Module 1 : Fundamentals of Logic

→ Basic connectives and truth tables.

→ Logic equivalence

→ The laws of logic.

→ Logical Implication.

→ Rules of Inference.

→ The use of Quantifiers.

→ Definitions and the proofs of theorems.

Fundamentals of Logic

* **Propositions** :- A proposition is a statement which in a given context is either true or false.

Ex :- i) when a coin is tossed, we get either head or tail — True

ii) one is greater than two — false.

Propositions are denoted by p, q, r, s, t (usually small letters)

The truth or the falsity of a proposition is called its truth value.

If a proposition is true, its value is 1.

If a proposition is false, its value is 0.

* **Logical connectives** :- The words or phrases like 'not', 'and', 'or', 'if and only if', 'if ... then' which are used to form new propositions using given proposition are called logical connectives. The new propositions obtained using logical connectives are called compound propositions.

QP

The original propositions from which a compound proposition is obtained are called primitives / components of the compound proposition.

The proposition which do not contain any logical connectives are called simple propositions.

1) **Negation** :- A proposition obtained by inserting the word 'not' at an appropriate place in a given proposition is called 'negation of the given proposition', denoted by $\neg p$ (read as not p) (or) $\sim p$. If p is true, then negation of p is false and viceversa.

2) <u>Conjunction</u> :- A compound proposition obtained by combining 2 given propositions by inserting the word 'and' in b/w them is called the conjunction of the given proposition, denoted by $p \wedge q$ (read as p and q).

<u>Rule</u> : The conjunction $p \wedge q$ is true only when p is true and q is true, in all other cases it is false.

3) <u>Disjunction</u> :- A compound proposition obtained by combining 2 given propositions by inserting the word 'or' in b/w them is called the disjunction of the given proposition, denoted by $p \vee q$ (read as p or q).

<u>Rule</u> : The disjunction $p \vee q$ is false only when p is false and q is false. In all other cases it is true.

4) <u>Implication (conditional)</u> :- A compound proposition obtained by combining 2 given propositions by using the words 'if' and 'then' at appropriate places is called a conditional (or) an implication.

If p then q is denoted by $p \rightarrow q$.

<u>Rule</u> :- $p \rightarrow q$ is false only when p is true and q is false. In all other cases, it is true.

5) <u>Biconditional</u> :- Let p and q be 2 propositions. The conjunction of the conditionals $p \rightarrow q$ and $q \rightarrow p$ is called biconditional of p and q and is denoted by $p \leftrightarrow q$.

$p \leftrightarrow q$ is read as if p then q and if q then p.

<u>Rule</u> :- $p \leftrightarrow q$ is true only if both p and q are true (or) both p and q are false.

6) **Exclusive disjunction (XOR)** :- A compound proposition "p or q is said to be true only when either p is true or q is true, but not both" is called exclusive disjunction, denoted by $p \veebar q$ (read as either p or q but not both).

Truth table for the above is as below :-

write $q \to p$ here

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$	$p \veebar q$
0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1
1	0	0	0	1	0	0	1
1	1	0	1	1	1	1	0

* QP **Tautology** :- A compound proposition which is always true regardless of truth value of its components is called a tautology.

* QP **Contradiction** :- A compound proposition which is always false regardless of truth value of its components is called contradiction.

* **Contingency** :- A compound proposition which is neither a tautology nor a contradiction is called contingency.

* **Logical Equivalence** :- Two propositions u and v are said to be logically equivalent whenever u and v have the same truth values or equivalently $u \leftrightarrow v$ is a tautology. We write $u \Leftrightarrow v$ (read as u logically equivalent to v)

Note :- Logically equivalent propositions are treated as identical propositions. Hence the symbol $\equiv$ is also used in place of $\Leftrightarrow$

Problems :-

1) Consider the following propositions concerned with a certain triangle ABC. p : ABC is isosceles. q : ABC is equilateral. r : ABC is equiangular. write down the following propositions in words:

i) $p \wedge \sim q$ ii) $\sim p \vee q$ iii) $p \rightarrow q$

iv) $q \rightarrow p$ v) $\sim r \rightarrow \sim q$ vi) $p \leftrightarrow \sim q$.

Soln :- i) ABC is isosceles and is not equilateral

ii) ABC is not isosceles or ABC is equilateral.

iii) If ABC is isosceles, then it is equilateral.

iv) If ABC is equilateral, then it is isosceles.

v) If ABC is not equiangular, then it is not equilateral.

vi) If ABC is isosceles then it is not equilateral and if ABC is not equilateral then it is isosceles.

2) Construct the truth tables for the following compound propositions:

i) $(p \vee q) \wedge r$ ii) $p \vee (q \wedge r)$ iii) $(p \wedge q) \rightarrow \sim r$

iv) $q \wedge (\sim r \rightarrow p)$.

Soln :-

p	q	r	$p \vee q$	$(p \vee q) \wedge r$	$q \wedge r$	$p \vee (q \wedge r)$	$p \wedge q$	$\sim r$	$(p \wedge q) \rightarrow \sim r$
0	0	0	0	0	0	0	0	1	1
0	0	1	0	0	0	0	0	0	1
0	1	0	1	0	0	0	0	1	1
0	1	1	1	1	1	1	0	0	1
1	0	0	1	0	0	1	0	1	1
1	0	1	1	1	0	1	0	0	1
1	1	0	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1	0	0

$\sim r \to p$	$q \wedge (\sim r \to p)$
0	0
1	0
0	0
1	1
1	0
1	0
1	1
1	1

3) Let p and q be premitive statements for which the conditional $p \to q$ is false. Determine the truth values of the following compound propositions.

i) $p \wedge q$ ii) $\sim p \vee q$ iii) $q \to p$ iv) $\sim q \to \sim p$.

Soln :- Since $p \to q$ is false, p has to be true and q has to be false. $\Rightarrow \sim p$ is false and $\sim q$ is true.

i) $p \wedge q$ is false $\Rightarrow$ truth value is 0.

ii) $\sim p \vee q$ is false $\Rightarrow$ truth value is 0.

iii) $q \to p$ is true $\Rightarrow$ truth value is 1.

iv) $\sim q \to \sim p$ is false $\Rightarrow$ truth value is 0.

split all 6 & do.

4) Let p, q, r be propositions having truth values 0, 0 and 1 resp. Find the truth values of the following compound proposition.

i) $(p \vee q) \vee r$ ii) $(p \wedge q) \wedge r$ iii) $(p \wedge q) \to r$

iv) $p \to (q \wedge r)$ v) $p \wedge (r \to q)$ vi) $p \to (q \to \sim r)$

Soln :-

p	q	r	$p \vee q$	i) $(p \vee q) \vee r$	$p \wedge q$	ii) $(p \wedge q) \wedge r$	iii) $(p \wedge q) \to r$
0	0	1	0	1	0	0	1

$q \wedge r$	iv) $p \to (q \wedge r)$	$r \to q$	v) $p \wedge (r \to q)$	$\sim r$	$q \to \sim r$	vi) $p \to (q \to \sim r)$
0	1	0	0	0	1	1

5) Find the truth value of p, q, r, s, t for which the compound proposition $((p \wedge q) \wedge r) \rightarrow (s \vee t)$ is false.

<u>soln</u> :- Given statement is in the form $u \rightarrow v$ where

$u : (p \wedge q) \wedge r$

$v : s \vee t$.

Given : $u \rightarrow v$ is false $\Rightarrow$ u must be true and v must be false.

<u>ie</u> $(p \wedge q) \wedge r$ is true $\Rightarrow$ p, q, r are true.

and $s \vee t$ is false $\Rightarrow$ s, t are false.

$\therefore$ truth values of p, q, r, s, t are $1, 1, 1, 0, 0$.

6) Find the truth value of the following proposition if p and r are true and q, s, t are false

$(p \vee \sim q) \longleftrightarrow [(\sim r \vee s) \rightarrow t]$

<u>soln</u> :- given p, r are true and q, s, t are false.

$\Rightarrow$ $\sim q$ is true and $\sim r$ is false.

Now $p \vee \sim q$ is true $\rightarrow$ ①.

$\sim r \vee s$ is false

$\Rightarrow$ $(\sim r \vee s) \rightarrow t$ is true $\longrightarrow$ ②

by ① & ②, the given compound proposition is true.

$\Rightarrow$ truth value is 1.

7) If a proposition q has the truth value 1, determine all the truth value assignments for the primitive propositions p, r and s for which the truth value of the following compound proposition is 1.

QP

$$[q \rightarrow \{(\sim p \vee r) \wedge \sim s\}] \wedge \{\sim s \rightarrow (\sim r \wedge q)\}$$

Sol^n :- The given compound proposition is of the form $u \wedge v$

where $u \equiv q \rightarrow \{(\sim p \vee r) \wedge \sim s\}$

$v \equiv \sim s \rightarrow (\sim r \wedge q)$

Given $u \wedge v$ is true $\Rightarrow$ both u and v are true.

Since truth value of q is 1 and truth value of u is 1,

$(\sim p \vee r) \wedge \sim s$ must have truth value 1.

$\Rightarrow (\sim p \vee r)$ and $\sim s$ both have truth values 1.

$\therefore$ Truth value of s is 0.

Now, since truth value of $\sim s$ is 1 and truth value of v is 1, $\sim r \wedge q$ must have truth value 1

Since q has truth value 1, $\sim r$ must have truth value 1.

$\Rightarrow$ Truth value of r is 0

Since r has truth value 0, and $(\sim p \vee r)$ has truth value 1, $\sim p$ must have the truth value 1

$\Rightarrow$ truth value of p is 0.

(OR) In simple words

give this

Since $u \wedge v$ is 1, both u and v are 1.

Since q is 1, u is 1, $(\sim p \vee r) \wedge \sim s$ must be 1.

$\Rightarrow (\sim p \vee r)$ must be 1 $\rightarrow$ (1)

and $\sim s$ must be 1

$\Rightarrow$ s is 0.

Now, Since $\sim s$ is 1, v is 1, $(\sim r \wedge q)$ must be 1

$\Rightarrow \sim r$ is 1 (since q is 1)

$\Rightarrow r$ is 0 $\rightarrow$ (2)

from (1) & (2), $\sim p$ must be 1.

$\Rightarrow p$ is 0.

Thus the truth values of p, r, s are all 0.

- The same in the form of truth table can be constructed as below: (for reference)

$u \equiv q \rightarrow \{(\sim p \vee r) \wedge (\sim s)\}$

$v \equiv \sim s \rightarrow (\sim r \wedge q)$.

$u \wedge v$	u	v	q	$(\sim p \vee r) \wedge \sim s$	$(\sim p \vee r)$	$\sim s$	s	$(\sim r \wedge q)$	$\sim r$	r
1	1	1	1	1	1	1	0	1	1	0

$\sim p$	p
1	0

8) Show that, for any propositions p and q, the compound proposition $p \rightarrow (p \vee q)$ is a tautology and the compound proposition $p \wedge (\sim p \wedge q)$ is a contradiction.

Soln:-

p	q	$p \vee q$	$p \rightarrow (p \vee q)$	$\sim p$	$\sim p \wedge q$	$p \wedge (\sim p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0

$\therefore$ $p \rightarrow (p \vee q)$ is a tautology

$p \wedge (\sim p \wedge q)$ is a contradiction.

9) Show that, for any 2 propositions p and q,

i) $(p \veebar q) \vee (p \leftrightarrow q)$ is a tautology

ii) $(p \veebar q) \wedge (p \leftrightarrow q)$ is a contradiction

iii) $(p \veebar q) \wedge (p \rightarrow q)$ is a contingency.

Soln:-

p	q	$p \veebar q$	$p \leftrightarrow q$	$(p \veebar q) \vee (p \leftrightarrow q)$ ✓	$(p \veebar q) \wedge (p \leftrightarrow q)$ ✓	$p \rightarrow q$	$(p \veebar q) \wedge (p \rightarrow q)$ ✓
0	0	0	1	1	0	1	0
0	1	1	0	1	0	1	1
1	0	1	0	1	0	0	0
1	1	0	1	1	0	1	0

From the above, we have all 3 satisfied.

10) Prove that, for any propositions p, q, r, the compound proposition $[(p \vee q) \to r] \leftrightarrow [\sim r \to \sim(p \vee q)]$ is a tautology.

QP

Soln:

p	q	r	$p \vee q$	$(p \vee q) \to r$	$\sim r$	$\sim(p \vee q)$	$\sim r \to \sim(p \vee q)$	$[(p \vee q) \to r] \leftrightarrow [\sim r \to \sim(p \vee q)]$
0	0	0	0	1	1	1	1	1
0	0	1	0	1	0	1	1	1
0	1	0	1	0	1	0	0	1
0	1	1	1	1	0	0	1	1
1	0	0	1	0	1	0	0	1
1	0	1	1	1	0	0	1	1
1	1	0	1	0	1	0	0	1
1	1	1	1	1	0	0	1	1

∴ Given compound proposition is a tautology.

11) Prove that, for any propositions p, q, r, the compound proposition $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$ is a tautology.

QP

Soln:

p	q	r	$p \vee q$ (x)	$p \to r$ (y)	$q \to r$ (z)	$x \wedge y \wedge z$	$(x \wedge y \wedge z) \to r$
0	0	0	0	1	1	0	1
0	0	1	0	1	1	0	1
0	1	0	1	1	0	0	1
0	1	1	1	1	1	1	1
1	0	0	1	0	1	0	1
1	0	1	1	0	1	0	1
1	1	0	1	1	0	0	1
1	1	1	1	1	1	1	1

∴ given compound proposition is a tautology.

