Interpolating log*

Imagine that we wish to find a function perfectly in between x and e^x. That is, we desire a function f such that $f(f(x)) = e^x$, at least asymptotically. There are slight technical difficulties with finding a function which exactly satisfies $f(f(x)) = e^x$, but it turns out that we can find a nice bijective function $f : [0, \infty) \to [0, \infty)$ which satisfies

$$f(f(x)) = e^x - 1.$$

The advantage of using $e^x - 1$ here is that $e^0 - 1 = 0$, so we can set $f(0) = 0$.

We define a pair of functions $\varepsilon(x)$ and $\ell(x)$ by

$$\varepsilon(x) = e^x - 1$$

and

$$\ell(x) = \ln(1 + x),$$

and note that $\varepsilon, \ell : [0, \infty) \to [0, \infty)$ are inverse bijections.

For each $n \in \mathbb{N}$, we define $\varepsilon^n(x)$ and $\ell^n(x)$ to be the nth iterates of ε and ℓ, so that $\varepsilon^0(x) = \ell^0(x) = x$ and $\varepsilon^{n+1}(x) = \varepsilon(\varepsilon^n(x)), \ell^{n+1}(x) = \ell(\ell^n(x))$. The strategy is to start by defining a bijective function $\ell^* : (0, \infty) \to \mathbb{R}$ such that $\ell^*(1) = 0$,

$$\ell^*(\varepsilon(x)) = \ell^*(x) + 1,$$

and

$$\ell^*(\ell(x)) = \ell^*(x) - 1.$$

Intuitively, $\ell^*(x)$ is "the number of times we have to apply ℓ to reach 1". Using ℓ^*, we can then construct a function $\varepsilon^{1/2}$ which satisfies $\varepsilon^{1/2}(\varepsilon^{1/2}(x)) = e^x - 1$.

Proposition 1. *For all $x > 0$, we have $\varepsilon(x) > x$ and $\ell(x) < x$. In particular, for any $x > 0$, we have $\lim_{n\to\infty} \ell^n(x) = 0$.*

The intuition for computing $\ell^*(x)$ is that we may use the identity

$$\ell^*(\ell^n(x)) = \ell^*(x) - n$$

to reduce the computation of $\ell^*(x)$ to the computation of $\ell^*(\ell^n(x))$. Since $\ell^n(x)$ is eventually quite close to 0, we just need to understand how ℓ acts on numbers close to 0. We can approximate $\ell(x)$ for small x by the Taylor series

$$\ell(x) = x - \frac{x^2}{2} + O(x^3).$$

Comparing $\frac{1}{\ell(x)}$ to $\frac{1}{x}$, we get the following estimate.

Proposition 2. *For x small, we have*

$$\frac{1}{\ell(x)} = \frac{1}{x} + \frac{1}{2} - \frac{x}{12} + O(x^2).$$

Additionally, we have

$$\frac{1}{2} - \frac{x}{12} < \frac{1}{\ell(x)} - \frac{1}{x} < \frac{1}{2}$$

for all $x > 0$.

Proof. The first statement follows from standard power series manipulation:

$$\frac{1}{x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \cdots} = \frac{1}{x} + \frac{1}{2} - \frac{x}{12} + \frac{x^2}{24} - \frac{19x^3}{720} + \cdots .$$

The inequality $\frac{1}{\ell(x)} - \frac{1}{x} < \frac{1}{2}$ is equivalent to

$$\ell(x) > \frac{1}{1/x + 1/2} = 2 - \frac{4}{2+x},$$

and since this is true for x sufficiently close to 0, we just need to check that the derivative of the left hand side is at least the derivative of the right hand side. Thus we just need to check that

$$\frac{1}{1+x} > \frac{4}{(2+x)^2},$$

which follows by multiplying out.

We only need to check the inequality $\frac{1}{2} - \frac{x}{12} < \frac{1}{\ell(x)} - \frac{1}{x}$ in the range $0 < x < 6$, and in this range it is equivalent to

$$\ell(x) < \frac{1}{1/x + 1/2 - x/12} = \frac{x}{1 + x/2 - x^2/12}.$$

Again, this is true for x sufficiently close to 0, so we may compare the derivatives instead. We see that we just need to check that

$$\frac{1}{1+x} < \frac{(1 + x/2 - x^2/12) - x(1/2 - x/6)}{(1 + x/2 - x^2/12)^2} = \frac{1 + x^2/12}{(1 + x/2 - x^2/12)^2}$$

for $0 < x < 6$. Multiplying out, this becomes

$$(1 + x/2 - x^2/12)^2 < (1+x)(1 + x^2/12),$$

or

$$1 + x + \frac{x^2}{12} - \frac{x^3}{12} + \frac{x^4}{144} < 1 + x + \frac{x^2}{12} + \frac{x^3}{12},$$

which holds for $0 < x < 24$. □

Corollary 1. *For $x \le 1$, we have*

$$\frac{5n}{12} < \frac{1}{\ell^n(x)} - \frac{1}{x} < \frac{n}{2}.$$

Corollary 2. *For $x \leq 1$, we have*

$$\frac{1}{\ell^n(x)} = \frac{1}{x} + \frac{n}{2} - \sum_{i<n} \frac{\ell^i(x)}{12} + O(x).$$

