

$$a \wedge ((a \wedge b) \vee (b \wedge c) \vee (c \wedge a)) = (a \wedge b) \vee (a \wedge c). \qquad \square$$

Proposition A.4.3. *A variety is congruence distributive iff it is congruence modular and none of its algebras has a nontrivial abelian congruence. In particular, the commutator is given by $[\alpha, \beta] = \alpha \wedge \beta$ and $p(x, y, z) = z$ is a Gumm difference term in any congruence distributive variety.*

Proof. If α is an abelian congruence, then $\ker \pi_1, \ker \pi_2, \Delta_\alpha^\alpha \in \mathrm{Con}(\alpha)$ generate a sublattice isomorphic to $\mathcal{M}_3$, with top element $\alpha \times \alpha|_\alpha$. The other direction follows from Proposition 1.9.32, since $\mathcal{M}_3$ does not satisfy the meet-semidistributive law $\mathrm{SD}(\wedge)$. □

Example A.4.1. The variety of unital rings is not congruence distributive, even though it is congruence modular (in fact, congruence permutable, since it has a Mal'cev term $x - y + z$) and contains no nontrivial abelian algebras (any such algebra would have $x \cdot y = 0$ for all x, y, and plugging in $y = 1$ would give $x = 0$ for all x). The reason for this is that the congruence on the ring $\mathbb{Z}/p^2$ corresponding to the ideal (p) is abelian, but no congruence class of this ideal forms a unital subring of $\mathbb{Z}/p^2$.

Theorem A.4.4 (Jónsson terms)**.** *A variety is congruence distributive iff it has ternary terms $q_0, ..., q_n$ satisfying the system of identities*

$$\begin{aligned} q_0(x,y,z) &\approx x, \\ q_i(x,y,x) &\approx x \text{ for all } i, \\ q_i(x,y,y) &\approx q_{i+1}(x,y,y) \text{ for } i \text{ odd,} \\ q_i(x,x,y) &\approx q_{i+1}(x,x,y) \text{ for } i \text{ even,} \\ q_n(x,y,z) &\approx z. \end{aligned}$$

Proof. Consider the congruences $\theta_{x,y}, \theta_{y,z}, \theta_{x,z}$ corresponding to identifying pairs of variables on the free algebra $\mathcal{F}(x,y,z)$ in a congruence distributive variety. From $(x,z) \in \theta_{x,z} \wedge (\theta_{x,y} \vee \theta_{y,z})$ and distributivity, we have

$$x \ (\theta_{x,z} \wedge \theta_{x,y}) \vee (\theta_{x,z} \wedge \theta_{y,z}) \ z.$$

Thus there exist $q_0, ..., q_n \in \mathcal{F}(x,y,z)$ with $q_0 = x$, $q_i \ (\theta_{x,z} \wedge \theta_{x,y}) \ q_{i+1}$ for i even, $q_i \ (\theta_{x,z} \wedge \theta_{y,z}) \ q_{i+1}$ for i odd, and $q_n(x,y,z) = z$. In particular, we have $q_i \ \theta_{x,z} \ x$ for all i by induction on i. Thus $q_0, ..., q_n$ satisfy the desired system of identities.

For the converse, suppose that α, β, γ are congruences on any algebra and that $(a,c) \in \alpha \wedge (\beta \vee \gamma)$. We need to show that $(a,c) \in (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$.

From $(a,c) \in \beta \vee \gamma$, there is a sequence $b_0, ..., b_m$ with $a = b_0$, $b_j \ \beta \cup \gamma \ b_{j+1}$ for all j, and $b_m = c$. Since $q_i(a, b_j, c) \ \alpha \ q_i(a, b_j, a) = a$ for all i, j, we then have

$$q_i(a, b_j, c) \ (\alpha \wedge \beta) \cup (\alpha \wedge \gamma) \ q_i(a, b_{j+1}, c)$$

for each i, j, so $q_i(a,a,c) \ (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \ q_i(a,c,c)$ for all i. Stringing these together with the identities relating q_i to q_{i+1}, we see that $a = q_0(a,c,c) \ (\alpha \wedge \beta) \vee (\alpha \wedge \gamma) \ q_n(a,a,c) = c$. □

Example A.4.2. The variety of lattices is congruence distributive. For the Jónsson terms, we may take $n = 2$ and $q_1(x,y,z)$ to be the majority term $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$. More generally, any variety with a near-unanimity term is congruence distributive.

We now prove a permutability result which is directly related to the fact that every congruence modular variety has a sequence of ternary terms known as *Gumm terms*, which look like Jónsson terms "glued to" a Mal'cev term.

Theorem A.4.5. *If α, β are any two congruences in a congruence modular variety, then*

$$\alpha \circ \beta \subseteq [\alpha, \alpha] \circ \beta \circ \alpha.$$

Proof. If $(a, c) \in \alpha \circ \beta$, then there is some b with $a\ \alpha\ b\ \beta\ c$. Applying the Gumm difference term p, we have

$$a\ [\alpha, \alpha]\ p(a, b, b)\ \beta\ p(a, b, c)\ \alpha\ p(b, b, c) = c. \qquad \square$$

Corollary A.4.6. *If α, β, γ are congruences in a congruence modular variety, then*

$$(\alpha \circ \beta) \cap \gamma \subseteq ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) \circ \beta \circ \alpha.$$

Proof. We have $(\alpha \circ \beta) \cap \gamma = ((\alpha \wedge (\beta \vee \gamma)) \circ \beta) \cap \gamma$, and

$$[\alpha \wedge (\beta \vee \gamma), \alpha \wedge (\beta \vee \gamma)] \leq [\alpha, \beta \vee \gamma] = [\alpha, \beta] \vee [\alpha, \gamma] \leq (\alpha \wedge \beta) \vee (\alpha \wedge \gamma).$$

Thus, by the previous theorem we have

$$(\alpha \circ \beta) \cap \gamma \subseteq (\alpha \wedge (\beta \vee \gamma)) \circ \beta \subseteq ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)) \circ \beta \circ \alpha. \qquad \square$$

A very similar argument shows that

$$(\alpha \circ \beta) \cap \gamma \subseteq ((\alpha \wedge \gamma) \vee (\beta \wedge \gamma)) \circ \beta \circ \alpha,$$

which we will use to prove the following result (the corollary above could also be used to prove it, but there is an extra step of reordering the variables if we do it that way). Note that this containment can be viewed as a combination of a distributivity result with a permutability result.

Theorem A.4.7 (Gumm terms)**.** *A variety is congruence modular iff it has ternary terms $q_0, ..., q_n, p$ satisfying the system of identities*

$$\begin{aligned}
q_0(x, y, z) &\approx x,\\
q_i(x, y, x) &\approx x \text{ for all } i,\\
q_i(x, y, y) &\approx q_{i+1}(x, y, y) \text{ for } i \text{ odd},\\
q_i(x, x, y) &\approx q_{i+1}(x, x, y) \text{ for } i \text{ even},\\
q_n(x, y, y) &\approx p(x, y, y),\\
p(x, x, y) &\approx y.
\end{aligned}$$

Furthermore, a ternary term p is a Gumm difference term iff there exist terms $q_0, ..., q_n$ satisfying the above system of identities.

Proof. Consider the congruences $\theta_{x,y}, \theta_{y,z}, \theta_{x,z}$ corresponding to identifying pairs of variables on the free algebra $\mathcal{F}(x, y, z)$ in a congruence distributive variety. From $(x, z) \in \theta_{x,z} \wedge (\theta_{x,y} \vee \theta_{y,z})$ and

$$[\theta_{x,z} \wedge (\theta_{x,y} \vee \theta_{y,z}), \theta_{x,z} \wedge (\theta_{x,y} \vee \theta_{y,z})] \leq (\theta_{x,z} \wedge \theta_{x,y}) \vee (\theta_{x,z} \wedge \theta_{y,z}),$$

which is proved as in the previous corollary, we see that for any Gumm difference term p we have

$$x\ (\theta_{x,z} \wedge \theta_{x,y}) \vee (\theta_{x,z} \wedge \theta_{y,z})\ p(x, z, z).$$

Thus there exist $q_0, ..., q_n \in \mathcal{F}(x, y, z)$ with $q_0 = x$, $q_i\ (\theta_{x,z} \wedge \theta_{x,y})\ q_{i+1}$ for i even, $q_i\ (\theta_{x,z} \wedge \theta_{y,z})\ q_{i+1}$ for i odd, and $q_n(x, y, z) = p(x, z, z)$. Therefore $q_0, ..., q_n, p$ satisfy the desired system of identities.

To see that Gumm terms imply congruence modularity, we just need to show that they imply the existence of Day terms. If we assume without loss of generality that n is odd and take

$$\begin{aligned}
m_0(x, y, z, w) &= x,\\
m_{2i-1}(x, y, z, w) &= q_i(x, w, y) \text{ for } i \text{ even},\\
m_{2i}(x, y, z, w) &= q_i(x, z, y) \text{ for } i \text{ even},\\
m_{2i-1}(x, y, z, w) &= q_i(x, z, y) \text{ for } i \text{ odd},\\
m_{2i}(x, y, z, w) &= q_i(x, w, y) \text{ for } i \text{ odd},\\
m_{2n+1}(x, y, z, w) &= p(z, w, y),\\
m_{2n+2}(x, y, z, w) &= y,
\end{aligned}$$

then we have $m_i(x, x, z, z) \approx x$ for all i, $m_i(x, y, x, y) \approx m_{i+1}(x, y, x, y)$ for i even, and $m_i(x, y, z, z) \approx m_{i+1}(x, y, z, z)$ for i odd, so $m_0, ..., m_{2n+2}$ are Day terms.

To show that any such p is a Gumm difference term, we just need to show that if $(x, y) \in \theta$, then $p(x, y, y)\ [\theta, \theta]\ x$. We will show by induction that $q_i(x, y, y)\ [\theta, \theta]\ x$ for all i. For the inductive step, we just need to show that for all i, we have $q_i(x, y, y)\ [\theta, \theta]\ q_i(x, x, y)$. This follows from the term condition for $[\theta, \theta]$:

$$q_i(x, y, \boxed{x}) = q_i(x, x, \boxed{x}) \implies q_i(x, y, \boxed{y})\ [\theta, \theta]\ q_i(x, x, \boxed{y}). \qquad \square$$

The need to constantly divide into cases for even vs. odd i can be eliminated by the main result of [110], which establishes the existence of *directed* Jónsson and Gumm terms. The idea behind the directed variants is that if we have idempotent ternary terms f, g which satisfy

$$f(x, y, y) \approx g(x, x, y),$$

then they can also be indirectly connected by a ternary term h which satisfies $h(x, y, x) \approx x$ and joins f, g by $f\ \theta_{y,z}\ h\ \theta_{x,y}\ g$, that is,

$$\begin{aligned}
f(x, y, y) &\approx h(x, y, y),\\
h(x, x, y) &\approx g(x, x, y).
\end{aligned}$$

In fact, we can just take $h(x, y, z) = f(x, z, z)$: then we will have $h(x, y, y) = h(x, x, y) = f(x, y, y) = g(x, x, y)$, and $h(x, y, x) = f(x, x, x) = x$. The goal of the directed Jónsson and Gumm terms is to cut out the middleman h, to obtain a substantially stronger system of identities.

Another reason to prefer the directed equations $f_i(x, y, y) \approx f_{i+1}(x, x, y)$ is that they have a clearer connection to higher arity terms, especially near-unanimity terms. Suppose that ϕ is an n-ary operation, and define terms f_i by

$$f_i(x, y, z) = \phi(x, ..., x, y, z, ..., z),$$

where the lone y occurs in the i-th position from the right (so there are $i - 1$ zs). Then the f_i will automatically satisfy

$$f_i(x, y, y) = \phi(x, ..., x, y, y, ..., y) = f_{i+1}(x, x, y),$$

and if ϕ is idempotent they will satisfy $f_1(x, x, y) \approx x$ and $f_n(x, y, y) \approx y$. Finally, ϕ will be a near-unanimity term iff each f_i satisfies $f_i(x, y, x) \approx x$.

Theorem A.4.8 (Directed Gumm terms [110]). *A variety is congruence modular iff it has ternary terms $f_1, ..., f_m, p$ with*

$$\begin{aligned} f_1(x,x,y) &\approx x, \\ f_i(x,y,x) &\approx x \text{ for all } i, \\ f_i(x,y,y) &\approx f_{i+1}(x,x,y) \text{ for all } i, \\ f_m(x,y,y) &\approx p(x,y,y), \\ p(x,x,y) &\approx y, \end{aligned}$$

and if the variety is congruence distributive then we can take $f_m(x,y,y) \approx y$ (directed Jónsson terms).

Proof. Assume without loss of generality that our variety is idempotent. Suppose that there are Gumm terms $q_1, ..., q_{2k+1}, p_1$ with

$$\begin{aligned} q_1(x,x,y) &\approx x, \\ q_i(x,y,x) &\approx x \text{ for all } i, \\ q_{2i-1}(x,y,y) &\approx q_{2i}(x,y,y) \text{ for all } i, \\ q_{2i}(x,x,y) &\approx q_{2i+1}(x,x,y) \text{ for all } i, \\ q_{2k+1}(x,y,y) &\approx p_1(x,y,y), \\ p_1(x,x,y) &\approx y. \end{aligned}$$

Let $\mathcal{F}$ be the free algebra on x, y. Let $\rightsquigarrow$ be the transitive closure of the binary relation on $\mathcal{F}$ generated by $x \rightsquigarrow x, x \rightsquigarrow y, y \rightsquigarrow y$, so binary terms $a(x,y), b(x,y)$ have $a \rightsquigarrow b$ iff there is a sequence of ternary terms t_i with $t_1(x,x,y) = a(x,y)$, $t_i(x,y,y) = t_{i+1}(x,x,y)$, and $t_n(x,y,y) = b(x,y)$.

Additionally, let $\rightarrow$ be the relation on $\mathcal{F}$ with $a \rightarrow b$ iff there is a sequence of ternary terms t_i with $t_1(x,x,y) = a(x,y)$, $t_i(x,y,y) = t_{i+1}(x,x,y)$, $t_n(x,y,y) = b(x,y)$, and additionally $t_i(x,y,x) = x$ for all i. Then for any ternary term q satisfying $q(x,y,x) = x$, we have

$$q(\rightarrow, \rightsquigarrow, \rightarrow) \subseteq \rightarrow .$$

For any binary term $a(x,y)$, we define $a^n(x,y)$ recursively by $a^0(x,y) = y, a^1(x,y) = a(x,y)$, and

$$a^{n+1}(x,y) = a(x, a^n(x,y))$$

for each n.

Setting $b_k(x,y) = q_{2k+1}(x,y,y) = p_1(x,y,y)$, our goal will be to prove that

$$\exists b \in \mathcal{F} \quad x \rightarrow b_k^{2^k}(b(x,y), b_k^{2^k-1}(x,y)).$$

It will then be easy to construct a ternary term p with $p(x,y,y) = b_k^{2^k}(b, b_k^{2^k-1})$ and $p(x,x,y) = y$, by recursively plugging p_1 into itself in a similar way to the way we constructed Mal'cev terms on solvable algebras.

Claim 1: If $a \rightsquigarrow b$ and $c(x,y) \rightarrow d(x,y)$, then $c(a,b) \rightarrow d(a,b)$.

Proof of Claim 1: We just have to check this in the case where $c \rightarrow d$ in one step. So suppose that $t(x,x,y) = c(x,y), t(x,y,y) = d(x,y), t(x,y,x) = x$. Then

$$\begin{bmatrix} c(a,b) \\ d(a,b) \end{bmatrix} = t\left(\begin{bmatrix} a \\ a \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ b \end{bmatrix} \right) \in t(\rightarrow, \rightsquigarrow, \rightarrow) \subseteq \rightarrow .$$

Claim 1.5: If $a \rightsquigarrow b$ and $c(x,y) \leftarrow d(x,y)$, then $c(b,a) \rightarrow d(b,a)$.

Proof of Claim 1.5: This follows from Claim 1 and the fact that $c(x,y) \leftarrow d(x,y) \iff c(y,x) \rightarrow d(y,x)$.

Claim 2: If $a \rightarrow b$, then $a^n \rightarrow b^n$ for every n.

Proof of Claim 2: Induct on n. For the inductive step, we have

$$a^{n+1}(x,y) = a(x, a^n(x,y)) \rightarrow b(x, a^n(x,y)) \rightarrow b(x, b^n(x,y)) = b^{n+1}(x,y),$$

where the first $\rightarrow$ follows from $x = a^n(x,x) \rightsquigarrow a^n(x,y)$ and Claim 1, while the second $\rightarrow$ follows from the fact that $\rightarrow$ is preserved by b and the inductive hypothesis.

The sequence of Gumm terms $q_1, ..., q_{2k+1}$ gives us a *k-fence*:

$$x = a_0 \rightarrow b_0 \leftarrow a_1 \rightarrow b_1 \leftarrow a_2 \rightarrow \cdots \leftarrow a_k \rightarrow b_k,$$

where $a_i(x,y) = q_{2i+1}(x,x,y) = q_{2i}(x,x,y)$, $b_i(x,y) = q_{2i+1}(x,y,y) = q_{2i+2}(x,y,y)$. Our strategy will be to use Claims 1 and 1.5 to iteratively reduce the length of the fence.

Claim 3: If $x \rightarrow b \leftarrow a \rightarrow c$ is a 1-fence, then $x \rightarrow b_k(b, c(b,c))$.

Proof of Claim 3: We define a sequence of terms d_i by $d_0 = x$ and

$$d_{i+1} = b(d_i, a),$$

and define terms e_i by

$$e_i = a(d_i, a).$$

We claim that for each i we have

- $d_i \rightsquigarrow a$, $d_i \rightarrow d_{i+1}$, $d_i \rightsquigarrow e_i$, $d_i \rightarrow b$,
- $e_i \rightarrow d_{i+1}$, $e_i \rightarrow e_{i+1}$, $e_i \rightarrow c(b,c)$.

$$\begin{tikzcd}
x = d_0 \arrow[rr] \arrow[dr, rightsquigarrow] & & d_1 \arrow[rr] \arrow[dr, rightsquigarrow] & & d_2 \arrow[rr] \arrow[dr, rightsquigarrow] & & b & \\
& e_0 \arrow[rr] \arrow[ur] & & e_1 \arrow[rr] \arrow[ur] & & e_2 \arrow[rr] & & c(b,c)
\end{tikzcd}$$

To see this, note first that $d_0 = x \rightsquigarrow a$, so by induction on i we have $d_{i+1} = b(d_i, a) \rightsquigarrow b(a,a) = a$ for each i. So from $x \rightarrow b \leftarrow a$ we get $d_i \rightarrow b(d_i, a) \leftarrow a(d_i, a)$ by Claim 1, that is, $d_i \rightarrow d_{i+1} \leftarrow e_i$ for each i.

Then we have $e_i = a(d_i, a) \rightarrow a(d_{i+1}, a) = e_{i+1}$ for each i, and $d_i = a(d_i, d_i) \rightsquigarrow a(d_i, a) = e_i$ for each i. This finishes up all of the arrows other than the rightmost two in the picture.

For $d_i \rightarrow b$, note that $d_0 = x \rightarrow b$ by assumption, and $d_{i+1} = b(d_i, a) \rightarrow b(b,b) = b$ inductively. Finally, for each i we have

$$e_i = a(d_i, a) \rightarrow c(d_i, a) \rightarrow c(b,c),$$

where the first arrow follows from Claim 1.

Now we can use all these arrows to see that

$$x = d_0 = a_0(d_0, e_0) \rightarrow b_0(d_0, e_0) \rightarrow b_0(d_1, e_0) \rightarrow a_1(d_1, e_0) \rightarrow a_1(d_1, e_1) \rightarrow b_1(d_1, e_1) \rightarrow \cdots,$$

where we have used Claim 1 and Claim 1.5 several times. Chaining these together, we get

$$x \rightarrow b_k(d_k, e_k) \rightarrow b_k(b, c(b,c)).$$

This completes the proof of Claim 3.

Claim 4: For each $i < k$, there is a $k-i$-fence

$$x \to b_{0,i} \leftarrow a_{1,i} \to b_{1,i} \leftarrow a_{2,i} \to \cdots \leftarrow a_{k-i,i} \to b_{k-i,i} = b_k^{2^{i+1}-1}.$$

Proof of Claim 4: We prove this by induction on i. The base case $i = 0$ comes from the Gumm terms. Suppose it is known for i, then by Claim 3 we have

$$x \to b_k(b_{0,i}, b_{1,i}(b_{0,i}, b_{1,i})),$$

and from $b_{0,i} \leftarrow x$ we have

$$b_k(b_{0,i}, b_{1,i}(b_{0,i}, b_{1,i})) \leftarrow b_k(x, b_{1,i}(x, b_{1,i})) = b_k(x, b_{1,i}^2).$$

By Claim 2, we have $b_{1,i}^2 \leftarrow a_{2,i}^2 \to b_{2,i}^2 \leftarrow \cdots$, so if we take

$$b_{0,i+1} = b_k(b_{0,i}, b_{1,i}(b_{0,i}, b_{1,i}))$$

and

$$a_{j,i+1} = b_k(x, a_{j+1,i}^2), \quad b_{j,i+1} = b_k(x, b_{j+1,i}^2),$$

we get

$$x \to b_{0,i+1} \leftarrow a_{1,i+1} \to b_{1,i+1} \leftarrow a_{2,i+1} \to \cdots \leftarrow a_{k-i-1,i+1} \to b_{k-i-1,i+1},$$

and

$$b_{k-i-1,i+1} = b_k(x, b_{k-i,i}^2) = b_k(x, (b_k^{2^{i+1}-1})^2) = b_k^{2^{i+2}-1}.$$

This completes the proof of Claim 4.

By Claim 4 applied with $i = k-1$, we get a 1-fence

$$x \to b_{0,k-1} \leftarrow a_{1,k-1} \to b_{1,k-1} = b_k^{2^k-1}.$$

Applying Claim 3, we get

$$x \to b_k(b_{0,k-1}, b_k^{2^k-1}(b_{0,k-1}, b_k^{2^k-1})) = b_k^{2^k}(b_{0,k-1}, b_k^{2^k-1}).$$

Letting $b = b_{0,k-1}$, we see that we have succeeded in showing that $x \to b_k^{2^k}(b, b_k^{2^k-1})$. Thus there exist ternary terms f_i with

$$\begin{aligned} f_1(x,x,y) &\approx x, \\ f_i(x,y,x) &\approx x \text{ for all } i, \\ f_i(x,y,y) &\approx f_{i+1}(x,x,y) \text{ for all } i, \\ f_m(x,y,y) &\approx b_k^{2^k}(b(x,y), b_k^{2^k-1}(x,y))). \end{aligned}$$

Note that if $b_k(x,y) = y$, then we also have $b_k^{2^k}(b(x,y), b_k^{2^k-1}(x,y))) = y$, so the above becomes a sequence of directed Jónsson terms.

To finish, we just need to construct p with $p(x,y,y) = b_k^{2^k}(b(x,y), b_k^{2^k-1}(x,y)))$ and $p(x,x,y) = y$. Recall that p_1 satisfied $p_1(x,y,y) = b_k(x,y)$ and $p_1(x,x,y) = y$. We construct terms p_i inductively. For $2 \le i+1 < 2^k$, we set

$$p_{i+1}(x,y,z) = p_1(x, p_i(x,y,y), p_i(x,y,z)),$$

and for $2^k \le i+1$, we set

$$p_{i+1}(x,y,z) = p_1(b(x,y), p_i(x,y,y), p_i(x,y,z)),$$

and finally we set $p(x,y,z) = p_{2^{k+1}-1}(x,y,z)$. □

A.5 Subdirectly irreducible algebras, ultraproducts, and residually small varieties

In this section, we go over the proof of an extension of Jónsson's Lemma [105], which shows that subdirectly irreducible algebras in a finitely generated congruence distributive variety have bounded size, to the congruence modular case. The key technical tool is the concept of an ultraproduct, and the fact that any ultrapower of a finite algebra $\mathbb{A}$ is isomorphic to $\mathbb{A}$.

Before we discuss ultraproducts, we first review some basic results about subdirect representations of algebras due to Birkhoff [29]. The following result is elementary.

Proposition A.5.1. *If $\mathbb{A} \le_{sd} \prod_{i\in I} \mathbb{A}_i$ is a subdirect product, then $\bigwedge_{i\in I} \ker \pi_i = 0_{\mathbb{A}}$. In particular, if no π_i is an isomorphism then the congruence $0_{\mathbb{A}}$ can be written as a meet of some family of nontrivial congruences.*

Conversely, if $0_{\mathbb{A}}$ can be written as a meet of congruences $\alpha_i \in \mathrm{Con}(\mathbb{A})$ for $i \in I$, then $\mathbb{A} \le_{sd} \prod_{i\in I} \mathbb{A}/\alpha_i$.

Definition A.5.2. An algebraic structure $\mathbb{A}$ is *subdirectly irreducible* if every way of writing $\mathbb{A}$ as a subdirect product $\mathbb{A} \le_{sd} \prod_{i\in I} \mathbb{A}_i$ has at least one coordinate i such that the projection map $\pi_i : \mathbb{A} \to \mathbb{A}_i$ is an isomorphism. The least nontrivial congruence on a subdirectly irreducible algebra is called its *monolith*.

The preceeding proposition can now be rephrased as saying that $\mathbb{A}$ is subdirectly irreducible iff $0_{\mathbb{A}}$ is *meet-irreducible*.

Definition A.5.3. An element α of a complete lattice $\mathcal{L}$ is *meet-irreducible* if for any set of elements $\alpha_i \in \mathcal{L}$ with $\bigwedge_{i\in I} \alpha_i = \alpha$, some α_i is equal to α. In this case, we define the *cover* of α, written α^*, to be the least element of $\mathcal{L}$ with $\alpha < \alpha^*$.

In particular, the monolith of a subdirectly irreducible algebra is the cover $0_{\mathbb{A}}^*$ of $0_{\mathbb{A}}$.

Theorem A.5.4 (Birkhoff's Subdirect Representation Theorem)**.** *Any algebraic structure $\mathbb{A}$ can be represented as a subdirect product of subdirectly irreducible algebras.*

Proof. For any $a \ne b \in \mathbb{A}$, Zorn's Lemma implies that there is a maximal congruence $\theta'_{a,b}$ such that $(a, b) \not\in \theta'_{a,b}$. Any such $\theta'_{a,b}$ is necessarily meet-irreducible, since any congruence which properly contains $\theta'_{a,b}$ necessarily contains (a, b), and therefore contains the congruence generated by $\theta'_{a,b}$ and the pair (a, b).

Since we clearly have $0_{\mathbb{A}} = \bigwedge_{a\ne b} \theta'_{a,b}$, we have the subdirect representation $\mathbb{A} \le_{sd} \prod_{a\ne b} \mathbb{A}/\theta'_{a,b}$. □

Birkhoff's subdirect representation theorem has a purely lattice-theoretic generalization to *algebraic* lattices.

Definition A.5.5. An element α of a complete lattice is called *compact* if for any family α_i such that $\alpha \le \bigvee_{i\in I} \alpha_i$, there is some finite subset $\{i_1, ..., i_k\} \subseteq I$ such that $\alpha \le \alpha_{i_1} \vee \cdots \vee \alpha_{i_k}$. A complete lattice is called *algebraic* if every element can be written as a join of compact elements.

Every congruence lattice $\mathrm{Con}(\mathbb{A})$ is an algebraic lattice, since for any $a, b \in \mathbb{A}$ the congruence $\theta_{a,b}$ generated by (a, b) is compact, and every congruence is a join of such congruences.

Proposition A.5.6. *Let $\mathcal{L}$ be an algebraic lattice. Then every element α of $\mathcal{L}$ can be written as a meet of some family of meet-irreducible elements of $\mathcal{L}$.*

Proof. Let θ be any compact element of $\mathcal{L}$ with $\alpha \not\geq \theta$. By Zorn's Lemma and the compactness of θ, there is some $\theta' \geq \alpha$ which is maximal such that $\theta' \not\geq \theta$, and this θ' is necessarily meet-irreducible with cover $\theta' \vee \theta$. Then $\bigwedge_{\theta \not\leq \alpha} \theta'$ is $\geq \alpha$, and is not $\geq$ any compact element θ with $\alpha \not\geq \theta$, so it must be equal to α. □

Corollary A.5.7. *If $\alpha < \beta$ in an algebraic lattice, then there is a meet-irreducible γ such that $\gamma \geq \alpha$ but $\gamma \not\geq \beta$.*

Now we can briefly discuss ultrafilters and ultraproducts before moving on to the main result of this section.

Definition A.5.8. If I is a set, then a collection of subsets $\mathcal{U} \subseteq \mathcal{P}(I)$ is a *filter* if $\mathcal{U}$ does not contain $\emptyset$, $U, V \in \mathcal{U} \implies U \cap V \in \mathcal{U}$, and $U \subseteq V, U \in \mathcal{U} \implies V \in \mathcal{U}$. We say that $\mathcal{U}$ is an *ultrafilter* if additionally for every $U \subseteq I$, one of $U, I \setminus U$ is in $\mathcal{U}$.

Proposition A.5.9. *Any filter is contained in an ultrafilter.*

Proof. We apply Zorn's Lemma to see that any filter is contained in a maximal filter. To finish, we just need to show that any maximal filter is an ultrafilter. Suppose that $U, I \setminus U \notin \mathcal{U}$, and let $\mathcal{U}'$ be the collection of $V \subseteq I$ such that $V \cup U \in \mathcal{U}$. Then $\mathcal{U}'$ is a filter which strictly contains $\mathcal{U}$. □

Definition A.5.10. If $\mathbb{A}_i$ is a collection of structures which share a common signature σ and are indexed by $i \in I$, and if $\mathcal{U}$ is an ultrafilter on I, then we define the *ultraproduct* $\prod_i \mathbb{A}_i/\mathcal{U}$ to be the quotient of $\prod_i \mathbb{A}_i$ by the congruence defined by

$$a \equiv_{\mathcal{U}} b \iff \{i \mid a_i = b_i\} \in \mathcal{U}.$$

That $\equiv_{\mathcal{U}}$ is compatible with functions $f \in \sigma$ follows from the fact that $\mathcal{U}$ is a filter. If $R \in \sigma$ is an m-ary relation, then R is interpreted on $\prod_i \mathbb{A}_i/\mathcal{U}$ by

$$R(a^1/\mathcal{U}, ..., a^m/\mathcal{U}) \iff \{i \mid R(a_i^1, ..., a_i^m)\} \in \mathcal{U}.$$

If all the $\mathbb{A}_i$ are isomorphic to $\mathbb{A}$, then we call $\mathbb{A}^I/\mathcal{U}$ an *ultrapower* of $\mathbb{A}$.

Note that in terms of the congruence lattice $\operatorname{Con}(\prod_i \mathbb{A}_i)$, the congruence $\equiv_{\mathcal{U}}$ is equal to the join

$$\bigvee_{U \in \mathcal{U}} \ker \pi_U,$$

where $\pi_U : \prod_{i \in I} \mathbb{A}_i \to \prod_{i \in U} \mathbb{A}_i$ is projection onto the coordinates in U. That this join is equal to the union $\bigcup_{U \in \mathcal{U}} \ker \pi_U$ follows from the fact that $\mathcal{U}$ is a filter.

Proposition A.5.11. *If $\mathcal{U}$ is an ultrafilter on I and $U_1, ..., U_k$ partition I into k disjoint parts, then exactly one of $U_1, ..., U_k$ is in $\mathcal{U}$.*

Corollary A.5.12. *If $|\mathbb{A}_i| \leq n$ for all $i \in I$, then $|\prod_i \mathbb{A}_i/\mathcal{U}| \leq n$ as well. If each $\mathbb{A}_i$ is finite and only finitely many isomorphism classes occur among the $\mathbb{A}_i$, then $\prod_i \mathbb{A}_i/\mathcal{U}$ is isomorphic to some $\mathbb{A}_i$.*

In fact, much more is true about ultraproducts, and the corollary above also follows from the following result from model theory.

Theorem A.5.13 (Łoś's Theorem)**.** *Let $\varphi(x_1, ..., x_n)$ be any first order formula in the signature σ with parameters $x_1, ..., x_n$, then for any $a^1, ..., a^n \in \prod_{i\in I} \mathbb{A}_i$ and any ultrafilter $\mathcal{U}$ on I, we have*

$$\prod_i \mathbb{A}_i/\mathcal{U} \models \varphi(a^1/\mathcal{U}, ..., a^n/\mathcal{U}) \iff \{i \mid \mathbb{A}_i \models \varphi(a^1_i, ..., a^n_i)\} \in \mathcal{U}.$$

Proof. If φ is atomic, then this follows directly from the definitions. Otherwise, φ can be built up from atomic formulas via $\neg, \wedge, \exists$, and we can induct on the structure of φ: for $\neg$, we use the ultrafilter property that exactly one of $U, I \setminus U$ is in $\mathcal{U}$ for each U, for $\wedge$ we use the filter property that intersections of sets in $\mathcal{U}$ are in $\mathcal{U}$, and for $\exists$ we just need the fact that supersets of sets in $\mathcal{U}$ are in $\mathcal{U}$. □

Now for the main result. We extend Birkhoff's H, S, P notation by the operation P_u, where $P_u(\{\mathbb{A}_i\})$ is the collection of ultraproducts of the $\mathbb{A}_i$s. Recall that for β a congruence, the centralizer $(0 : \beta)$ of β is defined as the largest α such that $[\alpha, \beta] = 0$, and more generally $(\delta : \beta)$ is defined as the largest α such that $[\alpha, \beta] \le \delta$.

Theorem A.5.14. *Let $\{\mathbb{A}_i\}$ be a family of algebras, and let $\mathcal{V} = \mathcal{V}(\{\mathbb{A}_i\})$ be the variety they generate. If $\mathcal{V}$ is congruence modular, $\mathbb{B} \in \mathcal{V}$ is subdirectly irreducible, and $\alpha = (0_{\mathbb{B}} : 0^*_{\mathbb{B}})$ is the centralizer of the monolith $0^*_{\mathbb{B}}$ of $\mathbb{B}$, then $\mathbb{B}/\alpha$ is a homomorphic image of a subalgebra of an ultraproduct of the $\mathbb{A}_i$s, that is, $\mathbb{B}/\alpha \in HSP_u(\{\mathbb{A}_i\})$.*

Proof. (From [80], where a stronger statement is proved.) By Birkhoff's HSP Theorem, we can write $\mathbb{B} = \mathbb{C}/\theta$ for $\mathbb{C} \le \prod_i \mathbb{A}_i$. Then $\mathbb{B}$ will be subdirectly irreducible iff θ is meet-irreducible in $\mathrm{Con}(\mathbb{C})$, so θ will have a cover θ^*. The preimage φ of α under $\mathbb{C} \to \mathbb{B}$ is the largest congruence on $\mathbb{C}$ such that $[\varphi, \theta^*] \le \theta$ (i.e. $\varphi = (\theta : \theta^*)$), and we have $\mathbb{B}/\alpha = \mathbb{C}/\varphi$.

The main step of the proof is the following **claim:** if $\beta \wedge \gamma \le \theta$ but $\gamma \not\le \theta$, then $\beta \le \varphi$.

Proof of claim: We have

$$[\beta, \theta^*] \le [\beta, \gamma \vee \theta] = [\beta, \gamma] \vee [\beta, \theta] \le (\beta \wedge \gamma) \vee \theta = \theta,$$

so $\beta \le \varphi$ by $\varphi = (\theta : \theta^*)$.

Using the claim, we can now argue as follows: let $\mathcal{F}$ be a maximal filter such that $U \in \mathcal{F}$ implies $\ker \pi_U \le \theta$, and let $\mathcal{U}$ be any ultrafilter which extends $\mathcal{F}$. Then for any $U \in \mathcal{U}$, we were unable to adjoin its complement to $\mathcal{F}$, so there is some $V \in \mathcal{F}$ such that $\ker \pi_{V\setminus U} \not\le \theta$. Then

$$\ker \pi_U \wedge \ker \pi_{V\setminus U} = \ker \pi_{U\cup V} \le \ker \pi_V \le \theta,$$

so by the claim we have $\ker \pi_U \le \varphi$. Thus the congruence $\bigvee_{U\in\mathcal{U}} \ker \pi_U$ corresponding to $\mathcal{U}$ is also $\le \varphi$, and we see that $\mathbb{B}/\alpha = \mathbb{C}/\varphi$ is a quotient of $\mathbb{C}/\mathcal{U} \le \prod_i \mathbb{A}_i/\mathcal{U}$. □

Corollary A.5.15 (Jónsson's Lemma [105])**.** *Let $\{\mathbb{A}_i\}$ be a family of algebras, and let $\mathcal{V} = \mathcal{V}(\{\mathbb{A}_i\})$ be the variety they generate. If $\mathcal{V}$ is congruence distributive and $\mathbb{B} \in \mathcal{V}$ is subdirectly irreducible, then $\mathbb{B} \in HSP_u(\{\mathbb{A}_i\})$. In particular, if $\{\mathbb{A}_i\}$ is a finite set of finite algebras, then $\mathbb{B} \in HS(\{\mathbb{A}_i\})$.*

Corollary A.5.16. *For any two finite subdirectly irreducible algebras $\mathbb{A}, \mathbb{B}$ with the same signature which generate congruence distributive varieties, we have $\mathbb{A} \cong \mathbb{B}$ iff the set of identities that hold in $\mathbb{A}$ is the same as the set of identities that hold in $\mathbb{B}$.*

Example A.5.1. Consider the variety of distributive lattices, and the two-element lattice $(\{0,1\}, \max, \min)$. It is easy to see that every identity that holds in the two-element lattice is implied by the lattice axioms together with distributivity (since these allow us to put every term into conjunctive normal form), so the variety of distributive lattices is generated by the two-element lattice.

By Jónsson's Lemma, the only subdirectly irreducible distributive lattice is the two-element lattice itself, so we see that in fact every distributive lattice is a sublattice of $\{0,1\}^I$ for some index set I, that is, every distributive lattice is a sublattice of the lattice of subsets of some set I.

In order to understand subdirectly irreducible algebras in congruence modular varieties, we need to combine the above results with an understanding of subdirectly irreducible modules over rings.

Proposition A.5.17. *Let $\mathbb{G}, \mathbb{M}$ be abelian groups and let $\mathbb{R}$ be a finite subgroup of* $\mathrm{Hom}(\mathbb{G}, \mathbb{M})$*, such that there is a nonzero element $a \in \mathbb{M}$ so that for all $x \in \mathbb{G}$ there is an $r \in \mathbb{R}$ with $rx = a$. Then $|\mathbb{G}|$ is a prime power dividing $|\mathbb{R}|$.*

Proof. First we show that $\mathbb{G}$ is finite, following [80]. Let $r_1, ..., r_k$ be the nontrivial elements of $\mathbb{R}$.

We will show by induction on k that $|\mathbb{G}| \le (k+1)!$. For the base case, if $k = 0$ then $\mathbb{G}$ can have no nonzero elements, so $|\mathbb{G}| = 1 = (k+1)!$. For the inductive step, note that by the pigeonhole principle there is some r_i such that at least $\frac{|\mathbb{G}|-1}{k}$ elements are mapped to a by r_i, so $|\ker r_i| \ge \frac{|\mathbb{G}|-1}{k}$ (this is the ordinary group theoretic kernel), and every nonzero element of $\ker r_i$ can be mapped to a by some r_j with $j \ne i$, so $|\ker r_i| \le k!$ by the induction hypothesis. Thus $|\mathbb{G}| \le 1 + k \cdot k! \le (k+1)!$.

Now that we know that $\mathbb{G}$ is finite, we know that every element of $\mathbb{G}$ has finite order, so some element x has order p for some prime p. Then there is some $r \in \mathbb{R}$ with $rx = a$, so a must also have order p. From this argument, we see that every element of $\mathbb{G}$ must have order a power of p, so $|\mathbb{G}|$ is also a power of p.

We may assume without loss of generality that $\mathbb{M}$ is generated by the image of $\mathbb{G}$ under all elements of $\mathbb{R}$, so in particular that $\mathbb{M}$ is finite. Then there exists an element $\pi \in \hat{\mathbb{M}} = \mathrm{Hom}(\mathbb{M}, \mathbb{Q}/\mathbb{Z})$ such that $\pi(a) \ne 0$.

