FYS4460: Project 3
Percolation

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Abstract

The aim of this project is to study scaling in numerical, experimental and real-world data, and gain experience with image analysis, discrete models for phase transitions, finite size scaling models, the geometry of percolation clusters, and dynamic processes on fractals.

Keywords: Percolation, Random walker, Flow on fractals, Cluster number density, Mass scaling, Finite size scaling, Singly connected bonds

Generating percolation cluster

In this exercise we will use MATLAB to generate and visualize percolation cluster. What we do is to generate a $L \times L$ matrix of uniformly distributed random numbers. Then, for a given probability p, we define those components of the matrix that have a value that is less than p to be occupied sites, and the others to be unoccupied sites. In Fig. 1 we have visualized the percolation matrix for three different values of p.

When $p > p_c$, the probability $P(p, L)$ for a given site to belong to the percolation cluster, has the form

$$P(p, L) \sim (p - p_c)^{\beta}. \tag{1}$$

Taking the logarithm of both sides, we obtain

$$\log P(p, L) \sim \beta \log{(p - p_c)}. \tag{2}$$

In Fig. 2 we have plotted Eq. (2). Using linear fitting we have fund that for $L = 100$, $\beta \approx 0.289$.

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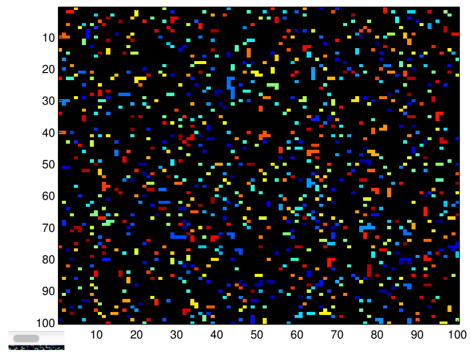
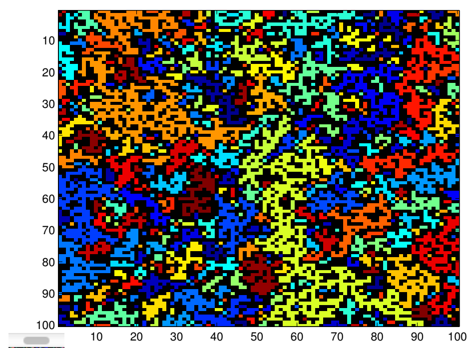
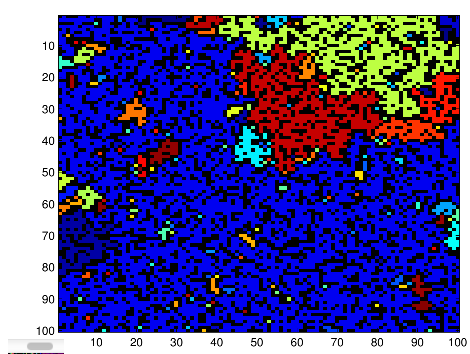


(a) $p = 0.1176$

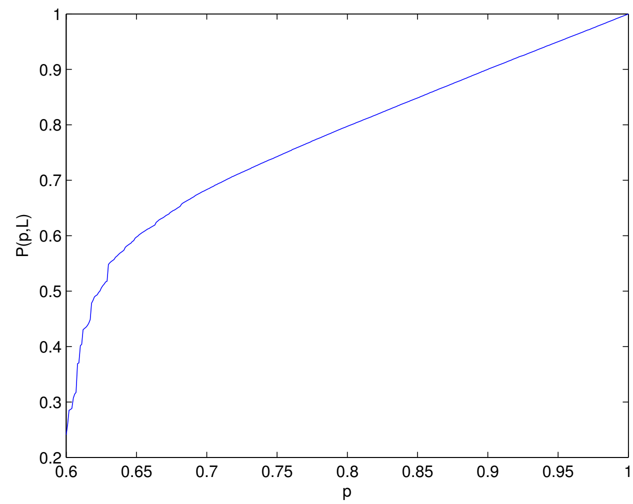


(b) $p = 0.50$

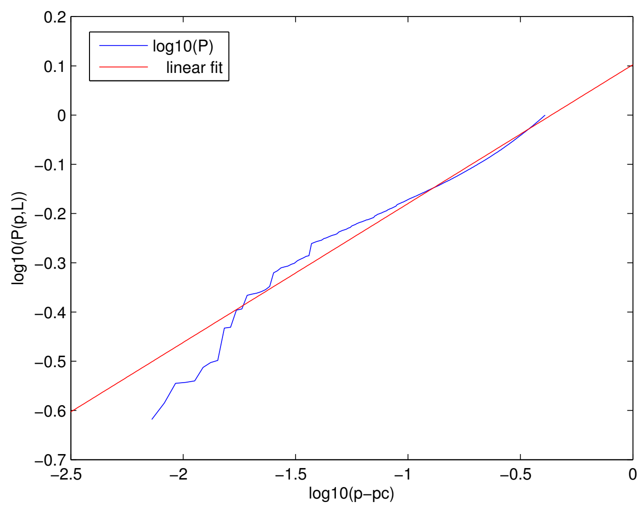


(c) $p = 0.610$

Figure 1: Visualization of the percolation matrix for three different values of p for a 100×100 lattice.



