Prices and Incomes in Linear Programming Models

John H. Duloy and Roger D. Norton

A procedure is developed for representing competitive and noncompetitive market structures in linear programming models. Arbitrarily close approximations to nonlinear forms—in both the objective function and constraint set—can be made without much loss of the computational efficiency of the simplex algorithm. The noncompetitive market structure may be used for measuring income at endogenous prices in a competitive model and may serve as a constraint on that measure of income to represent certain classes of economic policies. Product substitution effects in demand can be approximated by a linear program. The demand structure can be transformed to take account of any shift in demand which can be represented by a rotation of the demand function.

Key words: market structures, linear programming, substitution in demand.

The use of mathematical programming to simulate market behavior has been explored extensively in a number of studies since Samuelson first pointed out in 1952 that an objective function exists whose maximization guarantees fulfillment of the conditions of a competitive market. While some of the studies have been purely theoretical, his basic idea also has proven fruitful in the realm of empirical economics, particularly in the context of agricultural planning models which may contain rather detailed supply side specifications.

Nevertheless, in practice the existing empirical formulations of the idea are incomplete and awkward to use in a number of respects. This paper attempts to close some conceptual gaps and to make the idea more usable. Since the simplex algorithm is the most powerful computational programming algorithm available, linear programming (LP) is adopted as the context for the analysis. However, techniques are adapted by which some classes of nonlinear programming problems can be approximated almost arbitrarily closely at very little increase in computational difficulty.

The procedures described in this paper have been applied in policy analyses for Mexican agriculture (Bassoco and Norton, Duloy and Norton 1973b) and they have been extended in theory to the case of stochastic production levels (Hazell and Scandizzo), but a rigorous exposition of the approach has not been widely available heretofore.

Antecedents

Starting with Samuelson and Enke, a number of authors contributed over a span of twenty years to the conceptualization of static, single-product market demand structures with prices and quantities endogenous, in the context of mathematical programming models. Iterative LP procedures were proposed in Fox, Judge and Wallace, Schrader and King, King and Schrader, Tramel and Seale, and Yaron, but no progress had been made on efficient, noniterative solution procedures. Interdependence in demand had not been formulated in terms suitable for LP, nor had attention been paid to the problem of defining a variable, in either linear or nonlinear models, which represents producers' income as evaluated at the model's current endogenous prices. *Ex post* calculations can always be made, but if income levels are to be constrained in the solution they must be defined as variables at endogenous prices. This question has considerable empirical importance since many agricultural program parameters

John H. Duloy and Roger D. Norton are at the Development Research Center of the International Bank for Reconstruction and Development.

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are set according to target levels of producers' income.

Takayama and Judge (1964a, 1964b, 1971) established complete formulations for the case of quadratic programming, but as an algorithm it is not nearly as powerful as the simplex method. Consequently, it does not permit incorporation of the degree of detail regarding factor supplies and product demands which is usually necessary in an applied policy planning model.[1]

Without taking into account interdependence in demand, Martin spelled out in detail the piece-wise linear specification of product demand and factor supply functions in an LP model of competitive markets. He obtained, for the first time, equilibrium LP solutions without iterations. But regarding income at endogenous prices, he did not go beyond suggesting *ex post* evaluation and, if income is a relevant variable, an iterative sequence of solutions. Also, his stepped approximation technique is more costly computationally than the one described later in this paper. In this entire series of reports in the literature, one of the most significant empirical uses of LP (with separable, piece-wise linear demands) was that of Farhi and Vercueil for the French national agricultural planning model.

Given this background, this paper provides four additions to the earlier contributions, all in the context of LP: (*a*) a workable method of handling interdependence among products in demand, (*b*) a method for defining a producers' income variable at endogenous prices, without resort to iterations, (*c*) a simple method for adjusting the LP coefficients of the demand system to reflect changes in population and per capita income, which permits comparative statics analysis, and (*d*) an application of grid linearization techniques (Miller, Hadley) to permit arbitrarily close approximation to nonlinear demand functions without increasing the number of rows in the model. By this last technique, nonlinearities in both the constraint set and the objective function can be handled with LP algorithms. All of these elements may be viewed as steps toward making the Samuelson market determination idea usable in the context of large-scale optimization planning models. In the CHAC model for Mexico, there were thirty-three agricultural commodities, including several in the oilseeds and forage groups.[2] Consequently, it was necessary to develop some expression of interdependence in demand, and the number of crops ruled out an iterative equilibrium-seeking procedure. The sheer size of the model, combined with the need to make many solutions to explore alternative hypothetical policy packages made mandatory the use of the simplex algorithm instead of a nonlinear algorithm. For a full description of CHAC, including a brief outline of the demand structures presented here, see Duloy and Norton (1973a).

Motivation: Uses in Planning Models

This stream of the literature has had little impact on the analytic side of the national planning tradition. In the classic LP models of an economy or subeconomy, either infinitely elastic demand functions are assumed or an upper bound on the quantities to be traded is specified. Modifications sometimes are made for exports which constitute a significant share of the world market, leading to a formulation in which the optimizing unit equates marginal revenue and marginal cost on the export markets, but normally the existence of international markets is used to justify the price-taker assumption. However, for a large class of products, particularly agricultural commodities, the spread between CIF and FOB prices may be 20% or more, and for another group of products trading opportunities effectively do not exist. In these cases, domestic product demand functions are relevant in price determination.