Define a linear map $\phi : \mathbb{R} \to \hat{\mathbb{G}} = \mathrm{Hom}(\mathbb{G}, \mathbb{Q}/\mathbb{Z})$ by $\phi : r \mapsto \phi_r$, where ϕ_r is the linear map $\phi_r : x \mapsto \pi(rx)$. Then ϕ must be surjective, or else the image will be a proper subgroup of $\hat{\mathbb{G}}$ and so there will be some nonzero $x \in \mathbb{G}$ with $\phi_r(x) = 0$ for all $r \in \mathbb{R}$, which implies $rx \ne a$ for all r. Thus $|\mathbb{G}| = |\hat{\mathbb{G}}|$ divides $|\mathbb{R}|$. □

Corollary A.5.18. *Let $\mathbb{R}$ be a finite ring, and let $\mathbb{M}$ be a subdirectly irreducible module over $\mathbb{R}$. Then $|\mathbb{M}|$ is a prime power dividing $|\mathbb{R}|$.*

Proof. If $\mathbb{M}$ is subdirectly irreducible, then it has a least nontrivial submodule $\mathbb{N}$, which is generated by some nonzero element $a \in \mathbb{N}$. Then for each nonzero $x \in \mathbb{M}$ we have $\mathbb{N} \le \mathbb{R}x$, so there is some $r \in \mathbb{R}$ with $rx = a$. Thus we can apply the previous proposition with $\mathbb{G} = \mathbb{M}$. □

Now we can use this result to bound the sizes of subdirectly irreducible algebras in congruence modular varieties in the special case where the centralizer of the monolith is abelian.

Theorem A.5.19. *Suppose that $\mathbb{B} \in \mathcal{V}$ is subdirectly irreducible, and $\mathcal{V}$ is locally finite and congruence modular. If $\alpha \in$ Con$(\mathbb{B})$ is abelian and $|\mathbb{B}/\alpha| = k$, then every congruence class of α has size a prime power bounded by $|\mathcal{F}_{\mathcal{V}}(k+1)|$.*

Proof. (Adapted from [143].) Assume α is nontrivial, so $0_{\mathbb{B}}^* \le \alpha$. Let p be a Gumm difference term. By Corollary A.3.9, the restriction of the graph of p to the blocks of α is preserved by every operation of $\mathbb{B}$. Choose elements $0 \ne a$ with $(0,a) \in 0_{\mathbb{B}}^*$, and note that $0, a$ are in the same congruence block of α.

Pick constants $c_0, ..., c_{k-1}$ with $c_0 = 0$ such that each congruence class of α contains some c_i. We will treat each congruence class c_i/α of α as an abelian group with zero element c_i, addition given by $x +_i y = p(x, c_i, y)$, and subtraction given by $x -_i y = p(x, y, c_i)$.

Suppose that $x \ne y$ with $(x,y) \in \alpha$. Then since $0_{\mathbb{B}}^*$ is the least nontrivial congruence, the pair $(0,a)$ must be in the congruence generated by (x,y), so there must be a chain of unary polynomials f_i such that $0 = f_0(x)$, $f_i(y) = f_{i+1}(x)$, and $f_m(y) = a$. Note that this implies that $f_i(x), f_i(y)$ are all in the congruence class $0/\alpha$. Thus, it makes sense to define a unary polynomial f such that

$$f(z) = f_0(z) +_0 f_1(z) -_0 f_1(x) +_0 \cdots +_0 f_m(z) -_0 f_m(x)$$

for z in the congruence class x/α. One explicit way to construct such an f is given by

$$f(z) = p(p(\cdots p(p(f_0(z), f_1(x), f_1(z)), f_2(x), f_2(z)), \cdots), f_m(x), f_m(z)).$$

It's easy to check that we have $f(x) = 0$ and $f(y) = a$. Since f preserves the graph of p restricted to congruence classes of α, if $x, y \in c_i/\alpha$ then we have $f(x -_i y) -_0 f(c_i) = a$, and the unary polynomial $z \mapsto f(z) -_0 f(c_i)$ defines a linear map in $\mathrm{Hom}(c_i/\alpha, c_0/\alpha)$.

To finish, we just need to bound the size of the subgroup $\mathbb{R}_{i,0}$ of linear maps in $\mathrm{Hom}(c_i/\alpha, c_0/\alpha)$ which can be defined by unary polynomials f. Suppose that $f(z) = t(z, b_1, ..., b_m)$ for some term t and constants $b_1, ..., b_m \in \mathbb{B}$, such that $f(c_i) = c_0$. For each b_i, we choose j_i such that $b_i \in c_{j_i}/\alpha$. Define a unary polynomial f' by

$$f'(z) = t(z, c_{j_1}, ..., c_{j_m}) -_0 t(c_i, c_{j_1}, ..., c_{j_m}).$$

Then for $z \in c_i/\alpha$, we have $f'(z) \in c_0/\alpha$, and since t preserves the graph of p restricted to congruence classes of α, we have $f'(z) = f(z)$ for $z \in c_i/\alpha$ (alternatively, we could prove this by the term condition for $[\alpha, \alpha] = 0_{\mathbb{B}}$). Thus every element of $\mathrm{Hom}(c_i/\alpha, c_0/\alpha)$ which can be defined by a unary polynomial can also be defined by a polynomial f' which has the form $f'(z) = t'(z, c_0, ..., c_{k-1})$ for some $k+1$-ary term t', so

$$|\mathbb{R}_{i,0}| \le |\mathcal{F}_{\mathcal{V}}(k+1)|.$$

Applying the previous proposition, we see that $|c_i/\alpha|$ is a prime power dividing $|\mathbb{R}_{i,0}|$. □

Corollary A.5.20. *If $|\mathbb{A}| = m$ is finite and $\mathcal{V}(\mathbb{A})$ is congruence modular, and if $\mathbb{B} \in \mathcal{V}(\mathbb{A})$ is subdirectly irreducible with $(0_{\mathbb{B}} : 0_{\mathbb{B}}^*)$ abelian, then $|\mathbb{B}| \le m \cdot m^{m^{m+1}}$.*

Definition A.5.21. A variety $\mathcal{V}$ is called *residually small* if there is a cardinal κ such that every subdirectly irreducible algebra $\mathbb{B} \in \mathcal{V}$ has $|\mathbb{B}| < \kappa$, and *residually finite* if every subdirectly irreducible algebra in $\mathcal{V}$ is finite. An algebra $\mathbb{A}$ is called residually small if the variety $\mathcal{V}(\mathbb{A})$ generated by $\mathbb{A}$ is residually small.

First we show that if a locally finite variety contains an infinite subdirectly irreducible algebra, then it contains infinitely many distinct finite subdirectly irreducible algebras.

Theorem A.5.22. *If $\mathbb{B}$ is subdirectly irreducible, then $\mathbb{B}$ is a subalgebra of an ultraproduct of a family of finitely generated subdirectly irreducible algebras in $HS(\mathbb{B})$.*

Proof. (From [80].) Let the monolith $0^*_{\mathbb{B}}$ of $\mathbb{B}$ be generated (as a congruence) by the pair (a, b). Let I be the family of finitely generated subalgebras $\mathbb{S} \le \mathbb{B}$ with $a, b \in \mathbb{S}$, and for each $\mathbb{S} \in I$, pick a congruence $\alpha_{\mathbb{S}}$ on $\mathbb{S}$ which is maximal among all congruences which do not contain (a, b). Then each $\mathbb{S}/\alpha_{\mathbb{S}}$ is subdirectly irreducible, since every congruence which properly contains $\alpha_{\mathbb{S}}$ contains (a, b).

Let $\mathcal{U}$ be an ultrafilter on I such that the set $U_S = \{\mathbb{S} \mid S \subseteq \mathbb{S}\}$ is in $\mathcal{U}$ for every finite $S \subseteq \mathbb{B}$. Such an ultrafilter exists since for any S_1, S_2 we have $U_{S_1} \cap U_{S_2} = U_{S_1 \cup S_2}$, and for S finite U_S is nonempty since it contains $\mathrm{Sg}_{\mathbb{B}}(S \cup \{a, b\})$.

Define a map $\varphi : \mathbb{B} \to (\prod_{\mathbb{S} \in I} \mathbb{S}/\alpha_{\mathbb{S}})/\mathcal{U}$ as the ultraproduct of the family of maps $\varphi_{\mathbb{S}}$ given by $\varphi_{\mathbb{S}}(x) = x/\alpha_{\mathbb{S}}$ for $x \in \mathbb{S}$ and $\varphi_{\mathbb{S}}(x) = a/\alpha_{\mathbb{S}}$ for $x \notin \mathbb{S}$. Then for $x_1, ..., x_k \in \mathbb{B}$ and t a k-ary term of $\mathbb{B}$, we have

$$\{\mathbb{S} \mid \varphi_{\mathbb{S}}(t(x_1, ..., x_k)) = t(\varphi_{\mathbb{S}}(x_1), ..., \varphi_{\mathbb{S}}(x_k))\} \in \mathcal{U},$$

since it contains $U_{\{x_1,...,x_k\}}$. Thus φ is a homomorphism. To see that it is injective, just note that $\varphi_{\mathbb{S}}(a) \ne \varphi_{\mathbb{S}}(b)$ for all $\mathbb{S} \in I$. □

Corollary A.5.23. *If a locally finite variety contains an infinite subdirectly irreducible algebra, then it contains arbitrarily large finite subdirectly irreducible algebras.*

It turns out that finite residually small algebras can be understood in terms of a commutator condition. We say that an algebra $\mathbb{A}$ satisfies a commutator identity *hereditarily* if every congruence lattice of every subalgebra of $\mathbb{A}$ satisfies the identity.

Proposition A.5.24. *The commutator identity* $[\alpha \wedge \beta, \beta] = \alpha \wedge [\beta, \beta]$ *is equivalent to the implication* $\alpha \le [\beta, \beta] \implies [\alpha, \beta] = \alpha$.

Proof. The implication clearly follows from the identity. For the other direction, we apply the implication to $\alpha \wedge [\beta, \beta] \le [\beta, \beta]$ to see that

$$\alpha \wedge [\beta, \beta] = [\alpha \wedge [\beta, \beta], \beta] \le [\alpha \wedge \beta, \beta] \le \alpha \wedge [\beta, \beta].$$ □

Proposition A.5.25. *If* $\mathbb{A}$ *is in a congruence modular variety and satisfies the commutator identity* $[\alpha \wedge \beta, \beta] = \alpha \wedge [\beta, \beta]$ *hereditarily, then so does every quotient* $\mathbb{B}$ *of* $\mathbb{A}$.

Proof. Suppose $\mathbb{B} = \mathbb{A}/\gamma$ and $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$ with $\alpha, \beta \ge \gamma$. We need to check that if $\alpha \le [\beta, \beta]_\gamma$, then $\alpha = [\alpha, \beta]_\gamma$. By the modular law, if $\alpha \le [\beta, \beta] \vee \gamma$ then

$$\alpha = \alpha \wedge ([\beta, \beta] \vee \gamma) = (\alpha \wedge [\beta, \beta]) \vee \gamma = [\alpha, \beta] \vee \gamma = [\alpha, \beta]_\gamma.$$ □

Proposition A.5.26. *If* $\mathbb{A}_1, \mathbb{A}_2$ *are in a congruence modular variety and satisfy the commutator identity* $[\alpha \wedge \beta, \beta] = \alpha \wedge [\beta, \beta]$ *hereditarily, then so does their product* $\mathbb{A}_1 \times \mathbb{A}_2$.

Proof. Let $\mathbb{B} \le \mathbb{A}_1 \times \mathbb{A}_2$, we can assume without loss of generality that this inclusion is subdirect by replacing the $\mathbb{A}_i$ with $\pi_i(\mathbb{B})$. Suppose $\alpha, \beta \in \mathrm{Con}(\mathbb{B})$ with $\alpha \le [\beta, \beta]$, we will show that $[\alpha, \beta] = \alpha$. We have

$$\alpha \vee \ker \pi_1 \le [\beta \vee \ker \pi_1, \beta \vee \ker \pi_1]_{\ker \pi_1},$$

so from the assumption on $\mathbb{A}_1$ we get

$$\alpha \vee \ker \pi_1 = [\alpha \vee \ker \pi_1, \beta \vee \ker \pi_1]_{\ker \pi_1} = [\alpha, \beta] \vee \ker \pi_1.$$

Thus by the modular law and $[\alpha, \beta] \le \alpha$, we have

$$\alpha = \alpha \wedge (\ker \pi_1 \vee [\alpha, \beta]) = (\alpha \wedge \ker \pi_1) \vee [\alpha, \beta].$$

Similarly, we have $\alpha = (\alpha \wedge \ker \pi_2) \vee [\alpha, \beta]$. Since $\alpha \wedge \ker \pi_2 \le \alpha \le [\beta, \beta]$, we may apply the same reasoning to $\alpha \wedge \ker \pi_2$ to see that

$$\alpha \wedge \ker \pi_2 = (\alpha \wedge \ker \pi_2 \wedge \ker \pi_1) \vee [\alpha \wedge \ker \pi_2, \beta],$$

so $\alpha \wedge \ker \pi_2 \le [\alpha, \beta]$, so

$$\alpha = (\alpha \wedge \ker \pi_2) \vee [\alpha, \beta] = [\alpha, \beta]. \qquad \square$$

Theorem A.5.27. *If $|\mathbb{A}| = m$ is finite and $\mathcal{V}(\mathbb{A})$ is congruence modular, and if $\mathbb{A}$ satisfies the commutator identity $[\alpha \wedge \beta, \beta] = \alpha \wedge [\beta, \beta]$ hereditarily, then every subdirectly irreducible algebra $\mathbb{B} \in \mathcal{V}(\mathbb{A})$ has $|\mathbb{B}| \le m \cdot m^{m^{m+1}}$.*

Proof. By Corollary A.5.23, we just need to check the bound in the case where $\mathbb{B}$ is finite. In this case, we have $\mathbb{B} \in HSP_{fin}(\mathbb{A})$, so $\mathbb{B}$ satisfies the commutator identity $[\alpha \wedge \beta, \beta] = \alpha \wedge [\beta, \beta]$ by the previous propositions. Let $0_{\mathbb{B}}^*$ be the monolith of $\mathbb{B}$, and let $\alpha = (0_{\mathbb{B}} : 0_{\mathbb{B}}^*)$ be its centralizer.

We claim that α is abelian. To see this, note that from $[\alpha, 0_{\mathbb{B}}^*] = 0_{\mathbb{B}}$ we have

$$0_{\mathbb{B}} = [0_{\mathbb{B}}^* \wedge \alpha, \alpha] = 0_{\mathbb{B}}^* \wedge [\alpha, \alpha],$$

so $[\alpha, \alpha] = 0_{\mathbb{B}}$. Now we can apply Corollary A.5.20 to see that $|\mathbb{B}| \le m \cdot m^{m^{m+1}}$. $\square$

Example A.5.2. The symmetric group S_3 on three letters is residually small, since it satisfies the commutator identity $[\alpha \wedge \beta, \beta] = \alpha \wedge [\beta, \beta]$ hereditarily: the only interesting case to check is that $[A_3, S_3] = A_3$, where A_3 is the alternating group on three letters. We have $HS(S_3) = \{1, \mathbb{Z}/2, \mathbb{Z}/3, S_3\}$, and all three nontrivial elements are subdirectly irreducible.

The general theory shows that every subdirectly irreducible $\mathbb{G} \in \mathcal{V}(S_3)$ has an abelian normal subgroup $\mathbb{N}$ with $\mathbb{G}/\mathbb{N} \in HS(S_3)$, with $|\mathbb{N}|$ a prime power bounded by $|\mathcal{F}_{\mathcal{V}(S_3)}(|\mathbb{G}/\mathbb{N}| + 1)| \le 6^{6^7}$. Since $\mathbb{N} \in \mathcal{V}(S_3)$ and every element of S_3 has order dividing 6, $\mathbb{N}$ has exponent 2 or 3. From here it is not too hard to check that the only nontrivial subdirectly irreducible algebras in $\mathcal{V}(S_3)$ are $\mathbb{Z}/2, \mathbb{Z}/3, S_3$, and all three of these are subgroups of S_3. Thus every group in $\mathcal{V}(S_3)$ is a subgroup of a power of S_3.

Proposition A.5.28. *If $\mathbb{A}$ is contained in a congruence modular variety but does not satisfy the commutator identity $[\alpha \wedge \beta, \beta] = \alpha \wedge [\beta, \beta]$ hereditarily, then there is some subdirectly irreducible $\mathbb{B} \in HS(\mathbb{A})$ such that the centralizer of the monolith of $\mathbb{B}$ is not abelian.*

Proof. Suppose that $\mathbb{A}$ fails to satisfy the commutator identity. In this case there must be $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$ with $\alpha \le [\beta, \beta]$ and $[\alpha, \beta] < \alpha$. Let θ be a meet-irreducible congruence such that $\theta \ge [\alpha, \beta]$ but $\theta \not\ge \alpha$, and let θ^* be its cover. Then

$$\theta^* \le \alpha \vee \theta \le [\beta, \beta] \vee \theta \le [\beta \vee \theta, \beta \vee \theta]_\theta$$

and

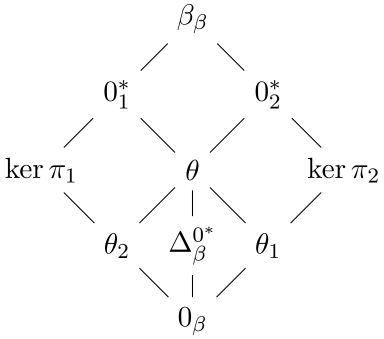
$$[\theta^*, \beta \vee \theta]_\theta \le [\alpha \vee \theta, \beta \vee \theta]_\theta = [\alpha, \beta] \vee \theta = \theta,$$

so if we take $\mathbb{B}$ to be $\mathbb{A}/\theta$, then the monolith of $\mathbb{B}$ is θ^*/θ, and $\beta \vee \theta/\theta$ is contained in the centralizer of the monolith of $\mathbb{B}$ but is not abelian. $\square$

Theorem A.5.29. *If $\mathbb{A}$ is contained in a congruence modular variety but does not satisfy the commutator identity $[\alpha \wedge \beta, \beta] = \alpha \wedge [\beta, \beta]$, then $\mathcal{V}(\mathbb{A})$ is not residually small. In fact, for every cardinal κ, $\mathcal{V}(\mathbb{A})$ contains a subdirectly irreducible algebra whose congruence lattice has size at least κ.*

Proof. (From [80].) By the proposition, we can reduce to the case where $\mathbb{A}$ is subdirectly irreducible and the centralizer β of the monolith 0^* is not abelian.

Consider β as a subalgebra of $\mathbb{A}^2$, and $\Delta_\beta^{0^*}$ as a congruence on β. From $[\beta, 0^*_{\mathbb{A}}] = 0_{\mathbb{A}}$ we have $\Delta_\beta^{0^*} \wedge \ker\pi_1 = \Delta_\beta^{0^*} \wedge \ker\pi_2 = 0_\beta$, and from the definition of $\Delta_\beta^{0^*}$ we have $\Delta_\beta^{0^*} \vee \ker\pi_i = \pi_i^{-1}(0^*)$. Set $0_i^* = \pi_i^{-1}(0^*)$ and $\theta = 0_1^* \wedge 0_2^*, \theta_i = 0_i^* \wedge \ker\pi_{\{1,2\}\setminus\{i\}}$, then (after several applications of the modular law - don't worry about the details just yet) we have the following sublattice in Con(β).



In the picture, we see that $\Delta_\beta^{0^*}$ appears to be meet-irreducible in Con(β), and the interval $[\![\Delta_\beta^{0^*}, \beta_\beta]\!]$ contains the incomparable elements $0_1^*, 0_2^*$. If $\Delta_\beta^{0^*}$ isn't meet-irreducible, we can still try to find a meet-irreducible congruence λ on β which is above $\Delta_\beta^{0^*}$ but not above θ, and then β/λ should give us a subdirectly irreducible algebra whose congruence lattice contains two distinct elements coming from $\ker\pi_1 \vee \lambda$ and $\ker\pi_2 \vee \lambda$ (that neither of these is equal to β_β will come from the assumption that β is not abelian). This is the basic idea behind the general construction, but we will need to scale up by considering higher dimensional analogues of $\beta \le \mathbb{A}^2$.

Let κ be any cardinal, considered as the set of all ordinals below κ. Define $\mathbb{B} \le \mathbb{A}^\kappa$ by

$$\mathbb{B} = \{a \in \mathbb{A}^\kappa \mid a_i \equiv_\beta a_j \ \forall i, j \in \kappa\}.$$

Then $\mathbb{B}$ has a natural map to $\mathbb{A}/\beta$, and we call the kernel of this map $\beta_{\mathbb{B}}$. Inside Con$(\mathbb{B})$, we have $\ker\pi_i \vee \ker\pi_j = \beta_{\mathbb{B}}$ for all $i \ne j \in \kappa$. The strategy is to construct a congruence λ on $\mathbb{B}$ such that $\mathbb{B}/\lambda$ is subdirectly irreducible and $\ker\pi_i \vee \lambda \not\ge \beta_{\mathbb{B}}$ for all i, which will guarantee that the congruences $\ker\pi_i \vee \lambda/\lambda \in$ Con$(\mathbb{B}/\lambda)$ are pairwise distinct. The congruence λ will be constructed by first constructing congruences Δ, θ with $\Delta < \theta$ and $\theta \vee \ker\pi_i \not\ge \beta_{\mathbb{B}}$.

We need a congruence on $\mathbb{B}$ generalizing $\Delta_\beta^{0^*}$ on β. We define Δ_i by

$$(a, b) \in \Delta_i \iff \begin{bmatrix} a_0 & b_0 \\ a_i & b_i \end{bmatrix} \in \Delta_\beta^{0^*} \wedge (a_j = b_j \ \forall j \ne 0, i),$$

and define Δ by

$$\Delta = \bigvee_{0<i<\kappa} \Delta_i.$$

We also define congruences θ_i by

$$(a,b) \in \theta_i \iff (a_i,b_i) \in 0_{\mathbb{A}}^* \wedge (a_j = b_j \ \forall j \neq i),$$

and define θ by

$$\theta = \bigvee_{i<\kappa} \theta_i.$$

We need to check some basic properties of these congruences, to see that they behave as in the picture of $\mathrm{Con}(\beta)$. First, we check that $\theta_0 \le \theta_i \vee \Delta_i$ for all i. Letting $\pi_{i'}$ be the projection onto all coordinates other than i, then it's easy to check that $\theta_0 \le \ker \pi_{i'} \vee \Delta$ by reasoning about just the two coordinates $0, i$ and keeping all other coordinates fixed:

$$\begin{bmatrix} a_0 \\ a_i \end{bmatrix} \ \ker \pi_{i'} \ \begin{bmatrix} a_0 \\ a_0 \end{bmatrix} \ \Delta_i \ \begin{bmatrix} b_0 \\ b_0 \end{bmatrix} \ \ker \pi_{i'} \ \begin{bmatrix} b_0 \\ b_i \end{bmatrix}.$$

Then by the modular law, if we let $0_i^* = \pi_i^{-1}(0_{\mathbb{A}}^*)$ and note that $\Delta_i \le 0_i^*$, we get

$$\theta_0 = \theta_0 \wedge 0_i^* \le (\Delta_i \vee \ker \pi_{i'}) \wedge 0_i^* = \Delta_i \vee (\ker \pi_{i'} \wedge 0_i^*) = \Delta_i \vee \theta_i.$$

Similarly, we get $\theta_i \le \theta_0 \vee \Delta_i$ for all i.

Next, for each i we have $\Delta \vee \theta_i = \theta$: for each $j \in \kappa$, we have

$$\Delta \vee \theta_i = \Delta \vee \Delta_i \vee \theta_i \ge \Delta \vee \theta_0 \ge \Delta_j \vee \theta_0 \ge \theta_j,$$

so $\Delta \vee \theta_i \ge \bigvee_{j\in\kappa} \theta_j = \theta$, while the other containment follows from $\Delta_i \le \theta_0 \vee \theta_i$ for all i.

We now check that $\Delta \neq \theta$. It's enough to check that $\theta_0 \not\le \Delta$, since $\Delta \vee \theta_0 = \theta$. Note first that θ_0 is compact, since $0_{\mathbb{A}}^*$ is compact. Thus we just need to check that $\theta_0 \not\le \bigvee_{j\le n} \Delta_{i_j}$ for all $i_1, ..., i_n$. In fact, we can assume that $i_1, ..., i_n$ are $1, ..., n$ by a symmetry argument.

We will show by induction on n that $\theta_0 \wedge (\Delta_1 \vee \cdots \vee \Delta_n) = 0_{\mathbb{B}}$ for all n. The base case follows from the fact that $[\beta, 0_{\mathbb{A}}^*] = 0_{\mathbb{A}} \implies \ker \pi_2 \wedge \Delta_\beta^{0^*} = 0_\beta$ in $\mathrm{Con}(\beta)$, which in turn implies $\theta_0 \wedge \Delta_1 = 0_{\mathbb{B}}$. For the inductive step, we argue as follows:

$$\begin{aligned} \theta_0 \wedge (\Delta_1 \vee \cdots \vee \Delta_n) &= \theta_0 \wedge (\theta_0 \vee \theta_n) \wedge (\Delta_1 \vee \cdots \vee \Delta_n) \\ &= \theta_0 \wedge (((\theta_0 \vee \theta_n) \wedge (\Delta_1 \vee \cdots \vee \Delta_{n-1})) \vee \Delta_n) \\ &= \theta_0 \wedge ((\theta_0 \wedge (\Delta_1 \vee \cdots \vee \Delta_{n-1})) \vee \Delta_n) \\ &= \theta_0 \wedge \Delta_n = 0_{\mathbb{B}}, \end{aligned}$$

where the second equality used the modular law and the fact that $\Delta_n \le \theta_0 \vee \theta_n$, the third equality used the fact that θ_n is independent of everything that happens on the coordinates $0, ..., n-1$, and the last two equalities used the inductive hypothesis.

We have shown that $\Delta < \theta$. We can now apply Corollary A.5.7 to see that there is some meet-irreducible congruence λ with $\lambda \ge \Delta$ but $\lambda \not\ge \theta$. To finish, we just need to check that $\lambda \vee \ker \pi_i \not\ge \beta_{\mathbb{B}}$. To see this, note that $\lambda \not\ge \theta_i$, since otherwise we would have $\lambda \ge \Delta \vee \theta_i = \theta$, a contradiction. Since θ_i is the minimal nonzero element of the interval $[\![0_{\mathbb{B}}, \ker \pi_{i'}]\!]$, this means that $\lambda \wedge \ker \pi_{i'} = 0_{\mathbb{B}}$. Thus if (for contradiction) $\lambda \vee \ker \pi_i \ge \beta_{\mathbb{B}}$, then we would have

$$[\beta_{\mathbb{B}}, \beta_{\mathbb{B}}] \le [\ker \pi_{i'} \vee \ker \pi_i, \lambda \vee \ker \pi_i] \le (\lambda \wedge \ker \pi_{i'}) \vee \ker \pi_i = \ker \pi_i,$$

and applying π_i we would get $[\beta, \beta] = 0_{\mathbb{A}}$, a contradiction to the assumption that β was not abelian.

Putting it all together, we have a meet-irreducible congruence λ such that $\lambda \vee \ker \pi_i \not\geq \beta_{\mathbb{B}}$ for each i, but $\ker \pi_i \vee \ker \pi_j \geq \beta_{\mathbb{B}}$ for all $i \neq j$. Thus $\mathbb{B}/\lambda$ is subdirectly irreducible, and the congruences $\ker \pi_i \vee \lambda/\lambda$ are mutually distinct elements of $\mathrm{Con}(\mathbb{B}/\lambda)$. □

Corollary A.5.30. *Let $\mathcal{V}$ be a finitely generated congruence modular variety. Then the following are equivalent:*

- *$\mathcal{V}$ is residually small,*
- *every algebra in $\mathcal{V}$ satisfies the commutator identity $[\alpha \wedge \beta, \beta] = \alpha \wedge [\beta, \beta]$,*
- *$\mathcal{V}$ is generated by a finite algebra $\mathbb{A}$ such that for every subdirectly irreducible $\mathbb{B} \in HS(\mathbb{A})$, the centralizer of the monolith of $\mathbb{B}$ is abelian,*
- *$\mathcal{V}$ is generated by a finite algebra which satisfies the commutator identity $[\alpha \wedge \beta, \beta] = \alpha \wedge [\beta, \beta]$ hereditarily,*
- *$\mathcal{V}$ has a finite bound on the size of its subdirectly irreducible elements.*

Corollary A.5.31. *If $\mathbb{A}$ is in a congruence modular variety and has size $|\mathbb{A}| \leq 3$, then $\mathcal{V}(\mathbb{A})$ is residually small.*

Proof. Suppose for contradiction that $\mathbb{A}$ is subdirectly irreducible with a monolith $0_{\mathbb{A}}^*$ whose centralizer $(0_{\mathbb{A}} : 0_{\mathbb{A}}^*)$ is not abelian. Then since $\mathrm{Con}(\mathbb{A})$ has height at most 2, we necessarily have

$$0_{\mathbb{A}} < 0_{\mathbb{A}}^* < (0_{\mathbb{A}} : 0_{\mathbb{A}}^*) = 1_{\mathbb{A}}.$$

Thus $|\mathbb{A}| = 3$, and we may name the elements of $\mathbb{A}$ as a, b, c, such that $0_{\mathbb{A}}^*$ corresponds to the partition $\{a, b\}, \{c\}$ of $\mathbb{A}$. Letting $p(x, y, z)$ be a Gumm difference term, we see from Theorem A.3.8 that

$$\begin{bmatrix} a & p(a, b, c) \\ b & c \end{bmatrix} \in \Delta_{0_{\mathbb{A}}^*}^{1_{\mathbb{A}}}.$$

Modulo $0_{\mathbb{A}}^*$, we have $p(a, b, c) \equiv_{0_{\mathbb{A}}^*} p(a, a, c) = c$, so we must have $p(a, b, c) = c$. Then by Theorem A.2.8 we have $(a, b) \in [1_{\mathbb{A}}, 0_{\mathbb{A}}^*]$, which contradicts $(0_{\mathbb{A}} : 0_{\mathbb{A}}^*) = 1_{\mathbb{A}}$. □

Proposition A.5.32. *If $\mathbb{A}$ satisfies the commutator identity $[\alpha \wedge \beta, \beta] = \alpha \wedge [\beta, \beta]$, then every nilpotent congruence on $\mathbb{A}$ is abelian.*

Proof. The commutator identity implies that

$$[[\alpha, \alpha], \alpha] = [[\alpha, \alpha] \wedge \alpha, \alpha] = [\alpha, \alpha] \wedge [\alpha, \alpha] = [\alpha, \alpha].$$ □

Proposition A.5.33 (Ol'šanskiĭ [148]). *If all the Sylow subgroups of a finite group $\mathbb{G}$ are abelian, then the center $Z(\mathbb{G})$ and the commutator subgroup $[\mathbb{G}, \mathbb{G}]$ intersect trivially, that is, $Z(\mathbb{G}) \wedge [\mathbb{G}, \mathbb{G}] = 0_{\mathbb{G}}$.*

Proof. Fix a Sylow subgroup $\mathbb{S}$ of $\mathbb{G}$, and consider the transfer map $\mathbb{G} \to \mathbb{S}/[\mathbb{S}, \mathbb{S}]$. Recall that the transfer homomorphism from a finite group to the abelianization of a subgroup is defined by making a choice of coset representatives x_i with $\mathbb{G} = \bigcup_i x_i\mathbb{S}$, and sending $g \in \mathbb{G}$ to $\prod_i s_i/[\mathbb{S}, \mathbb{S}]$, where for each i, $s_i \in \mathbb{S}$ is given by $gx_i = x_j s_i$ for some j. Since $\mathbb{S}$ is assumed to be abelian, this gives us a homomorphism from $\mathbb{G}$ to $\mathbb{S}$.

Now consider any $g \in Z(\mathbb{G}) \cap \mathbb{S}$. The transfer homomorphism sends g to $\prod_i g = g^{[\mathbb{G}:\mathbb{S}]}$ since $gx_i = x_i g$ for each i, and if $g \neq 1$ then $g^{[\mathbb{G}:\mathbb{S}]} \neq 1$ as well since $[\mathbb{G} : \mathbb{S}]$ is relatively prime to the order of g. Thus there is a map from $\mathbb{G}$ to an abelian group such that g is not in the kernel, so $g \notin [\mathbb{G}, \mathbb{G}]$. Since every nontrivial element of $Z(\mathbb{G}) \wedge [\mathbb{G}, \mathbb{G}]$ has a power which has prime order and is therefore contained in a Sylow subgroup of $\mathbb{G}$, we must have $Z(\mathbb{G}) \wedge [\mathbb{G}, \mathbb{G}] = 0_{\mathbb{G}}$ to avoid a contradiction. □

Corollary A.5.34 (Ol'šanskiĭ [148])**.** *A finite group is residually small iff all of its Sylow subgroups are abelian.*

Proof. By Proposition A.5.32, all nilpotent subgroups of a finite residually small group must be abelian, so in particular the Sylow subgroups must be abelian since all p-groups are nilpotent.

For the other direction, note that for any $\mathbb{B} \in HS(\mathbb{A})$, the Sylow subgroups of $\mathbb{B}$ are quotients of subgroups of the Sylow subgroups of $\mathbb{A}$ by the Sylow theorems. Thus we just have to check that if the Sylow subgroups of a subdirectly irreducible group are abelian, then the centralizer $\mathbb{C}$ of its monolith 0^* is abelian.

Note that if $\mathbb{C}$ centralizes 0^*, then $0^* \le Z(\mathbb{C})$. By Proposition A.5.33, we have $Z(\mathbb{C}) \wedge [\mathbb{C}, \mathbb{C}] = 0$, so $0^* \wedge [\mathbb{C}, \mathbb{C}] = 0$, which implies that $[\mathbb{C}, \mathbb{C}] = 0$. □

A.5.1 Similarity

Even if a finitely generated congruence modular variety is not residually small, we can still classify its subdirectly irreducible algebras by using the concept of *similarity* from Freese and McKenzie [80]. We will use a different definition of similarity than their definition, but which they prove to be equivalent.

Definition A.5.35. We say that subdirectly irreducible algebras $\mathbb{A}, \mathbb{B}$ in a congruence modular variety $\mathcal{V}$ are *similar* if there exists an algebra $\mathbb{C} \in \mathcal{V}$ with congruences $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $\mathbb{C}/\alpha \cong \mathbb{A}$, $\mathbb{C}/\beta \cong \mathbb{B}$, and

$$[\![\alpha, \alpha^*]\!] \searrow [\![\gamma, \delta]\!] \nearrow [\![\beta, \beta^*]\!].$$

If furthermore $\mathbb{C} \le_{sd} \mathbb{A} \times \mathbb{B}$ and α, β are the kernels of the projections to $\mathbb{A}, \mathbb{B}$, then we say that $\mathbb{C}$ is the *graph of a similarity* from $\mathbb{A}$ to $\mathbb{B}$.

Proposition A.5.36. *If $\mathbb{A}, \mathbb{B}$ are similar, then there is a witnessing algebra $\mathbb{C} \le_{sd} \mathbb{A} \times \mathbb{B}$ which is the graph of a similarity from $\mathbb{A}$ to $\mathbb{B}$. If α, β are the kernels of the projections to $\mathbb{A}, \mathbb{B}$, then $(\alpha : \alpha^*) = (\beta : \beta^*)$ and $\mathbb{C}/(\alpha : \alpha^*)$ is the graph of an isomorphism*

$$\mathbb{A}/(0_{\mathbb{A}} : 0_{\mathbb{A}}^*) \xrightarrow{\sim} \mathbb{B}/(0_{\mathbb{B}} : 0_{\mathbb{B}}^*).$$

If $\mathbb{A}, \mathbb{B}$ are similar but not isomorphic, then they must both have abelian monoliths.

Proof. For the first statement, let $\mathbb{C} \in \mathcal{V}$ and $\alpha, \beta, \gamma, \delta \in \mathrm{Con}(\mathbb{C})$ be as in the definition of similarity. It's enough to show that we have

$$[\![\alpha, \alpha^*]\!] \searrow [\![\alpha \wedge \beta, (\alpha \wedge \beta) \vee \delta]\!] \nearrow [\![\beta, \beta^*]\!],$$

since then we can replace $\mathbb{C}$ by $\mathbb{C}/(\alpha \wedge \beta)$, which is a subdirect product of $\mathbb{C}/\alpha \cong \mathbb{A}$ and $\mathbb{C}/\beta \cong \mathbb{B}$. We have

$$\alpha \vee ((\alpha \wedge \beta) \vee \delta) = \alpha \vee \delta = \alpha^*,$$

and by the modular law and the fact that $\gamma \le \alpha \wedge \beta$, we have

$$\alpha \wedge ((\alpha \wedge \beta) \vee \delta) = (\alpha \wedge \delta) \vee (\alpha \wedge \beta) = \gamma \vee (\alpha \wedge \beta) = (\alpha \wedge \beta),$$

so $[\![\alpha, \alpha^*]\!] \searrow [\![\alpha \wedge \beta, (\alpha \wedge \beta) \vee \delta]\!]$, and the other perspectivity follows by a symmetric argument.

The remaining statements follow from the Diamond Isomorphism Theorem A.2.5: if $[\![\alpha, \alpha^*]\!] \searrow [\![\gamma, \delta]\!] \nearrow [\![\beta, \beta^*]\!]$, then $(\alpha : \alpha^*) = (\gamma : \delta) = (\beta : \beta^*)$, so

$$\mathbb{A}/(0_\mathbb{A} : 0_\mathbb{A}^*) \cong \mathbb{C}/(\alpha : \alpha^*) = \mathbb{C}/(\beta : \beta^*) \cong \mathbb{B}/(0_\mathbb{B} : 0_\mathbb{B}^*),$$

and

$$[\alpha^*, \alpha^*]_\alpha = \alpha \iff [\delta, \delta]_\gamma = \gamma \iff [\beta^*, \beta^*]_\beta = \beta,$$

so $0_\mathbb{A}^*$ is abelian iff $0_\mathbb{B}^*$ is abelian, and if neither is abelian then $\alpha = (\alpha : \alpha^*) = (\beta : \beta^*) = \beta$ and $\mathbb{A} \cong \mathbb{C}/\alpha = \mathbb{C}/\beta \cong \mathbb{B}$. □

Proposition A.5.37. *If $\mathbb{A}, \mathbb{B}$ are similar such that σ is the corresponding isomorphism*

$$\sigma : \mathbb{A}/(0_\mathbb{A} : 0_\mathbb{A}^*) \xrightarrow{\sim} \mathbb{B}/(0_\mathbb{B} : 0_\mathbb{B}^*),$$

then they are similar via the algebra $\mathbb{R} = \{(x, y) \in \mathbb{A} \times \mathbb{B} \mid \sigma(x/(0_\mathbb{A} : 0_\mathbb{A}^*)) = y/(0_\mathbb{B} : 0_\mathbb{B}^*)\}$.

Proof. Suppose $\mathbb{C} \le \mathbb{R}$ is the graph of a similarity from $\mathbb{A}$ to $\mathbb{B}$, with

$$[\![\ker \pi_1, (\ker \pi_1)^*]\!] \searrow [\![0_\mathbb{C}, \delta]\!] \nearrow [\![\ker \pi_2, (\ker \pi_2)^*]\!]$$

in $\mathrm{Con}(\mathbb{C})$. We may assume that $\mathbb{A}, \mathbb{B}$ have abelian monoliths, so $[\delta, \delta] = 0_\mathbb{C}$ by the Diamond Isomorphism Theorem A.2.5. Then by Theorem A.4.5, δ permutes with all congruences in $\mathrm{Con}(\mathbb{C})$, so in particular $(\ker \pi_1)^* = \delta \circ \ker \pi_1$. In other words, for any $(a, b) \in \mathbb{C}$ and any $a' \in a/0_\mathbb{A}^*$, there exists a b' such that

$$\begin{bmatrix} a \\ b \end{bmatrix} \ \delta \ \begin{bmatrix} a' \\ b' \end{bmatrix}.$$

In fact, this b' is uniquely determined by a, b, a', since $\delta \wedge \ker \pi_1 = 0_\mathbb{C}$. Additionally, we must have $b' \in b/0_\mathbb{B}^*$, since $\delta \le (\ker \pi_2)^*$.

