MCTR911 - Milestone 2: 4-DoF Robotic Manipulator Kinematics

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Abstract

This report details the kinematic modeling of the selected 4-Degrees of Freedom (4-DoF) RRRR articulated robotic manipulator. The analysis includes the Denavit-Hartenberg (D-H) convention, the derivation of the Forward Kinematics (FK) equations, and the closed-form solution for the Inverse Position Kinematics (IK).

1 Denavit-Hartenberg (D-H) Convention

The robot is modeled as a 4-DoF RRRR articulated manipulator. The kinematic frames are assigned following the standard D-H procedure.

1.1 D-H Parameter Table

The following D-H parameters define the link transformations. **Note: a_2, a_3, d_1, and d_4 are placeholder values that MUST be updated with measurements from the CAD files.**

Link i	**Twist α_{i-1}**	**Length a_{i-1}**	**Offset d_i**	**Angle θ_i**
1 (Base)	$0°$	$a_0 = 0$	$\mathbf{d_1}$	θ_1^*
2 (Shoulder)	$-90°$	$a_1 = 0$	$d_2 = 0$	θ_2^*
3 (Elbow)	$0°$	$\mathbf{a_2}$	$d_3 = 0$	θ_3^*
4 (Wrist)	$90°$	$\mathbf{a_3}$	$\mathbf{d_4}$	θ_4^*

1.2 General Homogeneous Transformation Matrix (HTM)

The transformation from frame $\{i-1\}$ to $\{i\}$ is given by:

$$\mathbf{T}_{i-1}^{i} = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i)\cos\alpha_{i-1} & \sin(\theta_i)\sin\alpha_{i-1} & a_{i-1}\cos(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i)\cos\alpha_{i-1} & -\cos(\theta_i)\sin\alpha_{i-1} & a_{i-1}\sin(\theta_i) \\ 0 & \sin\alpha_{i-1} & \cos\alpha_{i-1} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2 Forward Kinematics (FK)

The FK solution is the product of the individual HTMs from the base $\{0\}$ to the end-effector $\{4\}$:

$$\mathbf{T}_0^4 = \mathbf{T}_0^1\mathbf{T}_1^2\mathbf{T}_2^3\mathbf{T}_3^4$$

The position of the end-effector $P = [X, Y, Z]^T$ is extracted from the last column of $\mathbf{T}_0^4$. Let $C_{23} = \cos(\theta_2 + \theta_3)$ and $S_{23} = \sin(\theta_2 + \theta_3)$.

$$\begin{aligned} X &= C_1[a_2C_2 + a_3C_{23} + d_4S_{23}] \\ Y &= S_1[a_2C_2 + a_3C_{23} + d_4S_{23}] \\ Z &= d_1 + a_2S_2 + a_3S_{23} - d_4C_{23} \end{aligned}$$

3 Inverse Position Kinematics (IK)

The IK solution solves for the required joint angles $(\theta_1, \theta_2, \theta_3)$ to achieve a target wrist position (X, Y, Z). The wrist roll θ_4 is typically solved using orientation kinematics.

3.1 Solving for θ_1

θ_1 is found by projection onto the base plane:

$$\theta_1 = \text{atan2}(Y, X)$$

3.2 Solving for θ_3

We define the projected coordinates of the wrist point $\mathbf{P}$ relative to the base joint: $P_{x\_23} = \sqrt{X^2 + Y^2}$ and $P_{z\_23} = Z - d_1$. The distance D is:

$$D = P_{x\_23}^2 + P_{z\_23}^2 - a_2^2 - a_3^2 - d_4^2$$

Let $R_k = \sqrt{a_3^2 + d_4^2}$ and $\gamma = \text{atan2}(d_4, a_3)$.

$$\theta_3 = \gamma \pm \text{acos}\left(\frac{D}{2a_2R_k}\right)$$

(The $\pm$ sign dictates the elbow-up or elbow-down configuration.)

3.3 Solving for θ_2

Using the solved θ_3, we define auxiliary constants A and B:

$$A = a_3 \cos\theta_3 + d_4 \sin\theta_3 \quad \text{and} \quad B = a_3 \sin\theta_3 - d_4 \cos\theta_3$$

θ_2 is then found via a two-argument arctangent:

$$\theta_2 = \text{atan2}\left(\frac{-BP_{x\_23} + (a_2 + A)P_{z\_23}}{(a_2 + A)^2 + B^2}, \frac{(a_2 + A)P_{x\_23} + BP_{z\_23}}{(a_2 + A)^2 + B^2}\right)$$