

Context-free Languages, Type Theoretically

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Abstract. Parsing is the process of recovering structure from strings, an essential part of implementing programming languages. Previous work has shown that formalizing languages and parsers using an idiomatic type theoretic approach can be simple and enlightening. Unfortunately, this approach has only been applied to regular languages, which are not expressive enough for many practical applications. We have extended the type theoretic formalization to context-free languages (without mutual recursion) which are substantially more expressive. We hope our formalization can serve as a foundation for reasoning about new disambiguation techniques and even more expressive formalisms such as data-dependent grammars.

Keywords: Parsing · Type Theory · Formal Languages

1 Introduction

Parsing—i.e., the process of recovering structure from strings—is an essential building block for many practical programming applications. While parsing has been extensively studied, it remains a relevant subject of research where new research questions continuously emerge. For example, how to compose grammars and parsers (e.g., [15]), dealing with ambiguous parse trees (e.g., [4,3,1]), and parsing grammar formalisms beyond context-free grammars (e.g., [1]). Research questions such as these serve a practical purpose, but answering them often requires a deep theoretical understanding of the semantics of parsing.

This theoretical understanding can be approached in different ways. Parsing is often studied using automata theory [10]. However, there is value in studying more *denotational* approaches to parsing. A main purpose of denotational semantics is to abstract away operational concerns, as such concerns tends to be a hindrance for equational reasoning. Such equational reasoning could be used to study and answer some of the open research questions in the parsers of today and tomorrow.

This paper studies the denotational semantics of parsing for context-free grammars. While the study is theoretical in nature, the motivation is to provide a foundation for practical future studies on proving the correctness of, e.g., parser optimizations and disambiguation techniques, as well as potentially providing a foundation for building and reasoning about parsers for more expressive grammar formalisms, such as data-dependent grammars [1].

We approach the question of giving a denotational semantics of parsing by building on existing work by Elliott [9]. In his recent work, Elliott demonstrated that regular grammars have a simple and direct denotational semantics. And that we can obtain parsers for such languages that are correct by construction, using *derivatives*. While it was well-known that we can parse regular grammars using Brzozowski derivatives [6], Elliott’s work provides a simple and direct mechanization in Agda’s type theory of the denotational semantics of these derivatives. This mechanization provides an implementation of parsing that is correct by construction, and that we can reason about without

relying on (bi-)simulation arguments. While the parsers obtained in this manner are not exactly performant, the denotational approach opens up the door to exploiting grammar structure to obtain optimized parsers.

Elliott leaves open the question of how the approach scales to more expressive grammar formalisms, such as context-free languages and beyond. However, the question of using derivatives to parse context-free grammars has been considered by others. Might et al. [13] demonstrate how to build parsers from context-free grammars using derivatives and optimizations applied to them, to obtain reasonable performance. Thiemann’s work [16] uses lattice theory and powerset semantics to formalize a notion of partial derivative for a variant of context-free grammars. In this work, we build on the approach of Elliott and study how to build a simple and direct mechanization in Agda’s type theory of the denotational semantics of derivatives for context-free grammars.

A main challenge for our mechanization is the question of how to deal with the recursive nature of context-free languages.

1.1 The Challenge with Automated Differentiation of Context-Free Grammars

Derivatives (or *language differentiation*) provide an automated procedure for parsing. We give an overview of what it means to take the derivative of a grammar, how this provides an approach to parsing, and consider the problem of automatically taking the derivative of a context-free grammar.

To illustrate, consider the following context-free grammar of palindromic bit strings:

$$\langle pal \rangle ::= 0 \mid 1 \mid 0 \langle pal \rangle 0 \mid 1 \langle pal \rangle 1$$

Say we want to use this grammar to parse the string 0110. The idea of automatic differentiation is this. We first compute the derivative of the grammar w.r.t. the first bit (0) of our bit string (0110); let us call this grammar $\langle pal_0 \rangle$. Then, we take the derivative of $\langle pal_0 \rangle$ w.r.t. the next bit (1). We continue this procedure until we either (a) get stuck because the derivative is invalid, in which case the bit string is not well-formed w.r.t. our grammar, or (b) we reach the end of our bit string, in which case the bit string is well-formed w.r.t. our grammar if the derived final grammar contains the empty production (we will use the symbol ϵ to denote the empty grammar).

Taking the derivative of the $\langle pal \rangle$ grammar w.r.t. the bit 0 yields the following derived grammar:

$$\langle pal_0 \rangle ::= \epsilon \mid \langle pal \rangle 0$$

This grammar essentially represents the residual parsing obligations after parsing a 0 bit. The derived grammar contains fewer productions than the original grammar because we have pruned those productions that started with the terminal symbol 1 (because the derivative of the bit 0 w.r.t. the terminal symbol 1 is invalid).