Corollary 3. *For $x \leq 1$, we have*

$$\frac{1}{\ell^n(x)} = \frac{1}{x} + \frac{n}{2} - O(\ln(n)).$$

Corollary 4. *For x fixed and n going to infinity, we have*

$$\frac{1}{\ell^n(x)} = \frac{n}{2} - \frac{\ln(n)}{6} + O_x(1).$$

So one natural path to computing $\ell^*(x)$ is to try to compute

$$\lim_{n\to\infty} n - \frac{\ln(n)}{3} - \frac{2}{\ell^n(x)}.$$

A simpler approach is to compare $\frac{2}{\ell^n(x)}$ to $\frac{2}{\ell^n(1)}$.

Proposition 3. *For any $x, y > 0$, we have*

$$\left|\frac{1}{x} - \frac{1}{y}\right| \leq \left|\frac{1}{\ell(x)} - \frac{1}{\ell(y)}\right| \leq \left|\frac{1}{x} - \frac{1}{y}\right| + \frac{|x-y|}{12}.$$

Proof. We just need to show that the function $f(x) = -1/\ell(x)$ has derivative bounded below by $\frac{1}{x^2}$ and above by $\frac{1}{x^2} + \frac{1}{12}$. We have

$$f'(x) = \frac{1}{1+x} \cdot \frac{1}{\ell(x)^2}.$$

Thus, for the left hand inequality, we just need to check that

$$\ell(x)^2 < \frac{x^2}{1+x},$$

or equivalently

$$\ell(x) < \frac{x}{(1+x)^{1/2}}.$$

Since equality holds at 0, it's enough to compare the derivatives: we just need to show that

$$\frac{1}{1+x} < \frac{1}{(1+x)^{1/2}} - \frac{x}{2(1+x)^{3/2}}.$$

Multiplying out, this becomes

$$2\sqrt{1+x} < 2 + x,$$

and squaring both sides shows that this holds for all $x > 0$.

For the right hand inequality, we need to check that

$$\ell(x)^2 > \frac{x^2}{(1+x)(1+x^2/12)},$$

or equivalently that

$$\ell(x) > \frac{x}{(1+x)^{1/2}(1+x^2/12)^{1/2}}.$$

Again, it's enough to compare the derivatives, so we just need to check that

$$\frac{1}{1+x} > \frac{1}{(1+x)^{1/2}(1+x^2/12)^{1/2}} - \frac{x}{2(1+x)^{3/2}(1+x^2/12)^{1/2}} - \frac{x^2}{12(1+x)^{1/2}(1+x^2/12)^{3/2}}.$$

Multiplying out, this becomes

$$(1+x)^{1/2}(1+x^2/12)^{3/2} > 1 + x/2 - x^3/24,$$

and on squaring both sides we get the inequality

$$(1+x)(1+x^2/12)^3 > 1 + x + x^2/4 - x^3/12 - x^4/24 + x^6/24^2,$$

which the reader may verify by using the inequality $x^5 + x^7 \geq 2x^6$. □

Corollary 5. *For any $x, y > 0$, the limit*

$$\lim_{n\to\infty} \frac{2}{\ell^n(y)} - \frac{2}{\ell^n(x)}$$

exists, and is equal to

$$\lim_{n\to\infty} \frac{n^2}{2}\Big(\ell^n(x) - \ell^n(y)\Big).$$

Proof. To see that the limit exists, note that if $x \geq y$, then the sequence

$$\frac{2}{\ell^n(y)} - \frac{2}{\ell^n(x)}$$

is increasing in n and is bounded above by

$$\frac{2}{y} - \frac{2}{x} + \sum_{m\geq 0} \frac{\ell^m(x) - \ell^m(y)}{6} \leq \frac{2}{y} - \frac{2}{x} + \frac{kx}{6},$$

where k is any integer which satisfies $y \geq \ell^k(x)$.