12) use truth tables to verify $[\sim p \wedge (\sim q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)] \Leftrightarrow r$.

Q.

Soln :-

p	q	r	$\sim p$	$\sim q$	$\sim q \wedge r$	x $\sim p \wedge (\sim q \wedge r)$	$q \wedge r$	$p \wedge r$	y $(q \wedge r) \vee (p \wedge r)$	$x \vee y$
0	0	0	1	1	0	0	0	0	0	0
0	0	1	1	1	1	1	0	0	0	1
0	1	0	1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	1	0	1	1
1	0	0	0	1	0	0	0	0	0	0
1	0	1	0	1	1	0	0	1	1	1
1	1	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	1	1	1	1

from columns 3 and 11, we see that truth values are same for all possible truth values of p, q, r.

$\therefore [\sim p \wedge (\sim q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)] \Leftrightarrow r$.

13) Prove that for any 3 propositions p, q, r, $[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)]$ is logically equivalent to $[(p \to q) \wedge (q \to r) \wedge (r \to p)]$

Q.

Soln :-

p	q	r	x $p \leftrightarrow q$	y $q \leftrightarrow r$	z $r \leftrightarrow p$	$x \wedge y \wedge z$	x_1 $p \to q$	y_1 $q \to r$	z_1 $r \to p$	$x_1 \wedge y_1 \wedge z_1$
0	0	0	1	1	1	1	1	1	1	1
0	0	1	1	0	0	0	1	1	0	0
0	1	0	0	0	1	0	1	0	1	0
0	1	1	0	1	0	0	1	1	0	0
1	0	0	0	1	0	0	0	1	1	0
1	0	1	0	0	1	0	0	1	1	0
1	1	0	1	0	0	0	1	0	1	0
1	1	1	1	1	1	1	1	1	1	1

from columns 7 and 11, we see that truth values are same for all possible truth values of p, q, r. Hence given compound propositions are logically equivalent.

The laws of Logic :-

Let T_0 denotes a tautology and F_0 denotes a contradiction.

1) Law of double negation : for any proposition p,

$$\sim\sim p \Leftrightarrow p$$

2) Idempotent laws : for any proposition p,

i) $(p \vee p) \Leftrightarrow p$ ii) $(p \wedge p) \Leftrightarrow p$.

3) Identity laws : for any proposition p,

(i) $(p \vee F_0) \Leftrightarrow p$ ii) $(p \wedge T_0) \Leftrightarrow p$.

4) Inverse laws : for any proposition p,

i) $(p \vee \sim p) \Leftrightarrow T_0$ ii) $(p \wedge \sim p) \Leftrightarrow F_0$

5) Domination laws : for any proposition p,

i) $(p \vee T_0) \Leftrightarrow T_0$ ii) $(p \wedge F_0) \Leftrightarrow F_0$

6) Commutative laws : for any 2 propositions p, q

i) $(p \vee q) \Leftrightarrow (q \vee p)$ ii) $(p \wedge q) \Leftrightarrow (q \wedge p)$

7) Absorption laws : for any 2 propositions p, q

i) $[p \vee (p \wedge q)] \Leftrightarrow p$ ii) $[p \wedge (p \vee q)] \Leftrightarrow p$.

8) DeMorgan laws : for any 2 propositions p, q

i) $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$ ii) $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$.

9) Associative laws : For any 3 propositions p, q, r

i) $p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ ii) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$.

10) Distributive laws : for any 3 propositions p, q, r

i) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ ii) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

Remark :-
For any 2 propositions p, q

i) $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

ii) $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$

iii) $\sim(p \to q) \Leftrightarrow (p \wedge \sim q)$

iv) $(p \to q) \Leftrightarrow \sim\sim(p \to q) \Leftrightarrow \sim(p \wedge \sim q) \Leftrightarrow \sim p \vee q$.

* Transitive and Substitution rules :

1) If $u \Leftrightarrow v$, $v \Leftrightarrow w$, then $u \Leftrightarrow w$ (Transitive rule)

2) Suppose u is a compound proposition which is a tautology and p is a proposition of u. If we replace each occurence of p in u by a proposition q then the resulting proposition v is also a tautology. (Substitution rule).

3) Suppose u is a compound proposition which contains a proposition p. Let $p \Leftrightarrow q$. Suppose we replace one or more occurences of p by q and obtain a compound proposition v, then $u \Leftrightarrow v$. (Substitution rule).

* Dual : Let s be a statement. If s contains no logical connectives other than $\wedge$ and $\vee$, then the dual of s, denoted as s^d is the statement obtained from s by replacing each occurence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$ resp and each occurence of T_0 and F_0 by F_0 and T_0 resp.

QP

Ex :- If $u : p \wedge (q \vee \sim r) \vee (s \wedge T_0)$, then

$u^d : p \vee (q \wedge \sim r) \wedge (s \vee F_0)$.

Remark :- 1) $(p^d)^d \Leftrightarrow p$ (ie dual of a dual of $u \Leftrightarrow u$)

2) If $p \Leftrightarrow q$, then $p^d \Leftrightarrow q^d$ (Principle of duality)

3) $(\sim p)^d \Leftrightarrow \sim p$, $(p \vee \sim p)^d \Leftrightarrow (p \wedge \sim p)$, $(p \wedge \sim p)^d \Leftrightarrow (p \vee \sim p)$

Problems :-

1) Let x be a specified number. Write down the negation of the following propositions:–
"If x is not a real number, then it is not a rational number and not an irrational number".

Soln :- Let p : x is a real number
q : x is a rational number
r : x is an irrational number.

then the given proposition reads : $\sim p \rightarrow (\sim q \wedge \sim r)$.

To find negation of this proposition.

$$\sim[\sim p \rightarrow (\sim q \wedge \sim r)] \Leftrightarrow \sim p \wedge [\sim(\sim q \wedge \sim r)]$$
$$\Leftrightarrow \sim p \wedge [\sim\sim q \vee \sim\sim r]$$
$$\Leftrightarrow \sim p \wedge (q \vee r)$$

Thus the negation of the given proposition is
" x is not a real number and it is a rational number or an irrational number".

2) Express each of the following statements in symbols, negate them and write in smooth english.
i) Vimala will get a good education if she puts her studies before her interest in cheer leading.
ii) Nirma is doing her homework and Kamala is practicing her music lessons.

Soln :- Let p : Vimala puts her studies before her interest in cheer leading.
q : Vimala will get a good education.
r : Nirma is doing her homework.
s : Kamala is practicing her music lessons.

i) $p \rightarrow q$ ii) $r \wedge s$.

To find negation of these statements

i) $\sim(p \to q) \equiv p \wedge \sim q$

Thus negation of the given proposition is

" Vimala puts her studies before her interest in cheer leading and she will not get a good education"

ii) $\sim(r \wedge s) \equiv \sim r \vee \sim s$.

" Nirma is not doing her homework or Kamala is not practicing her music lessons".

3) Without using the truth tables, P.T

$\{(p \to q) \to r\} \longleftrightarrow \{(\sim p \vee q) \to r\}$ is a tautology.

Soln :- Note :- To prove $u \leftrightarrow v$ is a tautology is equivalent of proving $u \Leftrightarrow v$ (see defn of logical equivalence)

we have $p \to q \Leftrightarrow \sim p \vee q$.

$(p \to q) \to r \Leftrightarrow (\sim p \vee q) \to r$.

$\therefore$ given proposition is a tautology.

4) Simplify the following compound propositions using the laws of logic :

i) $(p \vee q) \wedge [\sim\{(\sim p) \wedge q\}]$

ii) $\sim[\sim\{(p \vee q) \wedge r\} \vee \sim q]$

Soln :- i) $(p \vee q) \wedge [\sim\{\sim p \wedge q\}] \equiv (p \vee q) \wedge [\sim\sim p \vee \sim q]$ (D'Morgan law)

$\equiv (p \vee q) \wedge [p \vee \sim q]$ (law of double negation)

$\equiv p \vee \{q \wedge \sim q\}$ (distributive law)

$\equiv p \vee F_0$ (Inverse law)

$\equiv p$ (Identity law)

ii) $\sim[\sim\{(p \vee q) \wedge r\} \vee \sim q] \equiv \sim[\sim\{((p \vee q) \wedge r) \wedge q\}]$ (DeMorgan's law)

$\sim P \vee \sim Q \equiv \sim(P \wedge Q)$

$\equiv ((p \vee q) \wedge r) \wedge q$ (law of double negation)

$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$

$\equiv (p \vee q) \wedge (r \wedge q)$ (Associative law)

$\equiv (p \vee q) \wedge (q \wedge r)$ (commutative law)

$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$

$\equiv \{(p \vee q) \wedge q\} \wedge r$ (Associative law)

$\equiv q \wedge r$ (Absorption law)

5) Prove the following logical equivalence without using truth tables:

QP i) $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

ii) $[p \vee q \vee (\sim p \wedge \sim q \wedge r)] \Leftrightarrow (p \vee q \vee r)$

QP iii) $[(\sim p \vee \sim q) \to (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

QP iv) $(p \to q) \wedge [\sim q \wedge (r \vee \sim q)] \Leftrightarrow \sim (q \vee p)$

QP v) $[\sim p \wedge (\sim q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)] \Leftrightarrow r$

Sol^n:- i) $p \to (q \to r) \equiv p \to (\sim q \vee r)$ $(\because P \to Q \equiv \sim P \vee Q)$

$\equiv \sim p \vee (\sim q \vee r)$

$\equiv (\sim p \vee \sim q) \vee r$ (Associative law)

$\equiv \sim (p \wedge q) \vee r$ (D'Morgan's law)

$\equiv (p \wedge q) \to r$

ii) $[p \vee q \vee (\sim p \wedge \sim q \wedge r)]$

consider $\sim p \wedge \sim q \wedge r \equiv (\sim p \wedge \sim q) \wedge r$ (Associative law)

$\equiv \sim (p \vee q) \wedge r$ (D'Morgan's law)

$\therefore [p \vee q \vee (\sim p \wedge \sim q \wedge r)] \equiv (p \vee q) \vee \{\sim (p \vee q) \wedge r\}$

$\equiv \{(p \vee q) \vee \sim (p \vee q)\} \wedge \{(p \vee q) \vee r\}$ (Dist. law)

$\equiv T_0 \wedge \{(p \vee q) \vee r\}$ (Inverse law)

$\equiv T_0 \wedge (p \vee q \vee r)$ (Associative law)

$\equiv (p \vee q \vee r) \wedge T_0$ (commutative law)

$\equiv (p \vee q \vee r)$ (Identity law)

iii) $(\sim p \vee \sim q) \rightarrow (p \wedge q \wedge r) \equiv \sim(\sim p \vee \sim q) \vee (p \wedge q \wedge r)$

$(\because u \rightarrow v \Leftrightarrow \sim u \vee v)$

$\equiv (p \wedge q) \vee (p \wedge q \wedge r)$ (DeMorgan's law)

$\equiv (p \wedge q) \vee ((p \wedge q) \wedge r)$ (Associative law)

$P \vee (P \wedge Q) \Leftrightarrow P$

$\equiv p \wedge q$ (Absorption law).

iv) $(p \rightarrow q) \wedge [\sim q \wedge (r \vee \sim q)] \Leftrightarrow (p \rightarrow q) \wedge [\sim q \wedge (\sim q \vee r)]$ (comm. law)

$\equiv (p \rightarrow q) \wedge \sim q$ (absorption law)

$\equiv \sim[(p \rightarrow q) \rightarrow q]$ $(\because \sim(u \rightarrow v) \equiv u \wedge \sim v)$

$\equiv \sim[\sim(p \rightarrow q) \vee q]$ $(\because u \rightarrow v \equiv \sim u \vee v)$

$\equiv \sim[(p \wedge \sim q) \vee q]$ $(\because \sim(u \rightarrow v) \equiv u \wedge \sim v)$

$\equiv \sim[q \vee (p \wedge \sim q)]$ (commutative law)

$\equiv \sim[(q \vee p) \wedge (q \vee \sim q)]$ (distributive law)

$\equiv \sim[(q \vee p) \wedge T_0]$ (Inverse law)

$\equiv \sim(q \vee p)$ (Identity law).

v) consider $[\sim p \wedge (\sim q \wedge r)] \equiv (\sim p \wedge \sim q) \wedge r$ (Associative law)

$\equiv [\sim(p \vee q)] \wedge r$ (DeMorgan's law)

$\equiv r \wedge [\sim(p \vee q)]$ (commutative law)

consider $(q \wedge r) \vee (p \wedge r) \equiv (r \wedge q) \vee (r \wedge p)$ (commutative law)

$\equiv r \wedge (q \vee p)$ (Distributive law)

$\equiv r \wedge (p \vee q)$ (Commutative law)

$\therefore [\sim p \wedge (\sim q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)]$

$\equiv [r \wedge \{\sim(p \vee q)\}] \vee [r \wedge (p \vee q)]$

$\equiv r \wedge [\sim(p \vee q) \vee (p \vee q)]$ (Distributive law)

$\equiv r \wedge T_0$ (Inverse law)

$\equiv r$ (Identity law).

6) Prove that $[(p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))] \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$ is a tautology.

Soln:- Let w denote the given proposition. then $w \equiv u \vee v$,

where $u \equiv (p \vee q) \wedge \sim(\sim p \wedge (\sim q \vee \sim r))$

$v \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$.

$u \equiv (p \vee q) \wedge \sim\{\sim p \wedge (\sim q \vee \sim r)\}$.

$\equiv (p \vee q) \wedge (p \vee (q \wedge r))$ (DeMorgan's law).

