Bootstrapping realized multivariate volatility measures*

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Abstract

We study bootstrap methods for statistics that are a function of multivariate high frequency returns such as realized regression coefficients and realized covariances and correlations. For these measures of covariation, the Monte Carlo simulation results of Barndorff-Nielsen and Shephard (2004) show that finite sample distortions associated with their feasible asymptotic theory approach may arise if sampling is not too frequent. This motivates our use of the bootstrap as an alternative tool of inference for covariation measures.

We consider an i.i.d. bootstrap applied to the vector of returns. We show that the finite sample performance of the bootstrap is superior to the existing first-order asymptotic theory. Nevertheless, and contrary to the existing results in the bootstrap literature for regression models subject to heteroskedasticity in the error term, the Edgeworth expansion for the i.i.d. bootstrap that we develop here shows that this method is not second order accurate. We argue that this is due to the fact that the conditional mean parameters of realized regression models are heterogeneous under stochastic volatility.
Keywords: Realized regression, realized beta, realized correlation, bootstrap, Edgeworth expansions.

1 Introduction

Realized statistics based on high frequency returns have become very popular in financial economics. Realized volatility is perhaps the most well known example, providing a consistent estimator of the integrated volatility under certain conditions, including the absence of microstructure noise (see Jacod (1994), Jacod and Protter (1998), Barndorff-Nielsen and Shephard (2002) and Andersen, Bollerslev, Diebold and Labys (2003)). Its multivariate analogue is the realized covariance matrix, defined as the sum of the outer product of the vector of high frequency returns. Two economically interesting functions of the realized covariance matrix are the realized correlation and the realized regression

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coefficients. In particular, realized regression coefficients are obtained by regressing high frequency returns for one asset on high frequency returns for another asset. When one of the assets is the market portfolio, the result is a realized beta coefficient. A beta coefficient measures the asset's systematic risk as assessed by its correlation with the market portfolio. Recent examples of papers that have obtained empirical estimates of realized betas include Andersen, Bollerslev, Diebold and Wu (2005a, 2005b), Campbell, Sunderam and Viceira (2008), and Viceira (2007).

Recently, Barndorff-Nielsen and Shephard (2004) (henceforth BN-S(2004)) have proposed an asymptotic distribution theory for realized covariation measures based on multivariate high frequency returns. Their simulation results show that asymptotic theory-based confidence intervals for regression and correlation coefficients between two assets returns can be severely distorted if the sampling horizon is not small enough. To improve the finite sample performance of their feasible asymptotic theory approach, BN-S (2004) propose the Fisher-z transformation for realized correlation. This analytical transformation does not apply to realized regression coefficients, which in particular can be negative and larger than one in absolute value.

In this paper we propose bootstrap methods for statistics based on multivariate high frequency returns, including the realized covariance, the realized regression and the realized correlation coefficients. Our aim is to improve upon the first order asymptotic theory of BN-S (2004). We consider an i.i.d. bootstrap applied to the vector of realized returns. Gonçalves and Meddahi (2008a) have recently applied this method to realized volatility in the univariate context. They also proposed a wild bootstrap for realized volatility with the motivation that intraday returns are (conditionally on the volatility path) independent but heteroskedastic when log prices are driven by a stochastic volatility model. In this paper we focus only on the i.i.d. bootstrap for three reasons. First, the results in Gonçalves and Meddahi (2008a) show that the i.i.d. bootstrap dominates the wild bootstrap in Monte Carlo simulations even when volatility is time varying. Second, the i.i.d. bootstrap is easier to apply than the wild bootstrap: the wild bootstrap requires choosing an external random variable used to construct the bootstrap data whereas the i.i.d. bootstrap does not involve the choice of any tuning parameter. Third, the i.i.d. bootstrap is a natural candidate in the context of realized regressions driven by heteroskedastic errors. Indeed, the i.i.d. bootstrap applied to the vector of returns corresponds to a pairs bootstrap, as proposed by Freedman (1981). His results show that the pairs bootstrap is robust to heteroskedasticity in the error term of cross section regression models. Mammen (1993) shows that the pairs bootstrap is not only first order asymptotically valid under heteroskedasticity in the error term, but it is also second-order correct (i.e. the error incurred by the bootstrap approximation converges more rapidly to zero than the error incurred by the standard normal approximation).

We can summarize our main contributions as follows. We show the first order asymptotic validity of the i.i.d. bootstrap for estimating the distribution function of the realized covariance matrix and smooth functions of it such as the realized covariance, the realized regression and the realized correlation coefficients. Our simulation results show that the bootstrap outperforms the feasible first order asymptotic theory of BN-S (2004). For the realized regression estimator[1], we develop an Edgeworth expansion of the i.i.d. (or pairs) bootstrap distribution that allows us to study the ability of this method to provide an asymptotic refinement over the distribution theory of BN-S (2004).

Contrary to our expectations based on the existing theory for the pairs bootstrap in the statistics literature, we show that the pairs bootstrap does not provide an asymptotic refinement over the standard first order asymptotic theory in the context of realized regressions. We contrast our application of the pairs bootstrap to realized regressions with the application of the pairs bootstrap in standard cross section regressions. We show that there is a main difference between these two applications, namely the fact that the parameters describing the conditional mean high frequency returns model (i.e. the conditional mean of the high frequency returns of one asset conditional on the high frequency returns of another asset) are heterogeneous. This implies that the score of the underlying realized regression model is heterogeneous and does not have mean zero (although the mean of the sum of the scores is zero). This heterogeneity implies that the standard Eicker-White heteroskedasticity robust variance estimator is not consistent in the realized regression context, which justifies the need for the more involved variance estimator proposed by BN-S (2004). The pairs bootstrap variance coincides with the Eicker-White robust variance estimator and therefore it does not provide a consistent estimator of the variance of the scaled average of the scores. This is in contrast with the results of Freedman (1981) and Mammen (1993), where the score has mean zero by assumption. Nevertheless, the pairs bootstrap is first order asymptotically valid when applied to a bootstrap t-statistic which is studentized with a variance estimator that is consistent for the population bootstrap variance of the scaled average of the scores. Because the bootstrap scores have mean zero, the Eicker-White robust variance estimator can be used for this effect. This implies that the bootstrap statistic is not of the same form as the statistic based on the original data, which explains why we do not get second order refinements for the pairs bootstrap in our context.

An important characteristic of high frequency financial data that our theory ignores is the presence of microstructure effects: the prices are observed with contamination errors (the so-called noise) due to the presence of bid-ask bounds, rounding errors, etc, and prices are non-synchronous, i.e., the prices of two assets are often not observed at the same time, leading to the well known Epps effect. The first problem is well addressed by the literature in the univariate context, in particular, Zhang, Mykland, and Ait-Sahalia (2005a), Zhang (2006), and Barndorff-Nielsen, Hansen, Lunde and Shephard (2008a) provide consistent estimators of the integrated volatility. Likewise, Hayashi and Yoshida (2005) provide a consistent estimator of the covariation of two assets when they are non-synchronous, but their analysis rules out the presence of noise. Recently, Mykland (2006) provides

[1] We focus on the realized regression test statistic because existing results in the statistics literature (see Mammen (1993)) suggest that the pairs bootstrap may be second order correct in this case even under stochastic volatility. This is not the case for the two other statistics (covariance and correlation coefficients), where the i.i.d. bootstrap cannot be expected to provide second order refinements due to the fact that it does not replicate the conditional heteroskedasticity in the data. Thus, we do not analyze their higher order properties in this paper. A wild bootstrap could be used in this case, as in Gonçalves and Meddahi (2008a).

a consistent estimator of the variance of the Hayashi and Yoshida (2005) covariance estimator, thus allowing for feasible asymptotic inference. Less is known when the two effects are present; see however the analysis in Zhang (2006), Griffin and Oomen (2006), Voev and Lunde (2007), and more recently, Barndorff-Nielsen, Hansen, Lunde and Shephard (2008b). Another feature that our theory ignores is the possible presence of jumps and co-jumps. This is a difficult problem that the literature has only started recently to address (see Jacod and Todorov (2007), Bollerslev and Todorov (2007) and Gobbi and Mancini (2006)).

The bootstrap methods that we propose in this paper are not robust to the presence of microstructure noise (nor jumps) and apply only to synchronously observed multivariate returns. By abstracting from these complications, we can focus on the realized multivariate volatility measures proposed by BN-S (2004). These are very simple to compute and are often used as meaningful measures of covariation in applied work using moderate sampling frequencies (such as 30 or 15-minute returns, where the market microstructure noise and the Epps effect are less pronounced). Because this amounts to using a small to moderate sample size, the quality of the asymptotic approximations is less reliable, and we expect the bootstrap (in particular, the method we propose here) to be more useful in this empirically relevant case.

The remainder of this paper is organized as follows. In Section 2, we introduce the setup, review the existing first order asymptotic theory and state regularity conditions. In Section 3, we introduce the bootstrap methods and establish their first-order asymptotic validity for the three statistics of interest in this paper under the regularity conditions stated in Section 2. Section 4 contains a Monte Carlo study that compares the finite sample properties of the bootstrap with the feasible asymptotic theory of BN-S (2004). Section 5 provides a detailed study of the pairs bootstrap for realized regressions. We first revisit the first order asymptotic theory of the realized regression estimator, comparing the standard Eicker-White robust variance estimator with the more involved estimator of the variance proposed by BN-S (2004). We then contrast the theoretical properties of the pairs bootstrap, in particular its asymptotic variance, with the properties of the pairs bootstrap in a standard cross section regression. We also discuss the second order accuracy of this bootstrap method based on the Edgeworth expansions that we develop here. Section 5 contains one empirical application and Section 6 concludes. Appendix A contains the tables and figures. Appendix B contains the proofs.

A word on notation. In this paper, and as usual in the bootstrap literature, P^* (E^* and Var^*) denotes the probability measure (expected value and variance) induced by the bootstrap resampling, conditional on a realization of the original time series. In addition, letting h denote the sampling horizon, for a sequence of bootstrap statistics Z_h^*, we write $Z_h^* = o_{P^*}(1)$ in probability, or $Z_h^* \to^{P^*} 0$, as $h \to 0$, in probability, if for any $\varepsilon > 0$, $\delta > 0$, $\lim_{h \to 0} P\left[P^*\left(|Z_h^*| > \delta\right) > \varepsilon\right] = 0$. Similarly, we write $Z_h^* = O_{P^*}(1)$ as $h \to 0$, in probability if for all $\varepsilon > 0$ there exists a $M_\varepsilon < \infty$ such that $\lim_{h \to 0} P\left[P^*\left(|Z_h^*| > M_\varepsilon\right) > \varepsilon\right] = 0$. Finally, we write $Z_h^* \to^{d_*} Z$ as $h \to 0$, in probability, if conditional on the sample, Z_h^* weakly converges to Z under P^*, for all samples contained in a set with probability converging to one.

2 Setup and statistics of interest

2.1 The setup

Let $p(t)$, for $t \geq 0$, denote the log-price of a q dimensional vector of assets. We assume that $p(t)$ is defined on some filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ such that

$$p(t) = p(0) + \int_0^t \alpha(u)\, du + \int_0^t \Theta(u_-)\, dW_u, \tag{1}$$

where W is a q dimensional vector of independent Brownian motions, α is a q dimensional process whose elements are predictable and has locally bounded sample paths, and the spot covolatility process $q \times q$ dimensional matrix Θ has elements which have càdlàg sample paths.

We follow Barndorff-Nielsen, Graversen, Jacod and Shephard (2006) (henceforth BNGJS (2006)) and assume that the spot covariance matrix $\Sigma(t) = \Theta(t)\Theta'(t)$ is invertible and satisfies the following assumption

$$\Sigma(t) = \Sigma(0) + \int_0^t a(u)\, du + \int_0^t \sigma(u_-)\, dW_u + \int_0^t v(u_-)\, dZ_u, \tag{2}$$

where a, σ, and v are all adapted càdlàg processes, with a also being predictable and locally bounded, and Z is a vector Brownian motion independent of W.

The representation in (1) and (2) is rather general as it allows for leverage and drift effects. Although (2) rules out jumps in volatility, we could allow for these by relying on Assumption H1 of BNGJS (2006). Because it is rather complicated to describe, we prefer to postulate (2) and refer to BNGJS (2006) for the most general setup.

Given a sampling horizon h, we assume that we can compute $1/h$ equally spaced intraday returns

$$y_i = p(ih) - p((i-1)h) = \int_{(i-1)h}^{ih} \alpha(u)\, ds + \int_{(i-1)h}^{ih} \Theta(u_-)\, dW_u, \qquad i = 1, \ldots, 1/h,$$

where we will let y_{ki} to denote the i-th intraday return on asset k, $k = 1, \ldots, q$.

The parameters of interest in this paper are functions of the elements of the integrated covariance matrix measured over a fixed time interval $[0, 1]$ (which could represent a day, a month or a quarter, for instance) and defined as $\Gamma \equiv \int_0^1 \Sigma(u)\, du$, where we let Γ_{kl} denote the element (k, l) of Γ. When $k = l$, we write $\Gamma_k = \Gamma_{kk}$.

A consistent estimator of Γ (as $h \to 0$) is the realized covariance matrix defined as $\hat{\Gamma} = \sum_{i=1}^{1/h} y_i y_i'$. The l-th diagonal element of $\hat{\Gamma}$ is the realized volatility of asset l, whereas its (k, l)-th element is the realized covariance between the returns on assets l and k.

Recently, a distribution theory for $\hat{\Gamma}$ and smooth functions of its elements has been derived by BN-S (2004) and BNGJS (2006). In particular, under (1) and (2), we have that

$$V^{-1/2}\sqrt{h^{-1}}\left(vech\left(\hat{\Gamma}\right) - vech(\Gamma)\right) \to^d N\left(0, I_{q(q+1)/2}\right),$$

where $I_{q(q+1)/2}$ is a $q(q+1)/2$ dimensional identity matrix, $vech\left(\hat{\Gamma}\right)$ denotes the vector that stacks the lower triangular elements of the columns of the matrix $\hat{\Gamma}$ into a vector, and $V = \lim_{h\to 0} Var\left(\sqrt{h^{-1}} vech\left(\hat{\Gamma}\right)\right)$. BN-S (2004, Remark 5 (ii)) give the exact form of V.