(a) $P(p, L)$.



(b) $P(p, L)$ with linear fit.

Figure 2: The probability $P(p, L)$ for a given site to belong to a percolation site for a 100×100 lattice.

Determining the exponent of power distributions

We first generate the following data-points in Matlab

$$z = rand(1e6, 1)^{-2};$$

We want to determine the power-law probability distribution function, $f_Z(z) \propto z^\alpha$, for this distribution by making use of the fact that

$$f_Z(z) = \frac{dP(Z > z)}{dz},$$

where $P(Z > z)$ is the cumulative distribution. Using the above equation we obtain

$$\begin{gathered}
f_Z(z) \propto z^\alpha \propto \frac{dP(Z > z)}{dz}, \\
\Downarrow \\
\int z^\alpha dz \propto \int dP(Z > z) \\
\Downarrow \\
\frac{1}{\alpha + 1} z^{\alpha+1} \propto P(Z > z) \\
\Downarrow \\
(\alpha + 1) \log z \propto \log P(Z > z).
\end{gathered}$$

In Fig. 3 we have plotted the logarithm of the cumulative distribution vs. the logarithm of z. Based on linear fitting we have found that $\alpha \propto -3/2$. This means that $f_Z(z) \propto z^{-3/2}$.

Cluster number density $n(s, p)$

We will in this exercise generate the cluster number density $n(s, p)$ from the two dimensional data-set. We have estimated $n(s, p)$ for a sequence of p values approaching $p_c = 0.59275$ from above and blew for a 256×256 lattice, the result is shown in Figs. 4(a) and 4(b), respectively.

The next step is to estimate $n(s, p_c; L)$ for $L = 2^k$ for $k = 4, \cdots, 9$. $n(s, p_c)$ is given by

$$n(s, p_c) \sim s^{-\tau}. \tag{3}$$

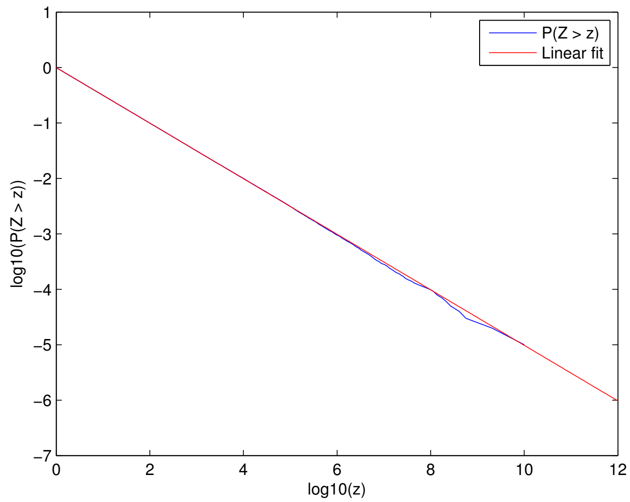


Figure 3: The logarithm of the cumulative distribution is plotted as a function of the logarithm of z.

Taking the logarithm of both sides we obtain

$$\log n(s, p_c) \sim -\tau \log s. \tag{4}$$

In Figs. 5(a) and 5(b) we have plotted $n(s, p_c; L)$, and based on a linear fitting we have found that $\tau \approx 1.97$.

In order to estimate the scaling of $s_\xi \sim |p - p_c|^{-1/\sigma}$, we plot $s^\tau n(s, p)$ as a function of $|p - p_c|^{1/\sigma}$ for an appropriate value of σ to get a data-collapse plot. The data-collapse plot, which is given in Fig. 6(a), is obtained for $\sigma \approx 0.37037$. In Fig. 6(b) we have plotted s_ξ as a function of p with $\sigma = 0.37037$. From this figure we see that when p approaches p_c, s_ξ diverges.