For the second statement, note that

$$\frac{2}{\ell^n(y)} - \frac{2}{\ell^n(x)} = \frac{2(\ell^n(x) - \ell^n(y))}{\ell^n(x)\ell^n(y)},$$

and use the asymptotic

$$\ell^n(x) = (1 + o_x(1))\frac{2}{n}$$

(and similarly for y) to replace the denominator by $4/n^2$. □

Definition 1. For $x > 0$, we define $\ell^*(x)$ by

$$\ell^*(x) = \lim_{n\to\infty} \frac{2}{\ell^n(1)} - \frac{2}{\ell^n(x)} = \lim_{n\to\infty} \frac{n^2}{2}\Big(\ell^n(x) - \ell^n(1)\Big).$$

Proposition 4. *For all $x > 0$, the function $\ell^*(x)$ satisfies*

$$\ell^*(e^x - 1) = \ell^*(x) + 1$$

and

$$\ell^*(\ln(1 + x)) = \ell^*(x) - 1.$$

Proof. It's enough to prove the second statement. By the definition of ℓ^*, we have

$$\ell^*(x) - \ell^*(\ell(x)) = \lim_{n\to\infty} \frac{2}{\ell^{n+1}(x)} - \frac{2}{\ell^n(x)}.$$

Setting $y_n = \ell^n(x)$, we have $y_n \to 0$, so the above is equal to

$$\lim_{y\to 0} \frac{2}{\ell(y)} - \frac{2}{y} = 1.$$

□

For the sake of concretely approximating ℓ^*, we have the following explicit bound.

Proposition 5. *If $x \geq y \geq \ell^k(x)$, then for any n we have*

$$\frac{2}{\ell^n(y)} - \frac{2}{\ell^n(x)} \leq \ell^*(x) - \ell^*(y) \leq \frac{2}{\ell^n(y)} - \frac{2}{\ell^n(x)} + \frac{k\ell^n(x)}{6}.$$

Of course, we'd like to know if the function ℓ^* is well-behaved: is it continuous, is it differentiable, etc. To answer this question, we use the theory of *completely monotone/Bernstein* functions.

Definition 2. A continuous function $f : [0, \infty) \to \mathbb{R}$ is called *completely monotone* if it satisfies

$$(-1)^n f^{(n)}(x) \geq 0$$

for all $x > 0$ and all $n \in \mathbb{N}$.

A function $g : [0, \infty) \to [0, \infty)$ whose derivative is completely monotone is called a *Bernstein function.*

A function h such that $h^{(n)}(x) \geq 0$ for all x and all $n \in \mathbb{N}$ is called *absolutely monotone.* If h is absolutely monotone on $(-\infty, 0]$, then $h(-x)$ is completely monotone, and conversely.

Proposition 6. *If f, g are Bernstein, then the composition $f \circ g$ is also a Bernstein function. If f is completely monotone and g is Bernstein, then $f \circ g$ is completely monotone.*

Corollary 6. *For every n, the function ℓ^n is a Bernstein function, and $1/\ell^n$ is a completely monotone function.*

The next result follows easily from standard facts about divided differences, but I haven't seen it explicitly stated anywhere (aside from the special cases we use here).

Proposition 7. *If f is a pointwise limit of functions f_i such that for each $n \geq 1$, the derivatives $f_i^{(n)}$ exist and have a fixed sign $s_n \in \{+,-\}$ not depending on i, then each derivative $f^{(n)}$ exists and has the same fixed sign s_n. In this case, we even have*

$$f^{(n)}(x) = \lim_{i\to\infty} f_i^{(n)}(x).$$

In particular, any pointwise limit of Bernstein functions is a Bernstein function, and the same holds for completely monotone functions.

Using direct arguments, one can show that if f is completely monotone on $(0,\infty)$ then for every $x > 0$, the Taylor series of f around x has radius of convergence at least as large as x, and converges to f on the interval $(0,x]$. This quickly leads to the following result.

Proposition 8 ([1])**.** *Every completely monotone function $f : (0,\infty) \to \mathbb{R}$ extends to an analytic function on the halfplane $\Re(x) > 0$, as does any Bernstein function. If f is completely monotone on $(0,\infty)$ and $\Re(x) > 0$, then $|f(x)| \leq f(\Re(x))$.*

Corollary 7. *The function ℓ^* has completely monotone derivative, and extends to an analytic function on the halfplane $\Re(x) > 0$. For all $n \geq 1$ and $\Re(x) > 0$, we have*

$$|\ell^{*(n)}(x)| \leq (-1)^{n-1}\ell^{*(n)}(\Re(x)).$$

Using standard facts about Taylor series, we can show that if $f(x) = \lim_{i\to\infty} f_i(x)$ is a pointwise limit of completely monotone (or Bernstein) functions f_i on $(0,\infty)$, and if we extend each f_i to a complex analytic function on $\Re(x) > 0$, then the extension of f to an analytic function on $\Re(x) > 0$ also satisfies

$$f(x) = \lim_{i\to\infty} f_i(x),$$

and the convergence is uniform on compact subsets of the halfplane $\Re(x) > 0$. As a consequence, we get the formula

$$\ell^*(x) = \lim_{n\to\infty} \frac{2}{\ell^n(1)} - \frac{2}{\ell^n(x)}$$

for all complex x with $\Re(x) > 0$.