BN-S (2004) propose the following consistent estimator of V:

$$\hat{V} = h^{-1}\sum_{i=1}^{1/h} x_i x_i' - \frac{1}{2}h^{-1}\sum_{i=1}^{1/h-1}\left(x_i x_{i+1}' + x_{i+1}x_i'\right),$$

where $x_i = vech\left(y_i y_i'\right)$ (see Corollary 2 of BN-S (2004), whose extension to the more general model assumed here can be obtained by applying Theorem 1 of BNGJS (2006)). Thus,

$$T_h \equiv \hat{V}^{-1/2}\sqrt{h^{-1}}\left(vech\left(\hat{\Gamma}\right) - vech\left(\Gamma\right)\right) \to^d N\left(0, I_{q(q+1)/2}\right). \tag{3}$$

As BN-S (2004) remark, $\hat{V}$ is a substantially different estimator than that used by Barndorff-Nielsen and Shephard (2002) in the univariate context, in which case letting $x_i = y_i^2$, it corresponds to

$$\hat{V} = h^{-1}\sum_{i=1}^{1/h} y_i^4 - h^{-1}\sum_{i=1}^{1/h-1} y_i^2 y_{i+1}^2,$$

as opposed to $\frac{2}{3}\sum_{i=1}^{1/h} y_i^4$, the estimator proposed by BN-S (2002). The main feature of notice is the presence of lags of returns in the second piece. One of our contributions is to provide a new interpretation for this estimator in the context of the realized regression estimator (see Section 5.1).

2.2 The statistics of interest

In this paper, we focus on three standard measures of dependence between two assets returns y_{ki} and y_{li}. One measure is the realized covariance between y_{li} and y_{ki} given by $\hat{\Gamma}_{lk}$, the (l,k)-th element of $\hat{\Gamma}$. The other two measures are the realized regression coefficient from regressing y_{li} on y_{ki}

$$\hat{\beta}_{lk} = \frac{\hat{\Gamma}_{kl}}{\hat{\Gamma}_k},$$

which consistently estimates $\beta_{lk} = \frac{\Gamma_{kl}}{\Gamma_k}$, and the realized correlation coefficient

$$\hat{\rho}_{lk} = \frac{\hat{\Gamma}_{kl}}{\sqrt{\hat{\Gamma}_k\hat{\Gamma}_l}},$$

which estimates $\rho_{lk} = \frac{\Gamma_{kl}}{\sqrt{\Gamma_k\Gamma_l}}$.

A distribution theory for each of these measures is readily available, given (3) and an application of the delta method. The details are in BN-S (2004). In particular, for the realized covariance measure, we have that

$$T_{\Gamma,h} \equiv \frac{\sqrt{h^{-1}}\left(\hat{\Gamma}_{lk} - \Gamma_{lk}\right)}{\sqrt{\hat{V}_\Gamma}} \to^d N\left(0,1\right),$$

where

$$\hat{V}_\Gamma = h^{-1}\sum_{i=1}^{1/h} y_{ki}^2 y_{li}^2 - h^{-1}\sum_{i=1}^{1/h-1} y_{ki}y_{li}y_{k,i+1}y_{l,i+1},$$

is a consistent estimator of V_Γ.

Similarly, for the realized regression,

$$T_{\beta,h} \equiv \frac{\sqrt{h^{-1}}(\hat{\beta}_{lk} - \beta_{lk})}{\sqrt{\hat{V}_\beta}} \to^d N(0,1),$$

where

$$\hat{V}_\beta = \hat{\Gamma}_k^{-2} h^{-1} \hat{g}_\beta,$$

with $\hat{g}_\beta = \sum_{i=1}^{1/h} x_{\beta i}^2 - \sum_{i=1}^{1/h-1} x_{\beta i} x_{\beta,i+1}$, and $x_{\beta i} = y_{li} y_{ki} - \hat{\beta}_{lk} y_{ki}^2 = y_{ki}\left(y_{li} - \hat{\beta}_{lk} y_{ki}\right)$.

For the realized correlation, the t-statistic is

$$T_{\rho,h} \equiv \frac{\sqrt{h^{-1}}(\hat{\rho}_{lk} - \rho_{lk})}{\sqrt{\hat{V}_\rho}} \to^d N(0,1),$$

where

$$\hat{V}_\rho = \left(\hat{\Gamma}_l \hat{\Gamma}_k\right)^{-1} h^{-1} \hat{g}_\rho,$$

with $\hat{g}_\rho = \sum_{i=1}^{1/h} x_{\rho i}^2 - \sum_{i=1}^{1/h-1} x_{\rho i} x_{\rho,i+1}$, $x_{\rho i} = y_{ki}(y_{li} - \hat{\beta}_{lk} y_{ki})/2 + y_{li}(y_{ki} - \hat{\beta}_{kl} y_{li})/2$, and $\hat{\beta}_{kl} = \sum_{i=1}^{1/h} y_{ki} y_{li} / \sum_{i=1}^{1/h} y_{li}^2$.

3 The bootstrap for realized covariation measures

Our bootstrap method consists of resampling the vector of returns y_i in an i.i.d. fashion from the set $\{y_i : i = 1, \ldots, 1/h\}$. Thus, if I_i is i.i.d. on $\{1, \ldots, 1/h\}$, we let $y_i^* = y_{I_i}$ for $i = 1, \ldots, 1/h$.

The bootstrap realized covariance matrix is $\hat{\Gamma}^* = \sum_{i=1}^{1/h} y_i^* y_i^{*\prime}$. Letting $x_i^* = vech\left(y_i^* y_i^{*\prime}\right)$, we can write $\sqrt{h^{-1}} vech\left(\hat{\Gamma}^*\right) = \sqrt{h^{-1}} \sum_{i=1}^{1/h} x_i^*$. It is easy to show that $E^*\left(vech\left(\hat{\Gamma}^*\right)\right) = vech\left(\hat{\Gamma}\right)$. Similarly,

$$V^* = Var^*\left(\sqrt{h^{-1}} vech\left(\hat{\Gamma}^*\right)\right) = h^{-1} \sum_{i=1}^{1/h} x_i x_i' - \left(\sum_{i=1}^{1/h} x_i\right)\left(\sum_{i=1}^{1/h} x_i\right)'.$$

We can show that

$$V^* \to^P V + \int_0^1 vech\left(\Sigma(u)\right) vech\left(\Sigma(u)\right)' du - \left(\int_0^1 vech\left(\Sigma(u)\right) du\right)\left(\int_0^1 vech\left(\Sigma(u)\right) du\right)',$$

which is not equal to V (one exception is when $\Sigma(u) = \Sigma$ for all u). Although V^* does not consistently estimate V, the i.i.d. bootstrap is still asymptotically valid when applied to the following studentized statistic

$$T_h^* \equiv \hat{V}^{*-1/2} \sqrt{h^{-1}} \left(vech\left(\hat{\Gamma}^*\right) - vech\left(\hat{\Gamma}\right)\right),$$

where

$$\hat{V}^* = h^{-1} \sum_{i=1}^{1/h} x_i^* x_i^{*\prime} - \left(\sum_{i=1}^{1/h} x_i^*\right)\left(\sum_{i=1}^{1/h} x_i^*\right)'$$

is a consistent estimator of V^*. The following theorem states formally these results.

Theorem 3.1 *Suppose (1) and (2) hold. Then, as $h \to 0$, (a) $\hat{V}^* - V^* \overset{P^*}{\to} 0$, in probability, and (b) $sup_{x \in \mathbb{R}^{q(q+1)/2}} \left|P^*\left(T_h^* \le x\right) - P\left(T_h \le x\right)\right| \to 0$ in probability.*

The statistics of interest in this paper can be written as smooth functions of the realized covariance matrix. The following theorem proves that the i.i.d. bootstrap is first order asymptotically valid when applied to smooth functions of the (appropriately centered and studentized version of) the vectorized realized covariance matrix.

Let $f(\theta) : \mathbb{R}^{q(q+1)/2} \to \mathbb{R}$ denote a real valued function with continuous derivatives, and let the $q \times 1$ vector $\nabla f(\theta)$ denote its gradient. We suppose that $\nabla f(\theta)$ is nonzero at θ_0, the true value of θ. The statistic of interest is defined as

$$T_{f,h} = \frac{\sqrt{h^{-1}}\left(f\left(vech\left(\hat{\Gamma}\right)\right) - f\left(vech\left(\Gamma\right)\right)\right)}{\sqrt{\hat{V}_f}},$$

where $\hat{V}_{f,h} = \left(\nabla' f\left(vech\left(\hat{\Gamma}\right)\right)\hat{V}\ \nabla f\left(vech\left(\hat{\Gamma}\right)\right)\right)$. The i.i.d. bootstrap version of $T_{f,h}$ is $T^*_{f,h}$, which replaces $\hat{\Gamma}$ with $\hat{\Gamma}^*$, Γ with $\hat{\Gamma}$, and $\hat{V}_f$ with $\hat{V}^*_f = \left(\nabla' f\left(vech\left(\hat{\Gamma}^*\right)\right)\hat{V}^* \nabla f\left(vech\left(\hat{\Gamma}^*\right)\right)\right)$, which is a consistent estimator of the bootstrap asymptotic variance $V^*_f \equiv \left(\nabla' f\left(vech\left(\hat{\Gamma}\right)\right) V^* \nabla f\left(vech\left(\hat{\Gamma}\right)\right)\right)$.

Theorem 3.2 *Under the same conditions of Theorem 3.1, as* $h \to 0$, $\sup_{x\in\mathbb{R}}\left|P^*\left(T^*_{f,h} \leq x\right) - P\left(T_{f,h} \leq x\right)\right| \to 0$, *in probability.*

We can apply Theorem 3.2 to prove the first order asymptotic validity of the bootstrap for each of the three measures of dependence of interest here. In particular, for the bootstrap realized covariance measure $\hat{\Gamma}^*_{lk} = \sum_{i=1}^{1/h} y^*_{li} y^*_{ki}$, the corresponding bootstrap t-statistic is

$$T^*_{\Gamma,h} \equiv \frac{\sqrt{h^{-1}}\left(\hat{\Gamma}^*_{lk} - \hat{\Gamma}_{lk}\right)}{\sqrt{\hat{V}^*_{\Gamma}}},$$

where $\hat{V}^*_{\Gamma} = h^{-1}\sum_{i=1}^{1/h} y^{*2}_{li} y^{*2}_{ki} - \left(\sum_{i=1}^{1/h} y^*_{li} y^*_{ki}\right)^2$.

Similarly, the bootstrap t-statistic associated with the bootstrap realized regression $\hat{\beta}^*_{lk} = \frac{\hat{\Gamma}^*_{lk}}{\hat{\Gamma}^*_k}$ is

$$T^*_{\beta,h} \equiv \frac{\sqrt{h^{-1}}(\hat{\beta}^*_{lk} - \hat{\beta}_{lk})}{\sqrt{\hat{V}^*_{\beta}}}, \tag{4}$$

where

$$\hat{V}^*_{\beta} = \hat{\Gamma}^{*-2}_k h^{-1} \sum_{i=1}^{1/h} y^{*2}_{ki}\left(y^*_{li} - \hat{\beta}^*_{lk} y^*_{ki}\right)^2 \equiv \left(\hat{\Gamma}^*_k\right)^{-2}\hat{B}^*_1. \tag{5}$$

Finally, the bootstrap realized correlation coefficient is $\hat{\rho}^*_{lk} = \frac{\hat{\Gamma}^*_{lk}}{\sqrt{\hat{\Gamma}^*_k}\sqrt{\hat{\Gamma}^*_l}}$ and the corresponding t-statistic is

$$T^*_{\rho,h} \equiv \frac{\sqrt{h^{-1}}\left(\hat{\rho}^*_{lk} - \hat{\rho}_{lk}\right)}{\sqrt{\hat{V}^*_{\rho}}}$$

where $\hat{V}^*_{\rho} = \left(\hat{\Gamma}^*_l \hat{\Gamma}^*_k\right)^{-1}\hat{B}^*_{\rho}$, $\hat{B}^*_{\rho} = h^{-1}\sum x^{*2}_{\rho i}$, and $x^*_{\rho i} = y^*_{ki}\left(y^*_{li} - \hat{\beta}^*_{lk} y^*_{ki}\right)/2 + y^*_{li}\left(y^*_{ki} - \hat{\beta}^*_{kl} y^*_{li}\right)/2$. Here $\hat{\beta}^*_{kl}$ denotes the bootstrap OLS regression estimator of the realized regression of y^*_k on y^*_l.

4 Monte Carlo simulation results

We compare the finite sample performance of the bootstrap with the first-order asymptotic theory for constructing confidence intervals for each of the three covariation measures. Our Monte Carlo design follows that of BN-S (2004). In particular, we assume that $dp(t) = \Theta(t)\, dW(t)$, with $\Sigma(t) = \Theta(t)\,\Theta'(t)$, where

$$\Sigma(t) = \begin{pmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{pmatrix} = \begin{pmatrix} \sigma_1^2(t) & \sigma_{12}(t) \\ \sigma_{21}(t) & \sigma_2^2(t) \end{pmatrix},$$

and $\sigma_{12}(t) = \sigma_1(t)\,\sigma_2(t)\,\rho(t)$. As in BN-S (2004), we let $\sigma_1^2(t) = \sigma_1^{2(1)}(t) + \sigma_1^{2(2)}(t)$, where for $s = 1, 2$, $d\sigma_1^{2(s)}(t) = -\lambda_s(\sigma_1^{2(s)}(t) - \xi_s)dt + \omega_s \sigma_1^{(s)}(t)\sqrt{\lambda_s} db_s(t)$, where b_i is the i-th component of a vector of standard Brownian motions, independent from W. We let $\lambda_1 = 0.0429$, $\xi_1 = 0.110$, $\omega_1 = 1.346$, $\lambda_2 = 3.74$, $\xi_2 = 0.398$, and $\omega_2 = 1.346$. Our model for $\sigma_2^2(t)$ is the GARCH(1,1) diffusion studied by Andersen and Bollerslev (1998): $d\sigma_2^2(t) = -0.035(\sigma_2^2(t) - 0.636)dt + 0.236\sigma_2^2(t)db_3(t)$. Finally, we follow BN-S (2004), and let $\rho(t) = (e^{2x(t)} - 1)/(e^{2x(t)} + 1)$, where x follows the GARCH diffusion: $dx(t) = -0.03(x(t) - 0.64)dt + 0.118x(t)db_4(t)$.

