Notes

Contents

Chapter 1

Algebra

1.1 Noncommutative rings

Definition 1. If R is a ring, then the *Jacobson radical* $J(R)$ (sometimes written $\mathrm{rad}(R)$) is the intersection of the annihilators of all simple left R-modules.

Definition 2. A submodule N of M is *superfluous*, written $N \subseteq_s M$ or $N \ll M$, if for all H we have $N + H = M \implies H = M$.

Theorem 1. *We can replace "left" by "right" in the definition of the Jacobson radical of a ring. Furthermore, we have the following equivalent definitions:*

- $J(R)$ *is the intersection of all maximal left ideals of* R*,*
- $J(R)$ *is the sum of all superfluous left ideals of* R*,*
- $J(R)$ *is the maximal left ideal of* R *such that for all* $x \in J(R)$*,* $1 - x$ *has a left inverse,*
- $J(R) = \{x \in R \mid 1 + RxR \subseteq R^\times\}$*.*

Lemma 1 (Nakayama's Lemma)**.** *If* M *is a finitely generated left* R*-module with* $M = J(R)M$*, then* $M = 0$*.*

Proof. Consider a minimal generating set $x_1, ..., x_n$ of M, and use $\sum x_i \in J(R)M$ to write x_n as a linear combination of $x_1, ..., x_{n-1}$. □

Proposition 1. $J(R/J(R)) = 0$*.*

1.1.1 Artinian Rings

Proposition 2. *If* R*, considered as a left* R*-module over itself, has a composition series of length* k*, then* $J(R)^k = 0$*.*

Theorem 2 (Hopkins' Theorem)**.** *If* M *is a left module over a left Artinian ring, then the following are equivalent:*

- M *is finitely generated,*

- *M has finite length,*
- *M is Noetherian,*
- *M is Artinian.*

Theorem 3 (Hopkins-Levitzki)**.** *If R is* semiprimary *- that is, if $R/J(R)$ is semisimple and $J(R)$ is nilpotent - then for left R-modules, being Noetherian, being Artinian, and having a composition series are equivalent.*

Proposition 3. *If $J(R) = 0$, then every minimal left ideal of R is a direct summand of R.*

Theorem 4. *R is semisimple if and only if it is left Artinian and has $J(R) = 0$.*

1.2 Commutative Algebra

Definition 3. If R is a commutative ring, then $I \lhd R$ means that I is an ideal of R.

Definition 4. If $I, J \lhd R$, set $(I : J) = \{r \in R \mid rJ \subseteq I\}$. If $a \in R$, we abbreviate $(I : (a))$ to $(I : a)$.

1.2.1 Primary Ideals

Definition 5. $Q \lhd R$ is *primary* if $\forall a, b \in R$ with $ab \in Q$, either $b \in Q$ or $\exists n$ such that $a^n \in Q$.

Definition 6. If $I \lhd R$, then $\mathrm{rad}(I) = \{r \in R \mid \exists n\ r^n \in I\}$.

Proposition 4. *Q is primary if and only if* $\mathrm{rad}(Q)$ *is prime. If Q_1, Q_2 are primary and* $\mathrm{rad}(Q_1) = \mathrm{rad}(Q_2)$*, then $Q_1 \cap Q_2$ is primary. If R is Noetherian and $Q \lhd R$, then $\exists n$ such that* $\mathrm{rad}(Q)^n \subseteq Q$.

Theorem 5 (Primary Decomposition)**.** *If R is Noetherian and $I \lhd R$, then $\exists k$ and $Q_1, ..., Q_k \lhd R$ primary such that $I = Q_1 \cap \cdots \cap Q_k$.*

Proof. By R Noetherian, $\forall a \in R\ \exists n$ with $(I : a^n) = (I : a^{n+1})$, and for this n we have $(I+(a^n))\cap(I : a) = I$, so either I is already primary or we can write I as an intersection of bigger ideals, and apply Noetherian induction. □

Lemma 2. *If R is Noetherian, then for any $I \lhd R$ and $r \in R \setminus I$, there exists $s \in R$ such that $(I : rs)$ is prime.*

Theorem 6 (Uniqueness of radicals)**.** *If R is Noetherian, $I = Q_1 \cap \cdots \cap Q_k$ with $Q_i \lhd R$ primary and no Q_i containing $\cap_{j \neq i} Q_j$, and if $\mathfrak{p} \lhd R$ is prime, then $\exists r \in R$ with $(I : r) = \mathfrak{p}$ if and only if there is an i with* $\mathrm{rad}(Q_i) = \mathfrak{p}$*. In particular, the set* $\{\mathrm{rad}(Q_i)\}_{i \leq k}$ *is uniquely determined by I.*

Theorem 7 (Uniqueness of primaries with minimal radical)**.** *If R is Noetherian, $I = Q_1 \cap \cdots \cap Q_k$ with $Q_i \lhd R$ primary and* $\mathrm{rad}(Q_i) \not\subseteq \mathrm{rad}(Q_1)$ *for $i > 1$, then for n sufficiently large we have* $(I : \mathrm{rad}(Q_2)^n \cdots \mathrm{rad}(Q_k)^n) = Q_1$*, so Q_1 is uniquely determined by I and* $\mathrm{rad}(Q_1)$.

Chapter 2

Analysis

2.1 Basic Facts

2.1.1 Metric Spaces

Definition 7. A metric space is *complete* if every Cauchy sequence has a limit. It is *totally bounded* if it can be covered by finitely many subsets of size ϵ, for every $\epsilon > 0$.

Theorem 8. *A metric space is compact iff it is complete and totally bounded.*

Definition 8. A metric space is *sequentially compact* if every sequence has a bounded subsequence.

Theorem 9 (Bolzano-Weierstrauss)**.** *A subset of $\mathbb{R}^n$ is sequentially compact iff it is closed and bounded.*

Proposition 5. *A closed subset of a complete space is complete, and a complete subset of a metric space is closed.*

Theorem 10 (Baire Category Theorem)**.** *If M is either a complete metric space or a locally compact Hausdorff space, then a union of countably many nowhere dense subsets of M has empty interior.*

Definition 9. A space is called a *Baire space* if the intersection of any countable collection of open dense sets is dense.

Theorem 11 (Banach Fixed Point)**.** *Contraction mappings on complete metric spaces have unique fixed points.*

Corollary 1 (Picard-Lindelöf)**.** *The initial value problem $y'(t) = f(t, y(t)), y(t_0) = y_0$ for $t \in [t_0 - \epsilon, t_0 + \epsilon]$ has a unique solution for some $\epsilon > 0$ if f is Lipschitz continuous in y and continuous in t.*

Definition 10. If X, Y are Banach spaces, $U \subseteq X$ open, then $f : U \to Y$ is called *Frechét differentiable* at x if there exists a bounded linear operator $A : X \to Y$ such that $\|f(x+h) - f(x) - Ah\|_Y = o(\|h\|_X)$ as $h \to 0$. In this case we write $Df_x = A$.