Now, how do we take the derivative of the grammar $\langle pal_0 \rangle$ w.r.t. the next bit (1) in our string? A simple solution is to recursively unfold the $\langle pal \rangle$ non-terminal. Doing so for the $\langle pal \rangle$ grammar yields the following derived grammar:

$$\langle pal'_0 \rangle ::= \epsilon \mid 00 \mid 10 \mid 0 \langle pal \rangle 00 \mid 1 \langle pal \rangle 10$$

By continuing this procedure, with additional recursive unfolding where needed, we eventually yield a grammar that contains the empty production ϵ , whereby we can conclude that 0110 is, in fact, a palindromic bit string.

However, the recursive unfolding we performed above is not safe to do for all grammars. Consider, for example, the infinitely recursive grammar:

$$\langle \text{rec} \rangle ::= \langle \text{rec} \rangle$$

We cannot ever unfold this grammar to expose a terminal symbol to derive w.r.t., akin to the informal procedure we applied above. While the $\langle \text{rec} \rangle$ grammar is contrived, similar issues arise for any *left-recursive* grammar, such as the following grammar of arithmetic expressions (left-recursive because of the $\langle \text{expr} \rangle$ non-terminal in the left-most position in the first production):

$$\langle \text{expr} \rangle ::= \langle \text{expr} \rangle + \langle \text{expr} \rangle \mid 0 \mid 1$$

Another challenge with context-free grammars is how to encode their recursive nature in a proof assistant such as Agda in a way that our encoding of grammars is *strictly positive*,³ and in a way that ensures that automated differentiation—that is, continuously applying the method we informally illustrated above for taking the derivative of a grammar w.r.t. a symbol—is guaranteed to terminate.

1.2 Contributions

This paper tackles the challenges discussed in the previous section by providing a mechanization in Agda of automated differentiation of a subset of context-free grammars. The subset of grammars that we consider corresponds to context-free grammars without mutually recursive grammars. For example, the following is an example of a mutually recursive grammar that does not fit into the subset of grammars we consider:

$$\begin{aligned} \langle \text{expr} \rangle &::= \langle \text{expr} \rangle + \langle \text{expr} \rangle \mid 0 \mid 1 \mid \langle \text{stmt} \rangle \\ \langle \text{stmt} \rangle &::= \langle \text{expr} \rangle \mid \langle \text{stmt} \rangle; \langle \text{stmt} \rangle \end{aligned}$$

The $\langle \text{pal} \rangle$, $\langle \text{rec} \rangle$, and $\langle \text{expr} \rangle$ grammars from the previous section are all examples of grammars that are in the subset. We conjecture that the approach we present later can be extended to deal with all context-free grammars, at the cost of some additional book-keeping during derivation. We leave verifying this conjecture as a challenge for future work.

We make the following technical contributions:

- We provide a semantics in Agda of context-free grammars without mutual recursion.
- We provide a derivative-based parser for this class of grammars, along with its simple and direct correctness proof.

The paper assumes basic familiarity with Agda. The rest of this paper is structured as follows. Section 2 recalls the essential definition from Elliott’s work which we subsequently extend in Section 3 to context-free grammars. Section 4 discusses expressiveness, performance, and simplicity of our approach, whereas Section 5 discusses related work, and Section 6 concludes.

2 Finite Languages

In this section, we introduce background information, namely how we define languages, basic language combinators, and parsers. Our exposition follows Elliott [9]. In Section 3, we extend these concepts to context free languages.

³ <https://agda.readthedocs.io/en/v2.6.1.3/language/data-types.html#strict-positivity>

2.1 Languages

We define languages as being functions from strings to types.⁴

`Lang = String → Type`

The result type can be thought of as the type of proofs that the string is in the language.

Remark 1. Note that a language may admit multiple different proofs for the same string. That is an important difference between the type theoretic approach and the more common set theoretic approach, which models languages as sets of strings.

This is a broad definition of what a language is; it includes languages that are outside the class of context-free languages.

Example 1. The language $a^n b^n c^n$ can be specified as follows:

```
abc : Lang
abc w = Σ[ n ∈ ℕ ] w ≡ repeat n 'a' ++ repeat n 'b' ++ repeat n 'c'
```

We can show that the string $aabbcc$ is in this language by choosing n to be 2, from which the required equality follows by reflexivity after normalization:

```
aabbcc : abc "aabbcc"
aabbcc = 2 , refl
```

Example 1 shows that it is possible to specify languages and prove that certain strings are in those languages, but for practical applications we do not want to be burdened with writing such proofs ourselves. In other words, we want a parser which can determine by itself whether a string is in the language or not.

Unfortunately, we cannot hope to write a parser for arbitrary languages defined in this way. A language could be defined, for example, such that the inclusion of a particular string is predicated on whether or not the Collatz conjecture holds. Therefore, we need to restrict ourselves to a subset of languages. Next, we explore basic language combinators for this purpose.