Table 1 contains the actual coverage probabilities of one-sided 95% confidence intervals for each of the three covariation measures across 10,000 replications for five different sample sizes: $1/h = 1152, 288, 48, 24$ and 12, corresponding to "1.25-minute", "5-minute", "15-minute", "half-hour", "1-hour", and "2-hour" returns for a market that is open 24 hours. For markets that are open only 8 hours, these correspond to "25-seconds", "1 minute and 40 seconds", "5-minute", "10-minute", "20-minute" and "40-minute", respectively. Bootstrap intervals are based on 999 bootstrap replications each. Both lower one-sided (where $\theta \leq a$ for some random variable a and θ the parameter of interest) and upper one-sided (where $\theta \geq b$ for some random variable b) intervals are considered. Table 2 contains results for two-sided intervals. For the bootstrap, both symmetric and equal tailed intervals are considered.

Table 1 shows that for the covariance and regression coefficients, lower one-sided intervals based on the existing asymptotic theory are quite severely distorted at the smaller sample sizes whereas the upper one-sided intervals are much less so. For instance, a lower 95% nominal level for the covariance measure between the two assets is equal to 80.76% when $h = 1/12$ (corresponding to a "2-hour" sampling frequency for a 24 hours open market or a "40-minute" frequency for an 8 hours open market) whereas it is equal to 86.01% for the regression coefficient. These numbers increase to 88.09% and 91.08% when $h = 1/48$ ("30-minute" and "10-minute" sampling frequencies, for 24 and 8 hours open markets, respectively). In contrast, the corresponding upper 95% nominal level intervals for the covariance and regression coefficients have probability rates equal to 98.4% and 92.64%, when $h = 1/12$, and 97.42% and 94.69%, when $h = 1/48$, respectively. The opposite is true for the correlation coefficient, where the BN-S (2004) lower one-sided intervals tend to be better behaved than upper one-sided intervals. The rates for $h = 1/12$ (cf. $h = 1/48$) are 92.02% (94.86%) and 83.51% (89.86%) for lower and upper 95% intervals, respectively. In this case, we also report the coverage probabilities of intervals based on the Fisher-z transform, as proposed by BN-S (2004). The Fisher transform implies coverage rates of 90.47% (93.75%) and 88.57% (92.35%) when $h = 1/12$ (cf. $h = 1/48$) for lower and upper 95% intervals respectively, thus improving upon the raw statistic only in the upper case. By comparison, Table 1 shows that the bootstrap intervals have coverage probabilities much closer to the desired 95% level than the intervals based on the asymptotic theory. This is especially true for the upper one-sided intervals, where the bootstrap essentially eliminates the finite sample distortions associated to the BN-S intervals. The bootstrap performance is quite remarkable for the correlation coefficient where it dominates both the raw and the Fisher transform based intervals of BN-S (2004).

Table 2 shows that the superior performance of the bootstrap carries over to two-sided intervals. Symmetric intervals are generally better than equal-tailed intervals (which is consistent with the theory based on Edgeworth expansions) and both improve upon the first order asymptotic theory. The gains associated with the i.i.d. bootstrap can be quite substantial, especially for the smaller sample sizes, when distortions of the BN-S intervals are larger. For instance, for the regression coefficient, the coverage rate for a symmetric bootstrap interval when $1/h = 12$ (cf. $h = 1/48$) is equal to 93.51% (94.05%), whereas it is equal to 85.20% (91.37%) for the feasible asymptotic theory of BN-S (2004) (the corresponding equal-tailed interval yields a coverage rate of 90.72% (93.04%), better than BN-S (2004) but worse than the symmetric bootstrap interval). The gains are especially important for the two-sided intervals for the correlation coefficient, when the asymptotic theory of BN-S (2004) does worst. For $1/h = 12$, the bootstrap symmetric interval has a rate of 93.82% (the equal tailed interval is in this case even better behaved, with a rate equal to 94.57%) whereas the BN-S interval based on the raw statistic has a rate of 81.47% and the interval based on the Fisher-z transform has a rate of 85.90%. These numbers increase to 93.97%, 94.43%, 90.24%, and 91.62%, for the bootstrap symmetric and equal-tailed intervals, the BN-S interval and the Fisher-z transform interval, respectively. For the correlation coefficient, the bootstrap essentially removes all finite sample bias associated with the first order asymptotic theory of BN-S (2004).

5 A detailed study of realized regressions

The realized regression estimator is one of the most popular measures of covariation between two assets. In this section we study in more detail the application of the i.i.d. bootstrap to realized regression. We first provide a new interpretation for the feasible approach of BN-S (2004). In particular, we establish a link between the standard Eicker-White heteroskedasticity robust variance estimator and the variance estimator proposed by BN-S (2004). We then exploit the special structure of the regression model to obtain the asymptotic distribution of the bootstrap realized regression estimator. We relate the bootstrap variance with the Eicker-White robust variance estimator. We end this section with a discussion of the second order accuracy of the i.i.d. bootstrap in this context.

5.1 The first order asymptotic theory revisited

Suppose $dp(t) = \Theta(t)\, dW(t)$ where Θ is independent of W.[2] Then, conditionally on Σ, we can write

$$y_{li} = \beta_{lki} y_{ki} + u_i, \tag{6}$$

where independently across $i = 1, \ldots, 1/h$, $u_i | y_{ki} \sim N(0, V_i)$, with $V_i \equiv \Gamma_{li} - \frac{\Gamma_{lki}^2}{\Gamma_{ki}}$, and $\beta_{lki} \equiv \frac{\Gamma_{lki}}{\Gamma_{ki}}$. Here $\Gamma_{lki} = \int_{(i-1)h}^{ih} \Sigma_{lk}(u)\, du$. Thus, the regression coefficient in the true DGP describing the relationship between y_{li} and y_{ki} is heterogeneous (it depends on i) and the true error term in this model is heteroskedastic.

When we regress y_{li} on y_{ki} to obtain $\hat{\beta}_{lk}$, we get that $\hat{\beta}_{lk} \xrightarrow{P} \beta_{lk} \equiv \frac{\Gamma_{lk}}{\Gamma_k}$. Thus, $\hat{\beta}_{lk}$ does not estimate β_{lki} but instead β_{lk}, which can be thought of as a weighted average of β_{lki}. We can write the underlying regression model as follows:

$$y_{li} = \beta_{lk} y_{ki} + \varepsilon_i, \tag{7}$$

where $\varepsilon_i = (\beta_{lki} - \beta_{lk}) y_{ki} + u_i$. It follows that $\varepsilon_i | y_{ki} \sim N((\beta_{lki} - \beta_{lk}) y_{ki}, V_i)$, independently across i. Moreover, noting that $E(y_{ki}) = 0$,

$$Cov(y_{ki}, \varepsilon_i) = E(y_{ki}\varepsilon_i) = (\beta_{lki} - \beta_{lk}) \Gamma_{ki} = \Gamma_{lki} - \beta_{lk}\Gamma_{ki},$$

which in general is not equal to zero (unless the volatility matrix is constant). However, $E\left(\sum_{i=1}^{1/h} y_{ki}\varepsilon_i\right) = 0$, and therefore $\hat{\beta}_{lk}$ converges in probability to β_{lk}. Because $E(y_{ki}\varepsilon_i) \neq 0$, $\hat{\beta}_{lk}$ does not consistently estimate β_{lki} but estimates β_{lk} instead. This is the parameter of interest, and therefore the endogeneity problem is not a concern here. Nevertheless, the fact that $E(y_{ki}\varepsilon_i) \neq 0$ and is heterogeneous has important consequences for the asymptotic inference on β_{lk}, as we now explain.

To find the asymptotic distribution of $\hat{\beta}_{lk}$, we can write

$$\sqrt{h^{-1}}\left(\hat{\beta}_{lk} - \beta_{lk}\right) = \frac{\sqrt{h^{-1}} \sum_{i=1}^{1/h} y_{ki}\varepsilon_i}{\sum_{i=1}^{1/h} y_{ki}^2} = (\Gamma_k)^{-1} \sqrt{h^{-1}} \sum_{i=1}^{1/h} y_{ki}\varepsilon_i + o_P(1).$$

The asymptotic variance of $\sqrt{h^{-1}}\hat{\beta}_{lk}$ is thus of the usual sandwich form $V_\beta \equiv Var\left(\sqrt{h^{-1}}\hat{\beta}_{lk}\right) = (\Gamma_k)^{-2} B$, where $B = \lim_{h\to 0} B_h$, and $B_h = Var\left(\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{ki}\varepsilon_i\right)$. Because $E(y_{ki}\varepsilon_i) \neq 0$, we have that

$$B_h = h^{-1}\sum_{i=1}^{1/h} E\left(y_{ki}^2\varepsilon_i^2\right) - h^{-1}\sum_{i=1}^{1/h} \left(E(y_{ki}\varepsilon_i)\right)^2 \equiv B_{1h} - B_{2h}.$$

We can easily show that

$$B = \lim_{h\to 0} B_h = \int_0^1 \left(\Sigma_{lk}^2(u) + \Sigma_l(u)\Sigma_k(u) - 4\beta_{lk}\Sigma_{lk}(u)\Sigma_k(u) + 2\beta_{lk}^2\Sigma_k^2(u)\right) du.$$

It follows that

$$S_{\beta,h} \equiv \frac{\sqrt{h^{-1}}(\hat{\beta}_{lk} - \beta_{lk})}{\sqrt{V_\beta}} \to^d N(0,1),$$

[2] We make the assumption of no leverage and no drift for notational simplicity and because this allows us to easily compute the moments of the intraday returns conditionally on the volatility path. The same arguments would follow under the presence of leverage and drift (for instance, by postulating a model for $\Theta(t)$ and $\alpha(t)$, as in Meddahi (2002)) but this would unnecessarily complicate the notation without any gain in the intuition.

where $V_\beta = (\Gamma_k)^{-2} B$. We can contrast this result with Proposition 1 of BN-S (2004). It is easy to check that $B = g_{(lk),i}$, where $g_{(lk),i}$ is defined as in Proposition 1 of BN-S (2004) (where we let $i = 1$ here given that we measure the integrated regression coefficient over the $[0,1]$ interval).

It is helpful to contrast the BN-S (2004) variance estimator of V_β with the Eicker-White heteroskedasticity robust variance estimator that one would typically use in a cross section regression context. Let $\hat{\varepsilon}_i$ denote the OLS residual underlying the regression model (7). Then, the Eicker-White robust variance estimator of B is given by $\hat{B}_{1h} = h^{-1}\sum_{i=1}^{1/h} y_{ki}^2 \hat{\varepsilon}_i^2$. In contrast, noting that $x_{\beta i} = y_{ki}\hat{\varepsilon}_i$, BN-S (2004)'s estimator of B corresponds to

$$h^{-1}\hat{g}_\beta = h^{-1}\sum_{i=1}^{1/h} y_{ki}^2\hat{\varepsilon}_i^2 - h^{-1}\sum_{i=1}^{1/h-1} y_{ki}\hat{\varepsilon}_i y_{k,i+1}\hat{\varepsilon}_{i+1} \equiv \hat{B}_{1h} - \hat{B}_{2h}. \tag{8}$$

We can see that $h^{-1}\hat{g}_\beta = \hat{B}_{1h} - \hat{B}_{2h}$, where $\hat{B}_{1h}$ is the usual Eicker-White robust variance estimator, and $\hat{B}_{2h} = h^{-1}\sum_{i=1}^{1/h-1} y_{ki}\hat{\varepsilon}_i y_{k,i+1}\hat{\varepsilon}_{i+1}$. This extra term is needed to correct for the fact that $E(y_{ki}\varepsilon_i) \neq 0$ and is heterogeneous, as we noted above. In particular, $\hat{B}_{1h} \to B_{1h}$ and $\hat{B}_{2h} \to B_{2h}$ in probability.

5.2 First order asymptotic properties of the pairs bootstrap

The i.i.d. bootstrap applied to the vector of returns y_i is equivalent to the so-called pairs bootstrap, a popular bootstrap method in the context of cross section regression models. Freedman (1981) proves the consistency of the pairs bootstrap for possibly heteroskedastic regression models when the dimension p of the regressor vector is fixed. Mammen (1993) treats the case where $p \to \infty$ as the sample size grows to infinity. Mammen (1993) also discusses the second order accuracy of the pairs bootstrap in this context. His results specialized to the case where p is fixed show that the pairs bootstrap is not only first order asymptotically valid under heteroskedasticity in the error term, but it is also second-order correct.

For the bivariate case, the pairs bootstrap corresponds to resampling the pairs (y_{li}, y_{ki}) in an i.i.d. fashion. Although we focus on this case here, our results follow straightforwardly when dealing with a multiple regression model where we regress the intraday returns on asset l on the returns of more than one asset. In this case, the pairs bootstrap corresponds to an i.i.d. bootstrap on the tuples that collect the dependent and all the explanatory variables.

Let $\hat{\beta}_{lk}^*$ denote the OLS bootstrap estimator from the regression of y_{li}^* on y_{ki}^*. It is easy to check that $\hat{\beta}_{lk}^*$ converges in probability (under P^*) to $\hat{\beta}_{lk} = \frac{\sum_{i=1}^{1/h} E^*(y_{li}^* y_{ki}^*)}{\sum_{i=1}^{1/h} E^*(y_{ki}^{*2})}$. The bootstrap analogue of the regression error ε_i in model (7) is thus $\varepsilon_i^* = y_{li}^* - \hat{\beta}_{lk} y_{ki}^*$, whereas the bootstrap OLS residuals are defined as $\hat{\varepsilon}_i^* = y_{li}^* - \hat{\beta}_{lk}^* y_{ki}^*$.

Our next theorem provides the first order asymptotic properties of $\hat{\beta}_{lk}^*$.