Incorporating product demand functions into a planning model designed for the purpose of analyzing policy alternatives, rather than assuming exogenously determined product prices, has three principal advantages. First, it allows the model to correspond to a market equilibrium. The effects of various policies, e.g., subsidizing or taxing product input prices or varying the exchange rate, can then be investigated. Second, it allows the model greater flexibility. For instance, substitution between capital and labor, corresponding to different factor price ratios, can occur not only directly through the technology set or through changes in the commodity mix of trade, but also through substitution in demand due to

[1] The relative ease of handling large-scale models by means of LP is especially relevant to the construction of sector-wide planning models, where incorporation of considerable detail by location and product is almost unavoidable in order to assure useful results.

[2] The word CHAC is the Mayan name for a rain god.

changing relative prices of products which are more or less labor-or capital-intensive. Third, it permits an appraisal of the distribution between consumers and producers of benefits accruing from changes in output. For example, in the common situation of agricultural production for the domestic market in the face of demand curves with elasticities which are less than unity in absolute value, the returns from increased output are negative to producers as a whole and positive to consumers.[3] Given these considerations, this paper attempts to facilitate wider use of market equilibria specifications in planning models.[4]

The Basic Model

Throughout, the exposition is developed in terms of a static LP model. Experience with these models amply indicates that the linearity does not prevent incorporation of significantly nonlinear behavior in the model.

The specification of the objective function follows from the choice of market form to be incorporated in the model. In the competitive case, producers act as price takers and equate marginal costs to the prices of products. In the monopolistic case, the sector maximizes its net income by equating marginal costs to the marginal revenues of products. For simplicity of exposition, the introduction of international trade is deferred to a later section.

In general terms, the static demand function may be written in inverted form as

$$(1) \qquad p = \phi(q, Y),$$

where p is an $N \times 1$ vector of prices, q is an $N \times 1$ vector of quantities, and Y is an exogenous scalar representing (lagged) permanent income.

For an unconstrained model, the objective function for the competitive case may be written as

$$(2) \quad Z = \int_0^{q_n} \ldots \int_0^{q_1} \phi(q, Y) dq - c(q) \rightarrow \max_q,$$

where $c(q)$ is an $N \times 1$ vector of total cost functions and $q \geq 0$. In this case, setting the derivative of equation (2) with respect to q equal to zero yields

$$(3) \qquad p - c'(q) = 0,$$

which is the equilibrium condition of price equals marginal cost.

The LP Formulation

To present an example of a LP formulation, the case of linear demand functions is used, although the procedure in general places no restriction on the shape of the demand function except that the Hessian matrix of detached coefficients of the joint demand functions be negative semidefinite in order to insure convexity of the program. The variable Y is dropped from the demand function since the model is static. Equation (1) may be rewritten as

$$(4) \qquad p = a + Bq,$$

where a is an $N \times 1$ vector of constants, and B is an $N \times N$ negative semidefinite matrix of demand coefficients.

The objective function, equation (2), then becomes

$$(5) \quad Z = q'(a + 0.5Bq) - c(q) \rightarrow \max_q.$$

The objective function can be decomposed into components[5] which correspond to consumer surplus (CS) and producer surplus (PS):[6]

$$(6) \quad CS = 0.5q'(a - p) = 0.5q'Bq,$$
$$(7) \quad PS = q'p - c(q) = q'(a + Bq) - c(q).$$

In addition to the explicit costs, $c(q)$, there usually are resources whose availability is constrained, so the model is extended by adding the conditions:

$$(8) \qquad Aq \leq b,$$

[3] None of these advantages accrue when a model is designed with fixed production targets and marginal supply prices for products are derived from the dual solution.

[4] The paper by Evans deals with a general equilibrium optimizing model with demand functions and endogenous prices at the economy-wide level but, for lack of a LP market structure specification, he falls back on the inefficient procedure of iterations of LP solutions.

[5] This objective function is essentially identical to Samuelson's "net social payoff" function, except that he includes interregional transportation costs whereas here only a single point in space is treated. The same objective function is elaborated in the multiproduct case by Takayama and Judge (1964b).

[6] Of course, the equation (2) may be interpreted merely as an equilibrium-seeking device, thus sidestepping the controversies surrounding the Marshallian surpluses. (See, for example, Mishan). An alternative interpretation of the objective function is possible; it can be interpreted as the profit function of a discriminating monopolist. Such an interpretation, of course, is hardly tenable for a sector-planning model, partly on account of problems of separability of markets but also because of the fact that the demand functions would require some reformulation on account of income effects.

where A is an $M \times N$ constraint matrix and b is an $M \times 1$ vector of resource availability levels.