2.2 Basic Language Combinators

Let's start with a simple example: POSIX file system permissions. These are usually summarized using the characters 'r', 'w', and 'x' if the permissions are granted, or '-' in place of the corresponding character if the permission is denied. For example the string "r-x" indicates that read and execute permissions are granted, but the write permission is denied. The full language can be expressed using the following grammar:

```
⟨permissions⟩ ::= ⟨read⟩ ⟨write⟩ ⟨execute⟩
⟨read⟩         ::= - | r
⟨write⟩        ::= - | w
⟨execute⟩       ::= - | x
```

$_ : \text{Char} \rightarrow \text{Lang}$ $(\text{' } c) w = w \equiv c :: []$	$\emptyset : \text{Lang}$ $\emptyset _ = \perp$
$_ \cup _ : \text{Lang} \rightarrow \text{Lang} \rightarrow \text{Lang}$ $(P \cup Q) w = P w \uplus Q w$	$\epsilon : \text{Lang}$ $\epsilon w = w \equiv []$
$_ * _ : \text{Lang} \rightarrow \text{Lang} \rightarrow \text{Lang}$ $(P * Q) w = \exists [u] \exists [v] w \equiv u ++ v \times P u \times Q v$	$_ \cdot _ : \text{Type} \rightarrow \text{Lang} \rightarrow \text{Lang}$ $(A \cdot P) w = A \times P w$

Fig. 1. Basic language combinators.

This grammar uses three important features: sequencing, choice, and matching character literals. We can define these features as combinators on languages in Agda as shown in the left column of Figure 1. Using these combinators we can define our permissions language as follows:

```
permissions = read * write * execute
read       = ' '-'' ∪ ' 'r'
write      = ' '-'' ∪ ' 'w'
execute    = ' '-'' ∪ ' 'x'
```

The right column of Figure 1 lists combinators whose purpose will become clear when we discuss how to write parsers for this simple language in the next section.

2.3 Parsers

We want to write a program which can prove for us that a given string is in a language. What should this program return for strings that are not in the language? We want to make sure our program does find a proof if it exists, so if it does not exist then we want a proof that the string is not in the language. We can capture this using a type called `Dec` from the Agda standard library. It can be defined as follows:

```
data Dec (A : Type) : Type where
  yes : A → Dec A
  no  : ¬ A → Dec A
```

A parser for a language, then, is a program which can tell us whether any given string is in the language or not.

```
Parser : Lang → Set
Parser P = (w : String) → Dec (P w)
```

⁴ We use `Type` as a synonym for Agda's `Set` to avoid confusion with set-theoretic sets.

Remark 2. Readers familiar with Haskell might see similarity between this type and the type `String -> Maybe a`, which is one way to implement parser combinators (although usually the return type is `Maybe (a, String)` giving parsers the freedom to consume only a prefix of the input string and return the rest). The differences are that the result of our `Parser` type depends on the language specification and input string, and that a failure carries with it a proof that the string cannot be part of the language. This allows us to separate the specification of our language from the implementation while ensuring correctness.

Remark 3. Note that the `Dec` type only requires our parsers to produce a single result; it does not have to exhaustively list all possible ways to parse the input string. In Haskell, one might write `String -> [(a, String)]`, which allows a parser to return multiple results but does nothing to ensure that it correctly produces all possible results. We could imagine requiring that the result type is in bijection with a finite or countably infinite set. However, that would introduce too many complications in our proofs. In practice, furthermore, we want our parsers to only give us a single result. Hence, our effort would be better spent in proving that our languages are unambiguous, meaning there is at most one valid way to parse each input string. Thus, in this paper, we use `Dec`.

To construct a parser for our permissions language, we start by defining parsers for each of the language combinators. Let us start by considering the character combinator. If the given string is empty or has more than one character, it can never be in a language formed by one character. If the string does consist of only one character, then it is in the language if that character is the same as from the language specification. In Agda, we can write such a parser for characters as follows:

```

'-parse_ : (x : Char) → Parser (' x)
('-parse _) [] = no λ ()
('-parse x) (c :: []) = Dec.map (mk⇔ (λ { refl → refl }) (λ { refl → refl })) (c  $\stackrel{?}{=}$  x)
('-parse _) (_ :: _ :: _) = no λ ()

```

This is a correct implementation of a parser for languages that consist of a single character, but the implementation seems ad hoc and it is hard to read, especially considering this is one of the simpler combinators.

Following the approach of parsing with derivatives, we can factor this parser into two cases: the empty string case and the case with at least one character. We call the former nullability and denote it with the greek character ν , and we call the latter derivative and denote it with the greek character δ .

Crucially, nullability deals only with (decidable) types, and derivatives deal only with languages. This clearly separates the level of abstraction between both cases.

Returning to our character parser, a single character language is not nullable. On the level of types we express this as \perp , the uninhabited type, which is trivially decidable as `no λ ()`.