Theorem 5.1 *Suppose (1) and (2) hold. As* $h \to 0$,

a) $\sqrt{h^{-1}}\left(\hat{\beta}_{lk}^* - \hat{\beta}_{lk}\right) \to^{d^*} N\left(0, V_\beta^*\right)$, *in probability, where* $V_\beta^* = \left(\hat{\Gamma}_k\right)^{-2} B_h^*$.

b) $B_h^* = Var^*\left(\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{ki}^* \varepsilon_i^*\right) = h^{-1}\sum_{i=1}^{1/h} y_{ki}^2 \hat{\varepsilon}_i^2 \equiv \hat{B}_{1h}$.

c) $V_\beta^* \to^P (\Gamma_k)^{-2} B^* \neq V_\beta$ *(except when the volatility matrix is constant), where*

$$B^* = B + \int_0^1 \left(\Sigma_{lk}(u) - \beta_{lk}\Sigma_k(u)\right)^2 du.$$

Part (a) of Theorem 5.1 states that the bootstrap OLS estimator has a first order asymptotic normal distribution with mean zero and covariance matrix V_β^*. Its proof follows from Theorem 3.2. Parts (b) and (c) show that the pairs bootstrap variance estimator is not consistent for V_β in the general context of stochastic volatility. One exception is when volatility is constant, in which case $B^* = B$ and $V_\beta^* \to^P V_\beta$.

To understand the form of V_β^*, note that we can write

$$\sqrt{h^{-1}}\left(\hat{\beta}_{lk}^* - \hat{\beta}_{lk}\right) = \left(\sum_{i=1}^{1/h} y_{ki}^{*2}\right)^{-1} \sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{ki}^* \varepsilon_i^*.$$

Since $\sum_{i=1}^{1/h} y_{ki}^{*2} \to^{P^*} \sum_{i=1}^{1/h} y_{ki}^2 = \hat{\Gamma}_k$, in probability, it follows that

$$\sqrt{h^{-1}}\left(\hat{\beta}_{lk}^* - \hat{\beta}_{lk}\right) = \left(\hat{\Gamma}_k\right)^{-1} \sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{ki}^* \varepsilon_i^* + o_{P^*}(1),$$

in probability. We can now apply a central limit theorem to $\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{ki}^* \varepsilon_i^*$ to obtain the limiting normal distribution for $\sqrt{h^{-1}}\left(\hat{\beta}_{lk}^* - \hat{\beta}_{lk}\right)$. It follows that

$$\sqrt{h^{-1}}\left(\hat{\beta}_{lk}^* - \hat{\beta}_{lk}\right) \to^{d^*} N\left(0, V_\beta^*\right),$$

in probability, where $V_\beta^* = \left(\hat{\Gamma}_k\right)^{-2} B_h^*$, with $B_h^* = Var^*\left(\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{ki}^* \varepsilon_i^*\right)$. Part (b) of Theorem 5.1 follows easily from the properties of the i.i.d. bootstrap. In particular, we can show that $B_h^* = h^{-1}\sum_{i=1}^{1/h} y_{ki}^2 \hat{\varepsilon}_i^2$, since $\sum_{i=1}^{1/h} y_{ki}\hat{\varepsilon}_i = 0$ by construction of $\hat{\beta}_{lk}$. Thus, the i.i.d. bootstrap variance of the scaled average of the bootstrap scores $y_{ki}^* \varepsilon_i^*$ is equal to $\hat{B}_{1h}$, the Eicker-White heteroskedasticity robust variance estimator of the scaled average of the scores $y_{ki}\varepsilon_i$.

Theorem 5.1 (part c) shows that the pairs bootstrap does not in general consistently estimate the asymptotic variance of $\hat{\beta}_{lk}$. An exception is when volatility is constant. This is in contrast with the existing results in the cross section regression context, where the pairs bootstrap variance estimator of the least squares estimator is robust to heteroskedasticity in the error term. This failure of the pairs bootstrap to provide a consistent estimator of the variance of $\hat{\beta}_{lk}$ is related to the fact that, as we explained in in the previous section, we cannot in general assume that $E(y_{ki}\varepsilon_i) = 0$, unless for instance when volatility is constant. When the the scores have mean zero, i.e. $E(y_{ki}\varepsilon_i) = 0$, the Eicker-White robust variance estimator, and therefore the pairs bootstrap variance estimator, are consistent estimators of the asymptotic variance of the scaled average of the scores. Both Freedman (1981) and Mammen (1993) make this assumption. The fact that $E(y_{ki}\varepsilon_i) \neq 0$ creates a bias term in $\hat{B}_{1h}$, which is estimated with the variance estimator proposed by BN-S (2004). Because $B_h^* = \hat{B}_{1h}$, the pairs bootstrap variance estimator is not a consistent estimator of $B_h = Var\left(\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{ki}\varepsilon_i\right)$. The heterogeneity (and non zero) mean property of the scores in our context is crucial to understanding the differences between the realized regression and the usual cross section regression.

The i.i.d. bootstrap is nevertheless first order asymptotically valid when applied to the t-statistic $T^*_{\beta,h}$ (defined in (4)), as our Theorem 3.2 proves. This first order asymptotic validity occurs despite the fact that V^*_β does not consistently estimate V_β. The key aspect is that we studentize the bootstrap OLS estimator with $\hat{V}^*_\beta$ (defined in (5)), a consistent estimator of V^*_β, implying that the asymptotic variance of the bootstrap t-statistic is one.

5.3 Second order asymptotic properties of the pairs bootstrap

In this section, we study the second order accuracy of the pairs bootstrap for realized regressions. In particular, we compare the rates of convergence of the error of the bootstrap and the normal approximation when estimating the distribution function of $T_{\beta,h}$. This is accomplished via a comparison of the Edgeworth expansion of the distribution of $T_{\beta,h}$ with the bootstrap Edgeworth expansion of $T^*_{\beta,h}$, which we derive here. See Gonçalves and Meddahi (2008b) and Zhang et al. (2005b) for two recent papers that have used Edgeworth expansions for realized volatility as a means to improve upon the first order asymptotic theory.

The results in this section are derived under the assumption of zero drift and no leverage (i.e. W is assumed independent of Σ). As in Gonçalves and Meddahi (2008a), a nonzero drift changes the expressions of the cumulants derived here. The no leverage assumption is mathematically convenient as it allows us to condition on the path of volatility when computing the cumulants of our statistics. Allowing for leverage is a difficult but promising extension of the results derived here.

For $i = 1, 3$, we denote by $\kappa_i\left(T_{\beta,h}\right)$ the first and third order cumulants of $T_{\beta,h}$, respectively. Conditionally on Σ, the second order Edgeworth expansion of the distribution of $T_{\beta,h}$ is given by (see e.g. Hall, 1992, p. 47)

$$P\left(T_{\beta,h} \leq x\right) = \Phi\left(x\right) + \sqrt{h} q\left(x\right)\phi\left(x\right) + o\left(\sqrt{h}\right),$$

where for any $x \in \mathbb{R}$, $\Phi\left(x\right)$ and $\phi\left(x\right)$ denote the cumulative distribution function and the density function of a standard normal random variable. The correction term $q\left(x\right)$ is defined as

$$q\left(x\right) = -\left(\kappa_1 + \frac{1}{6}\kappa_3\left(x^2-1\right)\right),$$

where κ_1 and κ_3 are the coefficients of the leading terms of $\kappa_1\left(T_{\beta,h}\right)$ and $\kappa_3\left(T_{\beta,h}\right)$, respectively. In particular, up to order $O\left(\sqrt{h}\right)$, as $h \to 0$, $\kappa_1\left(T_{\beta,h}\right) = \sqrt{h}\kappa_1$ and $\kappa_3\left(T_{\beta,h}\right) = \sqrt{h}\kappa_3$.

Given this Edgeworth expansion, the error (conditional on Σ) incurred by the normal approximation in estimating the distribution of $T_{\beta,h}$ is given by

$$\sup_{x\in\mathbb{R}}\left|P\left(T_{\beta,h} \leq x\right) - \Phi\left(x\right)\right| = \sqrt{h}\sup_{x\in\mathbb{R}}\left|q\left(x\right)\phi\left(x\right)\right| + O\left(h\right).$$

Thus, $\sup_{x\in\mathbb{R}}\left|q\left(x\right)\phi\left(x\right)\right|$ is the contribution of order $O\left(\sqrt{h}\right)$ to the normal error.

Similarly, we can write a one-term Edgeworth expansion for the conditional distribution of $T^*_{\beta,h}$ as follows

$$P^*(T^*_{\beta,h} \leq x) = \Phi(x) + \sqrt{h} q^*_h(x)\phi(x) + O_P(h),$$

where q^*_h is defined as

$$q^*_h(x) = -(\kappa^*_{1,h} + \kappa^*_{3,h}(x^2-1)/6),$$

and where $\kappa^*_{1,h}$ and $\kappa^*_{3,h}$ are the leading terms of the first and the third order cumulants of $T^*_{\beta,h}$. In particular, $\kappa^*_1\left(T^*_{\beta,h}\right) = \sqrt{h}\kappa^*_{1,h}$ and $\kappa^*_3\left(T^*_{\beta,h}\right) = \sqrt{h}\kappa^*_{3,h}$, up to order $O_P\left(\sqrt{h}\right)$.

The bootstrap error implicit in the bootstrap approximation of $P\left(T_{\beta,h} \leq x\right)$ (conditional on Σ) is given by

$$\begin{aligned}
P^*\left(T^*_{\beta,h} \leq x\right) - P\left(T_{\beta,h} \leq x\right) &= \sqrt{h}\ \left(q^*_h\left(x\right) - q\left(x\right)\right)\phi\left(x\right) + O_P\left(h\right) \\
&= \sqrt{h}\left(\operatorname*{plim}_{h\to 0} q^*_h\left(x\right) - q\left(x\right)\right)\phi\left(x\right) + o_P\left(\sqrt{h}\right) \\
&= -\sqrt{h}\left[\left(\kappa^*_1 - \kappa_1\right) + \frac{1}{6}\left(\kappa^*_3 - \kappa_3\right)\left(x^2-1\right)\right]\phi\left(x\right) + o_P\left(\sqrt{h}\right),
\end{aligned}$$

where $\kappa^*_1 \equiv \operatorname{plim}_{h\to 0}\kappa^*_{1,h}$ and $\kappa^*_3 \equiv \operatorname{plim}_{h\to 0}\kappa^*_{3,h}$. If $\kappa^*_1 = \kappa_1$ and $\kappa^*_3 = \kappa_3$, $P^*\left(T^*_{\beta,h} \leq x\right) - P\left(T_{\beta,h} \leq x\right) = o_P\left(\sqrt{h}\right)$, and the bootstrap error is of a smaller order of magnitude than the normal error which is equal to $O\left(\sqrt{h}\right)$. If this is the case, the bootstrap is said to be second-order correct and to provide an asymptotic refinement over the standard normal approximation.

The following result gives the expressions of κ_i and κ^*_i for $i = 1, 3$. We need to introduce some notation.

Let

$$\begin{aligned}
A_0 &= \int_0^1 \left(\Sigma_k\left(u\right)\Sigma_{lk}\left(u\right) - \beta_{lk}\Sigma^2_k\left(u\right)\right)du, \\
A_1 &= \int_0^1 \left(\begin{array}{c} 2\Sigma^3_{lk}\left(u\right) + 6\Sigma_l\left(u\right)\Sigma_{lk}\left(u\right)\Sigma_k\left(u\right) - 18\beta_{lk}\Sigma^2_{lk}\left(u\right)\Sigma_k\left(u\right) \\ -6\beta_{lk}\Sigma^2_k\left(u\right)\Sigma_l\left(u\right) + 24\beta^2_{lk}\Sigma_{lk}\left(u\right)\Sigma^2_k\left(u\right) - 8\beta^3_{lk}\Sigma^3_k\left(u\right) \end{array}\right)du, \\
B &= \int_0^1 \left(\Sigma^2_{lk}\left(u\right) + \Sigma_l\left(u\right)\Sigma_k\left(u\right) - 4\beta_{lk}\Sigma_{lk}\left(u\right)\Sigma_k\left(u\right) + 2\beta^2_{lk}\Sigma^2_k\left(u\right)\right)du, \\
H_1 &= \frac{4A_0}{\Gamma_k\sqrt{B}}, \quad \text{and} \quad H_2 = \frac{A_1}{B^{3/2}}.
\end{aligned}$$

Similarly, let

$$\begin{aligned}
B^* &= B + \int_0^1 \left(\Sigma_{lk}\left(u\right) - \beta_{lk}\Sigma_k\left(u\right)\right)^2 du, \\
A^*_1 &= A_1 + 2\int_0^1 \left(\Sigma_{lk}\left(u\right) - \beta_{lk}\Sigma_k\left(u\right)\right)^3 du, \\
H^*_1 &= \frac{4A_0}{\Gamma_k\sqrt{B^*}}, \text{ and } H^*_2 = \frac{A^*_1}{B^{*3/2}}.
\end{aligned}$$

Theorem 5.2 *Suppose (1) and (2) hold with $\alpha \equiv 0$ and W independent of Σ. Then, conditionally on Σ, (a) $\kappa_1 = \frac{1}{2}\left(H_1 - H_2\right)$ and $\kappa_3 = 3H_1 - 2H_2$; and $\kappa^*_1 = \frac{3}{4}\left(H^*_1 - H^*_2\right)$ and $\kappa^*_3 = \frac{3}{2}\left(3H^*_1 - 2H^*_2\right)$.*

Theorem 5.2 shows that the cumulants of $T^*_{\beta,h}$ and $T_{\beta,h}$ do not generally agree. Notice in particular that $B \neq B^*$ contributes to this discrepancy. B here denotes the limiting variance of the scaled average of the scores whereas B^* denotes its bootstrap analogue. As we noted before, under general stochastic volatility, the pairs bootstrap does not consistently estimate B and the bias term is exactly equal to the difference between B and B^*, i.e. $B^* - B = \int_0^1 (\Sigma_{lk}(u) - \beta_{lk}\Sigma_k(u))^2 du = \text{plim}_{h\to 0} B_{2h}$, where $B_{2h} = h^{-1}\sum_{i=1}^{1/h} (E(y_{ki}\varepsilon_i))^2$. An exception is when the volatility matrix is constant, where $B_{2h} = 0$ and therefore $B^* = B$. In this case, we also have that $A_1^* = A_1 = A_0 = 0$, implying that both the bootstrap and the normal approximations have an error of the order $o\left(\sqrt{h}\right)$. We need an expansion to order $O(h)$ to be able to discriminate the two approximations. In the general stochastic volatility case, the pairs bootstrap error is of order $O\left(\sqrt{h}\right)$, similar to the error incurred by the normal approximation.