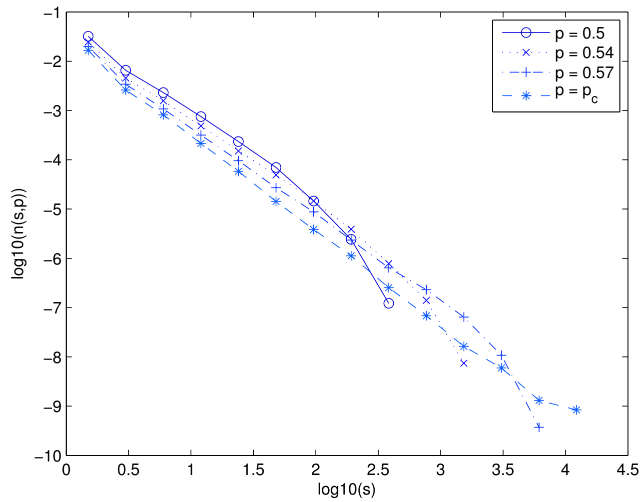
Mass scaling of percolating cluster

The mass of the percolating cluster is given by

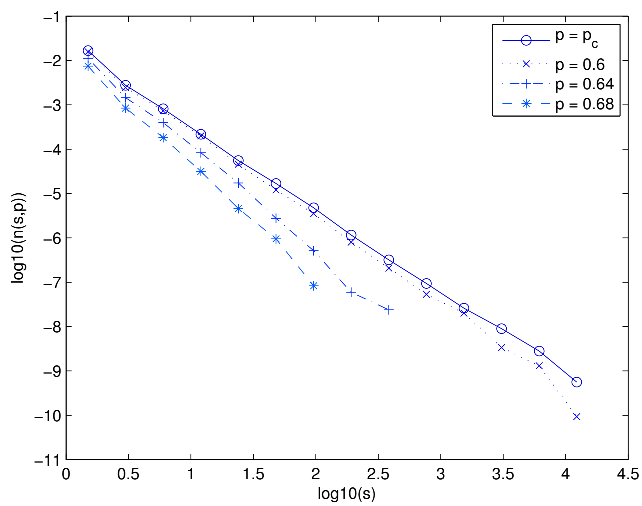
$$M(p, L) = L^d P(p, L). \tag{5}$$

In this exercise we want to find $M(p, L)$ at $p = p_c$ as a function of L for $L = 2^k$ for $k = 4, \cdots, 11$. We know (from lecture notes) that

$$M(p, L) \propto L^D, \tag{6}$$

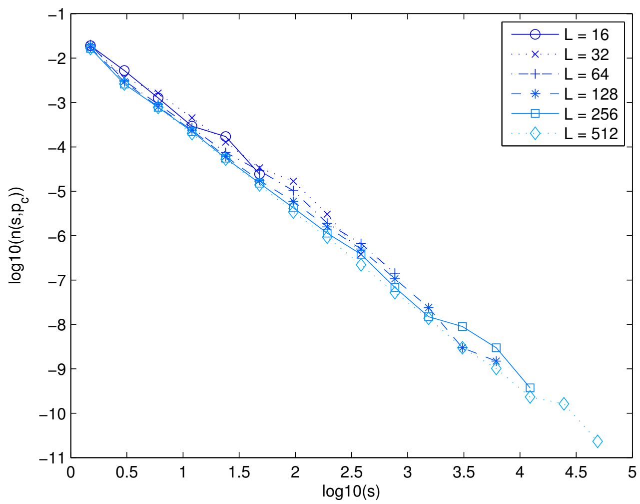


(a) $n(s,p)$ for different p-values approaching $p_c = 0.59275$ from blew.

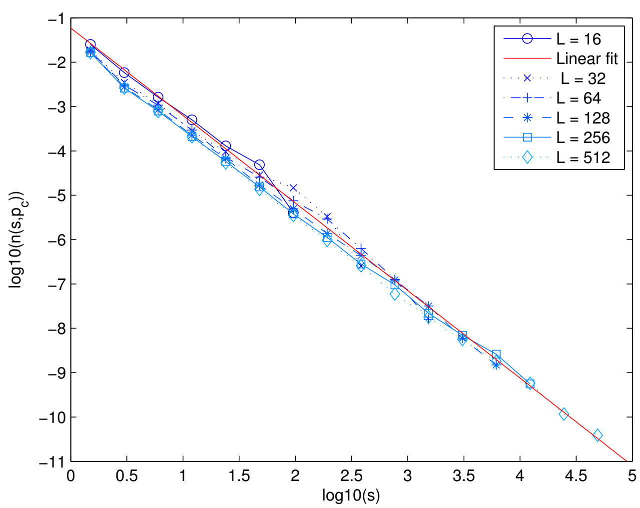


(b) $n(s,p)$ for different p-values approaching $p_c = 0.59275$ from above.

Figure 4: $n(s,p)$ as a function of s for different values of p for a 256×256 lattice.