The derivative of a single character language depends on whether the character of the derivative is the same as the character of the language. We might be tempted to define this condition externally in Agda, but that would break the abstraction of derivatives only dealing with languages. Instead, we are pushed toward defining a combinator, $_ \cdot _$, which allows us to express this conditional on the level of languages. If the condition holds then there is still a second condition which is that the remainder of the string needs to be empty. We use the epsilon language, ϵ , for that purpose. To conclude, the derivative of the character language $'c'$ with respect to the character c is $(c \stackrel{?}{=} c') \cdot \epsilon$ as shown in Figure 2.

$$\begin{array}{ll}
\nu P = P \square & (\delta \ c \ P) \ w = P \ (c :: w) \\
\\
A \Leftrightarrow B = (A \rightarrow B) \times (B \rightarrow A) & P \Leftrightarrow Q = \forall \{w\} \rightarrow P \ w \Leftrightarrow Q \ w \\
\\
\begin{array}{ll}
\nu \emptyset : \perp & \Leftrightarrow \nu \emptyset \\
\nu \epsilon : \top & \Leftrightarrow \nu \epsilon \\
\nu \cdot : (A \times \nu P) & \Leftrightarrow \nu (A \cdot P) \\
\nu ' : \perp & \Leftrightarrow \nu (' \ c') \\
\nu \cup : (\nu P \uplus \nu Q) & \Leftrightarrow \nu (P \cup Q) \\
\nu * : (\nu P \times \nu Q) & \Leftrightarrow \nu (P * Q)
\end{array}
\quad
\begin{array}{ll}
\delta \emptyset : \emptyset & \Leftrightarrow \delta \ c \ \emptyset \\
\delta \epsilon : \emptyset & \Leftrightarrow \delta \ c \ \epsilon \\
\delta \cdot : (A \cdot \delta \ c \ P) & \Leftrightarrow \delta \ c \ (A \cdot P) \\
\delta ' : ((c \equiv c') \cdot \epsilon) & \Leftrightarrow \delta \ c \ (' \ c') \\
\delta \cup : (\delta \ c \ P \cup \delta \ c \ Q) & \Leftrightarrow \delta \ c \ (P \cup Q) \\
\delta * : (\nu P \cdot \delta \ c \ Q \cup \delta \ c \ P * Q) & \Leftrightarrow \delta \ c \ (P * Q)
\end{array}
\end{array}$$

Fig. 2. Nullability, derivatives, and how they relate to the basic combinators.

Furthermore, Figure 2 shows the nullability and derivatives of all basic combinators using simple and self-contained equivalences. The implementation of parsers for our basic combinators follow completely from the decomposition into nullability and derivatives and these equivalences. For example, we can rewrite our character parser as follows:

```

'-parse_ : (c' : Char) → Parser (' c')
('-parse _) [] = Dec.map ν' ⊥-dec
('-parse c') (c :: w) = Dec.map δ' (((c ≐ c') · -parse ε-parse) w)

```

Parsers for the other basic combinators are equally straightforward and can be found in our source code artifact.

The parser for our full permissions language can now be implemented by simply mapping each of the language combinators onto their respective parser combinators.

```

permissions-parse = read-parse *-parse (write-parse *-parse execute-parse)
read-parse       = ('-parse '-' ) ∪-parse ('-parse 'r')
write-parse      = ('-parse '-' ) ∪-parse ('-parse 'w')
execute-parse    = ('-parse '-' ) ∪-parse ('-parse 'x')

```

This allows us to generate a parser for any language that is defined using the basic combinators from Figure 1. We mechanize this result later in Section 3.3, but we first consider extending the expressivity of our combinators.

2.4 Infinite Languages

This permissions language is very simple. In particular, it is finite. In practice, many languages are infinite, for which the basic combinators will not suffice. For example, file paths can be arbitrarily long on most systems. Elliott [9] defines a Kleene star combinator which enables him to specify regular languages such as file paths.

However, we want to go one step further, specifying and parsing context-free languages. Most practical programming languages are at least context-free, if not more complicated. An essential feature of many languages is the ability to recognize balanced brackets. A minimal example language with balanced brackets is the following:

$$\langle \text{brackets} \rangle ::= \epsilon \mid [\langle \text{brackets} \rangle] \mid \langle \text{brackets} \rangle \langle \text{brackets} \rangle$$

This is the language of all strings which consist of balanced square brackets. Many practical programming languages include some form of balanced brackets. Furthermore, this language is well known to be context-free and not regular. Thus, we need more powerful combinators.

We could try to naively transcribe the brackets grammar using our basic combinators, but Agda will justifiably complain that it is not terminating. Here we have added a `NON_TERMINATING` pragma to make Agda to accept it any way, but this is obviously not the proper way to define our brackets language.

```
{-# NON_TERMINATING #-}
brackets =  $\epsilon \cup ' [' * brackets * ' ] ' \cup brackets * brackets$ 
```

We need to find a different way to encode this recursive relation. A fixed point combinator could resolve this issue as follows:

```
postulate  $\mu : (\text{Lang} \rightarrow \text{Lang}) \rightarrow \text{Lang}$ 
brackets $\mu$  =  $\mu (\lambda P \rightarrow \epsilon \cup ' [' * P * ' ] ' \cup P * P)$ 
```

Unfortunately, such a fixed point combinator does not exist for arbitrary functions on languages. Luckily, we will see in the next section that we can define such a fixed point combinator if we restrict the functions we are allowed to take the fixed point of.