For this constrained maximization problem, the Kuhn-Tucker necessary conditions are equation (8) plus

$$(9) \qquad p - c'(q) - \lambda A \leq 0,$$
$$(10) \qquad [p - c'(q) - \lambda A]q = 0,$$

and

$$(11) \qquad \lambda[Aq - b] = 0,$$

where λ is the vector of dual variables to the LP.

Equation (9) says that profits must be nonpositive. Profits per unit are defined as prices minus marginal costs, where costs now have two components, the explicit (market) costs of inputs whose behavior is subsumed in the vector of cost functions $c(q)$ and the economic rents which accrue to the use of the fixed factors represented by the vector b. Equation (10) is the complementary slackness condition which says that for every activity of nonzero level in the optimal basis, profits are zero and that for an activity with nonzero profits at given levels of use, it cannot enter the optimal basis at any of those levels. Equation (11) is the complementary slackness condition for the dual solution; either a resource's rent is nonzero or its slack is nonzero, but not both.

Taken together with Euler's theorem, which guarantees equality in equation (9) for activities which enter positively in the optimal basis, these conditions describe the characteristics of a system of competitive markets when fixed factors are used in the productive process. As such, they constitute a generalization of equation (3) which is particularly relevant to agriculture.

To describe the monopolistic equilibrium with the LP model, the appropriate objective function would be

$$(12) \qquad M = q'(a + Bq) - c(q) \rightarrow \max_{q}.$$

The Kuhn-Tucker conditions for this version of the model are equation (8) and

$$(13) \quad a + 2Bq - c'(q) - \lambda A \leq 0,$$
$$(14) \quad [a + 2Bq - c'(q) - \lambda A]q = 0,$$

and

$$(15) \quad \lambda[Aq - b] = 0.$$

The only difference between equations (9)–(11) and equations (13)–(15) is that the vector p is replaced by the term $a + 2Bq$, which is the vector of marginal revenues. Therefore the previous interpretations of equations (9)–(11) are maintained subject to substitution of "marginal revenue" for "price." Hence, the model given by equations (12) and (8) guarantees the monopolistic equilibrium as the optimal solution.

Under the linear demand function, the competitive maximand, equation (5), and the monopolist's maximand, equation (12), both involve a quadratic form in p. With nonlinear demand functions, the maximands are polynomials of a higher order. Two linear approximation procedures have been developed; the first is the case where estimates of the coefficients of B are available (interdependence among products in demand) and the second where less information is known about the structure of demand (separability assumed). In this section the latter case is examined.

In order to set up the LP tableau, a function representing the area under the demand curve is defined as

$$(16) \qquad W = q'(a + 0.5Bq),$$

and also a total expenditure (= gross revenue) function is written as

$$(17) \qquad R = q'(a + Bq).$$

The demand function, equation (4), and the counterparts, equations (16) and (17), are shown diagramatically in figure 1.

Note that for a linear demand function, both W and R are nonlinear. The approximation procedure, however, involves direct segmentation of W and R instead of the demand function. Since W is the positive component of the maximand, any point below W (fig. 1) is inefficient and hence nonoptimal. In the piece-wise linear approximation to W, optimality guarantees that no more than two nodes (two points on the q axis) will enter the optimal basis. The representation of the piece-wise linear approximation in LP is shown for the two-good case, with additive separability in demands, in table 1, where it is assumed that the model may include a constraint value, $Y^* \geq 0$, on producers' income.[7]

Note that in table 1, no more than two adjacent activities from the set of selling activities (each corresponding to one segment in the approximation) will enter the optimal basis at

[7] This is an application of the grid-linearization technique of separable programming as proposed by Miller.

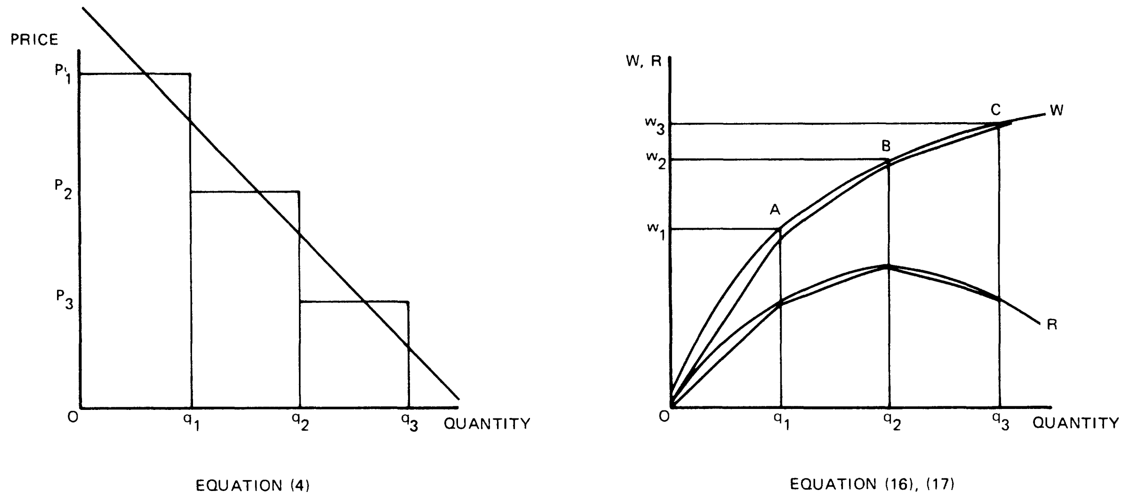

Figure 1. Demand and expenditure equations

positive levels. As argued above with reference to the W function in figure 1, a linear combination of more than two points is a line interior to the piece-wise efficiency frontier $OABA$. The principal advantage of this formulation over a more straightforward segmentation is that the demand function (or area function W) can be approximated as closely as desired without requiring additional constraints in the program. The number of selling activities increases as the number of linear segments increases, but the number of rows remains constant. And, by use of the function R in the constraint set, the model includes a measure of income at endogenous prices.