$\equiv p \vee \{q \wedge (q \wedge r)\}$ (Distributive law).

$\equiv p \vee (q \wedge r)$ (Associative and Idempotent laws).

and $v \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$

$\equiv \sim(p \vee q) \vee \sim(p \vee r)$ (DeMorgan's law)

$\equiv \sim[(p \vee q) \wedge (p \vee r)]$ (DeMorgan's law)

$\equiv \sim[p \vee (q \wedge r)]$ (Distributive law)

$\equiv \sim u$.

$\therefore$ $w \equiv u \vee v \equiv u \vee \sim u \equiv T_0$ (Inverse law).

$\therefore$ given compound proposition is a tautology.

7) write down the duals of the following propositions:

i) $(p \wedge q) \vee [(\sim p \vee q) \wedge (\sim r \vee s)] \vee (r \wedge s)$

QP ii) $[(p \vee T_0) \wedge (q \vee F_0)] \vee [(r \wedge s) \wedge T_0]$

iii) $(p \to q) \to r$

iv) $p \to (q \to r)$.

Soln:- i) $(p \vee q) \wedge [(\sim p \wedge q) \vee (\sim r \wedge s)] \wedge (r \vee s)$.

ii) $[(p \wedge F_0) \vee (q \wedge T_0)] \wedge [(r \vee s) \vee F_0]$.

iii) WKT $(u \to v) \Leftrightarrow (\sim u \vee v)$. by Principle of duality,

$$[(p \to q) \to r]^d \Leftrightarrow [\sim(p \to q) \vee r]^d$$
$$\Leftrightarrow [(p \wedge \sim q) \vee r]^d$$
$$\Leftrightarrow (p \vee \sim q) \wedge r.$$

iv) By principle of duality,

$$[p \to (q \to r)]^d \equiv [\sim p \vee (q \to r)]^d$$
$$\equiv [\sim p \vee (\sim q \vee r)]^d$$
$$\equiv \sim p \wedge (\sim q \wedge r).$$

8) Prove that $[(\sim p \vee q) \wedge (p \wedge (p \wedge q))] \Leftrightarrow p \wedge q$. Hence deduce that $[(\sim p \wedge q) \vee (p \vee (p \vee q))] \Leftrightarrow p \vee q$.

Soln:- $(\sim p \vee q) \wedge (p \wedge (p \wedge q)) \equiv (\sim p \vee q) \wedge ((p \wedge p) \wedge q)$

$$\equiv (\sim p \vee q) \wedge (p \wedge q)$$
$$\equiv [\sim p \wedge (p \wedge q)] \vee [q \wedge (p \wedge q)]$$
$$\equiv (F_0 \wedge q) \vee (q \wedge p)$$
$$\equiv F_0 \vee (p \wedge q)$$
$$\equiv p \wedge q.$$

taking dual on both sides of this, we get,

$$(\sim p \wedge q) \vee (p \vee (p \vee q)) \equiv p \vee q.$$

9) verify the principle of duality for:

QP $[\sim(p \wedge q) \to \sim p \vee (\sim p \vee q)] \Leftrightarrow (\sim p \vee q)$

Soln:- Given logical equivalence is $u \Leftrightarrow v$, where

$u \equiv \sim(p \wedge q) \to \sim p \vee (\sim p \vee q)$ and $v \equiv \sim p \vee q$.

Now, $u \equiv \sim\sim(p \wedge q) \vee (\sim p \vee (\sim p \vee q))$

$$\equiv (p \wedge q) \vee (\sim p \vee (\sim p \vee q))$$

$\therefore u^d \equiv (p \vee q) \wedge (\sim p \wedge (\sim p \wedge q))$

$$\equiv (p \vee q) \wedge (\sim p \wedge q).$$

$$u^d \equiv [p \wedge (\sim p \wedge q)] \vee [q \wedge (\sim p \wedge q)]$$
$$\equiv (F_0 \wedge q) \vee (q \wedge \sim p)$$
$$\equiv F_0 \vee (q \wedge \sim p)$$
$$\equiv q \wedge \sim p .$$

Also $v^d \equiv \sim p \wedge q \equiv q \wedge \sim p$.

$\therefore \quad u^d \equiv v^d$

This verifies the principle of duality for the given logica equivalence.

The Connectives NAND and NOR.

→ The compound proposition $\sim(p \wedge q)$, read as "Not p and q" is denoted by $(p \uparrow q)$. The symbol $\uparrow$ is called the NAND connective. (NAND is a combination of Not and and)

Thus $p \uparrow q = \sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$.

→ The compound proposition $\sim(p \vee q)$, read as "Not (p or q)" is denoted by $(p \downarrow q)$. The symbol $\downarrow$ is called the NOR connective. (NOR is a combination of Not and or).

Thus $p \downarrow q = \sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$.

Note :- $p \uparrow q$ and $p \downarrow q$ are duals of each other.

Truth table :-

p	q	$p \uparrow q$	$p \downarrow q$
0	0	1	1
0	1	1	0
1	0	1	0
1	1	0	0

Problems :-

1) for any propositions p, q, prove the following :-

i) $\sim(p \downarrow q) \Leftrightarrow (\sim p \uparrow \sim q)$

ii) $\sim(p \uparrow q) \Leftrightarrow (\sim p \downarrow \sim q)$

Soln :- i) $\sim(p \downarrow q) \equiv \sim[\sim(p \vee q)]$

$\equiv \sim[\sim p \wedge \sim q]$

$\equiv \sim p \uparrow \sim q$

ii) $\sim(p \uparrow q) \equiv \sim[\sim(p \wedge q)]$

$\equiv \sim[\sim p \vee \sim q]$

$\equiv \sim p \downarrow \sim q$.

2) For any propositions p, q, r, prove that

$[p \uparrow (q \uparrow r)] \not\Leftrightarrow [(p \uparrow q) \uparrow r]$ i.e. the connective $\uparrow$ is not associative.

Soln :- $p \uparrow (q \uparrow r) \equiv \sim [p \wedge (q \uparrow r)]$

$\equiv \sim [p \wedge \sim (q \wedge r)]$

$\equiv \sim p \vee (q \wedge r) \longrightarrow (1)$

$(p \uparrow q) \uparrow r \equiv \sim [(p \uparrow q) \wedge r]$.

$\equiv \sim [\{\sim (p \wedge q)\} \wedge r]$

$\equiv \sim \sim (p \wedge q) \vee \sim r$

$\equiv (p \wedge q) \vee \sim r \longrightarrow (2)$

from (1) & (2), $p \uparrow (q \uparrow r) \not\Leftrightarrow (p \uparrow q) \uparrow r$

$\Rightarrow \uparrow$ is not associative

3) Express $p \vee q$, $p \wedge q$, $p \rightarrow q$ using NAND only.

Soln:- i) $p \vee q \equiv \sim (\sim (p \vee q))$

$\equiv \sim [\underbrace{\sim p}_{P} \wedge \underbrace{\sim q}_{Q}]$ $\quad \sim [P \wedge Q] \equiv P \uparrow Q$

$\equiv \sim p \uparrow \sim q$.

ii) $p \wedge q \equiv \sim [\sim (p \wedge q)]$

$\equiv \sim (p \uparrow q)$

iii) $p \rightarrow q \equiv$

3) Express $\sim p$, $p \vee q$, $p \wedge q$, $p \rightarrow q$ using NAND only.

Q

Soln :- i) for any proposition p, we have $p \wedge p \equiv p$.

$\therefore \sim p \equiv \sim (p \wedge p) \equiv p \uparrow p$.

ii) $p \vee q \equiv \sim [\sim (p \vee q)]$

$\equiv \sim [\underbrace{\sim p}_{P} \wedge \underbrace{\sim q}_{Q}]$ $\quad \sim (P \wedge Q) \equiv P \uparrow Q$

$= (\sim p) \uparrow (\sim q) \equiv (p \uparrow p) \uparrow (q \uparrow q)$.

iii) $p \wedge q \equiv \sim[\sim(p \wedge q)]$

$\equiv \sim[\underbrace{(\sim p) \vee (\sim q)}_{P}]$ $\quad \sim P \equiv P \uparrow P$

$\equiv (\sim p \vee \sim q) \uparrow (\sim p \vee \sim q)$

$\equiv (p \uparrow q) \uparrow (p \uparrow q) \quad (\because \sim p \vee \sim q \equiv \sim(p \wedge q)$
$\equiv p \uparrow q)$

iv) $p \rightarrow q \equiv \sim(p \wedge \sim q)$

$\equiv p \uparrow \sim q$

$\equiv p \uparrow (q \uparrow q)$.

4) Express $\sim p$, $p \vee q$, $p \wedge q$, $p \rightarrow q$ using NOR only.
HW

Converse, Inverse and Contrapositive :-

QP Consider a conditional $p \to q$, then

i) $q \to p$ is called the converse of $p \to q$.

ii) $\sim p \to \sim q$ is called the inverse (or opposite) of $p \to q$.

iii) $\sim q \to \sim p$ is called the contrapositive of $p \to q$.

Truth Table :-

p	q	$\sim p$	$\sim q$	$p \to q$	$q \to p$	$\sim p \to \sim q$	$\sim q \to \sim p$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

Remark from truth table :-

i) $(p \to q) \Leftrightarrow (\sim q) \to (\sim p)$

ie a conditional and its contrapositive are logically equivalent.

ii) $(q \to p) \Leftrightarrow (\sim p) \to (\sim q)$.

ie the converse and the inverse of a conditional are logically equivalent.

iii) $(p \to q) \not\Leftrightarrow (q \to p)$

ie if a conditional is true, its converse need not be true and vice-versa.

Problems :- Write the Inverse, converse and contrapositive for the foll:

i) If a real number x^2 is greater than zero, then x is not equal to zero.

soln:- p : Real no. x^2 is greater than zero.

q : x is not equal to zero.

Converse : $q \to p$ ie If a real no. x is not equal to zero then x^2 is greater than zero.

Inverse : $\sim p \rightarrow \sim q$. i.e If a real no. x^2 is not greater than zero, then x is equal to zero.

Contrapositive : $\sim q \rightarrow \sim p$. i.e If a real no. x is equal to zero, then x^2 is not greater than zero.

2) If Kabir wears brown pant, then he will wear white shirt.

Soln :- Given statement is in the form $p \rightarrow q$

p : Kabir wears brown pant.

q : Kabir wears white shirt.

Converse : $q \rightarrow p$

If Kabir wears white shirt, then he wears brown pant.

Inverse : $\sim p \rightarrow \sim q$.

If Kabir does not wear brown pant, then he will not wear white shirt.

Contrapositive : $\sim q \rightarrow \sim p$.

If Kabir does not wear white shirt, then he will not wear brown pant.

3) If m divides n and n divides p, then m divides p.

Soln :- p : m divides n.

q : n divides p.

r : m divides p.

Given statement is $(p \wedge q) \rightarrow r$.

converse : $r \rightarrow (p \wedge q)$.

If m divides p, then m divides n and n divides p.

Inverse : $\sim (p \wedge q) \rightarrow \sim r$. i.e $(\sim p \vee \sim q) \rightarrow \sim r$.

If m does not divides n or n does not divides p, then m does not divides p.

Contrapositive : $\sim r \to \sim(p \wedge q)$. ie $\sim r \to (\sim p \vee \sim q)$ (15)

If m does not divides p, then m does not divides n or n does not divides p.

4) Write down the contrapositive of $[p \to (q \to r)]$ with
QP i) only one occurrence of the connective $\to$
ii) no occurrence of the connective $\to$.

Soln:- The contrapositive of $[p \to (q \to r)]$ is $[\sim(q \to r) \to (\sim p)]$

Now $[\sim(q \to r) \to (\sim p)] \Leftrightarrow \sim[\sim(q \to r)] \vee \sim p \to$ (i) $(\because P \to Q \Leftrightarrow \sim P \vee$

$\Leftrightarrow (q \to r) \vee \sim p$

$\Leftrightarrow (\sim q \vee r) \vee \sim p \to$ (ii)

Expressions (i) & (ii) are the required representations.

Logical Implication :-

The conditional $p \to q$ where p and q are related in such a way that the truth value of q depends on truth value of p and vice-versa. Such conditionals are called hypothetical or implicative statements.

Note :- 1) when a hypothetical statement $p \to q$ is such that q is true whenever p is true, we say that p (logically) implies q. Symbolically $p \Rightarrow q$ ($\Rightarrow$ denotes implies).

2) when a hypothetical statement $p \to q$ is such that q is not necessarily true whenever p is true, we say that p does not imply q. Symbolically $p \not\Rightarrow q$ ($\not\Rightarrow$ denotes does not imply).

Prove the following :-

1) $[(p \to q) \wedge \sim q] \Rightarrow \sim p$

Soln :-

p	q	$\sim p$	$\sim q$	$p \to q$	$(p \to q) \wedge \sim q$
0	0	1 ✓	1	1	1 ✓
0	1	1	0	1	0
1	0	0	1	0	0
1	1	0	0	1	0

from the table, we find that when $(p \to q) \wedge \sim q$ is true, then $\sim p$ is true. ie $[(p \to q) \wedge \sim q] \Rightarrow \sim p$.