The lack of second order refinements of the pairs bootstrap in the context of realized regressions is in contrast with the results available in the bootstrap literature for standard regression models (see Mammen (1993)). One explanation for this difference lies in the fact that $E(y_{ki}\varepsilon_i) \neq 0$, as we noted above. This implies that $T_{\beta,h}$ must rely on a variance estimator that contains a bias correction term, as proposed by BN-S (2004). Instead, in the bootstrap regression, $E^*(y_{ki}^*\varepsilon_i^*) = h\sum_{i=1}^{1/h} y_{ki}\hat{\varepsilon}_i = 0$, and therefore there is no need for the bias correction proposed by BN-S (2004). This implies that the bootstrap t-statistic $T_{\beta,h}^*$ is not of the same form as $T_{\beta,h}$, relying on a bootstrap variance estimator $\hat{V}_\beta^*$ that depends on an Eicker-White type variance estimator $\hat{B}_{1h}^*$.

6 Empirical application

A well documented empirical fact in finance is the time variability of bonds risk, as recently documented by Viceira (2007) and Campbell, Sunderam and Viceira (2008) for the US market. As suggested by the CAPM, the bond risk is often measured by its beta over the return on the market portfolio. With a positive beta, bonds are considered as risky as the market while a bond with a negative beta could be used to hedge the market risk.

Viceira (2007) studies the bond risk for the US market by considering the 3-month (monthly) rolling realized beta as measured by the ratio of the realized covariance of daily log-returns on bonds and stocks and the realized volatility of the daily log-return on stocks over the same period. Following the standard practice, the number of days in a month is normalized to 22 such that the 3-month realized beta is computed considering sub-samples of 66 days. From July 1962 through December 2003, Viceira (2007) reports a strong variability of US bond CAPM betas, which may switch sign even though the average over the full sample is positive. Nevertheless, in his analysis Viceira (2007) does not discuss the precision of the realized betas as a measure of the actual covariation between bonds and stock returns.

The aim of this section is to illustrate the usefulness of our approach as a method of inference for realized covariation measures in the context of measuring the time variation of bonds risk. We consider both the US bonds market, as in Viceira (2007), and the UK bonds market.

Our data set includes the daily 7-to-10-year maturity government bond index for the US and the UK markets as released by JP Morgan from January 2, 1986 through August 24, 2007. As a proxy for the US and the UK market portfolio returns, we consider the log-return on the S&P500 and the FTSE 100 indices, respectively. The S&P500 index is designed to measure performance of the broad domestic economy through changes in the aggregate market value of 500 stocks representing all major industries. The FTSE 100 index is a capitalization-weighted index of the 100 most highly capitalized companies traded on the London Stock Exchange. The first two series have a shorter history and therefore constrained the sample we consider in this study.

¿From the estimates presented in Table 3 (Appendix A), the full-sample beta for bonds in the US is about 0.024, slightly smaller than the UK bond beta, which is about 0.030. Both the bootstrap and the asymptotic theory based confidence intervals display support that the true values of the betas in both countries are positive.

A closer analysis of Figures 1 and 2 shows that the average positivity of the betas hides considerable time variation in both countries, a fact already documented by Viceira (2007) for the US market. Furthermore, the betas for these two countries follow similar dynamics. We can distinguish two patterns for the 3-month betas. For the period before April 1997, the betas are mostly significantly positive or, in few cases, non-significantly different from 0. This period is also characterized by betas of larger magnitude, with a maximum value of 0.500 at the end of July 1994 for the US and 0.438 in August 1994 for the UK. The period after April 1997 is characterized by a drop of the magnitude of the bonds betas in both countries. They are often not significantly different from 0. For this whole sub-period, the betas for the US and UK bonds are significantly negative only between June 2002 and July 2003, but in these cases their magnitude is small. We conclude that bonds are riskier in the period before April 1997, while in the recent periods they appear to be non risky or at most a hedging instrument against shocks on market portfolio returns.

A comparison of the bootstrap intervals with the intervals based on the asymptotic theory of BN-S (2004) suggests that the two types of intervals tend to be similar, but there are instances where the bootstrap intervals are wider than the asymptotic theory-based intervals (see Tables 4 and 5 for a detailed comparison of the two types of intervals for a selected set of dates). This is especially true for the first part of the sample for the UK bond market, where the width of the bootstrap intervals can be much larger than the width of the BN-S (2004) intervals. It turns out that these days correspond to days on which there is evidence for jumps, as determined by applying the test for jumps of Barndorff-Nielsen and Shephard (2006). Because none of the intervals discussed here (bootstrap or asymptotic theory-based) are robust to the presence of jumps, a different analysis should be pursued for these particular days.

7 Conclusion

This paper proposes bootstrap methods for inference on measures of multivariate volatility such as integrated covariance, integrated correlation and integrated regression coefficients. We prove the first order asymptotic validity of a particular bootstrap scheme, the i.i.d. bootstrap applied to the vector of returns, for the three statistics of interest. Our simulation results show that the bootstrap outperforms the feasible first order asymptotic approach of BN-S(2004).

For the special case of the realized regression estimator, our i.i.d. bootstrap corresponds to a pairs bootstrap as proposed by Freedman (1981) and further studied by Mammen (1993). We analyze the second order accuracy of this bootstrap method and conclude that it is not second order accurate. This contrasts with the existing literature on the pairs bootstrap for cross section models, which shows that this method is not only robust to heteroskedasticity in the error term but it is also second order accurate. We provide a detailed analysis of the pairs bootstrap in the context of realized regressions which allows us to highlight some key differences with respect to the usual application of the pairs bootstrap in standard cross section regression models. These differences explain why the pairs bootstrap does not provide second order refinements in this context.

As we noted in the Introduction, our theory does not take into account the complications that arise from the presence of market microstructure noise, non synchronicity, and jumps and co-jumps. The extension of the bootstrap to these important problems is left for future research.

Appendix A

Table 1. Coverage rates of one-sided nominal 95% intervals for covariation measures

	Covariance				Regression				Correlation					
	Lower		Upper		Lower		Upper		Lower			Upper		
$1/h$	BN-S	Boot	BN-S	Boot	BN-S	Boot	BN-S	Boot	BN-S	Fisher	Boot	BN-S	Fisher	Boot
12	80.76	87.30	98.40	97.34	86.01	90.05	92.64	95.46	92.02	90.47	94.91	83.51	88.57	94.67
24	84.74	89.55	98.04	96.08	89.34	91.90	94.63	95.27	93.55	92.15	94.51	87.46	91.13	94.24
48	88.09	92.28	97.42	95.15	91.08	93.05	94.69	94.27	94.86	93.75	95.15	89.86	92.35	94.21
288	92.22	94.61	96.61	94.62	93.57	94.80	95.34	94.55	95.42	94.75	94.94	93.58	94.45	94.81
1152	93.69	94.98	95.50	94.39	94.30	95.07	95.35	94.78	95.27	94.92	95.03	94.38	94.82	94.95

Note: 10,000 replications, with 999 bootstrap replications each.

Table 2. Coverage rates of two-sided nominal 95% intervals for covariation measures

	Covariance			Regression			Correlation			
$1/h$	BN-S	Boot Sym	Boot Eq-T	BN-S	Boot Sym	Boot Eq-T	BN-S	Fisher	Boot Sym	Boot Eq-T
12	83.98	90.58	89.22	85.20	93.51	90.72	81.47	85.90	93.82	94.57
24	87.59	91.37	90.65	89.57	93.84	91.91	86.90	89.15	93.59	93.96
48	90.39	93.01	92.36	91.37	94.05	93.04	90.24	91.62	93.97	94.43
288	93.89	94.76	94.44	94.00	94.73	94.52	93.87	94.12	94.71	94.69
1152	94.62	94.90	94.59	94.57	94.70	94.68	94.63	94.74	94.90	94.90

Note: 10,000 replications, with 999 bootstrap replications each.

Figure 1: Symmetric bootstrap and BN-S (2004) asymptotic theory based 95% two-sided confidence intervals for the CAPM 3-month (monthly) rolling realized beta of US bond. April 1986 through July 2007.

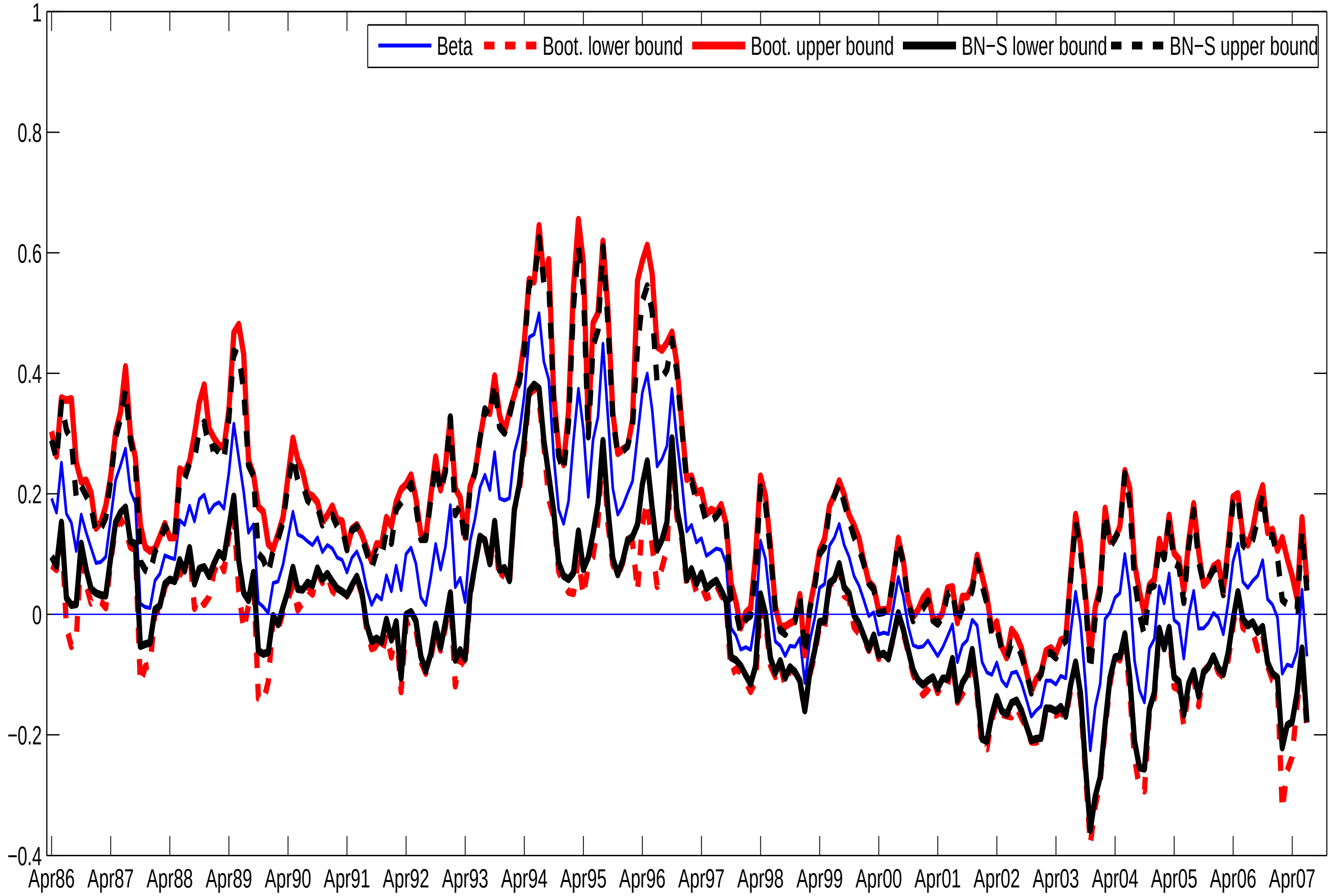

Figure 2: Symmetric bootstrap and BN-S (2004) asymptotic theory based 95% two-sided confidence intervals for the CAPM 3-month (monthly) rolling realized beta of UK bond. April 1986 through July 2007.