Corollary 2 (Inverse Function Theorem). *If X, Y are Banach spaces, U an open neighborhood of 0 in X, $F : U \to Y$ continuously (Fréchet) differentiable and $DF_0 : X \to Y$ a bounded isomorphism from X to Y (with bounded inverse), then there exists an open neighborhood $V \subseteq Y$ of $F(0)$ and a continuously differentiable map $G : V \to X$ such that $F(G(y)) = y$ for all $y \in V$.*

Definition 11. A topological space is called *separable* if it contains a countable dense set. It is called *second countable* if its topology has a countable base.

Proposition 6. *Every second countable space is separable, and every separable metric space is second countable.*

Definition 12. If X, Y are metric spaces, then $f : X \to Y$ is called *uniformly continuous* if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall x, y \in X$ such that $d_X(x, y) < \delta$, we have $d_Y(f(x), f(y)) < \epsilon$.

Definition 13. A family of functions F is called *equicontinuous* at $x_0 \in X$ if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall f \in F, x \in X$ such that $d(x_0, x) < \delta$ we have $d(f(x_0), f(x)) < \epsilon$. F is *uniformly equicontinuous* if $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall f \in F, x, y$ such that $d(x, y) < \delta$ we have $d(f(x), f(y)) < \epsilon$.

Theorem 12 (Arzelà-Ascoli). *If $(f_n)_{n \in \mathbb{N}}$ defined on $[a, b]$ is uniformly bounded and equicontinuous, then there is a subsequence which converges uniformly.*

Theorem 13 (Ascoli Version 2). *If X is compact Hausdorff, then a subset of $C(X)$ (with the uniform norm) is compact iff it is closed, pointwise bounded, and equicontinuous.*

Definition 14. The *Bernstein polynomials* are defined by

$$b_{\nu,n}(x) = \binom{n}{\nu} x^\nu (1 - x)^{n-\nu}.$$

Theorem 14 (Weierstrauss approximation). *If $f : [a, b] \to \mathbb{C}$ is continuous, then $\forall \epsilon > 0$ there exists a polynomial $p \in \mathbb{C}[x]$ such that $\forall x \in [a, b]$, we have $|f(x) - p(x)| < \epsilon$.*

Proof. Suppose $[a, b] = [0, 1]$, and define $B_n(f)$ by

$$B_n(f) = \sum_{\nu=0}^{n} f(\tfrac{\nu}{n}) b_{\nu,n}.$$

If k is the number of times we flip heads in n independent random coinflips with bias x, then

$$\mathbb{E}[f(\tfrac{k}{n})] = B_n(f)(x),$$

so the law of large numbers shows that $B_n(f)$ approximates f. □

Theorem 15 (Stone-Weierstrauss for $\mathbb{R}$). *X compact Hausdorff, A a subalgebra of $C(X, \mathbb{R})$ which contains a non-zero constant. Then A is dense in $C(X, \mathbb{R})$ iff it separates points.*

Theorem 16 (Stone-Weierstrauss for $\mathbb{C}$). *X compact Hausdorff, $S \subseteq C(X, \mathbb{C})$ separates points. Then the complex unital $*$-algebra generated by S is dense in $C(X, \mathbb{C})$.*

Theorem 17 (Stone-Weierstrauss, Boolean ring version). *If X compact Hausdorff, $B \subseteq C(X, \mathbb{R})$ separates points, contains 1, is an $\mathbb{R}$-vector space, and contains $\max(f, g)$ whenever it contains f, g, then B is dense in $C(X, \mathbb{R})$.*

Lemma 3 (Finite Vitali Covering Lemma)**.** *If $B_1, ..., B_n$ are balls in a metric space, then there is a subcollection $B_{j_1}, ..., B_{j_k}$ which are disjoint, and which satisfy*

$$B_1 \cup \cdots \cup B_n \subseteq 3B_{j_1} \cup \cdots \cup 3B_{j_k},$$

where $3B_j$ is the ball with the same center as B_j and three times the radius.

Proof. Keep adding the biggest ball which is disjoint from the ones you have chosen so far to your collection. Then every ball you haven't chosen will intersect a larger ball that you have chosen. □

Lemma 4 (Infinite Vitali Covering Lemma)**.** *If $(B_i)_{i\in I}$ is a collection of balls in a metric space such that $\sup_{i\in I} \operatorname{rad}(B_i) < \infty$, then for any $c > 1$ there is a subcollection $J \subseteq I$ such that the B_j with $j \in J$ are disjoint, and $\cup_{i\in I} B_i \subseteq \cup_{j\in J}(1 + 2c)B_j$.*

Proof. Let $R = \sup \operatorname{rad}(B_i)$, and for each n choose a maximal disjoint subcollection of the balls with radius between R/c^n and R/c^{n+1} which are disjoint from the balls you have already chosen so far. Then every ball you haven't chosen will intersect a ball you have chosen, whose radius is at most a factor of c smaller. □

2.1.2 Topologies on $C(X, Y)$

Definition 15. The *compact-open* topology on $C(X, Y)$ has a subbase given by

$$V(K, U) = \{f : X \to Y \mid f(K) \subseteq U\}$$

for K compact and U open.

Proposition 7. *If Y is a metric space then $f_n \to f$ in the compact-open topology iff $\forall K \subseteq X$ compact we have $f_n \to f$ uniformly on K, so in this case the compact-open topology is the "topology of compact convergence". If X is compact as well, this becomes the uniform convergence topology.*

Proposition 8. *If Y is locally compact Hausdorff, composition $\circ : C(Y, Z) \times C(X, Y) \to C(X, Z)$ is continuous in the compact-open topology.*

Definition 16. If X, Y Banach spaces, $U \subseteq X$ open, $\mathcal{C}^m(U, Y)$ the m-times continuously Frechét-differentiable functions $U \to Y$, then the "compact-open" topology on $\mathcal{C}^m(U, Y)$ is induced by the seminorms

$$\rho_K(f) = \sup\{\|D^j f_x\| \mid x \in K,\ 0 \le j \le m\}$$

for $K \subseteq U$ compact.

Definition 17. The topology of *compact convergence* is defined by $f_n \to f$ iff for all K compact, $f_n|_K \to f|_K$ converges uniformly.

Proposition 9. *A set F of functions is called* normal *if every sequence of functions from F contains a subsequence that converges compactly to a continuous function.*

Theorem 18 (Montel)**.** *Any uniformly bounded family of holomorphic functions defined on an open subset of $\mathbb{C}$ is normal.*

Definition 18. The topology of *pointwise convergence* is the product topology on Y^X - this has $f_n \to f$ iff $f_n(x) \to f(x)$ for all x.