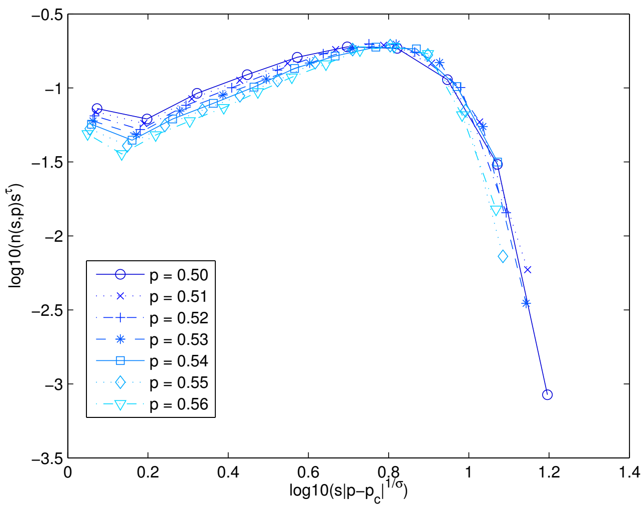


(a)

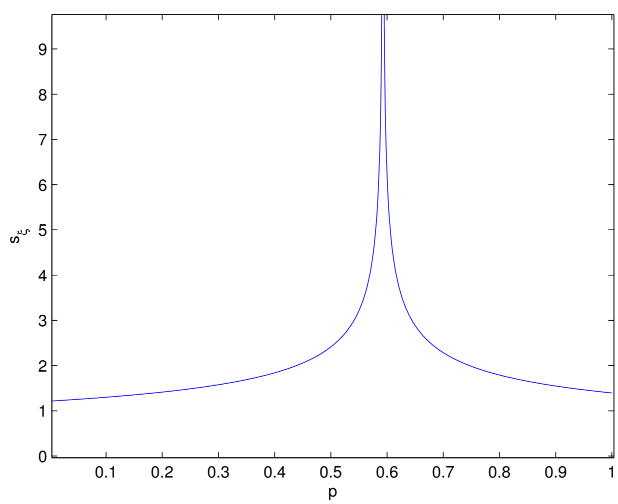


(b)

Figure 5: $n(s, p_c)$ as a function of s for different values of L.



(a) $s^{\tau}n(s,p)$ as a function of $|p-p_c|^{1/\sigma}$.



(b) s_{ξ} as a function of p.

Figure 6: The data-collapse plot for $s^{\tau}n(s,p)$ as a function of $|p-p_c|^{1/\sigma}$ in Fig. 6(a). Fig. 6(b) shows the plot of s_{ξ} as a function of p with $\sigma = 0.37037$.

which can be rewritten as

$$\log M(p, L) \propto D \log L. \tag{7}$$

Eq. (7) is plotted in Fig. 7. Based on linear fitting we have fund that $D \approx 1.8289$

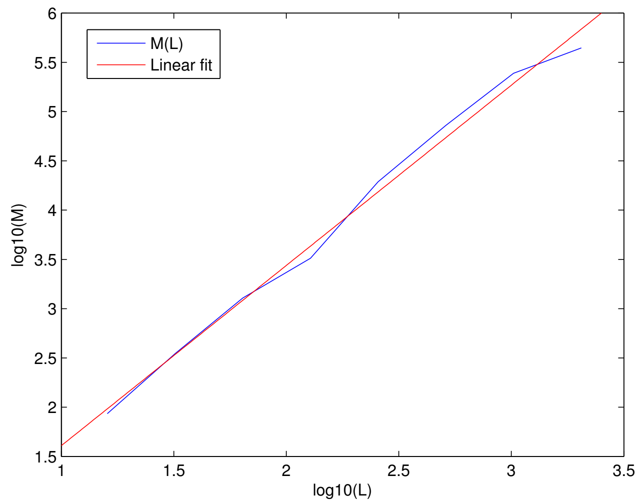


Figure 7: The mass $M(p_c, L)$ as a function of L.

Finite size scaling

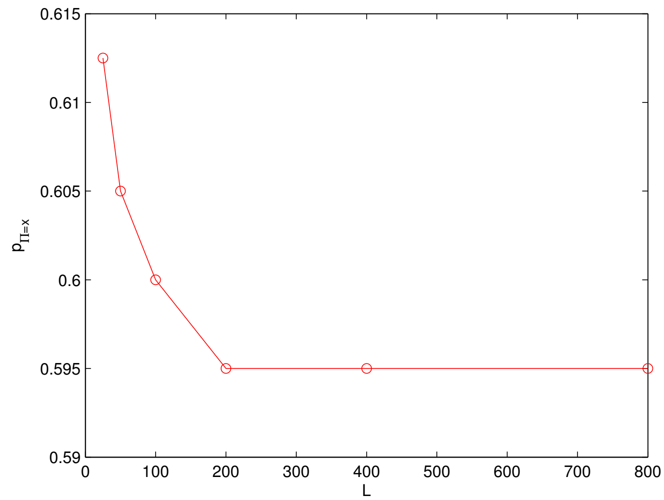
In this exercise we will use a finite size scaling ansatz to provide estimates of ν and p_c, and the average percolation probability $\langle p \rangle$ in a system of size L. We define $p_{\Pi=x}$ so that