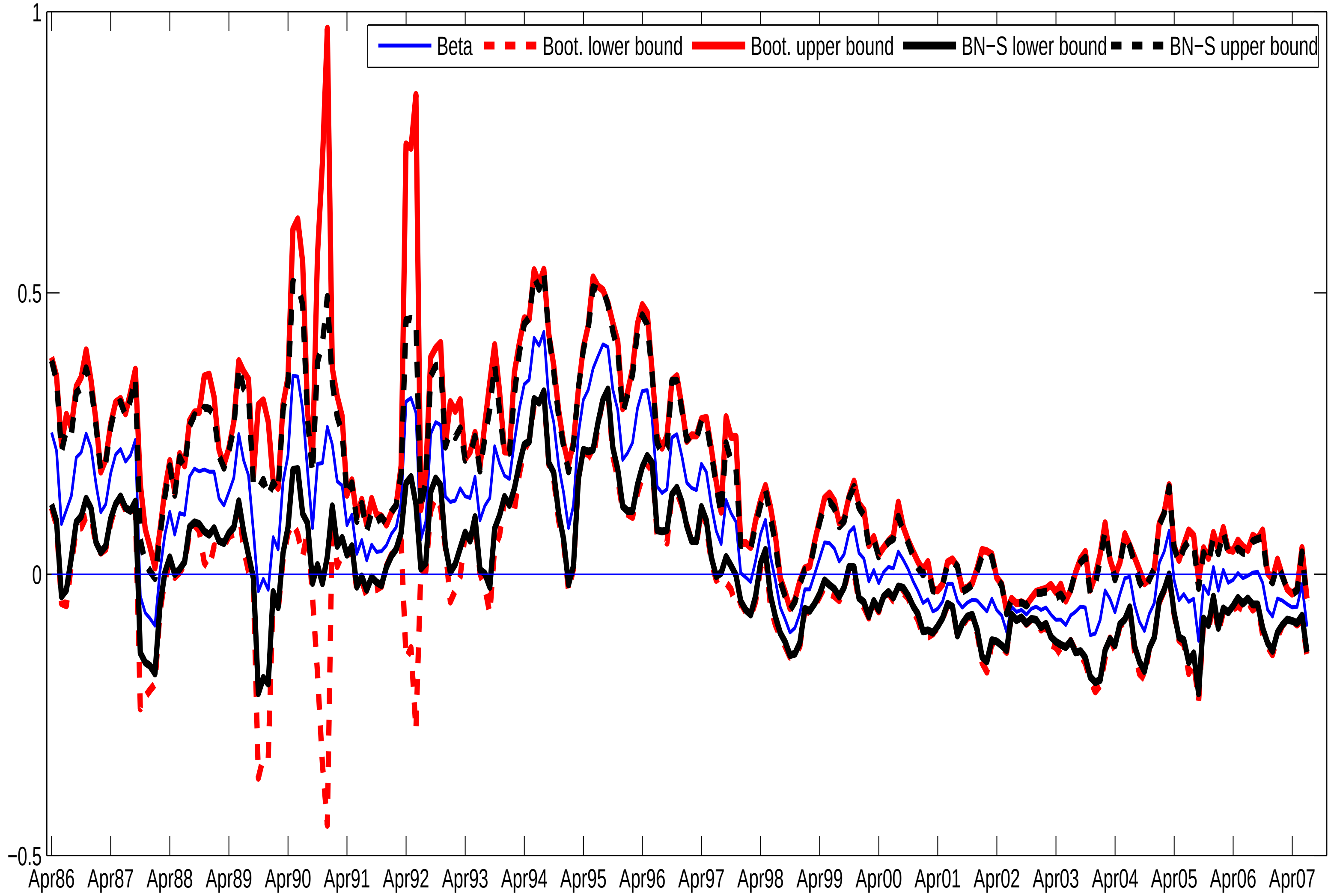

Table 3. Full-sample estimates of bonds betas for the US and the UK from January 2, 1986 through August 24, 2007

	Beta	BN-S 95% 2-sided CI	Boot. symm. 95% CI
US			
	0.024	$[0.010, 0.038]$	$[0.009, 0.038]$
UK			
	0.030	$[0.016, 0.045]$	$[0.015, 0.046]$

Table 4. Divergence between BN-S and bootstrap confidence intervals for the US

Date	Beta	BN-S	Bootstrap
31-Jul-86	0.167	$[0.027, 0.306]$	$[-0.022, 0.355]$
29-Aug-86	0.152	$[0.015, 0.289]$	$[-0.053, 0.357]$
30-Sep-86	0.106	$[0.017, 0.194]$	$[-0.041, 0.252]$
31-Jul-89	0.204	$[0.036, 0.371]$	$[-0.025, 0.432]$
29-May-92	0.111	$[0.004, 0.217]$	$[-0.010, 0.231]$
29-May-98	0.093	$[0.001, 0.184]$	$[-0.012, 0.197]$
31-Aug-00	0.062	$[0.002, 0.121]$	$[-0.002, 0.126]$
30-Jan-98	-0.054	$[-0.101, -0.008]$	$[-0.111, 0.003]$
27-Feb-98	-0.059	$[-0.115, -0.002]$	$[-0.128, 0.010]$
29-Dec-00	-0.055	$[-0.109, -0.000]$	$[-0.117, 0.008]$
31-May-01	-0.055	$[-0.107, -0.004]$	$[-0.113, 0.003]$
31-Dec-03	-0.154	$[-0.302, -0.005]$	$[-0.319, 0.011]$
29-Oct-04	-0.146	$[-0.256, -0.036]$	$[-0.293, 0.001]$

Table 5. Divergence between BN-S and bootstrap confidence intervals for the UK

Date	Beta	BN-S	Bootstrap
31-Mar-88	0.070	$[0.003, 0.137]$	$[-0.010, 0.150]$
31-Oct-90	0.197	$[0.016, 0.377]$	$[-0.175, 0.568]$
31-Dec-90	0.262	$[0.031, 0.493]$	$[-0.446, 0.970]$
30-Apr-92	0.307	$[0.162, 0.452]$	$[-0.151, 0.764]$
29-May-92	0.314	$[0.173, 0.454]$	$[-0.131, 0.758]$
30-Jun-92	0.288	$[0.125, 0.450]$	$[-0.277, 0.852]$
29-Jan-93	0.129	$[0.003, 0.254]$	$[-0.049, 0.306]$
26-Feb-93	0.131	$[0.018, 0.243]$	$[-0.029, 0.290]$
31-Mar-93	0.153	$[0.046, 0.259]$	$[-0.004, 0.309]$
31-Aug-93	0.122	$[0.001, 0.242]$	$[-0.025, 0.268]$
29-Aug-97	0.054	$[0.002, 0.105]$	$[-0.003, 0.111]$
30-Sep-97	0.132	$[0.031, 0.233]$	$[-0.015, 0.279]$
31-Oct-97	0.109	$[0.015, 0.202]$	$[-0.027, 0.244]$
29-Jan-88	-0.092	$[-0.177, -0.007]$	$[-0.195, 0.012]$
31-Jan-01	-0.052	$[-0.102, -0.001]$	$[-0.111, 0.008]$
30-Sep-04	-0.085	$[-0.156, -0.014]$	$[-0.177, 0.008]$
30-Nov-06	-0.064	$[-0.122, -0.005]$	$[-0.129, 0.002]$

Appendix B

This Appendix is divided in three parts. The first part (Appendix B.1) contains the proofs of Theorems 3.1, 3.2 and 5.1, as well as two auxiliary lemmas. The second part (Appendix B.2) contains the proof of Theorem 5.2 (a) and a list of lemmas useful for this proof. The third part (Appendix B.3) contains the proof of Theorem 5.2 (b) and a list of auxiliary lemmas.

Appendix B.1. Proofs of Theorems 3.1, 3.2 and 5.1.

Lemma B.1 *Under (1) and (2), for any $q_1, q_2 \geq 0$ such that $q_1 + q_2 > 0$, and for any $k, l = 1, \ldots, q$, $h^{1-(q_1+q_2)/2} \sum_{i=1}^{1/h} |y_{li}|^{q_1} |y_{ki}|^{q_2} = O_P(1)$.*

Proof of Lemma B.1. Apply Theorem 2.1 of Barndorff-Nielsen, Graversen, Jacod, Podolskij and Shephard (2006) (henceforth BNGJPS (2006)).

Lemma B.2 *Under (1) and (2), for k, l, k', l' $= 1, \ldots, q$, with probability approaching one, (i) $\sum_{i=1}^{1/h} y_{ki}^* y_{li}^* \xrightarrow{P^*} \sum_{i=1}^{1/h} y_{ki} y_{li}$, and (ii) $h^{-1} \sum_{i=1}^{1/h} y_{ki}^* y_{li}^* y_{k'i}^* y_{l'i}^* \xrightarrow{P^*} h^{-1} \sum_{i=1}^{1/h} y_{ki} y_{li} y_{k'i} y_{l'i}$.*

Proof of Lemma B.2. We show that the results hold in quadratic mean with respect to P^*, with probability approaching one. This ensures that the bootstrap convergence also holds in probability. For (i), we have $E^* \left(\sum_{i=1}^{1/h} y_{ki}^* y_{li}^* \right) = h^{-1} E^* \left(y_{k1}^* y_{l1}^* \right) = h^{-1} h \sum_{i=1}^{1/h} y_{ki} y_{li} = \sum_{i=1}^{1/h} y_{ki} y_{li}$. Similarly,

$$
\begin{aligned}
Var^* \left(\sum_{i=1}^{1/h} y_{ki}^* y_{li}^* \right) &= h^{-1} Var^* \left(y_{k1}^* y_{l1}^* \right) = h^{-1} \left(E^* (y_{k1}^* y_{l1}^*)^2 - (E^* y_{k1}^* y_{l1}^*)^2 \right) \\
&= h^{-1} \left(h \sum_{i=1}^{1/h} (y_{ki} y_{li})^2 - \left(h \sum_{i=1}^{1/h} y_{ki} y_{li} \right)^2 \right) = \sum_{i=1}^{1/h} (y_{ki} y_{li})^2 - h \left(\sum_{i=1}^{1/h} y_{ki} y_{li} \right)^2 = o_P(1),
\end{aligned}
$$

since Lemma B.1 implies that $\sum_{i=1}^{1/h} (y_{ki} y_{li})^2 = O_P(h) = o_P(1)$ and $\sum_{i=1}^{1/h} y_{ki} y_{li} = O_P(1)$. The proof of (ii) follows similarly and therefore we omit the details.

Proof of Theorem 3.1. (a) follows from Lemma B.2 by noting that the elements of $x_i^* x_i^{*\prime}$ are of all of the form $y_{ki}^* y_{li}^* y_{k'i}^* y_{l'i}^*$, for $k, l, k', l' = 1, \ldots, q$. To prove (b), first note that both $\hat{V}^*$ and V^* are non singular in large samples with probability approaching one, as $h \to 0$. Second, letting $S_h^* = V^{*-1/2} \sqrt{h^{-1}} (\sum_{i=1}^{1/h} x_i^* - \sum_{i=1}^{1/h} x_i)$, we have that $T_h^* = \hat{V}^{*-1/2} V^{*1/2} S_h^*$. Because $\hat{V}^{*-1} V^* \xrightarrow{P^*} I_{q(q+1)/2}$, in probability, the proof of (b) follows from showing that for any $\lambda \in \mathbb{R}^{q(q+1)/2}$ such that $\lambda' \lambda = 1$, $\sup_{x \in \mathbb{R}} |P^*(\sum_{i=1}^{1/h} \tilde{x}_i^* \leq x) - \Phi(x)| \xrightarrow{P} 0$, where $\tilde{x}_i^* = \left(\lambda' V^* \lambda \right)^{-1/2} \sqrt{h^{-1}} \lambda' (x_i^* - E^*(x_i^*))$. Clearly, $E^* \left(\sum_{i=1}^{1/h} \tilde{x}_i^* \right) = 0$ and $Var^* \left(\sum_{i=1}^{1/h} \tilde{x}_i^* \right) = 1$. Thus, by Katz's (1963) Berry-Essen Bound, for some small $\epsilon > 0$ and some constant $K > 0$, $\sup_{x \in \mathbb{R}} \left| P^* \left(\sum_{i=1}^{1/h} \tilde{x}_i^* \leq x \right) - \Phi(x) \right| \leq K \sum_{i=1}^{1/h} E^* |\tilde{x}_i^*|^{2+\epsilon}$. Next, we show that $\sum_{i=1}^{1/h} E^* |\tilde{x}_i^*|^{2+\epsilon} = o_p(1)$. We have that

$$
\begin{aligned}
\sum_{i=1}^{1/h} E^* |\tilde{x}_i^*|^{2+\epsilon} &= h^{-1} E^* |\tilde{x}_1^*|^{2+\epsilon} = h^{-1} E^* \left| \left(\lambda' V^* \lambda \right)^{-1/2} h^{-1/2} \lambda' (x_1^* - E^*(x_1^*)) \right|^{2+\epsilon} \\
&= h^{-1} h^{-(2+\epsilon)/2} |\lambda' V^* \lambda|^{-(2+\epsilon)/2} E^* |\lambda' (x_1^* - E^*(x_1^*))|^{2+\epsilon} \\
&\leq 2^{2+\epsilon} h^{-(2+\epsilon/2)} |\lambda' V^* \lambda|^{-(1+\epsilon/2)} E^* |\lambda' x_1^*|^{2+\epsilon} \leq 2^{2+\epsilon} h^{-(2+\epsilon/2)} |\lambda' V^* \lambda|^{-(1+\epsilon/2)} E^* |x_1^*|^{2+\epsilon} \\
&= 2^{2+\epsilon} h^{-1-\epsilon/2} |\lambda' V^* \lambda|^{-(1+\epsilon/2)} \sum_{i=1}^{1/h} |x_i|^{2+\epsilon},
\end{aligned}
$$

where the first inequality follows from the C_r and the Jensen inequalities, and the second inequality follows from the Cauchy-Schwarz inequality and the fact that $\lambda'\lambda = 1$. We let $|z| = (z'z)^{1/2}$ for any vector z. It follows that $\sum_{i=1}^{1/h} |x_i|^{2+\epsilon} = \sum_{i=1}^{1/h} |x_i|^{2(1+\epsilon/2)} \leq \sum_{i=1}^{1/h} \left(\sum_{j=1}^{q} y_{ji}^2\right)^{2(1+\epsilon/2)}$. Lemma B.1 and the Minkowski inequality imply that $\sum_{i=1}^{1/h} |x_i|^{2+\epsilon} = O_P(h^{1+\epsilon})$, so that $\sum_{i=1}^{1/h} E^*|\tilde{x}_i^*|^{2+\epsilon} = O_P(h^{\epsilon/2}) = o_P(1)$.

Proof of Theorem 3.2. Since $T_h \xrightarrow{d} N(0, I_{q(q+1)/2})$, by the standard delta method, $T_{f,h} \xrightarrow{d} N(0,1)$. Similarly, by a mean value expansion, and conditionally on the original sample,

$$\sqrt{h^{-1}}\left(f(vech(\hat{\Gamma}^*)) - f(vech(\hat{\Gamma}))\right) = \sqrt{h^{-1}}\nabla' f\left(vech(\hat{\Gamma})\right)\left(vech(\hat{\Gamma}^*) - vech(\hat{\Gamma})\right) + o_{P^*}(1),$$

since $\hat{\Gamma}^* \to^{P^*} \hat{\Gamma}$ in probability. Let

$$S_{f,h}^* \equiv \frac{\sqrt{h^{-1}}\left(f(vech(\hat{\Gamma}^*)) - f(vech(\hat{\Gamma}))\right)}{\sqrt{V_f^*}},$$

with $V_f^* \equiv \nabla' f(vech(\hat{\Gamma}))V^*\nabla f(vech(\hat{\Gamma}))$. It follows that $S_{f,h}^* \to^{d^*} N(0,1)$ in probability, given Theorem 3.1 (b). Next note that $T_{f,h}^* = \sqrt{\frac{V_f^*}{\hat{V}_f^*}} S_{f,h}^*$, where $\hat{V}_f^* \to^{P^*} V_f^*$. The result follows from Polya's theorem (e.g. Serfling, 1980) given that the normal distribution is continuous.