Using the functional equation $\ell^*(x) = \ell^*(\ell(x)) - 1$, we can extend ℓ^* to an analytic function on $\mathbb{C} \setminus (-\infty, 0]$. For this to make sense, we need to first extend ℓ to an analytic function on $\mathbb{C} \setminus (-\infty, -1]$ - we do this in the usual way, by integrating $x \mapsto \frac{1}{1+x}$ along paths contained in the region $\mathbb{C} \setminus (-\infty, -1]$. This extension of ℓ takes the halfplane $\Re(x) > 0$ into itself, satisfies

$$\ell\big(\mathbb{C} \setminus (-\infty, 0]\big) \subseteq \mathbb{C} \setminus (-\infty, 0],$$

and satisfies

$$|\Im(\ell(x))| < \pi$$

for all $x \in \mathbb{C} \setminus (-\infty, 0]$.

Proposition 9. *The function ℓ^* extends to an anaytic function on $\mathbb{C} \setminus (-\infty, 0]$, which satisfies the functional equation $\ell^*(x) = \ell^*(\ell(x)) + 1$ for all $x \in \mathbb{C} \setminus (-\infty, 0]$. This extension of ℓ^* is still given by the formula*

$$\ell^*(x) = \lim_{n\to\infty} \frac{2}{\ell^n(1)} - \frac{2}{\ell^n(x)}$$

on $\mathbb{C} \setminus (-\infty, 0]$.

Proof. Since we already have an extension of ℓ^* to the halfplane $\Re(x) > 0$, we just need to check that for every $x \in \mathbb{C} \setminus (-\infty, 0]$, there is some $n \in \mathbb{N}$ such that $\Re(\ell^n(x)) > 0$. For x such that $|1+x| > 1$, we have

$$\Re(\ell(x)) = \ln|1+x| > 0,$$

so we just need to check that for every x there is some n with $|1+\ell^n(x)| > 1$. To prove this, we will first show that for $|1+x| \le 1$ and $\Im(x) \neq 0$, we always have

$$|\Im(\ell(x))| > |\Im(x)|.$$

To see this, suppose that $\Im(x) > 0$, and consider the right triangle with vertices -1, $\Re(x)$, and x in the complex plane. If θ is the angle of this triangle at the vertex -1, then we have $\Im(\ell(x)) = \theta$, and

$$\Im(x) = |1+x|\sin(\theta) \le \sin(\theta) \le \theta = \Im(\ell(x)),$$

with equality only when $\theta = 0$.

Now suppose for a contradiction that $|1+\ell^n(x)| \le 1$ for all n. Suppose without loss of generality that $\Im(x) > 0$. Then the sequence $n \mapsto \Im(\ell^n(x))$ is an increasing sequence, so all of the points $\ell^n(x)$ are contained in the compact region

$$C = \{z : |1+z| \le 1, \Im(z) \ge \Im(x) > 0\}.$$

In particular, the sequence of points $\ell^n(x)$ must have some limit point $z \in C$, and since ℓ is continuous we must then have $\ell(z) = z$. But this is impossible, since we've proved that $\Im(\ell(z)) > \Im(z)$ for all $z \in C$. The contradiction proves that there must be some $n \in \mathbb{N}$ such that $|1+\ell^n(x)| > 1$, and then we have $\Re(\ell^{n+1}(x)) > 0$ for the same n. $\square$

The extension of ℓ^* to $\mathbb{C} \setminus (-\infty, 0]$ has

$$\begin{aligned}\lim_{\epsilon \to 0} \ell^*(-1+i\epsilon) &= 1 + \lim_{x\to\infty} \ell^*\Big(-x + \frac{\pi i}{2}\Big)\\ &= 2 + \lim_{x\to\infty} \ell^*(x + (\pi + o(1))i)\\ &= 2 + \lim_{x\to\infty} \ell^*(x) + O(\ell^{*\prime}(x))\\ &= +\infty.\end{aligned}$$

Using the functional equation $\ell^*(x) = \ell^*(\ell(x)) + 1$, we get

$$\lim_{\epsilon\to 0} \ell^*(\varepsilon^n(-1) + i\epsilon) = +\infty$$

for each $n \in \mathbb{N}$, and since the sequence $\varepsilon^n(-1)$ approaches 0 from below, we see that ℓ^* has an essential singularity at 0.