3 Context-free Languages

3.1 Fixed Points

To be able to specify the recursive structure of context-free languages, we need a fixed point. From type theory we know that a fixed point of a functor $F : \text{Type} \rightarrow \text{Type}$ is well-defined if it is strictly positive. So we could restrict the argument of our fixed point combinator to only accept strictly positive functors. We are dealing with languages and not types directly, but luckily our definition of language is based on types and our basic combinators are strictly positive. One catch is that Agda currently has no way of directly expressing that a functor is strictly positive.⁵ We can still make this evident to Agda by defining a data type of descriptions as shown by Chapman et al. [7].

```
data Desc : Type1 where
   $\emptyset$       : Desc
   $\epsilon$      : Desc
  ' _      : Char  $\rightarrow$  Desc
  _  $\cup$  _   : Desc  $\rightarrow$  Desc  $\rightarrow$  Desc
  _ * _    : Desc  $\rightarrow$  Desc  $\rightarrow$  Desc
```

⁵ There is work on implementing positivity annotations.[14]


```

_ · _ : {A : Type} → Desc A → Desc → Desc
var   : Desc

```

We can give semantics to our descriptions in terms of languages that we defined in Section 2. We use the \diamond prefix to refer to the language combinators defined in Section 2.

```

[ ]_o : Desc → Lang → Lang
[ ∅ ]_o      = ∅
[ ε ]_o      = ε
[ ' c ]_o    = c
[ D1 ∪ D2 ]_o  P = [ D1 ]_o P ∪ [ D2 ]_o P
[ D1 * D2 ]_o  P = [ D1 ]_o P * [ D2 ]_o P
[ _ · {A} _ D ]_o P = A · [ D ]_o P
[ var ]_o    P = P

```

Using these descriptions, we can define a fixed point as follows:

```

data [ ] (D : Desc) : Lang where
  roll : [ D ]_o [ D ] w → [ D ] w

```

```

unroll : [ D ] w → [ D ]_o [ D ] w
unroll (roll x) = x

```

With this fixed point, we can finally define the brackets language.⁶

```

bracketsD = ε ∪ ' '[' * var * ']' ∪ var * var
brackets = [ bracketsD ]

```

This representation is not modular, however. We cannot nest fixed points in descriptions. For example, we could not create a new language which contains the brackets language as a subexpression, because the fixed point is only taken over the whole descriptor. This problem comes up naturally when considering derivatives, which we discuss next.

3.2 Derivatives by Example

As we have seen with finite languages in Section 2, when writing parsers it is useful to consider how a language changes after one character has been parsed. We will call this the *derivative* of the original language. For example, we could consider what happens to our brackets languages after one opening brackets has been parsed: δ '[' brackets. In this section, we search for a description of this reduced language.

We can systematically deduce the derivative language from the brackets definition by going over each of the disjuncts. The first disjunct, ϵ , does not play a role because we know the string contains at least the opening bracket. The second disjunct, brackets surrounding a self-reference, is trickier. The opening bracket clearly matches, but it would be a mistake to say the new disjunct should be a self-reference followed by a closing bracket: `var * ' ']`.

⁶ We split this definition into two because we want to separately reuse the description later.

Note that the self-reference in the new language would refer to the derivative of the old language, not to the old language itself. We would like to refer to the original bracket language: `brackets * ' ']`, but we cannot nest the brackets language into another description.

There are cases where we do want to use self-reference in the new language. Consider the third disjunct, `var * var`. It is a sequence so we expect from the finite case of Section 2 that matching one character results in two new disjuncts: one where the first sequent matches the empty string and the second is reduced and one where the first is reduced and the second is unchanged. In this case both sequents are self-references, so we need to know three things:

1. Does the original language match the empty string?
2. What is the derivative of the language?
3. What does it mean for the language to remain the same?

At first glance, the last point seems obvious, but remember that we are reducing the language, so self-references will change meaning even if they remain unchanged. Similarly to the previous disjunct, we want to refer to the original brackets in this case. To resolve this issue of referring to the original brackets expression, we introduce a new combinator μ , which has the meaning of locally taking a fixed point of a subexpression.

```
data Desc : Type1 where
  - ...
   $\mu$  : Desc → Desc

[ ]o : Desc → Lang → Lang
- ...
[  $\mu$  D ]o _ = [ D ]
```

The first question is easy to answer: yes, the first disjunct of brackets is epsilon which matches the empty string.

```
 $\nu$ brackets : Dec ( $\nu$  brackets)
 $\nu$ brackets = yes (roll (inj1 refl))
```

The second question is where having a self-reference in the new language is useful. We can refer to the derivative of brackets by using self-reference.

Using these answers, we can write the derivative of brackets with respect to the opening bracket.

```
bracketsD' =  $\mu$  bracketsD * ' ' ]' ∪  $\nu$ brackets · var ∪ var *  $\mu$  bracketsD
brackets'  = [ bracketsD' ]
```

From this example, we have learned the following three lessons:

- We can reuse many of the results of finite languages (Section 2).
- We need a new μ combinator to nest fixed points in descriptions. This is necessary to refer back to the original language before reduction.
- Reducing a self-reference simply results in a self-reference again, because self-references in the reduct refer to the reduct.