2) $\{[p \vee (q \vee r)] \wedge \sim q\} \Rightarrow p \vee r$

p	q	r	$\sim q$	$q \vee r$	$p \vee (q \vee r)$	$(p \vee (q \vee r)) \wedge \sim q$	$p \vee r$
0	0	0	1	0	0	0	0
0	0	1	1	1	1	1	1
0	1	0	0	1	1	0	1
0	1	1	0	1	1	0	0
1	0	0	1	0	1	0	1
1	0	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	0	1	1	0	1

from the above table, If $[p \vee (q \vee r)] \wedge \sim q$ is true, then $p \vee r$ is true

$\therefore \{[p \vee (q \vee r)] \wedge \sim q\} \Rightarrow p \vee r$

3) $[p \wedge (p \to q)] \Rightarrow q$

4) $[p \wedge (p \to q) \wedge r] \Rightarrow [(p \vee q) \to r]$

Rules of Inference

Consider a set of propositions $p_1, p_2, \ldots p_n$ and a proposition Q, then a compound proposition of the form $(p_1 \wedge p_2 \wedge p_3 \wedge \ldots . p_n) \rightarrow Q$ is called an argument.

Here $p_1, p_2 \ldots p_n$ are called the premises of the argument and Q is called a conclusion of the argument.

we usually write the argument as

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore Q \end{array}$$

This argument is valid if each of $p_1, p_2 \ldots . p_n$ are true and Q is also true.

ie $(p_1 \wedge p_2 \wedge p_3 \cdots \wedge p_n) \Rightarrow Q$.

Note :- In an argument, premises are always taken to be true whereas conclusion may be true or false. The conclusion is true only in the case of a valid argument.

Rules of Inference are:

1) Rule of Conjuctive Simplification :-

For any 2 propositions p and q, if $p \wedge q$ is true, then p is true. ie $(p \wedge q) \Rightarrow p$.

2) Rule of Disjunctive Amplification :-

$$p \Rightarrow (p \vee q)$$

3) Rule of syllogism :-

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \Rightarrow (p \rightarrow r).$$ ie $$\begin{array}{c} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

4) Modus pones :- (Rule of detachment)

$$[p \wedge (p \rightarrow q)] \Rightarrow q$$ ie $$\begin{array}{c} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

5) Modus Tollens :-

$$[(p \to q) \wedge \sim q] \Rightarrow [\sim p]$$ i.e.

$$\begin{array}{l} p \to q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

If $p \to q$ is true and q is false, then p is false.

6) Rule of Disjunctive Syllogism :-

$$[(p \vee q) \wedge \sim p] \Rightarrow q$$ i.e.

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array}$$

7) Rule of Contradiction :-

$$(\sim p \to F_0) \Rightarrow p$$ (F_0 is a contradiction)

i.e.

$$\begin{array}{l} \sim p \to F_0 \\ \hline \therefore p \end{array}$$

Problems

I Test whether the following statements are Valid :-

1) If interest rate falls then stock market will rise. The stock Market will not rise, therefore the interest rate will not fall.

Soln :- Let p : Interest rate falls.

q : Stock market will rise.

$\therefore$ given argument reads

$$\begin{array}{l} p \to q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

In view of Modus Tollens rule, this is a valid argument.

2) If I study, then I do not fail in the examination. If I do not fail in the examination, my father gifts a two-wheeler to me. Therefore, if I study then my father gifts a two-wheeler to me.

P.T.O.

Soln:- Let p : I study

q : I do not fail in the Examination

r : My father gifts a two-wheeler to me.

The given argument reads

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

In view of the rule of syllogism, this is a valid statement

3) If Ravi goes out with friends, he will not study.
If Ravi does not study, his father becomes angry.
His father is not angry.

∴ Ravi has not gone out with friends.

Soln:- Let p : Ravi goes out with friends

q : Ravi does not study.

r : His father gets angry.

∴ given arguments reads

$$\begin{array}{l} p \to q \\ q \to r \\ \sim r \\ \hline \therefore \sim p \end{array}$$

$$\Leftrightarrow \begin{array}{l} p \to r \quad \text{(by rule of syllogism)} \\ \sim r \\ \hline \therefore \sim p \end{array}$$

In view of Modus Tollens rule, this is a valid argument.

4) I will become famous or I will not become a musician.
I will become a musician.

∴ I will become famous.

P.T.O.

Soln :- Let p : I will become famous.

q : I will not become a musician.

then the given argument reads

$$\begin{array}{l} p \vee q \\ \sim q \\ \hline \therefore p \end{array}$$

In view of rule of disjunctive syllogism, this argument is valid

(OR)

Let p : I will become famous.

q : I will become a musician.

given argument reads

$$\begin{array}{l} p \vee \sim q \\ \quad q \\ \hline \therefore p \end{array}$$

$$\Leftrightarrow \begin{array}{l} q \rightarrow p \\ q \\ \hline \therefore p \end{array} \quad (\because p \vee \sim q \equiv \sim q \vee p \equiv q \rightarrow p)$$

In view of modus pones rule, this argument is valid.

5) I will get grade A in this course or I will not graduate

If I do not graduate, I will join the army.

I got grade A.

$\therefore$ I will not join the army.

Soln :- Let p : I get grade A in this course.

q : I do not graduate.

r : I join the army.

The given argument reads

$$\begin{array}{l} p \vee q \\ q \rightarrow r \\ p \\ \hline \therefore \sim r \end{array}$$

consider $(p \vee q) \wedge (q \rightarrow r) \wedge p$

$\equiv (\sim p \rightarrow q) \wedge (q \rightarrow r) \wedge p$. $(\because p \rightarrow q \equiv \sim p \vee q \Rightarrow \sim p \rightarrow q \equiv p \vee q)$

$\Rightarrow (\sim p \to r) \wedge p$ (by rule of syllogism)

$\Rightarrow (p \vee r) \wedge p$

$\Rightarrow p \vee r$ (by the rule of conjunctive Simplification)

$\not\Rightarrow \sim r$.

$\therefore (p \vee q) \wedge (q \to r) \wedge p \not\Rightarrow \sim r$.

$\therefore$ given argument is not valid.

6) If I study, I will not fail in the examination.
If I do not watch TV in the evenings, I will study.
I failed in the examination.

$\therefore$ I must have watched TV in the evenings.

Soln:- Let p : I Study
q : I fail in the examination.
r : I watch TV in the evenings.

Then the given argument reads

$p \to \sim q$

$\sim r \to p$

q

$\therefore r$

consider $(p \to \sim q) \wedge (\sim r \to p) \wedge q$

$\Leftrightarrow (\sim\sim q \to \sim p) \wedge (\sim p \to r) \wedge q$ (by contrapositive)

$\Leftrightarrow (q \to \sim p) \wedge (\sim p \to r) \wedge q$

$\Rightarrow (q \to r) \wedge q$ (by rule of syllogism)

$\Rightarrow r$ (by rule of Modus ponens).

Thus $(p \to \sim q) \wedge (\sim r \to p) \wedge q \Rightarrow r$.

$\therefore$ given argument is Valid.

7) If A gets supervisor position and work hard then he will get arise. If he gets arise, he will buy a new car. He has not bought a new car. $\therefore$ A did not get Supervisor position or he did not work hard.

Soln:- Let p : A gets supervisor position.

q : A works hard

r : He will arise.

s : He buy a new car.

given argument reads

$$\begin{array}{l} (p \wedge q) \to r \\ r \to s \\ \sim s \\ \hline \therefore \sim p \vee \sim q \end{array}$$

consider $(p \wedge q) \to r \wedge (r \to s) \wedge \sim s$

$\Rightarrow (p \wedge q) \to r \wedge \sim r$ (by rule of Modus Tollens)

$\Rightarrow \sim (p \wedge q)$ (" ——— " ——— ")

$\equiv \sim p \vee \sim q$.

Thus the given argument is valid.

8) i)
$$\begin{array}{l} p \to r \\ q \to r \\ \hline \therefore (p \vee q) \to r \end{array}$$

ii)
$$\begin{array}{l} p \to q \\ r \to s \\ \sim q \vee \sim s \\ \hline \therefore \sim (p \wedge r) \end{array}$$

iii)
$$\begin{array}{l} (\sim p \vee \sim q) \to (r \wedge s) \\ r \to t \\ \sim t \\ \hline \therefore p \end{array}$$

Soln:- i) $(p \to r) \wedge (q \to r)$

$\Leftrightarrow (\sim p \vee r) \wedge (\sim q \vee r)$

$\Leftrightarrow (r \vee \sim p) \wedge (r \vee \sim q)$ (commutative law)

$\Leftrightarrow r \vee (\sim p \wedge \sim q)$ (distributive law)

$\Leftrightarrow r \vee \sim (p \vee q)$ (De Morgan's law)

$\Leftrightarrow \sim (p \vee q) \vee r$ (commutative law).

$\Leftrightarrow (p \vee q) \to r$

$\therefore$ given argument is valid.

ii) $(p \to q) \wedge (r \to s) \wedge (\sim q \vee \sim s)$

$\Leftrightarrow (p \to q) \wedge (r \to s) \wedge (q \to \sim s)$

$\Leftrightarrow (p \to q) \wedge (q \to \sim s) \wedge (r \to s)$ (commutative law)

$\Leftrightarrow (p \to \sim s) \wedge (r \to s)$ (rule of syllogism)

$\Leftrightarrow (p \to \sim s) \wedge (\sim s \to \sim r)$ (by using contrapositive)

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$\Leftrightarrow \quad p \to \sim r$ (rule of syllogism)

$\Leftrightarrow \quad \sim p \vee \sim r$

$\Leftrightarrow \quad \sim (p \wedge r)$

$\therefore$ given argument is valid.

iii) $[(\sim p \vee \sim q) \to (r \wedge s)] \wedge (r \to t) \wedge (\sim t)$

$\Rightarrow [(\sim p \vee \sim q) \to (r \wedge s)] \wedge \sim r$ (Modus Tollen's rule)

$\Rightarrow [(\sim p \vee \sim q) \to (r \wedge s)] \wedge (\sim r \vee \sim s)$ (rule of disjunctive amplification)

$\Leftrightarrow [(\sim p \vee \sim q) \to (r \wedge s)] \wedge (\sim (r \wedge s))$ (Demorgan's rule)

$P \to Q \wedge \sim Q \Longrightarrow \sim P$

$\Rightarrow \sim (\sim p \vee \sim q)$ (Modus Tollen's rule)

$\Leftrightarrow p \wedge q$ (DeMorgan's law)

$\Rightarrow p$ (rule of conjunctive simplification).

$\therefore$ given argument is valid.

9) i) $\sim p \leftrightarrow q$
$q \to r$
$\sim r$

$\therefore p$

ii) $(\sim p \vee q) \to r$
$r \to (s \vee t)$
$\sim s \wedge \sim u$
$\sim u \to \sim t$

$\therefore p$

iii) QP
p
$p \to q$
$s \vee r$
$r \to \sim q$

$\therefore s \vee t$

Soln :- i) $(\sim p \leftrightarrow q) \wedge (q \to r) \wedge \sim r$

$\Leftrightarrow (\sim p \leftrightarrow q) \wedge \sim q$ (Modus Tollens rule)

$\Leftrightarrow [(\sim p \to q) \wedge (q \to \sim p)] \wedge \sim q$

$\Leftrightarrow [(\sim p \to q) \wedge \sim q] \wedge (q \to \sim p)$

$\Leftrightarrow (\sim p \to q) \wedge \sim q$ (rule of conjunctive simplification)

$\Leftrightarrow \sim\sim p$ (Modus Tollens rule)

$\Leftrightarrow p$.

$\therefore$ given argument is valid.

ii) $[(\sim p \vee q) \to r] \wedge [r \to (s \vee t)] \wedge (\sim s \wedge \sim u) \wedge (\sim u \to \sim t)$

$\Rightarrow [(\sim p \vee q) \to (s \vee t)] \wedge \sim s \wedge [\sim u \wedge (\sim u \to \sim t)]$ (by rule of syllogism & Associative law)

$\Rightarrow$ $[(\sim p \vee q) \rightarrow (s \vee t)] \wedge (\sim s \wedge \sim t)$ (by modus Pones rule)

$\Leftrightarrow$ $[(\sim p \vee q) \rightarrow (s \vee t)] \wedge [\sim (s \vee t)]$ (DeMorgan's law).

$\Rightarrow$ $\sim (\sim p \vee q)$ (Modus Tolkn's rule)

$\Leftrightarrow$ $p \wedge \sim q$

$\Rightarrow$ p (rule of conjunctive simplification).

$\therefore$ given argument is valid.

iii) $p \wedge (p \rightarrow q) \wedge (s \vee r) \wedge (r \rightarrow \sim q)$

$\Leftrightarrow$ $p \wedge (p \rightarrow q) \wedge (\sim s \rightarrow r) \wedge (r \rightarrow \sim q)$

$\Rightarrow$ $p \wedge (p \rightarrow q) \wedge (\sim s \rightarrow \sim q)$ (rule of syllogism)

$\Leftrightarrow$ $p \wedge \{(p \rightarrow q) \wedge (q \rightarrow s)\}$ (contrapositive)

$\Leftrightarrow$ $p \wedge (p \rightarrow s)$ (rule of syllogism)

$\Rightarrow$ s (rule of Modus pones)

$\Rightarrow$ $s \vee t$ (rule of disjunctive amplification).

$\therefore$ given argument is valid.

10) Prove that the following arguments are not valid:

i) $p \wedge \sim q$
$p \rightarrow (q \rightarrow r)$
$\therefore \sim r$

ii) $p \vee q$
$q \rightarrow (r \rightarrow s)$
$t \rightarrow r$
$\therefore \sim s \rightarrow \sim t$

iii) $p \rightarrow q$
$r \rightarrow s$
$\sim q \rightarrow \sim s$
$\therefore \sim (p \wedge r)$

Soln :- i) $p \wedge \sim q \wedge (p \rightarrow (q \rightarrow r))$

$\Rightarrow$ $p \wedge (p \rightarrow (q \rightarrow r))$ ($\because$ $p \wedge q \Rightarrow p$ ie rule of conjunctive simplificn)

$\Rightarrow$ $q \rightarrow r$ (Rule of Modus Pones)

$\not\Rightarrow$ $\sim r$.