$$\Pi(p_{\Pi=x}) = x.$$

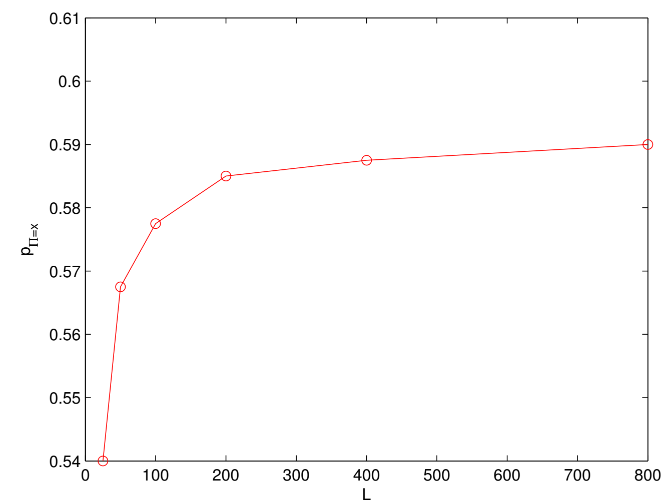
We first find $p_{\Pi=x}$ for $x = 0.8$ and $x = 0.3$ for $L = 25, 50, 100, 200, 400, 800$. The plot of $p_{\Pi=x}$ as a function of L for these two cases is given is Figs. 8(a) and 8(b), respectively.

According to the scaling theory we have

$$p_{x_1} - p_{x_2} = (C_{x_1} - C_{x_2}) L^{-1/\nu}.$$



(a)



(b)

Figure 8: $p_{\Pi=x}$ as a function of L for $x = 0.8$ in Fig.13(a) and for $x = 0.3$ in Fig.13(b).

Taking the logarithm of both sides of the above equation, we get

$$\log{(p_{x_1} - p_{x_2})} = \log{(C_{x_1} - C_{x_2})} - \frac{1}{\nu}\log{(L)}.$$

Plotting this equation we expect to get a straight line with a slope given by $-\frac{1}{\nu}$. In Fig. 9 we have plotted $\log{(p_{x_1} - p_{x_2})}$ as function of $\log{(L)}$, and based on linear fitting we have fund that $\nu = 1.2762$, which is very close to the exact value $\nu = 4/3$.

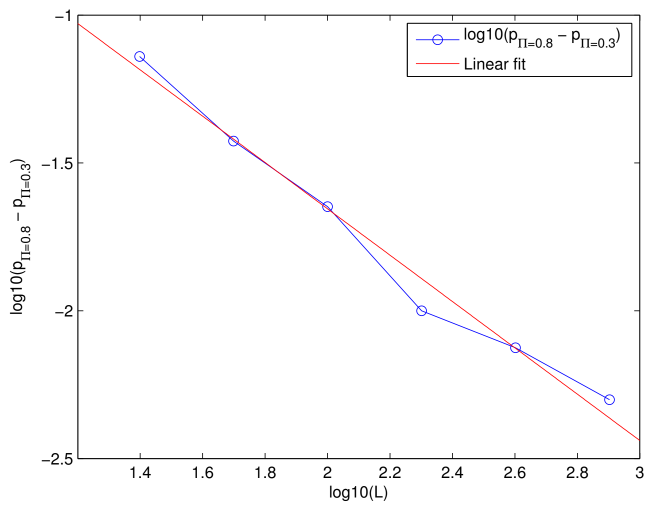


Figure 9: $\log{(p_{x_1} - p_{x_2})}$ as function of $\log{(L)}$.

The scaling theory also predicts that

$$p_{\Pi=x} = p_c + C_x L^{-1/\nu}.$$

In Fig. 10 we have plotted $p_{\Pi=x}$ as a function of $L^{-1/\nu}$ with $\nu = 4/3$ for $x = 0.8$ and $x = 0.3$, to estimate p_c. Using the best fit we have found that $p_c \in [0.59042, 0.59306]$, which is reasonable compared to the given value $p_c = 0.59275$. In Fig. 11 we have a data-collapse plot of $\Pi(p, L)$.

Singly connected bonds

We first run the given program *exwalk.m* to visualize the singly connected bonds. The visualization of the singly connected bonds is given in Fig. 12.