Now we can extend δ to a congruence $\delta_\mathbb{R} \in \mathrm{Con}(\mathbb{R})$ as follows. For $(a, b), (a', b') \in \mathbb{R}$ with $a \ 0_\mathbb{A}^* \ a'$ and $b \ 0_\mathbb{B}^* \ b'$, we pick any $(u, v) \in \mathbb{C}$ with $u \ (0_\mathbb{A} : 0_\mathbb{A}^*) \ a$ and write

$$\begin{bmatrix} a & a' \\ b & b' \end{bmatrix} \in \delta_\mathbb{R} \iff \begin{bmatrix} p(a, a', u) & u \\ p(b, b', v) & v \end{bmatrix} \in \delta,$$

where p is a Gumm difference term. Note that by Corollary A.3.9, this choice of $\delta_\mathbb{R}$ is preserved by the operations of $\mathbb{A}$ so long as it is well-defined. To check that this is in independent of the choice of $(u, v) \in \mathbb{C}$, suppose $(u', v') \in \mathbb{C}$ with $u' \ (0_\mathbb{A} : 0_\mathbb{A}^*) \ a$, and apply Corollary A.3.9 again to see that

$$p\left(\begin{bmatrix} p(a, a', u) & u \\ p(b, b', v) & v \end{bmatrix}, \begin{bmatrix} p(a, a, u) & u \\ p(b, b, v) & v \end{bmatrix}, \begin{bmatrix} p(a, a, u') & u' \\ p(b, b, v') & v' \end{bmatrix}\right) = \begin{bmatrix} p(a, a', u') & u' \\ p(b, b', v') & v' \end{bmatrix},$$

where we have used $0_{\mathbb{A}}^*, 0_{\mathbb{B}}^*$ abelian to see that $p(a', a, a) = a'$ and $p(b', b, b) = b'$.

We need to check that $\delta_{\mathbb{R}}$ is a congruence on $\mathbb{R}$. It clearly contains the equality relation on $\mathbb{R}$. For symmetry and transitivity, note that

$$p\left(\begin{bmatrix} p(a,a',u) \\ p(b,b',v) \end{bmatrix}, \begin{bmatrix} p(a'',a',u) \\ p(b'',b',v) \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}\right) = p\left(\begin{bmatrix} p(a,a',u) \\ p(b,b',v) \end{bmatrix}, \begin{bmatrix} p(a'',a',u) \\ p(b'',b',v) \end{bmatrix}, \begin{bmatrix} p(a'',a'',u) \\ p(b'',b'',v) \end{bmatrix}\right) = \begin{bmatrix} p(a,a'',u) \\ p(b,b'',v) \end{bmatrix}.$$

Finally, we need to check that $\delta_{\mathbb{R}} \wedge \ker \pi_1 = 0_{\mathbb{R}}$ and $\delta_{\mathbb{R}} \vee \ker \pi_1 = (\ker \pi_1)^*$. That $\delta_{\mathbb{R}} \wedge \ker \pi_1 = 0_{\mathbb{R}}$ follows from the fact that if we pick u such that $(u, b') \in \mathbb{C}$, then

$$\begin{bmatrix} a & a \\ b & b' \end{bmatrix} \in \delta_{\mathbb{R}} \iff \begin{bmatrix} p(a,a,u) & u \\ p(b,b',b') & b' \end{bmatrix} = \begin{bmatrix} u & u \\ b & b' \end{bmatrix} \in \delta,$$

and so this can only occur when $b = b'$ since $\delta \wedge \ker \pi_1 = 0_{\mathbb{C}}$ (by assumption). That $\delta_{\mathbb{R}} \vee \ker \pi_1 = (\ker \pi_1)^*$ follows from $\delta \subseteq \delta_{\mathbb{R}} \subseteq (\ker \pi_1)^*$ and $\delta \not\subseteq \ker \pi_1$. □

Corollary A.5.38. *A similarity from $\mathbb{A}$ to $\mathbb{B}$ can be described by the following data: an isomorphism*

$$\sigma : \mathbb{A}/(0_{\mathbb{A}} : 0_{\mathbb{A}}^*) \xrightarrow{\sim} \mathbb{B}/(0_{\mathbb{B}} : 0_{\mathbb{B}}^*)$$

together with a congruence $\delta \in \mathrm{Con}(\mathbb{R})$, where $\mathbb{R} = \{(x, y) \in \mathbb{A} \times \mathbb{B} \mid \sigma(x/(0_{\mathbb{A}} : 0_{\mathbb{A}}^*)) = y/(0_{\mathbb{B}} : 0_{\mathbb{B}}^*)\}$, such that for every $(a, b) \in \mathbb{R}$ and every $a' \in a/0_{\mathbb{A}}^*$, there exists a unique $b' \in b/0_{\mathbb{B}}^*$ such that

$$\begin{bmatrix} a & a' \\ b & b' \end{bmatrix} \in \delta.$$

In particular, if $\mathbb{A}, \mathbb{B}$ are idempotent, then for any $(a, b) \in \mathbb{R}$ the congruence classes $a/0_{\mathbb{A}}^*$ and $b/0_{\mathbb{B}}^*$ are isomorphic to each other.

Corollary A.5.39. *Similarity is an equivalence relation on subdirectly irreducible algebras.*

Proof. Suppose we have similarities from $\mathbb{A}$ to $\mathbb{B}$ and from $\mathbb{B}$ to $\mathbb{C}$, described by isomorphisms

$$\mathbb{A}/(0_{\mathbb{A}} : 0_{\mathbb{A}}^*) \xrightarrow{\sigma} \mathbb{B}/(0_{\mathbb{B}} : 0_{\mathbb{B}}^*) \xrightarrow{\sigma'} \mathbb{C}/(0_{\mathbb{C}} : 0_{\mathbb{C}}^*)$$

and congruences δ, δ'. We define a congruence $\delta \circ \delta'$ by

$$\begin{bmatrix} a & a' \\ c & c' \end{bmatrix} \in \delta \circ \delta' \iff \exists (b, b') \in 0_{\mathbb{B}}^* \left(\begin{bmatrix} a & a' \\ b & b' \end{bmatrix} \in \delta\right) \wedge \left(\begin{bmatrix} b & b' \\ c & c' \end{bmatrix} \in \delta'\right).$$

We need to check that for each a, c, a' there exists a unique c' satisfying the above. Existence is easy: for each b, we can fill in a unique b' to satisfy δ, and then there is a unique c' which satisfies δ'. We just need to show that the choice of b doesn't affect the final c' we get. Suppose that instead of b we had picked v. Then the claim is that if we leave a, a', c, c' unchanged and replace b by v and b' by $p(b', b, v)$, we get another valid solution. For δ, this follows from

$$p\left(\begin{bmatrix} a & a' \\ b & b' \end{bmatrix}, \begin{bmatrix} a & a \\ b & b \end{bmatrix}, \begin{bmatrix} a & a \\ v & v \end{bmatrix}\right) = \begin{bmatrix} a & a' \\ v & p(b', b, v) \end{bmatrix},$$

and it follows for δ' similarly. □

We will show that every subdirectly irreducible algebra $\mathbb{A}$ with abelian monolith is similar to a subdirectly irreducible algebra $D(\mathbb{A})$ such that the monolith of $D(\mathbb{A})$ is equal to its own centralizer. The size of the algebra $D(\mathbb{A})$ can then be bounded using Theorem A.5.14 and the following proposition.

Proposition A.5.40. *If $\mathbb{B} \in \mathcal{V}(\mathbb{A})$ is subdirectly irreducible, $\mathbb{A}$ is finite, and $\mathcal{V}(\mathbb{A})$ is congruence modular, then every congruence class of $0_{\mathbb{B}}^*$ has size at most $|\mathbb{A}|$.*

Proof. By Theorem A.5.22 and Corollary A.5.12, we may assume without loss of generality that $\mathbb{B}$ is finite. By Theorem A.5.14, we may also assume that $0_{\mathbb{B}}^*$ is abelian. Take m minimal such that there exists $\mathbb{C} \le \mathbb{A}^m$ and $\theta \in \mathrm{Con}(\mathbb{C})$ with $\mathbb{B} \cong \mathbb{C}/\theta$, so $[\theta^*, \theta^*] \le \theta$.

Let $\pi_{1'}$ be the projection onto all but the first coordinate, then by the minimality of m we have $\ker \pi_{1'} \not\le \theta$. Thus we have

$$[\![\theta, \theta^*]\!] \searrow [\![\theta \wedge \ker \pi_{1'}, \theta^* \wedge \ker \pi_{1'}]\!].$$

By Theorem A.4.5, the congruences θ and $\theta^* \wedge \ker \pi_{1'}$ permute. Thus for every congruence class C^* of θ^* containing some $c \in \mathbb{C}$, the size of C^*/θ is equal to the size of $C'/(\theta \wedge \ker \pi_{1'})$, where C' is the congruence class of $\theta^* \wedge \ker \pi_{1'}$ containing c. But $|C'/(\theta \wedge \ker \pi_{1'})| \le |\mathbb{C}/\ker \pi_{1'}| = |\mathbb{A}|$, so every congruence class of $0_{\mathbb{B}}^*$ has size bounded by $|\mathbb{A}|$. □

Definition A.5.41. Suppose $\mathbb{A}$ is a subdirectly irreducible algebra in a congruence modular variety. If $0_{\mathbb{A}}^*$ is nonabelian, define $D(\mathbb{A})$ to be $\mathbb{A}$. Otherwise, consider $0_{\mathbb{A}}^*$ as a subalgebra of $\mathbb{A}^2$ and $\Delta_{0_{\mathbb{A}}^*}^{(0:0^*)}$ as a congruence on $0_{\mathbb{A}}^*$, and define $D(\mathbb{A}) = 0_{\mathbb{A}}^*/\Delta_{0_{\mathbb{A}}^*}^{(0:0^*)}$.

Recall that by Theorem A.3.8, if $0_{\mathbb{A}}^*$ is abelian and p is a Gumm difference term, then $(0_{\mathbb{A}} : 0_{\mathbb{A}}^*) \ge 0_{\mathbb{A}}^*$ and $[(0_{\mathbb{A}} : 0_{\mathbb{A}}^*), 0_{\mathbb{A}}^*] = 0_{\mathbb{A}}$, so we have

$$\begin{bmatrix} x & w \\ y & z \end{bmatrix} \in \Delta_{0_{\mathbb{A}}^*}^{(0:0^*)} \iff (p(x,y,z) = w) \wedge (x \ \equiv_{0_{\mathbb{A}}^*} \ y \ \equiv_{(0:0^*)} \ z).$$

In this case, the subalgebra $\{(x,x)/\Delta_{0_{\mathbb{A}}^*}^{(0:0^*)}\} \le D(\mathbb{A})$ meets every congruence class of $(0_{\mathbb{A}} : 0_{\mathbb{A}}^*)_{D(\mathbb{A})}$ (that is, the congruence $(0_{\mathbb{A}} : 0_{\mathbb{A}}^*)$ considered as a congruence on $D(\mathbb{A})$) exactly once, and is isomorphic to $\mathbb{A}/(0_{\mathbb{A}} : 0_{\mathbb{A}}^*)$.

Proposition A.5.42. *If $\mathbb{A}$ is a subdirectly irreducible algebra in a congruence modular variety with an abelian monolith, then $D(\mathbb{A})$ is subdirectly irreducible with monolith $(0_{\mathbb{A}} : 0_{\mathbb{A}}^*)_{D(\mathbb{A})}$, and $\mathbb{A}, D(\mathbb{A})$ are similar via the algebra $0_{\mathbb{A}}^*$ and the congruences $\ker \pi_1, \Delta_{0_{\mathbb{A}}^*}^{(0:0^*)} \in \mathrm{Con}(0_{\mathbb{A}}^*)$. Furthermore, the monolith $(0_{\mathbb{A}} : 0_{\mathbb{A}}^*)_{D(\mathbb{A})}$ of $D(\mathbb{A})$ is its own centralizer.*

Proof. Note that $\ker \pi_1$ is covered by $\ker \pi_1 \vee \ker \pi_2$, since $\pi_1(\ker \pi_1 \vee \ker \pi_2) = 0_{\mathbb{A}}^*$. First we check that in $\mathrm{Con}(0_{\mathbb{A}}^*)$ we have the perspectivities

$$[\![\ker \pi_1, \ker \pi_1 \vee \ker \pi_2]\!] \searrow [\![0_{0_{\mathbb{A}}^*}, \ker \pi_2]\!] \nearrow [\![\Delta_{0_{\mathbb{A}}^*}^{(0:0^*)}, (0_{\mathbb{A}} : 0_{\mathbb{A}}^*)_{0_{\mathbb{A}}^*}]\!].$$

The hardest step here is checking that $\ker \pi_2 \vee \Delta_{0_{\mathbb{A}}^*}^{(0:0^*)} = (0_{\mathbb{A}} : 0_{\mathbb{A}}^*)_{0_{\mathbb{A}}^*}$: if $(x,y),(w,z) \in 0_{\mathbb{A}}^*$ with $(y,z) \in (0_{\mathbb{A}} : 0_{\mathbb{A}}^*)$, then we have

$$\begin{bmatrix} x \\ y \end{bmatrix} \ \Delta_{0_{\mathbb{A}}^*}^{(0:0^*)} \ \begin{bmatrix} p(x,y,z) \\ z \end{bmatrix} \ \ker \pi_2 \ \begin{bmatrix} w \\ z \end{bmatrix}.$$

To see that $\ker\pi_2 \wedge \Delta_{0^*_{\mathbb{A}}}^{(0:0^*)} = 0_{0^*_{\mathbb{A}}}$, note that by Theorem A.2.8 the inequality $\ker\pi_2 \wedge \Delta_{0^*_{\mathbb{A}}}^{(0:0^*)} \le \ker\pi_1$ is equivalent to $[(0_{\mathbb{A}} : 0^*_{\mathbb{A}}), 0^*_{\mathbb{A}}] = 0_{\mathbb{A}}$.

Next we show that $(0_{\mathbb{A}} : 0^*_{\mathbb{A}})_{0^*_{\mathbb{A}}}$ is the unique cover of $\Delta_{0^*_{\mathbb{A}}}^{(0:0^*)}$ in $\mathrm{Con}(0^*_{\mathbb{A}})$. Note first that $(0_{\mathbb{A}} : 0^*_{\mathbb{A}})_{0^*_{\mathbb{A}}}$ is *a* cover of $\Delta_{0^*_{\mathbb{A}}}^{(0:0^*)}$, since the interval $[\![\Delta_{0^*_{\mathbb{A}}}^{(0:0^*)}, (0_{\mathbb{A}} : 0^*_{\mathbb{A}})_{0^*_{\mathbb{A}}}]\!]$ is isomorphic to $[\![\ker\pi_1, \ker\pi_1 \vee \ker\pi_2]\!] \cong [\![0_{\mathbb{A}}, 0^*_{\mathbb{A}}]\!]$ by the Diamond Isomorphism Theorem A.2.5.

Suppose that ψ is any congruence in $\mathrm{Con}(0^*_{\mathbb{A}})$ with $\psi > \Delta_{0^*_{\mathbb{A}}}^{(0:0^*)}$. If $\psi \ge \ker\pi_2$, then $\psi \ge \Delta_{0^*_{\mathbb{A}}}^{(0:0^*)} \vee \ker\pi_2 = (0_{\mathbb{A}} : 0^*_{\mathbb{A}})_{0^*_{\mathbb{A}}}$, and we are done. Otherwise, since $\ker\pi_2$ is a cover of $0_{0^*_{\mathbb{A}}}$, we must have $\psi \wedge \ker\pi_2 = 0_{0^*_{\mathbb{A}}}$. Then we have

$$[\psi \vee \ker\pi_1, \ker\pi_2 \vee \ker\pi_1]_{\ker\pi_1} \le [\psi, \ker\pi_2] \vee \ker\pi_1 \le (\psi \wedge \ker\pi_2) \vee \ker\pi_1 = \ker\pi_1.$$

Applying π_1 to both sides, we see that $\pi_1(\psi \vee \ker\pi_1) \le (0_{\mathbb{A}} : 0^*_{\mathbb{A}})$, so $\psi \vee \ker\pi_1 \le (0_{\mathbb{A}} : 0^*_{\mathbb{A}})_{0^*_{\mathbb{A}}}$. Thus $\psi \in [\![\Delta_{0^*_{\mathbb{A}}}^{(0:0^*)}, (0_{\mathbb{A}} : 0^*_{\mathbb{A}})_{0^*_{\mathbb{A}}}]\!]$, so again we must have $\psi = (0_{\mathbb{A}} : 0^*_{\mathbb{A}})_{0^*_{\mathbb{A}}}$. We have finished showing that $D(\mathbb{A})$ is subdirectly irreducible.

To see that the monolith $(0_{\mathbb{A}} : 0^*_{\mathbb{A}})_{D(\mathbb{A})}$ of $D(\mathbb{A})$ is its own centralizer, note that by the Diamond Isomorphism Theorem A.2.5 we have

$$(\Delta_{0^*_{\mathbb{A}}}^{(0:0^*)} : (0_{\mathbb{A}} : 0^*_{\mathbb{A}})_{0^*_{\mathbb{A}}}) = (\ker\pi_1 : \ker\pi_1 \vee \ker\pi_2) = \pi_1^{-1}((0_{\mathbb{A}} : 0^*_{\mathbb{A}})) = (0_{\mathbb{A}} : 0^*_{\mathbb{A}})_{0^*_{\mathbb{A}}}. \qquad \square$$

Proposition A.5.43. *If $\mathbb{A}, \mathbb{B}$ are subdirectly irreducible algebras in a congruence modular variety, then $\mathbb{A}$ is similar to $\mathbb{B}$ iff $D(\mathbb{A}) \cong D(\mathbb{B})$.*

Proof. Since similarity is an equivalence relation, we may as well replace $\mathbb{A}, \mathbb{B}$ by $D(\mathbb{A}), D(\mathbb{B})$. Thus we just need to prove that if $\mathbb{A}, \mathbb{B}$ have monoliths equal to their own centralizers, and have subalgebras $X_{\mathbb{A}}, X_{\mathbb{B}}$ which intersect their monoliths transversely, then they are similar iff they are isomorphic.

Let $\sigma : \mathbb{A}/0^*_{\mathbb{A}} \to \mathbb{B}/0^*_{\mathbb{B}}$ be the isomorphism and $\delta \in \mathrm{Con}(\mathbb{R})$, where $\mathbb{R} = \{(x,y) \in \mathbb{A} \times \mathbb{B} \mid \sigma(x/(0_{\mathbb{A}} : 0^*_{\mathbb{A}})) = y/(0_{\mathbb{B}} : 0^*_{\mathbb{B}})\}$, be the data describing a similarity from $\mathbb{A}$ to $\mathbb{B}$. Then σ induces an isomorphism $\sigma_X : X_{\mathbb{A}} \to X_{\mathbb{B}}$, and the graph of σ_X is a subalgebra of $\mathbb{R}$. Let $\mathbb{S}$ be the subalgebra of $(a,b) \in \mathbb{R}$ such that (a,b) is congruent to some element of σ_X modulo δ. Then $\mathbb{S}$ must be the graph of an isomorphism from $\mathbb{A}$ to $\mathbb{B}$. □

Theorem A.5.44. *If $\mathbb{B} \in \mathcal{V}(\mathbb{A})$ is subdirectly irreducible, $\mathbb{A}$ is finite, and $\mathcal{V}(\mathbb{A})$ is congruence modular, then $\mathbb{B}$ is similar to a subdirectly irreducible algebra in $HS(\mathbb{A})$.*

Proof. We may as well replace $\mathbb{B}$ by $D(\mathbb{B})$, so assume without loss of generality that the monolith of $\mathbb{B}$ is either nonabelian or equal to its own centralizer. If the monolith of $\mathbb{B}$ is nonabelian, then $\mathbb{B} \in HS(\mathbb{A})$ by Theorem A.5.14, so we just need to handle the case where $0^*_{\mathbb{B}} = (0_{\mathbb{B}} : 0^*_{\mathbb{B}})$. In this case, Theorem A.5.14 implies that $\mathbb{B}/0^*_{\mathbb{B}} \in HS(\mathbb{A})$, so by Proposition A.5.40 we have $|\mathbb{B}| \le |\mathbb{A}|^2 < \infty$.

Since $\mathbb{B}$ is finite, we can write $\mathbb{B} = \mathbb{R}/\theta$ for some $\mathbb{R} \le \mathbb{A}^n$ and $\theta \in \mathrm{Con}(\mathbb{R})$. Then we can write $\mathbb{R}$ as a subdirect product $\mathbb{R} \le_{sd} \mathbb{A}_1 \times \cdots \times \mathbb{A}_m$ of finitely many subdirectly irreducible algebras $\mathbb{A}_i \in HS(\mathbb{A})$. We assume that the $\mathbb{A}_i$ are chosen such that none of them can be replaced by a subdirect product of some number of proper quotients of $\mathbb{A}_i$ while still keeping the isomorphism $\mathbb{R}/\theta \cong \mathbb{B}$.

Then for any i, we must have $\theta \wedge \ker \pi_{[m]\setminus\{i\}} = 0_{\mathbb{R}}$: if not, we could replace $\mathbb{A}_i$ with a subdirect representation of $\mathbb{R}/(\ker \pi_i \vee (\theta \wedge \ker \pi_{[m]\setminus\{i\}}))$, since by the modular law we have

$$\ker \pi_{[m]\setminus\{i\}} \wedge (\ker \pi_i \vee (\theta \wedge \ker \pi_{[m]\setminus\{i\}})) = (\ker \pi_{[m]\setminus\{i\}} \wedge \ker \pi_i) \vee (\theta \wedge \ker \pi_{[m]\setminus\{i\}}) \le \theta.$$

Since $\ker \pi_{[m]\setminus\{i\}} \ne 0_{\mathbb{R}}$, we have $\theta \vee \ker \pi_{[m]\setminus\{i\}} \ge \theta^*$, so $\theta^* \wedge \ker \pi_{[m]\setminus\{i\}}$ is a cover of $0_{\mathbb{R}}$, and we have

$$[\![\theta, \theta^*]\!] \searrow [\![0_{\mathbb{R}}, \theta^* \wedge \ker \pi_{[m]\setminus\{i\}}]\!] \nearrow [\![\ker \pi_i, (\ker \pi_i)^*]\!],$$

so $\mathbb{B} = \mathbb{R}/\theta$ is similar to $\mathbb{A}_i = \mathbb{R}/\ker \pi_i$. □

Example A.5.3. Let's work out what $D(\mathbb{G})$ is when $\mathbb{G}$ is a subdirectly irreducible group. Let $\mathbb{M} \lhd \mathbb{G}$ be the normal subgroup corresponding to the monolith $0_{\mathbb{G}}^*$, and let $\mathbb{N} = C_{\mathbb{G}}(\mathbb{M}) \lhd \mathbb{G}$ be the normal subgroup corresponding to the centralizer $(0_{\mathbb{G}} : 0_{\mathbb{G}}^*)$. First off, what is the group structure on the congruence $0_{\mathbb{G}}^*$?

By definition, we have

$$0_{\mathbb{G}}^* = \{(x, y) \in \mathbb{G}^2 \mid x^{-1}y \in \mathbb{M}\}.$$

We have a natural exact sequence of groups

$$0 \to \mathbb{M} \hookrightarrow 0_{\mathbb{G}}^* \twoheadrightarrow \mathbb{G} \to 0,$$

where the inclusion is the map $m \mapsto (1, m)$ and the quotient map is the first projection π_1. The quotient $0_{\mathbb{G}}^* \twoheadrightarrow \mathbb{G}$ has a section $\Delta : \mathbb{G} \hookrightarrow 0_{\mathbb{G}}^*$ given by $g \mapsto (g, g)$. Thus we can write $0_{\mathbb{G}}^*$ as a semidirect product

$$0_{\mathbb{G}}^* \cong \mathbb{M} \rtimes \mathbb{G},$$

where the action of $\mathbb{G}$ on $\mathbb{M}$ is the standard conjugation action.

How about the congruence $\Delta_{0_{\mathbb{G}}^*}^{(0:0^*)} \in \mathrm{Con}(0_{\mathbb{G}}^*)$? By Theorem A.3.8, we have

$$\begin{bmatrix} x & w \\ y & z \end{bmatrix} \in \Delta_{0_{\mathbb{G}}^*}^{(0:0^*)} \iff (xy^{-1}z = w) \wedge (x \ \equiv_{\mathbb{M}} \ y \ \equiv_{\mathbb{N}} \ z).$$

Since this is a congruence on a group, we just need to understand the congruence class of the identity, so we plug in $x = y = 1$ and ask what values (w, z) can take. We find that $\Delta_{0_{\mathbb{G}}^*}^{(0:0^*)}$ corresponds to the normal subgroup

$$\{(n, n) \mid n \in \mathbb{N}\},$$

so under the isomorphism $0_{\mathbb{G}}^* \cong \mathbb{M} \rtimes \mathbb{G}$ it corresponds to $\mathbb{N}$, considered as a subgroup of $\mathbb{G}$. Thus we have

$$D(\mathbb{G}) = 0_{\mathbb{G}}^*/\Delta_{0_{\mathbb{G}}^*}^{(0:0^*)} \cong (\mathbb{M} \rtimes \mathbb{G})/\mathbb{N} \cong \mathbb{M} \rtimes (\mathbb{G}/\mathbb{N}).$$

That any of this makes sense follows from $\mathbb{N} = C_{\mathbb{G}}(\mathbb{M})$. We see that $\mathbb{M}$ is the normal subgroup corresponding to the monolith of $D(\mathbb{G})$, that $\mathbb{M}$ is equal to its own centralizer in $D(\mathbb{G})$, and that the natural map $\mathbb{G}/\mathbb{N} \hookrightarrow D(\mathbb{G})$ has image transverse to the monolith, and induces an isomorphism

$$\sigma : \mathbb{G}/\mathbb{N} \xrightarrow{\sim} D(\mathbb{G})/\mathbb{M}.$$

To complete the description of the similarity from $\mathbb{G}$ to $D(\mathbb{G})$, we let $\mathbb{R}$ be the fiber product of $\mathbb{G}$ and $D(\mathbb{G})$ over $\mathbb{G}/\mathbb{N}$, and define the congruence $\delta \in \mathrm{Con}(\mathbb{R})$ as the 4-ary relation

$$\begin{bmatrix} a & a' \\ b & b' \end{bmatrix} \in \delta \iff \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a' \\ b' \end{bmatrix} \in \mathbb{R} \ \wedge \ a^{-1}a' = b^{-1}b' \in \mathbb{M}.$$

That δ is closed under multiplication must be checked - it follows from the fact that $\mathbb{N}$ centralizes $\mathbb{M}$, and the fact that for any a, b, a', b' satisfying the above conditions all of a, b, a', b' must necessarily map to the same element of $\mathbb{G}/\mathbb{N}$.

What are the possible values for $D(\mathbb{G})$, assuming the monolith is abelian? Note that if we consider $\mathbb{M}$ as a module via the $\mathbb{G}/\mathbb{N}$ action, then it must be a *simple* module, since if it has any nontrivial submodule $\mathbb{M}'$, then $\mathbb{M}'$ will be a smaller normal subgroup of $\mathbb{G}$. Thus the general situation is that $\mathbb{M}$ is some simple module over the ring $\mathbb{Z}[\mathbb{G}/\mathbb{N}]$ (where $\mathbb{G}/\mathbb{N}$ acts faithfully on $\mathbb{M}$), and $D(\mathbb{G}) \cong \mathbb{M} \rtimes (\mathbb{G}/\mathbb{N})$.

Example A.5.4. If we take $\mathbb{G} = S_3$ in the above, we find that $D(S_3) \cong \mathbb{Z}/3 \rtimes \mathbb{Z}/2 \cong S_3$. The 4-ary relation $\delta \le S_3^{2\times 2}$ corresponding to the trivial similarity from S_3 to itself is given by

$$\begin{bmatrix} a & a' \\ b & b' \end{bmatrix} \in \delta \iff s(a) = s(b) = s(a') = s(b') \ \wedge \ a^{-1}a' = b^{-1}b',$$

where $s : S_3 \to \{\pm 1\}$ is the sign homomorphism.

We can think of the relation δ as having two "strands" corresponding to the two possible signs of permutations, and if we restrict to either strand then δ becomes an affine relation over $\mathbb{Z}/3$. The fact that we can multiply elements of δ which come from different strands and still get an element of δ is worth thinking about.

Now suppose that $\mathbb{G}$ is some other subdirectly irreducible group such that $D(\mathbb{G}) \cong S_3$, with monolith corresponding to $\mathbb{M} \lhd \mathbb{G}$ and $\mathbb{N} = C_G(\mathbb{M})$. Then since $\mathbb{G}$ is similar to S_3, we must have $\mathbb{M} \cong \mathbb{Z}/3$ and $\mathbb{G}/\mathbb{N} \cong \mathbb{Z}/2$ by Corollary A.5.38, with $\mathbb{G}/\mathbb{N}$ acting on $\mathbb{M}$ by negation since $D(\mathbb{G}) \cong \mathbb{M} \rtimes (\mathbb{G}/\mathbb{N}) \cong S_3$. If the action of $\mathbb{G}/\mathbb{N}$ on $\mathbb{N}$ is given by an involution τ, then for any $n \in \mathbb{N} \setminus \{1\}$ we must have $\mathbb{M}$ contained in the normal subgroup of $\mathbb{N}$ generated by n, n^τ.

In particular, if $\mathbb{N}$ is abelian then we see that $n + n^\tau, n - n^\tau \in \mathbb{M}$ for all $n \in \mathbb{N}$, and additionally in this case $\mathbb{N}$ must have prime power order by Theorem A.5.19. Thus if $\mathbb{N}$ is abelian then we must actually have $\mathbb{N} = \mathbb{M}$, and $\mathbb{G} \cong S_3$.

Example A.5.5. If we take $\mathbb{G} = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ the quaternion group with $i^2 = j^2 = k^2 = ijk = -1$, then the monolith is equal to the center, corresponding to the normal subgroup $\{\pm 1\}$, and the centralizer of the monolith is the full congruence 1_{Q_8}. Thus

$$D(Q_8) \cong \{\pm 1\} \cong \mathbb{Z}/2.$$

The relation $\delta \le (Q_8 \times \mathbb{Z}/2)^2$ is then given by

$$\begin{bmatrix} a & a' \\ b & b' \end{bmatrix} \in \delta \iff a' = (-1)^{b+b'}a.$$

This relation closely resembles an affine relation over $\mathbb{Z}/2$.

Appendix B

Tame Congruence Theory

Tame congruence theory was introduced by Hobby and McKenzie [95] in order to answer questions about congruence lattices of finite algebras. Since every congruence contains the diagonal, every congruence is automatically invariant under the *polynomial* clone of our algebra, and in fact the congruence lattice of an algebra is completely determined by the collection of *unary* polynomials of the algebra.

Proposition B.0.1. *An equivalence relation θ on an algebra $\mathbb{A}$ is a congruence of $\mathbb{A}$ iff θ is preserved by every unary polynomial operation of $\mathbb{A}$. More generally, a quasiorder $\preceq$ on $\mathbb{A}$ is a subalgebra of $\mathbb{A}^2$ if and only if $\preceq$ is preserved by every unary polynomial of $\mathbb{A}$.*

Proof. Recall that a quasiorder is just a binary relation which contains the diagonal and is transitively closed, so any quasiorder which is preserved by every basic operation of $\mathbb{A}$ will also be preserved by any unary polynomial of $\mathbb{A}$.

Conversely, suppose that the quasiorder $\preceq$ is closed under all unary polynomials of $\mathbb{A}$. Let t be any k-ary term of $\mathbb{A}$, and suppose that $a_i \preceq b_i$ for $i \in [k]$. Then for each i, we have

$$t(b_1, ..., b_{i-1}, a_i, a_{i+1}, ..., a_k) \preceq t(b_1, ..., b_{i-1}, b_i, a_{i+1}, ..., a_k),$$

since $\preceq$ is closed under the unary polynomial

$$x \mapsto t(b_1, ..., b_{i-1}, x, a_{i+1}, ..., a_k).$$

Since $\preceq$ is transitively closed, we can string these inequalities together to show that

$$\begin{aligned} t(a_1, a_2, ..., a_k) &\preceq t(b_1, a_2, ..., a_k) \\ &\preceq t(b_1, b_2, ..., a_k) \\ &\preceq \cdots \\ &\preceq t(b_1, b_2, ..., b_k). \end{aligned}$$

□

So tame congruence theory is really about the how the identities satisfied in a (usually locally finite) variety affect the behavior of unary polynomials, and how this in turn affects the behavior of congruences. Because of the important role of polynomial operations in tame congruence theory, we will use $\mathrm{Pol}_n(\mathbb{A})$ to represent the set of n-ary polynomial operations of $\mathbb{A}$ throughout this appendix (hopefully this doesn't cause any confusion with the notation for the polymorphism clone of a relational structure).

The material in this appendix is mostly taken from Hobby and McKenzie's wonderful book [95] (some of it is my solutions to various exercises from their book).

B.1 Shrinking algebras with unary polynomials, minimal sets, and traces

As soon as you have a unary operation φ on a finite set, the natural thing to do with it is to iterate it until we get the compositionally idempotent operation

$$\varphi^\infty := \lim_{n\to\infty} \varphi^{\circ n!}.$$

This gives us a large collection of (compositionally) idempotent unary polynomial operations on any finite algebra. For unary operations, I will drop the qualifier "compositionally" on "idempotent", since idempotent unary operations in the usual sense are not very interesting.

Definition B.1.1. For any algebra $\mathbb{A}$, we define $E(\mathbb{A})$ to be the set of unary polynomials $e \in \mathrm{Pol}_1(\mathbb{A})$ such that $e \circ e = e$. Elements of $E(\mathbb{A})$ might be called the *idempotents* or *projections* of $\mathbb{A}$.

Recall from Section 3.2 that for any idempotent $e \in E(\mathbb{A})$, the clone of restrictions to $e(\mathbb{A})$ of polynomial operations of $\mathbb{A}$ which preserve $e(\mathbb{A})$ is essentially the same as the clone of operations of the form

$$(x_1, ..., x_n) \mapsto e(f(e(x_1), ..., e(x_n))),$$

for $f \in \mathrm{Pol}_n(\mathbb{A})$ (strictly speaking, the es on the inside are not really necessary, they are only there to stop us from caring about how f behaves outside the set $e(\mathbb{A})$). This gives us a rich enough source of polynomial operations which preserve $e(\mathbb{A})$ to make it worth studying the restricted clone and introducing notation for it.

Definition B.1.2. If $\mathbb{A}$ is an algebra and $U \subseteq \mathbb{A}$, then we define the restriction $\mathrm{Pol}(\mathbb{A})|_U$ to be the set of restrictions $f|_U$ of polynomial operations $f \in \mathrm{Pol}(\mathbb{A})$ which preserve the subset U, and we define the *induced algebra* $\mathbb{A}|_U$ to be $(U, \mathrm{Pol}(\mathbb{A})|_U)$ (up to term equivalence).

Restrictions are related to the congruence lattice using the following result.

Lemma B.1.3 (Pálfy and Pudlák [151], [95])**.** *For any idempotent $e \in E(\mathbb{A})$, if we set $U = e(\mathbb{A})$, then the map taking the congruence $\theta \in \mathrm{Con}(\mathbb{A})$ to $e(\theta) = \theta|_U$ defines a surjective lattice homomorphism:*

$$\theta \mapsto \theta|_U : \mathrm{Con}(\mathbb{A}) \twoheadrightarrow \mathrm{Con}(\mathbb{A}|_U).$$

More generally, if $N \subseteq U$, then:

- *$\mathbb{A}|_N = (\mathbb{A}|_U)|_N$,*
- *if N is a union of $\theta|_U$ congruence classes, then the map $\alpha \mapsto \alpha|_N$ defines a lattice homomorphism from the interval $[\![0_\mathbb{A}, \theta]\!]$ of $\mathrm{Con}(\mathbb{A})$ to the interval $[\![0_N, \theta|_N]\!]$ of $\mathrm{Con}(\mathbb{A}|_N)$, and*
- *if N is equal to a congruence class of $\theta|_U$, then the map $\alpha \mapsto \alpha|_N$ is a surjective lattice homomorphism $[\![0_\mathbb{A}, \theta]\!] \twoheadrightarrow \mathrm{Con}(\mathbb{A}|_N)$.*

Proof. (Following [95]) First, we will prove the statements about the map $\theta \mapsto \theta|_U$. For any $\alpha \in \mathrm{Con}(\mathbb{A}|_U)$, we define the equivalence relation $\hat{\alpha}$ on $\mathbb{A}$ by

$$\hat{\alpha} := \{(x, y) \mid \forall f \in \mathrm{Pol}_1(\mathbb{A}), (e(f(x)), e(f(y))) \in \alpha\}.$$

Then for any $\alpha \in \mathrm{Con}(\mathbb{A}|_U)$ and $\theta \in \mathrm{Con}(\mathbb{A})$ we have $\hat{\alpha} \in \mathrm{Con}(\mathbb{A})$, $\hat{\alpha}|_U = \alpha$, and

$$\theta|_U \le \alpha \iff \theta \le \hat{\alpha}.$$

Since restriction obviously preserves meets of congruences, we just need to check that it preserves joins. For this, let $\alpha = \theta_1|_U \vee \theta_2|_U$, and note that

$$\theta_i|_U \le \alpha \implies \theta_i \le \hat{\alpha} \implies \theta_1 \vee \theta_2 \le \hat{\alpha} \implies (\theta_1 \vee \theta_2)|_U \le \alpha,$$

while the inequality $\alpha \le (\theta_1 \vee \theta_2)|_U$ is obvious.

To see that $\mathbb{A}|_N = (\mathbb{A}|_U)|_N$, note that if $f \in \mathrm{Pol}(\mathbb{A})$ preserves N, then $e \circ f$ preserves U and $(e \circ f)|_N = f|_N$. Since we have $\alpha|_N = (\alpha|_U)|_N$ for $\alpha \in \mathrm{Con}(\mathbb{A})$, to prove the remaining claims we just need to think about the restriction map $[\![0_U, \theta|_U]\!] \to [\![0_N, \theta|_N]\!]$.

If N is a union of congruence classes of $\theta|_U$, then for any $\theta_1, \theta_2 \le \theta|_U$ neither θ_i connects any element inside N to any element outside N, so $(\theta_1 \vee \theta_2)|_N = \theta_1|_N \vee \theta_2|_N$. Thus the map $[\![0_U, \theta|_U]\!] \to [\![0_N, \theta|_N]\!]$ is a lattice homomorphism.

To finish, we need to show that if N is a congruence class of $\theta|_U$ then this map is surjective. For this, we extend a congruence $\alpha \in \mathrm{Con}(\mathbb{A}|_N)$ to a congruence on $\mathbb{A}|_U$ by

$$\breve{\alpha} \coloneqq \{(x, y) \mid \forall f \in \mathrm{Pol}_1(\mathbb{A}|_U), f(x) \in N \text{ or } f(y) \in N \implies (f(x), f(y)) \in \alpha\}.$$

Since every $f \in \mathrm{Pol}_1(\mathbb{A}|_U)$ preserves $\theta|_U$, if $f(x) \in N$ for any $x \in N$ then f must preserve N, so we have $\breve{\alpha}|_N = \alpha$. □

Definition B.1.4. Write $\mathbb{B} \preceq_| \mathbb{A}$ if there is some idempotent $e \in E(\mathbb{A})$, congruence $\theta \in \mathrm{Con}(\mathbb{A})$, and $a \in e(\mathbb{A})$, such that if we define

$$U = e(\mathbb{A})$$

and

$$N = U \cap (a/\theta),$$

then $\mathbb{B}$ is polynomially equivalent to $\mathbb{A}|_N$. (Note that in general $\mathbb{B}$ will have a different signature than $\mathbb{A}$.)