In table 1, for illustration only, the demand functions for the two commodities are segmented into two and three steps, but in the operational versions of the Mexican model CHAC, up to thirty-five segments per commodity were used.

Substitution in Demand

In the event that two or more products are not separable in demand, the nonlinear demand set can be linearized directly, to an arbitrarily close approximation, by specification of activity vectors representing points on the demand surface and by incorporating an appropriate convex combination constraint. An example of the tableau in such a case, for two products and six segments per term in the objective function, is shown in table 2.

In the treatment of this second tableau, it is assumed that the elements of the matrix B, including off-diagonal elements, in equation (4) are known or can be estimated. Frequently, the available information consists only of estimates of own-price elasticities for a number of individual commodities and commodity groups, so an alternative approach is required.

Table 1. LP Tableau with Separable Demands

	Production Activities		Selling Activities					
	Good 1	Good 2						RHS
Objective function	$-c_{1j}$	$-c_{2j}$	w_{11}	w_{12}	w_{21}	w_{22}	w_{23}	(max)
Income constraint	$-c_{1j}$	$-c_{2j}$	r_{11}	r_{12}	r_{21}	r_{22}	r_{23}	$\geq Y^*$
Commodity balance 1	y_{1j}		$-q_{11}$	$-q_{12}$				≥ 0
Commodity balance 2		y_{2j}			$-q_{21}$	$-q_{22}$	$-q_{23}$	≥ 0
Demand constraint 1			1	1				≤ 1
Demand constraint 2					1	1	1	≤ 1

Note: Costs for the ith product in the jth activity producing it are represented by c_{ij}; unit outputs of the ith product in the jth activity producing it are given by y_{ij}. The quantities sold of the ith product corresponding to the endpoint of the jth segment are defined as q_{ij}. Values of W for the ith commodity corresponding to the amount sold, q_{ij}, are given by w_{ij}. Values of R for the ith commodity corresponding to the amount sold, q_{ij}, are represented by r_{ij}. The target level of producer's income is denoted by Y^*. It is set equal to zero (or at any nonnegative value such that the constraint is nonbinding) for the competitive case. It becomes the objective function in the monopolistic case.

Table 2. LP Tableau with Substitution in Demand, Two Products

	Production Activities								
	Good 1	Good 2	Selling Activities						RHS
Objective function	$-c_{1j}$	$-c_{2j}$	w_{11}	w_{12}	w_{13}	w_{21}	w_{22}	w_{23}	(max)
Income constraint	$-c_{1j}$	$-c_{2j}$	r_{11}	r_{12}	r_{13}	r_{21}	r_{22}	r_{23}	$\geq Y^*$
Commodity balance 1	y_{1j}		$-q_{11}$	$-q_{11}$	$-q_{11}$	$-q_{12}$	$-q_{12}$	$-q_{12}$	≥ 0
Commodity balance 2		y_{2j}	$-q_{21}$	$-q_{22}$	$-q_{23}$	$-q_{21}$	$-q_{22}$	$-q_{23}$	≥ 0
Convex combination constraint			1	1	1	1	1	1	≤ 1

Note: The symbols are as defined in table 1.

The basis of the approximation procedure developed for this situation of limited information is the assumption that commodities can be classified into groups, such that the marginal rate of substitution (MRS) is zero between all groups but nonzero and constant within each group. Clearly this assumption is only an approximation to reality. A group may consist of one or more commodities, and limits are defined on the variability of the commodity mix within each group. The relevant portions of the indifference surface with respect to two commodities in a group are shown in figure 2. The rays *OC* and *OD* in the figure define the limits on the composition of the commodity bundle. If sufficient information is available, the approach can be extended to more linear segments per indifference curve, each segment representing a different value of the MRS.