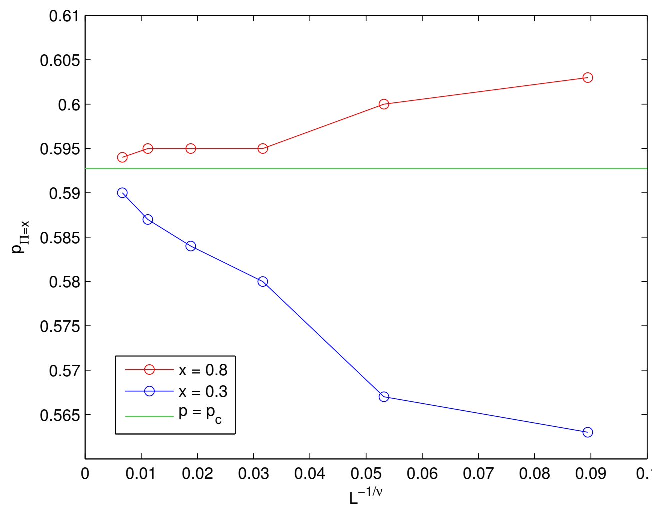


Figure 10: $p_{\Pi=x}$ as a function of $L^{-1/\nu}$ with $\nu = 4/3$ for $x = 0.8$ and $x = 0.3$.

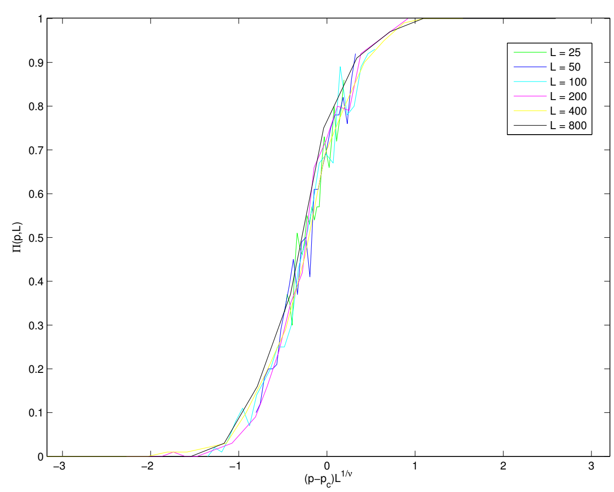


Figure 11: The data collapse plot of $\Pi(p, L)$.

We find the mass, M_{SC}, of the singly connected bonds as a function of system size L for $p = p_c$ and use this to estimate the exponent $D_{SC} : M_{SC} \propto L^{D_{SC}}$. In Fig. 13(a) we have plotted the mass as a function of L, and based on linear fitting we have found that $D_{SC} = 0.74928$. In Fig. 13(b) we have plotted the behavior of $P_{SC} = M_{SC}/L^d$ as a function of $p - p_c$. From this figure we see that P_{SC} drops off as p approaches 1.

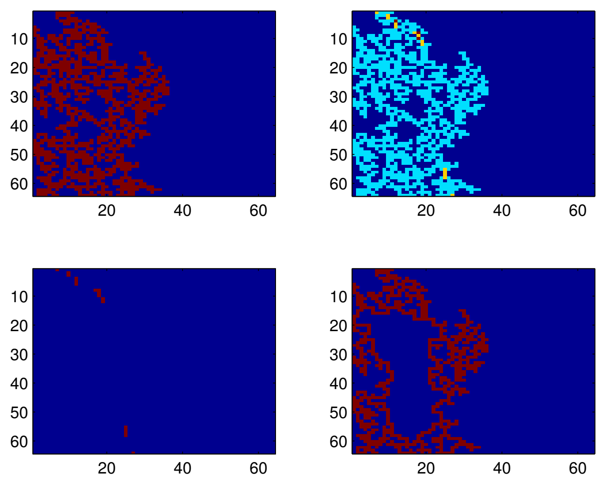
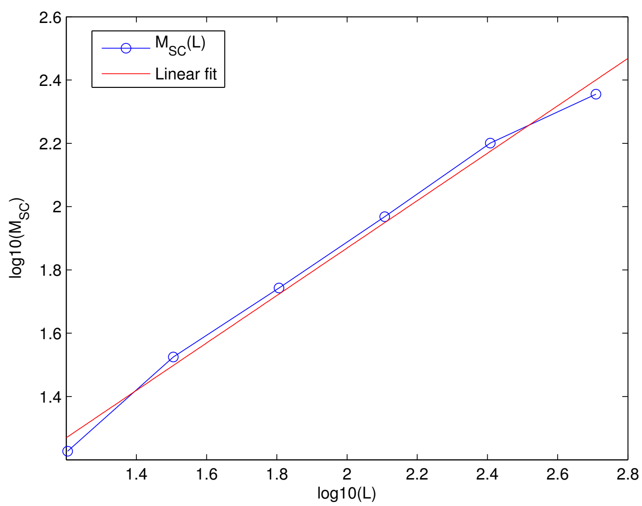


Figure 12: The visualization of the singly connected bonds.