Proposition B.1.5. *The relation $\preceq_|$ is transitively closed on finite algebras: if $\mathbb{C} \preceq_| \mathbb{B} \preceq_| \mathbb{A}$ and $\mathbb{A}$ is finite, then $\mathbb{C} \preceq_| \mathbb{A}$.*

Proof. Suppose that $\mathbb{B} = \mathbb{A}|_N$ for $N = U \cap (a/\theta)$, $U = e(\mathbb{A})$, $e \in E(\mathbb{A})$. Additionally, suppose that $\mathbb{C} = \mathbb{B}|_{N'}$, for $N' = U' \cap (a'/\alpha)$, $\alpha \in \mathrm{Con}(\mathbb{B})$, $U' = e'(\mathbb{B})$, $e' \in E(\mathbb{B})$.

Since α is a congruence on $\mathbb{B}$, by Lemma B.1.3 there is some congruence $\bar{\alpha} \in [\![0_{\mathbb{A}}, \theta]\!]$ such that $\bar{\alpha}|_N = \alpha$. Additionally, e' is the restriction of some unary polynomial $\hat{e}'$ of $\mathbb{A}$ to N, and by composing $\hat{e}' \circ e$ and iterating it, we get $\bar{e}' \in E(\mathbb{A})$ such that

$$\bar{e}'(\mathbb{A}) \subseteq U \quad \text{and} \quad \bar{e}'(N) = U'.$$

Thus we have

$$N' \subseteq \bar{e}'(\mathbb{A}) \cap (a'/\bar{\alpha}) \subseteq \bar{e}'(U \cap (a'/\bar{\alpha})) \cap (a'/\bar{\alpha}) \subseteq \bar{e}'(N) \cap (a'/\bar{\alpha}) \subseteq N'.$$

To finish, we need to check that for every polynomial $f \in \mathrm{Pol}(\mathbb{A})$ which preserves N', there is a polynomial $\bar{f} \in \mathrm{Pol}(\mathbb{A}|_N)$ such that $\bar{f}|_{N'} = f|_{N'}$. If we take $\bar{f} = e \circ f$, then $\bar{f}$ automatically preserves U, and since f preserves N', we have

$$\bar{f}(a', ..., a') \in a'/\bar{\alpha} \subseteq a/\theta,$$

so $\bar{f}$ also preserves a/θ, and therefore $\bar{f}$ preserves $U \cap (a/\theta) = N$. □

Proposition B.1.6. *If $\mathbb{A}$ is finite and $\mathbb{B} \preceq_| \mathbb{A}$ is such that every constant of $\mathbb{B}$ is a term of $\mathbb{B}$, and if $\mathbb{D} \in HSP_{fin}(\mathbb{B})$, then there is some $\mathbb{C} \in HSP_{fin}(\mathbb{A})$ such that $\mathbb{D} \preceq_| \mathbb{C}$.*

Proof. Suppose that $\mathbb{B} = \mathbb{A}|_B$ with $B = e(\mathbb{A}) \cap (a/\alpha)$, where $e \in E(\mathbb{A})$, $\alpha \in \mathrm{Con}(\mathbb{A})$, and $a \in e(\mathbb{A})$. We handle quotients and subpowers separately - for quotients, we will not need to assume that every constant of $\mathbb{B}$ is a term of $\mathbb{B}$. In fact, if $\mathbb{D} = \mathbb{B}/\theta$, we just choose $\bar{\theta} \in [\![0_\mathbb{A}, \alpha]\!]$ such that $\bar{\theta}|_\mathbb{B} = \theta$ using Lemma B.1.3, and take $\mathbb{C} = \mathbb{A}/\bar{\theta}$.

Now suppose that $\mathbb{D} \le \mathbb{B}^n$. Note that since every constant operation of $\mathbb{B}$ is a term of $\mathbb{B}$, every $\mathbb{D} \le \mathbb{B}^n$ must contain the diagonal $\mathbb{B}^{(n)} = \{b^{(n)} \mid b \in \mathbb{B}\}$, where $b^{(n)} = (b, b, ..., b)$. Let $\mathbb{C} \le \mathbb{B}^n$ be given by

$$\mathbb{C} = \mathrm{Sg}_{\mathbb{A}^n}(\mathbb{A}^{(n)} \cup \mathbb{D}).$$

Note that $\mathbb{C}$ is exactly the closure of $\mathbb{D}$ under coordinatewise application of polynomials of $\mathbb{A}$. Define $e^{(n)} \in \mathrm{Pol}(\mathbb{A}^n)$ by replacing each constant c in the definition of e by $c^{(n)}$. Since each $c^{(n)}$ is also an element of $\mathbb{C}$, we see that $e^{(n)}$ is also a polynomial of $\mathbb{C}$, which acts like e on each coordinate, so $e^{(n)} \in E(\mathbb{C})$. We need to check that if we set

$$D = e^{(n)}(\mathbb{C}) \cap (a^{(n)}/\alpha^n),$$

then D is the underlying set of $\mathbb{D}$ and $\mathbb{C}|_D$ is polynomially equivalent to $\mathbb{D}$. This follows from the facts that $e^{(n)}(\mathbb{C})$ is the closure of $\mathbb{D}$ under coordinatewise application of polynomials of $\mathbb{A}$ which have been composed with e, and that a polynomial $f \in \mathrm{Pol}(\mathbb{A})$ has $f^{(n)}(\mathbb{D}, ..., \mathbb{D}) \cap (a^{(n)}/\alpha^n) \ne \emptyset$ iff $e \circ f$ preserves B. □

In order to get any use out of this to study an interval $[\![\alpha, \beta]\!]$ of the congruence lattice, we need to find idempotents e such that $e(\alpha) \ne e(\beta)$, or equivalently such that $e(\beta) \not\subseteq \alpha$.

Definition B.1.7. If $\alpha < \beta \in \mathrm{Con}(\mathbb{A})$, then we define $U_\mathbb{A}(\alpha, \beta)$ to be the collection of sets of the form $f(\mathbb{A})$ for $f \in \mathrm{Pol}_1(\mathbb{A})$ such that $f(\beta) \not\subseteq \alpha$. We define $M_\mathbb{A}(\alpha, \beta)$ to be the collection of minimal sets in $U_\mathbb{A}(\alpha, \beta)$, and we call the sets in $M_\mathbb{A}(\alpha, \beta)$ the (α, β)-*minimal sets* of $\mathbb{A}$.

Proposition B.1.8. *If β is a cover of α in* $\mathrm{Con}(\mathbb{A})$ *and $\mathbb{A}$ is finite, then for each (α, β)-minimal set $U \in M_\mathbb{A}(\alpha, \beta)$, there is some $e \in E(\mathbb{A})$ such that $U = e(\mathbb{A})$.*

Proof. Pick any $g \in \mathrm{Pol}_1(\mathbb{A})$ such that $g(\mathbb{A}) = U$ and $g(\beta) \not\subseteq \alpha$. Then since β covers α and $g(\beta) \subseteq \beta$, the congruence generated by $g(\beta) \cup \alpha$ must be β. Thus for any $(x, y) \in \beta$, there must be some $h_i \in \mathrm{Pol}_1(\mathbb{A})$ and $(u_i, v_i) \in g(\beta) \cup \alpha$ such that $x = h_1(u_1)$, $h_i(v_i) = h_{i+1}(u_{i+1})$, and $h_n(v_n) = y$ for some n.

If we choose $(x, y) \in \beta$ such that $(g(x), g(y)) \notin \alpha$, we see that there must be some i such that $(g(h_i(u_i)), g(h_i(v_i))) \not\subseteq \alpha$, and for this i we must have $(u_i, v_i) \in g(\beta)$. Setting $f = g \circ h_i$, we see that

$$f(g(\beta)) \not\subseteq \alpha$$

and

$$f(g(\mathbb{A})) \subseteq U.$$

Since U is (α,β)-minimal, we must have $f(g(\mathbb{A})) = U$, so $f(U) = U$. Iterating f gives us $e = f^\infty \in E(\mathbb{A})$ with $e(\mathbb{A}) = U$. □

We will also want to prove similar results for certain other pairs of congruences $\alpha < \beta$. Precisely stating what we are going for requires a bit more work.

Definition B.1.9. A $0,1$-*lattice* is defined to be a lattice with constants 0 and 1 which satisfy $0 \le x \le 1$ for all x. Note that every interval $[\![\alpha,\beta]\!]$ in a lattice can be regarded as a $0,1$-lattice, with 0 interpreted as α and 1 interpreted as β.

A $0,1$-*separating homomorphism* is a lattice homomorphism such that $f^{-1}(f(0)) = \{0\}$ and $f^{-1}(f(1)) = \{1\}$.

Definition B.1.10. A *congruence quotient* is defined to be an ordered pair of congruences (α,β) such that $\alpha < \beta$. A congruence quotient (α,β) is called *prime* if β covers α.

A congruence quotient (α,β) is called *tame* if there is some $U \in M_{\mathbb{A}}(\alpha,\beta)$ and some $e \in E(\mathbb{A})$ such that $e(\mathbb{A}) = U$, and such that the restriction homomorphism

$$[\![\alpha,\beta]\!] \twoheadrightarrow [\![\alpha|_U,\beta|_U]\!]$$

is a $0,1$-separating homomorphism, that is, for $\alpha \le \gamma \le \beta$ we have $\gamma|_U = \alpha|_U \implies \gamma = \alpha$ and $\gamma|_U = \beta|_U \implies \gamma = \beta$. An algebra $\mathbb{A}$ is called *tame* if $(0_{\mathbb{A}}, 1_{\mathbb{A}})$ is tame.

So far we have shown that every prime quotient on a finite algebra is tame. There is a more general lattice theoretic condition that implies tameness, but first we should try to see what being tame is good for. The first important result is that all of the minimal sets for a tame quotient look the same as each other.

Definition B.1.11. If $U, V \subseteq \mathbb{A}$, then we say that U, V are *polynomially isomorphic* in $\mathbb{A}$ if there are unary polynomials $f, g \in \mathrm{Pol}_1(\mathbb{A})$ such that

$$f(U) = V,\ g(V) = U,\ g \circ f|_U = \mathrm{id}_U,\ f \circ g|_V = \mathrm{id}_V.$$

In this case we write $f : U \simeq V$.

Proposition B.1.12. *If $f : U \simeq V$ in $\mathbb{A}$, then $f|_U$ defines an isomorphism from $\mathbb{A}|_U$ to $\mathbb{A}|_V$ (up to term equivalence). Furthermore, for any $\theta \in \mathrm{Con}(\mathbb{A})$ we have $f|_U(\theta|_U) = \theta|_V$.*

Proposition B.1.13. *If $U, V \subseteq \mathbb{A}$ and $\mathbb{A}$ is finite, then $U \simeq V$ iff there are $f, g \in \mathrm{Pol}_1(\mathbb{A})$ such that $f(U) = V$ and $g(V) = U$. If additionally we have $f(\mathbb{A}) = V$, then there is some idempotent $e \in E(\mathbb{A})$ such that $e(\mathbb{A}) = U$.*

Proof. Take $g' = (g \circ f)^{\infty - 1} \circ g$ and $e = g' \circ f = (g \circ f)^\infty$, so $e \in E(\mathbb{A})$. Then $g'(V) = U$, $g' \circ f|_U = \mathrm{id}_U$, $f \circ g'|_V = \mathrm{id}_V$, and if $f(\mathbb{A}) = V$ then $e(\mathbb{A}) = g'(f(\mathbb{A})) = g'(V) = U$. □

Theorem B.1.14 (Minimal sets for tame quotients [95])**.** *If (α,β) is a tame congruence quotient on a finite algebra $\mathbb{A}$, then all of the following are true.*

(a) For all $U, V \in M_{\mathbb{A}}(\alpha,\beta)$, we have $U \simeq V$ in $\mathbb{A}$.

(b) *For all $U \in M_{\mathbb{A}}(\alpha,\beta)$, there is some $e \in E(\mathbb{A})$ such that $e(\mathbb{A}) = U$, and the restriction homomorphism $[\![\alpha,\beta]\!] \twoheadrightarrow [\![\alpha|_U,\beta|_U]\!]$ is a $0,1$-separating homomorphism.*

(c) *For all $U \in M_{\mathbb{A}}(\alpha,\beta)$ and $(x,y) \in \beta \setminus \alpha$, there is some $f \in \mathrm{Pol}_1(\mathbb{A})$ such that $f(\mathbb{A}) = U$ and $(f(x),f(y)) \not\in \alpha$.*

(d) *For all $U \in M_{\mathbb{A}}(\alpha,\beta)$, β is the transitive closure of $\alpha \cup \bigcup_{g\in\mathrm{Pol}_1(\mathbb{A})} g(\beta|_U)$.*

(e) *For all $U \in M_{\mathbb{A}}(\alpha,\beta)$ and $f \in \mathrm{Pol}_1(\mathbb{A})$ such that $f(\beta|_U) \not\subseteq \alpha$, we have $f(U) \in M_{\mathbb{A}}(\alpha,\beta)$ and $f : U \simeq f(U)$.*

(f) *For any $f \in \mathrm{Pol}_1(\mathbb{A})$ such that $f(\beta) \not\subseteq \alpha$, there is some $U \in M_{\mathbb{A}}(\alpha,\beta)$ such that $f : U \simeq f(U)$.*

(g) *For any $(x,y) \in \beta \setminus \alpha$, there is some $U \in M_{\mathbb{A}}(\alpha,\beta)$ and $e \in E(\mathbb{A})$ such that $e(\mathbb{A}) = U$ and $(e(x),e(y)) \not\in \alpha$.*

Proof. (Following [95]) By the definition of tameness, there is *some* $U \in M_{\mathbb{A}}(\alpha,\beta)$ that satisfies (b). We will first show that (c) and (d) hold for this U, and then use this to prove (a), which will imply that (b) is true in general. Then we will use these to prove (e), (f), and (g).

Suppose that (b) holds for U. To prove (c), let $\gamma \in [\![\alpha,\beta]\!]$ be the congruence generated by α and (x,y). Then since $\gamma \ne \alpha$ we must have $\gamma|_U \ne \alpha|_U$ (since restriction is $0,1$-separating). Since $\gamma|_U = e(\gamma)$, this means that $\gamma \not\subseteq e^{-1}(\alpha)$, so there must be some $g \in \mathrm{Pol}_1(\mathbb{A})$ such that $(g(x),g(y)) \not\in e^{-1}(\alpha)$. Taking $f = e \circ g$ proves (c).

To see that (b) implies (d), let $\gamma \in [\![\alpha,\beta]\!]$ be the transitive closure of $\alpha \cup \bigcup_{g\in\mathrm{Pol}_1(\mathbb{A})} g(\beta|_U)$. Then $\gamma|_U = \beta|_U$, so we must have $\gamma = \beta$ (since restriction is $0,1$-separating).

Now suppose that $U,V \in M_{\mathbb{A}}(\alpha,\beta)$ and that V satisfies (b), (c), (d). Since $U \in M_{\mathbb{A}}(\alpha,\beta)$, there must be some $h \in \mathrm{Pol}_1(\mathbb{A})$ and $(x,y) \in h(\beta) \setminus \alpha$ with $h(\mathbb{A}) = U$, so by (c) applied to V there is some $f \in \mathrm{Pol}_1(\mathbb{A})$ such that $f(\mathbb{A}) = V$ and $(f(x),f(y)) \not\in \alpha$. Then from $f(h(\mathbb{A})) \subseteq V$ and $f(h(\beta)) \not\subseteq \alpha$ we have $f(h(\mathbb{A})) = V$ by (α,β)-minimality of V, so $f(U) = V$. Next, by (d) applied to V we see that there must be some $g \in \mathrm{Pol}_1(\mathbb{A})$ such that $g(\beta|_V) \not\subseteq h^{-1}(\alpha)$. Then if $e \in E(\mathbb{A})$ has $e(\mathbb{A}) = V$, we see that $h(g(e(\beta))) \not\subseteq \alpha$ and $h(g(e(\mathbb{A}))) \subseteq U$, so since U is (α,β)-minimal we see that $h(g(V)) = U$. Now we can apply the previous proposition to see that $f : U \simeq V$ and that U satisfies (b) as well.

To prove (e), we use (b) to see that there is some $e \in E(\mathbb{A})$ with $e(\mathbb{A}) = U$, and note that $f(e(\beta)) = f(\beta|_U) \not\subseteq \alpha$, so $f(U) = f(e(\mathbb{A}))$ must contain some minimal $V \in M_{\mathbb{A}}(\alpha,\beta)$. By (a) we see that $|V| = |U|$, so we must in fact have $f(U) = V$ and $f : U \simeq V$.

To prove (f), we apply (d) to any $V \in M_{\mathbb{A}}(\alpha,\beta)$ to see that there is some $g \in \mathrm{Pol}_1(\mathbb{A})$ such that $g(\beta|_V) \not\subseteq f^{-1}(\alpha)$. Then by applying (e) twice we see that we can take $U = g(V)$.

To prove (g), we apply (c) to any $V \in M_{\mathbb{A}}(\alpha,\beta)$ to see that there is some $f \in \mathrm{Pol}_1(\mathbb{A})$ such that $f(\mathbb{A}) = V$ and $(f(x),f(y)) \not\in \alpha$. By (f), there is some $U \in M_{\mathbb{A}}(\alpha,\beta)$ such that $f : U \simeq f(U)$, and since $f(U) \subseteq f(\mathbb{A}) = V$, we must have $f(U) = V$ by (α,β)-minimality. Thus there is some $g \in \mathrm{Pol}_1(\mathbb{A})$ such that $g \circ f|_U = \mathrm{id}_U$, and we can take $e = g \circ f$. □

Corollary B.1.15. *If (α,β) is a tame congruence quotient on a finite algebra $\mathbb{A}$ and $U \in M_{\mathbb{A}}(\alpha,\beta)$, then every unary polynomial f of $\mathbb{A}|_U$ is either a permutation of U or has $f(\beta|_U) \subseteq \alpha|_U$.*

If we restrict to a congruence class of $\beta|_U$, we get an even stronger result.

Definition B.1.16. If (α, β) is a tame congruence quotient on $\mathbb{A}$ and $U \in M_{\mathbb{A}}(\alpha, \beta)$, then a set $N \subseteq U$ is called an (α, β)-*trace* in U if N is a congruence class of $\beta|_U$ which is not also a congruence class of $\alpha|_U$.

We define the *body* of the (α, β)-minimal set U to be the union of the (α, β)-traces, and we define the *tail* of U to be the set of congruence classes of $\beta|_U$ which are also congruence classes of $\alpha|_U$.

Since $\beta|_U \not\subseteq \alpha|_U$ by the definition of an (α, β)-minimal set, we see that every (α, β)-minimal set U has a nonempty body, i.e. there is at least one (α, β)-trace N in U.

Corollary B.1.17. *If (α, β) is a tame congruence quotient on a finite algebra $\mathbb{A}$ and N is an (α, β)-trace, then every unary polynomial f of $\mathbb{A}|_N$ is either a permutation, or has $f(N)$ contained in some congruence class of $\alpha|_N$. In particular, every unary polynomial of $\mathbb{A}|_N/\alpha|_N$ is either a permutation or is constant.*

Definition B.1.18. We say that an algebra is *permutational* or *minimal* if every unary polynomial is either a permutation or is constant.

More generally, if (α, β) is a congruence quotient on $\mathbb{A}$, we say that $\mathbb{A}$ is (α, β)-*minimal* if every unary polynomial f of $\mathbb{A}$ is either a permutation or has $f(\beta) \subseteq \alpha$. Note that in this case, (α, β) is necessarily tame.

The general strategy will be to understand an algebra by first understanding the structure of the traces, and then reconstructing the algebra from the traces. When we apply unary polynomials of $\mathbb{A}$ to (α, β)-traces, the result will often also be an (α, β)-trace.

Corollary B.1.19. *If (α, β) is a tame congruence quotient on a finite algebra $\mathbb{A}$ and N is an (α, β)-trace, then for every unary polynomial f of $\mathbb{A}$ either $f(N)$ is contained in some congruence class of α, or $f(N)$ is another (α, β)-trace and $f : N \simeq f(N)$.*

Proof. This follows directly from Theorem B.1.14(e). □

For the purpose of stitching together the traces to reconstruct the algebra, we have another consequence of Theorem B.1.14.

Corollary B.1.20. *If (α, β) is a tame congruence quotient on a finite algebra $\mathbb{A}$, then β is the transitive closure of*

$$\alpha \cup \{N^2 \mid N \text{ is an } (\alpha, \beta)\text{-trace}\}.$$

Proof. This follows from Theorem B.1.14(d) and the previous corollary, once we note that for each (α, β)-minimal set U, every congruence class of $\beta|_U$ is either a congruence class of $\alpha|_U$ or is an (α, β)-trace. □

Proposition B.1.21. *If (α, β) is a prime congruence quotient of a finite algebra $\mathbb{A}$ (i.e., if β is a cover of α), then all of the (α, β)-traces are polynomially isomorphic in $\mathbb{A}$.*

Proof. Let U be any (α, β)-minimal set. By Theorem B.1.14(a) it's enough to show that any pair of (α, β)-traces N, K contained in U are polynomially isomorphic in $\mathbb{A}|_U$. By Lemma B.1.3, the restriction homomorphism $[\![\alpha, \beta]\!] \to [\![\alpha|_U, \beta|_U]\!]$ is surjective, so $\beta|_U$ covers $\alpha|_U$ in $\mathbb{A}|_U$. Then since N is not contained in a congruence class of α, the congruence of $\mathbb{A}|_U$ generated by $\alpha|_U \cup N^2$ must be $\beta|_U$. In particular, there must be some unary polynomial $f \in \operatorname{Pol}_1(\mathbb{A}|_U)$ such that $f(N) \subseteq K$ and $f(N)$ is not contained in any congruence class of α. Then f must be a permutation of U, and so we have $f : N \simeq K$. □

Tameness can also be derived from some of its consequences.

Proposition B.1.22. *If $\alpha < \beta \in \mathrm{Con}(\mathbb{A})$ and there is some finite (α, β)-minimal set U which satisfies Theorem B.1.14(c) and B.1.14(d), then (α, β) is tame.*

Proof. If we make a digraph on $(x, y) \in \beta|_U \setminus \alpha|_U$ with an edge from (x, y) to (u, v) whenever there is some $f \in \mathrm{Pol}_1(\mathbb{A})$ with $f(\mathbb{A}) = U$ and $f(x) = u, f(y) = v$, then Theorem B.1.14(c) for U implies that every vertex in this digraph has outdegree at least one, so there must be a directed cycle. By composing the unary polymorphisms corresponding to the edges of this directed cycle, we find an $f \in \mathrm{Pol}_1(\mathbb{A})$ such that $f(\mathbb{A}) \subseteq U$ and $f(x) = x, f(y) = y$ for some $(x, y) \in \beta|_U \setminus \alpha|_U$. Taking $e = f^\infty$, we see that $e \in E(\mathbb{A})$ with $e(\mathbb{A}) \subseteq U$ and $e(\beta) \not\subseteq \alpha$. Finally, Theorem B.1.14(c) and B.1.14(d) for U directly imply that the restriction homomorphism $[\![\alpha, \beta]\!] \twoheadrightarrow [\![\alpha|_U, \beta|_U]\!]$ is $0, 1$-separating. □

Many of the previous results simplify if $\alpha = 0_{\mathbb{A}}$. To relate the general case to that situation, we need to know how tameness behaves when we pass to a quotient.

Proposition B.1.23. *If $\delta \le \alpha < \beta$ are congruences on a finite algebra $\mathbb{A}$, then (α, β) is tame on $\mathbb{A}$ iff $(\alpha/\delta, \beta/\delta)$ is tame on $\mathbb{A}/\delta$. If (α, β) is tame, then we have*

$$M_{\mathbb{A}/\delta}(\alpha/\delta, \beta/\delta) = \{U/\delta \mid U \in M_{\mathbb{A}}(\alpha, \beta)\}.$$

In particular, the $(\alpha/\delta, \beta/\delta)$-traces are exactly the quotients of the (α, β)-traces by δ.

Proof. (Following [95]) For any unary polynomial $f \in \mathrm{Pol}_1(\mathbb{A})$, we have $f(\beta) \subseteq \alpha$ iff $f(\beta/\delta) \subseteq \alpha/\delta$ in $\mathbb{A}/\delta$, and for any $\gamma, \gamma' \in [\![\alpha, \beta]\!]$ we have $\gamma = \gamma' \iff \gamma/\delta = \gamma'/\delta$ and similarly for restrictions. The challenge is showing that the minimal sets correspond.

First we show that if $U \in M_{\mathbb{A}}(\alpha, \beta)$ and there is an $e \in E(\mathbb{A})$ with $e(\mathbb{A}) = U$, then $U/\delta \in M_{\mathbb{A}/\delta}(\alpha/\delta, \beta/\delta)$. To see this, suppose that some $f \in \mathrm{Pol}_1(\mathbb{A})$ has $f(\beta) \not\subseteq \alpha$ and $f(\mathbb{A}/\delta) \subseteq U/\delta$. Then $e(f(\mathbb{A})) \subseteq U$ and $e(f(\beta)) \not\subseteq \alpha$, so $e(f(\mathbb{A})) = U$, so $f(\mathbb{A}/\delta) = U/\delta$. This shows that if (α, β) is tame then $(\alpha/\delta, \beta/\delta)$ is tame.

Now suppose $(\alpha/\delta, \beta/\delta)$ is tame, and let U be any (α, β)-minimal set. Pick $f \in \mathrm{Pol}_1(\mathbb{A})$ with $f(\mathbb{A}) = U$ and $f(\beta) \not\subseteq \alpha$. By Theorem B.1.14(f) we see that there is some $V \in M_{\mathbb{A}/\delta}(\alpha/\delta, \beta/\delta)$ such that $f : V \simeq f(V)$. Pick $g \in \mathrm{Pol}_1(\mathbb{A})$ such that $g : f(V) \to V$ inverts $f : V \to f(V)$, then by iterating $f \circ g$ we get an idempotent $e \in E(\mathbb{A})$ with $e(f(V)) = f(V)$. Thus $e(\beta) \not\subseteq \alpha$, and from $e(\mathbb{A}) \subseteq f(\mathbb{A}) = U$ we get $e(\mathbb{A}) = U$. Then by the previous paragraph we see that U/δ is $(\alpha/\delta, \beta/\delta)$-minimal, which allows us to conclude that the restriction homomorphism $[\![\alpha, \beta]\!] \twoheadrightarrow [\![\alpha|_U, \beta|_U]\!]$ is $0, 1$-separating.

To finish, we need to show that any $V \in M_{\mathbb{A}/\delta}(\alpha/\delta, \beta/\delta)$ is a quotient of an (α, β)-minimal set when (α, β) is tame. Pick any $U \in M_{\mathbb{A}}(\alpha, \beta)$, then since U/δ is $(\alpha/\delta, \beta/\delta)$-minimal we can apply Theorem B.1.14(a) to see that there is an $f \in \mathrm{Pol}_1(\mathbb{A})$ with $f : U/\delta \simeq V$. Then $f(\beta|_U) \not\subseteq \alpha$, so by Theorem B.1.14(e) we have $f(U) \in M_{\mathbb{A}}(\alpha, \beta)$, and $V = f(U)/\delta$. □

Proposition B.1.24. *If $\alpha \le \gamma < \beta$ and $(\alpha, \beta), (\gamma, \beta)$ are both tame quotients on a finite algebra $\mathbb{A}$, then $M_{\mathbb{A}}(\alpha, \beta) = M_{\mathbb{A}}(\gamma, \beta)$.*

Proof. If $U \in M_{\mathbb{A}}(\alpha, \beta)$, then $\beta|_U \not\subseteq \gamma|_U$ since the restriction map is $0, 1$-separating (by Theorem B.1.14(b)). If $f(\mathbb{A}) \subseteq U$ and $f(\beta) \not\subseteq \gamma$, then we have $f(\beta) \not\subseteq \alpha$, so $f(\mathbb{A}) = U$ by (α, β)-minimality, so $U \in M_{\mathbb{A}}(\gamma, \beta)$. Conversely, if $V \in M_{\mathbb{A}}(\gamma, \beta)$, then by Theorem B.1.14(a) we have $V \simeq U$ for some $U \in M_{\mathbb{A}}(\alpha, \beta)$, so $V \in M_{\mathbb{A}}(\alpha, \beta)$ by Theorem B.1.14(e). □

Recall that two intervals are perspective, written $[\![\alpha, \beta]\!] \searrow [\![\gamma, \delta]\!]$, if $\alpha \wedge \delta = \gamma$ and $\alpha \vee \delta = \beta$.

Proposition B.1.25. *If $[\![\alpha, \beta]\!] \searrow [\![\gamma, \delta]\!]$ in $\mathrm{Con}(\mathbb{A})$, then $M_{\mathbb{A}}(\alpha, \beta) = M_{\mathbb{A}}(\gamma, \delta)$ and $M_{\mathbb{A}}(\gamma, \beta) \subseteq M_{\mathbb{A}}(\gamma, \alpha) \cup M_{\mathbb{A}}(\gamma, \delta)$.*

Proof. For any $f \in \mathrm{Pol}_1(\mathbb{A})$, we have $f(\beta) = f(\alpha \vee \delta) \subseteq \alpha$ iff $f(\alpha) \cup f(\delta) \subseteq \alpha$ iff $f(\delta) \subseteq \alpha$ iff $f(\delta) \subseteq \alpha \cap \delta = \gamma$, so $M_{\mathbb{A}}(\alpha, \beta) = M_{\mathbb{A}}(\gamma, \delta)$. Similarly, we have $f(\beta) \subseteq \gamma$ iff $f(\beta) \subseteq \alpha$ and $f(\beta) \subseteq \delta$, so $M_{\mathbb{A}}(\gamma, \beta) \subseteq M_{\mathbb{A}}(\alpha, \beta) \cup M_{\mathbb{A}}(\delta, \beta) = M_{\mathbb{A}}(\gamma, \alpha) \cup M_{\mathbb{A}}(\gamma, \delta)$. □

Corollary B.1.26. *If α, β are congruences on a finite algebra and the interval $[\![\alpha, \beta]\!]$ is isomorphic to the diamond lattice $\mathcal{M}_n$ for some $n \geq 3$, then (α, β) is tame and every congruence quotient contained in $[\![\alpha, \beta]\!]$ has the same collection of minimal sets.*

Proposition B.1.27. *If (α, β) is a tame congruence quotient on a finite algebra $\mathbb{A}$ and U is an (α, β)-minimal set, then for any $\gamma' < \delta' \in [\![\alpha|_U, \beta|_U]\!]$, there are lifts $\gamma < \delta \in [\![\alpha, \beta]\!]$ such that $\gamma|_U = \gamma', \delta|_U = \delta'$, with (γ, δ) a tame quotient. For any such γ, δ we have $M_{\mathbb{A}}(\gamma, \delta) = M_{\mathbb{A}}(\alpha, \beta)$.*

Proof. Since any unary polynomial which collapses β into α necessarily collapses δ into γ for any $\gamma, \delta \in [\![\alpha, \beta]\!]$, we see that U is (γ, δ)-minimal for any $\gamma < \delta$ which restrict to γ', δ', so we just need to ensure that the restriction homomorphism from $[\![\gamma, \delta]\!]$ is $0, 1$-separating. Since restriction to U is a lattice homomorphism, we can take γ to be maximal among congruences which restrict to γ' and are $\leq \beta$, and take δ to be minimal among congruences which restrict to δ' and are $\geq \gamma$. By Theorem B.1.14(a) and B.1.14(e) every (γ, δ)-minimal set V has $U \simeq V$ and so is also (α, β)-minimal, and similarly every (α, β)-minimal set is (γ, δ)-minimal. □

Some basic examples to keep in mind follow.

Example B.1.1. If $\mathbb{A}$ is a finite lattice and $\alpha < \beta \in \mathrm{Con}(\mathbb{A})$, then the (α, β)-minimal sets all have the form $\{a, b\}$ with $a < b$ and $(a, b) \in \beta \setminus \alpha$, and any pair $\{a, b\}$ which satisfies those conditions and additionally has b covering a is an (α, β)-minimal set. Each such set with b covering a is the image of the idempotent unary polynomial $x \mapsto (x \wedge b) \vee a$, however, in order for the restriction homomorphism to be $0, 1$-separating, β must be a cover of α. Thus the tame congruence quotients of $\mathbb{A}$ are exactly the same as the prime quotients. Additionally, every (α, β)-minimal set consists of just a single (α, β)-trace (i.e., the minimal sets have no tails), and every trace is the image of some pair $\{a, b\}$ with b covering a under some unary polynomial of $\mathbb{A}$, and is polynomially equivalent to a two element lattice.

Example B.1.2. If $\mathbb{A}$ is a finite module over a ring $\mathbb{R}$ (which we may assume acts faithfully on $\mathbb{A}$ without loss of generality) and $\alpha < \beta \in \mathrm{Con}(\mathbb{A})$, then we can represent the congruences α, β by the submodules $\mathbb{M}_\alpha = 0/\alpha$, $\mathbb{M}_\beta = 0/\beta$ of $\mathbb{A}$. Every unary polynomial f of $\mathbb{A}$ has the form $f : x \mapsto rx + c$, and we have $f(\beta) \subseteq \alpha$ iff $r\mathbb{M}_\beta \subseteq \mathbb{M}_\alpha$. The set of $r \in \mathbb{R}$ such that $r\mathbb{M}_\beta \subseteq \mathbb{M}_\alpha$ is called the *annihilator* of $\mathbb{M}_\beta/\mathbb{M}_\alpha$, and forms a (two-sided) ideal $\mathbb{I}$ of $\mathbb{R}$. Then $\mathbb{M}_\beta/\mathbb{M}_\alpha$ is a module over $\mathbb{R}/\mathbb{I}$, and $\mathbb{R}/\mathbb{I}$ acts faithfully on $\mathbb{M}_\beta/\mathbb{M}_\alpha$.

We will show that (α, β) is a tame congruence quotient if and only if $\mathbb{I}$ is maximal among two-sided ideals of $\mathbb{R}$, that is, iff $\mathbb{R}/\mathbb{I}$ is a simple ring. By the classification of finite simple rings, this is equivalent to proving that $\mathbb{R}/\mathbb{I}$ is isomorphic to a matrix ring $M_n(\mathbb{F}_{p^k})$ over a finite field $\mathbb{F}_{p^k}$ for some n and some prime power p^k. Additionally, we will show that $\mathbb{M}_\beta/\mathbb{M}_\alpha$ is one of the modules $\mathbb{F}_{p^k}^{n \times m}$ for some m, with the action of $\mathbb{R}/\mathbb{I}$ on $\mathbb{M}_\beta/\mathbb{M}_\alpha$ given by matrix multiplication. To prove this, it is simpler to think only about the module $\mathbb{M}_\beta/\mathbb{M}_\alpha$ - note that $\mathbb{M}_\beta/\mathbb{M}_\alpha$ is a tame algebra, with

$(e\mathbb{A} \cap \mathbb{M}_\beta)/\mathbb{M}_\alpha = e\mathbb{M}_\beta/\mathbb{M}_\alpha$ as a $(0,1)$-minimal set, for any $e \in \mathbb{R}$ such that $e^2 = e$ and $e\mathbb{A}$ is an (α, β)-minimal set.

Proposition B.1.28. *Suppose that $\mathbb{M}$ is a finite module over a finite ring $\mathbb{R}$ which acts faithfully on $\mathbb{M}$, and suppose that $\mathbb{M}$ is a tame algebra. Then $\mathbb{R}$ is isomorphic to a matrix ring $M_n(\mathbb{F}_{p^k})$ over a finite field $\mathbb{F}_{p^k}$ for some n and some prime power p^k, and $\mathbb{M}$ is $\mathbb{F}_{p^k}^{n\times m}$ for some m.*

Proof. By Corollary B.1.17, every $(0_\mathbb{M}, 1_\mathbb{M})$-trace N has $\mathbb{M}|_N$ a permutational algebra. If $N = e\mathbb{M}$ for some $e \in E(\mathbb{A})$, the restriction $\mathrm{Pol}(\mathbb{M})|_N$ consists of the linear functions with coefficients in the ring $e\mathbb{R}e$, so we see that every nonzero element of the ring $e\mathbb{R}e$ is invertible, that is, $e\mathbb{R}e$ is a division ring. Since every finite division ring is a field by Wedderburn's little theorem, we have $e\mathbb{R}e \cong \mathbb{F}_{p^k}$ for some finite field $\mathbb{F}_{p^k}$ (note, the rest of the argument still works over a division ring rather than a field).

We claim that we can pick $e_1, ..., e_n \in \mathbb{R}$ idempotent such that $\sum_i e_i = 1$, $e_ie_j = e_je_i = 0$ for $i \neq j$, and each $e_i\mathbb{M}$ is $(0_\mathbb{M}, 1_\mathbb{M})$-minimal. Suppose that $e_1, ..., e_n$ is a maximal collection of idempotents such that $e_ie_j = e_je_i = 0$ for $i \neq j$ and each $e_i\mathbb{M}$ is $(0_\mathbb{M}, 1_\mathbb{M})$-minimal, and let $f = 1 - \sum_i e_i$ (that such a maximal set exists and is finite follows from the fact that $e_i\mathbb{M} \cap e_j\mathbb{M} = \emptyset$ for $i \neq j$, since $e_ie_j = 0$). Then we have $f^2 = f$, and if $f \neq 0$ then there must be some $(0_\mathbb{M}, 1_\mathbb{M})$-minimal set $U \subseteq f\mathbb{M}$. If $e' \in E(\mathbb{M})$ has $e'\mathbb{M} = U$, then we set $e_{n+1} = e'f$, and note that we have $e_{n+1}\mathbb{M} = e'\mathbb{M} = U$, and $e_{n+1}e_i = e_ie_{n+1} = 0$ for each $i \leq n$, contradicting the maximalilty of the collection $e_1, ..., e_n$. Therefore we must have $f = 0$, that is, $\sum_i e_i = 1$. Note that every element $x \in \mathbb{M}$ has a unique decomposition

$$x = \sum_i e_ix_i,$$

so as a group, $\mathbb{M}$ is the direct sum of the $e_i\mathbb{M}$s.

By Theorem B.1.14(a), for each pair $i, j \leq n$ we have $e_i\mathbb{M} \simeq e_j\mathbb{M}$. Pick $f_i : e_1\mathbb{M} \simeq e_i\mathbb{M}$ for each i, and pick inverses $g_i : e_i\mathbb{M} \simeq e_1\mathbb{M}$ to each f_i (with $f_1 = g_1 = e_1$ for $i = 1$). Then for any i, j, define the matrix element e_{ij} by

$$e_{ij} = f_ig_je_j,$$

and note that each e_{ij} is an isomorphism $e_{ij} : e_j\mathbb{M} \simeq e_i\mathbb{M}$, with $e_ie_{ij} = e_{ij}$ and $e_{ij}e_j = e_{ij}$, with $e_{ij}e_{ji} = e_{ii} = e_i$, with $e_{ij}e_{jk} = e_{ik}$, and $e_{ij}e_{kl} = 0$ for $j \neq k$. For each $r_1 \in e_1\mathbb{R}e_1$ we can additionally define the corresponding scalar $r \in \mathbb{R}$ by

$$r_1 \in e_1\mathbb{R}e_1 \mapsto r = \sum_i e_{i1}r_1e_{1i},$$

and we identify the set of such scalars r with $\mathbb{F}_{p^k}$, noting that $re_{ij} = e_{ij}r$ for all $r \in \mathbb{F}_{p^k}$ and $i, j \leq n$, and that the multiplication in $\mathbb{F}_{p^k}$ is the same as the multiplication in $e_1\mathbb{R}e_1$. We claim that every element $m \in \mathbb{R}$ can be written uniquely in the form

$$m = \sum_{i,j} r_{i,j}e_{ij}$$

for some $r_{i,j} \in \mathbb{F}_{p^k}$. To prove this, note that since $\sum_i e_i = 1$, we have

$$m = \sum_{i,j} e_ime_j,$$

and each e_ime_j defines a map $e_j\mathbb{M} \to e_i\mathbb{M}$. If we define the element $r_{i,j} \in \mathbb{F}_{p^k}$ by

$$r_{i,j} \coloneqq \sum_k e_{ki}me_{jk},$$

then we have

$$r_{i,j}e_{ij} = \sum_k e_{ki}me_{jk}e_{ij} = e_{ii}me_{ji}e_{ij} = e_ime_j,$$

so $m = \sum_{i,j} r_{i,j}e_{ij}$. For the uniqueness, note that $\sum_{i,j} r_{i,j}e_{ij} = 0$ implies that each $r_{i,j}e_{ij} = 0$, and if $r_{i,j} \neq 0$ then $r_{i,j}$ is invertible, since $\mathbb{F}_{p^k}$ is a field. Thus we have an explicit isomorphism $M_n(\mathbb{F}_{p^k}) \cong \mathbb{R}$. Finally, $e_1\mathbb{M}$ is a vector space of some dimension m over $\mathbb{F}_{p^k}$, and since $\mathbb{M}$ is the direct sum of n copies of $e_1\mathbb{M}$ as a vector space over $\mathbb{F}_{p^k}$ we have $\mathbb{M} \cong \mathbb{F}_{p^k}^{n\times m}$, with the action of $\mathbb{R}$ on $\mathbb{M}$ corresponding to matrix multiplication. □

For the sake of completeness, we include Witt's proof of Wedderburn's little theorem here.