The function ℓ^* can be further extended to a multivalued function on

$$\mathbb{C} \setminus (\{0\} \cup \{\varepsilon^n(-1) : n \in \mathbb{N}\}).$$

More precisely, we can extend ℓ^* to an analytic function on a infinitely branched cover of $\mathbb{C}$ with branch points lying above $\varepsilon^n(-1)$ for $n \in \mathbb{N}$ (we make branch cuts along $(-\infty, -1), (-1, \varepsilon(-1)), \ldots$). One way to make this precise is to consider the set of continuous paths

$$p : [0,1] \to \mathbb{C} \setminus \{\varepsilon^n(-1) : n \in \mathbb{N}\}$$

with $p(0) \in (0, \infty)$ and $p(1) = x$ instead of just points x. We can apply ℓ to any such path p to get another such path $\ell(p)$, and as we will soon see, if we apply ℓ sufficiently many times then eventually the path $\ell^n(p)$ will be entirely contained in the halfplane $\Re(x) > 0$.

As it turns out the set of paths p we have to consider (or equivalently, the set of sheets of the branched cover) is simpler than expected: if a path p starts out by looping counterclockwise around -1 even one time, we will have $\Im(\ell(p)) > \pi$ from there on, so

$$\ell^*(p) = \ell^*(\ell(p)) + 1$$

can be defined straightforwardly from that point on without worrying about p hitting any other branch points. Similar reasoning applied to $\ell^n(p)$ shows that if p begins by looping counterclockwise around any $\varepsilon^n(-1)$, we don't have to worry about hitting any $\varepsilon^{n+k}(-1)$ for any $k \geq 1$, although we do still have to worry about hitting $\varepsilon^m(-1)$ for $m \leq n$. Representative paths p for the various sheets of the cover can be described by sequences

$$(w_0, w_1, ..., w_n, 0, ...)$$

of integer winding numbers which are eventually 0, with the following interpretation: if w_n is the last nonzero winding number in the sequence, then the sequence of winding numbers describes the path that *first* winds w_n times counterclockwise around $\varepsilon^n(-1)$, *second* winds w_{n-1} times counterclockwise around $\varepsilon^{n-1}(-1)$, ..., and *finally* winds w_0 times counterclockwise around -1. When we apply ℓ to a path p, the winding number sequence simplifies:

$$p \leftrightarrow (w_0, w_1, w_2, ..., w_n, 0, ...) \quad \Longrightarrow \quad \ell(p) \leftrightarrow (w_1, w_2, ..., w_n, 0, 0, ...).$$

To see this, just note that $\Im(x)$ and $\Im(\ell(x))$ always have the same sign as long as we stay in the halfplane $\Re(x) > -1$, so every loop of p around $\varepsilon^n(-1)$ gets mapped by ℓ to a loop of $\ell(p)$ around $\varepsilon^{n-1}(-1)$. After applying ℓ to p exactly $n+1$ times, we get a path $\ell^{n+1}(p)$ which is entirely contained in $\mathbb{C} \setminus (-\infty, 0]$, and we have already extended ℓ^* to an analytic function on this region.

Now we turn to the task of computing values of ℓ^* to higher accuracy. Since applying ℓ repeatedly always takes us close to 0, and since we have the reference values $\ell^*(\ell^n(1)) = -n$, we just need to find more accurate approximations to $\ell^{*\prime}(x)$ for x close to 0. Unfortunately, the essential singularity of ℓ^* at 0 implies that there is no Laurent series which computes ℓ^* (or any of its derivatives) in any punctured disk around 0. Nevertheless, we will show that there is an asymptotic series for the derivative of ℓ^* around 0, beginning with

$$\frac{d}{dx}\ell^*(x) = \frac{2}{x^2} + \frac{1}{3x} - \frac{1}{36} + \frac{x}{270} + \frac{x^2}{2592} - \frac{71x^3}{108864} + \frac{8759x^4}{32659200} + \frac{31x^5}{3499200} + O(x^6)$$

for $x > 0$. Further terms of the asymptotic series can be computed by expanding the functional equation

$$\ell^{*\prime}(x) = \frac{\ell^{*\prime}(\ell(x))}{1+x}$$

as if both sides were formal Laurent series and equating coefficients. Explicit error terms in the asymptotic series can be computed using the theory of divided differences - this can be useful for getting the most accurate possible approximation when numerically computing ℓ^*.