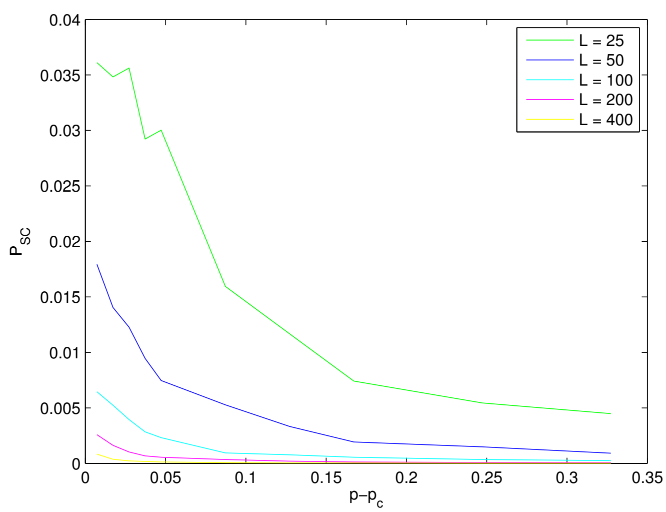
Flow on fractals

In this exercise we will use and modify the program *exflow.m* to study flow on a spanning percolation cluster. We first run the program to visualize the currents on the spanning cluster. The visualization of the currents on the spanning cluster is given in Fig. 14.

We find the masses, M_{SC}, for the singly connected bonds, M_{BB} for the backbone, M_{DE} for dangling ends and the total mass, M_{total}, as a function of system size L for $p = p_c$ and use this to estimate the exponents $D_{SC} : M_{SC} \propto L^{D_{SC}}$, $D_{BB} : M_{BB} \propto L^{D_{BB}}$, $D_{DE} : M_{DE} \propto L^{D_{DE}}$ and $D_{total} : M_{total} \propto L^{D_{total}}$. In Fig. 15 we have plotted the masses as a function of L, and based on linear fitting we have found that $D_{SC} = 0.819$, $D_{BB} = 1.6332$, $D_{DE} = 2.1138$ and $D_{total} = 1.8361$.



(a)



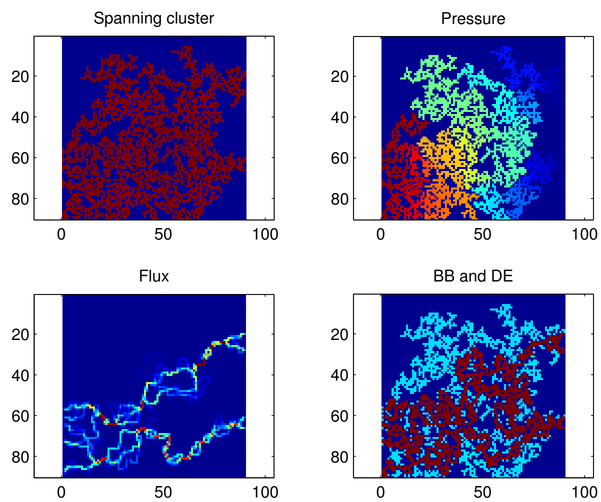
(b)

Figure 13: M_{SC} as a function of L is displayed in Fig. 13(a). Fig. 13(b) shows the behavior of $P_{SC} = M_{SC}/L^d$ as a function of $p - p_c$ for different values of L.

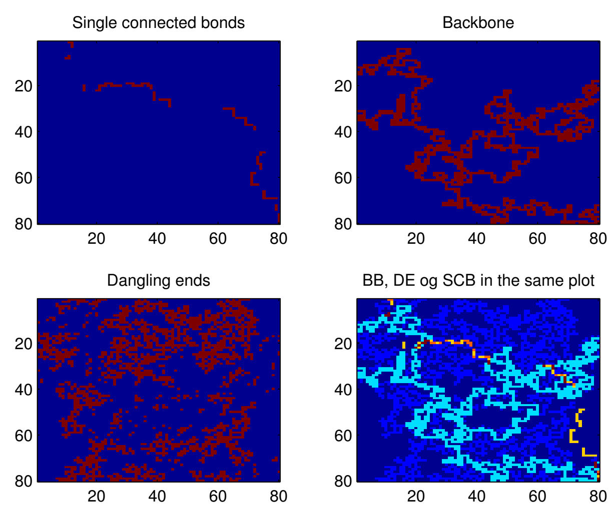
We know that

$$G \propto L^{-\tilde{\zeta}_R}.$$

In Fig. 17 we have plotted conductivity as a function of L at $p = p_c$. Based on a linear fitting we have found that $\tilde{\zeta}_R = 0.9292$. The conductivity as a function of $p - p_c$ for a 20×20 lattice is displayed in Fig. 16.