$\therefore$ given argument is not valid.

ii) $(p \vee q) \wedge (q \rightarrow (r \rightarrow s)) \wedge (t \rightarrow r)$.

$\Rightarrow$ $(\sim p \rightarrow q) \wedge [q \rightarrow (r \rightarrow s)] \wedge (t \rightarrow r)$.

$\Rightarrow [\sim p \to (r \to s)] \wedge (t \to r)$ (rule of Syllogism)

$\Rightarrow [p \vee (r \to s)] \wedge (t \to r)$

$\Rightarrow [p \vee (\sim r \vee s)] \wedge (\sim t \vee r)$

$\Rightarrow [(p \vee s) \vee (\sim r)] \wedge (\sim t \vee r)$

$\Rightarrow [\{(p \vee s) \vee \sim r\} \wedge \sim t] \vee \{[(p \vee s) \vee \sim r] \wedge r\}$

$\not\Rightarrow \sim s \to \sim t$.

$\therefore$ given argument is not valid.

iii) $(p \to q) \wedge (r \to s) \wedge (\sim q \to \sim s)$

$\Rightarrow (p \to q) \wedge (\sim s \to \sim r) \wedge (\sim q \to \sim s)$ (contrapositive)

$\Rightarrow (p \to q) \wedge (\sim q \to \sim s) \wedge (\sim s \to \sim r)$

$\Rightarrow (p \to q) \wedge (\sim q \to \sim r)$ (rule of Syllogism)

$\Rightarrow (p \to q) \wedge (r \to q)$ (contrapositive)

$\Rightarrow (\sim p \vee q) \wedge (\sim r \vee q)$

$\Rightarrow (q \vee \sim p) \wedge (q \vee \sim r)$

$\Rightarrow q \vee (\sim p \wedge \sim r)$

$\Rightarrow q \vee \sim (p \vee r)$

$\not\Rightarrow \sim p \vee \sim r$. $\Rightarrow$ given argument is not valid.

11) Show that $R \vee S$ follows logically from the premises
QP $C \vee D$, $(C \vee D) \to \sim H$, $\sim H \to (A \wedge \sim B)$ and $(A \wedge \sim B) \to R \vee S$.

Soln :- $(C \vee D) \wedge [(C \vee D) \to \sim H] \wedge [\sim H \to (A \wedge \sim B)] \wedge [(A \wedge \sim B) \to (R \vee S)]$

$\Rightarrow (C \vee D) \wedge [(C \vee D) \to (A \wedge \sim B)] \wedge [(A \wedge \sim B) \to (R \vee S)]$ (syllogism)

$\Rightarrow (C \vee D) \wedge [(C \vee D) \to (R \vee S)]$ (syllogism)

$\Rightarrow R \vee S$ (rule of modus Pones).

This Proves the required result.

12) Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \rightarrow R$, $P \rightarrow M$ and $\sim M$.

Soln :- $(P \vee Q) \wedge (Q \rightarrow R) \wedge (P \rightarrow M) \wedge (\sim M)$

$\Rightarrow (\sim P \rightarrow Q) \wedge (Q \rightarrow R) \wedge (\sim P)$ (Modus Tollens rule)

$\Rightarrow (\sim P \rightarrow R) \wedge (\sim P)$ (rule of syllogism)

$\Rightarrow R$ (Modus Pones rule)

$\Rightarrow R \wedge (P \vee Q)$ ($\because$ $P \vee Q$ is true (premise)).

This proves the required result.

13) Show that $S \vee R$ is tautologically implied by $(P \vee Q)$, $(P \rightarrow R)$ and $(Q \rightarrow S)$.

Soln :- $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$

$\Leftrightarrow (\sim P \rightarrow Q) \wedge (Q \rightarrow S) \wedge (P \rightarrow R)$

$\Rightarrow (\sim P \rightarrow S) \wedge (P \rightarrow R)$ (syllogism)

$\Leftrightarrow (\sim R \rightarrow \sim P) \wedge (\sim P \rightarrow S)$ (contrapositive)

$\Rightarrow (\sim R \rightarrow S)$ (syllogism)

$\Leftrightarrow R \vee S$

$\equiv S \vee R$.

This proves the required result.

Application of switching Circuits

A switching network is made up of switches and wire connecting to terminals. If a switch is open, we assign symbol 0 to it and if it is close, we assign symbol 1 to it.

fig (i)

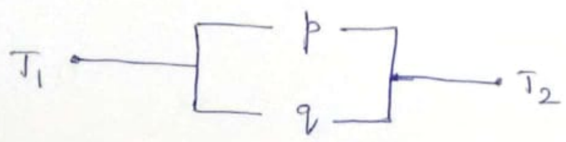


fig (ii)

fig(i) shows series network consisting of network p and q in which current flows from T_1 to T_2 if both p and q are closed. This network is represented by $p \wedge q$.

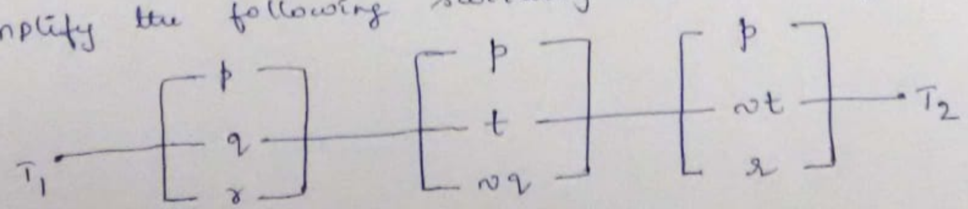
fig (ii) shows parallel network consisting of switches p and q where current flows from T_1 to T_2 either p is closed or q is closed or both are closed. This network is represented by $p \vee q$.

Note :- If every switching network is a combination of these networks, then using laws of logic, we can obtain new networks containing less no. of switches.

Problem

Simplify the following switching network using laws of logic.

1)



soln :- Given network can be represented as

$(p \vee q \vee r) \wedge (p \vee t \vee \sim q) \wedge (p \vee \sim t \vee r)$ (distributive law)

$\equiv [p \vee \{(q \vee r) \wedge (t \vee \sim q)\}] \wedge (p \vee \sim t \vee r)$

$\equiv p \vee [(q \vee r) \wedge (t \vee \sim q) \wedge (\sim t \vee r)]$ (distributive law)

$\equiv p \vee [(q \vee r) \wedge (\sim t \vee r) \wedge (t \vee \sim q)]$ (commutative law)

$\equiv p \vee [(r \vee q) \wedge (r \vee \sim t) \wedge (t \vee \sim q)]$

$\equiv p \vee [\{r \vee (q \wedge \sim t)\} \wedge (t \vee \sim q)]$ (distributive law)

$\equiv p \vee [\{r \vee (q \wedge \sim t)\} \wedge \sim(\sim t \wedge q)]$ (DeMorgan's law)

$\equiv p \vee [\{r \vee (\sim t \wedge q)\} \wedge \sim(\sim t \wedge q)]$

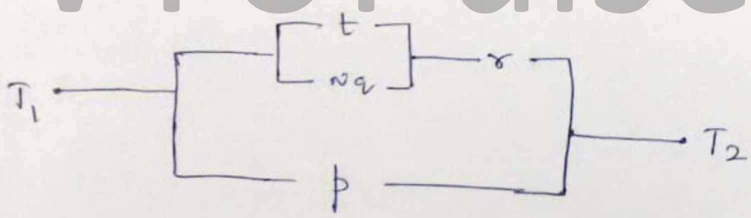
$\equiv p \vee [(r \vee u) \wedge \sim u]$ where $u = \sim t \wedge q$

$\equiv p \vee [(r \wedge \sim u) \vee (u \wedge \sim u)]$ (distributive law)

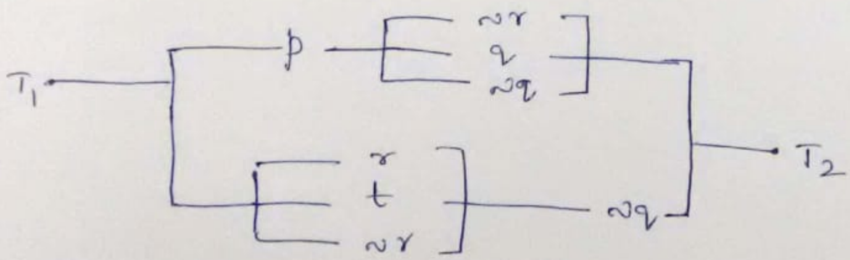
$\equiv p \vee [(r \wedge \sim u) \vee F_0]$ (Inverse law)

$\equiv p \vee [r \wedge \sim u]$ (Identity law)

$\equiv p \vee \{r \wedge (t \vee \sim q)\}$



2)

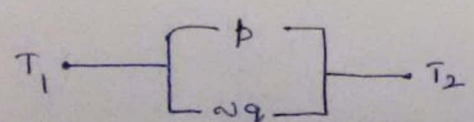


Soln:- $[p \wedge (\sim r \vee q \vee \sim q)] \vee [\sim q \wedge (r \vee t \vee \sim r)]$

$\equiv [p \wedge (\sim r \vee T_0)] \vee [\sim q \wedge (t \vee T_0)]$ (Inverse law)

$\equiv [p \wedge T_0] \vee [\sim q \wedge T_0]$ (Identity law)

$\equiv p \vee \sim q$



Open statements : Consider the declarative sentences $\frac{x}{3}$, $x^2 = 2$, $x+5 = 8$ etc which are not propositions unless the symbol x is specified. Such statements are called open statements (or) open sentences. The unspecified symbol present in open statement such as 'x' are called free-variables. Open statements containing variable x are denoted by $p(x), q(x), r(x) \ldots$

If U is set of elements such that $p(x)$ becomes a proposition when x is replaced by elements of U, then U is called Universe. If $a \in U$, then the proposition obtained by replacing x by a is denoted by $p(a)$.

Ex: consider the open statement $p(x) : x+3 = 6$, then $p(2) = 2+3 = 6$, which is false ; $p(3) = 3+3 = 6$, which is true.

Note :- An open statement becomes a proposition only when x is replaced by some element of the universe. For a given open statement and given element of universe, we can determine truth or false of the given open statement using same laws as we used in case of compound propositions.

Ex :- $p(x) : x \geq 0$ $\quad q(x) : x^2 \geq 0$ $\quad r(x) : x^2 - 5x - 4 = 0$

then find i) $p(2) \wedge q(2)$ ii) $p(-1) \vee q(-2)$ iii) $p(3) \rightarrow r(3)$

i) $p(2) : 2 \geq 0 \wedge q(2) : 4 \geq 0 \Rightarrow$ It is true.

ii) $-1 \geq 0 \vee 4 \geq 0 \Rightarrow$ It is true.

iii) $3 \geq 0 \rightarrow 9 - 15 - 4 = 0 \Rightarrow$ It is false.

Quantifiers :- The words 'all', 'every', 'some', 'any', 'there exists' etc which are associated with open statements, with the idea of a quantity are called quantifiers.

Note :-

1) The phrases for all, for every, for any etc are called universal quantifiers. The phrases for some, there exists, for atleast one etc are called existential quantifiers.

2) universal quantifiers are denoted by symbol $\forall$.
Existential quantifiers are denoted by symbol $\exists$.

3) A proposition involving the universal or the existential quantifier is called a quantified statement.

4) The variable present in a quantified statement is called a bound variable - it is bound by a quantifier.

Rules to find truth value of a quantified statement :

1) $\forall x, p(x)$ is true only when $p(x)$ is true for each and every value of the universe.

2) $\exists x, p(x)$ is false only when $p(x)$ is false for each and every value of the universe.

3) If $\forall x \in S$, $p(x)$ is true and if $a \in S$, then $p(a)$ is true. This rule is called the Rule of universal specification.

4) If $a \in S$ and $p(a)$ is true, then $\exists x, p(x)$ is true. This rule is called the Rule of Universal Generalization.

5) Rules of Negation : i) $\sim\{\forall x, p(x)\} \equiv \exists x, \sim p(x)$
ii) $\sim\{\exists x, p(x)\} \equiv \forall x, \sim p(x)$.

Note :- 1) Two quantified statements are said to be logically equivalent whenever they have same truth values in all possible situations.

2) $\forall x, [p(x) \wedge q(x)] \equiv (\forall x, p(x)) \wedge (\forall x, q(x))$

3) $\exists x, [p(x) \vee q(x)] \equiv (\exists x, p(x)) \vee (\exists x, q(x))$

4) $\exists x, [p(x) \rightarrow q(x)] \equiv \exists x, [\sim p(x) \vee q(x)]$

5) $\forall x, \sim p(x) \equiv$ for no x, $p(x)$.

Problems :-

1) For the universe of all integers, let

QP

$p(x) : x > 0$

$q(x) : x$ is even

$r(x) : x$ is a perfect square

$s(x) : x$ is divisible by 3

$t(x) : x$ is divisible by 7.

Write down the following quantified statements in symbolic form

i) Atleast one integer is even

ii) There exists a positive integer that is even.

iii) Some even integers are divisible by 3.

iv) Every integer is either even or odd.

v) If x is even and a perfect square, then x is not divisible by 3.

vi) If x is odd or is not divisible by 7, then x is divisible by 3.