Proposition 10. *For each $k \in \mathbb{N}$, define $A_k(x)$ by*

$$A_k(x) \coloneqq \sum_{i=1}^{k} \frac{i}{x - \ell^i(x)} \prod_{\substack{j \leq k \\ j \neq i}} \frac{x - \ell^j(x)}{\ell^i(x) - \ell^j(x)}.$$

Then for every $k \in \mathbb{N}$ we have

$$\ell^{*\prime}(x) = A_k(x) + O(x^{k-2})$$

as x approaches 0 from above, and for all $x \in (0, \infty)$ we have

$$A_{2k}(x) \leq \ell^{*\prime}(x) \leq A_{2k+1}(x).$$

In particular, since each $A_k(x)$ has a Laurent series with rational coefficients which converges in a punctured disk around 0 of positive radius, $\ell^{*\prime}(x)$ has an asymptotic series with rational coefficients as x approaches 0 from above.

Proof. We use the theory of divided differences. For f a function on $(0, \infty)$, we set $f[x] \coloneqq f(x)$, and recursively

$$f[x_1, ..., x_{n+1}] \coloneqq \frac{f[x_1, ..., x_{n-1}, x_n] - f[x_1, ..., x_{n-1}, x_{n+1}]}{x_n - x_{n+1}}.$$

If two entries x_i, x_j are equal, then we define the divided difference by taking a suitable limit (this will be well-defined as long as f is sufficiently differentiable).

Using $\ell^*(x) - \ell^*(\ell^i(x)) = i$, a standard computation gives

$$\ell^*[x, x, \ell(x), ..., \ell^k(x)] = \frac{\ell^{*\prime}(x)}{\prod_{i=1}^{k}(x - \ell^i(x))} - \sum_{i=1}^{k} \frac{i}{(x - \ell^i(x))^2 \prod_{j \neq i}(\ell^i(x) - \ell^j(x))}.$$

By the mean value theorem for divided differences, there is some $\xi \in [\ell^k(x), x]$ such that

$$\ell^*[x, x, \ell(x), ..., \ell^k(x)] = \ell^{*(k+1)}(\xi).$$

By the fact that $\ell^{*\prime}$ is completely monotone, we have

$$(-1)^k \ell^*[x, x, \ell(x), ..., \ell^k(x)] = (-1)^k \ell^{*(k+1)}(\xi) \geq 0,$$

so depending on whether k is even or odd, $\ell^{*\prime}(x)$ is either bounded below or above by

$$A_k(x) \coloneqq \sum_{i=1}^{k} \frac{i}{x - \ell^i(x)} \prod_{j \neq i} \frac{x - \ell^j(x)}{\ell^i(x) - \ell^j(x)}.$$

To finish the proof, we just need to check that

$$\ell^{*\prime}(x) - A_k(x) = \ell^*[x, x, \ell(x), ..., \ell^k(x)] \cdot \prod_{i=1}^{k}(x - \ell^i(x))$$

is $O(x^{k-2})$. Since $x - \ell^i(x) \propto x^2$, this is equivalent to proving that

$$\ell^*[x, x, \ell(x), ..., \ell^k(x)] = \ell^{*(k+1)}(\xi) \stackrel{?}{=} O(x^{-k-2}).$$

By complete monotonicity of $\ell^{*\prime}$, we have

$$\ell^{*\prime}(\xi/2) \geq \left|\ell^{*(k+1)}(\xi)\right| \cdot \frac{(\xi/2)^k}{k!},$$

and we already know that $\ell^{*\prime}(\xi/2) = O(\xi^{-2})$ by the concavity of ℓ^* and the fact that $\ell^n(1) = \frac{2+o(1)}{n}$. Since $\xi = x - O(x^2)$, we have

$$\left|\ell^{*(k+1)}(\xi)\right| = O(\xi^{-k-2}) = O(x^{-k-2}),$$

so we are done. ☐

We can also prove a uniqueness result for ℓ^*, using only the assumption of concavity together with the functional equation.