2.1.3 Measure

Definition 19. A set of subsets Σ of X is a *σ-algebra* over X if Σ staisfies: $\emptyset \in \Sigma$, $\forall A \in \Sigma$ we have $X \setminus A \in \Sigma$, and for any sequence $(A_n)_{n\in\mathbb{N}}$ of elements of Σ we have $\cup_n A_n \in \Sigma$.

Definition 20. If X is a topological space, the *Borel* σ-algebra is the smallest σ-algebra containing the open subsets of X (some authors replace "open" by "compact" in this definition).

Proposition 10. *If X is metric, then the Borel σ-algebra can be generated from the open sets by iterating the obvious construction (taking closure under countable unions and intersections) at most ω_1 times.*

Proof. Every open subset of X is a countable union of closed subsets of X, and ω_1 has uncountable cofinality. □

Corollary 3. *The Borel σ-algebra on $\mathbb{R}$ has cardinality $2^{\aleph_0}$.*

Definition 21. $\mu : \Sigma \to [0, \infty]$ is a *measure* if $\mu(\emptyset) = 0$ and $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ whenever $E_i \in \Sigma$ and $E_i \cap E_j = \emptyset$ for all $i \neq j$. (X, Σ, μ) is called a *measure space* if Σ is a σ-algebra over X and $\mu : \Sigma \to [0, \infty]$ is a measure.

Proposition 11. *If μ is a measure and $E_1 \subseteq E_2 \subseteq \cdots$ are measurable, then $\mu(\cup_{i=1}^{\infty} E_i) = \sup_i \mu(E_i)$. If $F_1 \supseteq F_2 \supseteq \cdots$ are measurable and $\mu(F_1) < \infty$, then $\mu(\cap_{i=1}^{\infty} F_i) = \inf_i \mu(F_i)$.*

Definition 22. A *signed measure* is a map $\mu : \Sigma \to [-\infty, \infty]$ which is countably additive (and doesn't take both $\infty, -\infty$ as values).

Theorem 19 (Hahn decomposition Theorem)**.** *If μ is a signed measure, then there exist measurable sets P, N such that $P \cup N = X, P \cap N = \emptyset$, and for all $E \subseteq P$ measurable we have $\mu(E) \geq 0$, while for all $E \subseteq N$ measurable we have $\mu(E) \leq 0$. This decomposition is unique up to null sets.*

Proof. Assume WLOG that μ doesn't take the value $-\infty$. Say a measurable set is *negative* if every measurable subset has measure ≤ 0. First we show that for any measurable D with $\mu(D) \leq 0$ there is a negative set $A \subseteq D$ with $\mu(A) \leq \mu(D)$: define a sequence of sets A_n, $A_0 = D$, each A_{n+1} given by removing a set of positive measure from A_n whose measure is at least half as large as the sup of measures of subsets (if finite), or at least 1 otherwise, and take $A = \cap_n A_n$. Next, we define N by making a sequence N_n with $N_0 = \emptyset$, and N_{n+1} given by adding a negative set to N_n whose measure is at least half as negative as the inf of measure of subsets (if finite), or at most -1 otherwise, and take $N = \cup_n N_n$. □

Theorem 20 (Jordan decomposition Theorem)**.** *If μ is a signed measure, there is a unique decomposition $\mu = \mu^+ - \mu^-$ where μ^+, μ^- are positive measures (at least one of which is finite), such that $\mu^+(E)$ is 0 for any negative set E and μ^- is 0 for any positive set E.*

Definition 23. If μ is a signed measure and $\mu = \mu^+ - \mu^-$ is its Jordan decomposition, then we set $|\mu| = \mu^+ + \mu^-$.

Definition 24. A *complex measure* is a countably additive function $\mu : \Sigma \to \mathbb{C}$. Equivalently, it is a complex combination of finite measures.

Definition 25. If μ, ν are (possibly signed) measures, then μ is *absolutely continuous* with respect to ν, written $\mu \ll \nu$, if $|\nu|(A) = 0 \implies |\mu|(A) = 0$.

Definition 26. We say that two (possibly signed or complex) measures μ, ν on X are *singular*, written $\mu \perp \nu$, if there are measurable sets A, B with $A \cup B = X$ such that B is μ-null and A is ν-null.

Theorem 21 (Lebesgue decomposition Theorem)**.** *If μ, ν are (possibly signed) σ-finite measures over X, then there is a unique pair of σ-finite measure μ_{ac}, μ_s such that $\mu = \mu_{ac} + \mu_s$, $\mu_{ac} \ll \nu$, and $\mu_s \perp \nu$.*

Proof. We just need to prove this in the finite, unsigned case. Let $\mathcal{N}$ be the collection of ν-null sets. Define μ_{ac} by

$$\mu_{ac}(A) = \inf_{N \in \mathcal{N}} \mu(A \setminus N).$$

μ_{ac} is clearly nonnegative and countably additive, and we clearly have $\mu_{ac} \ll \nu$. Set $\mu_s = \mu - \mu_{ac}$, taking $A = X$ and noting that the infimum must actually be attained, we see that there is a ν-null set N such that $\mu_s(X \setminus N) = 0$, so $\mu_s \perp \nu$.

For uniqueness, suppose that $\mu = \mu_1 + \mu_2$ with $\mu_1 \ll \nu, \mu_2 \perp \nu$. Since $\mu_1 \le \mu$ and $\mu_1 \ll \nu$, we have

$$\mu_1(A) = \inf_{N \in \mathcal{N}} \mu_1(A \setminus N) \le \inf_{N \in \mathcal{N}} \mu(A \setminus N) = \mu_{ac}(A),$$

so $\mu_1 \le \mu_{ac}$. Thus $\mu_{ac} - \mu_1 = \mu_2 - \mu_s$ is both ν-absolutely continuous and ν-singular, so $\mu_1 = \mu_{ac}$. □

Constructing measures

Definition 27. On any set, the *counting measure* takes every finite set to its size and every infinite set to ∞.

Definition 28. A measure space (X, Σ, μ) is *complete* if every subset of a null set (that is, a set with measure 0) is in Σ. If Z is the collection of all subsets of null sets, then define Σ_0 to be the σ-algebra generated by Σ and Z, and $\mu_0(C) = \inf\{\mu(D) \mid C \subseteq D \in \Sigma\}$, and define the *completion* of (X, Σ, μ) to be (X, Σ_0, μ_0).

Proposition 12. *The completion of a measure space is always a complete measure space, and in fact $\Sigma_0 = \{A \cup B \mid A \in \Sigma, B \in Z\}$.*

Definition 29. $\varphi : 2^X \to [0, \infty]$ is an *outer measure* if $\varphi(\emptyset) = 0$, $A \subseteq B \implies \varphi(A) \le \varphi(B)$, and for any sequence $(A_n)_{n \in \mathbb{N}}$ we have have $\varphi(\cup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \varphi(A_i)$.