Again, we do not want to have to manually construct these derivatives. Instead, we show how to do it in general for any description in the next section.

3.3 Parsing in General

Our goal is to define a parse function for every description fixed point.

$\text{parse} : \forall D \rightarrow \text{Parser } \llbracket D \rrbracket$

We approach this by decomposing parsing into computing nullability and derivatives separately as follows:

$\nu D : \forall D \rightarrow \text{Dec } (\nu \llbracket D \rrbracket)$
 $\delta D : \text{Char} \rightarrow \text{Desc} \rightarrow \text{Desc}$

The νD function can easily be written to be correct by construction, however δD must be proven correct separately as follows:

$\delta D\text{-correct} : \llbracket \delta D \ c \ D \rrbracket \iff \delta \ c \ \llbracket D \rrbracket$

The actual parsing can now be done character by character:

$\text{parse } D \llbracket \rrbracket = \nu D \ D$
 $\text{parse } D \ (c :: w) = \text{Dec.map } \delta D\text{-correct} \ (\text{parse } (\delta D \ c \ D) \ w)$

That is the main result of this paper. The remainder of the paper concerns the implementation of νD , δD , $\delta D\text{-correct}$.

3.4 Nullability

If we know the nullability of a language, P , then the nullability of a description functor applied to P is the same as the empty string parsers for our finite languages, but with the nullability of the variables given by the nullability of P . For the μ case we use the nullability of the fixed point, which we will implement shortly. The cases for the basic combinators are the same as in Figure 2. We only use this explicitly in the sequencing case because the other cases are simple enough to implement directly.

$\nu_o : \text{Dec } (\nu \ P) \rightarrow \forall D \rightarrow \text{Dec } (\nu \ (\llbracket D \rrbracket_o \ P))$
 $\nu_o _ \emptyset = \text{no } \lambda \ ()$
 $\nu_o _ \epsilon = \text{yes refl}$
 $\nu_o _ (' \ c) = \text{no } \lambda \ ()$
 $\nu_o \ z \ (D \cup D_1) = \nu_o \ z \ D \uplus\text{-dec } \nu_o \ z \ D_1$
 $\nu_o \ z \ (D * D_1) = \text{Dec.map } \nu * \ (\nu_o \ z \ D \times\text{-dec } \nu_o \ z \ D_1)$
 $\nu_o \ z \ (x \cdot D) = x \times\text{-dec } \nu_o \ z \ D$
 $\nu_o \ z \ \text{var} = z$
 $\nu_o _ (\mu \ D) = \nu D \ D$

Naively, we might try $\nu D \ D = \nu_o \ (\nu D \ D) \ D$ But that will not terminate. Consider the language $\llbracket \text{var} \rrbracket$, to determine the nullability of this language we would need to know its nullability. Instead we use the following lemma.

Lemma 1. *The nullability of a fixed point is determined completely by a single application of the underlying functor to the empty language.*

$$\nu D \emptyset \Leftrightarrow \nu D : \nu ([D]_o \diamond \emptyset) \Leftrightarrow \nu [D]$$

Proof. The forward direction is easily proven by noting that nullability and the semantics of a description are functors and that the empty language is initial. It is also straightforward to write the proof directly.

$$\nu D \emptyset \rightarrow \nu D : \forall D \rightarrow \nu ([D]_o \diamond \emptyset) \rightarrow \nu ([D]_o [D_0])$$

The backwards direction is more difficult. We prove a more general lemma from which our desired result follows. The generalized lemma states that, if the application of a descriptor functor to a fixed point of another descriptor is nullable, then either the fixed point plays no role and the descriptor functor is also nullable if applied to the empty language, or the other descriptor (that we took the fixed point of) is nullable when applied to the empty language.

$$\nu D \emptyset \leftarrow \nu D : \forall D \rightarrow \nu ([D]_o [D_0]) \rightarrow \nu ([D]_o \diamond \emptyset) \uplus \nu ([D_0]_o \diamond \emptyset)$$

If we choose $D_0 = D$ then both cases of the resulting disjoint union have the same type, so we can just pick whichever of the two we get as a result using the `reduce` : $A \uplus A \rightarrow A$ function. Modulo wrapping and unwrapping of the fixed point (using the `roll` constructor), we now have the two functions which prove the lemma:

$$\nu D \emptyset \Leftrightarrow \nu D \{D\} = \text{mk} \Leftrightarrow (\text{roll} \circ \nu D \emptyset \rightarrow \nu D) (\text{reduce} \circ \nu D \emptyset \leftarrow \nu D \{D_0 = D\} D \circ \text{unroll})$$

From Lemma 1, we know that it is sufficient to only look one layer deep when determining the nullability of a fixed point. We can safely assume the argument is not nullable.

$$\nu D = \text{Dec.map } \nu D \emptyset \Leftrightarrow \nu D \circ \nu_o (\text{no } \lambda ())$$

Remark 4. Lemma 1 does not define an isomorphism on types. In particular, the backwards direction is not injective. Consider the brackets language. It has the following null element, where we first choose the third disjunct, `var * var`, and then the first disjunct `ε` for both branches.