Proposition 11. *If $f : (0, \infty) \to \mathbb{R}$ is a concave function which satisfies $f(\ell(x)) = f(x) - 1$ for all $x > 0$, and if $f(1) = 0$, then $f = \ell^*$.*

Proof. By the functional equation, it's enough to show that $f(x) - \ell^*(x) = O(x)$ as x approaches 0 from above. Since $f(\ell^n(1)) = -n = \ell^*(\ell^n(1))$ for $n \in \mathbb{N}$, it's enough to show that the difference between $f(x) - f(y)$ and $\ell^*(x) - \ell^*(y)$ is $O(x)$ for $x > y > \ell(x)$. By the concavity of f, we have

$$f[x, y, \ell(x)] \leq 0$$

and

$$f[\varepsilon(x), x, y] \leq 0,$$

and expanding these inequalities out in the case $x > y > \ell(x)$ we get

$$\frac{f(\varepsilon(x)) - f(x)}{\varepsilon(x) - x} \leq \frac{f(x) - f(y)}{x - y} \leq \frac{f(x) - f(\ell(x))}{x - \ell(x)}.$$

By the functional equation $f(\ell(x)) = f(x) - 1$, we get

$$\frac{1}{\varepsilon(x) - x} \leq \frac{f(x) - f(y)}{x - y} \leq \frac{1}{x - \ell(x)},$$

and expanding the upper and lower bounds as Laurent series in x, we get

$$\frac{f(x) - f(y)}{x - y} = \frac{2}{x^2} + O(1/x)$$

as x approaches 0 from above. Since

$$x - y \leq x - \ell(x) = O(x^2),$$

we have

$$f(x) - f(y) = \frac{2(x - y)}{x^2} + O(x).$$

Since the same reasoning applies to ℓ^* as well, we have $f(x) - f(y) = \ell^*(x) - \ell^*(y) + O(x)$ for all $x > y > \ell(x)$. ☐

We now define a real-analytic tetration function ε^*.

Definition 3. We define $\varepsilon^* : \mathbb{R} \to (0, \infty)$ to be the inverse function to ℓ^*.

Some of the nice properties of ℓ^* immediately imply nice properties of ε^*. Since ℓ^* is increasing and concave, ε^* will be increasing and convex. Since $\ell^*(x)$ extends to a complex analytic function on $\mathbb{C} \setminus (-\infty, 0]$ which satisfies the functional equation $\ell^*(x) = \ell^*(\ell(x)) + 1$, ε^* extends to a complex analytic function on some neighborhood of $\mathbb{R}$ which satisfies the functional equation

$$\varepsilon^*(x+1) = \varepsilon(\varepsilon^*(x)).$$

The uniqueness property of ℓ^* implies that ε^* is the unique convex function on $\mathbb{R}$ which satisfies this functional equation together with the initial condition $\varepsilon^*(0) = 1$.

Proposition 12. *For every $x \in \mathbb{R}$, we have*

$$\varepsilon^*(x) = \lim_{n\to\infty} \varepsilon^n\Big(\frac{2}{\frac{2}{\ell^n(1)} - x}\Big).$$

Proof. If $y = \varepsilon^*(x)$, then from $\ell^*(y) = x$ we see that

$$\lim_{n\to\infty} \frac{2}{\ell^n(1)} - \frac{2}{\ell^n(y)} = x.$$

By induction on k, using the inequality $|1/a - 1/b| \le |1/\ell(a) - 1/\ell(b)|$ which we proved earlier, we have

$$\Big|\frac{2}{\ell^{n-k}(y)} - \frac{2}{\varepsilon^k(\frac{2}{\frac{2}{\ell^n(1)}-x})}\Big| \le \Big|\frac{2}{\ell^n(y)} - \Big(\frac{2}{\ell^n(1)} - x\Big)\Big|,$$

so in particular we have

$$\Big|\frac{2}{y} - \frac{2}{\varepsilon^n(\frac{2}{\frac{2}{\ell^n(1)}-x})}\Big| \le \Big|\frac{2}{\ell^n(y)} - \Big(\frac{2}{\ell^n(1)} - x\Big)\Big| = \Big|x - \Big(\frac{2}{\ell^n(1)} - \frac{2}{\ell^n(y)}\Big)\Big|.$$

Taking the limit of both sides as $n \to \infty$ proves that

$$\frac{2}{\varepsilon^*(x)} = \frac{2}{y} = \lim_{n\to\infty} \frac{2}{\varepsilon^n(\frac{2}{\frac{2}{\ell^n(1)}-x})}. \qquad \square$$

Corollary 8. *The function ε^* is absolutely monotone, that is, ${\varepsilon^*}^{(n)}(x) \ge 0$ for every $n \in \mathbb{N}$ and every $x \in \mathbb{R}$.*

Proof. From the fact that the function $x \mapsto \frac{2}{c-x}$ is absolutely monotone on $(-\infty, c)$ for every constant c, the fact that $\varepsilon(x)$ is absolutely monotone on $(0, \infty)$, and the fact that compositions of absolutely monotone functions are absolutely monotone, each function

$$\varepsilon^n\Big(\frac{2}{\frac{2}{\ell^n(1)} - x}\Big)$$

is absolutely monotone on $(-\infty, 2/\ell^n(1))$. Since pointwise limits of absolutely monotone functions are absolutely monotone, we see that ε^* is absolutely monotone as well. $\square$