Theorem B.1.29 (Wedderburn's little theorem)**.** *If $\mathbb{R}$ is a finite division ring, then $\mathbb{R}$ is a field.*

Proof. (Following Witt [187]) Let $\mathbb{R}^\times$ be the group of nonzero elements of $\mathbb{R}$, and let $Z(\mathbb{R}^\times)$ be the center of the group $\mathbb{R}^\times$. Then $\mathbb{F} = Z(\mathbb{R}) = Z(\mathbb{R}^\times) \cup \{0\}$ is a finite field, of some prime power order $q = p^k$. Since $\mathbb{R}$ is an $\mathbb{F}$-algebra, $\mathbb{R}$ is in particular a vector space over $\mathbb{F}$ of some dimension n, so $|\mathbb{R}| = q^n$ and $|\mathbb{R}^\times| = q^n - 1$.

We consider the conjugation action of $\mathbb{R}^\times$ on itself: if $x \in \mathbb{R} \setminus \mathbb{F}$, then the centralizer $C_{\mathbb{R}}(x) = \{r \in \mathbb{R} \mid rx = xr\}$ is a proper $\mathbb{F}$-subalgebra of $\mathbb{R}$, so $|C_{\mathbb{R}}(x)| = q^k$ for some $k < n$, and since $\mathbb{R}$ can be thought of as a module over the division ring $C_{\mathbb{R}}(x)$, we have $k \mid n$. Then the conjugacy class of x in $\mathbb{R}^\times$ has size $\frac{q^n-1}{q^k-1}$, so we have

$$q^n - 1 = |\mathbb{R}^\times| = |Z(\mathbb{R}^\times)| + \sum_{\text{conj. classes of } \mathbb{R}\backslash\mathbb{F}} \frac{q^n-1}{q^{k_i}-1} = q - 1 + \sum_{\text{conj. classes of } \mathbb{R}\backslash\mathbb{F}} \frac{q^n-1}{q^{k_i}-1}.$$

If we let $\Phi_n(x)$ be the nth cyclotomic polynomial, then we have $\Phi_n(q) \mid \frac{q^n-1}{q^{k_i}-1}$ for each conjugacy class, so $q-1$ must be a multiple of $\Phi_n(q)$. However, $|\Phi_n(q)|$ is the product of $|q-\zeta|$ over various nth roots of unity ζ, so $|\Phi_n(q)| > q-1$ for $n > 1$, a contradiction. □

B.1.1 Tight lattices produce tame quotients

The purpose of this subsection is to give a purely lattice-theoretic criterion which we can use to prove that certain congruence quotients are tame. As we will see later, nontrivial occurences of this sort of sublattice imply the existence of abelian congruence quotients.

Definition B.1.30. Suppose $\mathcal{L}$ is a lattice with a 0 and a 1. A lattice homomorphism $f : \mathcal{L} \to \mathcal{L}'$ is $0,1$-*separating* if we have

$$f^{-1}(f(0)) = \{0\}, \quad f^{-1}(f(1)) = \{1\}.$$

A lattice $\mathcal{L}$ is $0,1$-*simple* if it has a 0 and a 1 which are not equal to each other, and if every nonconstant lattice homomorphism $\mathcal{L} \to \mathcal{L}'$ is $0,1$-separating.

A *meet endomorphism* of a lattice $\mathcal{L}$ is a function $\mu : \mathcal{L} \to \mathcal{L}$ which preserves $\wedge$, i.e. such that

$$\mu(x \wedge y) = \mu(x) \wedge \mu(y).$$

A function $\mu : \mathcal{L} \to \mathcal{L}$ is called *increasing* if

$$\mu(x) \geq x$$

for all $x \in \mathcal{L}$, and is called *strictly increasing* if

$$\mu(x) > x$$

for all $x \in \mathcal{L} \setminus \{1\}$.

A lattice $\mathcal{L}$ is called *tight* if $\mathcal{L}$ is $0,1$-simple and every strictly increasing meet endomorphism of $\mathcal{L}$ is constant.

Theorem B.1.31. *If $\mathbb{A}$ is a finite algebra and if the interval $[\![\alpha, \beta]\!]$ of $\mathrm{Con}(\mathbb{A})$ is tight, then the congruence quotient (α, β) is tame: for every $U \in M_{\mathbb{A}}(\alpha, \beta)$, there is an idempotent unary polynomial $e \in E(\mathbb{A})$ such that $e(\mathbb{A}) = U$.*

Proof. (Following [95]) Since $\mathcal{L}$ is $0,1$-simple, the restriction homomorphism will automatically be $0,1$-separating once we show that such an e exists. It's enough to show that there is some $f \in \mathrm{Pol}_1(\mathbb{A})$ such that $f(\mathbb{A}) = U$ and $f(U) = U$, since then we can iterate f to produce e. To find such an f, we just need to find a pair $f, g \in \mathrm{Pol}_1(\mathbb{A})$ such that $f(\mathbb{A}), g(\mathbb{A}) \subseteq U$ and $f(g(\beta)) \not\subseteq \alpha$.

Let K be the set of unary polynomials $f \in \mathrm{Pol}_1(\mathbb{A})$ such that $f(\mathbb{A}) \subseteq U$. One way to check whether there is some $f \in K$ with $f(\beta) \not\subseteq \alpha$ is to try to find the largest congruence μ below β such that $f(\mu) \subseteq \alpha$ for all $f \in K$, and then to check if $\mu = \beta$. This leads to defining the following mapping on congruences:

$$\mu(\theta) \coloneqq \{(x, y) \in \beta \mid \forall f \in K,\ (f(x), f(y)) \in \theta\}.$$

It's easy to see that $\mu(\theta)$ is automatically a congruence, that $\theta \leq \mu(\theta)$, and that

$$\mu(\theta_1 \wedge \theta_2) = \mu(\theta_1) \wedge \mu(\theta_2).$$

Thus μ is an increasing meet endomorphism of $[\![\alpha, \beta]\!]$.

Since $U \in M_{\mathbb{A}}(\alpha, \beta)$, there must be some $f \in K$ such that $f(\beta) \not\subseteq \alpha$, so

$$\mu(\alpha) < \beta.$$

Thus μ is not constant. By the assumption that $[\![\alpha, \beta]\!]$ is tight, μ must not be *strictly* increasing, so there must be some $\theta < \beta$ such that

$$\mu(\theta) = \theta.$$

Thus we have

$$\mu(\mu(\alpha)) \leq \mu(\mu(\theta)) = \theta < \beta.$$

The point is that $\mu \circ \mu$ is what we would get if we replaced K by K^2 in the definition of μ, that is,

$$\mu(\mu(\alpha)) = \{(x, y) \in \beta \mid \forall f, g \in K,\ (f(g(x)), f(g(y))) \in \alpha\},$$

so from $\mu(\mu(\alpha)) \neq \beta$ we conclude that there must be some $f, g \in K$ such that $f(g(\beta)) \not\subseteq \alpha$, and we are done. □

At first it may seem that the proof only needs us to require that $[\![\alpha, \beta]\!]$ has no nonconstant increasing meet endomorphisms μ such that $\mu \circ \mu$ is constant. However, this is actually equivalent to having no nonconstant strictly increasing meet endomorphisms: if μ is a nonconstant strictly increasing meet endomorphism, then there is some minimal $k > 1$ such that $\mu^{\circ k}(\alpha) = \beta$, and then $\mu^{\circ(k-1)}$ will be nonconstant but $\mu^{\circ(k-1)} \circ \mu^{\circ(k-1)}$ will be constant.

In the remainder of this subsection, we will give alternative lattice-theoretic characterizations of what it means for a finite lattice to tight. We start by examining what it means for a lattice to be $0,1$-simple.

Proposition B.1.32. *A lattice $\mathcal{L}$ is $0,1$-simple iff there is a unique dual atom $\theta \prec 1_{\mathcal{L}} \in \mathrm{Con}(\mathcal{L})$ and the associated map $\mathcal{L} \twoheadrightarrow \mathcal{L}/\theta$ is $0,1$-separating.*

Proof. Call a congruence η on $\mathcal{L}$ $0,1$-separating if the quotient map $\mathcal{L} \twoheadrightarrow \mathcal{L}/\eta$ is $0,1$-separating, that is, if $0/\eta = \{0\}$ and $1/\eta = \{1\}$. Then any join of $0,1$-separating congruences is $0,1$-separating, so there is always a unique maximal $0,1$-separating congruence $\theta \in \mathrm{Con}(\mathcal{L})$. Then $\mathcal{L}$ is $0,1$-simple iff all congruences $\eta < 1_{\mathcal{L}}$ satisfy $\eta \le \theta$. □

Proposition B.1.33. *If a complete lattice $\mathcal{L}$ is $0,1$-simple and θ is a proper congruence on $\mathcal{L}$, then $\mathcal{L}$ is tight iff $\mathcal{L}/\theta$ is tight.*

Proof. Let $f : \mathcal{L}/\theta \to \mathcal{L}$ be the meet homomorphism given by

$$f(a/\theta) = \bigvee_{b \in a/\theta} b.$$

Then f is a section of the quotient map $\pi : \mathcal{L} \to \mathcal{L}/\theta$, i.e. $\pi \circ f$ is the identity on $\mathcal{L}/\theta$, and furthermore $f \circ \pi$ is an increasing meet endomorphism of $\mathcal{L}$ which maps 0 to 0 and 1 to 1. Then for any nonconstant strictly increasing meet endomorphism μ of $\mathcal{L}$, the map

$$\pi \circ \mu \circ f$$

is a strictly increasing meet endomorphism of $\mathcal{L}/\theta$ which sends $0/\theta$ to $\mu(0)/\theta \ne 1/\theta$, and similarly for any nonconstant strictly increasing meet endomorphism μ' of $\mathcal{L}/\theta$, the map

$$f \circ \mu' \circ \pi$$

is a strictly increasing meet endomorphism of $\mathcal{L}$ which sends 0 to $f(\mu'(0/\theta)) \ne 1$. □

From this we see that we only need to characterize *simple* tight lattices. Recall that a *tolerance* on an algebraic structure is a compatible binary relation which is symmetric and which contains the diagonal, and that a tolerance is called *connected* if its transitive closure is the full congruence.

Proposition B.1.34. *A simple lattice $\mathcal{L}$ of finite length is tight iff it has no nontrivial tolerances. Equivalently, a $0,1$-simple lattice of finite length is tight iff it has no proper connected tolerances.*

Proof. If $\mathbb{S} \le_{sd} \mathcal{L} \times \mathcal{L}$ is a tolerance, then we can define a corresponding increasing meet endomorphism $\mu_{\mathbb{S}}$ by

$$\mu_{\mathbb{S}}(a) = \bigvee_{(a,b) \in \mathbb{S}} b.$$

The tolerance $\mathbb{S}$ is connected iff $\mu_{\mathbb{S}}$ is strictly increasing: the largest element 0 can be connected to via $\mathbb{S}$ in k steps is $\mu_{\mathbb{S}}^{\circ k}(0)$.

Conversely, if μ is an increasing meet endomorphism, then we can define a corresponding tolerance $\mathbb{S}_\mu$ by

$$(a, b) \in \mathbb{S}_\mu \iff (a \le \mu(b)) \wedge (b \le \mu(a)).$$

In fact, the constructions $\mathbb{S} \mapsto \mu_{\mathbb{S}}$ and $\mu \mapsto \mathbb{S}_\mu$ invert each other. □

We say that a lattice is *order polynomially complete* if every monotone operation $\mathcal{L}^n \to \mathcal{L}$ is a polynomial of $\mathcal{L}$.

Proposition B.1.35. *If $\mathcal{L}$ is a finite lattice, then the following are equivalent:*

(a) $\mathcal{L}$ is simple and tight,

(b) for any $a < b \in \mathcal{L}$, there is a unary polynomial f of $\mathcal{L}$ such that $f(a) = 0$ and $f(b) = 1$,

(c) the only compatible binary relations on $\mathcal{L}$ which contain the diagonal are the diagonal $\Delta_{\mathcal{L}}$, the partial orders $\le_{\mathcal{L}}$ and $\ge_{\mathcal{L}}$, and the full relation $\mathcal{L}^2$,

(d) $\mathcal{L}$ is order polynomially complete.

Proof. For (a) $\implies$ (b), note that if $\mathcal{L}$ has no nontrivial tolerances, then the tolerance generated by the diagonal and $\{(a, b), (b, a)\}$ must contain $(0, 1)$, so there is some binary polynomial g such that

$$g\Big(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix}\Big) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since g is monotone, we must also have

$$g\Big(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a \\ a \end{bmatrix}\Big) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

so we can take $f(x) = g(x, a)$.

For (b) $\implies$ (c), we just have to prove that the binary relation $\mathbb{R}$ generated by the diagonal and $\{(a, b)\}$ contains $\le_{\mathcal{L}}$ as long as $a \not\ge b$. Note that $a \not\ge b$ implies $a < a \vee b$, so by (b) there is some unary polynomial f such that $f(a) = 0$ and $f(a \vee b) = 1$. Since

$$(a, a \vee b) = (a, a) \vee (a, b) \in \mathbb{R},$$

we see that $(0, 1) \in \mathbb{R}$. But then for any $c \le d$, we have

$$(c, d) = ((c, c) \vee (0, 1)) \wedge (d, d) \in \mathbb{R},$$

so $\le_{\mathcal{L}}$ is contained in $\mathbb{R}$.

For (c) $\implies$ (d), note that the collection of n-ary polynomials of $\mathcal{L}$ is equal to the sublattice $\mathbb{R} \le \mathcal{L}^{\mathcal{L}^n}$ which is generated by the constant tuples and the projections $\pi_i : x \mapsto x_i$. Since every lattice has a majority term, we see that $\mathbb{R}$ is equal to the intersection of its binary projections, each of which contains the diagonal. Applying (c), we see that $f \in \mathbb{R}$ if and only if for every $x, y \in \mathcal{L}^n$ such that

$$\pi_i(x) = x_i \le \pi_i(y) = y_i$$

for all i, we have $f(x) \le f(y)$.

For (d) $\implies$ (b), we check that the map $f : \mathcal{L} \to \mathcal{L}$ given by

$$f : x \mapsto \begin{cases} 0 & x \le a, \\ 1 & x \not\le a \end{cases}$$

is monotone. Finally, (c) $\implies$ (a) follows from the previous proposition. □

Proposition B.1.36. *If $\mathcal{L}$ is a lattice of finite length, then the smallest connected tolerance on $\mathcal{L}$ is generated by the diagonal and the pairs (x, y) such that either y covers x or x covers y.*

Proof. Let $\mathbb{S}$ be any connected tolerance of $\mathcal{L}$. Suppose that y covers x, and consider an increasing path $x = x_0 < x_1 < \cdots < x_n = 1$ from x to 1 through $\mathbb{S}$. Then there must be some first i such that $x_i \ge y$, and we see that

$$(x, y) = (y, y) \wedge (x_{i-1}, x_i) \in \mathbb{S}.$$ □

Proposition B.1.37. *If the join of the atoms of a $0, 1$-simple lattice $\mathcal{L}$ of finite length is equal to 1, or if the meet of the co-atoms is equal to 0, then $\mathcal{L}$ is tight.*

Proof. We will check that in either case $\mathcal{L}$ has no proper connected tolerances. Suppose that $\mathbb{S}$ is a connected tolerance, and let a be any atom of $\mathcal{L}$. In order for a to be connected to 0 via $\mathbb{S}$ in any number of steps, a must be connected to something strictly less than a via $\mathbb{S}$ in one step, so we must have

$$(0, a) \in \mathbb{S}.$$

Since this is true for all atoms of $\mathcal{L}$, joining them together we see that $(0, 1) \in \mathbb{S}$ if the join of the atoms is 1. □

Proposition B.1.38. *The lattice $\mathcal{L}_{\mathbb{M}}$ of subspaces of a finite-dimensional vector space $\mathbb{M}$ over a field $\mathbb{F}$ is always tight.*

Proof. By the previous result, we just need to check that $\mathcal{L}_{\mathbb{M}}$ is in fact simple. If θ is a nontrivial congruence on $\mathcal{L}_{\mathbb{M}}$ which identifies subspaces $u \ne v$, then by taking meets with a one-dimensional subspace which is contained in one of u, v but not the other, we see that 0 is congruent to some atom $a = \mathrm{Sg}_{\mathbb{M}}\{x\}$ of $\mathcal{L}_{\mathbb{M}}$.

Now let $b = \mathrm{Sg}_{\mathbb{M}}\{y\}$ be any other atom, and note that $c = \mathrm{Sg}_{\mathbb{M}}\{x + y\}$ is necessarily different from both a and b. Then $0, a, b, c$, and $a \vee b = \mathrm{Sg}_{\mathbb{M}}\{x, y\}$ form a sublattice of $\mathcal{L}_{\mathbb{M}}$ isomorphic to the diamond lattice $\mathcal{M}_3$. Since $\mathcal{M}_3$ is simple, we see that $(0, a) \in \theta$ implies $(0, b) \in \theta$ - so in fact, any nontrivial congruence θ on $\mathcal{L}_{\mathbb{M}}$ must contain every atom in $0/\theta$. Since the join of the atoms is the whole space, we have $(0, 1) \in \theta$, so θ was not a proper congruence on $\mathcal{L}_{\mathbb{M}}$. □

Proposition B.1.39. *If A is a finite set, then the lattice $\mathcal{L}_A$ of equivalence relations on A is tight.*

Proof. The proof is very similar to the previous proof - this time we use the fact that for any distinct $x, y, z \in A$, the equivalence relations $0_A, \mathrm{Cg}_A\{(x, y)\}, \mathrm{Cg}_A\{(y, z)\}, \mathrm{Cg}_A\{(x, z)\}, \mathrm{Cg}_A\{(x, y), (y, z)\}$ form a sublattice of $\mathcal{L}_A$ which is isomorphic to $\mathcal{M}_3$. □

We say that a $0,1$-lattice $\mathcal{L}$ is *complemented* if for all $x \in \mathcal{L}$ there is some $x' \in \mathcal{L}$ such that

$$x \vee x' = 1, \quad x \wedge x' = 0.$$

Such an x' is called a *complement* of x (and in general there may be more than one complement). Both types of lattices just considered (the subspaces of a finite dimensional vector space and the equivalence relations on a finite set) are complemented.

Proposition B.1.40. *If a lattice $\mathcal{L}$ of finite length is complemented, then the join of the atoms of $\mathcal{L}$ is 1 and the meet of the co-atoms is 0.*

Proof. Let x be the join of the atoms of $\mathcal{L}$, and suppose that x' is a complement of x. Then since $x \wedge x' = 0$, x' is not greater than any atom of $\mathcal{L}$, so $x' = 0$. Thus we have $1 = x \vee x' = x$. □

Proposition B.1.41. *If $\mathcal{L} \twoheadrightarrow \mathcal{L}'$ is $0,1$-separating, then $\mathcal{L}$ is complemented iff $\mathcal{L}'$ is complemented, and the atoms of $\mathcal{L}$ join to 1 iff the atoms of $\mathcal{L}'$ join to 1.*

The theory of tight lattices simplifies dramatically when we restrict our attention to modular lattices.

Proposition B.1.42. *If $\mathcal{L}$ is a modular lattice of finite length, then the map μ given by*

$$\mu : x \mapsto x \vee \bigvee \{y \mid x \prec y\},$$

which takes x to the join of the collection of covers of x, is a strictly increasing meet endomorphism.

Proof. First we check that μ is monotone, i.e. that $x \le z$ implies $\mu(x) \le \mu(z)$. If $x \le z$ and $x \prec y$, then modularity of $\mathcal{L}$ implies that either $y \le z$ or $y \vee z$ is a cover of z. Thus we have

$$y \le y \vee z \le \mu(z)$$

for all $x \prec y$, so $\mu(x) \le \mu(z)$.

Define a dual map σ by

$$\sigma : x \mapsto x \wedge \bigwedge \{y \mid y \prec x\}.$$

Note that σ is also monotone (by a dual argument to the above). Our strategy is to prove that

$$x \le \mu(y) \iff \sigma(x) \le y. \tag{$*$}$$

If we prove ($*$), then we will have

$$x \le \mu(a \wedge b) \iff \sigma(x) \le a \wedge b \iff x \le \mu(a) \wedge \mu(b),$$

which will prove that μ is a meet endomorphism.

By the monotonicity of σ, we just need to check that we have $\sigma(x) \le y$ when $x = \mu(y)$ in order to verify the forward direction of ($*$). Let $y_1, ..., y_k$ be a minimal collection of covers of y such that

$$x = y \vee y_1 \vee \cdots \vee y_k.$$

For each i, define x_i by

$$x_i = y \vee y_1 \vee \cdots \vee y_{i-1} \vee y_{i+1} \vee \cdots \vee y_k.$$

By modularity of $\mathcal{L}$ and the choice of k, we have $x_i \prec x$ for all i. It's now easy to prove by induction that for any $I \subset [k]$, we have

$$x \wedge \bigwedge_{i\in I} x_i = y \vee \bigvee_{j\in[k]\setminus I} y_j,$$

so

$$\sigma(x) \le x \wedge \bigwedge_{i\in[k]} x_i = y. \qquad \square$$

Proposition B.1.43. *If $\mathcal{L}$ is a modular lattice of finite length, then $\mathcal{L}$ is tight iff $\mathcal{L}$ is simple and complemented.*

Proof. We've already proven that if $\mathcal{L}$ is simple and complemented then $\mathcal{L}$ is tight, so suppose that $\mathcal{L}$ is tight. Let μ be the strictly increasing meet endomorphism from the previous result. Since $\mathcal{L}$ is tight, we must have $\mu(0) = 1$, so 1 must be a join of atoms.

By the Jordan-Hölder Theorem A.0.10 and the fact that 1 is a join of atoms, we see that any cover $x \prec y$ of $\mathcal{L}$ is projective to $0 \prec a$ for some atom a. Thus any nontrivial congruence θ of $\mathcal{L}$ which includes (x, y) also includes $(0, a)$, so in order for $\mathcal{L}$ to be $0,1$-simple $\mathcal{L}$ must actually be simple.

To finish, we just need to check that $\mathcal{L}$ is complemented. Letting x be any element of $\mathcal{L}$, pick a minimal set of atoms $a_1, ..., a_k$ such that

$$x \vee a_1 \vee \cdots \vee a_k = 1,$$

and let $x' = a_1 \vee \cdots \vee a_k$. We claim that $x \wedge x' = 0$. Suppose for contradiction that there is some atom a' with

$$a' \le x \wedge x'.$$

Then since

$$a' \vee a_1 \vee \cdots \vee a_k = x'$$

is not a cover of x', modularity of $\mathcal{L}$ implies that there must be some i such that

$$a' \vee a_1 \vee \cdots \vee a_{i-1} = a_1 \vee \cdots \vee a_i.$$

But then we can leave a_i out of the list of atoms and we still have

$$x \vee a_1 \vee \cdots \vee a_{i-1} \vee a_{i+1} \vee \cdots \vee a_k = 1,$$

contradicting the choice of k. $\square$

B.2 Pálfy's classification of finite permutational algebras: the five types

In the last section we proved that if (α, β) is a tame congruence quotient of a finite algebra $\mathbb{A}$, then for every (α, β)-trace N the restriction $\mathbb{A}|_N/\alpha|_N$ is permutational, i.e. every unary polynomial of $\mathbb{A}|_N$ is either a constant (modulo $\alpha|_N$) or a permutation. In [150], Pálfy gave a complete classification of the finite permutational algebras (up to polynomial equivalence), which was one of the key ingredients needed for tame congruence theory.

The classification splits into two very different cases: algebras of size 2, and algebras of size ≥ 3. Since every unary operation on a set of size 2 is either constant or is a permutation, the classification of permutational algebras on a set of size 2 is the same as the classification of *all* algebras on a set of size 2, up to polynomial equivalence. There turn out to be exactly 7 of these. On the other hand, the number of polynomial clones on any set of size ≥ 3 is uncountable [194] - but as we will see, the permutational algebras on a set of size ≥ 3 are all either unary or affine algebras, so they end up being much simpler than general algebras.

We start by giving some definitions in order to rule out the least interesting case - the case of unary operations only.

Definition B.2.1. An operation f of arity n *depends on* its ith input if there is some tuple $a_1, ..., a_n$ and some b_i such that

$$f(a_1, ..., a_n) \neq f(a_1, ..., a_{i-1}, b_i, a_{i+1}, ..., a_n).$$

An operation f is *essentially unary* if it only depends on one of its inputs - equivalently, f is essentially unary if it can be written as the composition of a projection π_i and a unary operation.

If f does not depend on its ith input, then we can express f in terms of the function we get by replacing its ith input by some other input, such as its first input. So there is no need to ever think too deeply about functions which do not depend on all their inputs. In order to gain a foothold, it is helpful to start by considering the case of a binary operation which depends on all of its inputs - for this, we will replace one of the inputs of a higher arity polynomial with some constant to get a lower arity polynomial which also depends on all its inputs.

The next result is much stronger than what we will need: all we really need is the fact that if f depends on at least two of its inputs, then there is a way to plug in constants for some subset of the inputs to f to get a polynomial in two variables that depends on both of its inputs.

Proposition B.2.2 (Salomaa [168])**.** *If a polynomial f of arity n depends on all of its inputs, then it is possible to substitute a constant for one of its inputs to get a polynomial of arity $n-1$ which also depends on all of its inputs.*

In fact, if $n \geq 2$, then it is possible to find at least two different inputs to f where constants can be substituted to get polynomials depending on all $n-1$ of their inputs.

Proof. Following [95], we write $f[a, i]$ for the polynomial we get by substituting a for the ith input of f. Suppose that for some a and i, j, k, $f[a, i]$ does not depend on the jth input but does depend on the kth input. Then for *every* b we see that $f[b, j]$ depends on the kth input, by considering the case where we plug in a in the ith input and b in the jth input. Additionally, since f depends on all its inputs there must be some a' such that $f[a', i]$ depends on its jth input, so there must be some b such that $f[b, j]$ depends on the ith coordinate (consider plugging in a tuple with an a' in the ith input such that varying the jth input changes the value, and note that if we change a' to a in the ith position then varying the jth input no longer changes the value of f). Thus if $f[a, i]$ does not depend on its jth input, then there is some b such that $f[b, j]$ depends on a strictly larger subset of its inputs than $f[a, i]$ does, which proves the first claim.

For the second claim, note that for each $i \leq n$ and each $j \neq i$, there is some a such that $f[a, j]$ depends on the ith input, as long as f depends on its ith input. Then if we choose a pair a, j such that $j \neq i$, $f[a, j]$ depends on the ith input, and $f[a, j]$ depends on as many inputs as possible subject to the previous constraints, then the argument of the previous paragraph shows that $f[a, j]$ must depend on all of its inputs. □

Lemma B.2.3. *Suppose that $f \in \mathrm{Pol}_2(\mathbb{A})$ is a binary polynomial of a finite permutational algebra $\mathbb{A}$ which depends on both of its inputs, and suppose that $|\mathbb{A}| \geq 3$. Then f is a quasigroup operation, that is, every unary polynomial of the form $f(a, \cdot)$ or $f(\cdot, b)$ is a permutation.*

Proof. Suppose for the sake of contradiction that f depends on both of its inputs, but that there is some a such that $f(a, \cdot)$ is constant, with $f(a, y) = e$ for all $y \in \mathbb{A}$. Since f depends on its second coordinate, there must be some $a' \neq a$ such that $f(a', \cdot)$ is a permutation, which implies that there is some $b \in \mathbb{A}$ such that $f(a', b) = e$. Then since $f(a, b) = e = f(a', b)$, we must have $f(x, b) = e$ for all $x \in \mathbb{A}$ as well.

For any $a' \neq a$, if $f(a', \cdot)$ is constant, then since $f(a', b) = e$, we must have $f(a', y) = e$ for all $y \in \mathbb{A}$, and then for each y from $f(a', y) = e = f(a, y)$ we conclude that $f(x, y) = e$ for all x, so f is constant. This contradicts the assumption that f depends on its inputs, so for all $a' \neq a$ the unary polynomial $f(a', \cdot)$ must be a permutation.

So far we have not used the fact that $|\mathbb{A}| \geq 3$, and we have not fully exploited the fact that $\mathbb{A}$ is finite and permutational. For this, we iterate f on its second argument: define $f^1 = f$, and for each n define $f^{n+1}(x, y)$ by

$$f^{n+1}(x, y) = f(x, f^n(x, y)),$$

and take $f^\infty(x, y) = \lim_{n\to\infty} f^{n!}(x, y)$, so

$$f^\infty(x, y) = f^\infty(x, f^\infty(x, y)).$$

Then $f^\infty(a, \cdot)$ is constant, while for $a' \neq a$ we have $f^\infty(a', y) = y$ for all $y \in \mathbb{A}$ since each $f(a', \cdot)$ is a permutation. But then for any distinct $a', a'' \neq a$, we have $f^\infty(a', y) = y = f^\infty(a'', y)$, so $f^\infty(\cdot, y)$ must be constant for all y, and in particular $f^\infty(a, y) = y$ for all y, which contradicts the fact that $f^\infty(a, \cdot)$ is constant. □

Corollary B.2.4. *Suppose $\mathbb{A}$ is a finite permutational algebra with $|\mathbb{A}| \geq 3$, and suppose that some operation of $\mathbb{A}$ is not essentially unary. Then $\mathbb{A}$ has a Mal'cev polynomial $p(x, y, z)$.*

Proof. This follows from the previous lemma and Proposition 1.7.10. □

Corollary B.2.5. *Suppose $\mathbb{A}$ is a finite permutational algebra with $|\mathbb{A}| \geq 3$, and suppose $f \in \mathrm{Pol}_n(\mathbb{A})$ has $f(a_1, ..., a_n) = f(a_1, ..., a_{i-1}, b_i, a_{i+1}, ..., a_n)$ for some $a_1, ..., a_n$ and some $b_i \neq a_i$. Then f does not depend on its ith coordinate.*

Proof. We will show that for any $j \neq i$, any a_j', and any b_i', we have

$$f(a_1, ..., a_{j-1}, a_j', a_{j+1}, ..., a_n) = f(a_1, ..., a_{i-1}, b_i', a_{i+1}, ..., a_{j-1}, a_j', a_{j+1}, ..., a_n).$$

For this, we define a two-variable polynomial from f by substituting the kth input of f with a_k for all $k \neq i, j$, and apply the previous lemma to this two variable polynomial to see that it can't depend on its ith input. Applying this repeatedly, we can mutate the tuple $a_1, ..., a_n$ into any tuple $a_1', ..., a_n'$, so f does not depend on its ith coordinate. □

Lemma B.2.6. *Suppose $\mathbb{A}$ is a finite permutational algebra with $|\mathbb{A}| \geq 3$, $f, g \in \mathrm{Pol}_n(\mathbb{A})$, and suppose that for some $0 \in \mathbb{A}$ we have $f(x_1, ..., x_n) = g(x_1, ..., x_n)$ for all $x_1, ..., x_n \in \mathbb{A}$ such that all but one x_i is 0. Then $f = g$.*

Proof. If every operation of $\mathbb{A}$ is essentially unary, then this is obvious. Otherwise, let $p \in \mathrm{Pol}_3(\mathbb{A})$ be the Mal'cev operation from Corollary B.2.4. Then the polynomial

$$h(x_1, ..., x_n) = p(f(x_1, ..., x_n), g(x_1, ..., x_n), 0)$$

is 0 whenever $f(x_1, ..., x_n) = g(x_1, ..., x_n)$, and since any Mal'cev operation must depend on its second input, we can apply Corollary B.2.5 to $p(x, y, z)$ to see that $h(x_1, ..., x_n) = 0$ if and only if $f(x_1, ..., x_n) = g(x_1, ..., x_n)$. For any input i, since we have

$$h(0, ..., 0, x_i, 0, ..., 0) = 0$$

for all x_i, we can apply Corollary B.2.5 to see that h does not depend on any of its inputs, so h must be constantly 0, which implies that $f = g$. □

Theorem B.2.7 (Pálfy [150]). *Suppose $\mathbb{A}$ is a finite permutational algebra with $|\mathbb{A}| \geq 3$, and suppose that some operation of $\mathbb{A}$ is not essentially unary. Then $\mathbb{A}$ is affine, and in fact $\mathbb{A}$ is polynomially equivalent to a vector space over a finite field.*

Proof. Let $p(x, y, z) \in \mathrm{Pol}_3(\mathbb{A})$ be the Mal'cev operation from Corollary B.2.4, pick an element to call 0 in $\mathbb{A}$, and define a binary polynomial $+ \in \mathrm{Pol}_2(\mathbb{A})$ by

$$x + y = p(x, 0, y).$$

Then since p is Mal'cev, we have $0+x = x = x+0$ for all x, so by Lemma B.2.6 we have $x+y = y+x$ for all x, y. Similarly, from

$$0 + (0 + x) = x = (0 + 0) + x, \;\; 0 + (x + 0) = x = (0 + x) + 0, \;\; x + (0 + 0) = x = (x + 0) + 0,$$

we can apply Lemma B.2.6 to conclude that $x + (y + z) = (x + y) + z$ for all x, y, z. Since $x + 0 = x = 0 + x$ for all x, $+$ depends on both of its arguments, so by Lemma B.2.3, we see that every element $x \in \mathbb{A}$ has an inverse $-x$ such that $x + (-x) = 0$. Thus $+$ defines an abelian group structure on $\mathbb{A}$ with identity element 0.

For any $f \in \mathrm{Pol}_1(\mathbb{A})$, if $f(0) = c$, then we can define $r \in \mathrm{Pol}_1(\mathbb{A})$ by $r(x) = f(x) - c$, so that $r(0) = 0$. We will show that any such r distributes over addition: since

$$r(x + 0) = r(x) = r(x) + r(0), \;\; r(0 + y) = r(y) = r(0) + r(y),$$

we can apply Lemma B.2.6 to conclude that $r(x + y) = r(x) + r(y)$ for all x, y. Thus every unary polynomial $f \in \mathrm{Pol}_1(\mathbb{A})$ can be written in the form $f(x) = r(x) + c$, where r distributes over addition and takes 0 to 0. Letting $\mathbb{F}$ be the ring of $r \in \mathrm{Pol}_1(\mathbb{A})$ such that $r(0) = 0$, we see that every nonzero element of $\mathbb{F}$ is invertible, so $\mathbb{F}$ is a finite division ring, and therefore $\mathbb{F}$ is a finite field by Wedderburn's little theorem B.1.29.

Now suppose $f \in \mathrm{Pol}_n(\mathbb{A})$ is any n-ary polynomial. Then if we define unary polynomials r_i by

$$r_i(x_i) = f(0, ..., 0, x_i, 0, ..., 0) - f(0, ..., 0),$$

then each r_i has $r_i(0) = 0$, so $r_i \in \mathbb{F}$ for all i. If we define $g \in \mathrm{Pol}_n(\mathbb{A})$ by

$$g(x_1, ..., x_n) = r_1(x_1) + \cdots + r_n(x_n) + f(0, ..., 0),$$

then we can apply Lemma B.2.6 to see that $f = g$, so every operation of $\mathbb{A}$ is linear over the finite field $\mathbb{F}$. □

To complete the classification, we just need to classify the polynomial clones on the two-element set $\{0,1\}$. We use $\neg$ to denote the unary negation operation on $\{0,1\}$, $\oplus$ to denote xor, and of course $\wedge, \vee$ to denote and and or.

Proposition B.2.8. *If $\mathbb{A}$ has underlying set $\{0,1\}$, then $\mathbb{A}$ is polynomially complete iff $\neg, \wedge \in \mathrm{Pol}(\mathbb{A})$. Additionally, we have $\oplus \in \mathrm{Pol}(\mathbb{A}) \implies \neg \in \mathrm{Pol}(\mathbb{A})$.*

Lemma B.2.9. *If $\mathbb{A}$ has underlying set $\{0,1\}$ and if there is some $f \in \mathrm{Pol}(\mathbb{A})$ which is not monotone, then the unary negation $\neg$ is a polynomial of $\mathbb{A}$.*

Proof. If f is not monotone, then there is some tuple $a_1, ..., a_n \in \{0,1\}$ and some i such that

$$f(a_1, ..., a_{i-1}, 0, a_{i+1}, ..., a_n) > f(a_1, ..., a_{i-1}, 1, a_{i+1}, ..., a_n).$$

Then the left hand side of the displayed inequality must be 1 and the right hand side must be 0, so we have

$$\neg(x) = f(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_n). \qquad \square$$

Lemma B.2.10. *If $\mathbb{A}$ has underlying set $\{0,1\}$ and $\oplus \in \mathrm{Pol}(\mathbb{A})$, and if there is any $f \in \mathrm{Pol}(\mathbb{A})$ which is not affine-linear over $\mathbb{Z}/2$, then $\wedge \in \mathrm{Pol}(\mathbb{A})$, so $\mathbb{A}$ is polynomially complete.*

Proof. By xoring with an affine-linear function over $\mathbb{Z}/2$, we may assume without loss of generality that $f(x_1, ..., x_n) = 0$ whenever at most one x_i is nonzero. Since f is not identically 0, there must be some $a_1, ..., a_n \in \{0,1\}$ with $f(x_1, ..., x_n) = 1$, and we may suppose that $\sum_i a_i$ is minimal. By our assumption on f, the number of nonzero a_i must be at least 2, so there is some pair of coordinates $i \neq j$ such that $a_i = a_j = 1$. If we decrease either a_i or a_j, then by the minimality assumption f becomes 0, so we have

$$x \wedge y = f(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_{j-1}, y, a_{j+1}, ..., a_n). \qquad \square$$

Proposition B.2.11. *The polynomial clone of $(\{0,1\}, \wedge, \vee)$ is exactly the clone of all monotone functions.*

Proof. We prove that every monotone function f of arity n is in the clone generated by $\wedge, \vee$ by induction on n:

$$f(x_1, ..., x_n) = f(x_1, ..., x_{n-1}, 0) \vee (f(x_1, ..., x_{n-1}, 1) \wedge x_n). \qquad \square$$

Lemma B.2.12. *If $\mathbb{A}$ has underlying set $\{0,1\}$ and $\vee \in \mathrm{Pol}(\mathbb{A})$, and if $f \in \mathrm{Pol}(\mathbb{A})$ is monotone but is not contained in the clone generated by $\vee$, then $\wedge \in \mathrm{Pol}(\mathbb{A})$, so $\mathrm{Pol}(\mathbb{A})$ contains the clone of all monotone functions.*

Proof. We may suppose without loss of generailty that f depends on all of its inputs. If f is not contained in the clone generated by $\vee$, then in particular $f(x_1, ..., x_n) \neq x_1 \vee \cdots \vee x_n$, so since f is monotone there must be some input i such that $f(0, ..., 0, 1, 0, ..., 0) = 0$. Since f is monotone and depends on its ith input, there must be some $a_1, ..., a_n \in \{0,1\}$ such that

$$f(a_1, ..., a_{i-1}, 0, a_{i+1}, ..., a_n) = 0, \quad f(a_1, ..., a_{i-1}, 1, a_{i+1}, ..., a_n) = 1.$$

Choose the $a_1, ..., a_n$ such that $\sum_j a_j$ is minimized subject to the displayed equations above. Then there is at least one j such that $a_j = 1$, by the choice of i, and for this j we have

$$x \wedge y = f(a_1, ..., a_{i-1}, x, a_{i+1}, ..., a_{j-1}, y, a_{j+1}, ..., a_n). \qquad \square$$

Putting everything together, we have the following classification of finite permutational algebras.