$$\begin{aligned} \text{brackets}_0 &: \nu \text{ brackets} \\ \text{brackets}_0 &= \text{roll} (\text{inj}_2 (\text{inj}_2 ([], [], \text{refl}), \text{roll} (\text{inj}_1 \text{refl}), \text{roll} (\text{inj}_1 \text{refl}))) \end{aligned}$$

When we round-trip this through our lemma, we get a different result:

$$\begin{aligned} \text{brackets}_0' &: \nu D \emptyset \Leftrightarrow \nu D \{\text{bracketsD}\} .\text{to} (\nu D \emptyset \Leftrightarrow \nu D \{\text{bracketsD}\} .\text{from } \text{brackets}_0) \\ &\equiv \text{roll} (\text{inj}_1 \text{refl}) \\ \text{brackets}_0' &= \text{refl} \end{aligned}$$

It now directly takes the first disjunct, `ε`.

In practice, such problems should be avoided by using unambiguous languages, ensuring that there is only one valid parse result for each string. In that case, only one of the two solutions are possible and we would have to specify exactly which one we intend to allow when defining the language.

3.5 Reduction

The final piece of the puzzle is reduction. This tells us how the language descriptions change after parsing each input character.

In Section 3.2, we established that the meaning of self-references changes and thus they need to be replaced by local fixed points of the original language. We define a function σD to perform this substitution. It is a simple recursive function which replaces the **var** constructor with a given D' description.

$$\begin{aligned}
\sigma : \text{Desc} &\rightarrow \text{Desc} \rightarrow \text{Desc} \\
\sigma \emptyset & \quad D' = \emptyset \\
\sigma \epsilon & \quad D' = \epsilon \\
\sigma (' c) & \quad D' = ' c \\
\sigma (D \cup D_1) & \quad D' = \sigma D D' \cup \sigma D_1 D' \\
\sigma (D * D_1) & \quad D' = \sigma D D' * \sigma D_1 D' \\
\sigma (x \cdot D) & \quad D' = x \cdot \sigma D D' \\
\sigma \text{var} & \quad D' = D' \\
\sigma (\mu D) & \quad D' = \mu D
\end{aligned}$$

It turns out that the only the sequencing case, $*$ leaves the variables untouched, thus we only need to apply the substitution there. This substitution does mean we need to keep track of the original description, D_0 , through the recursion. Most other cases follow the structure we uncovered in Figure 2. For the self-reference case, **var**, we produce a self-reference again, which works because it now refers to the reduct. Finally, for the internal fixed point, μ , we can simply recursively call the reduction function. Thus, our reduction helper function is defined as follows:

$$\begin{aligned}
\delta_o : \text{Desc} &\rightarrow \text{Char} \rightarrow \text{Desc} \rightarrow \text{Desc} \\
\delta_o D_0 c \emptyset & = \emptyset \\
\delta_o D_0 c \epsilon & = \emptyset \\
\delta_o D_0 c (' c') & = (c \stackrel{?}{=} c') \cdot \epsilon \\
\delta_o D_0 c (D \cup D_1) & = \delta_o D_0 c D \cup \delta_o D_0 c D_1 \\
\delta_o D_0 c (D * D_1) & = \nu_o (\nu D D_0) D \cdot \delta_o D_0 c D_1 \cup \delta_o D_0 c D * \sigma D_1 (\mu D_0) \\
\delta_o D_0 c (x \cdot D) & = x \cdot \delta_o D_0 c D \\
\delta_o D_0 c \text{var} & = \text{var} \\
\delta_o D_0 c (\mu D) & = \mu (\delta D c D)
\end{aligned}$$

At the top level, we simply delegate to the helper by passing $D_0 = D$.

$$\delta D c D = \delta_o D c D$$

Lemma 2. *Substitution of a local fixed point into a description is the same as applying the corresponding functor to the semantic fixed point.*

$$\sigma \mu : \forall D \rightarrow \llbracket \sigma D (\mu D_0) \rrbracket_o P w \equiv \llbracket D \rrbracket_o \llbracket D_0 \rrbracket w$$

The proof follows directly by induction and computation.

To prove the correctness of the derivative, we consider both directions of the equivalence separately. Furthermore, it is important that we separate the top-level description, D_0 , from the current description, D , as they need to change independently throughout the induction.

$$\delta\text{D-to} : \forall D \rightarrow \llbracket \delta_o D_0 \ c \ D \rrbracket_o \llbracket \delta D \ c \ D_0 \rrbracket w \rightarrow \delta \ c \ (\llbracket D \rrbracket_o \llbracket D_0 \rrbracket) w$$

$$\delta\text{D-from} : \forall D \rightarrow \delta \ c \ (\llbracket D \rrbracket_o \llbracket D_0 \rrbracket) w \rightarrow \llbracket \delta_o D_0 \ c \ D \rrbracket_o \llbracket \delta D \ c \ D_0 \rrbracket w$$

The proofs follow from induction, the equivalences in Figure 2, and Lemma 2. Finally, we combine these two direction into our desired equivalence:

$$\delta\text{D-correct} \{D = D\} = \text{mk} \Leftrightarrow (\text{roll} \circ \delta\text{D-to} \ D \circ \text{unroll}) (\text{roll} \circ \delta\text{D-from} \ D \circ \text{unroll})$$

This completes our proof.