Proof of Theorem 5.1. Take $q = 2$ and $l = 1$ and $k = 2$. Part (a) follows from Theorem 3.2 with $f(\theta) = \theta_2/\theta_3$. V_β^* and part (b) are proven in the text. Part (c) follows from Theorem 1 of BNGJS (2006) and the fact that $\hat{\beta}_{12} \to^P \beta_{12}$.

Appendix B.2. Asymptotic expansions of the cumulants of $T_{\beta,h}$

Notation

Throughout this Appendix, we use the convention that $z_{1+1/h} = 0$ for any random variable z. Focusing on the pair $(l,k) = (1,2)$ without loss of generality, we can write

$$T_{\beta,h} \equiv \frac{\sqrt{h^{-1}}(\hat{\beta}_{12} - \beta_{12})}{\sqrt{\left(\sum_{i=1}^{1/h} y_{2i}^2\right)^{-2} h^{-1}\hat{g}_\beta}} = \frac{\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i}{\sqrt{h^{-1}\hat{g}_\beta}} = S_h\left(\frac{h^{-1}\hat{g}_\beta}{B_h}\right)^{-1/2},$$

where $\hat{g}_\beta$ and $B_h = Var\left(\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right)$ are defined in the text, and $S_h = \frac{\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i}{\sqrt{B_h}}$. Recall that $\Gamma_{lki} \equiv \int_{(i-1)h}^{ih} \Sigma_{lk}(u)\,du$ (where $\Gamma_{ki} = \Gamma_{kki}$ when $l = k$). We let

$$\begin{aligned}
u_i &= h^{-1}\left(y_{2i}^2\varepsilon_i^2 - E\left(y_{2i}^2\varepsilon_i^2\right)\right), \\
u_{i,i+1} &= h^{-1}\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1} - E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right)\right), \\
A_{1h}^1 &= h^{-2}\sum_{i=1}^{1/h}\left(2\Gamma_{12i}^3 - 18\beta_{12}\Gamma_{2i}\Gamma_{12i}^2 + 24\beta_{12}^2\Gamma_{2i}^2\Gamma_{12i} + 6\Gamma_{1i}\Gamma_{2i}\Gamma_{12i} - 8\beta_{12}^3\Gamma_{2i}^3 - 6\beta_{12}\Gamma_{1i}\Gamma_{2i}^2\right) \\
A_{1h}^2 &= h^{-2}\sum_{i=1}^{1/h}(-12\Gamma_{2i}^3\beta_{12}^3 + 2\Gamma_{2i}\Gamma_{2,i+1}^2\beta_{12}^3 + 2\Gamma_{2i}^2\Gamma_{2,i+1}\beta_{12}^3 + 36\Gamma_{12i}\Gamma_{2i}^2\beta_{12}^2 \\
&\quad -2\Gamma_{12,i+1}\Gamma_{2i}^2\beta_{12}^2 - 2\Gamma_{12i}\Gamma_{2,i+1}^2\beta_{12}^2 - 4\Gamma_{12i}\Gamma_{2i}\Gamma_{2,i+1}\beta_{12}^2 - 4\Gamma_{12,i+1}\Gamma_{2i}\Gamma_{2,i+1}\beta_{12}^2 \\
&\quad -8\Gamma_{1i}\Gamma_{2i}^2\beta_{12} - 28\Gamma_{12i}^2\Gamma_{2i}\beta_{12} + \Gamma_{12,i+1}^2\Gamma_{2i}\beta_{12} + 4\Gamma_{12i}\Gamma_{12,i+1}\Gamma_{2i}\beta_{12} + \Gamma_{12i}^2\Gamma_{2,i+1}\beta_{12} \\
&\quad +4\Gamma_{12i}\Gamma_{12,i+1}\Gamma_{2,i+1}\beta_{12} + \Gamma_{1i}\Gamma_{2i}\Gamma_{2,i+1}\beta_{12} + \Gamma_{1,i+1}\Gamma_{2i}\Gamma_{2,i+1}\beta_{12} + 4\Gamma_{12i}^3 - \Gamma_{12i}\Gamma_{12,i+1}^2 \\
&\quad -\Gamma_{12i}^2\Gamma_{12,i+1} + 8\Gamma_{1i}\Gamma_{12i}\Gamma_{2i} - \Gamma_{1i}\Gamma_{12,i+1}\Gamma_{2i} - \Gamma_{1,i+1}\Gamma_{12i}\Gamma_{2,i+1}).
\end{aligned}$$

Similarly, let

$$
\begin{aligned}
A^1_{0h} &= h^{-1}\sum_{i=1}^{1/h} E(y^3_{2i}\varepsilon_i), \quad A^2_{0h} = h^{-1}\sum_{i=1}^{1/h} E(y^2_{2i}y_{2,i+1}\varepsilon_{i+1}), \quad A^3_{0h} = h^{-1}\sum_{i=1}^{1/h} E(y^2_{2,i+1}y_{2i}\varepsilon_i), \\
A_{0h} &= \frac{1}{4}(2A^1_{0h} - A^2_{0h} - A^3_{0h}).
\end{aligned}
$$

Auxiliary Lemmas

Lemma B.3 *Let $k, l, k', l', k'', l'', m, n, m', n', m'', n'' = 1, \dots, q$ and let n_1, n_2, n_3, n_4, n_5 and n_6, be any non negative integers. Suppose (1) and (2) hold. Conditionally on Σ,*

$$
\begin{aligned}
& h^{1-(n_1+n_2+n_3+n_4+n_5+n_6)}\sum_{i=1}^{1/h} \Gamma^{n_1}_{kli}\Gamma^{n_2}_{k'l'i}\Gamma^{n_3}_{k'l'i}\Gamma^{n_4}_{mn,i+1}\Gamma^{n_5}_{m'n',i+1}\Gamma^{n_6}_{m''n'',i+1} \\
\to & \int_0^1 \Sigma^{n_1}_{kl}(u)\Sigma^{n_2}_{k'l'}(u)\Sigma^{n_3}_{k'l'}(u)\Sigma^{n_4}_{mn}(u)\Sigma^{n_5}_{m'n'}(u)\Sigma^{n_6}_{m''n''}(u)du,
\end{aligned}
$$

as $h \to 0$.

Lemma B.4 *Suppose (1) and (2) hold with $\alpha \equiv 0$ and W independent of Σ. Then, conditionally on Σ, as $h \to 0$, (a1) $A^j_{1h} \to A_1$, for $j = 1, 2$; (a2) $B_h = h^{-1}\sum_{i=1}^{1/h}\left(\Gamma_{12i} - 4\beta_{12}\Gamma_{2i}\Gamma_{12i} + 2\beta^2_{12}\Gamma^2_{2i} + \Gamma_{1i}\Gamma_{2i}\right) \to B$; (a3) $A^1_{0h} = 3h^{-1}\sum_{i=1}^{1/h}(\Gamma_{12i}\Gamma_{2i} - \beta_{12}\Gamma^2_{2i}) \to 3A_0$; (a4) $A^2_{0h} = h^{-1}\sum_{i=1}^{1/h}(\Gamma_{12,i+1}\Gamma_{2i} - \beta_{12}\Gamma_{2i}\Gamma_{2,i+1}) \to A_0$; (a5) $A^3_{0h} = h^{-1}\sum_{i=1}^{1/h}(\Gamma_{12i}\Gamma_{2,i+1} - \beta_{12}\Gamma_{2i}\Gamma_{2,i+1}) \to A_0$.*

Lemma B.5 *Suppose (1) and (2) hold with $\alpha \equiv 0$ and W independent of Σ. Then, conditionally on Σ, (a1) $E\left(\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right) = 0$; (a2) $E\left(\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right)^2 = hB_h$; (a3) $E\left(\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right)^3 = h^2 A^1_{1h}$; (a4) $E\left(\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right)^4 = 3h^2B^2_h + O(h)$, as $h \to 0$; (a5) $E\left(\sum_{i=1}^{1/h} y_{2i}\varepsilon_i \sum_{i=1}^{1/h}(u_i - u_{i,i+1})\right) = hA^2_{1h}$; (a6) $E\left(\left(\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right)^2 \sum_{i=1}^{1/h}(u_i - u_{i,i+1})\right) = O\left(h^2\right)$, as $h \to 0$; (a7) $E\left(\left(\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right)^3 \sum_{i=1}^{1/h}(u_i - u_{i,i+1})\right) = 3h^2B_hA^2_{1h} + O(h^3)$, as $h \to 0$.*

Lemma B.6 *Suppose (1) and (2) hold with $\alpha \equiv 0$ and W independent of Σ. Then, conditionally on Σ, (a1) $E\left(S_{\beta,h}\right) = 0$; (a2) $E\left(S^2_{\beta,h}\right) = 1$; (a3) $E\left(S^3_{\beta,h}\right) = \sqrt{h}\frac{A^1_{1h}}{B^{3/2}_h}$; (a4) $E\left(S^4_{\beta,h}\right) = 3 + O(h)$, as $h \to 0$; (a5) $E\left(S_{\beta,h}\sqrt{h^{-1}}\sum_{i=1}^{1/h}(u_i - u_{i,i+1})\right) = \frac{A^2_{1h}}{\sqrt{B_h}}$; (a6)$E\left(S^2_{\beta,h}\sqrt{h^{-1}}\sum_{i=1}^{1/h}(u_i - u_{i,i+1})\right) = O\left(\sqrt{h}\right)$, as $h \to 0$; and (a7) $E\left(S^3_{\beta,h}\sqrt{h^{-1}}\sum_{i=1}^{1/h}(u_i - u_{i,i+1})\right) = 3\frac{A^2_{1h}}{\sqrt{B_h}} + O(h)$,as $h \to 0$.*

Lemma B.7 *Suppose (1) and (2) hold with $\alpha \equiv 0$ and W independent of Σ. Then, conditionally on Σ, as $h \to 0$,*

$$
h^{-1}\hat{g}_\beta = B_h\left(1 + \frac{1}{B_h}\sum_{i=1}^{1/h}(u_i - u_{i,i+1}) - \frac{4A_{0h}}{B_h\Gamma_2}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right) + o_P(\sqrt{h}).
$$

Proof of Lemma B.3. This result follows from the boundedness of $\Sigma_k(u)$ and the Reimann integrability of $\Sigma^n_{kl}(u)$ for any $k, l = 1, \dots, q$ and for any non negative integer n_i.

Proof of Lemma B.4. We derive the appropriate moments as a function of integrals of Γ_{kli} and then apply Lemma B.3. To derive the expressions of the moments, we rely on the fact that conditionally on Σ, independently across $i = 1, \dots, 1/h$, $y_i \sim N(0, \Gamma_i)$ with $\Gamma_i = \int_{(i-1)h}^{ih}\Sigma(u)du$. Let C_i be the

Cholesky decomposition of Γ_i. Note that $y_i \overset{d}{=} C_i u_i$: $u_i \sim iidN(0, I_2)$ where I_2 is the 2×2 identity matrix and '$\overset{d}{=}$' denotes equivalence in distribution. Then,

$$C_i = \begin{pmatrix} \sqrt{\Gamma_{1i}} & 0 \\ \frac{\Gamma_{12,i}}{\sqrt{\Gamma_{1i}}} & \sqrt{\Gamma_{2i} - \frac{\Gamma_{12i}^2}{\Gamma_{1i}}} \end{pmatrix},$$

and $y_{1i} \overset{d}{=} C_{1i} u_{1i}$ and $y_{2i} \overset{d}{=} C_{21i} u_{1i} + C_{2i} u_{2i}$. For instance, to obtain the expression for B_h, let $z_i = y_{2i}\varepsilon_i - E(y_{2i}\varepsilon_i)$ and note that by definition, the $z_i's$ are independent with $E(z_i) = 0$. It follows that

$$B_h = Var\left(\sqrt{h^{-1}} \sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right) = h^{-1} E\left(\sum_{i=1}^{1/h} (y_{2i}\varepsilon_i - E(y_{2i}\varepsilon_i))\right)^2 = h^{-1} \sum_{i=1}^{1/h} E\left(z_i^2\right).$$

Now, $E\left(z_i^2\right) = E\left(y_{2i}^2 \varepsilon_i^2\right) - \left(E\left(y_{2i}\varepsilon_i\right)\right)^2$. Since $\varepsilon_i = y_{1i} - \beta_{12} y_{2i}$, we get that

$$\begin{aligned} E\left(y_{2i}\varepsilon_i\right) &= E\left(y_{1i}y_{2i}\right) - \beta_{12} E\left(y_{2i}^2\right) = \Gamma_{12i} - \beta_{12}\Gamma_{2i}, \\ E\left(y_{2i}^2 \varepsilon_i^2\right) &= E\left(y_{2i}^2 \left(y_{1i} - \beta_{12} y_{2i}\right)^2\right) = E\left(y_{2i}^2 y_{1i}^2\right) - 2\beta_{12} E\left(y_{1i} y_{2i}^3\right) + \beta_{12}^2 E\left(y_{2i}^4\right). \end{aligned}$$

We now use the Cholesky decomposition to get that

$$\begin{aligned} E\left(y_{2i}^2 y_{1i}^2\right) &= E\left(\left(C_{1i}u_{1i}\right)^2 \left(C_{21i}u_{1i} + C_{2i}u_{2i}\right)^2\right) = E\left(C_{1i}^2 u_{1i}^2\right)\left(C_{21i}^2 u_{1i}^2 + 2C_{21i}C_{2i}u_{1i}u_{2i} + C_{2i}^2 u_{2i}^2\right) \\ &= 3C_{1i}^2 C_{21i}^2 + C_{1i}^2 C_{2i}^2 = 2\Gamma_{12i}^2 + \Gamma_{1i}\Gamma_{2i}; \\ E\left(y_{1i} y_{2i}^3\right) &= E\left(\left(C_{1i}u_{1i}\right)\left(C_{21i}u_{1i} + C_{2i}u_{2i}\right)^3\right) = 3C_{1i}C_{21i}^3 + 3C_{1i}C_{21i}C_{2i}^2 = 3\Gamma_{12i}\Gamma_{2i}; \text{ and} \\ E\left(y_{2i}^4\right) &= E\left(\left(C_{21i}u_{1i} + C_{2i}u_{2i}\right)^4\right) = 3C_{21i}^4 + 6C_{21i}^2 C_{2i}^2 + 3C_{2i}^4 = 3\Gamma_{2i}. \end{aligned}$$

Thus, $E\left(y_{2i}^2 \varepsilon_i^2\right) = 2\Gamma_{12i}^2 + \Gamma_{1i}\Gamma_{2i} - 6\beta_{12}\Gamma_{12i}\Gamma_{2i} + 3\beta_{12}^2\Gamma_{2i}^2$ and

$$\begin{aligned} E\left(z_i^2\right) &= 2\Gamma_{12i}^2 + \Gamma_{1i}\Gamma_{2i} - 6\beta_{12}\Gamma_{12i}\Gamma_{2i} + 3\beta_{12}^2\Gamma_{2i}^2 - \left(\Gamma_{12i} - \beta_{12}\Gamma_{2i}\right)^2 \\ &= \Gamma_{12i}^2 + \Gamma_{1i}\Gamma_{2i} - 4\beta_{12}\Gamma_{12i}\Gamma_{2i} + 2\beta_{12}^2\Gamma_{2i}^2, \end{aligned}$$

which implies $B_h = h^{-1}\sum_{i=1}^{1/h}\left(\Gamma_{12i}^2 + \Gamma_{1i}\Gamma_{2i} - 4\beta_{12}\Gamma_{12i}\Gamma_{2i} + 2\beta_{12}^2\Gamma_{2i}^2\right)$. To conclude the proof of the second result, we then apply Lemma B.3. The proof of the remaining results follows similarly and therefore we omit the details.

