NATIONAL INSTITUTE OF TECHNOLOGY CALICUT

DEPARTMENT OF MATHEMATICS

Winter Semester 2022-23

MA2003D: Mathematics IV · Tutorial sheet 3

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1 Complex differentiability, Cauchy-Riemann equations, Laplace's equation

1. Show that $\lim_{x+iy \to 0} \left[xy / \left(x^2 + y^2\right) \right]$ does not exist.

2. Show that $\lim_{x+iy \to 0} \left[x^2 y / \left(x^4 + y^2\right) \right]$ does not exist even though this function approaches the same limit along every straight line through the origin.

3. If
$$f(x, y) = \begin{cases} x \sin\left(\frac{1}{y}\right) & y \neq 0 \\ 0 & y = 0 \end{cases}$$
show that $\lim_{y \to 0} \left[\lim_{x \to 0} f(x, y)\right]$ and $\lim_{x \to 0} f(x, y)$ exist and are equal, but that $\lim_{x \to 0} \left[\lim_{y \to 0} f(x, y)\right]$ does not exist.

4. Show that the various values approached by the difference quotient of $f(z) = \bar{z}$ as $\Delta z \to 0$ along the lines $y = mx$ all lie on a circle.

5. Check whether the following functions analytic?
   - (a) $f(z) = \operatorname{Im}\left(z^2\right)$
   - (b) $f(x + iy) = e^{2x}(\cos y + i \sin y)$
   - (c) $f(x + iy) = e^{-x}(\cos y - i \sin y)$
   - (d) $f(z) = \operatorname{Arg} z$
   - (e) $f(z) = \operatorname{Re} z + \operatorname{Im} z$
   - (f) $f(z) = \ln|z| + i \operatorname{Arg} z$
   - (g) $f(z) = i/z^8$
   - (h) $f(z) = z^2 + 1/z^2$

6. Prove that the function $f(x + iy) = \sqrt{|x||y|}$ satisfies the Cauchy-Riemann equations at the origin, but f is not holomorphic at origin.

7. Find the set points on which each of the following functions satisfies the Cauchy-Riemann equations. Also check whether the function is holomorphic and if so find its derivative.
   - (a) $f(x + iy) = x$
   - (b) $f(x + iy) = x^2 + iy^2$
   - (c) $f(x + iy) = x^3 + i(y - 1)^3$
   - (d) $f(z) = \bar{z}$.
   - (e) $f(z) = \bar{z}^2$.
   - (f) $f(z) = |z|^2$.

8. Determine a, b, c such that given the functions are harmonic and find a harmonic conjugate.
   - (a) $u = e^{3x} \cos ay$
   - (b) $u = \sin x \cosh cy$
   - (c) $u = ax^3 + by^3$

9. Are the following functions harmonic? If yes, find a corresponding analytic function $f(z) = u(x, y) + iv(x, y)$
   - (a) $u = xy$
   - (b) $v = xy$
   - (c) $v = -y/\left(x^2 + y^2\right)$
   - (d) $v = \ln|z|$
   - (e) $u = x^3 - 3xy^2$
   - (f) $u = e^{-x} \sin 2y$

10. Let f be a holomorphic function in an open set Ω. Prove that f reduces to a constant if any of the following is true:

(a) $\text{Re}(f)$ is a constant
(b) $\text{Im}(f)$ is a constant
(c) $|f|$ is a constant
(d) $\text{Arg}(f)$ is a constant
(e) $f' = 0$.
(f) $\bar{f}$ is analytic.

11. Let u be a real valued harmonic function in an open disc $\mathbb{D}$. Prove that any two harmonic conjugates of u differ by a constant.

12. Let u be a real valued harmonic function in an open disc $\mathbb{D}$. If u^2 is also harmonic, prove that u is a constant.

13. Let u be a real valued harmonic function in an open disc $\mathbb{D}$ and v its harmonic conjugate. Prove that $-u$ is a harmonic conjugate of v.

14. Let u be a real valued harmonic function in an open set Ω and v its harmonic conjugate. Prove that uv and $u^2 - v^2$ are also harmonic.

15. Let u be a real valued harmonic function in an open set Ω . Prove that $\frac{\partial u}{\partial z}$ is holomorphic in Ω.

16. Check whether the following functions u are harmonic in some domain and if so find their harmonic conjugates . Also find the holomorphic function $f(z)$ such that $u = Re(f)$.