Definition 30. If φ is an outer measure over X, we say that E is *φ-measurable* if $\forall A \subseteq X$, we have $\varphi(A) = \varphi(A \cap E) + \varphi(A \cap E^c)$. We write Σ_φ for the collection of all φ-measurable sets.

Theorem 22. *If φ is an outer measure, then Σ_φ is a σ-algebra, and the restriction of φ to Σ_φ is a complete measure.*

Proof. If $E_i \in \Sigma_\varphi$ are pairwise disjoint and $E = \cup_{i=1}^{\infty} E_i$, then for any A we have

$$\varphi(A) \le \varphi(A \cap E^c) + \varphi(A \cap E) \le \varphi(A \cap E^c) + \sum_{i=1}^{\infty} \varphi(A \cap E_i) = \sup_n \left(\varphi(A \cap E^c) + \sum_{i=1}^{n} \varphi(A \cap E_i) \right) \le \varphi(A).$$

Taking $A = E$ shows that $\varphi(E) = \sum_{i=1}^{\infty} \varphi(E_i)$. □

Definition 31. If X is a metric space and φ is an outer measure over X, we say that φ is a *metric outer measure* if $d(E,F) > 0 \implies \varphi(E \cup F) = \varphi(E) + \varphi(F)$.

Theorem 23. *If φ is a metric outer measure, then all Borel sets are φ-measurable.*

Proof. If U is open, let $U_n = \{x \in U \mid B(x, \frac{1}{n}) \subseteq U\}$, and note that for any n, $d(U_n, U_{n+1}^c) \geq \frac{1}{n(n+1)} > 0$. For any A with $\varphi(A) < \infty$ we then have

$$\sum_{n \text{ odd}} \varphi(A \cap (U_{n+1} \setminus U_n)) \leq \varphi(A) < \infty,$$

and similarly for n even, so the tails of the sum go to zero. Then for any A we have

$$\varphi(A) \leq \varphi(A \cap U^c) + \varphi(A \cap U) \leq \inf_n \Big(\varphi(A \cap U^c) + \varphi(A \cap U_n) + \sum_{m \geq n} \varphi(A \cap (U_{m+1} \setminus U_m))\Big) \leq \varphi(A). \quad \square$$

Definition 32. A collection of sets S is a *semi-ring* if $\emptyset \in S$, for any $A, B \in S$ we have $A \cap B \in S$, and for any $A, B \in S$ there exists n and pairwise disjoint $C_1, ..., C_n \in S$ such that $A \setminus B = \cup_{i=1}^n C_i$.

Definition 33. If S is a collection of sets, then a map $\mu : S \to [0, \infty]$ is a *pre-measure* if $\mu(\emptyset) = 0$ and for any sequence A_n of pairwise disjoint sets in S such that $\cup_{i=1}^\infty A_i \in S$, we have $\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$.

Theorem 24 (Carathéodory Extension Theorem)**.** *If S is a semi-ring of subsets of X and $\mu_0 : S \to [0, \infty]$ is a pre-measure, then if we define μ^* by*

$$\mu^*(E) = \inf\Big\{\sum_{i=1}^\infty \mu_0(A_i) \mid A_i \in S,\ E \subseteq \bigcup_{i=1}^\infty A_i\Big\},$$

then μ^* is an outer measure over X with $\mu^*(A) = \mu_0(A)$ for all $A \in S$, and $S \subseteq \Sigma_{\mu^*}$.

Definition 34. A pre-measure $\mu : S \to [0, \infty]$ with S a collection of subsets of X is *σ-finite* if there exists a sequence $A_n \in S$ with $\mu(A_i) < \infty$ and $X = \cup_{i=1}^\infty A_i$.

Theorem 25 (Hahn-Kolmogorov)**.** *If μ_0 is a pre-measure on a semi-ring S, then it extends to a measure μ on the σ-algebra Σ generated by S. If μ_0 is σ-finite, then this extension is unique.*

Proof. Let μ^* be the associated outer measure from the Carathéodory extension theorem, and suppose μ' is a different measure extending μ on $\Sigma' \supseteq S$. Then for any $E \in \Sigma' \cap \Sigma_{\mu^*}$, we clearly have $\mu'(E) \leq \mu^*(E)$. By σ-finiteness and the fact that μ' is countably additive, we can assume WLOG that $\mu^*(X) = \mu'(X) < \infty$, but then $\mu'(E^c) \leq \mu^*(E^c)$ implies $\mu'(E) = \mu^*(E)$ since E is μ^*-measurable. $\square$

Proposition 13. *Let $\mu_0, \mu^*, \mu, S, \Sigma, \Sigma_{\mu^*}$ be as above. If μ_0 is σ-finite, then Σ_{μ^*} is the completion of Σ - in fact, for any $E \in \Sigma_{\mu^*}$, there is a countable intersection of countable unions of elements of S which contains E and differs from it in a null set.*

Theorem 26 (Lebesgue outer measure)**.** *Let S be the collection of half-open intervals $[a, b)$ for $a \leq b \in \mathbb{R}$, and define $\lambda_0 : S \to [0, \infty)$ by $\lambda_0([a, b)) = b - a$. Then S is a semi-ring, λ_0 is a pre-measure, and the associated outer measure λ^* is a translation-invariant metric outer measure over $\mathbb{R}$ with $\lambda^*([0, 1]) = 1$.*

Proof. Suppose that $[a,b) = \cup_{i=1}^{\infty} A_i$, where the A_i are pairwise disjoint half-open intervals. Then the set of left endpoints of the A_i is well-ordered (any descending sequence must have a limit in $[a,b)$, and this limit must be contained in some A_i), so we can show by well-founded induction that if $A_i = [c,d)$, then $\sum_{A_j < A_i} \lambda_0(A_j) = c - a$.

Alternate proof: Let $A' = [a, b-\epsilon]$, and if $A_i = [c_i, d_i)$ let $A_i' = (c_i - \epsilon/2^i, d_i)$. Then by compactness, some finite subset of the A_i's cover A'. □

Definition 35. If λ^* is constructed as above, then a set is called *Lebesgue-measurable* if it is in Σ_{λ^*}, and $\lambda^* |_{\Sigma_{\lambda^*}}$ is called the *Lebesgue measure*, and written as λ.

Theorem 27 (Lebesgue-Stieltjes measure)**.** *If I is an interval and $g : I \to \mathbb{R}$ is monotone increasing, set $g_-(x) = \sup_{y<x} g(y)$, then there is a unique Borel measure μ_g such that $\mu_g([a,b)) = g_-(b) - g_-(a)$.*

Definition 36. If g has bounded variation, then we define the *signed Lebesgue-Stieltjes measure* μ_g by writing $g = g_1 - g_2$ with g_1, g_2 monotone increasing, and $\mu_g = \mu_{g_1} - \mu_{g_2}$.