Soln :- i) $\exists x, q(x)$ ii) $\exists x, [p(x) \wedge q(x)]$

iii) $\exists x, [q(x) \wedge s(x)]$ iv) $\forall x, [q(x) \underline{\vee} \sim q(x)]$

v) $\forall x, [\{q(x) \wedge r(x)\} \rightarrow \sim s(x)]$ vi) $\forall x, [\{\sim q(x) \vee t(x)\} \rightarrow s(x)]$

2) Let $p(x)$ be an open statement $x^2 = 2x$, where the universe comprises all integers. Determine whether each of the statement is true or false.

QP

i) $p(0)$ ii) $p(1)$ iii) $p(2)$ iv) $p(-2)$ v) $\exists x, p(x)$ vi) $\forall x, p(x)$

Soln :- i) $p(0) : 0^2 = 2(0) \rightarrow$ true

ii) $p(1) : 1^2 = 2(1) \rightarrow$ false.

iii) $p(2) : 2^2 = 2(2) \rightarrow$ true

iv) $p(-2) : (-2)^2 = 2(-2) \rightarrow$ false

v) $\exists x, p(x)$ read as 'for some x', $p(x) \rightarrow$ true.

vi) $\forall x, p(x)$ read as 'for every x', $p(x) \rightarrow$ false.

P.T.O.

3) Let the universe comprise of all integers.

QP i) Given $p(x)$: x is odd, $q(x)$: x^2-1 is even.
Express the statement "If x is odd then x^2-1 is even"
in symbolic form using quantifiers and negate it.

ii) If $r(x) : 2x+1=5$, $s(x) : x^2=9$ are open sentences,
obtain the negation of the quantified statement.
$\exists x, [r(x) \wedge s(x)]$.

Soln :- Let Z denote the set of all integers.

i) Given conditional reads :

$$\forall x \in Z, [p(x) \rightarrow q(x)].$$

negation of this is : $\exists x \in Z, [p(x) \wedge \sim q(x)]$

$[\because$ WKT $p \rightarrow q \equiv \sim p \vee q$

$\therefore \sim[p \rightarrow q] \equiv \sim[\sim p \vee q] \equiv p \wedge \sim q.]$

In words : for some integer x, x is odd and x^2-1 is not even.

ii) Given : $\exists x, [r(x) \wedge s(x)]$

negation of this is : $\forall x \in Z, [\sim r(x) \vee \sim s(x)]$.

In words : for all integers x, $2x+1 \neq 5$ or $x^2 \neq 9$.

4) write the following sentences in symbolic form and find its
QP negation :

i) "If all triangles are right angled, then no triangle is equiangular".

ii) "All integers are rational no's and some rational numbers are not integers".

iii) "Some straight lines are parallel or all straight line intersect".

Soln :- i) Let T denote the set of all triangles.

Let $p(x)$: x is right-angled, $q(x)$: x is equiangular.

In symbolic form, given proposition reads :

$$\{\forall x \in T, p(x)\} \rightarrow \{\forall x \in T, \sim q(x)\}.$$

The negation of this is:

$\{\forall x \in T,\ p(x)\} \wedge \{\exists x \in T,\ q(x)\}$ $\left\{ \begin{array}{l} \because p \to \sim q \equiv \sim p \vee \sim q \\ \Rightarrow \sim[p \to \sim q] \equiv \sim[\sim p \vee \sim q] \\ \equiv p \wedge q \end{array} \right.$

In words: "All triangles are right angled and some triangles are equiangular".

ii) Let Z denote the set of all integers. Let Q denote the set of all rational no's.

Let $p(x)$: x is a rational no.

$q(x)$: x is an integer.

Given: $\{\forall x \in Z, p(x)\} \wedge \{\exists x \in Q, \sim q(x)\}$

Negation: $\{\exists x \in Z, \sim p(x)\} \vee \{\forall x \in Q,\ q(x)\}$ $(\because \sim[p \wedge \sim q] \equiv \sim p \vee q$.

In words: "Some integers are not rational numbers or all rational no's are integers".

iii) Let S denote the set of all straight lines.

$p(x)$: x is parallel.

$q(x)$: x intersects.

Given: $\{\exists x \in S,\ p(x)\} \vee \{\forall x \in S,\ q(x)\}$

Negation: $\{\forall x \in S, \sim p(x)\} \wedge \{\exists x \in S, \sim q(x)\}$

In words: "All straight lines are not parallel and some straight lines do not intersect".

iv) For all real numbers x, y, if $x^2 > y^2$, then $x > y$.

HW

Soln:- Let $\mathbb{R}$ denote the set of all real numbers.

Let $p(x)$: $x^2 > y^2$

$q(x)$: $x > y$

Given: $\{\forall x, y \in \mathbb{R},\ p(x)\} \to \{\forall x, y \in \mathbb{R},\ q(x)\}$.

Negation: $\{\exists x, y \in \mathbb{R},\ p(x)\} \wedge \{\exists x, y \in \mathbb{R}, \sim q(x)\}$ $\left\{ \begin{array}{l} \because p \to q \equiv \sim p \vee q \\ \Rightarrow \sim(p \to q) \equiv p \wedge \sim q \end{array} \right.$

In words: There exists real no's x, y such that $x^2 > y^2$ and $x \leq y$.

5) Consider the following open statements with the set
Q8 of all real numbers as the universe.

$p(x) : x \geq 0$ $\quad$ $q(x) : x^2 \geq 0$

$r(x) : x^2 - 3x - 4 = 0$ $\quad$ $s(x) : x^2 - 3 > 0$

Determine the truth values of the following statements:

i) $\exists x, \; p(x) \wedge q(x)$ $\quad$ ii) $\forall x, \; p(x) \rightarrow q(x)$

iii) $\forall x, \; q(x) \rightarrow s(x)$ $\quad$ iv) $\forall x, \; r(x) \vee s(x)$

v) $\exists x, \; p(x) \wedge r(x)$ $\quad$ vi) $\forall x, \; r(x) \rightarrow p(x)$.

Soln:- i) $\exists x, \; p(x) \wedge q(x)$ is true ($\because$ for $x = 1$,
$p(1) : 1 \geq 0 \rightarrow T$ and $q(1) : 1^2 \geq 0 \rightarrow T$)

$\therefore$ It's truth value is 1.

ii) $\forall x, \; p(x) \rightarrow q(x)$ is true ($\because$ for all real no. x, $q(x)$ is true, i.e. $q(x) : x^2 \geq 0$ cannot be false for any real x)

$\therefore$ It's truth value is 1.

iii) $\forall x, \; q(x) \rightarrow s(x)$ is false ($\because$ $s(x) : x^2 - 3 > 0$ is false for $x = 1$ i.e. $q(x) \rightarrow s(x)$ is not always true)

$\therefore$ It's truth value is 0.

iv) $\forall x, \; r(x) \vee s(x)$ is false ($\because$ $r(x)$ is false for $x = 1$ and $s(x)$ is false for $x = 1$)

$\therefore$ It's truth value is 0.

v) $\exists x, \; p(x) \wedge r(x)$ is true ($\because$ $p(4) : 4 \geq 0$, $r(4) : 0 = 0$) is true
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad T \qquad\qquad\qquad T$

$\therefore$ for some x, $p(x) \wedge r(x)$ is true.

$\therefore$ It's truth value is 1.

vi) $\forall x, \; r(x) \rightarrow p(x)$ is false ($\because$ for $x = -1$, $p(-1) : -1 \geq 0$ is false and $r(-1) : 0 = 0$ is true $\Rightarrow r(x) \rightarrow p(x)$ is false for $x = -1$)

$\therefore$ It's truth value is 0.

P.T.O.

6) Let $p(x) : x^2 - 7x + 10 = 0$, $q(x) : x^2 - 2x - 3 = 0$, $r(x) : x < 0$

QP Determine the truth or falsity of the following statements when the universe U contains only the integers 2 and 5. If a statement is false, provide a counter example or explanation.

i) $\forall x,\ p(x) \to \sim r(x)$ ii) $\forall x,\ q(x) \to r(x)$

iii) $\exists x,\ q(x) \to r(x)$ iv) $\exists x,\ p(x) \to r(x)$.

Soln:- Here $U = \{2, 5\}$

consider $p(x): x^2 - 7x + 10 = (x-5)(x-2)$

$\therefore$ $p(x)$ is true for $x = 5$ and $x = 2$.

ie $p(x)$ is true for all $x \in U$.

Consider $q(x) : x^2 - 2x - 3 = (x-3)(x+1)$

$\therefore$ $q(x)$ is true only for $x = 3, x = -1$ (which are not in U)

ie $q(x)$ is false for all $x \in U$.

obviously $r(x)$ is false for all $x \in U$

i) $\forall x,\ p(x) \to \sim r(x)$ is true; since $\sim r(x)$ is true $\forall x \in U$. and $p(x)$ is true $\forall x \in U$.

ii) $\forall x,\ q(x) \to r(x)$ is true; since $q(x)$ is false for all $x \in U$ and $r(x)$ is false for all $x \in U$.

iii) $\exists x,\ q(x) \to r(x)$ is true; since $q(x)$ and $r(x)$ are false for $x = 2$.

iv) $\exists x,\ p(x) \to r(x)$ is false; since $p(x)$ is true $\forall x \in U$ and $r(x)$ is false for all $x \in U$. $\therefore$ $p(x) \to r(x)$ is false for every $x \in U$.

7) Write down the negation of each of the following statements:

QP i) If k, m, n are any integers where $(k-m)$ and $(m-n)$ are odd, then $(k-n)$ is even.

ii) If x is a real number where $x^2 > 16$, then $x < -4$ or $x > 4$.

Soln :- i) Let Z denote the set of all integers.

Given : $\forall\ k, m, n \in Z$, $[p(x) \wedge q(x)] \rightarrow r(x)$

Let $p(x)$: $(k-m)$ is odd.

$q(x)$: $(m-n)$ is odd.

$r(x)$: $(k-n)$ is even.

Negation : $\exists\ k, m, n \in Z$, $[p(x) \wedge q(x)] \wedge \sim r(x)$

$(\because\ P \rightarrow Q \equiv \sim p \vee Q$

$\sim(P \rightarrow Q) \equiv p \wedge \sim Q)$

In words: There exists integers k, m, n such that $(k-m)$, $(m-n)$ are odd and $(k-n)$ is not even.

ii) Let R denote the set of all real numbers.

Let $p(x)$: $x^2 > 16$

$q(x)$: $x < -4$

$r(x)$: $x > 4$

Given : $\forall\ x \in R$, $p(x) \rightarrow (q(x) \vee r(x))$

Negation : $\exists\ x \in R$, $[p(x) \wedge \sim q(x) \wedge \sim r(x)]$

$\because\ P \rightarrow Q \equiv \sim P \vee Q$

$\sim(P \rightarrow Q) \equiv P \wedge \sim Q$

In words : For some real no. x, $x^2 > 16$ and $x \geq -4$ and $x \leq 4$.

8) Negate and simplify each of the following:

i) $\exists\ x, [p(x) \vee q(x)]$

ii) $\forall\ x, [p(x) \wedge \sim q(x)]$

iii) $\forall\ x, [p(x) \rightarrow q(x)]$

iv) $\exists\ x, [\{p(x) \vee q(x)\} \rightarrow r(x)]$

Soln :- i) $\sim [\exists\ x, \{p(x) \vee q(x)\}] \equiv \forall\ x, \{\sim p(x) \wedge \sim q(x)\}$

ii) $\sim[\forall\ x, \{p(x) \wedge \sim q(x)\}] \equiv \exists\ x, \{\sim p(x) \vee q(x)\}$.

iii) $\sim[\forall\ x, \{p(x) \rightarrow q(x)\}] \equiv \sim[\forall\ x, \{\sim p(x) \vee q(x)\}]$

$\equiv \exists\ x, \{p(x) \wedge \sim q(x)\}$

iv) $\sim[\exists\ x, [\{p(x) \vee q(x)\} \rightarrow r(x)]] \equiv \sim[\exists\ x, \{\sim\{p(x) \vee q(x)\}\} \vee r(x)]$

$\equiv \forall\ x, [\{p(x) \vee q(x)\} \wedge \sim r(x)]$

P.T.O.

9) Write down the following statement in symbolic form & negate them

i) There exist a matrix whose transpose is itself.

ii) Every element of a group has inverse.

iii) Atleast one parallelogram is a rhombus.

Soln:- i) let $p(x)$: Transpose of x is itself.

Sym. form : $\exists x \in S, p(x)$

Negation : $\sim[\exists x \in S, p(x)] \equiv \forall x \in S, \sim p(x)$.

In words : " None of the matrix has its own transpose".

ii) let $p(x)$: x has an inverse ; S : group.

Sf : $\forall x \in S, p(x)$

Negation : $\sim\{\forall x \in S, p(x)\} \equiv \exists x \in S, \sim p(x)$

"There exists some elements of a group, which does not have inverse".

iii) Let S : set of all parallelogram

$p(x)$: x is a rhombus.

Sf : $\exists x \in S, p(x)$

Negation : $\sim\{\exists x \in S, p(x)\} \equiv \forall x \in S, \sim p(x)$.

" None of the parallelogram is a rhombus".

10) For the universe of all polygons with 3 or 4 sides, given

Q) $i(x)$: all interior angles of x are equal.

$h(x)$: all sides of x are equal.

$s(x)$: x is a square

$t(x)$: x is a triangle.