Theorem B.2.13. *If $\mathbb{A}$ is a finite permutational algebra, then up to isomorphism and polynomial equivalence, $\mathbb{A}$ is one of the following:*

(1) a unary algebra, with the set of unary operations equal to a finite permutation group,

(2) a vector space over a finite field,

(3) the boolean algebra $(\{0,1\},\vee,\wedge,\neg)$,

(4) the lattice $(\{0,1\},\vee,\wedge)$, or

(5) the semilattice $(\{0,1\},\vee)$.

Proof. If every polynomial of $\mathbb{A}$ is essentially unary, then we are in case (1). Otherwise, by Proposition B.2.2 there is some binary polynomial $f \in \mathrm{Pol}_2(\mathbb{A})$ which depends on both of its inputs. If $|\mathbb{A}| \geq 3$, then Theorem B.2.7 shows that we are in case (2). Otherwise, we assume that the underlying set of $\mathbb{A}$ is $\{0,1\}$.

If $\oplus \in \mathrm{Pol}_2(\mathbb{A})$, then Lemma B.2.10 shows that we are either in case (2) or case (3). If $\oplus \not\in \mathrm{Pol}_2(\mathbb{A})$, then we must also have $\neg \not\in \mathrm{Pol}_2(\mathbb{A})$, since $\oplus$ is in the clone generated by $\neg$ and any binary operation f which depends on both its inputs. Then by Lemma B.2.9 every polynomial operation of $\mathbb{A}$ is monotone, and f is either $\vee$ or $\wedge$. By possibly swapping 0 and 1, we may assume without loss of generality that $f = \vee$. Then Lemma B.2.12 shows that we are either in case (4) or (5). □

Corollary B.2.14. *If $\mathbb{A}$ is a finite permutational algebra, then* $\mathrm{Pol}(\mathbb{A})$ *is generated by the binary polynomials of $\mathbb{A}$.*

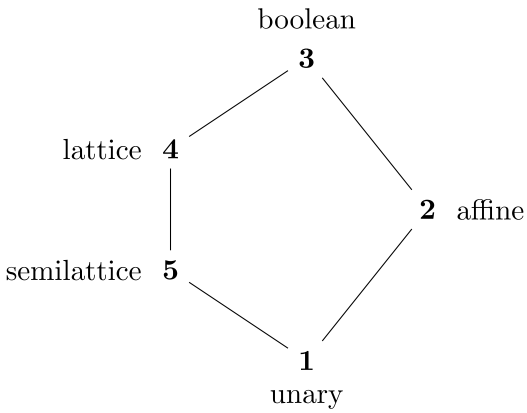
The previous corollary is a general feature of tame congruence theory: most concrete computations in tame congruence theory depend only on the set of binary polynomials.

Definition B.2.15. If (α,β) is a tame congruence quotient of a finite algebra $\mathbb{A}$, and if N is an (α,β)-trace, then we say that N has

- *unary type*, or type **1**, if $\mathbb{A}|_N/\alpha|_N$ is polynomially equivalent to a unary algebra,
- *affine type*, or type **2**, if $\mathbb{A}|_N/\alpha|_N$ is polynomially equivalent to a vector space over a finite field,
- *boolean type*, or type **3**, if $\mathbb{A}|_N/\alpha|_N$ is polynomially equivalent to a boolean algebra,
- *lattice type*, or type **4**, if $\mathbb{A}|_N/\alpha|_N$ is polynomially equivalent to a lattice,
- *semilattice type*, or type **5**, if $\mathbb{A}|_N/\alpha|_N$ is polynomially equivalent to a semilattice.

We say that the tame congruence quotient (α,β) has type **i** if there is any (α,β)-trace with type **i**. As we will see in the next section, all of the traces of a tame congruence quotient (α,β) have the same type as each other.

The numbering of the five types can be remembered with the following visual mnemonic lattice, where the ordering corresponds to the richness of the operations in the polynomial clone.



B.3 The structure of minimal sets

So far we have classified the traces of tame congruence quotients, by classifying the permutational algebras. In order to classify the minimal sets of an algebra, we note that for any (α, β)-minimal set U, the restriction $\mathbb{A}|_U$ has the property that for each unary polynomial $f \in \mathrm{Pol}_1(\mathbb{A}|_U)$, either f is a permutation or $f(\beta|_U) \subseteq \alpha|_U$. Thus the restricted algebra $\mathbb{A}|_U$ is $(\alpha|_U, \beta|_U)$-minimal:

Definition B.3.1. A finite algebra $\mathbb{A}$ is called (α, β)-*minimal*, for $\alpha < \beta \in \mathrm{Con}(\mathbb{A})$, if for every unary polynomial $f \in \mathrm{Pol}_1(\mathbb{A})$, either f is a permutation or $f(\beta) \subseteq \alpha$.

By Proposition B.1.23, an algebra $\mathbb{A}$ is (α, β)-minimal iff $\mathbb{A}/\alpha$ is $(0_{\mathbb{A}/\alpha}, \beta/\alpha)$-minimal, and for each (α, β)-trace N there is a corresponding $(0_{\mathbb{A}/\alpha}, \beta/\alpha)$-trace N/α of the same type, so we can often reduce to the case $\alpha = 0$ without loss of generality. We can simplify some of the arguments of [95] about types **3**, **4**, and **5** by using the concept of a partial semilattice operation from Section 3.2.

Definition B.3.2. We say that an idempotent binary operation s is a *partial semilattice* if it satisfies the identity

$$s(x, s(x, y)) \approx s(s(x, y), x) \approx s(x, y).$$

Equivalently, s is a partial semilattice if for all x, y, the set $\{x, s(x, y)\}$ is closed under s, and acts like a semilattice subalgebra with absorbing element $s(x, y)$ under s.

We write $a \rightarrow_s b$ if s is a partial semilattice and $s(a, b) = b$.

Proposition B.3.3. *If $\mathbb{A}$ is (α, β)-minimal and has a trace N of type **3**, **4**, or **5** (that is, of either boolean, lattice, or semilattice type), then $\mathbb{A}$ has a partial semilattice polynomial $s \in \mathrm{Pol}_2(\mathbb{A})$ such that N is closed under s, and such that $(N/\alpha, s)$ is a two-element semilattice. Furthermore, if N has type **3** or **4** (i.e. boolean or lattice type), then there is another partial semilattice $s' \in \mathrm{Pol}_2(\mathbb{A})$ such that (N, s, s') is a two-element lattice.*

If s is a partial semilattice term and $a, b \in N$ have $s(a, b) \not\equiv_\alpha a$, then $a \rightarrow_s x$ for all $x \in \mathbb{A}$ and $a/\alpha = \{a\}$.

Proof. Let $t \in \mathrm{Pol}_2(\mathbb{A})$ be such that N is closed under t and $(N/\alpha, t)$ is a two-element semilattice. Since the unary polynomial $t(x, x)$ is not constant on N/α, (α, β)-minimality implies the unary polynomial $t(x,x)$ must be invertible, so we may assume without loss of generality that t is idempotent. Then we may apply the semilattice iteration argument from Proposition 3.2.9 to produce a partial semilattice polynomial $s \in \mathrm{Clo}(t)$ such that the restriction of s and t to N/α agree.

For the last statement, if $a, b \in N$ have $s(a,b) \not\equiv_\alpha s(a,a) = a$, then by (α,β)-minimality the unary polynomial $x \mapsto s(a,x)$ must be a permutation, and since $s(a,s(a,x)) = s(a,x)$, it must be the identity, so $s(a,x) = x$ for all $x \in \mathbb{A}$. If $a' \in a/\alpha$, then the same argument shows that $s(a',x) = x$ for all x, so $a \rightarrow_s a'$ and $a' \rightarrow_s a$, which is only possible if $a' = a$. □

Hobby and McKenzie [95] like to think of their semilattices as meet-semilattices, so they call the partial semilattice polynomial s from Proposition B.3.3 a *pseudo-meet* operation. If the type is **3** or **4**, then they call the second partial semilattice operation s' a *pseudo-join* operation. If the type is **3** (i.e. boolean), then you can additionally find a unary polynomial f which preserves N and swaps the elements of N, and any such f will be invertible. We can assume without loss of generality that this f has even order, and we might call such an f a *pseudo-negation* operation.

Proposition B.3.4. *If $\mathbb{A}$ is (α,β)-minimal and has at least two different (α,β)-traces, then all of the (α,β)-traces have type **1** or **2** (that is, they all have either unary or affine type).*

Proof. We assume without loss of generality that $\alpha = 0_\mathbb{A}$. Suppose that N is a $(0_\mathbb{A},\beta)$-trace of type **3**, **4**, or **5** (that is, of either boolean, lattice, or semilattice type). Then N has two elements, call them a, b, and by Proposition B.3.3 there is some partial semilattice polynomial $s \in \mathrm{Pol}_2(\mathbb{A})$ such that $N = \{a,b\}$ is closed under s and such that s acts as a semilattice operation on $\{a,b\}$, say with $s(a,b) = b$. By the second part of Proposition B.3.3, we then have $s(a,x) = s(x,a) = x$ for all $x \in \mathbb{A}$.

Now suppose, for the sake of a contradiction, that K is a different $(0_\mathbb{A},\beta)$-trace. Since $s(a,b) = s(b,b) = b$, $(0_\mathbb{A},\beta)$-minimality implies that the unary polynomial $f : x \mapsto s(x,b)$ has $f(\beta) \subseteq 0_\mathbb{A}$, so $s(K,b)$ is a singleton. Thus there must be some $c \in K$ such that $s(c,b) \ne c$. Since $s(c,a) = c \ne s(c,b)$, the unary polynomial $g : x \mapsto s(c,x)$ must be a permutation by $(0_\mathbb{A},\beta)$-minimality. However, we have $g(K) = s(c,K) \subseteq s(c,c)/\beta = c/\beta = K$ and $g(N) = s(c,N) \subseteq s(c,a)/\beta = c/\beta = K$, so g can't be a permutation, which is a contradiction. □

Proposition B.3.5. *If $\mathbb{A}$ is (α,β)-minimal, then β is abelian over α if and only if all of the (α,β)-traces have type **1** or **2** (i.e., unary or affine type).*

Proof. We assume without loss of generality that $\alpha = 0_\mathbb{A}$. If some (α,β)-trace N has type **3**, **4**, or **5** (i.e., boolean, lattice, or semilattice), then there is a partial semilattice polynomial $s \in \mathrm{Pol}_2(\mathbb{A})$ which acts nontrivially on N by Proposition B.3.3. Since semilattices aren't abelian, $\mathbb{A}|_N$ is not abelian, and therefore β isn't abelian either (since N is a congruence class of β).

Now suppose for contradiction that all the traces have type **1** or **2**, but that β is not abelian. The plan is to transport the nonabelianness of β into one of the $(0_\mathbb{A},\beta)$-traces to contradict the fact that traces of type **1** or **2** must be abelian. Recall that β not being abelian means that there is some polynomial $g \in \mathrm{Pol}(\mathbb{A})$ and some $(u,v),(a_1,b_1),...,(a_n,b_n) \in \beta$ such that

$$g(u,a_1,...,a_n) = g(u,b_1,...,b_n)$$

but

$$g(v,a_1,...,a_n) \ne g(v,b_1,...,b_n),$$

and we may assume without loss of generality that n is minimal. By the minimality of n, we have $a_i \neq b_i$ for all i (else we could just substitute a_i for the ith argument), and we clearly have $u \neq v$, so there are $(0_{\mathbb{A}}, \beta)$-traces $N_0, N_1, ..., N_n$ such that $u, v \in N_0$ and $a_i, b_i \in N_i$ for each i. Let K be the $(0_{\mathbb{A}}, \beta)$-trace which contains $g(u, a_1, ..., a_n)$, then since g is compatible with β we have

$$g(N_0, N_1, ..., N_n) \subseteq K.$$

The restriction of g to $N_0 \times N_1 \times \cdots \times N_n$ must depend on all of its inputs by the minimality of n, so for each i there are $c_j \in N_j$ such that the unary polynomial

$$f_i : x \mapsto g(c_0, ..., c_{i-1}, x, c_{i+1}, ..., c_n)$$

is not constant on N_i. Each such f_i must be a permutation by $(0_{\mathbb{A}}, \beta)$-minimality, so we have $f_i : N_i \simeq K$ for each i. Then the polynomial h given by

$$h(x_0, ..., x_n) = g(f_0^{-1}(x_0), ..., f_n^{-1}(x_n))$$

preserves K, and we have

$$h(f_0(u), f_1(a_1), ..., f_n(a_n)) = h(f_0(u), f_1(b_1), ..., f_n(b_n))$$

but

$$h(f_0(v), f_1(a_1), ..., f_n(a_n)) \neq h(f_0(v), f_1(b_1), ..., f_n(b_n)).$$

Thus $\mathbb{A}|_K$ is not abelian, so $\mathbb{A}|_K$ can't be polynomially equivalent to a unary or affine algebra, which is a contradiction. □

The next challenge is to construct a *pseudo-Mal'cev* operation when the type is **2**, and to use it to prove that all of the (α, β)-traces are isomorphic when at least one of them has type **2**.

Lemma B.3.6. *If $\mathbb{A}$ is (α, β)-minimal and has an (α, β)-trace N of type 2, then there is a ternary polynomial $p \in \mathrm{Pol}_3(\mathbb{A})$ such that, if B is the union of all (α, β)-traces (the "body"), we have*

(a) N is closed under p, and the restriction of $p(x, y, z)$ to $\mathbb{A}|_N/\alpha$ is the Mal'cev operation $x-y+z$,

(b) p is idempotent, that is $p(a, a, a) = a$ for all $a \in \mathbb{A}$,

(c) for all $a \in \mathbb{A}, b \in B$ we have $p(a, b, b) = a$, and

(d) for all $a \in \mathbb{A}, b \in B$ we have $p(b, b, a) = a$.

Proof. (Following [95]) We construct p in stages, in each step getting a ternary polynomial which satisfies one more of (a), (b), (c), (d). To start, since N has type **2**, there is a polynomial $f \in \mathrm{Pol}_3(\mathbb{A})$ satisfying (a). Next, since the restriction of the unary polynomial $g(x) = f(x, x, x)$ to $\mathbb{A}|_N/\alpha$ is nonconstant, (α, β)-minimality implies that $g(x)$ is a permutation, and since the restriction of g to $\mathbb{A}|_N/\alpha$ is the identity, the polynomial $h(x, y, z) = g^{-1}(f(x, y, z))$ satisfies (a) and (b).

Claim. Suppose that f is any polynomial satisfying (a) and (b). For any $b \in B$, the polynomials $x \mapsto f(x, b, b)$ and $x \mapsto f(b, b, x)$ are permutations.

Proof of claim. Suppose not - suppose for contradiction that the unary polynomial $x \mapsto f(x, b, b)$ is not a permutation for some $b \in B$, and let K be the (α, β)-trace which contains b. Iterate f on its first argument to get $f^\infty \in \mathrm{Pol}_3(\mathbb{A})$ such that

$$f^\infty(f^\infty(x, y, z), y, z) = f^\infty(x, y, z)$$

for all x, y, z, define a unary polynomial g by

$$g(x) = f^\infty(x, b, b),$$

and note that if $x \mapsto f(x, b, b)$ is not a permutation, then g is also not a permutation. By (a) and (α, β)-minimality, K can't be N: for any $a \in N$, the restriction of $x \mapsto f^\infty(x, a, a)$ to $\mathbb{A}|_N/\alpha$ is the identity, so (α, β)-minimality and the identity $f^\infty(f^\infty(x, a, a), a, a) = f^\infty(x, a, a)$ imply that

$$f^\infty(x, a, a) = x$$

for all $x \in \mathbb{A}$ and $a \in N$. Since g is not a permutation, (α, β)-minimality implies that $g(K)$ is contained in a single α-congruence class, so in particular there is some $c \in K$ such that $g(c) \not\equiv_\alpha c$. Then if we define the unary polynomial h by $h(x) = f^\infty(c, x, x)$, we have

$$h(c) = f^\infty(c, c, c) = c \not\equiv_\alpha g(c) = f^\infty(c, b, b) = h(b),$$

so by (α, β)-minimality h is a permutation. But then

$$h(c) = f^\infty(c, c, c) = c = f^\infty(c, a, a) = h(a)$$

for any $a \in N$, so h is not injective, a contradiction.

Now we use the claim to upgrade an f satisfying (a) and (b) to one which also satisfies (c). Let $t(x, y) = f(x, y, y)$, and iterate t on its first argument, to get $t^\infty \in \mathrm{Pol}_2(\mathbb{A})$ with $t^\infty(t^\infty(x, y), y) = t^\infty(x, y)$. By the claim, for any $b \in B$ we have $t^\infty(x, b) = x$. Now define $g \in \mathrm{Pol}_3(\mathbb{A})$ by

$$g(x, y, z) = t^{\infty-1}(f(x, y, z), z).$$

The restriction of t to $\mathbb{A}|_N/\alpha$ is just first projection, so g satisfies (a), since f is idempotent g will be idempotent as well, and by construction we have $g(x, b, b) = t^\infty(x, b) = x$ for any $b \in B$.

Finally, we use the claim to upgrade an f satisfying (a), (b), (c) to one which also satisfies (d). By swapping the first and third inputs to f, this is equivalent to upgrading an f which satisfies (a), (b), (d) to one which also satisfies (c). We use the exact same construction for this as in the previous step - we just have to check that the resulting g also satisfies (d): for $b \in B$, we have

$$g(b, b, x) = t^{\infty-1}(f(b, b, x), x) = t^{\infty-1}(x, x) = x,$$

where the last step follows from idempotence. □

Definition B.3.7. If $\mathbb{A}$ is (α, β)-minimal and has a trace of type **2** (i.e. affine type), and if B is the union of the (α, β)-traces, then we call any idempotent ternary polynomial $p \in \mathrm{Pol}_3(\mathbb{A})$ such that $p(a, b, b) = p(b, b, a) = a$ for all $a \in \mathbb{A}$ and all $b \in B$ a *pseudo-Mal'cev* polynomial for $\mathbb{A}$.

Theorem B.3.8. *Suppose that $\mathbb{A}$ is (α, β)-minimal and has a trace of type **2** (i.e. affine type), let B be the union of the (α, β)-traces, and let p be any pseudo-Mal'cev polynomial for $\mathbb{A}$. Then*

- *for all $a, b \in B$, the unary polynomials $x \mapsto p(x, a, b), p(a, x, b), p(a, b, x)$ are all permutations,*
- *B is closed under p and the restriction of p to B is Mal'cev, and*
- *all of the (α, β)-traces are polynomially isomorphic.*

*In particular, if one of the (α, β)-traces has type **2** then they all do.*

Proof. (Following [95]) We assume without loss of generality that $\alpha = 0_{\mathbb{A}}$. First we show that for $a, b \in B$, the unary polynomial $x \mapsto p(x, a, b)$ is a permutation iff $x \mapsto p(a, x, b)$ is. For this, let c be any element of the $(0_{\mathbb{A}}, \beta)$-trace $N = a/\beta$, and note that since β is abelian (by Proposition B.3.5) and $(a, c) \in \beta$, we have

$$p(c, a, b) = p(a, a, b) = b \iff p(a, c, b) = p(c, c, b) = b,$$

so $x \mapsto p(x, a, b)$ is not a permutation iff $p(N, a, b) = \{b\}$, which happens iff $p(a, N, b) = \{b\}$, which happens iff $x \mapsto p(a, x, b)$ is not a permutation (and in fact, these all occur iff $p(N, N, b) = \{b\}$). Similarly, $x \mapsto p(a, x, b)$ is a permutation iff $x \mapsto p(a, b, x)$ is a permutation.

Now suppose for a contradiction that $x \mapsto p(a, x, b)$ and $x \mapsto p(a, b, x)$ are not permutations, and consider the unary poynomial $f(x) = p(a, p(a, x, b), x)$. If $x \in N = a/\beta$, then we have

$$f(x) = p(a, p(a, a, b), x) = p(a, b, x) = p(a, b, a),$$

so f is not a permutation. If $x \in b/\beta$, then we have

$$f(x) = p(a, p(a, b, b), x) = p(a, a, x) = x,$$

so by $(0_{\mathbb{A}}, \beta)$-minimality f must be a permutation, which is a contradiction.

To see that B is closed under p, let $a, b \in B$, then since the unary polynomial $x \mapsto p(a, b, x)$ is a permutation and therefore takes $(0_{\mathbb{A}}, \beta)$-traces to $(0_{\mathbb{A}}, \beta)$-traces, we see that it takes B to B. Finally, if N, K are two $(0_{\mathbb{A}}, \beta)$-traces and $a \in N, b \in K$, then the unary polynomial $g(x) = p(a, x, b)$ takes N to K bijectively. □

Putting these results together, we have proved the main result of this section.

Theorem B.3.9. *If (α, β) is a tame congruence of a finite algebra $\mathbb{A}$, then all of the (α, β)-traces have the same type. If this type is not **1**, or if (α, β) is a prime quotient, then all of the (α, β)-traces are polynomially isomorphic to each other. If the type is not **1** or **2**, then each (α, β)-minimal set has just one trace, and if the type is **3** or **4** then every (α, β)-trace has size two.*

In order to rule out type **1** in most cases, we introduce a stronger version of abelianness which is characteristic of unary algebras.

Definition B.3.10. If $\alpha \le \beta \in \mathrm{Con}(\mathbb{A})$, then β is *strongly abelian* over α if for all $f \in \mathrm{Pol}(\mathbb{A})$, all $(u, v) \in \beta$ and all tuples x, y, z with $x_i \equiv y_i \equiv z_i \pmod{\beta}$, we have

$$f(u, x_1, ..., x_n) \equiv_\alpha f(v, y_1, ..., y_n) \implies f(u, z_1, ..., z_n) \equiv_\alpha f(v, z_1, ..., z_n).$$

An algebra $\mathbb{A}$ is *strongly abelian* if $1_{\mathbb{A}}$ is strongly abelian over $0_{\mathbb{A}}$.

It's easy to see that every unary algebra is strongly abelian, while any group or semilattice is not strongly abelian. We can now characterize type **1** in terms of strong abelianness.

Proposition B.3.11. *If $\mathbb{A}$ is (α, β)-minimal, then β is strongly abelian over α iff all of the (α, β)-traces have type **1** (i.e. unary type).*

Proof. The proof is almost identical to the proof of Proposition B.3.5. □

Proposition B.3.12. *Suppose that* $\mathbb{A}$ *has a Taylor polynomial and that* $\alpha < \beta \in \mathrm{Con}(\mathbb{A})$. *Then* β *is not strongly abelian over* α. *As a consequence, if* $\mathbb{A}$ *is a finite Taylor algebra then no tame congruence quotient of* $\mathbb{A}$ *has type* ***1***.

Proof. Suppose for contradiction that β is strongly abelian over α, and pick some $(a, b) \in \beta \setminus \alpha$. Let t be a Taylor term, then for each input of t we have an equation of the form

$$t(?, ..., ?, a, ?, ..., ?) = t(?, ..., ?, b, ?, ..., ?),$$

where each ? is either an a or a b, so strong abelianness of β over α implies that

$$t(a, ..., a, a, b, ..., b) \equiv_\alpha t(a, ..., a, b, b, ..., b)$$

at each input. Stringing these equations together and using the idempotence of t, we get

$$a = t(a, ..., a) \equiv_\alpha \cdots \equiv_\alpha t(b, ..., b) = b,$$

which contradicts the assumption $(a, b) \notin \alpha$.

For the last statement, note that if t is a Taylor polynomial for $\mathbb{A}$ and $e \in E(\mathbb{A})$ has $e(\mathbb{A}) = U$ for some (α, β)-minimal set U, then $e(t(x_1, ..., x_n))$ is a Taylor polynomial for $\mathbb{A}|_U$. In fact, by the idempotence of t, the restriction of $e \circ t$ to any (α, β)-trace N would be a Taylor polynomial for the unary algebra $\mathbb{A}|_N/\alpha|_N$, which gives an even simpler contradiction. □

The following reformulation of strong abelianness from [142] should give a more concrete idea of just how strong it is.

Proposition B.3.13. *An algebra* $\mathbb{A}$ *is strongly abelian iff, for each* n*-ary polynomial* t *of* $\mathbb{A}$*, there are equivalence relations* R_i *on* $\mathbb{A}$ *such that*

$$t(a_1, ..., a_n) = t(b_1, ..., b_n) \quad \iff \quad \forall i \le n,\ (a_i, b_i) \in R_i.$$

In particular, if $\mathbb{A}$ *is finite and strongly abelian then every polynomial of* $\mathbb{A}$ *depends on at most* $\log_2 |\mathbb{A}|$ *of its inputs.*

Corollary B.3.14. *Every finite, idempotent, strongly abelian algebra can be written as a product of algebras where every operation is a projection.*

Proof. Let t be any m-ary operation of $\mathbb{A}$, and let the equivalence relations R_i be as in the previous proposition. If t is idempotent, then t is the graph of a bijection between $\prod_i \mathbb{A}/R_i$ and $\mathbb{A}$: the inverse map takes $a \in \mathbb{A}$ to $(a/R_1, ..., a/R_m)$. If any R_i is $0_\mathbb{A}$, then t must be the ith projection, otherwise each $\mathbb{A}/R_i$ is smaller than $\mathbb{A}$. To finish the proof, we just need to verify that each R_i is a congruence of $\mathbb{A}$. It's enough to prove this for R_1.

Let s be any other operation of $\mathbb{A}$, say of arity n, and consider the term

$$t(s(y_1, ..., y_n), x_2, ..., x_m).$$

Then by the previous proposition, there are equivalence relations $S_1, ..., S_n$ on $\mathbb{A}$ such that

$$t(s(a_1, ..., a_n), c_2, ..., c_m) = t(s(b_1, ..., b_n), d_2, ..., d_m)$$

iff each $(a_i, b_i) \in S_i$ and each $(c_j, d_j) \in R_j$. Taking all a_i to be equal to a and all b_i to be equal to b, by idempotence we see that $(a, b) \in R_1$ iff $(a, b) \in S_i$ for all i. In particular, we have $(a, b) \in R_1 \implies (a, b) \in S_i$, so

$$\forall i \ \ (a_i, b_i) \in R_1 \quad \implies \quad \forall i \ \ (a_i, b_i) \in S_i \quad \implies \quad (s(a_1, ..., a_n), s(b_1, ..., b_n)) \in R_1.$$

Since s was arbitrary, R_1 is a congruence of $\mathbb{A}$. □

Example B.3.1. A *rectangular band* is an idempotent semigroup which satisfies the identity

$$xyx \approx x.$$

This identity implies the apparently stronger identity

$$xyz \approx xz,$$

as follows:

$$xyz \approx xy(zxz) \approx (xyzx)z \approx xz.$$

Every rectangular band $\mathbb{A}$ is strongly abelian, and is therefore isomorphic to a product of two semigroups $\mathbb{A}_1, \mathbb{A}_2$ such that $\cdot^{\mathbb{A}_1} = \pi_1$ and $\cdot^{\mathbb{A}_2} = \pi_2$. The multiplication on the rectangular band $\mathbb{A}_1 \times \mathbb{A}_2$ is explicitly given by the rule

$$(a, b) \cdot (c, d) = (a, d).$$

Example B.3.2 (From [119]). There is a 5-element strongly abelian algebra $\mathbb{A} = (\{a, b, c, d, e\}, \cdot_1, \cdot_2)$ which is not quasiaffine (of course, this algebra is not idempotent). The basic binary operations of $\mathbb{A}$ are given below.

$\cdot_1$	a	b	c	d	e
a	a	b	a	b	b
b	a	b	a	b	b
c	c	d	c	d	d
d	c	d	c	d	d
e	c	d	c	d	d

$\cdot_2$	a	b	c	d	e
a	a	b	a	b	b
b	a	b	a	b	b
c	c	e	c	e	e
d	c	e	c	e	e
e	c	e	c	e	e

This algebra fails to be quasiaffine because it fails to satisfy the two term condition:

$$a \cdot_1 a = a \cdot_2 a, \quad a \cdot_1 b = a \cdot_2 b, \quad c \cdot_1 a = c \cdot_2 a, \quad \text{but} \quad c \cdot_1 b \neq c \cdot_2 b.$$

To see that it is strongly abelian, note that for each i, j we have the identities

$$x \cdot_i (y \cdot_j z) \approx x \cdot_i z, \quad (x \cdot_i y) \cdot_j z \approx x \cdot_j z,$$

so every term of $\mathbb{A}$ is one of $x, x \cdot_i x, x \cdot_i y$ for some $i \in \{1, 2\}$, up to permuting its inputs.

Example B.3.3. A *p-cyclic groupoid* is an idempotent binary operation $\cdot$ which satisfies the following identities:

$$\begin{aligned} x(yz) &\approx xy, \\ (xy)z &\approx (xz)y, \\ (...((x\underbrace{y)y)\cdots)y}_{p\ ys} &\approx x. \end{aligned}$$

See [156] for the theory of p-cyclic groupoids for arbitrary primes p.

The free 2-cyclic groupoid on two generators a, b is isomorphic to the idempotent algebra $\mathbb{A} = (\{a, b, c, d\}, \cdot)$ with basic operation $\cdot$ given below.

$\cdot$	a	b	c	d
a	a	c	a	c
b	d	b	d	b
c	c	a	c	a
d	b	d	b	d

This algebra has a congruence θ corresponding to the partition $\{a, c\}, \{b, d\}$, such that $\cdot$ on $\mathbb{A}/\theta$ is first projection - in particular, $1_{\mathbb{A}}$ is strongly abelian over θ. Additionally, θ is strongly abelian over $0_{\mathbb{A}}$, so $\mathbb{A}$ is strongly solvable. The algebra $\mathbb{A}$ is $(0_{\mathbb{A}}, \theta)$-minimal, and the $(0_{\mathbb{A}}, \theta)$-traces are $\{a, c\}$ and $\{b, d\}$. The reader may check that the $(0_{\mathbb{A}}, \theta)$-traces $\{a, c\}$ and $\{b, d\}$ are *not* polynomially isomorphic in $\mathbb{A}$. The algebra $\mathbb{A}|_{\{a,c\}}$ is polynomially equivalent to the unary algebra with the unary operation which swaps a and c, corresponding to the polynomial $x \mapsto x \cdot b$.

The reader may check that $\mathbb{A}$ is abelian (and even quasiaffine) as well. If we let η be the congruence corresponding to the partition $\{a\}, \{c\}, \{b, d\}$, however, then we see that $\mathbb{A}/\eta$ is *not* abelian - so quotients of idempotent abelian algebras are not necessarily abelian. This is one of the senses in which type **1** can be pathological.

More generally, the free p-cyclic groupoid on n generators is (up to isomorphism) the subalgebra of

$$\big((\mathbb{Z}/p^2)^n,\ (x, y) \mapsto x + p(y - x)\big)$$

generated by the basis vectors $(1, 0, ..., 0), (0, 1, ..., 0), ..., (0, ..., 0, 1)$ - this algebra has np^{n-1} elements. The free p-cyclic groupoid on n generators is always quasiaffine and strongly solvable in 2 steps via the congruence corresponding to reduction modulo p, and for $p, n \geq 2$ it always has a quotient which is *not* abelian.

B.4 The abelian types: type 1 (unary) and 2 (affine)

In case the reader has lost track, we briefly recap what we have done so far before moving on.

- For any $\alpha < \beta \in \mathrm{Con}(\mathbb{A})$, we defined $M_{\mathbb{A}}(\alpha, \beta)$ to be the collection of minimal sets $U \subseteq \mathbb{A}$ such that there is some unary polynomial $f \in \mathrm{Pol}_1(\mathbb{A})$ with $f(\mathbb{A}) = U$ and $f(\beta) \not\subseteq \alpha$.

- We showed that if β is a cover of α in $\mathrm{Con}(\mathbb{A})$, then the congruence quotient (α, β) is automatically *tame*, that is, for every $U \in M_{\mathbb{A}}(\alpha, \beta)$ there is some idempotent unary polynomial $e \in E(\mathbb{A})$ with $e(\mathbb{A}) = U$, and the restriction homomorphism $[\![\alpha, \beta]\!] \twoheadrightarrow [\![\alpha|_U, \beta|_U]\!]$ is a 0, 1-separating homomorphism (Proposition B.1.8).

- We showed that if (α, β) is tame, then any two minimal sets $U, V \in M_{\mathbb{A}}(\alpha, \beta)$ are polynomially isomorphic (Theorem B.1.14).

- We defined an (α, β)-trace N to be any congruence class of $\beta|_U$ which is not contained in a congruence class of $\alpha|_U$, for any minimal set $U \in M_{\mathbb{A}}(\alpha, \beta)$.

- We showed that if (α, β) is tame, then β is the transitive closure of $\alpha \cup \{N^2 \mid N \text{ is an } (\alpha, \beta)\text{-trace}\}$ (Corollary B.1.20).

- We showed that if (α, β) is tame, then each trace N has $\mathbb{A}|_N/\alpha|_N$ a permutational algebra (Corollary B.1.17), and we classified the permutational algebras into five types (Theorem B.2.13).
- We showed that if (α, β) is tame and $U \in M_{\mathbb{A}}(\alpha, \beta)$ is a minimal set, then all of the (α, β)-traces $N \subseteq U$ have the same type, and if that type is not **1** (i.e. unary) or if (α, β) is prime, then in fact all of the (α, β)-traces are polynomially isomorphic (Theorem B.3.9).
- We showed that if (α, β) is tame with type **1** or **2** (i.e. unary or affine), then $\beta|_U$ is abelian over $\alpha|_U$ for any minimal set $U \in M_{\mathbb{A}}(\alpha, \beta)$ (Proposition B.3.5), and if the type is **1** then $\beta|_U$ is strongly abelian over $\alpha|_U$ (Proposition B.3.11).

We would like to have some results which don't directly reference minimal sets or traces. In this section, we will upgrade the results in the last bullet point to the claims that if (α, β) is tame with type **1** or **2**, then β is abelian over α, and if the type is **1** then β is strongly abelian over α. We start with the strongly abelian case, but the reader may prefer to read the next two results in the opposite order (or even to skip the strongly abelian case entirely, if they only care about Taylor algebras).

Theorem B.4.1. *If $\mathbb{A}$ is a finite algebra and (α, β) is a tame congruence quotient, then β is strongly abelian over α if and only if the type of (α, β) is **1** (i.e., unary).*

Proof. (Following [95]) We assume without loss of generality that $\alpha = 0_{\mathbb{A}}$. If the type of $(0_{\mathbb{A}}, \beta)$ is not **1**, then every $(0_{\mathbb{A}}, \beta)$-trace N has $\mathbb{A}|_N$ not strongly abelian, so in this case β definitely can't be strongly abelian. We just need to prove that if the type is **1** then β is strongly abelian.

Suppose for contradiction that β is *not* strongly abelian, i.e. that there is some $f \in \mathrm{Pol}_n(\mathbb{A})$ and $a_i \equiv b_i \equiv c_i \pmod{\beta}$ such that

$$f(a_1, ..., a_n) = f(b_1, ..., b_n)$$

but

$$f(a_1, c_2, ..., c_n) \ne f(b_1, c_2, ..., c_n).$$

Since

$$f(a_1, c_2, ..., c_n) \equiv f(b_1, c_2, ..., c_n) \pmod{\beta},$$

by Theorem B.1.14(c) or (g) there is some $e \in \mathrm{Pol}_1(\mathbb{A})$ and $U \in M_{\mathbb{A}}(0_{\mathbb{A}}, \beta)$ such that $e(\mathbb{A}) = U$ and

$$e(f(a_1, c_2, ..., c_n)) \ne e(f(b_1, c_2, ..., c_n)) \in U.$$

Let f' be the restriction of $e \circ f$ to $C = \prod_i c_i/\beta$. Then if we let N be the $(0_{\mathbb{A}}, \beta)$-trace in U which contains $f'(c)$, we have

$$f'(C) \subseteq U \cap f'(c)/\beta = N.$$

Since

$$f'(a_1, c_2, ..., c_n) \ne f'(b_1, c_2, ..., c_n),$$

we see that f' must depend on its first variable, and since

$$f'(a) = e(f(a)) = e(f(b)) = f'(b),$$

we see that f' must depend on at least one other variable. By Salomaa's Proposition B.2.2, there must be some binary polynomial $g \in \mathrm{Pol}_2(\mathbb{A})$ which we get by fixing some of the coordinates of f' to constants which depends on both of its inputs (with the inputs restricted to the relevant c_i/βs).

In other words, we have a binary polynomial $g \in \mathrm{Pol}_2(\mathbb{A})$ and a pair of congruence classes $c_i/\beta, c_j/\beta$ such that

$$g(c_i/\beta, c_j/\beta) \subseteq N,$$

such that the restriction of g to $C_{ij} = (c_i/\beta) \times (c_j/\beta)$ depends on both of its inputs. We will show that this already gives us a contradiction.

Claim. If N_1, N_2 are a pair of $(0_{\mathbb{A}}, \beta)$-traces such that $g(N_1, N_2) \subseteq N$, then the restriction of g to $N_1 \times N_2$ depends on at most one of its arguments.

Proof of Claim. Suppose not, for a contradiction. Then by plugging in constants to the first and second argument of g, we can apply Corollary B.1.19 to see that $N_i \simeq N$. Thus we may assume without loss of generality that $N_1 = N_2 = N$. But in this case, g preserves N, so $g|_N$ must be unary since $\mathbb{A}|_N$ is a unary algebra. This contradiction proves the claim.

Now note that if $(0_{\mathbb{A}}, \beta)$-traces N_2, N_2' overlap, and if the restriction of g to $N_1 \times N_2$ depends on its first input, then by the claim g restricts to a nonconstant unary function of its first input on $N_1 \times N_2$, so the restriction of g to $N_1 \times N_2'$ also depends on its first input, and is equal to the same nonconstant unary function of its first input.

Since every congruence class of β is connected through $(0_{\mathbb{A}}, \beta)$-traces by Corollary B.1.20, we see that if the restriction of g to C_{ij} depends on its first input, then there is some trace $N_i \subseteq c_i/\beta$ such that the restriction of g to $N_1 \times (c_j/\beta)$ is a nonconstant unary function of its first input. Similarly, there is some trace $N_2 \subseteq c_j/\beta$ such that the restriction of g to $(c_i/\beta) \times N_2$ is a nonconstant unary function of its second input. But then the restriction of g to $N_1 \times N_2$ depends on both of its inputs, which is a contradiction. □

Theorem B.4.2. *If $\mathbb{A}$ is a finite algebra and (α, β) is a tame congruence quotient, then β is abelian over α if and only if the type of (α, β) is **1** or **2** (i.e., unary or affine type).*

Proof. (Following Pálfy's argument from [95]) We assume without loss of generality that $\alpha = 0_{\mathbb{A}}$. If the type of $(0_{\mathbb{A}}, \beta)$ is **3**, **4**, or **5**, then every $(0_{\mathbb{A}}, \beta)$-trace N has $\mathbb{A}|_N$ nonabelian, so in this case β definitely can't be abelian. We just need to prove that if the type is **1** or **2** then β is abelian. We handled the case where the type is **1** in Theorem B.4.1, so from here on we assume that $(0_{\mathbb{A}}, \beta)$ has type **2**.

Suppose for contradiction that β is *not* abelian, i.e. that there is some $f \in \mathrm{Pol}_{n+1}(\mathbb{A})$ and $a \equiv b \pmod{\beta}$, $c_i \equiv d_i \pmod{\beta}$, such that

$$f(a, c_1, ..., c_n) = f(a, d_1, ..., d_n),$$

but

$$f(b, c_1, ..., c_n) \neq f(b, d_1, ..., d_n).$$

Since every congruence class of β is connected through $(0_{\mathbb{A}}, \beta)$-traces by Corollary B.1.20, we may assume without loss of generality that a, b are both contained in some $(0_{\mathbb{A}}, \beta)$-trace N.