Definition 37. A Borel measure μ is *locally finite* if every point has an open neighborhood of finite measure. It is *inner regular* if for every Borel set B, $\mu(B)$ is the supremum of $\mu(K)$ over all compact $K \subseteq B$. It is *outer regular* if for all B, $\mu(B)$ is the infimum of $\mu(U)$ over all open U containing B. A measure is *Radon* if it is inner regular, outer regular, and locally finite.

Proposition 14. *Every locally finite Borel measure over $\mathbb{R}$ is a Lebesgue-Stieltjes measure, and every Lebesgue-Stieltjes measure is a Radon measure. More generally, every locally finite Borel measure on $\mathbb{R}^n$ is Radon.*

Theorem 28 (Product measures)**.** *If μ, ν are pre-measures on semi-rings S, T, respectively, then the collection of rectangles $S \times T$ is a semi-ring, and $\mu \times \nu$ is a pre-measure on $S \times T$.*

Proof. Suppose $E \times F \in S \times T$ is a countable union of disjoint rectangles $E_i \times F_i$. We'll show that for any $M < \mu(E)$ and $N < \nu(F)$, we have $MN \le \sum_i \mu(E_i)\nu(F_i)$. Let $A_n = \{x \in E \mid \sum_{i=1}^{n} 1_{x \in E_i} \cdot \nu(F_i) \ge N\}$. Each A_n is a finite union of elements of S, and $\cup_n A_n = E$ since for each $x \in E$, the collection of F_is with $x \in E_i$ is disjoint and covers F, so some finite subset of them must have measure at least N. Thus there is some n such that $\mu(A_n) \ge M$, and for this n we have $MN \le \sum_{i=1}^{n} \mu(E_i)\nu(F_i)$. □

Theorem 29 (Infinite products)**.** *Let I be any index set. If μ_i are pre-measures on semi-rings S_i, such that each S_i has an element X_i with $\mu_i(X_i) = 1$, and if we let $S = \prod'_{i\in I} S_i$ be the set of rectangles $\prod_{i \in I} A_i$ such that $A_i = X_i$ for all but finitely many i and define $\mu = \prod_i \mu_i$, then S is a semi-ring and μ is a pre-measure on S.*

Proof. Suppose that $A = \cup_{n=1}^{\infty} A_n$ with $A, A_n \in S$ and the A_ns disjoint, but that $\mu(A) > \sum_n \mu(A_n)$. Each A_n only has finitely many coordinates i which are not equal to X_i, so at most countably many coordinates in I are relevant - rename these relevant coordinates as $1, 2, \ldots$. Write $A = E \times F$, $A_n = E_n \times F_n$, with $E, E_n \in S_1$ and $F, F_n \in \prod'_{i \ne 1} S_i$, and write $\mu^1 = \prod_{i \ne 1} \mu_i$. By the argument for the finite case, there is some $x_1 \in E$ such that $\mu^1(F) > \sum_n 1_{x_1 \in E_n} \cdot \mu^1(F_n)$. Continuing inductively, we find a sequence of coordinates $x_1, x_2, \ldots$ such that for each k, when we restrict the first k coordinates to be $x_1, \ldots, x_k$, the two sides don't add up. But then no point with $(x_1, x_2, \ldots)$ as the relevant countably many coordinates can be an element of any A_n (take k to be larger than the finitely many coordinates i of A, A_n which are not equal to X_i), contradicting the assumption $A = \cup_n A_n$. $\square$

Corollary 4 (Lebesgue measure on $\mathbb{R}^n$). *For every n, there is a translation-invariant metric outer measure λ^* on $\mathbb{R}^n$ with $\lambda^*([0,1]^n) = 1$. If T is a linear transformation and $A \subseteq \mathbb{R}^n$, then $\lambda^*(T(A)) = |\det(T)|\lambda^*(A)$. The associated measure λ is a Radon measure.*

Proof. For the statement about linear transformations, it's enough to check this for shear and stretch transformations in the case A is a box, and this can done done using a standard dissection argument (the pieces are Borel sets). $\square$

Definition 38. If X, Y are measure spaces with measures μ, ν, then $X \times Y$ has a measure $\mu \times \nu$ given by applying the Carathéodory extension Theorem 24 to the product pre-measure contructed in Theorem 28 - this measure is called the *maximal product measure* on $X \times Y$.

Proposition 15. *If $A \subseteq X \times Y$ is $\mu \times \nu$-null, then the set of $y \in Y$ such that $A_y = \{x \in X \mid (x, y) \in A\}$ is not μ-null is ν-null.*

Proof. Pick $\epsilon > 0$, and let E be the set of $y \in Y$ such that $\mu(A_y) > \epsilon$. If $A \subseteq \cup_{n=1}^{\infty} R_n$ such that the R_n are measurable rectangles, and E_k is the set of y such that $\mu((\cup_{n=1}^{k} R_k)_y) > \epsilon$, then $\cup_k E_k = E$, so if $\nu(E) > \delta$ then some $\nu(E_k) > \delta/2$, so $\mu \times \nu(\cup_n R_n) > \epsilon\delta/2$. $\square$

Theorem 30 (Cavalieri Principle). *If X, Y are σ-finite measure spaces and $A, B \subseteq X \times Y$ are measurable with $\mu(A_y) = \mu(B_y)$ for ν-almost every $y \in Y$, then $\mu \times \nu(A) = \mu \times \nu(B)$.*

Proof. TODO: find a proof that doesn't use integrals. $\square$

Example 1. To see σ-finiteness is necessary, take X to be $[0,1]$ with counting measure, Y to be $[0,1]$ with Lebesgue measure, A to be $\{0\} \times Y$, and B to be the diagonal.

Theorem 31 (Lebesgue Density Theorem). *If $E \subseteq \mathbb{R}^n$, then for Lebesgue-a.e. x in E we have*

$$\lim_{r \to 0} \frac{\lambda^*(E \cap B_r(x))}{\lambda(B_r(x))} = 1.$$

Proof. Let A_t be the set of points such that the left hand side (with a $\liminf$ instead) is less than $1 - t$, and let U_ϵ be an open set containing A_t with $\lambda^*(U \setminus A) \le \epsilon$. Then for each point x in A_t, we can find an r such that the left hand side of the above is at most $1 - t$ and such that $B_r(x) \subseteq U_\epsilon$. Now apply the Vitali Covering Lemma to get a collection $(B_i)_{i \in I}$ is disjoint balls contained in U_ϵ such that $A_t \subseteq \cup_i 5B_i$. Then since $\cup_i B_i \subseteq U_\epsilon$, we have

$$\lambda(\cup_i B_i) - \epsilon \le \lambda^*(A \cap (\cup_i B_i)) \le \lambda^*(E \cap (\cup_i B_i)) \le \sum_i (1-t)\lambda(B_i) = (1-t)\lambda(\cup_i B_i),$$

so $\lambda(\cup_i B_i) \le \epsilon/t$, and since $A_t \subseteq \cup_i 5B_i$ we get $\lambda^*(A_t) \le 5^n\epsilon/t$. Since $\epsilon > 0$ was arbitrary, $\lambda^*(A_t) = 0$. $\square$

Definition 39. If X is a metric space and $S \subseteq X$, we set

$$H^d_\delta(S) = \inf\Big\{ \sum_{i=1}^{\infty} \operatorname{diam}(U_i)^d \mid S \subseteq \bigcup_{i=1}^{\infty} U_i,\ \operatorname{diam}(U_i) < \delta \Big\}$$

and

$$H^d(S) = \sup_{\delta>0} H^d_\delta(S).$$

This is a metric outer measure, called the *Hausdorff measure*.