4 Discussion

Zooming back out, we want to discuss three aspects of our work: expressiveness, performance, and simplicity.

Expressiveness Our language descriptions are similar to μ -regular expressions, which have been shown to be equivalent to context-free grammars [16]. The major difference is that we only allow variables to refer to the nearest enclosing fixed point and not back to earlier fixed points. Each fixed point roughly corresponds to a nonterminal in the context-free grammar, so we conjecture this limitation means we only support context-free grammars without mutual recursion. We hope to expand our work to cover mutually recursive context-free grammars in future work.

Furthermore, an anonymous reviewer of a draft of this paper have pointed out that simply showing an equivalence of expressive power does not mean μ -regular expressions are as easy to use as context-free grammars in practice. In future work, we hope to explore this further by specifying and parsing more representative and practical languages.

Going beyond context-free languages, many practical programming languages cannot be adequately described as context-free languages. For example, features such as associativity, precedence, and indentation sensitivity cannot be expressed directly using context-free grammars. Recent work by Afrozeh and Izmaylova [1] shows that all these advanced features can be supported if we extend our grammars with data-dependencies. In future work, we want to explore using our framework as a foundation for such extensions.

Performance For a parser to practically useful, it must at least have linear asymptotic complexity for practical grammars. Might et al. [13] show that naively parsing using derivatives does not achieve that bound, but optimizations might make it possible. In particular, they argue that we could achieve $O(n|G|)$ time complexity (where $|G|$ is the grammar size) if the grammar size stays approximately constant after every derivative. By compacting the grammar, they conjecture it is possible to achieve this bound for all unambiguous grammars. In future work, we want to investigate if similar optimizations could be applied to our parser and if we can prove that we achieve this bound.

Simplicity One of the main contributions of Elliott’s type theoretic formalization of languages [9] is its simplicity of implementation and proof. To be able to extend his approach to context-free languages we have had to introduce language descriptors, which are a slight complication. For example, descriptors made it harder to parameterize the symbolic representation of a language with its semantics as Elliott does in his paper, allowing him to write a correct by construction derivative function. Instead, we have split the derivative function into a syntactic transformation and an external correctness proof. Nevertheless, we think our approach retains much of the simplicity of Elliott’s work.

5 Related Work

Formal languages have a long history. We refer the interested reader to Hocroft et al. [10] which is an overview of traditional formal language theory.

The main inspiration for our work is the work by Elliott on automatic and symbolic differentiation of languages [9]. As the title suggests, it shows a duality between two styles of implementing language derivatives in Agda. In this paper, we focus on the symbolic approach to differentiation, but we still benefit from Elliott’s clear and concise presentation. Our work is an extension of Elliott’s symbolic differentiation to a more expressive subset of context-free languages.

Previous work has shown that context-free (or similar) languages can be implemented in Agda. For example, Danielsson and Anders [8] and Allais [2] show how to implement a form of parser combinators in Agda. Both ensure termination by requiring that parsers consume at least some input each recursion. In our work, we lift this restriction, freeing programmers from having to ensure their parsers consume input.

Another approach to writing context-free grammars in Agda is to first convert arbitrary context-free grammars to a form more amenable to parsing. For example, Brink et al. [5] show how to formalize the left-corner transformation in Agda, which removes left-recursion from the grammar, thus allowing the use of a more naive parsing algorithm. Another example of this approach is by Bernardy and Jansson, who first transform the grammar into Chomsky Normal Form and subsequently formalize and the efficient Valiant’s algorithm. For the sake of simplicity, we avoid such pre-processing transformations in our work.

Perhaps the closest related work to ours can be found outside of Agda formalizations, namely Thiemann’s [16] work on partial derivatives for context-free languages. His work does cover mutually recursive nonterminals and furthermore relates derivative-based parsers to pushdown automata. In contrast to our type-theoretic approach, Thiemann’s approach is based on set theory which means languages are just sets of strings and the result of parsing is only the boolean which tells you whether the input string is in the language or not. In this way, the information about the tree structure that naturally results from parsing—and which is often desired in practice—remains implicit. Furthermore, our proofs are mechanized in Agda, which gives us confidence in the correctness, but also facilitates computer-aided experimentation.

Finally, there are some works which focus less on formalization and more on the practical implementation of parsing μ -regular languages. Might et al. [13] show how to parse μ -regular languages using derivatives. They use laziness and memoization to avoid nontermination. Krishnaswami and Yallop [12] propose an alternative approach to parsing μ -regular expressions. They introduce a type system which enforces their languages to be in $LL(1)$, which they parse efficiently using staging.

6 Conclusion

In conclusion, we have formalized (acyclic) context-free grammars using a type theoretic approach to provide fertile ground for further formalizations of disambiguation strategies and parsers that are both correct and performant.

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