By Theorem B.1.14(c) or (g), we see that there is some unary polynomial e with $e(\mathbb{A}) \in M_{\mathbb{A}}(0_{\mathbb{A}}, \beta)$ such that

$$e(f(b, c_1, ..., c_n)) \neq e(f(b, d_1, ..., d_n)).$$

For this choice of e, we see that we have

$$e(f(b/\beta, c_1/\beta, ..., c_n/\beta)) \subseteq e(\mathbb{A}) \cap e(f(b, c))/\beta = N'$$

for some $(0_{\mathbb{A}}, \beta)$-trace N'. Since the type of $(0_{\mathbb{A}}, \beta)$ is not **1**, we can apply Theorem B.3.9 to see that the traces N and N' are polynomially isomorphic. Thus we may assume without loss of generality that $N' = N$, and to simplify the notation we replace f with $e \circ f$, so that we have

$$f(b/\beta, c_1/\beta, ..., c_n/\beta) \subseteq N.$$

The purpose of ensuring that the output of f is in the same trace as the elements a, b which we used in the first input is as follows. Suppose that we have traces $N_i \subseteq c_i/\beta$ for each i. Then by Theorem B.3.9 there are unary polynomials $g_i \in \mathrm{Pol}_1(\mathbb{A})$ such that

$$g_i : N \simeq N_i$$

for each i. Then the function

$$f(x, g_1(y_1), ..., g_n(y_n))$$

has

$$f(N, g_1(N), ..., g_n(N)) = f(N, N_1, ..., N_n) \subseteq N,$$

so it preserves N. Since $\mathbb{A}|_N$ is affine, we can fix once and for all a vector space structure on $\mathbb{A}|_N$ with coefficients in some fixed finite field $\mathbb{F}$. Then there are coefficients $r, r_1, ..., r_n, c \in \mathbb{F}$ such that

$$f(x, g_1(y_1), ..., g_n(y_n))|_N \approx rx + r_1y_1 + \cdots + r_ny_n + c.$$

The coefficients r_i depend on the choice of the maps $g_i : N \simeq N_i$, but the coefficient r on x does not - this is what we will exploit to complete the proof.

Claim. Suppose that $N_i, N_i' \subseteq c_i/\beta$ are $(0_{\mathbb{A}}, \beta)$-traces for each i, such that each N_i overlaps with N_i'. If we choose unary polynomials $g_i, g_i' \in \mathrm{Pol}_1(\mathbb{A})$ with

$$g_i : N \simeq N_i, \quad g_i' : N \simeq N_i',$$

and if we let $r, r', r_i, ..., r_i', c, c' \in \mathbb{F}$ be the coefficients which satisfy

$$\begin{aligned} f(x, g_1(y_1), ..., g_n(y_n))|_N &\approx rx + r_1y_1 + \cdots + r_ny_n + c, \\ f(x, g_1'(y_1), ..., g_n'(y_n))|_N &\approx r'x + r_1'y_1 + \cdots + r_n'y_n + c', \end{aligned}$$

then $r = r'$.

Proof of Claim. Since N_i and N_i' overlap, we can find $u_i, u_i' \in N$ such that

$$g_i(u_i) = g_i'(u_i') \in N_i \cap N_i'.$$

Pluggin in u_i, u_i' for the y_is, we get

$$rx + \sum_i r_iu_i + c = f(x, g(u)) = f(x, g'(u')) = r'x + \sum_i r_i'u_i' + c'$$

for all $x \in N$. Thus the difference $(r - r')x$ is a constant function on N, so $r = r'$, which proves the claim.

Since every congruence class of β is connected through $(0_{\mathbb{A}}, \beta)$-traces by Corollary B.1.20, we can apply the claim repeatedly to see that if we let $N_i, N_i' \subseteq c_i/\beta$ be any $(0_{\mathbb{A}}, \beta)$-traces with $c_i \in N_i$ and $d_i \in N_i'$ (with N_i, N_i' not necessarily overlapping any more), and if we define unary polynomials g_i, g_i' and coefficients $r, r', r_i, ..., r_i', c, c' \in \mathbb{F}$ as in the claim, then we must still have $r = r'$.

Let $u_i, u_i' \in N$ have $g_i(u_i) = c_i, g_i'(u_i') = d_i$. Then from

$$f(a, c_1, ..., c_n) = f(a, d_1, ..., d_n)$$

we conclude that

$$ra + \sum_i r_i u_i + c = f(a, g(u)) = f(a, g'(u')) = ra + \sum_i r_i' u_i' + c'.$$

Adding $r(b - a)$ to both sides, we see that

$$f(b, g(u)) = rb + \sum_i r_i u_i + c = rb + \sum_i r_i' u_i' + c' = f(b, g'(u')),$$

and this contradicts our assumption that

$$f(b, c_1, ..., c_n) \neq f(b, d_1, ..., d_n),$$

completing the proof. □

Corollary B.4.3. *If $\mathbb{A}$ is a finite algebra and if the interval $[\![\alpha, \beta]\!]$ is a tight sublattice of* $\mathrm{Con}(\mathbb{A})$ *of size at least* 3*, then β is abelian over α.*

If additionally $[\![\alpha, \beta]\!]$ does not have a $0, 1$*-separating homomorphism onto the congruence lattice of a vector space, then β is strongly abelian over α.*

B.5 The basic tolerance, and orderability

The main idea of this section is to take a prime congruence quotient (α, β) in $\mathrm{Con}(\mathbb{A})$, and try to study the simplest binary relations $\mathbb{R}$ with $\alpha \leq \mathbb{R} \leq \beta$. Actually, we want to study this in a way that doesn't give a different answer for the pair (α, β) on $\mathbb{A}$ from the answer it gives for the pair $(0_{\mathbb{A}/\alpha}, \beta/\alpha)$ on $\mathbb{A}/\alpha$. So we mainly focus on relations which are compatible with α in the following sense.

Definition B.5.1. If α is a congruence on $\mathbb{A}$ and $\mathbb{R}$ is a binary relation on $\mathbb{A}$, then we say that $\mathbb{R}$ is *α-closed* if whenever $(a, b) \in \mathbb{R}$ and $a \equiv c \pmod{\alpha}, b \equiv d \pmod{\alpha}$, we also have $(c, d) \in \mathbb{R}$.

We define the *α-closure* of $\mathbb{R}$ to be the binary relation

$$\alpha \circ \mathbb{R} \circ \alpha.$$

Proposition B.5.2. *For $\alpha \in$* $\mathrm{Con}(\mathbb{A})$ *and $\mathbb{R} \leq \mathbb{A}^2$, the α-closure of $\mathbb{R}$ is the smallest α-closed relation on $\mathbb{A}$ which contains $\mathbb{R}$. There is a bijection between α-closed binary relations on $\mathbb{A}$ and binary relations on $\mathbb{A}/\alpha$.*

So we can mainly focus on the case $\alpha = 0_{\mathbb{A}}$. In this case, we are studying the simplest binary relations $\mathbb{R}$ which contain the diagonal (and are contained in some atomic congruence β) - but relations which contain the diagonal are exactly the same as relations which are preserved by $\mathrm{Pol}(\mathbb{A})$, so we are really studying the simplest binary relations on the algebraic structure $(A, \mathrm{Pol}(\mathbb{A}))$.

The simplest thing we can do is to take some pair $(a, b) \in \beta$ with $a \neq b$, and consider the binary relation generated by $\Delta_{\mathbb{A}} \cup \{(a, b)\}$, where $\Delta_{\mathbb{A}} = 0_{\mathbb{A}}$ is the diagonal of $\mathbb{A}$. This can be written down explicitly as

$$\mathrm{Sg}_{\mathbb{A}^2}(\Delta_{\mathbb{A}} \cup \{(a, b)\}) = \{(f(a), f(b)) \mid f \in \mathrm{Pol}_1(\mathbb{A})\}.$$

So we really just need to know what *unary* polynomials do to the pair (a, b). Now we can see how tame congruence theory will be helpful: each trace N of $(0_{\mathbb{A}}, \beta)$ contains an image of every pair $(a, b) \in \beta \setminus 0_{\mathbb{A}}$ under some unary polynomial by Theorem B.1.14(c) and Proposition B.1.21.

We start by studying tolerances - recall that a tolerance on $\mathbb{A}$ is just a symmetric reflexive relation which is compatible with the algebraic structure of $\mathbb{A}$.

Theorem B.5.3. *If (α, β) is a prime congruence quotient of a finite algebra $\mathbb{A}$ with type different from **1**, then there is a unique minimal α-closed tolerance τ with*

$$\alpha \subsetneq \tau \subseteq \beta.$$

This tolerance τ is the α-closure of the relation

$$\mathrm{Sg}_{\mathbb{A}^2}\left(\Delta_{\mathbb{A}} \cup \{N^2 \mid N \text{ is an } (\alpha, \beta)\text{-trace}\}\right).$$

*Furthermore, if the type is **2** or **3**, then τ is also minimal among reflexive α-closed relations which properly contain α and are contained in β.*

Proof. We can assume without loss of generality that $\alpha = 0_{\mathbb{A}}$. We just need to prove that every nontrivial tolerance $\tau \subseteq \beta$ contains N^2 for every $(0_{\mathbb{A}}, \beta)$-trace N.

Since τ is nontrivial, it must contain some $(a, b) \in \beta \setminus 0_{\mathbb{A}}$, and by Theorem B.1.14(c) and Proposition B.1.21 we can assume without loss of generality that $a, b \in N$, for any particular (α, β)-trace N. Thus $\tau \cap N^2$ is a nontrivial tolerance on $\mathbb{A}|_N$, and we just have to check that $\mathbb{A}|_N$ has no nontrivial proper tolerances to finish the proof.

If the type is **3**, **4**, or **5**, then $|N| = 2$, so in this case we have $N = \{a, b\}$, and

$$N^2 = \Delta_N \cup \{(a, b), (b, a)\} \subseteq \tau,$$

since $(a, b) \in \tau$ and since τ is symmetric and reflexive.

If the type is **2** or **3**, then $\mathbb{A}|_N$ is a Mal'cev algebra, so every reflexive relation on $\mathbb{A}|_N$ is a congruence, and we see that $N^2 \subseteq \tau$ in these cases, even without the assumption that τ is symmetric. □

Example B.5.1. If the type is equal to **1**, then there might not be a unique minimal α-closed tolerance containing α. Consider the unary algebra $\mathbb{A} = (\mathbb{Z}/5, x \mapsto x + 1 \pmod 5)$, which is simple and permutational. The minimal tolerances of $\mathbb{A}$ are

$$\tau_1 = \{(x, y) \mid x - y \in \{-1, 0, 1\} \pmod 5\}$$

and

$$\tau_2 = \{(x, y) \mid x - y \in \{-2, 0, 2\} \pmod 5\}.$$

Definition B.5.4. If (α, β) is a prime congruence quotient with type different from **1**, then we define the *basic tolerance* of (α, β) to be the minimal α-closed tolerance τ with $\alpha \subsetneq \tau \subseteq \beta$.

If the type is **2** or **3** the situation simplifies - in these cases, the basic tolerance really is basic.

Theorem B.5.5. *If (α, β) is a prime congruence quotient of a finite algebra $\mathbb{A}$ with type **2** or **3**, then the basic tolerance of (α, β) is just the α-closure of*

$$\Delta_{\mathbb{A}} \cup \{N^2 \mid N \textit{ is an } (\alpha, \beta)\textit{-trace}\},$$

without needing to apply $\mathrm{Sg}_{\mathbb{A}^2}$.

Proof. We can assume without loss of generality that $\alpha = 0_{\mathbb{A}}$. By the argument of Theorem B.5.3, for any $(0_{\mathbb{A}}, \beta)$-trace N and for any $a \neq b \in N$, the basic tolerance τ is given by

$$\begin{aligned} \tau &= \mathrm{Sg}_{\mathbb{A}^2}(\Delta_{\mathbb{A}} \cup \{(a, b)\}) \\ &= \{(f(a), f(b)) \mid f \in \mathrm{Pol}_1(\mathbb{A})\}. \end{aligned}$$

By Corollary B.1.19, for every $f \in \mathrm{Pol}_1(\mathbb{A})$ either $f(a) = f(b)$ or $f(N)$ is another $(0_{\mathbb{A}}, \beta)$-trace and $f : N \simeq f(N)$. In other words, we have

$$\tau = \Delta_{\mathbb{A}} \cup \{f(N)^2 \mid f(N) \text{ is a } (0_{\mathbb{A}}, \beta)\text{-trace}\}. \qquad \square$$

If the type is **4** or **5**, then we can find smaller reflexive relations within the basic tolerance.

Theorem B.5.6. *If (α, β) is a prime congruence quotient of a finite algebra $\mathbb{A}$ with type **4** or **5**, then there are exactly two minimal α-closed reflexive relations ρ_0, ρ_1 which strictly contain α and are contained in β. These relations have the following properties:*

- $\rho_1 = \rho_0^-$*, that is,* $\rho_1 = \{(y, x) \mid (x, y) \in \rho_0\}$,
- $\rho_0 \cap \rho_1 = \alpha$,
- $\rho_0 \cup \rho_1$ *is the α-closure of* $\Delta_{\mathbb{A}} \cup \{N^2 \mid N \textit{ is an } (\alpha, \beta)\textit{-trace}\}$,
- *the basic tolerance of (α, β) is the α-closure of* $\mathrm{Sg}_{\mathbb{A}^2}(\rho_0 \cup \rho_1)$.

Proof. We can assume without loss of generality that $\alpha = 0_{\mathbb{A}}$. By the argument of Theorem B.5.3, if ρ is a nontrivial reflexive relation contained in β, then for any $(0_{\mathbb{A}}, \beta)$-trace N the restriction $\rho \cap N^2$ is a nontrivial reflexive relation on $\mathbb{A}|_N$.

Since the type is **4** or **5**, N has size 2, say $N = \{a, b\}$. Then we see that ρ must either contain (a, b) or (b, a), so the minimal α-closed relations are

$$\rho_0 = \mathrm{Sg}_{\mathbb{A}^2}(\Delta_{\mathbb{A}} \cup \{(a, b)\})$$

and

$$\rho_1 = \mathrm{Sg}_{\mathbb{A}^2}(\Delta_{\mathbb{A}} \cup \{(b, a)\}) = \rho_0^-.$$

As in the previous argument, we have

$$\rho_0 = \{(f(a), f(b)) \mid f \in \mathrm{Pol}_1\} \subseteq \Delta_{\mathbb{A}} \cup \{f(N)^2 \mid f(N) \text{ is a } (0_{\mathbb{A}}, \beta)\text{-trace}\},$$

and

$$\rho_0 \cup \rho_1 = \Delta_{\mathbb{A}} \cup \{f(N)^2 \mid f(N) \text{ is a } (0_{\mathbb{A}}, \beta)\text{-trace}\}.$$

To finish the proof, we just need to check that $\rho_0 \cap \rho_1 = \Delta_{\mathbb{A}}$, or equivalently that $(b, a) \not\in \rho_0$. To see this, note that if there was a unary polynomial f such that

$$(f(a), f(b)) = (b, a),$$

then $f|_N$ would be a unary operation of $\mathbb{A}|_N$ which swaps the elements of N. In this case, $\mathbb{A}|_N$ would actually have type **3** (i.e. boolean type), contradicting our assumption that the type was **4** or **5** (i.e. lattice or semilattice type, respectively). □

The "α-antisymmetry" of the relation ρ_0 is intriguing, and leads us to wonder if we can produce a nice quasiorder by taking the transitive closure of ρ_0 when the type is **4** or **5**.

Definition B.5.7. We say that a compatible binary relation $\zeta \le \mathbb{A}^2$ is an (α, β)-*preorder* if

- ζ is a quasiorder on $\mathbb{A}$,
- $\zeta \cap \zeta^- = \alpha$, and
- the transitive closure of $\zeta \cup \zeta^-$ is β.

Note that every compatible quasiorder ζ on $\mathbb{A}$ is an (α, β)-preorder for some pair of congruences (α, β), since the transitive closure of $\zeta \cup \zeta^-$ is exactly the linking congruence of ζ.

We say that a congruence quotient (α, β) is *orderable* if an (α, β)-preorder exists.

Theorem B.5.8. *If (α, β) is a tame congruence quotient with type different from **1**, then (α, β) is orderable if and only if the type of (α, β) is **4** or **5**.*

*In fact, if the type of (α, β) is **4** or **5**, then there are exactly two minimal (α, β)-preorders ζ_0, ζ_1 and two maximal (α, β)-preorders ξ_0, ξ_1 such that every (α, β)-preorder η satisfies*

$$\zeta_i \subseteq \eta \subseteq \xi_i$$

for either $i = 0$ or $i = 1$.

Proof. (Following [95]) Once again, we assume without loss of generality that $\alpha = 0_{\mathbb{A}}$. Theorem B.5.3 shows that if the type is **2** or **3** then every nontrivial reflexive relation η contained in β contains the basic tolerance, and therefore can't be a $(0_{\mathbb{A}}, \beta)$-preorder.

Now suppose that the type is **4** or **5**. Let ρ_0, ρ_1 be the minimal nontrivial reflexive relations contained in β from Theorem B.5.6. Note that any compatible relation η which contains both ρ_0 and ρ_1 also contains the basic tolerance, and therefore can't be a $(0_{\mathbb{A}}, \beta)$-quasiorder. Clearly we need to let ζ_i be the transitive closure of ρ_i, but the difficulty lies in verifying that $\zeta_0 \cap \rho_1 = \Delta_{\mathbb{A}}$. To pull this off, we need to understand the maximal quasiorder $\xi_0 \subseteq \beta$ which satisfies $\xi_0 \cap \rho_1 = \Delta_{\mathbb{A}}$.

Let $N = \{a, b\}$ be a $(0_{\mathbb{A}}, \beta)$-trace, and suppose that $(a, b) \in \rho_0$. Then we define ξ_0 by

$$\xi_0 = \{(x, y) \in \beta \mid \forall f \in \mathrm{Pol}_1(\mathbb{A}) \text{ s.t. } f(x/\beta) \subseteq N \text{ and } f(x) = b, \text{ we also have } f(y) = b\}.$$

Since $(x, y) \in \xi_0$ is defined in terms of an implication from x to y, we see that ξ_0 is a quasiorder. By the definition of ξ_0 and the fact that $(0_{\mathbb{A}}, \beta)$ is tame, we have

$$(b, a) \not\in \xi_0,$$

so $\xi_0 \cap \rho_1 = \Delta_{\mathbb{A}}$, and ξ_0 is clearly maximal among quasiorders which are contained in β and only meet ρ_1 at $\Delta_{\mathbb{A}}$. Additionally, we have

$$(a, b) \in \xi_0$$

since there is no unary polynomial which swaps a and b if $(0_{\mathbb{A}}, \beta)$ has type **4** or **5**, so $\zeta_0 \subseteq \xi_0$. By Proposition B.0.1, to finish we just need to check that ξ_0 is closed under unary polynomials - but this follows directly from the definition of ξ_0. □

Example B.5.2. If the type of (α, β) is **1**, then (α, β) can sometimes be orderable and sometimes not. To see this, consider the unary algebra $\mathbb{A}_1 = (\{0, 1\})$ with no operations, and the unary algebra $\mathbb{A}_2 = (\{0, 1\}, 1 - x)$ with just a single operation which swaps the two elements. Then $(0_{\mathbb{A}_1}, 1_{\mathbb{A}_1})$ is orderable but $(0_{\mathbb{A}_2}, 1_{\mathbb{A}_2})$ is not.

B.6 Snags and (strong) solvability

Recall that Theorem B.4.2 says that a tame congruence quotient (α, β) of $\mathbb{A}$ is abelian iff it has type **1** or **2**. Since the type of a tame congruence quotient is determined by the collection of binary polynomials $\mathrm{Pol}_2(\mathbb{A})$, we should be able to tell if (α, β) is abelian by examining $\mathrm{Pol}_2(\mathbb{A})$. We can make this more explicit by recalling that minimal sets of congruence quotients of type **3**, **4**, and **5** all have a binary pseudo-meet polynomial s, by Proposition B.3.3. This naturally leads to the concept of a *snag*.

Definition B.6.1. If $\mathbb{A}$ is an algebra, then an ordered pair of elements $(a, b) \in \mathbb{A}^2$ is a *2-snag* if there is a binary polynomial $s \in \mathrm{Pol}_2(\mathbb{A})$ such that

$$s(a, a) = s(a, b) = s(b, a) = a, \quad s(b, b) = b.$$

In other words, we require $(\{a, b\}, s)$ to be a semilattice with $b \to_s a$. We write $\mathrm{Sn}_2(\mathbb{A}) \subseteq \mathbb{A}^2$ for the set of 2-snags of $\mathbb{A}$.

Similarly, Theorem B.4.1 says that a tame congruence quotient (α, β) is strongly abelian iff it has type **1**. If we already know that (α, β) is abelian, then we can use the fact that the minimal sets for congruence quotients of type **2** all have a ternary pseudo-Mal'cev polynomial p, by Lemma B.3.6. The trick to deal with this case is to pick an element b in the body of one of the minimal sets (recall that the "body" of a minimal set is defined to be the union of the traces contained in it), and to examine the binary polynomial

$$s(x, y) = p(x, b, y).$$

The pseudo-Mal'cev property ensures that for any a in the minimal set, we have $s(a, b) = p(a, b, b) = a$ and $s(b, a) = p(b, b, a) = a$, while $s(b, b) = p(b, b, b) = b$, so s depends on both of its arguments in a way that can't occur in a strongly abelian algebra.

Definition B.6.2. If $\mathbb{A}$ is an algebra, then an ordered pair of elements $(a, b) \in \mathbb{A}^2$ is a *1-snag* if there is a binary polynomial $s \in \mathrm{Pol}_2(\mathbb{A})$ such that

$$s(a, b) = s(b, a) = a, \quad s(b, b) = b.$$

We write $\mathrm{Sn}_1(\mathbb{A})$ for the set of 1-snags of $\mathbb{A}$, and note that $\mathrm{Sn}_2(\mathbb{A}) \subseteq \mathrm{Sn}_1(\mathbb{A})$.

Theorem B.6.3. *If (α, β) is a tame congruence quotient of a finite algebra $\mathbb{A}$, then*

- *β is abelian over α iff $\beta \cap \mathrm{Sn}_2(\mathbb{A}) = \alpha \cap \mathrm{Sn}_2(\mathbb{A})$, and*
- *β is strongly abelian over α iff $\beta \cap \mathrm{Sn}_1(\mathbb{A}) = \alpha \cap \mathrm{Sn}_1(\mathbb{A})$.*

As a consequence, for any *$\alpha \le \beta \in \mathrm{Con}(\mathbb{A})$, we have*

- *β is solvable over α iff $\beta \cap \mathrm{Sn}_2(\mathbb{A}) = \alpha \cap \mathrm{Sn}_2(\mathbb{A})$, and*
- *β is strongly solvable over α iff $\beta \cap \mathrm{Sn}_1(\mathbb{A}) = \alpha \cap \mathrm{Sn}_1(\mathbb{A})$.*

This motivates the definition of two equivalence relations $\overset{s}{\sim}$, $\overset{ss}{\sim}$ on $\mathrm{Con}(\mathbb{A})$.

Definition B.6.4. If $\mathbb{A}$ is an algebra and $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$, then we write

$$\alpha \overset{s}{\sim} \beta$$

when $\beta \cap \mathrm{Sn}_2(\mathbb{A}) = \alpha \cap \mathrm{Sn}_2(\mathbb{A})$, and

$$\alpha \overset{ss}{\sim} \beta$$

when $\beta \cap \mathrm{Sn}_1(\mathbb{A}) = \alpha \cap \mathrm{Sn}_1(\mathbb{A})$.

Theorem B.6.5. *If $\mathbb{A}$ is a finite algebra, then each of the equivalence relations $\overset{s}{\sim}, \overset{ss}{\sim}$ defines a congruence on the lattice* $\mathrm{Con}(\mathbb{A})$*.*

In particular, we have $\alpha \overset{s}{\sim} \beta$ iff $\alpha \vee \beta$ is solvable over $\alpha \wedge \beta$, and similarly $\alpha \overset{ss}{\sim} \beta$ iff $\alpha \vee \beta$ is strongly solvable over $\alpha \wedge \beta$.

Proof. That $\overset{s}{\sim}, \overset{ss}{\sim}$ are compatible with $\wedge$ is immediate from the definition, so we just have to prove that they are compatible with $\vee$. Note that the compatibility with $\wedge$ immediately implies that

$$\alpha \overset{s}{\sim} \beta \quad \iff \quad \alpha \overset{s}{\sim} \alpha \wedge \beta \text{ and } \alpha \wedge \beta \overset{s}{\sim} \beta,$$

so we just have to check that

$$\alpha \overset{s}{\sim} \beta \text{ and } \gamma \overset{s}{\sim} \delta \quad \implies \quad \alpha \vee \gamma \overset{s}{\sim} \beta \vee \delta$$

in the special case where $\alpha \le \beta$ and $\gamma \le \delta$ (and similarly for $\overset{ss}{\sim}$). In fact, we just have to check this in the special case where $\delta = \gamma$ and (α, β) is a prime congruence quotient, and we may as well assume further that $\alpha \le \gamma$. By taking γ as large as possible among potential counterexamples, we see that we just need to prove the following claim.

Claim. If (α, β) and (γ, η) are tame congruence quotients such that

$$\alpha \le \gamma < \eta \le \beta \vee \gamma,$$

and if $\eta \setminus \gamma$ contains a snag, then $\beta \setminus \alpha$ contains a snag of the same type.

Proof of Claim. Let $U \in M_{\mathbb{A}}(\gamma, \eta)$ be a (γ, η)-minimal set. By Lemma B.1.3, we have

$$\eta|_U \subseteq \beta|_U \vee \gamma|_U.$$

If $\eta \setminus \gamma$ contains a snag, then (γ, η) fails to be abelian (or strongly abelian), so $\eta|_U \setminus \gamma|_U$ will also contain a snag (a, b) of the same type, with a, b contained in some (γ, η)-trace N. From $(a, b) \in \beta|_U \vee \gamma|_U$, we see that there must be some $(a', b') \in \beta|_U$ such that

$$b' \in b/\gamma, \quad a' \notin b/\gamma.$$

We clearly have $(a', b') \in \beta \setminus \gamma \subseteq \beta \setminus \alpha$, so we just need to check that (a', b') is a snag of the same type as (a, b).

If (γ, η) has type **3**, **4**, or **5** (which must always occur if (a, b) is a 2-snag), then by Proposition B.3.3 we see that $b' = b$, and that $\mathbb{A}|_U$ has a partial semilattice polynomial s such that $b \rightarrow_s x$ for all $x \in U$. In particular, $(a', b') = (a', b)$ is a 2-snag via s.

If (γ, η) has type **2** (which may only occur if (a, b) is a 1-snag), then by Lemma B.3.6 we see that $\mathbb{A}|_U$ has a pseudo-Mal'cev operation p which satisfies

$$p(x, b', b') = p(b', b', x) = x$$

for all $x \in U$, since $b' \in b/\gamma \cap U \subseteq N$ is contained in the body of U. In particular, defining the binary polynomial s by

$$s(x, y) = p(x, b', y),$$

we see that (a', b') is a 1-snag via s. □

The equivalence relations $\overset{s}{\sim}, \overset{ss}{\sim}$ still make sense on infinite algebras $\mathbb{A}$, but they lose some of their meaning. We can still make use of them for *locally finite* algebras - recall that an algebra $\mathbb{A}$ is locally finite if every finitely generated subalgebra of $\mathbb{A}$ is finite.

Corollary B.6.6. *If $\mathbb{A}$ is a locally finite algebra, then each of the equivalence relations $\overset{s}{\sim}, \overset{ss}{\sim}$ defines a congruence on the lattice* $\mathrm{Con}(\mathbb{A})$.

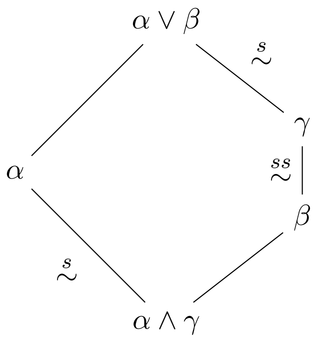
In particular, we have $\alpha \overset{s}{\sim} \beta$ iff $\alpha \vee \beta|_{\mathbb{B}}$ is solvable over $\alpha \wedge \beta|_{\mathbb{B}}$ for every finite subalgebra $\mathbb{B}$ of $\mathbb{A}$, and similarly for $\overset{ss}{\sim}$.

It therefore makes sense to read $\overset{s}{\sim}$ as "locally solvably equivalent", and $\overset{ss}{\sim}$ as "locally strongly solvably equivalent" when studying locally finite algebras. In the infinite case, we may want to know slightly more than just the fact that $\overset{s}{\sim}, \overset{ss}{\sim}$ are congruences - we want to know if they are compatible with infinite meets and joins, for instance. Recall from Definition A.5.5 that a complete lattice is called *algebraic* if every element can be written as a join of compact elements.

Proposition B.6.7. *If $\mathbb{A}$ is locally finite, then the congruences $\overset{s}{\sim}, \overset{ss}{\sim}$ are compatible with arbitrary meets and joins, and the lattices* $\mathrm{Con}(\mathbb{A})/\overset{s}{\sim}, \mathrm{Con}(\mathbb{A})/\overset{ss}{\sim}$ *are algebraic.*

Proof. The only tricky claim to check is that the quotient lattices are algebraic. For this, we use the fact that each $\overset{s}{\sim}$-class $\alpha/\overset{s}{\sim}$ is determined by the intersection $\alpha \cap \mathrm{Sn}_2(\mathbb{A})$, and for any 2-snag $(a, b) \in \mathrm{Sn}_2(\mathbb{A})$, we can prove that $\mathrm{Cg}_{\mathbb{A}}\{(a, b)\}/\overset{s}{\sim}$ is a compact element of $\mathrm{Con}(\mathbb{A})/\overset{s}{\sim}$. The argument for $\overset{ss}{\sim}$ is similar. □

We would like to claim that as long as we avoid type **1**, locally solvable algebras behave like Mal'cev algebras - i.e., that they are congruence modular. We can actually prove a much stronger claim about copies of the pentagon lattice $\mathcal{N}_5$ in $\mathrm{Con}(\mathbb{A})$.



Theorem B.6.8. *If $\mathbb{A}$ is locally finite and $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbb{A})$ form a copy of the pentagon lattice $\mathcal{N}_5$ with*

$$\alpha \wedge \gamma \le \beta \le \gamma \le \alpha \vee \beta,$$

then

$$\alpha \overset{s}{\sim} \alpha \wedge \gamma \quad \implies \quad \beta \overset{ss}{\sim} \gamma.$$

Proof. (Following [95]) We may assume without loss of generality that $\mathbb{A}$ is finite and that (β, γ) is a prime congruence quotient. Since $\overset{s}{\sim}$ is a congruence on $\mathrm{Con}(\mathbb{A})$, we know that (β, γ) must be abelian, so assume for the sake of contradiction that (β, γ) has type **2**.

Let U be a (β, γ)-minimal set, let B be the body of U (i.e., B is the union of the (β, γ)-traces), and $T = U \setminus B$ the "tail" of U. Let p be a pseudo-Mal'cev operation for $\mathbb{A}|_U$. Then (B, p) is a Mal'cev algebra by Theorem B.3.8, so

$$\gamma|_B \not\subseteq \alpha|_B \vee \beta|_B,$$

since otherwise we would have a copy of the pentagon lattice $\mathcal{N}_5$ in the congruence lattice of (B, p), contradicting Proposition 1.7.8. Since $\gamma|_U \le \alpha|_U \wedge \beta|_U$ by Lemma B.1.3, we see that we must have

$$\alpha \cap (B \times T) \ne \emptyset.$$

We will use this to show that α can't possibly be solvable over $\alpha \wedge \gamma$, which will give us our desired contradiction. We just need to prove the following claim.

Claim. If $\delta < \theta$ is a pair of congruences such that

$$\delta \cap (B \times T) = \emptyset \quad \text{and} \quad \theta \cap (B \times T) \ne \emptyset,$$

then θ is not abelian over δ.

Proof of Claim. Pick $(b, t) \in \theta \cap (B \times T)$, and assume for contradiction that θ is abelian over δ. Since p is pseudo-Mal'cev, we have

$$p(b, b, t) = p(t, t, t) = t,$$

so abelianness of θ over δ implies that

$$b = p(b, b, b) \equiv p(t, t, b) \pmod{\delta},$$

so

$$p(t, t, b) \in B$$

by the assumption $\delta \cap (B \times T) = \emptyset$. Let a be any element in the (β, γ)-trace $U \cap b/\gamma$ which is not in b/β. Then we have

$$p(t, a, b) \equiv p(t, b, b) = t \pmod{\gamma},$$

and since t is in the tail of U, we have $U \cap t/\gamma = U \cap t/\beta$, so

$$p(t, a, b) \equiv t \pmod{\beta}.$$

Then if we define the unary polynomial f by

$$f(x) = p(x, p(t, p(t, x, b), b), b),$$

we have

$$\begin{aligned} f(b) &= p(b, p(t, t, b), b), \\ f(a) &\equiv p(a, p(t, t, b), b) \pmod{\beta}, \\ f(t) &\equiv p(t, p(t, b, b), b) = p(t, t, b) \equiv b \pmod{\delta}. \end{aligned}$$

Since $p(t, t, b) \in B$, Theorem B.3.8 implies that $f(a) \not\equiv f(b) \pmod{\beta}$, so $f|_U$ is a permutation of U by the (β, γ)-minimality of U. But then we must have $f(T) = T$, so

$$(b, f(t)) \in \delta \cap (B \times T),$$

which is a contradiction. □

Corollary B.6.9. *If $\mathbb{A}$ is locally finite, then every equivalence class of $\overset{s}{\sim}/\overset{ss}{\sim}$ is a modular sublattice of* $\operatorname{Con}(\mathbb{A})/\overset{ss}{\sim}$.

Definition B.6.10. If $\mathcal{V}$ is a locally finite variety, then we say that $\mathcal{V}$ *omits* type **i** if for every finite $\mathbb{A} \in \mathcal{V}$ and every tame congruence quotient (α, β) of $\mathbb{A}$, the type of (α, β) is not **i**. We write $\operatorname{typ}(\mathcal{V})$ for the set of types which $\mathcal{V}$ does not omit.

Proposition B.6.11. *If $\mathcal{V}$ is a locally finite variety, then the locally solvable algebras in $\mathcal{V}$ form a subvariety $\mathcal{V}_s$, and similarly the locally strongly solvable algebras in $\mathcal{V}$ form a subvariety $\mathcal{V}_{ss}$.*

Corollary B.6.12. *If $\mathcal{V}$ is a locally finite variety which omits type **1**, then the subvariety $\mathcal{V}_s$ of locally solvable algebras in $\mathcal{V}$ has a Mal'cev term.*

Proof. We've already shown that in this case $\mathcal{V}_s$ is congruence modular, so by Corollary A.3.7 applied to the free algebra on two generators in $\mathcal{V}_s$ (which is finite, and therefore solvable), the variety $\mathcal{V}_s$ has a Mal'cev term. □

If we are only studying idempotent algebras, then this result is satisfying - but for general algebras, we need to be able to "restrict to a congruence class" if we want to get the most use out of this. Recall the partial order $\preceq_|$ from Definition B.1.4.

Proposition B.6.13. *If $\mathbb{A}$ is finite and $\mathbb{B} \preceq_| \mathbb{A}$ is such that every constant of $\mathbb{B}$ is a term of $\mathbb{B}$, then* $\operatorname{typ}(\mathcal{V}(\mathbb{B})) \subseteq \operatorname{typ}(\mathcal{V}(\mathbb{A}))$.

Proof. If there is some finite $\mathbb{C} \in \mathcal{V}(\mathbb{B})$ which has a congruence quotient (α, β) of type $\mathbf{i}$, then we pick an (α, β)-trace N and note that $\mathbb{C}|_N/\alpha|_N$ is a permutational algebra (of type $\mathbf{i}$) by Corollary B.1.17. Then by Propositions B.1.5 and B.1.6, we see that there is some finite $\mathbb{A}' \in \mathcal{V}(\mathbb{A})$ such that $\mathbb{C}|_N/\alpha|_N \preceq_| \mathbb{A}'$.

Pick $e \in E(\mathbb{A}'), \theta \in \mathrm{Con}(\mathbb{A}')$, and $a \in e(\mathbb{A}')$ such that the permutational algebra $\mathbb{C}|_N/\alpha|_N$ is polynomially equivalent to $\mathbb{A}'|_{N'}$, where N' is given by

$$N' = e(\mathbb{A}') \cap (a/\theta).$$

Pick $\eta \le \theta$ maximal such that $\eta|_{N'} \ne \theta|_{N'}$, and let $\theta' \le \theta$ be a cover of η. Then (η, θ') is tame by Proposition B.1.8, so to finish we just need to check that N' is an (η, θ')-trace.

Note that $e(a/\theta') \not\subseteq a/\eta$, so by Corollary B.1.20 there must be some (η, θ')-trace $N'' \subseteq a/\theta'$ and some $b, c \in N''$ such that $e(b)/\eta \ne e(c)/\eta$. Then by Corollary B.1.19 $e(N'')$ is also an (η, θ')-trace, and we have

$$e(N'') \subseteq e(\mathbb{A}') \cap (e(a)/\theta') = e(\mathbb{A}') \cap (a/\theta) = N'.$$

Since $e(N'')$ is an (η, θ')-trace contained in a/θ', there is some $e' \in e(\mathbb{A}')$ such that

$$e(N'') = e'(\mathbb{A}') \cap (a/\theta'),$$

and we may assume without loss of generality that $e' = e \circ e'$. But then $e'|_{N'}$ is a polynomial of $\mathbb{A}'|_{N'}$, which is permutational, so in fact we have $e(N'') = N'$, so N' is an (η, θ')-trace. □

Putting this together with the previous results, we can prove the existence of a Mal'cev-like term which behaves nicely on every locally solvable congruence.

Theorem B.6.14. *If $\mathcal{V}$ is a locally finite variety which omits type $\mathbf{1}$, then $\mathcal{V}$ has an idempotent ternary term p such that for any $\mathbb{A} \in \mathcal{V}$ and any $a, b \in \mathbb{A}$,*

$$\mathrm{Cg}_{\mathbb{A}}\{(a,b)\} \overset{s}{\sim} 0_{\mathbb{A}} \quad \implies \quad p(a,b,b) = p(b,b,a) = a.$$

Proof. Let $\mathbb{F} = \mathcal{F}_{\mathcal{V}}(x,y)$ be the (finite) free algebra on two generators in $\mathcal{V}$. Define $\beta \in \mathrm{Con}(\mathbb{F})$ to be the congruence $\mathrm{Cg}_{\mathbb{F}}\{(x,y)\}$, that is, the least congruence which identifies x with y, so that $\mathbb{F}/\beta \cong \mathcal{F}_{\mathcal{V}}(x)$. Then x/β consists of all binary terms $t(x,y)$ of $\mathcal{V}$ which satisfy $t(x,x) \approx x$, that is, x/β corresponds exactly to the set of idempotent binary terms of $\mathcal{V}$. Additionally, let $\alpha \in \mathrm{Con}(\mathbb{F})$ be minimal such that $\alpha \overset{s}{\sim} \beta$.