Theorem 32. *In $\mathbb{R}^n$, we have $H^n(B) = 2^n$, where B is the unit ball.*

Proof. This follows from the isodiametric inequality: the volume of a set of diameter 2 is at most the volume of the unit ball. Suppose that K has diameter 2, then $K - K \subseteq 2B$, so by Brunn-Minkowski we have $\lambda(K) \le \lambda(\frac{1}{2}(K-K)) \le \lambda(B)$. $\square$

Definition 40. If X is a locally compact Hausdorff space, then a Borel measure μ is called a *Borel regular measure* if it is locally finite, outer regular, and inner regular on open sets (note that the only difference from a Radon measure is that we only require inner regularity on open sets).

Definition 41. A *field* of sets is a collection of sets which is closed under finite intersections, unions and complements. A *content* on a field of sets $\mathcal{A}$ is a function $\lambda : \mathcal{A} \to [0,\infty]$, such that $\lambda(A)$ is increasing in A, and such that for any $A_1, A_2 \in \mathcal{A}$ disjoint, we have $\lambda(A_1 \cup A_2) = \lambda(A_1) + \lambda(A_2)$.

Definition 42. A *content* on a locally compact Hausdorff space is a function $\lambda : \mathcal{K} \to [0,\infty)$, where $\mathcal{K}$ is the collection of compact subsets of X, such that $\lambda(K)$ is increasing in K, $\lambda(K_1 \cup K_2) \le \lambda(K_1) + \lambda(K_2)$, and such that for any $K_1, K_2 \in \mathcal{K}$ disjoint, we have $\lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$. A content λ is *regular* if for any $K \in \mathcal{K}$, we have $\lambda(K) = \inf\{\lambda(L) \mid K \subseteq \mathrm{int}(L)\}$.

Lemma 5. *For every content λ on a locally compact Hausdorff space X, there is a unique Borel regular measure μ on X such that for all open sets U we have $\mu(U) = \sup\{\lambda(K) \mid K \subseteq U\}$. If λ is a regular content, then μ extends λ.*

Proof. Define μ on open sets as in the theorem statement, and define $\mu^* : \mathcal{P}(X) \to [0,\infty]$ by $\mu^*(A) = \inf\{\mu(U) \mid A \subseteq U\}$. μ is finite on the interior of any compact set, so μ^* is locally finite.

First we show that μ^* is an outer measure: If $A = \cup_{n=1}^\infty A_n$, then pick U_n open with $A_n \subseteq U_n$ and $\mu^*(U_n) \le \mu^*(A_n) + \epsilon/2^n$, and let $U = \cup_n U_n$. Pick $K \subseteq U$ compact with $\mu(U) \le \lambda(K) + \epsilon$, then some finite subset of the U_n cover K, say $U_1, ..., U_k$. We just need to show that $\lambda(K) \le \sum_{i=1}^k \mu(U_i)$, and this follows if we can construct compact $K_i \subseteq U_i$ with $K \subseteq \cup_i K_i$, and for this, we may assume that $k = 2$. Let $L_1 = K \setminus U_2, L_2 = K \setminus U_1$, then L_1, L_2 are disjoint compact sets of a Hausdorff space, so there are disjoint open sets V_1, V_2 with $L_i \subseteq V_i$. Now take $K_1 = K \setminus V_2, K_2 = K \setminus V_1$.

Now we show that open sets are μ^*-measurable. Let U be open and $A \subseteq X$ be arbitrary. We want to show that for any open $V \supseteq A$, we have $\mu(V) \ge \mu^*(A \cap U) + \mu^*(A \cap U^c)$, so we just need to show that $\mu(V) \ge \mu(V \cap U) + \mu^*(V \setminus U)$. For any compact $K \subseteq V \cap U$, let $W = V \setminus K$, then for any compact $L \subseteq W$ we have $\mu(V) \ge \lambda(K \cup L) = \lambda(K) + \lambda(L)$, so $\mu(V) \ge \lambda(K) + \mu(W) \ge \lambda(K) + \mu^*(V \setminus U)$, so $\mu(V) \ge \mu(V \cap U) + \mu^*(V \setminus U)$. $\square$

Definition 43. If G is a locally compact Hausdorff group and μ is a Borel measure on G, then μ is a *left Haar measure* on G if $\mu(gE) = \mu(E)$ for $g \in G$ and E Borel, and μ is Borel regular.

Theorem 33 (Haar measure)**.** *If G is a locally compact Hausdorff group, then there is a unique (up to scale) left Haar measure on G.*

Sketch. For K compact and V with nonempty interior, let $(K : V)$ be the minimum number of left translates of V that are needed to cover K. Pick K_0 compact with nonempty interior. For every U, define μ_U on compact sets by

$$\mu_U(K) = \frac{(K : U)}{(K_0 : U)}.$$

Then for all K, U we have $0 \le \mu_U(K) \le (K : K_0)$. We consider each μ_U as a point in $\prod_K [0, (K : K_0)]$. For each open V, let $C(V)$ be the closure of the set of μ_Us with $U \subseteq V$. By compactness, there exists $\mu \in \cap_V C(V)$. For K_1, K_2 disjoint, find V open such that $K_1 V^{-1} \cap K_2 V^{-1} = \emptyset$, then from $\mu \in C(V)$ we see that $\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$. Thus μ defines a left-invariant content on the compact sets of G, so there is a left-invariant Borel regular measure on G by Lemma 5.

To prove uniqueness, suppose μ, ν are left Haar measures and K, L are compact, L with nonempty interior. Since $x \mapsto \frac{1}{\mu(Lx)}$ is integrable on compact sets (to see this, note for any continuous g supported in L, the function $x \mapsto \int_G g(tx) d\mu(t)$ is continuous in x), then by a version of Fubini and left-invariance we have

$$\begin{aligned}
\nu(K) &= \int_G \int_G \frac{1_{x \in K, yx \in L}}{\mu(Lx^{-1})} d\mu(y) d\nu(x) \\
&= \int_G \int_G \frac{1_{y^{-1}x \in K, x \in L}}{\mu(Lx^{-1}y)} d\mu(y) d\nu(x) \\
&= \int_G \int_G \frac{1_{y^{-1} \in K, x \in L}}{\mu(Ly)} d\mu(y) d\nu(x) \\
&= \nu(L) \int_G \frac{1_{y^{-1} \in K}}{\mu(Ly)} d\mu(y),
\end{aligned}$$

so $\nu(K)/\nu(L)$ does not depend on ν. TODO: find a proof that doesn't require Fubini, or even integration.