Consider a group consisting of *C* commodities. The appropriate LP tableau may be represented in general form as in table 3. In table 3, each of the blocks of activities $[W'_s R'_s - Q'_s\ 1]$ constitutes a set of "mixing" activities for one segment of the composite demand function for the commodity group. This block of activities can be written as

$$(18) \qquad \begin{bmatrix} W_s \\ R_s \\ -Q_s \\ 1 \end{bmatrix} = \begin{bmatrix} w_s & w_s & \cdots & w_s & \cdots & w_S \\ r_s & r_s & \cdots & r_s & \cdots & r_S \\ -q_{s11} & -q_{s12} & \cdots & -q_{s1m} & \cdots & -q_{S1M} \\ -q_{s21} & -q_{s22} & \cdots & -q_{s2m} & \cdots & -q_{S2M} \\ -q_{sc1} & -q_{sc2} & \cdots & -q_{scm} & \cdots & -q_{ScM} \\ \vdots & \vdots & & \vdots & & \vdots \\ -q_{sC1} & -q_{sC2} & \cdots & -q_{sCm} & \cdots & -q_{SCM} \\ 1 & 1 & \cdots & 1 & \cdots & 1 \end{bmatrix},$$

where the elements are as defined below and the 1 represents the unit vector.

The derivation of formulae for the elements

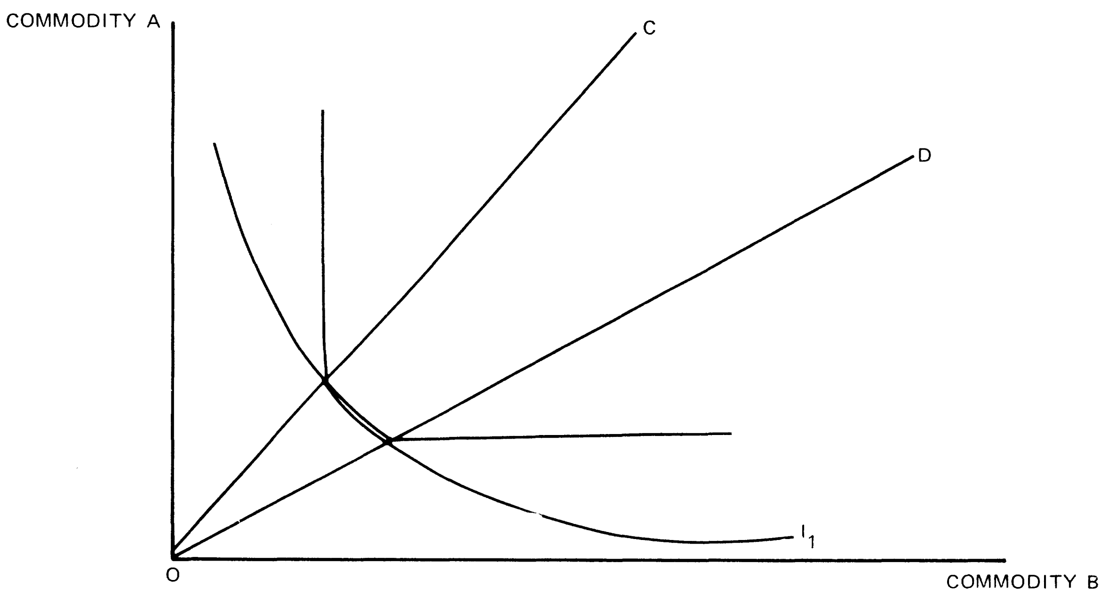

Figure 2. Indifference surface depicting limited commodity substitution

Table 3. General LP Tableau with Substitution in Demand

	Production Activities	Selling Activities						RHS
Objective function	$-C$	W_1	. . .	W_s	. . .	W_S	. . .	(max)
Income constraint	$-C$	R_1	. . .	R_s	. . .	R_S	. . .	$\geq Y^*$
Commodity balance	X	$-Q_1$	. . .	$-Q_s$	. . .	$-Q_S$	. . .	≥ 0
Convex combination constraint		1	. . .	1	. . .	1	. . .	≤ 1

Note: The segment index is $s = 1, 2, \ldots, S$. The row vector of production costs is C. A C-rowed matrix of production coefficients entering the commodity balances is given by X. Vectors $(1 \times C)$ of areas under the demand functions and gross revenue functions, respectively, are given by W_s and R_s. A $C \times C$ matrix of adjusted quantities, as defined below, is Q_s, and 1 is the unit row vector on the left-hand side of the equation.

of equation (18) is tedious because they take account of shifts both between and among segments. The starting point is a set of observed prices $\{p_1{}^o, \ldots, p_c{}^o, \ldots, p_C{}^o\}$ and a set of quantities $\{q_1{}^o, \ldots, q_c{}^o, \ldots, q_C{}^o\}$. Relative prices of commodities in the group are assumed fixed, both within and between segments, and are defined by

$$\rho_c = p_c{}^o / \sum_c p_c{}^o. \tag{19}$$

Corresponding to the observed sets of prices and quantities are a quantity index,

$$q^o = \sum_c q_c{}^o \rho_c{}^o, \tag{20}$$

and a price index,

$$p^o = \sum_c p_c{}^o q_c{}^o / V^o, \tag{21}$$

where

$$V^o = \sum_c q_c{}^o. \tag{22}$$

It is assumed that an estimate of a demand function exists for the group with a price index as a function of a quantity index, as in equations (20) and (21). Assume for a moment that no substitution occurs among commodities (i.e., that they are consumed in the fixed observed proportions), and that the demand function is segmented in S segments. Then this case corresponds to the tableau in table 4, which is a simple extension of the single product case. Only the selling activities are shown.