(a) $u(x, y) = c$
(b) $u(x, y) = y^3 - 3x^2y$
(c) $u(x, y) = 2x(1 - y)$
(d) $u(x, y) = \frac{y}{x^2+y^2}$
(e) $u(x, y) = \sinh x \sin y$

17. If f is holomorphic on a region Ω, prove that f is harmonic there. When is $|f|^2$ harmonic?

18. Let f be a complex function on a region Ω and both f and f^2 are harmonic in Ω. Prove that either f or $\bar{f}$ is harmonic in Ω.

19. If u and v are harmonic in a region $\Omega \subset \mathbb{C}$, show that

$$\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

is analytic in Ω.

20. If $f(z)$ is an analytic function, show that

(a) $\left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = |f'(z)|^2$
(b) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\,|f'(z)|^2$

21. If u and v are respectively the real and imaginary parts of a holomorphic function f, prove that the level curves $u(x, y) = k$ and $v(x, y) = c$ are orthogonal to each other at all points (x, y) where $f'(x + iy) \neq 0$.

2 Geometry of analytic functions

1. What is the image of the circle $x^2 + y^2 = a^2$ under the mapping $w = z^2$?

2. Discuss the transformation between the z plane and the w plane defined by $w = x - iy$:

3. Discuss the transformation defined by $w = z^3$. Plot the image of the line $u = 1$. What is the equation of the image of the line $x = 1$?

4. Find the equations of the transformation defined by the function $w = (z - i)/z$, and show that every circle through the origin in the z plane is transformed into a straight line. 11 If $w = (2+i)z - (1+3i)$, what are the equations of the mapping of the w plane onto the z plane?

5. If $w = e^z$, (a) What are the images of the lines $x = c$? What are the images of the lines $y = k$? (b) Show that the strip $-\pi < y \leq \pi$ is mapped onto the entire w plane. What is the preimage of the line $r = 0$? (c) What are the images of the positive halves of the strips $-\pi < y \leq -\pi/2$. $-\pi/2 < y \leq 0, 0 < y \leq \pi/2, \pi/2 < y \leq \pi$?

6. Show that if a transformation of the form $w = (az + b)/(cz + d)$ maps z_1 into w_1 and z_2 into w_1, then either $z_1 = z_2$ or else $ad - bc = 0$.

7. What is the cross ratio of the four fourth roots of -1 ?

8. What is the cross ratio of the four complex sixth roots of 1 ?

9. Show that in general there are two points which are left invariant by a bilinear transformation. thought of as a mapping of the z plane onto itself. Are there any bilinear transformations which leave only one point invariant? No points invariant?

10. Find the invariant points of the transformation $z' = -(2z + 4i)(iz + 1)$, and prove that these two points, together with an arbitrary point z and its image z', form a set of four points whose cross ratio is independent of z.

11. Find the invariant points of the transformation

$$T(z) = \frac{2iz + 1}{(-3 + 4i)z + 4}$$

12. What is the bilinear transformation which sends the points $z = 0, -1, \infty$ into the points $w = -1, -2 - i$, i. respectively? What is the image of the circle $|z| = 1$ under this transformation?

13. What is the bilinear transformation which sends the points $z = 0, -i, 2i$ into the points $w = 5i, \alpha, -i/3$, respectively? What are the invariant points of this transformation?

14. Find the equations of the transformation of inversion in the circle $x^2 + y^2 = 1$, and show that under this transformation a circle is mapped into itself if and only if it is perpendicular to the circle defining the inversion.

15. What is the most general bilinear transformation which maps the upper half of the z plane onto the lower half of the w plane?

16. Prove that $w = z/(1 - z)$ maps the upper half of the z plane onto the upper half of the w plane. What is the image of the circle $|z| = 1$ under this transformation?

17. Find a Schwarz-Christoffel transformation that maps the upper half-plane onto the sector $\{w \in \mathbb{C} : 0 < \mathrm{Arg}(w) < \alpha\}$ for some $\alpha < \pi$.

18. Derive a Schwarz-Christoffel transformation mapping the upper half-plane onto the triangle with vertices $0, 1, i$.

19. Determine a Schwarz-Christoffel transformation that maps the upper half-plane onto the semi-infinite strip $\{w \in \mathbb{C} : |\,\mathrm{Re}(w)| < 1; \mathrm{Im}(w) > 0\}$.

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