If G is σ-compact, then we can prove uniqueness as follows instead: the Radon-Nikodym derivative $h = \frac{d\mu}{d(\mu+\nu)}$ is left-invariant up to null sets, so by Fubini applied to $\int_{G \times G} |h(gx) - h(x)|\, d(g, x)$, we see that there is some x such that $h(gx) = h(x)$ for almost all g, so μ and ν differ by a constant factor. □

2.1.4 Integration

Definition 44. If $f : X \to Y$ and $\mathcal{B}$ is a σ-algebra on Y, then $\sigma(f)$ is the σ-algebra on X generated by $f^{-1}(S)$ for $S \in \mathcal{B}$. We say that $f : (X, \Sigma) \to (Y, \mathcal{B})$ is Σ-*measurable*, or just *measurable* if Σ is clear, if $\sigma(f) \subseteq \Sigma$ (if unspecified, $\mathcal{B}$ is usually taken to be the Borel sets of Y).

Proposition 16. *$f : (X, \Sigma) \to [-\infty, \infty]$ is measurable iff $f^{-1}([-\infty, a]) \in \Sigma$ for all $a \in \mathbb{R}$. If $f_1, ..., f_n$ are measurable and $g : \mathbb{R}^n \to [-\infty, \infty]$ is Borel measurable, then $g(f_1, ..., f_n)$ is measurable. If f_k is a sequence of measurable functions, then $\sup f_k$ is measurable.*

Proposition 17. *If $f_k : X \to Y$ is a sequence of measurable functions to a metric space and $f_k \to f$ pointwise, then f is measurable.*

Proof. For any open set U the collection of $x \in X$ such that $f_k(x)$ are eventually all in U is measurable, and this set contains $f^{-1}(U)$ and is contained $f^{-1}(\overline{U})$. Since every open set in a metric space is a countable union of open subsets whose closures are contained in it, the preimage of every open set is measurable. □

Definition 45. A *simple function* is a function which can be written as a finite linear combination of measurable sets.

Definition 46. For $f \geq 0$ measurable (up to a null set), we define the *integral* of f with respect to a measure $\mu : \Sigma \to [0, \infty]$ to be

$$\int f \, d\mu = \sup \Big\{ \sum_{i=1}^{k} c_i \mu(A_i) \mid c_1, ..., c_k \geq 0, \; A_1, ..., A_k \in \Sigma, \; \sum_{i=1}^{k} c_i \cdot 1_{x \in A_i} \leq f(x) \Big\}.$$

A measurable (up to a null set) complex-valued function f is *integrable* if $\int |f| d\mu < \infty$. We extend the integral to all integrable functions by linearity.

Proposition 18. *For $f, g \geq 0$ measurable, we have $\int f + g \; d\mu = \int f \; d\mu + \int g \; d\mu$.*

Proof. For any finite $S \subset [0, \infty]$, define f_S by

$$f_S(x) = \max\{s \in S \mid s \leq f(x)\}.$$

Note f_S is a simple function and $\int f \; d\mu = \sup_S \int f_S \; d\mu$. For any S and any n, if we let $S_n = \{\frac{k}{n} s \mid k \leq n, \; s \in S\}$, then $(f + g)_S \leq \frac{n-1}{n}(f_{S_n} + g_{S_n})$. □

Proposition 19. *Any Riemann integrable function $f : [0, 1] \to \mathbb{C}$ is Lebesgue integrable, with the same integral.*

Proposition 20. *If $f : X \to [0, \infty]$ is measurable, then $\{(x, t) \mid 0 \leq t \leq f(x)\}$ is measurable in $X \times [0, \infty]$, with $\mu \times \lambda$-measure $\int_X f \; d\mu = \int_0^\infty \mu(\{x \mid f(x) \geq t\}) \; dt$.*

Theorem 34 (Monotone Convergence Theorem)**.** *If f_k is a sequence of measurable functions with $0 \leq f_k \leq f_{k+1}$ for all k and f is the pointwise limit of the f_k, then f is measurable and $\int f \; d\mu = \lim_k \int f_k \; d\mu$.*

Proof. It's enough to prove this when f is the characteristic function of a measurable set A. Fix $\epsilon > 0$, and for each k set $A_k = \{x \mid f_k(x) \geq 1 - \epsilon\}$, then from $\cup_k A_k = A$, we have $\lim_k \mu(A_k) = \mu(A)$, so $\lim_k \int f_k \; d\mu \geq (1 - \epsilon)\mu(A)$. □

Lemma 6 (Fatou's Lemma)**.** *If $f_k \geq 0$ are measurable, then $\int \liminf_k f_k \; d\mu \leq \liminf_k \int f_k \; d\mu$.*

Proof. $\int \liminf_k f_k \; d\mu = \lim_k \int \inf_{l \geq k} f_l \; d\mu \leq \liminf_k \int f_k \; d\mu$. □

Corollary 5. *If f_k measurable, $|f_k| \leq g$, g integrable, then*

$$\int \liminf f_k \; d\mu \leq \liminf \int f_k \; d\mu \leq \limsup \int f_k \; d\mu \leq \int \limsup f_k \; d\mu.$$

Theorem 35 (Dominated Convergence Theorem)**.** *If f_k measurable, $|f_k| \leq g$, g integrable, $f_k \to f$ pointwise, then $\lim_k \int f_k \; d\mu = \int f \; d\mu$, and $\lim_k \int |f_k - f| \; d\mu = 0$.*

Theorem 36 (Radon-Nikodym Theorem)**.** *If μ, ν are σ-finite measures on X (ν possibly signed or complex) and $\nu \ll \mu$, then there exists a measurable function f (unique up to a μ-null set) such that for any measurable set A, $\nu(A) = \int_A f \; d\mu$.*

Proof. We just need to prove this in the positive, finite case. Let $\mathcal{F}$ be the family of measurable functions f such that for all measurable A, $\nu(A) \geq \int_A f\ d\mu$. Note that $\mathcal{F}$ is closed under maximum, and by the Monotone Convergence Theorem 34 $\mathcal{F}$ is closed under countable monotone limits, so there is some $f \in \mathcal{F}$ with $\int_X f\ d\mu = \sup_{g\in\mathcal{F}} \int_X g\ d\mu$. Let $\nu_0 = \nu - \int f\ d\mu$. If $\nu_0(X) > 0$, take $\epsilon > 0$ such that $\nu_0(X) > \epsilon\mu(X)$, and let (N, P) be a Hahn decomposition 19 of $\nu_0 - \epsilon\mu$. But then $f + \epsilon \cdot 1_P \in \mathcal{F}$ and $\mu(P) > 0$, contradicting our choice of f. □

Definition 47. If μ, ν have $\nu = \int f\ d\mu$, then the *Radon-Nikodym derivative* $\frac{d\nu}{d\mu}$ is defined to be the equivalence class of f when we quotient by μ-null functions.