Corollary 9. *The function ε^* extends to an entire function on $\mathbb{C}$ which satisfies the functional equation $\varepsilon^*(x+1) = \varepsilon(\varepsilon^*(x))$. For all $x \in \mathbb{C}$, we have*

$$\varepsilon^*(x) = \lim_{n\to\infty} \varepsilon^n\Big(\frac{2}{\frac{2}{\ell^n(1)} - x}\Big).$$

By Picard's little theorem, ε^* can only avoid a single value on $\mathbb{C}$, and since $\varepsilon(x) = e^x - 1 \neq -1$ for all $x \in \mathbb{C}$, the functional equation for ε^* shows that

$$\varepsilon^*(x) \neq -1$$

for all $x \in \mathbb{C}$ as well. Interestingly, this implies that ε^* must instead take values such as 0 and $\varepsilon(-1)$ - and in fact, Picard's great theorem implies that ε^* takes these values infinitely often. This may not be such a surprise if you think back to our description of the extension of ℓ^* to a function on a branched cover of $\mathbb{C}$ with infinitely many sheets, and use the fact that

$$\varepsilon^*(\ell^*(x)) = x.$$

For instance, the zeros of ε^* are in one-to-one correspondence with the set of sheets of this branched covering, excluding the single starting sheet where ℓ^* has an essential singularity at 0.

Now we can finally define the fractional compositional powers of the function $e^x - 1$.

Definition 4. For every $n \in \mathbb{C}$, we define the function $\varepsilon^n : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ by

$$\varepsilon^n(x) = \varepsilon^*(\ell^*(x) + n).$$

We define ℓ^n by $\ell^n(x) = \varepsilon^{-n}(x)$.

Proposition 13. *For any $m, n \in \mathbb{R}$ and any $x > 0$, we have*

$$\varepsilon^m(\varepsilon^n(x)) = \varepsilon^{m+n}(x).$$

In particular, we have

$$\varepsilon^{1/2}(\varepsilon^{1/2}(x)) = e^x - 1.$$

We can compute ε^m with a direct limit formula, using the fact that the limit formulas for ℓ^* and ε^* converge uniformly on compact subsets of their domains.

Proposition 14. *For $m \in \mathbb{C}$ and $x \in \mathbb{C} \setminus (-\infty, 0]$, we have*

$$\varepsilon^m(x) = \lim_{n\to\infty} \varepsilon^n\Big(\frac{2}{\frac{2}{\ell^n(x)} - m}\Big).$$

Just for fun, we can also define an asymptotic measurement of "how exponentially" a function grows.

Definition 5. We say that a function $f : (0, \infty) \to (0, \infty)$ has *exponentiality* $\alpha(f)$ if

$$\alpha(f) = \lim_{x\to\infty} \ell^*(f(x)) - \ell^*(x) = \lim_{x\to\infty} \ell^*(f(\varepsilon^*(x))) - x.$$

Under this definition, we have $\alpha(1) = -\infty$, $\alpha(x) = 0$, $\alpha(\varepsilon) = 1$, and $\alpha(\ell) = -1$. Additionally, we have $\alpha(\varepsilon^n) = n$ for all $n \in \mathbb{R}$, $\alpha\ell^* = -\infty$, and $\alpha(\varepsilon^*) = +\infty$.

Proposition 15. *If $f, g : (0,\infty) \to [\epsilon, \infty)$ are functions with exponentialities $\alpha(f), \alpha(g)$, then*

$$\alpha(fg) = \alpha(f+g) = \max(\alpha(f), \alpha(g)).$$

Proposition 16. *If $f, g : (0,\infty) \to (0,\infty)$ have exponentialities $\alpha(f), \alpha(g) > -\infty$, then*

$$\alpha(f \circ g) = \alpha(f) + \alpha(g).$$

Proof. For any x, we have

$$\ell^*(f(g(x))) - \ell^*(x) = \ell^*(f(g(x))) - \ell^*(g(x)) + \ell^*(g(x)) - \ell^*(x).$$

Since $g(x)$ must go to ∞ as $x \to \infty$ if $\alpha(g) > -\infty$, we see that the limit of the above expression is $\alpha(f) + \alpha(g)$. □

Corollary 10. *Every function which can be constructed (in finitely many steps) out of positive polynomials by addition, multiplication, exponentiation, and taking logarithms has an exponentiality in $\mathbb{Z} \cup \{-\infty\}$.*

References

[1] David Vernon Widder. *The Laplace Transform.* Princeton University Press, Princeton, 1946.