In table 4, it is evident that

$$q_{sc} = a_c{}^o V_s, \tag{23}$$

where $a_c = q_c{}^o / V_s$, the observed proportion in physical units of the cth commodity, and V_s is the total quantity sold in the sth segment, in physical units; W_s and R_s are, of course, computed from the demand function with appropriate price and quantity indexes although, in table 4, the weights are all constant. The price-weighted total quantity is

$$q_s{}^{o*} = \sum_c \rho_c q_{sc} = V_s \Sigma a_c{}^o \rho_c. \tag{24}$$

To extend the case of demand in fixed proportions within a group, it is supposed that for C commodities, the set of feasible alternative mixes as proportions in physical terms is given by the matrix A, assumed for simplicity to be invariant across segments:

$$A = [a_{cm}], \tag{25}$$

where $c = 1, \ldots, C$ commodities in the group; $m = 1, \ldots, M$ mixes of the commodities; and a_{cm} = the proportion in physical terms of the cth commodity in the mth mix, such that $\sum_c a_{cm} = 1$. The elements, a_{cm}, define the rays shown in figure 2.

The elements in matrix Q_s in equation set (18) can now be defined as

$$q_{scm} = a_{cm} V_s \sum_c a_c{}^o \rho_c / \sum_c a_{cm} \rho_c, \tag{26}$$

Table 4. LP Tableau with C Commodities in Alternative Fixed Proportions

	Selling Activities					
Objective function	W_1	. . .	W_s	. . .	W_S	(max)
Income constraint	R_1	. . .	R_s	. . .	R_S	$\geq Y^*$
Commodity balances	$-q_{11}$	. . .	$-q_{s1}$	. . .	$-q_{S1}$	≥ 0
	$-q_{12}$	. . .	$-q_{s2}$	. . .	$-q_{S2}$	≥ 0
	.		.		.	.
	.		.		.	.
	.		.		.	.
	$-q_{1c}$	. . .	$-q_{sc}$	. . .	$-q_{Sc}$	≥ 0
Convex combination constraint	1	. . .	1	. . .	1	≤ 1

which differs from the expression for q_{sc} (consumption in fixed proportions) in equation (23) by the factor $\sum_c a_c{}^o \rho_c / \sum_c a_{cm} \rho_c$, which reflects the changing commodity weights. Using equation (24), equation (26) can be rewritten as

$$(27) \qquad q_{scm} = a_{cm} q_s{}^{o*} / \sum_c a_{cm} \rho_c.$$

The price-weighted total quantity, $q^*{}_{sm}$, is given by

$$(28) \qquad q^*{}_{sm} = \sum_c \rho_c q_{scm} = q_s{}^{o*},$$

that is, the price-weighted quantity of the aggregate commodity is independent of the commodity mix, and it can be written as $q^*{}_s$. Using this result, equation (26) can be simplified as follows:

$$(29) \qquad q_{scm} = a_{cm} q^*{}_s / \sum_c a_{cm} \rho_c.$$

This completes the definition of the elements of the matrix Q_s in equation set (18). By equation (28), $q^*{}_s$ is invariant with respect to the commodity mix, so that the elements of w_s and r_s are invariant over the mixing activities. They are computed exactly as in the single product case, using $q^*{}_s$ in place of q_s. To recapitulate, if the demand function is linear,

$$(30) \qquad w_s = q^*{}_s (a - \frac{1}{2} b q^*{}_s),$$

and

$$(31) \qquad r_s = q^*{}_s (a - b q^*{}_s).$$

The demand side of a model may be constructed to incorporate a number of product groups, some of which can consist of a single commodity. Between product groups, the MRS is zero, and within, it is constant and given by the inverse of the price ratio. It is this last property which leads to the constancy of consumer surplus $(w_s - r_s)$ and of consumer expenditure (r_s) within a commodity group.

The constancy of the MRS can readily be shown for the case of two products (table 5), where again only the selling activities are included.

By the constancy of w_s and r_s, movement along a given indifference function requires changes in the activity levels, x_1 and x_2, which are equal but of opposite sign. Without loss of generality, consider the two cases ($x_1 = 1$, $x_2 = 0$) and ($x_1 = 0$, $x_2 = 1$). Then the MRS is given by equations (32), dropping the subscript s and the parameter $q^*{}_s$, which are common to all terms:

Table 5. LP Tableau for Limited Substitution with Two Commodities in a Segment

Activity level	x_1	x_2	
Objective function	w_s	w_s	(max)
Income constraint	r_s	r_s	≥ 0
Commodity balances	$-q_{s11}$	$-q_{s12}$	≥ 0
	$-q_{s21}$	$-q_{s12}$	≥ 0
Convex combination constraint	1	1	≤ 1

$$(32) \qquad MRS = \frac{\Delta_1}{\Delta_2} = \frac{q_{11} - q_{12}}{q_{21} - q_{22}} = \frac{a_{11}/\sum_c a_{c1}\rho_c - a_{12}/\sum a_{c2}\rho_c}{a_{21}/\sum_c a_{c1}\rho_c - a_{22}/\sum a_{c2}\rho_c}.$$

By expanding and rearranging equation (32),

$$(33) \qquad \frac{\Delta_1}{\Delta_2} = \frac{-\rho_2}{\rho_1},$$

which is the required result.