Proposition 21. *Where the relevant Radon-Nikodym derivatives make sense, we have* $\frac{d(\nu+\mu)}{d\lambda} = \frac{d\mu}{d\lambda} + \frac{d\nu}{d\lambda}$, $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$, $\frac{d|\nu|}{d\mu} = |\frac{d\nu}{d\mu}|$, *and* $\int g\ d\mu = \int g\frac{d\mu}{d\lambda}\ d\lambda$.

Proposition 22. *If $E \subseteq X \times Y$ is measurable and $\mu \times \nu(E) < \infty$, then for μ-almost every $x \in X$ E_x is measurable up to a ν-null set, the function $g(x) = \mu(E_x)$ is measurable up to a μ-null set, and $\int g\ d\mu = \mu \times \nu(E)$.*

Proof. By definition of $\mu \times \nu$, there is an $F \supseteq E$ which is a countable decreasing intersection of countable unions of measurable rectangles, such that $\mu \times \nu(E) = \mu \times \nu(F)$. Since $\mu \times \nu(E) < \infty$, $F \setminus E$ is $\mu \times \nu$-null, so we may replace E by F without changing g (aside from on a μ-null set) by Proposition 15 and then apply monotone 34 and dominated 35 convergence to reduce to the case of a finite union of measurable rectangles. □

Theorem 37 (Fubini's Theorem)**.** *If $\int_{X\times Y} |f(x, y)|\ d(x, y) < \infty$, where $d(x, y)$ is the maximal product measure on $X \times Y$, then for a.e. $x \in X$ $f(x, y)$ is integrable in y, and we have $\int_{X\times Y} f(x, y)\ d(x, y) = \int_X \int_Y f(x, y)\ dy\ dx$.*

Theorem 38 (Tonelli's Theorem)**.** *If X, Y are σ-finite, then $\int_{X\times Y} |f(x, y)|\ d(x, y) = \int_X \int_Y |f(x, y)|\ dy\ dx$.*

Proof. Assume $f \geq 0$. The assumptions of either Fubini or Tonelli imply that f can be written as the pointwise limit of an increasing sequence ϕ_n of nonnegative simple functions that each vanish outside a set of finite measure. Thus, using Proposition 22, for almost every fixed x the function $y \mapsto f(x, y) = \lim_n \phi_n(x, y)$ is measurable up to a null set, and by monotone convergence 34 the function $x \mapsto \int_Y f(x, y)\ dy = \lim_n \int_Y \phi_n(x, y)\ dy$ is measurable up to a null set. Applying monotone convergence and Proposition 22 again, we get

$$\begin{aligned} &\int_X \int_Y f(x, y)\ dy\ dx = \lim_n \int_X \int_Y \phi_n(x, y)\ dy\ dx \\ &= \lim_n \int_{X\times Y} \phi_n(x, y)\ d(x, y) = \int_{X\times Y} f(x, y)\ d(x, y). \end{aligned}$$

□

2.1.5 Banach spaces

Definition 48. If V is a vector space (over $\mathbb{R}$ or $\mathbb{C}$), then $p : V \to [0, \infty)$ is a *seminorm* if $p(0) = 0$, $p(cv) = |c|p(v)$ for c a scalar and $v \in V$, and $p(v + w) \leq p(v) + p(w)$ for $v, w \in V$. p is a *norm* if additionally $p(v) = 0 \iff v = 0$.

Lemma 7 (Zabreiko's Lemma)**.** *If X is a Banach space and $p : X \to [0, \infty)$ is a seminorm such that for all absolutely convergent series $\sum_{n=1}^{\infty} x_n$ in X we have $p(\sum_n x_n) \leq \sum_n p(x_n)$, then p is continuous, that is, $p(x) \ll \|x\|$.*

Proof. Let $A_n = p^{-1}([0,n])$, then since $X = \cup_n \overline{A_n}$, there is some n such that $\overline{A_n}$ has nonempty interior by the Baire category theorem. Since $\overline{A_n}$ is convex and symmetric, some open ball $B_R(0)$ around 0 is contained in $\overline{A_n}$. We claim that $B_R(0) \subseteq A_n$ as well: if $\|x\| < R$, pick $0 < q < 1$ such that $\frac{\|x\|}{1-q} < R$, set $y = \frac{R}{\|x\|}x$, then since $y \in \overline{A_n}$ there exists $y_0 \in A_n$ with $\|y - y_0\| < qR$, and then inductively we find $y_0, y_1, ... \in A_n$ such that for each k, we have $\|y - \sum_{i<k} y_i\| < q^k R$: y_k is taken to be a point in A_n with $\|q^{-k}(y - \sum_{i<k} y_i) - y_k\| < qR$. Since $\|y_k\| < R + qR$ for each k, the sum $\sum_k q^k y_k = y$ is absolutely convergent, so by hypothesis $p(y) \le \sum_k q^k p(y_k) \le \frac{n}{1-q}$, so $p(x) \le \frac{\|x\|}{R}\frac{n}{1-q} < n$, so $x \in A_n$. □

Theorem 39 (Open Mapping Theorem)**.** *If X, Y Banach spaces, $A : X \to Y$ surjective and continuous, then A takes open sets to open sets.*

Proof. For $y \in Y$, set $p(y) = \inf\{\|x\| \mid Ax = y\}$ in Zabreiko's Lemma. □

Theorem 40 (Bounded Inverse Theorem)**.** *If X, Y Banach spaces, $A : X \to Y$ bijective and continuous, then A^{-1} is also bounded.*

Theorem 41 (Closed Graph Theorem)**.** *If X, Y Banach spaces, then $A : X \to Y$ is bounded iff the graph is closed in $X \times Y$.*

Proof. For $x \in X$, set $p(x) = \|Ax\|$ in Zabreiko's Lemma. □

Theorem 42 (Uniform Boundedness Theorem/Banach Steinhaus)**.** *If X is Banach, Y a normed vector space, F a set of continuous linear functions $T : X \to Y$. If $\forall x \in X$ $\sup_{T\in F} \|T(x)\| < \infty$, then $\sup_{T\in F} \|T\| < \infty$.*

Proof. Set $p(x) \in \sup_{T\in F} \|T(x)\|$ in Zabreiko's Lemma. □

Corollary 6. *If a sequence of bounded operators from a Banach space to a normed space converges pointwise, then the pointwise limit is a bounded operator.*

Bibliography