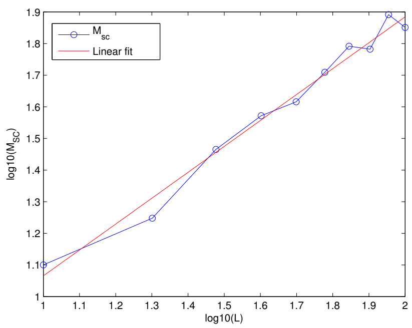


(a)

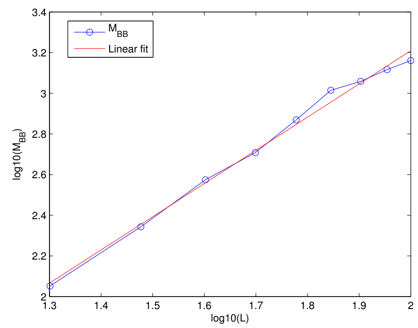


(b)

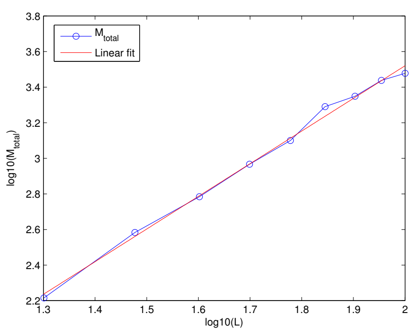
Figure 14: The visualization of the currents on the spanning cluster.



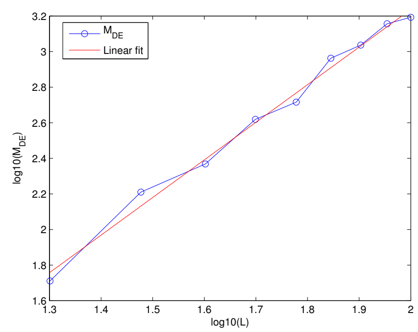
(a) The mass M_{SC} for singly connected bonds.



(b) The mass M_{BB} for the backbone.



(c) The mass M_{total}.



(d) The mass M_{DE} for dangling ends.

Figure 15: M_{SC}, M_{BB}, M_{DE} and M_{total} as functions of L.

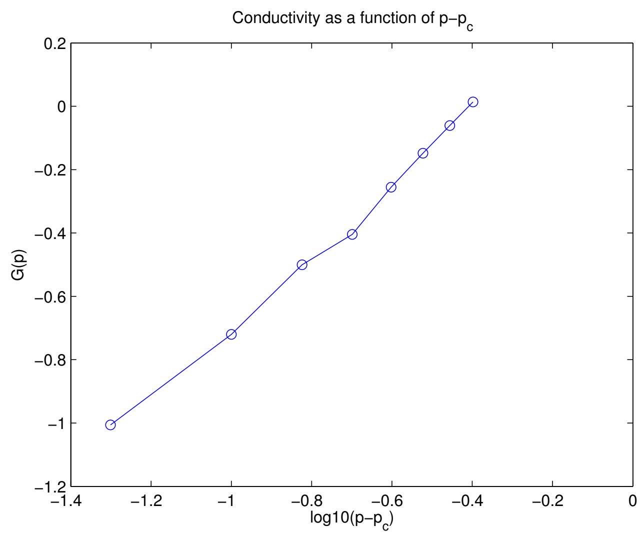


Figure 16: The conductivity as a function of $p - p_c$ for a 20×20 lattice.

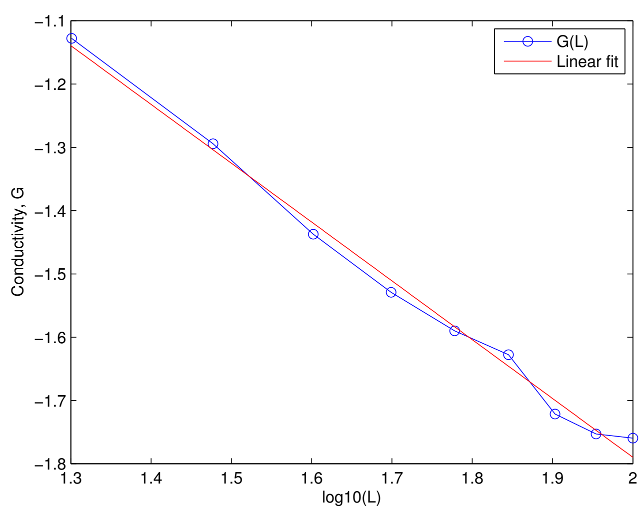


Figure 17: The conductivity as a function of L at $p = p_c$.