Translate each of the following into an English sentence and determine its truth value :

i) $\forall x, [s(x) \leftrightarrow (i(x) \wedge h(x))]$ ii) $\exists x, [t(x) \rightarrow (i(x) \leftrightarrow h(x))]$

Soln :- i) Every polygon with 3 or 4 sides is a square if and only if all its interior angles are equal and all its sides are equal. → False Ex :- Equilateral Δ^{le}.

P.T.O.

ii) $\exists x, [t(x) \rightarrow (i(x) \leftrightarrow h(x))] \equiv \exists x, \{\sim t(x) \vee (i(x) \leftrightarrow h(x))\}$.

Some polygon with 3 or 4 sides is not a Δ^{le} or all its interior angles are equal iff all its sides are equal.

$\rightarrow$ True EX:- a square $\rightarrow T \vee T$; Equilateral $\Delta^{le} \rightarrow F \vee T$.

11) For the above defined open statement, write the following statements symbolically and determine its truth values.

i) Any polygon with 3 or 4 sides is either a Δ^{le} or a square.

ii) For any Δ^{le} if all the interior angles are not equal, then all its sides are not equal.

Soln:- i) $\forall x, (t(x) \veebar s(x)) \rightarrow$ false $\because$ if x is a rectangle then both $s(x)$ and $t(x)$ are false.

ii) Given statement - "If a polygon with 3 or 4 sides is a Δ^{le} and all its interior angles are not equal, then all its sides are not equal."

$\forall x, [\{t(x) \wedge \sim i(x)\} \rightarrow \sim h(x)] \rightarrow$ true.

Logical Implication involving Quantifiers :

A quantified statement P is said to logically imply a quantified statement Q if Q is true whenever P is true. we write $P \Rightarrow Q$.

Note :- Given a set of quantified statements $P_1, P_2, \ldots P_n$ and Q, we say that "Q is a valid conclusion from the premises $P_1, P_2 \ldots P_n$ if $P_1 \wedge P_2 \wedge \ldots \wedge P_n \Rightarrow Q$.

Problems

1) Prove the following :

i) $\forall x,\ p(x) \Rightarrow \exists x,\ p(x)$ ii) $\forall x, [p(x) \vee q(x)] \Rightarrow \forall x, p(x) \vee \exists x, q(x)$.

Soln :- Let S denote the universe.

i) $\forall x,\ p(x) \Rightarrow p(x)$ is true for every $x \in S$.

$\Rightarrow p(a)$ is true for $x = a \in S$

$\Rightarrow p(x)$ is true for some $x \in S$

$\Rightarrow \exists x,\ p(x)$.

ii) $\forall x\, [p(x) \vee q(x)] \Rightarrow p(x) \vee q(x)$ is true for every $x \in S$.

$\Rightarrow \{p(x)$ is true for every $x \in S\}$
$\vee \{q(x)$ is true for every $x \in S\}$

$\Rightarrow \forall x,\ p(x) \vee q(a)$ is true for $x = a \in S$.

$\Rightarrow \forall x,\ p(x) \vee \exists x, q(x)$

2) Prove that $\{\forall x, p(x) \vee \forall x, q(x)\} \Rightarrow \forall x, [p(x) \vee q(x)]$. the converse is not true. Through a counterexample, show that

Soln :- Let S denote the universe.

Take any $a \in S$. then $\forall x, [p(x) \vee q(x)]$ is true whenever $p(a) \vee q(a)$ is true.

ie whenever $p(a)$ is true or $q(a)$ is true

ie " $p(x)$ is true for any x or $q(x)$ is true for any x.

ie " $\forall x, p(x)$ is true or $\forall x, q(x)$ is true.

ie " $\forall x,\ p(x) \vee \forall x, q(x)$ is true.

i.e. $[\forall x, p(x) \vee \forall x, q(x)] \Rightarrow \forall x, [p(x) \vee q(x)]$

Conversely

Let $p(x) : x^2 - 4 = 0$; $q(x) : x^2 - 1 = 0$; with $S = \{1, 2\}$.

For $x = 1$; $p(x)$ is false but $q(x)$ is true so that $p(x) \vee q(x)$ is true.

for $x = 2$; $p(x)$ is true but $q(x)$ is false so that $p(x) \vee q(x)$ is true.

Thus for every $x \in S$, $p(x) \vee q(x)$ is true

i.e. $\forall x, \{p(x) \vee q(x)\}$ is true.

But $\forall x, p(x)$ is false and $\forall x, q(x)$ is false.

$\therefore$ $[\forall x, p(x) \vee \forall x, q(x)]$ is false.

Thus, $\forall x, [p(x) \vee q(x)] \not\Rightarrow [\forall x, p(x) \vee \forall x, q(x)]$

i.e. Converse of the given open statement is not true.

3) For each of the following, determine whether the following arguments are valid or not :-

i) All men are mortal.
Sachin is a man

$\therefore$ Sachin is mortal.

Soln :- Let S denote the set of all men.
Let $p(x)$: x is mortal
a : Sachin.

Given argument reads :

$$\begin{array}{c} \forall x \in S, p(x) \\ a \in S \\ \hline \therefore p(a). \end{array}$$

By the rule of universal specification, $p(a)$ is true.

$\therefore$ given argument is valid.

ii) All mathematics professors have studied Calculus.
Ramanujan is a mathematics professor.

$\therefore$ Ramanujan has studied Calculus.

Given the universe is the set of all people.

Soln :- Let $p(x)$: x is a Mathematics professor.

$q(x)$: x studies Calculus.

a : Ramanujan.

Given argument reads :

$$\begin{array}{l} \forall x, \quad p(x) \rightarrow q(x) \\ p(a) \\ \hline \therefore\ q(a) \end{array}$$

$[\forall x, p(x) \rightarrow q(x)] \wedge p(a)$

$\Rightarrow [p(a) \rightarrow q(a)] \wedge p(a)$

$\Rightarrow q(a)$ (Rule of Modus ponens).

$\therefore$ given argument is valid.

iii) For the universe of set of all students,

No engineering student is bad in studies.

Anil is not bad in studies

$\therefore$ Anil is an Engg. student.

Soln :- Let $p(x)$: x is an engg. student.

$q(x)$: x is not bad in studies.

a : Anil

Given argument reads :

$$\begin{array}{l} \forall x, [p(x) \rightarrow q(x)] \\ q(a) \\ \hline \therefore\ p(a) \end{array}$$

$[\forall x, p(x) \rightarrow q(x)] \wedge q(a) \Rightarrow [p(a) \rightarrow q(a)] \wedge q(a)$

$\not\Rightarrow p(a)$

$\therefore$ given argument is not valid.

iv) No Engg. student of I or II semester studies logic.

QP Anil is an Engg. student who studies logic.

$\therefore$ Anil is not in II semester.

Soln :- Let the universe S be the set of all engg. students.

Let $p(x)$: x is in 1st semester

$q(x)$: x is in 2nd semester.

$r(x)$: x studies logic.

a : Anil.

Given :

$$\begin{array}{l} \forall x \left[\{ p(x) \vee q(x) \} \rightarrow \sim r(x) \right] \\ r(a) \\ \hline \therefore \sim q(a). \end{array}$$

$\forall x \left[\{ p(x) \vee q(x) \} \rightarrow \sim r(x) \right] \wedge r(a)$

$\Rightarrow \left[\{ p(a) \vee q(a) \} \rightarrow \sim r(a) \right] \wedge r(a)$

$\Rightarrow r(a) \wedge \left[r(a) \rightarrow \sim \{ p(a) \vee q(a) \} \right]$ (commutative law & contrapositive)

$\Rightarrow \sim \left[p(a) \vee q(a) \right]$ (Rule of Modus pones)

$\Rightarrow \sim p(a) \wedge \sim q(a)$ (DeMorgan's law)

$\Rightarrow \sim q(a)$ (Rule of conjuctive simplification)

Thus the given argument is valid.

v) $\forall x, \left[p(x) \rightarrow q(x) \right]$

$$\begin{array}{l} \forall x, \left[q(x) \rightarrow r(x) \right] \\ \hline \therefore \forall x, \left[p(x) \rightarrow r(x) \right] \end{array}$$

Soln :- Let S be the universe and $a \in S$.

$\left[\forall x, \{ p(x) \rightarrow q(x) \} \right] \wedge \left[\forall x, \{ q(x) \rightarrow r(x) \} \right]$

$\Rightarrow \left[p(a) \rightarrow q(a) \right] \wedge \left[q(a) \rightarrow r(a) \right]$

$\Rightarrow p(a) \rightarrow r(a)$ (Rule of syllogism)

$\Rightarrow \forall x, \left[p(x) \rightarrow r(x) \right]$ (Rule of universal Generalization)

$\therefore$ given argument is valid.

P.T.O.

vi) $\forall x, [p(x) \vee q(x)]$

QP $\forall x, [\{\sim p(x) \wedge q(x)\} \rightarrow r(x)]$

$\therefore \forall x, [\sim r(x) \rightarrow p(x)]$

Soln :- consider

$\{\sim p(x) \wedge q(x)\} \rightarrow r(x) \Leftrightarrow \sim r(x) \rightarrow \{p(x) \vee \sim q(x)\}$ (using contrapositive & DeMorgan's law)

$\Rightarrow p(x) \vee \sim q(x)$ (Modus pones rule).

$\therefore$ for any x,

$[p(x) \vee q(x)] \wedge [\{\sim p(x) \wedge q(x)\} \rightarrow r(x)]$

$\Rightarrow [p(x) \vee q(x)] \wedge [p(x) \vee \sim q(x)]$

$\Leftrightarrow p(x) \vee [q(x) \wedge \sim q(x)]$ (distributive law)

$\Rightarrow p(x) \vee F_0$

$\Rightarrow p(x) \rightarrow$ true

$\Rightarrow \sim r(x) \rightarrow \underset{T}{p(x)}$

irrespective of T or F

vii) $\forall x, [p(x) \vee q(x)]$

QP $\exists x, \sim p(x)$

$\forall x, [\sim q(x) \vee r(x)]$

$\forall x, [s(x) \rightarrow \sim r(x)]$

$\therefore \exists x, \sim s(x)$

Let S be the universe and $a \in S$.

Soln :- $[\forall x, \{p(x) \vee q(x)\}] \wedge [\exists x, \sim p(x)] \wedge [\forall x, \{\sim q(x) \vee r(x)\}] \wedge [\forall x, \{s(x) \rightarrow \sim r(x)\}]$

$\Rightarrow \{p(a) \vee q(a)\} \wedge \{\sim p(a)\} \wedge \{\sim q(a) \vee r(a)\} \wedge \{s(a) \rightarrow \sim r(a)\}$.

$\Rightarrow q(a) \wedge \{\sim q(a) \vee r(a)\} \wedge \{s(a) \rightarrow \sim r(a)\}$ (rule of disjunctive syllogism)

$\Rightarrow r(a) \wedge \{s(a) \rightarrow \sim r(a)\}$ (rule of disjunctive syllogism)

$\Rightarrow \sim s(a)$ (Rule of Modus Tollens)

$\Rightarrow \exists x, \sim s(x)$

$\therefore$ given argument is valid.

Open Statements with more than one Variable :-

Open statements containing two variables x and y are usually denoted by $p(x,y)$, $q(x,y)$ etc. Open statements containing three variables are denoted by $p(x,y,z)$, $q(x,y,z)$ etc. Universe can be same for all variables or different for different variables.

Ex :- $p(x,y)$: $x+2y=0$.

$p(x,y,z)$: $x^2+y^2+z^2=a^2$.

If U is the Universe for x and V is the Universe for y in an open statement $p(x,y)$ and if $a \in U$ and $b \in V$, then the proposition got by replacing x by a and y by b in $p(x,y)$ is denoted by $p(a,b)$.

Quantified Statements with more than one variable :-

If $p(x,y)$ is an open statement with variables x, y, we can have quantified statements of the following forms:

i) $\forall x, \forall y, p(x,y)$

ii) $\exists x, \exists y, p(x,y)$

iii) $\forall x, \exists y, p(x,y)$

iv) $\exists x, \forall y, p(x,y)$.

If x and y belong to the same universe, then i) & ii) becomes

$\forall x, \forall y, p(x,y) \equiv \forall x,y, p(x,y)$.

$\exists x, \exists y, p(x,y) \equiv \exists x,y, p(x,y)$.

Note :- i) $\forall x, \forall y, p(x,y) \equiv \forall y, \forall x, p(x,y)$.

ii) $\exists x, \exists y, p(x,y) \equiv \exists y, \exists x, p(x,y)$.

Problems :

1) Let x and y denote integers. Consider the statement

$p(x,y)$: $x+y$ is even.

write the following statements in words :

i) $\forall x, \exists y, p(x,y)$ ii) $\exists x, \forall y, p(x,y)$.

Soln :- i) For every integer x, there exists an integer y such that $x+y$ is even.

ii) There exists an integer x such that $x+y$ is even, for every (for all) integer y.

2) write down the following statements in symbolic form using quantifiers :

i) Every real number has an additive inverse.

ii) The integer 58 is equal to sum of 2 perfect squares.

Soln :- i) Given statement is same as :

" Given any real number x, there is a real number y such that $x+y = y+x = 0$ ".

In symbols : $\forall x, \exists y, [x+y = y+x = 0]$. Here $x, y \in \mathbb{R}$.

ii) Given statement is same as :

" There exists integers x and y such that $58 = x^2+y^2$ ".

In symbols : $\exists x, \exists y, 58 = x^2+y^2$. Here $x, y \in \mathbb{Z}$.