Taking $N = x/\beta$, we have $\mathbb{F}|_N \preceq_| \mathbb{F}$, so the variety generated by $\mathbb{F}|_N$ omits type $\mathbf{1}$. Additionally, $\mathbb{F}|_N/\alpha|_N$ is solvable, so

$$\mathrm{typ}(\mathcal{V}(\mathbb{F}|_N/\alpha|_N)) = \{\mathbf{2}\}.$$

In particular, we see that $\mathbb{F}|_N/\alpha|_N$ has a ternary Mal'cev term p_0. By the definition of $\mathbb{F}|_N$, p_0 is the restriction to N of some polynomial p_1 of $\mathbb{F}$ which preserves N. Since $\mathbb{F}$ is generated by x and y, we see that there is some 5-ary term t of $\mathcal{V}$ such that

$$p_1(u,v,w) = t(u,v,w,x,y)$$

for all $u, v, w \in \mathbb{F}$. Since p_1 preserves $N = x/\beta$, we have

$$t(x,x,x,x,y) = p_1(x,x,x) \in x/\beta,$$

so t is idempotent. Define an idempotent ternary term p of $\mathcal{V}$ by

$$p(u, v, w) = t(u, v, w, u, w).$$

Then we have

$$p(x, x, y) = t(x, x, y, x, y) = p_1(x, x, y) \equiv_\alpha y$$

and

$$p(x, y, y) = t(x, y, y, x, y) = p_1(x, y, y) \equiv_\alpha x.$$

Now suppose that $\mathbb{A} \in \mathcal{V}$ and $a, b \in \mathbb{A}$ have $\mathrm{Cg}_{\mathbb{A}}\{(a, b)\} \overset{s}{\sim} 0_{\mathbb{A}}$. Let $\pi : \mathbb{F} \to \mathbb{A}$ be the unique map with $\pi(x) = a, \pi(y) = b$. Then

$$\pi^{-1}(\mathrm{Cg}_{\mathbb{A}}\{(a, b)\}) \supseteq \beta,$$

so we have $\beta \overset{s}{\sim} \ker \pi$, which implies that $\alpha \le \ker \pi$ by our choice of α. In particular, we have

$$p(x, x, y) \equiv_{\ker \pi} y, \quad p(x, y, y) \equiv_{\ker \pi} x,$$

so $p(a, a, b) = b$ and $p(a, b, b) = a$. Interchanging a and b in the argument gives $p(b, b, a) = a$ as well, so we are done. □

Definition B.6.15. An idempotent ternary term p is called a *weak difference term* for $\mathcal{V}$ if for any $\mathbb{A} \in \mathcal{V}$, any $a, b \in \mathbb{A}$, and any $\theta \in \mathrm{Con}(\mathbb{A})$ with $(a, b) \in \theta$, we have

$$p(a, b, b) \equiv_{[\theta,\theta]} p(b, b, a) \equiv_{[\theta,\theta]} a.$$

Corollary B.6.16. *A locally finite variety omits type **1** iff it has a weak difference term. In particular, every locally finite Taylor variety has a weak difference term.*

Proof. We always have $\theta \overset{s}{\sim} [\theta, \theta]$, so any term p as in Theorem B.6.14 is automatically a weak difference term. Conversely, we need to show that if $\mathcal{V}$ has a weak difference term p, then $\mathcal{V}$ omits type **1**.

Suppose for contradiction that $\mathbb{A} \in \mathcal{V}$ has a tame congruence quotient (α, β) of type **1**. We may assume without loss of generality that $\alpha = 0_{\mathbb{A}}$, in which case Theorem B.4.2 implies that $[\beta, \beta] = 0_{\mathbb{A}}$. Letting $U = e(\mathbb{A})$ (with $e \in E(\mathbb{A})$) be a $(0_{\mathbb{A}}, \beta)$-minimal set, we see that $e \circ p$ restricts to a Mal'cev operation on any $(0_{\mathbb{A}}, \beta)$-trace N, contradicting the assumption that $\mathbb{A}|_N$ is a unary algebra. □

Corollary B.6.17. *If $\mathcal{V}$ is a locally finite variety which omits type **1**, $\mathbb{A} \in \mathcal{V}$, and $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$, then*

$$\alpha \overset{s}{\sim} \beta \quad \Longrightarrow \quad \alpha \vee \beta = \alpha \circ \beta = \beta \circ \alpha.$$

Proof. By symmetry, we just need to check that $\alpha \circ \beta \subseteq \beta \circ \alpha$. Pick p as in Theorem B.6.14, and suppose that $x, y, z \in \mathbb{A}$ satisfy

$$x \ \alpha \ y \ \beta \ z.$$

Then we have

$$x \ \beta \ p(x, y, z) \ \alpha \ z,$$

where

$$p(x, y, z)/\beta = p(x, y, y)/\beta = x/\beta$$

follows from $(\alpha \vee \beta)/\beta \overset{s}{\sim} 0_{\mathbb{A}/\beta}$, and

$$p(x, y, z)/\alpha = p(x, x, z)/\alpha = z/\alpha$$

follows from $(\alpha \vee \beta)/\alpha \overset{s}{\sim} 0_{\mathbb{A}/\alpha}$. □

Corollary B.6.18. *If $\mathcal{V}$ is a locally finite variety which omits type **1**, $\mathbb{A} \in \mathcal{V}$, and $\alpha, \beta \in \mathrm{Con}(\mathbb{A})$, then*

$$\beta \overset{s}{\sim} \alpha \wedge \beta \quad \implies \quad \alpha \vee \beta = \alpha \circ \beta \circ \alpha.$$

Proof. Note that the assumption is equivalent to $\alpha \vee \beta \overset{s}{\sim} \alpha$, or equivalently $(\alpha \vee \beta)/\alpha \overset{s}{\sim} 0_{\mathbb{A}/\alpha}$. We just need to check that $\beta \circ \alpha \circ \beta \subseteq \alpha \circ \beta \circ \alpha$. Pick p as in Theorem B.6.14, and suppose that $w, x, y, z \in \mathbb{A}$ satisfy

$$w \ \beta \ x \ \alpha \ y \ \beta \ z.$$

Then we have

$$w \ \alpha \ p(w, y, y) \ \beta \ p(x, y, z) \ \alpha \ z,$$

with the α congruences following as in the previous corollary, while the β congruence follows directly from $w \equiv_\beta x$ and $y \equiv_\beta z$. □

B.7 Pseudocomplements and semidistributivity

In this section we start investigating the consequences of avoiding the abelian types on the congruence lattices of finite algebras. We start with some lattice-theoretic preliminaries.

Definition B.7.1. If $\mathcal{L}$ is a lattice and $\alpha \le \beta \in \mathcal{L}$, then we say that δ is the *weak pseudocomplement of β over α* if δ is the greatest element of $\mathcal{L}$ such that $\beta \wedge \delta = \alpha$, that is, if

$$\beta \wedge \gamma = \alpha \quad \iff \quad \gamma \in [\![\alpha, \delta]\!].$$

A closely related concept is the relative pseudocomplement: for any $\alpha, \beta \in \mathcal{L}$, δ is called the *relative pseudocomplement* of β with respect to α if

$$\beta \wedge \gamma \le \alpha \quad \iff \quad \gamma \le \delta,$$

and this is written in symbols as $\delta = \beta \to \alpha$ or $\delta = \beta \supset \alpha$. If $\alpha = 0$, then a weak or relative pseudocomplement δ of β over 0 is just called a *pseudocomplement* of β, and written in symbols as $\delta = \neg\beta$ or $\delta = \beta^*$.

Similarly, for $\alpha \le \beta \in \mathcal{L}$ we say that δ is the *dual weak pseudocomplement of α under β* if

$$\alpha \vee \gamma = \beta \quad \iff \quad \gamma \in [\![\delta, \beta]\!].$$

Additionally, for any $\alpha, \beta \in \mathcal{L}$, δ is called the *dual relative pseudocomplement* of α with respect to β if

$$\alpha \vee \gamma \ge \beta \quad \iff \quad \gamma \ge \delta,$$

and some authors write this in symbols as $\delta = \beta - \alpha$ or $\delta = \beta \backslash \alpha$.

Of course, pseudocomplements don't always exist - for instance, the diamond lattice $\mathcal{M}_3$ is not pseudocomplemented. Note that there can be a weak pseudocomplement of β over α even if there is no relative pseudocomplement of β with respect to α - this situation occurs in the pentagon lattice $\mathcal{N}_5$. We at least have the following implication between the two concepts.

Proposition B.7.2. *If $\alpha \le \beta \in \mathcal{L}$ and the relative pseudocomplement of β with respect to α exists and is equal to δ, then δ is also the weak pseudocomplement of β over α.*

To put these concepts in context, we recall the definition of a Heyting algebra, from intuitionistic logic.

Definition B.7.3. A *Heyting algebra* is an algebraic structure $\mathcal{H} = (H, \wedge, \vee, \supset, 0, 1)$ such that $(H, \wedge, \vee, 0, 1)$ is a $0,1$-lattice and for every pair of elements $\alpha, \beta \in H$, $\beta \supset \alpha$ is the relative pseudocomplement of β with respect to α.

Proposition B.7.4. *A complete lattice is the lattice reduct of a Heyting algebra iff it satisfies the infinite distributive law*

$$\alpha \wedge \Big(\bigvee_{\beta \in S} \beta\Big) = \bigvee_{\beta \in S} (\alpha \wedge \beta). \qquad (\mathrm{D}_\infty(\wedge))$$

Proof. First we check that any complete lattice $\mathcal{L}$ which satisfies the infinite distributive law $(\mathrm{D}_\infty(\wedge))$ can be expanded to a Heyting algebra. For $\alpha, \beta \in \mathcal{L}$, the least possible value for the relative pseudocomplement $\beta \supset \alpha$ is given by

$$\beta \supset \alpha = \bigvee_{\beta \wedge \gamma \le \alpha} \gamma.$$

To check that this definition works, we just need to check that it actually satisfies $\beta \wedge (\beta \supset \alpha) \le \alpha$, which follows from

$$\beta \wedge \Big(\bigvee_{\beta \wedge \gamma \le \alpha} \gamma\Big) = \bigvee_{\beta \wedge \gamma \le \alpha} (\beta \wedge \gamma) \le \bigvee_{\beta \wedge \gamma \le \alpha} \alpha = \alpha,$$

where the first equality is a special case of $(\mathrm{D}_\infty(\wedge))$.

Conversely, we need to check that any complete Heyting algebra satisfies the infinite distributive law $(\mathrm{D}_\infty(\wedge))$. For this, we argue as follows:

$$\begin{aligned}
\alpha \wedge \Big(\bigvee_{\beta \in S} \beta\Big) \le \gamma &\iff \bigvee_{\beta \in S} \beta \le \alpha \supset \gamma \\
&\iff \forall \beta \in S,\ \beta \le \alpha \supset \gamma \\
&\iff \forall \beta \in S,\ \alpha \wedge \beta \le \gamma \\
&\iff \bigvee_{\beta \in S} (\alpha \wedge \beta) \le \gamma.
\end{aligned}$$

□

Now we compare this to the relationship between weak pseudocomplements and semidistributivity.

Definition B.7.5. A lattice $\mathcal{L}$ is *meet-semidistributive*, written $\mathrm{SD}(\wedge)$, if for all $\alpha, \beta, \gamma \in \mathcal{L}$ we have

$$\alpha \wedge \beta = \alpha \wedge \gamma \quad \implies \quad \alpha \wedge (\beta \vee \gamma) = \alpha \wedge \beta.$$

Similarly, a lattice $\mathcal{L}$ is *join-semidistributive*, written SD($\vee$), if for all $\alpha, \beta, \gamma \in \mathcal{L}$ we have

$$\alpha \vee \beta = \alpha \vee \gamma \quad \Longrightarrow \quad \alpha \vee (\beta \wedge \gamma) = \alpha \vee \beta.$$

A lattice is called *semidistributive* if it is both meet-semidistributive and join-semidistributive.

Recall from Definition A.5.5 that a lattice is called *algebraic* if it is complete and every element can be written as a join of compact elements.

Proposition B.7.6 ([55])**.** *An algebraic lattice $\mathcal{L}$ is meet-semidistributive iff for all $\alpha \le \beta \in \mathcal{L}$, there is a weak pseudocomplement of β over α. In this case, $\mathcal{L}$ also satisfies the following infinite form of meet-semidistributivity:*

$$\forall i, j\ \alpha \wedge \beta_i = \alpha \wedge \beta_j \implies \forall i\ \alpha \wedge \Big(\bigvee_j \beta_j\Big) = \alpha \wedge \beta_i. \qquad (\mathrm{SD}_\infty(\wedge))$$

Proof. First we prove that every meet-semidistributive lattice has weak pseudocomplements. Note that the sublattice $[\![\alpha, 1]\!]$ of elements of $\mathcal{L}$ which are above α also forms an algebraic lattice: if θ is compact in $\mathcal{L}$, then $\alpha \vee \theta$ is compact as an element of $[\![\alpha, 1]\!]$. Thus we may assume without loss of generality that $\alpha = 0$, in which case we just need to prove that every element β has a pseudocomplement $\neg\beta$.

If $\alpha = 0$, then the least possible value for $\neg\beta$ is given by

$$\neg\beta = \bigvee_{\beta\wedge\gamma=0} \gamma.$$

Note that meet-semidistributivity implies that every join of finitely many elements γ_i satisfying $\beta \wedge \gamma_i = 0$ will satisfy

$$\beta \wedge \Big(\bigvee_{i\le n} \gamma_i\Big) = 0.$$

We reduce the infinite case to the finite case by using the algebraicity of the lattice $\mathcal{L}$. Suppose for the sake of contradiction that $\beta \wedge (\neg\beta) \ne 0$, then since $\mathcal{L}$ is algebraic there is some nonzero compact element θ of $\mathcal{L}$ such that

$$\theta \le \beta \wedge (\neg\beta) \le \bigvee_{\beta\wedge\gamma=0} \gamma.$$

Since θ is compact, there is a finite collection of γ_i satisfying $\beta \wedge \gamma_i = 0$ such that $\theta \le \bigvee_{i\le n} \gamma_i$. But then we have

$$\theta \le \beta \wedge \Big(\bigvee_{i\le n} \gamma_i\Big) = 0,$$

contradicting the assumption that θ is nonzero.

Now suppose that for all $\alpha \le \beta \in \mathcal{L}$ there is a weak pseudocomplement of β over α. We will prove the infinite form of the meet-semidistributivity property. Suppose that there is a family β_i such that

$$\alpha \wedge \beta_i = \gamma$$

for all i. Let δ be a weak pseudocomplement of α over γ. Then by the definition of a weak pseudocomplement, we have $\beta_i \in [\![\gamma, \delta]\!]$ for all i, so

$$\bigvee_i \beta_i \in [\![\gamma, \delta]\!],$$

which in turn implies that

$$\alpha \wedge \Big(\bigvee_i \beta_i\Big) = \gamma. \qquad \square$$

Remark B.7.1. A similar argument can be used to show that an algebraic lattice satisfies the finite distributive law if and only if it satisfies the infinite distributive law ($\mathrm{D}_\infty(\wedge)$). In particular, if a variety $\mathcal{V}$ is congruence distributive, then for every $\mathbb{A} \in \mathcal{V}$ the congruence lattice $\mathrm{Con}(\mathbb{A})$ forms a Heyting algebra.

For lattices of finite length, we can show that meet-semidistributivity is a consequence of the existence of weak pseudocomplements for covers $\alpha \prec \beta$.

Proposition B.7.7. *If $\alpha \wedge \beta = \alpha \wedge \gamma = \delta$ but $\alpha \wedge (\beta \vee \gamma) \ne \delta$, then for any ϵ such that*

$$\delta \prec \epsilon \le \alpha \wedge (\beta \vee \gamma),$$

there is no weak pseudocomplement of ϵ over δ.

Proof. Suppose for the sake of contradiction that there was some weak pseudocomplement θ of ϵ over δ. Then from

$$\delta \le \epsilon \wedge \beta \le \alpha \wedge \beta = \delta$$

we see that $\beta \le \theta$, and similarly $\gamma \le \theta$. But then $\beta \vee \gamma \le \theta$, so we have

$$\epsilon \le \epsilon \wedge \alpha \wedge (\beta \vee \gamma) = \epsilon \wedge (\beta \vee \gamma) = \delta,$$

contradicting the assumption $\delta \prec \epsilon$. $\square$

With the lattice-theoretic preliminaries out of the way, our task is now to show that weak pseudocomplements exist when we avoid the abelian types. We will use the concept of the relative centralizer $(\alpha : \beta)$ from Definition 1.9.35.

Proposition B.7.8. *If (α, β) is a nonabelian prime congruence quotient on $\mathbb{A}$, then the relative centralizer $(\alpha : \beta)$ is the weak pseudocomplement of β over α in* $\mathrm{Con}(\mathbb{A})$. *In particular, in this case the weak pseudocomplement of β over α exists.*

More generally, if $\alpha \le \beta$ then $(\alpha : \beta)$ is the weak pseudocomplement of β over α if and only if

$$\beta \wedge (\alpha : \beta) = \alpha,$$

and this occurs if the lattice $[\![\alpha, \beta]\!]$ is atomic and every prime congruence quotient (α, δ) with $\alpha \prec \delta \le \beta$ is nonabelian.

Proof. We may assume without loss of generality that $\alpha = 0_{\mathbb{A}}$, so we just have to prove that if $\beta \in \mathrm{Con}(\mathbb{A})$ is a nonabelian atomic congruence, then the centralizer $(0_{\mathbb{A}} : \beta)$ is the pseudocomplement of β. We just need to check that

$$C(\gamma, \beta; 0_{\mathbb{A}}) \quad \iff \quad \beta \wedge \gamma = 0_{\mathbb{A}}.$$

By Proposition 1.9.30(b) we see that

$$\beta \wedge \gamma = 0_{\mathbb{A}} \quad \implies \quad C(\gamma, \beta; 0_{\mathbb{A}})$$

without any assumptions on β. For the other direction, since β is an atom we have

$$\beta \wedge \gamma \neq 0_{\mathbb{A}} \quad \iff \quad \gamma \geq \beta,$$

and by Proposition 1.9.30(c) we see that if β is nonabelian and $\gamma \geq \beta$ then $C(\gamma, \beta; 0_{\mathbb{A}})$ can't be true.

For the more general statement, note that by the argument above every γ which satisfies $\beta \wedge \gamma = \alpha$ also satisfies $\gamma \leq (\alpha : \beta)$. If $\beta \wedge (\alpha : \beta) \neq \alpha$, then picking any δ with $\alpha \prec \delta \leq \beta \wedge (\alpha : \beta)$ we see that $C(\delta, \beta; \alpha)$ holds, so δ is abelian over α by Proposition 1.9.30(c). □

In [95], the following alternative tame congruence theoretic characterization of the weak pseudocomplement of β over α is given, based on Proposition B.3.3.

Proposition B.7.9. *Suppose (α, β) is a nonabelian prime congruence quotient on a finite algebra $\mathbb{A}$. Let $U \in M_{\mathbb{A}}(\alpha, \beta)$ be any (α, β)-minimal set, and as in Proposition B.3.3 let $a \in U$ be an element of the unique (α, β)-trace N, such that there is a partial semilattice polynomial $s \in \mathrm{Pol}(\mathbb{A}|_U)$ with $s(a, x) = x$ for all $x \in U$.*

Then the weak pseudocomplement of β over α is equal to the largest congruence $\delta \in \mathrm{Con}(\mathbb{A})$ such that $\delta|_U$ has $\{a\}$ as a congruence class (and this δ exists).

Proof. To see that such a δ exists, we first let δ' be the largest congruence on $\mathbb{A}|_U$ with $\{a\}$ as a congruence class, and then we apply Lemma B.1.3 to see that the restriction map $\theta \mapsto \theta|_U$ is a surjective homomorphism from $\mathrm{Con}(\mathbb{A})$ to $\mathrm{Con}(\mathbb{A}|_U)$, and we take δ to be the join of all preimages of δ' under this map.

To see that δ is the weak pseudocomplement of β over α, first we note that Proposition B.3.4 implies that N is the unique (α, β)-trace contained in U, so since $\{a\}$ is a congruence class of $\delta|_U$ we must have $(\beta \wedge \delta)|_U \subseteq \alpha|_U$. Additionally, since $\{a\}$ is a congruence class of $\alpha|_U$ we must have $\alpha \leq \beta \wedge \delta$. Then since the restriction map $[\![\alpha, \beta]\!] \twoheadrightarrow [\![\alpha|_U, \beta|_U]\!]$ is $0,1$-separating we see that we must in fact have $\beta \wedge \delta = \alpha$.

Additionally, for any $\alpha \leq \gamma \not\leq \delta$, $\{a\}$ is not a congruence class of $\gamma|_U$, so there is some $c \in U \setminus \{a\}$ such that $(a, c) \in \gamma$. If $c \in N \setminus \{a\}$, then $(a, c) \in \beta \setminus \alpha$, so $\beta \wedge \gamma \neq \alpha$. Otherwise, let b be any element of $N \setminus \{a\}$, so we have $N/\alpha|_N = \{a, b\}/\alpha|_N$. Then we have

$$c = s(a, c) \equiv_{\beta|_U} s(b, c),$$

and since $c/\beta|_U = c/\alpha|_U$ (since $c \notin N$ and N is the unique (α, β)-trace contained in U), we have

$$a \equiv_\gamma c \equiv_{\alpha|_U} s(b, c) \equiv_\gamma s(b, a) = b.$$

Thus, in this case we have $(a, b) \in (\beta \wedge \gamma) \setminus \alpha$, so $\beta \wedge \gamma \neq \alpha$. Either way, $\gamma \not\leq \delta$ implies $\beta \wedge \gamma \neq \alpha$. □

There is a corresponding result for dual weak pseudocomplements, as long as we exclude both the abelian types and the semilattice type.

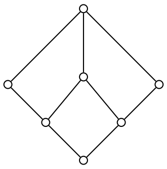
Proposition B.7.10. *Suppose (α, β) is a prime congruence quotient of type **3** or **4** (i.e. boolean or lattice type) on a finite algebra $\mathbb{A}$, and let N be any (α, β)-trace. Then the dual weak pseudocomplement of α under β exists and is equal to $\mathrm{Cg}_{\mathbb{A}}(N^2)$.*

Proof. By Proposition B.3.3, $|N| = 2$, and by Lemma B.1.3, the restriction map $[\![0_{\mathbb{A}}, \beta]\!] \to \mathrm{Con}(\mathbb{A}|_N)$ is a surjective lattice homomorphism. Since $|N| = 2$ we have $\alpha|_N = 0_{\mathbb{A}|_N}$, so we have

$$\begin{aligned} \alpha \vee \gamma = \beta \quad &\Longrightarrow \quad \gamma \in [\![0_{\mathbb{A}}, \beta]\!] \text{ and } 0_{\mathbb{A}|_N} \vee \gamma|_N = 1_{\mathbb{A}|_N} \\ &\Longrightarrow \quad \gamma \le \beta \text{ and } N^2 \subseteq \gamma \\ &\Longrightarrow \quad \gamma \le \beta \text{ and } \gamma \not\le \alpha \\ &\Longrightarrow \quad \alpha \vee \gamma = \beta, \end{aligned}$$

where the last implication follows from the fact that (α, β) is a prime congruence quotient. □

The fact that we had to exclude type **5** from the last result isn't just an artifact of the proof: if $\mathbb{A} = (\{0,1\}, \vee)$ is a two-element semilattice, then $\mathrm{Con}(\mathbb{A}^2)$ is depicted in Example 2.2.3, and we can see that the congruence $\Theta = \mathrm{Cg}_{\mathbb{A}^2}\{((0,1),(1,0))\}$ has no dual weak pseudocomplement under $1_{\mathbb{A}^2}$. As an abstract lattice, $\mathrm{Con}(\mathbb{A}^2)$ is isomorphic to the lattice pictured below, which is called $\mathcal{D}_2$.



The occurence of the lattice $\mathcal{D}_2$ in $\mathrm{Con}(\mathbb{A}^2)$ is not restricted to this particular example - the next result from [95] shows that something like this occurs whenever we have a prime congruence quotient of type **5**.

Proposition B.7.11 (Theorem 5.27 of [95])**.** *Suppose (α, β) is a nonabelian prime quotient on a finite algebra $\mathbb{A}$ and let $\mathbb{R} \le \mathbb{A}^2$ be the basic tolerance for (α, β). Consider the sublattice*

$$\mathcal{L} = [\![(\alpha \times \alpha)|_{\mathbb{R}}, (\beta \times \beta)|_{\mathbb{R}}]\!]$$

*of $\mathrm{Con}(\mathbb{R})$. If (α, β) has type **3** or **4** then $\mathcal{L}$ is isomorphic to the four-element diamond lattice $\mathcal{M}_2$, and if (α, β) has type **5** then $\mathcal{L}$ is isomorphic to the lattice $\mathcal{D}_2$ depicted above.*

Proof. We can assume without loss of generality that $\alpha = 0_{\mathbb{A}}$. Let N be a $(0_{\mathbb{A}}, \beta)$-trace, then $|N| = 2$ and $\mathbb{A}|_N$ is polynomially equivalent to either a boolean algebra, a lattice, or a semilattice according to the type of $(0_{\mathbb{A}}, \beta)$. Suppose $N = \{a, b\}$ and pick $e \in E(\mathbb{A})$ such that

$$N = e(\mathbb{A}) \cap a/\beta.$$

Additionally, if $(0_{\mathbb{A}}, \beta)$ has type **5** then assume that a is the neutral element of $\mathbb{A}|_N$ and that b is the absorbing element.

First we check that each congruence on $(\mathbb{A}|_N)^2$ extends to a congruence on $\mathbb{R}$ which is contained in $(\beta \times \beta)|_{\mathbb{R}}$. By Theorem B.5.3, we have

$$\mathbb{R} = \mathrm{Sg}_{\mathbb{A}^2}(\Delta_{\mathbb{A}} \cup N^2).$$

Defining the unary polynomial $e^{(2)}$ on $\mathbb{R}$ as in the proof of Proposition B.1.6, we see that

$$N^2 = e^{(2)}(\mathbb{R}) \cap (a,a)/(\beta \times \beta)|_{\mathbb{R}},$$

and $(\mathbb{A}|_N)^2$ is polynomially equivalent to $\mathbb{R}|_{N^2}$ by the argument of Proposition B.1.6, so Lemma B.1.3 shows that restriction to N^2 defines a surjective lattice homomorphism from $[\![0_{\mathbb{R}}, (\beta \times \beta)|_{\mathbb{R}}]\!]$ to $\mathrm{Con}((\mathbb{A}|_N)^2)$.

The main difficulty is to check that every congruence θ on $\mathbb{R}$ which is contained in $(\beta \times \beta)|_{\mathbb{R}}$ is equal to $\mathrm{Cg}_{\mathbb{R}}(\theta|_{N^2})$ - this requires some tedious casework. It's helpful to note that since the transitive closure of the tolerance $\mathbb{R}$ is β, the congruence $(\beta \times \beta)|_{\mathbb{R}}$ is actually the linking congruence of $\mathbb{R} \le_{sd} \mathbb{A} \times \mathbb{A}$. In other words, we have

$$(\beta \times \beta)|_{\mathbb{R}} = \ker \pi_1 \vee \ker \pi_2.$$

First we will show that the containment

$$\mathrm{Cg}_{\mathbb{R}}(\ker \pi_1|_{N^2}) \subseteq \ker \pi_1 = (0_{\mathbb{A}} \times 1_{\mathbb{A}})|_{\mathbb{R}}$$

is an equality. Consider any $(c,d) \in \mathbb{R}$. Since $\mathbb{R}$ is generated by $\Delta_{\mathbb{A}} \cup N^2$, there is some binary polynomial $p \in \mathrm{Pol}_2(\mathbb{A})$ such that $p(a,b) = c, p(b,a) = d$. Then we have

$$\begin{bmatrix} c \\ d \end{bmatrix} = p\Big(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix} \Big) \equiv p\Big(\begin{bmatrix} a \\ a \end{bmatrix}, \begin{bmatrix} b \\ b \end{bmatrix} \Big) = \begin{bmatrix} c \\ c \end{bmatrix} \pmod{\mathrm{Cg}_{\mathbb{R}}(\ker \pi_1|_{N^2})}.$$

Since this is true for any $(c,d) \in \mathbb{R}$, we see that $\mathrm{Cg}_{\mathbb{R}}(\ker \pi_1|_{N^2}) = \ker \pi_1$.

Now for any $\theta \le (\beta \times \beta)|_{\mathbb{R}}$, if $\pi_1(\theta) \ne \beta$ then $\theta \subseteq \ker \pi_1$ since β is atomic. If $\pi_1(\theta) = \beta$, then $(a,b) \in \pi_1(e^{(2)}(\theta)) = \pi_1(\theta|_{N^2})$, so we have the implication

$$\theta|_{N^2} \subseteq \ker \pi_1|_{N^2} \quad \implies \quad \theta \subseteq \ker \pi_1 = \mathrm{Cg}_{\mathbb{R}}(\ker \pi_1|_{N^2}).$$

Together with $\mathrm{Cg}_{\mathbb{R}}(N^2) \supseteq \ker \pi_1 \vee \ker \pi_2$, this shows that $\theta = \mathrm{Cg}_{\mathbb{R}}(\theta|_{N^2})$ if $\theta|_{N^2}$ is one of $0_N \times 0_N, 0_N \times 1_N, 1_N \times 0_N, 1_N \times 1_N$. This handles the cases where $(0_{\mathbb{A}}, \beta)$ has type **3** or **4** (lattices and boolean algebras are congruence distributive, so congruences on $(\mathbb{A}|_N)^2$ are determined by their first and second projections in these cases), so from here on we may assume that $(0_{\mathbb{A}}, \beta)$ has type **5** (i.e. semilattice type).

By the analysis of the congruences on the square of the two-element semilattice from Example 2.2.3, we have two remaining cases: either $\theta|_{N^2}$ is the congruence generated by $((a,b),(b,a))$, or (possibly after swapping coordinates) $\theta|_{N^2}$ is the congruence generated by $((b,a),(b,b))$. In the first case, $\theta|_{N^2}$ contains both $((b,a),(b,b))$ and $((a,b),(b,b))$, so for any $(c,d) \in \mathbb{R}$, if we choose the binary polynomial p satisfying $p(a,b) = c, p(b,a) = d$ as before, we get

$$\begin{bmatrix} c \\ d \end{bmatrix} = p\Big(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix} \Big) \equiv p\Big(\begin{bmatrix} b \\ b \end{bmatrix}, \begin{bmatrix} b \\ b \end{bmatrix} \Big) \in \Delta_{\mathbb{A}} \pmod{\mathrm{Cg}_{\mathbb{R}}(\theta|_{N^2})}.$$

Thus in this case, every element of $\mathbb{R}$ is congruent modulo θ to a diagonal element, so θ is determined by its restriction to $\Delta_{\mathbb{A}} \cong \mathbb{A}$. Since $\theta|_{\Delta_{\mathbb{A}}} \subseteq (\beta \times \beta)|_{\Delta_{\mathbb{A}}}$ and (a, a) is not congruent to (b, b) modulo θ, we see that $\theta|_{\Delta_{\mathbb{A}}}$ is trivial, so every pair of elements of $\mathbb{R}$ which are congruent modulo θ are congruent to the same diagonal element of $\Delta_{\mathbb{A}}$ via the congruence $\mathrm{Cg}_{\mathbb{R}}\{((a, b), (b, a))\}$.

To finish the proof, we consider the case where $\theta|_{N^2}$ is the congruence generated by $((b, a), (b, b))$, so $\theta \subsetneq \ker \pi_1$. Suppose that the pairs $(c, d_1), (c, d_2) \in \mathbb{R}$ are congruent modulo θ, and choose binary polynomials p_1, p_2 such that $p_i(a, b) = c$ and $p_i(b, a) = d_i$. Then we have

$$\begin{bmatrix} c \\ d_i \end{bmatrix} = p_i\Big(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix} \Big) \equiv p_i\Big(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ b \end{bmatrix} \Big) = \begin{bmatrix} c \\ p_i(b, b) \end{bmatrix} \pmod{\mathrm{Cg}_{\mathbb{R}}(\theta|_{N^2})}.$$

We claim that $p_1(b, b) = p_2(b, b)$. Suppose not, for the sake of contradiction. Since $p_1(b, b) \neq p_2(b, b) \in c/\beta$, we can apply Theorem B.1.14(c) to see that there is some unary $f \in \mathrm{Pol}_1(\mathbb{A})$ such that $f(p_1(b, b)) \neq f(p_2(b, b))$ and $f(c/\beta) = N$. Suppose without loss of generality that $f(p_1(b, b)) = a$, and note that since (a, a) is not congruent to (a, b) modulo θ we must have $f(c) = b$. Then the unary polynomial $g(x) = f(p_1(x, b))$ preserves N and satisfies

$$g(b) = f(p_1(b, b)) = a, \quad g(a) = f(p_1(a, b)) = f(c) = b.$$

Then $g|_N$ is not monotone, which contradicts the assumption that $\mathbb{A}|_N$ is polynomially equivalent to a semilattice, so we must have had $p_1(b, b) = p_2(b, b)$ after all. Therefore (c, d_1) is congruent to (c, d_2) modulo $\mathrm{Cg}_{\mathbb{R}}(\theta|_{N^2})$, and since this is true for any $(c, d_1), (c, d_2)$ which are congruent modulo θ, we are done. □

The fact that prime congruences of types **3** and **4** have dual weak pseudocomplements has a nice concrete consequence.

Proposition B.7.12. *If $\mathbb{B}$ is a finite simple algebra of boolean or lattice type (i.e., if $(0_{\mathbb{B}}, 1_{\mathbb{B}})$ has type **3** or **4**), then for any finite collection of finite algebras $\mathbb{A}_i$, if $\mathbb{B} \in \mathcal{V}(\mathbb{A}_1, ..., \mathbb{A}_n)$ then $\mathbb{B} \in HS(\mathbb{A}_i)$ for some i.*

Proof. Since $\mathbb{B}, \mathbb{A}_i$ are finite, if $\mathbb{B} \in \mathcal{V}(\mathbb{A}_1, ..., \mathbb{A}_n)$ then $\mathbb{B} \in HSP_{fin}(\mathbb{A}_1, ..., \mathbb{A}_n)$, so there is some $\mathbb{R} \le \prod_i \mathbb{A}_i^{k_i}$ and some congruence $\theta \in \mathrm{Con}(\mathbb{R})$ such that $\mathbb{B} \cong \mathbb{R}/\theta$. Assume for simplicity that the k_i are all 1, by repeating some of the $\mathbb{A}_i$s if necessary. We just need to prove that $\ker \pi_i \le \theta$ for some i to complete the proof, since then $\mathbb{B}$ will be isomorphic to a quotient of $\pi_i(\mathbb{R}) \le \mathbb{A}_i$.

By Proposition B.1.23, the prime quotient $(\theta, 1_{\mathbb{R}})$ has the same type as $(0_{\mathbb{B}}, 1_{\mathbb{B}})$, so by Proposition B.7.10 we see that θ has a dual weak pseudocomplement δ under $1_{\mathbb{R}}$. If every i has $\ker \pi_i \not\le \theta$, then each i has $\theta \vee \ker \pi_1 = 1_{\mathbb{R}}$, in which case we must have $\delta \le \ker \pi_i$ for all i. But then we have $\delta = 0_{\mathbb{R}}$, which contradicts $\theta \vee \delta = 1_{\mathbb{R}}$. □

Even though prime congruences of type **5** might not have dual weak pseudocomplements in general, the fact that they always have weak pseudocomplements can be use to prove that they have dual weak pseudocomplements in some special cases.

Proposition B.7.13. *If (α, β) is a nonabelian prime quotient on a finite algebra $\mathbb{A}$, and if there is some γ such that $\alpha \vee \gamma = \beta$ and $\alpha \wedge \gamma = 0_{\mathbb{A}}$, then α has a dual weak pseudocomplement under β.*

More generally, any nonabelian prime quotient (α, β) of $\mathbb{A}$ has the following property: for all γ such that $\alpha \vee \gamma = \beta$, there is a least δ such that $\alpha \vee \delta = \beta$ and $\alpha \wedge \gamma \le \delta$.

Proof. The more general statement follows from the first statement by replacing $\mathbb{A}$ by $\mathbb{A}/(\alpha \wedge \gamma)$, so suppose that $\alpha \vee \gamma = \beta$ and $\alpha \wedge \gamma = 0_{\mathbb{A}}$.

Let δ be any atom of the lattice $[\![0_{\mathbb{A}}, \gamma]\!]$. Then we have

$$\alpha \wedge \delta \le \alpha \wedge \gamma = 0_{\mathbb{A}}$$

and

$$\delta \not\le \alpha, \quad \delta \le \gamma \le \beta \quad \implies \quad \alpha \vee \delta = \beta.$$

Thus the prime congruence quotient $(0_{\mathbb{A}}, \delta)$ is perspective to (α, β), so it must be nonabelian since $\overset{s}{\sim}$ is a congruence on $\mathrm{Con}(\mathbb{A})$ by Theorem B.6.5 (in fact $(0_{\mathbb{A}}, \delta)$ has the same type as (α, β) by Proposition B.1.25).

By Proposition B.7.8, δ has a pseudocomplement $(0_{\mathbb{A}} : \delta)$. Then for any θ such that $\delta \not\le \theta$ we have

$$\delta \wedge \theta = 0_{\mathbb{A}},$$

and together with $\alpha \wedge \delta = 0_{\mathbb{A}}$ we see that

$$\alpha \vee \theta \le (0_{\mathbb{A}} : \delta).$$

Since $\delta \le \beta$ we have $\beta \not\le (0_{\mathbb{A}} : \delta)$, so we have proven that

$$\delta \not\le \theta \quad \implies \quad \alpha \vee \theta \ne \beta.$$

Thus δ is the dual weak pseudocomplement of α under β. □

Putting together the results we have shown so far, we can give a sufficient condition for intervals in $\mathrm{Con}(\mathbb{A})$ to be congruence semidistributive.

Proposition B.7.14. *If $\mathbb{A}$ is a finite algebra and $\alpha \le \beta \in \mathrm{Con}(\mathbb{A})$, then*

- *if no prime congruence quotient (γ, δ) with $\alpha \le \gamma \prec \delta \le \beta$ has type **1** or **2**, then $[\![\alpha, \beta]\!]$ is meet-semidistributive, and*

- *if no prime congruence quotient (γ, δ) with $\alpha \le \gamma \prec \delta \le \beta$ has type **1**, **2**, or **5**, then $[\![\alpha, \beta]\!]$ is join-semidistributive.*

Proof. This follows from Proposition B.7.7, Proposition B.7.8, and Proposition B.7.10. □

We can prove much stronger results by making use of the congruence $\overset{s}{\sim}$ on $\mathrm{Con}(\mathbb{A})$.

Theorem B.7.15. *If $\mathbb{A}$ is locally finite, then $\mathrm{Con}(\mathbb{A})/\overset{s}{\sim}$ satisfies the infinite meet-semidistributivity law* (SD$_\infty(\wedge)$).

Proof. Suppose that $\alpha, \beta_i \in \mathrm{Con}(\mathbb{A})$ satisfy $\alpha \wedge \beta_i \overset{s}{\sim} \alpha \wedge \beta_j$ for all i, j. By Proposition B.6.7, we may assume without loss of generality that each β_i is the join of all the elements of its $\overset{s}{\sim}$-class, in which case we must actually have

$$\alpha \wedge \beta_i = \alpha \wedge \beta_j$$

for all i, j. Let δ be the common value of $\alpha \wedge \beta_i$. By Proposition 1.9.30(b), we have

$$\alpha \wedge \beta_i = \delta \quad \implies \quad C(\beta_i, \alpha; \delta)$$

for all i, so by Proposition 1.9.30(e) we have

$$C\big(\bigvee_i \beta_i, \alpha; \delta\big).$$

Then by Proposition 1.9.30(c) $\alpha \wedge \left(\bigvee_i \beta_i\right)$ is abelian over δ, which implies $\alpha \wedge \left(\bigvee_i \beta_i\right) \overset{s}{\sim} \delta$. □

Theorem B.7.16. *If $\mathbb{A}$ is finite and a convex sublattice $\mathcal{L} \le \mathrm{Con}(\mathbb{A})$ contains no prime congruence quotients of type **5**, then $\mathcal{L}/\overset{s}{\sim}$ is semidistributive (i.e. both meet-semidistributive and join-semidistributive).*

Proof. We've already shown that $\mathcal{L}/\overset{s}{\sim}$ is meet-semidistributive, so we only need to check that it is join-semidistributive. Suppose that $\alpha, \beta, \gamma \in \mathcal{L}$ satisfy $\alpha \vee \beta \overset{s}{\sim} \alpha \vee \gamma$. We can assume without loss of generality that α is minimal in $\alpha/\overset{s}{\sim} \cap \mathcal{L}$, and similarly for β and γ, in which case we actually have

$$\alpha \vee \beta = \alpha \vee \gamma,$$

and if we call the common value δ then δ is minimal in $\delta/\overset{s}{\sim} \cap \mathcal{L}$. If we assume for the sake of contradiction that $\alpha \vee (\beta \wedge \gamma) \ne \delta$, then there is some prime congruence quotient (ϵ, δ) such that

$$\alpha \vee (\beta \wedge \gamma) \le \epsilon \prec \delta,$$

and by the dual to Proposition B.7.7 there can't be any dual weak pseudocomplement to ϵ under δ. Since $\epsilon \in \mathcal{L}$ and δ is minimal in $\delta/\overset{s}{\sim} \cap \mathcal{L}$, we see that $\epsilon \overset{s}{\not\sim} \delta$, and by our assumption on $\mathcal{L}$ the type of (ϵ, δ) must therefore be **3** or **4**, contradicting Proposition B.7.10. □

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