Comparative Statics and International Trade

This specification of commodity demand structures incorporates one characteristic which makes it particularly convenient for obtaining comparative statics solutions. The demand function for any commodity group can be rotated merely by an appropriate change in the constraint value of the convex combination inequality, i.e., the matrices W_s, R_s, Q_s are invariant under this class of transformations of the commodity demand function.

The transformation of the demand function for a single product is illustrated in figure 3, assuming that the function in linear. The original demand function and corresponding W function are shown as D_1D_1 and OW_1, respectively, in figure 3, and the rotated demand function and corresponding W function are shown as D_1D_2 and OW_2, respectively. If the original demand function is

$$(34) \qquad p = f(q),$$

it is required that the transformed function can be expressed as

$$(35) \qquad p = f(kq).$$

Such a formulation readily accommodates shifts in the demand function due to changes in population and/or per capita incomes. The rotation upwards of the demand function is

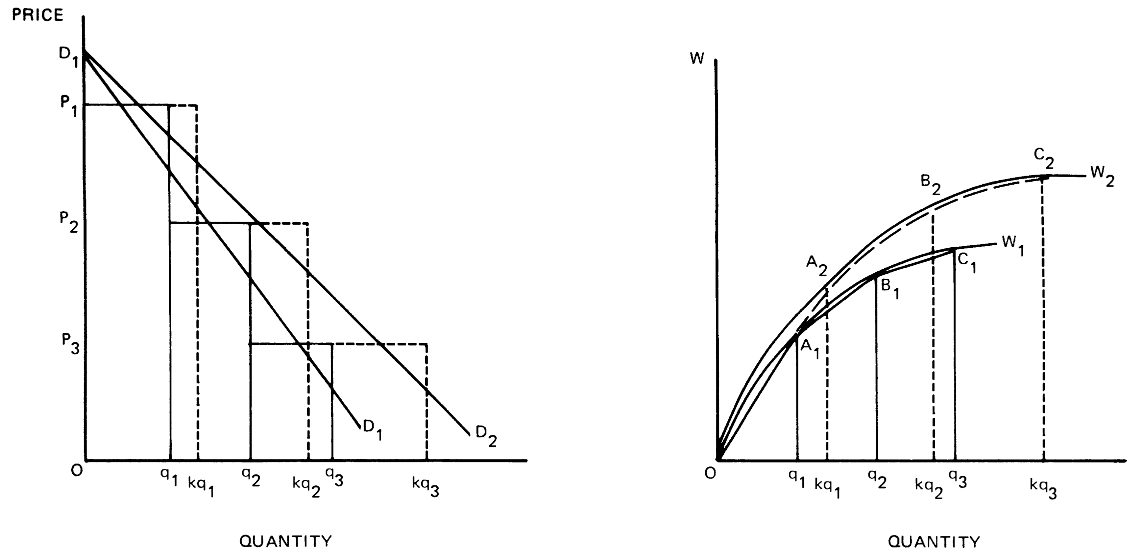

Figure 3. Transformation of a demand function

expressed as a proportional lengthening of the segments with price held constant. For the segmented W function, the slope of the linearized function in each segment, being the approximation to price within that segment, is equal for both W_1 and W_2 for corresponding segments. A similar condition holds for the linearized R function where the slopes are approximations to marginal revenue within the segment. Given linearity and the constancy of the slopes of the segmented functions within each segment, the coefficients in the W_s and R_s matrices can be expressed as simple multiples of the corresponding quantities. This is done for the transformed demand function in table 6, where again only the selling activities appear.

Simply by dividing all the elements of each activity by k and multiplying through the convex combination constraint by k, the program with the transformed demand function in table 6 can be seen to reduce to a program with coefficients in the constraint matrix identical to those before the demand transformation, but with k replacing unity on the right-hand side of the convex combination constraint. This result is readily extended to the commodity group case by replacing q_s with q^*_s and, in the objective function and income constraint, by replacing w'_s and r'_s by the corresponding vectors W'_s and R'_s, and by recalling that the matrices Q_s in the commodity balances can be written as scalar multiples of q^*_s.[8] This characteristic of the demand structure permits computationally simple parametric variation of the position of the demand function. It also opens the possibility in a larger system of endogenously determining both the position of the demand functions and the position on it.

A representation of international trade can readily be incorporated into the structures developed in this article in the usual way in which it is incorporated into planning models, that is, by adding commodity-specific importing activities as additional "production" activities and similarly by adding exporting activities as additional selling activities. Again, it is possible to specify import supply (export demand) as being infinitely elastic, but bounded, or as being represented by an up-

[8] Notice that q^*_s, being invariant over mixing activities, is a scalar.