3) Determine the truth value of each of the following quantified statements, the universe being the set of all non-zero integers.

i) $\exists x, \exists y, [xy = 1]$ ii) $\exists x, \forall y, [xy = 1]$.

iii) $\forall x, \exists y, [xy = 1]$ iv) $\exists x, \exists y, [(2x+y=5) \wedge (x-3y = -8)]$

v) $\exists x, \exists y, [(3x-y = 17) \wedge (2x+4y=3)]$.

Soln :- i) True (Take $x=1, y=1$)

ii) False ($\because$ for a specified x, $xy=1$ for every y is not true).

iii) False ($\because$ for $x=2$, there is no integer y such that $xy=1$).

iv) True. (Take $x=1$, $y=3$).

v) False ($\because$ Eqns $3x-y=17$ and $2x+4y=3$ do not have a common integer soln).

4) Prove that the following argument is valid, where a and b are some particular members of the universe.

$$\forall x, \forall y, [p(x,y) \rightarrow q(x,y)]$$
$$\sim q(a,b)$$
$$\overline{\therefore \exists x, \exists y, \{\sim p(x,y)\}}$$

Soln:- $\{\forall x, \forall y, [p(x,y) \rightarrow q(x,y)]\} \wedge \sim q(a,b)$

$\Rightarrow \{p(a,b) \rightarrow q(a,b)\} \wedge \sim q(a,b)$

$\Rightarrow \sim p(a,b)$ (Modus Tollens rule)

$\Rightarrow \exists x, \exists y, \{\sim p(x,y)\}$

$\therefore$ given argument is valid.

5) Find the negation of the following quantified statement:

$\forall x, \exists y, [\{p(x,y) \wedge q(x,y)\} \rightarrow r(x,y)]$.

Soln:- $\sim[\forall x, \exists y, \{p(x,y) \wedge q(x,y)\} \rightarrow r(x,y)\}$.

$\Leftrightarrow \exists x, \sim\{\exists y, \{p(x,y) \wedge q(x,y)\} \rightarrow r(x,y)]$

$\Leftrightarrow \exists x, \forall y, \sim[\{p(x,y) \wedge q(x,y)\} \rightarrow r(x,y)]$

$\Leftrightarrow \exists x, \forall y, [p(x,y) \wedge q(x,y) \wedge \sim r(x,y)]$.

Methods of Proof and Methods of Disproof

Given $p \to q$, where p and q are simple / compound propositions, the process of establishing that the conditional is true by using laws of logic / using rules etc constitutes a proof of the conditional. The process of establishing that a proposition is false constitutes a disproof.

1) Direct Proof :- To Prove $p \to q$ is true.
 i) Hypothesis : Assume p is true.
 ii) Analysis : Employ the rules / laws of logic, infer that q is true.
 iii) conclusion : $p \to q$ is true.

2) Indirect proof :- Recall that a conditional and its contrapositive (ie $p \to q$ and $\sim q \to \sim p$) are logically equivalent. In some situations, given a conditional $p \to q$, a direct proof of the contrapositive $\sim q \to \sim p$ is easier. On the basis of this proof, we infer that the conditional $p \to q$ is true. This method of proving a conditional is called an indirect method of proof.

3) Proof by contradiction :-
 i) Hypothesis : Assume $p \to q$ is false. ie p is true & q is false.
 ii) Analysis : Employ the rules / laws of logic and infer that p is false. This contradicts the assumption that p is true.
 iii) conclusion : Because of the contradiction arrived in analysis, we infer that $p \to q$ is true.

4) Proof by Exhaustion :- Recall that a proposition of the form "$\forall x \in S, p(x)$" is true if $p(x)$ is true for each x in S. If S consists of only a limited no. of elements, we can prove that the statement "$\forall x \in S, p(x)$" is true by considering $p(a)$ for each a in S, and verifying that $p(a)$ is true (in each case). Such a method of proof is called the method of exhaustion.

5) Proof by Existence :-

A proposition of the form "$\exists x \in S, p(x)$" is true if any one element $a \in S$ such that $p(a)$ is true is exhibited. Hence proving a proposition of the form "$\exists x \in S, p(x)$" is by exhibiting the existence of one $a \in S$ such that $p(a)$ is true. The method of proof is called the proof of existence.

6) Disproof by Contradiction :-

To disprove a conditional $p \rightarrow q$ (i.e to prove $p \rightarrow q$ is false), we take the hypothesis that p is true and q is true, and end up with a contradiction. This concludes that the conditional $p \rightarrow q$ is false. This method of disproving $p \rightarrow q$ is called Disproof by contradiction.

7) Disproof by counterexample :

A proposition of the form "$\forall x \in S, p(x)$" is false if any one element $a \in S$ such that $p(a)$ is false is exhibited. Hence disproving a proposition involving the universal quantifier is by exhibiting just one case where the proposition is false. This method of disproof is called Disproof by counterexample.
(A particular case where the proposition is false is called a counterexample).

Problems

1) Give a direct proof of the statement:
QP " The square of an odd integer is an odd integer".

Soln:- To Prove: If n is an odd integer, then n^2 is an odd integer.

Assume n is an odd integer.

then $n = 2k+1$, $k \in \mathbb{Z}$.

$\therefore$ $n^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$

if $2k^2+2k = k'$, then

$n^2 = 2k'+1$

$\therefore$ n^2 is odd.

2) Give a direct proof for each of the following:
QP i) For all integers k and l, if k and l are both even, then $k+l$ is even.

ii) For all integers k and l, if k and l are both even, then $k \cdot l$ is even.

Soln:- Take any 2 integers k and l, and assume both these are even, then

$k = 2m$, $l = 2n$ for $m, n \in \mathbb{Z}$.

$\therefore$ $k+l = 2m+2l = 2(m+l)$, divisible by 2.

and $kl = 2m \times 2n = 4mn$, divisible by 2.

$\therefore$ $k+l$ and kl are both even.

Since k and l are arbitrary integers, in view of rule of universal generalisation, the proof is complete.

3) Provide an indirect proof for following argument:
for all positive integers x and y, If the product exceeds 25, then $x>5$ or $y>5$.

Soln:- Let p : $xy > 25$

q : $x > 5$

r : $y > 5$.

Then the given statement reads:

$$p \to (q \vee r).$$

contrapositive of this statement is:

$$\sim(q \vee r) \to \sim p.$$

$$\Rightarrow (\sim q \wedge \sim r) \to \sim p.$$

suppose $(\sim q \wedge \sim r)$ is true, then $\sim q$ is true and $\sim r$ is true; i.e. $x \le 5$ and $y \le 5$.

$\Rightarrow xy \le 25$, so that $\sim p$ is true.

Thus the proof by contrapositive is true.

4) For each of the following statements, provide an indirect proof of the given statements:

i) For all integers k and l, if kl is odd, then both k and l are odd.

ii) For all integers k and l, if $k+l$ is even, then k and l are both even or both odd.

Soln:- i) Let p: kl is odd; $k, l \in Z$.
q: k is odd and l is odd

Given statement reads: $p \to q$.

It's contrapositive is: $\sim q \to \sim p$. (to be proved).

i.e. TPT if k is even or l is even, then $k.l$ is even.

suppose k is even, then $k = 2m$, $m \in Z$.

$\therefore kl = (2m)l = 2(ml)$ which is even.

$|||^{ly}$ suppose l is even, then $l = 2n$, $n \in Z$.

$\therefore kl = k(2n) = 2(kn)$ which is even.

This proves the contrapositive.

ii) Let p: $k+l$ is even.
q: k and l are both even or k and l are both odd
$k, l \in Z$

Given statement reads: $p \to q$.

It's contrapositive is: $\sim q \to \sim p$ (to be proved).

i.e. TPT, if one of k and l is odd and the other is even, then k+l is odd.

Suppose k is odd and l is even.

i.e. $k = 2m+1, \quad l = 2n, \quad m,n \in Z$.

then $k+l = 2m+1+2n = 2(m+n)+1$, which is odd.

III^{ly} Suppose k is even and l is odd.

i.e. $k = 2m, \quad l = 2n+1, \quad m,n \in Z$.

then $k+l = 2m+2n+1 = 2(m+n)+1$, which is odd.

This proves the contrapositive.

5) Provide a proof by contradiction of the following statement:
QP for every integer n, if n^2 is odd, then n is odd.

Sol^n :- Let $p: n^2$ is odd $\quad q: n$ is odd.

Given statement reads: $p \to q$.

Assume $p \to q$ is false, then p is true and q is false.

q is false $\Rightarrow$ n is even

$\Rightarrow n = 2k, \quad k \in Z$.

$\therefore n^2 = (2k)^2 = 4k^2$, which is even.

$\Rightarrow$ p is false.

This contradicts the assumption that p is true.

$\therefore p \to q$ is true.

6) Prove by contradiction that, for all real no.'s x and y,
QP if $x+y \geq 100$, then $x \geq 50$ or $y \geq 50$.

Sol^n :- Let $p: x+y \geq 100, \quad q: x \geq 50, \quad r: y \geq 50 \quad \forall x,y \in R$.

Given statement reads: $p \to (q \vee r)$.

Assume that $p \to (q \vee r)$ is false, then p is true and $q \vee r$ is false $\Rightarrow$ q is false and r is false.

$\Rightarrow x < 50$ and $y < 50$.

$\Rightarrow x+y < 100 \quad \Rightarrow p$ is false.

This contradicts the assumption that p is true.

$\therefore$ $p \to q$ is true.

7) Give (i) a direct proof (ii) an indirect proof and (iii) proof by contradiction for the following statement:
"If n is an odd integer, then $n+9$ is an even integer".

QP

Soln:- i) Direct Proof :- Assume that n is an odd integer.

then $n = 2k+1$, $k \in \mathbb{Z}$.

$\therefore$ $n+9 = (2k+1)+9 = 2k+10 = 2(k+5) = 2k'$ (say)

$\Rightarrow$ $n+9$ is even.

ii) Indirect proof :- Let p : n is odd q : $n+9$ is even.

Given statement reads: $p \to q$.

It's contrapositive is: $\sim q \to \sim p$.

Suppose $\sim q$ is true i.e. $n+9$ is odd.

then $n+9 = 2k+1$, $k \in \mathbb{Z}$.

$\Rightarrow$ $n = 2k+1-9$ $\Rightarrow$ $n = 2k-8$

$\Rightarrow$ $n = 2(k-4) = 2k'$ (say)

$\Rightarrow$ n is even.

i.e. $\sim p$ is true.

This proves the contrapositive.

iii) Proof by contradiction :-

Let p : n is odd, q : $n+9$ is even.

Given statement reads : $p \to q$.

Assume that $p \to q$ is false. i.e. p is true and q is false.

Since q is false, $n+9$ is odd

$\Rightarrow$ $n+9 = 2k+1$, $k \in \mathbb{Z}$.

$\Rightarrow$ $n = 2k+1-9$ $\Rightarrow$ $n = 2k-8$

$\Rightarrow$ $n = 2(k-4) = 2k'$ (say)

$\Rightarrow$ n is even.

This contradicts the assumption that n is odd.

$\therefore$ $p \to q$ is true.

8) Prove that every even integer n with $2 \le n \le 26$ can be written as a sum of atmost 3 perfect squares.

Sol^n:- Let $S = \{2, 4, 6 \ldots 24, 26\}$.

Let $p(x)$: x is a sum of atmost 3 perfect squares.

TPT : the statement $\forall x \in S, p(x)$ is true

(Proof by exhaustion).

we have
$2 = 1^2 + 1^2$
$4 = 2^2$
$6 = 2^2 + 1^2 + 1^2$
$8 = 2^2 + 2^2$
$10 = 3^2 + 1^2$
$12 = 2^2 + 2^2 + 2^2$
$14 = 3^2 + 2^2 + 1^2$
$16 = 4^2$
$18 = 4^2 + 1^2 + 1^2$
$20 = 4^2 + 2^2$
$22 = 3^2 + 3^2 + 2^2$
$24 = 4^2 + 2^2 + 2^2$
$26 = 5^2 + 1^2$.

$\therefore$ each x in S is a sum of atmost 3 perfect squares.

9) Prove that there exists a real no. x such that $x^3 + 2x^2 - 5x - 6 = 0$.

Sol^n:- It is sufficient to exhibit one real no. x such that

$x^3 + 2x^2 - 5x - 6 = 0$.

For $x = 2$; $2^3 + 2(2)^2 - 5(2) - 6 = 0$.
$\Rightarrow 8 + 8 - 10 - 6 = 0$
$\Rightarrow 0 = 0$.

10) Disprove the statement :
" The sum of 2 odd integers is an odd integer".

Sol^n:- Let p: a and b are odd integers
q: $a+b$ is an odd integer.

then the proposition to be disproved is $p \rightarrow q$.

Assume that p is true and q is true.

then $a = 2k_1 + 1$; $b = 2k_2 + 1$ $\longrightarrow (1)$

$a + b = 2k_3 + 1$ $\longrightarrow (2)$. $k_1, k_2, k_3 \in \mathbb{Z}$.

From (1), $a+b = 2(k_1+k_2+1)$.

$\Rightarrow$ $a+b$ is even.

This contradicts the assumption (2).

$\therefore$ $p \to q$ is false, which disproves the given statement.

11) Disprove by counterexample that the sum of squares of any 4 non-zero integers is an even integer.

Soln :- Proposition is : " for any 4 non-zero integers a, b, c, d, $a^2+b^2+c^2+d^2$ is an even integer".

for $a=1$, $b=1$, $c=1$, $d=2$, the proposition is false.

$\therefore$ The proposition is disproved through the counterexample.

12) Give a direct proof of the statement " the square of an even integer is an even integer".