Table 6. LP Tableau with a Transformed Demand Function

Objective function	$kq_1q'_1$	. . .	$kq_sw'_s$	. . .	$kq_Sw'_S$	(max)
Income constraint	$kq_1r'_1$	. . .	$kq_sr'_s$	. . .	$kq_Sr'_S$	$\geq Y^*$
Commodity balance	$-kq_1$	. . .	$-kq_s$	. . .	$-kq_S$	≥ 0
Convex combination constraint	1	. . .	1	. . .	1	≤ 1

Note: w'_s and r'_s are simply w_s and r_s divided by q_s, and k is the factor of proportionality by which the quantity demanded increases at a given price.

ward sloping supply (downward sloping demand) schedule. In this last case, it is possible to approximate the nonlinearities involved by the methods developed above. Notice, however, that it is only possible to specify a monopolistic formulation of export supply, or a monopsonistic formulation of import demand, if the objective function and the scope of the model represents multicountry welfare.

When trading opportunities are included as outlined above, the model captures the different trading positions posited by price theory and depending on relative domestic and foreign supply and demand functions and on whether the objective function is chosen to reflect competitive or monopolistic behavior. For example, in the monopolist case, final product importing activities never enter the optimal basis, and the model reproduces the expected two-price behavior when the foreign marginal revenue function lies above the domestic marginal revenue function.[9]

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References

Bassoco, L. M., and R. D. Norton. "A Quantitative Approach to Agricultural Policy and Planning." *Ann. Econ. & Soc. Meas.* 4 (1975).

Duloy, J. H., and R. D. Norton. "CHAC, A Programming Model of Mexican Agriculture." *Multi-level Planning: Case Studies in Mexico*, eds. L. Goreux and A. Manne, pp. 291–337. Amsterdam: North-Holland Publishing Co., 1973a.

———. "CHAC Results: Economic Alternatives for Mexican Agriculture." *Multi-level Planning: Case Studies in Mexico*, eds. L. Goreux and A. Manne, pp. 373–99. Amsterdam: North-Holland Publishing Co., 1973b.

Enke, Stephen. "Equilibrium among Spatially Separated Markets: Solution by Electric Analogue." *Econometrica* 19 (1951):40–47.

Evans, H. David. "On a General Equilibrium Trade Model for Australia." *Input-Output Techniques, Proceedings of the Fifth International Conference on Input-Output Techniques*, eds. A. Brody and A. P. Carter, pp. 127–44. Amsterdam: North Holland Publishing Co., 1972.

Farhi, L., and J. Vercueil. *Recherche pour une Planification Coherente: Le Model de Prevision du Ministere de L'Agriculture*. Paris: Editions du Centre National de la Recherche Scientifique, 1969.

Fox, Karl A. "A Spatial Equilibrium Model of the Livestock Feed Economy in the U.S." *Econometrica* 21 (1953):547–66.

Hadley, G. *Nonlinear and Dynamic Programming*. Reading, Mass.: Addison-Wesley Publishing Co., 1964.

Hazell, P. B. R., and P. L. Scandizzo. "Competitive Demand Structures under Risk in Agricultural Linear Programming Models." *Amer. J. Agr. Econ.* 56 (1974):235–44.

Judge, G. G., and T. D. Wallace. "Estimation of Spatial Price Equilibrium Models." *J. Farm Econ.* 40 (1958): 801–20.

King, G. A., and Lee F. Schrader. "Regional Location of Cattle Feeding: A Spatial Equilibrium Analysis." *Hilgardia* 34 (1963):331–416.

Martin, N. R., Jr. "Stepped Product Demand and Factor Supply Functions in Linear Programming Analyses." *Amer. J. Agr. Econ.* 54 (1972):116–20.

Miller, C. "The Simplex Method for Local Separable Programming." *Recent Advances in Mathematical Programming*, eds. R. L. Graves and P. Wolfe, pp. 88–100. New York: John Wiley & Sons, 1963.

Mishan, E. J. "What is Producer's Surplus?" *Amer. Econ. Rev.* 58 (1968):1269–82.

Samuelson, P. A. "Spatial Price Equilibrium and Linear Programming." *Amer. Econ. Rev.* 42 (1952):283–303.

Schrader, Lee F., and G. A. King. "Regional Location of Beef Cattle Feeding." *J. Farm Econ.* 44 (1962): 64–81.

Takayama, T., and G. G. Judge. "Equilibrium among Spatially Separated Markets: A Reformulation." *Econometrica* 32 (1964a):510–24.

———. "Spatial Equilibrium and Quadratic Programming." *J. Farm Econ.* 46 (1964b):67–93.

———. *Spatial and Temporal Price and Allocation Models*. Amsterdam: North-Holland Publishing Co., 1971.

Tramel, Thomas E., and A. D. Seale, Jr. "Reactive Programming of Supply and Demand Relations—Applications to Fresh Vegetables." *J. Farm Econ.* 41 (1959):1012–22.

Yaron, Dan. "Incorporation of Income Effects into Mathematical Programming Models." *Metroeconomica* 19 (1967):141–64.

[9] One case which the structure will not handle is the monopolist case where either of the demand functions is of the double-log form and where the elasticity of demand is less than unity in absolute value. In this case, marginal revenue is negative, but increasing, i.e., the function is nonconvex.