Proof of Lemma B.5. (a1) follows by definition of β_{12} whereas (a2) follows by the definition of B_h. For the remaining results, write $z_i = y_{2i}\varepsilon_i - E(y_{2i}\varepsilon_i)$ and note that by definition, the $z_i's$ are independent with $E(z_i) = 0$. Note also that $\sum_{i=1}^{1/h} z_i = \sum_{i=1}^{1/h} y_{2i}\varepsilon_i$ since $\sum_{i=1}^{1/h} E\left(y_{2i}\varepsilon_i\right) = 0$. For (a3), note that

$$E\left(\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right)^3 = E\left(\sum_{i=1}^{1/h} z_i\right)^3 = \sum_{i,j,k=1}^{1/h} E\left(z_i z_j z_k\right) = \sum_{i=1}^{1/h} E\left(z_i^3\right).$$

We now compute $E\left(z_i^3\right)$ using the Cholesky decomposition as in the proof of Lemma B.4 to show that $\sum_{i=1}^{1/h} E\left(z_i^3\right) = h^2 A_{1h}^1$, with A_{1h}^1 as defined above. For (a4), note that $E\left(\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right)^4 = \sum_{i=1}^{1/h} E\left(z_i^4\right) + 3\sum_{i \neq j} E\left(z_i^2\right) E\left(z_j^2\right) = 3\left(\sum_{i=1}^{1/h} E\left(z_i^2\right)\right)^2 + O(h^3)$ and use the definition of B_h to prove the result. For (a5), note that

$$E\left(\left(\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right) \sum_{i=1}^{1/h}\left(u_i - u_{i,i+1}\right)\right) = \sum_{i=1}^{1/h} E(z_i u_i) - \sum_{i=1}^{1/h} E(z_i u_{i,i+1}) - \sum_{i=1}^{1/h} E(z_{i+1} u_{i,i+1}).$$

Useing the definitions of u_i and $u_{i,i+1}$, the result follows from simple but tedious algebra using the Cholesky decomposition. The remaining results follow similarly and therefore we omit the details.
Proof of Lemma B.6. We apply Lemma B.5, given the definition of $S_{\beta,h}$.
Proof of Lemma B.7. Using the definition of $\hat{g}_\beta$ in the text, we can write

$$\begin{aligned} h^{-1}\hat{g}_\beta &= h^{-1}\sum_{i=1}^{1/h}\left(y_{2i}^2\varepsilon_i^2 + (\hat{\beta}_{12}-\beta_{12})^2 y_{2i}^4 - 2(\hat{\beta}_{12}-\beta_{12})y_{2i}^3\varepsilon_i\right) \\ &\quad - h^{-1}\sum_{i=1}^{1/h}\left(y_{2i}y_{2,i+1}\varepsilon_i\varepsilon_{i+1} + (\hat{\beta}_{12}-\beta_{12})^2 y_{2i}^2 y_{2,i+1}^2 - (\hat{\beta}_{12}-\beta_{12})(y_{2i}^2 y_{2,i+1}\varepsilon_{i+1} + y_{2,i+1}^2 y_{2i}\varepsilon_i)\right). \end{aligned}$$

Adding and subtracting appropriately, it follows that

$$\begin{aligned} h^{-1}\hat{g}_\beta &= h^{-1}\sum_{i=1}^{1/h}E(y_{2i}\varepsilon_i)^2 - h^{-1}\sum_{i=1}^{1/h}E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right) + \left(h^{-1}\sum_{i=1}^{1/h}((y_{2i}\varepsilon_i)^2 - E(y_{2i}\varepsilon_i)^2)\right) \\ &\quad - \left(h^{-1}\sum_{i=1}^{1/h}\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1} - E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right)\right)\right) - (\hat{\beta}_{12}-\beta_{12})h^{-1}\sum_{i=1}^{1/h}E(2y_{2i}^3\varepsilon_i) \\ &\quad + (\hat{\beta}_{12}-\beta_{12})h^{-1}\sum_{i=1}^{1/h}E(y_{2i}^2 y_{2,i+1}\varepsilon_{i+1}) + (\hat{\beta}_{12}-\beta_{12})h^{-1}\sum_{i=1}^{1/h}E(y_{2,i+1}^2 y_{2i}\varepsilon_i) + O_P(h), \\ &= B_h + h^{-1}\sum_{i=1}^{1/h}\left(Ey_{2i}\varepsilon_i\right)^2 - h^{-1}\sum_{i=1}^{1/h}E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right) + \sum_{i=1}^{1/h}(u_i - u_{i,i+1}) - (\hat{\beta}_{12}-\beta_{12})2A_{0h}^1 \\ &\quad + (\hat{\beta}_{12}-\beta_{12})A_{0h}^2 + (\hat{\beta}_{12}-\beta_{12})A_{0h}^3 + O_P\left(h\right) \\ &= B_h + \sum_{i=1}^{1/h}(u_i - u_{i,i+1}) - \frac{2A_{0h}^1}{\Gamma_2}\sum_{i=1}^{1/h}y_{2i}\varepsilon_i + \frac{A_{0h}^2}{\Gamma_2}\sum_{i=1}^{1/h}y_{2i}\varepsilon_i + \frac{A_{0h}^3}{\Gamma_2}\sum_{i=1}^{1/h}y_{2i}\varepsilon_i + o_P\left(\sqrt{h}\right). \end{aligned}$$

To show that the remainder term is of order $o_P(\sqrt{h})$ we have used the following facts: $\hat{\beta}_{12} - \beta_{12} = O_P(\sqrt{h})$; $h^{-1}\sum_{i=1}^{1/h} y_{2i}^2 y_{2,i+1}^2 = O_P(1)$; $h^{-1}\sum_{i=1}^{1/h}\left(y_{2i}^3\varepsilon_i - E\left(y_{2i}^3\varepsilon_i\right)\right) = O_P(\sqrt{h})$; $h^{-1}\sum_{i=1}^{1/h}\left(y_{2,i+1}^2 y_{2i}\varepsilon_i - E\left(y_{2,i+1}^2 y_{2i}\varepsilon_i\right)\right) = O_P(\sqrt{h})$; $h^{-1}\sum_{i=1}^{1/h}\left(Ey_{2i}\varepsilon_i\right)^2 - \operatorname{p\,lim}_{h\to 0} h^{-1}\sum_{i=1}^{1/h}\left(Ey_{2i}\varepsilon_i\right)^2 = o_P\left(\sqrt{h}\right)$, and $h^{-1}\sum_{i=1}^{1/h}E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right) - \operatorname{p\,lim}_{h\to 0} h^{-1}\sum_{i=1}^{1/h}E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right) = o_p\left(\sqrt{h}\right)$, whose proof relies on Theorem 2.3 of BNGJPS (2006). In addition, by Lemma B.3, $h^{-1}\sum_{i=1}^{1/h}\left(Ey_{2i}\varepsilon_i\right)^2$ and $h^{-1}\sum_{i=1}^{1/h}E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right)$ have the same probability limit and therefore, $h^{-1}\sum_{i=1}^{1/h}\left(Ey_{2i}\varepsilon_i\right)^2 - h^{-1}\sum_{i=1}^{1/h}E\left(y_{2i}\varepsilon_i y_{2,i+1}\varepsilon_{i+1}\right) = o_P\left(\sqrt{h}\right)$.

Proof of Theorem 5.2(a).

Given Lemma B.7, we can write

$$T_{\beta,h} = S_{\beta,h}\left(1 + \frac{1}{B_h}\sum_{i=1}^{1/h}(u_i - u_{i,i+1}) - \frac{4A_{0h}}{B_h\Gamma_2}\sum_{i=1}^{1/h}y_{2i}\varepsilon_i + o_P(\sqrt{h})\right)^{-1/2}.$$

The first and third cumulants of $T_{\beta,h}$ are given by (see e.g., Hall, 1992, p. 42) $\kappa_1(T_{\beta,h}) = E(T_{\beta,h})$ and

$$\kappa_3(T_{\beta,h}) = E(T_{\beta,h}^3) - 3E(T_{\beta,h}^2)E(T_{\beta,h}) + 2[E(T_{\beta,h})]^3.$$

Our goal is to identify the terms of order up to $O(\sqrt{h})$ of the asymptotic expansions of these two cumulants. We will first provide asymptotic expansions through order $O(\sqrt{h})$ for the first three moments of $T_{\beta,h}$. Note that for a given fixed value of k, a first-order Taylor expansion of $f(x) = (1+x)^{-k/2}$ around 0 yields $f(x) = 1 - \frac{k}{2}x + O(x^2)$. Provided that $\sum_{i=1}^{1/h}(u_i - u_{i,i+1}) = O_P(\sqrt{h})$, we have for any fixed integer k,

$$T_{\beta,h}^k = S_{\beta,h}^k \left(1 - \sqrt{h}\frac{k}{2}\frac{\sqrt{h^{-1}}}{B_h}\sum_{i=1}^{1/h}(u_i - u_{i,i+1}) + \sqrt{h}k\frac{2A_{0h}}{B_h\Gamma_2}\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}\varepsilon_i\right) + o(\sqrt{h}) = \tilde{T}_{\beta,h}^k + o(\sqrt{h}).$$

For $k = 1, 2, 3$, the moments of $\tilde{T}_{\beta,h}^k$ are given by

$$\begin{aligned}
E(\tilde{T}_{\beta,h}) &= E\left(S_{\beta,h}\right) - \sqrt{h}\frac{1}{2}\frac{1}{B_h}E\left(S_{\beta,h}\sqrt{h^{-1}}\sum_{i=1}^{1/h}(u_i - u_{i,i+1})\right) + \sqrt{h}\frac{2A_{0h}}{\sqrt{B_h}\Gamma_2}E(S_{\beta,h}^2), \\
E(\tilde{T}_{\beta,h}^2) &= E\left(S_{\beta,h}^2\right) - \sqrt{h}\frac{1}{B_h}E\left(S_{\beta,h}^2\sqrt{h^{-1}}\sum_{i=1}^{1/h}(u_i - u_{i,i+1})\right) + \sqrt{h}\frac{4A_{0h}}{\sqrt{B_h}\Gamma_2}E(S_{\beta,h}^3), \\
E(\tilde{T}_{\beta,h}^3) &= E(S_{\beta,h}^3) - \frac{3}{2}\sqrt{h}\frac{1}{B_h}E\left(S_{\beta,h}^3\sqrt{h^{-1}}\sum_{i=1}^{1/h}(u_i - u_{i,i+1})\right) + \sqrt{h}\frac{6A_{0h}}{\sqrt{B_h}\Gamma_2}E(S_{\beta,h}^4).
\end{aligned}$$

Given Lemma B.6, $E(\tilde{T}_{\beta,h}) = -\sqrt{h}\frac{1}{2B_h}\frac{A_{1h}^2}{\sqrt{B_h}} + \sqrt{h}\frac{2A_{0h}}{\sqrt{B_h}\Gamma_2}$; $E(\tilde{T}_h^2) = 1 + O(h)$; and $E(\tilde{T}_{\beta,h}^3) = \sqrt{h}\frac{A_{1h}^1}{B_h^{3/2}} - \frac{3}{2B_h}\sqrt{h}\,3 \times \frac{A_{1h}^2}{\sqrt{B_h}} + \sqrt{h}\frac{18A_{0h}}{\sqrt{B_h}\Gamma_2} + O(h)$. Thus

$$\begin{aligned}
\kappa_1(T_{\beta,h}) &= \sqrt{h}\left(-\frac{1}{2B_h}\frac{A_{1h}^2}{\sqrt{B_h}} + \frac{2A_{0h}}{\sqrt{B_h}\Gamma_2}\right) + o(\sqrt{h}) \equiv \sqrt{h}\kappa_{1,h} + o\left(\sqrt{h}\right), \\
\kappa_3\left(T_{\beta,h}\right) &= \sqrt{h}\left(\frac{A_{1h}^1}{B_h^{3/2}} - \frac{3}{B_h}\frac{A_{1h}^2}{\sqrt{B_h}} + \frac{12A_{0h}}{\sqrt{B_h}\Gamma_2}\right) + o(\sqrt{h}) \equiv \sqrt{h}\kappa_{3,h} + o\left(\sqrt{h}\right).
\end{aligned}$$

By Lemma B.4, we can now show that $\lim_{h\to 0}\kappa_{1,h} = -\frac{1}{2}\frac{A_1}{B^{3/2}} + \frac{1}{2}\frac{4A_0}{\sqrt{B}\Gamma_2} \equiv \frac{1}{2}\left(H_1 - H_2\right)$ and $\lim_{h\to 0}\kappa_{3,h} = -2\frac{A_1}{B^{3/2}} + 3\frac{4A_0}{\sqrt{B}\Gamma_2} \equiv 3H_1 - 2H_2$, where A_0, A_1, B, H_1 and H_2 are as defined in the text.

Appendix B.3. Asymptotic expansions of the bootstrap cumulants of $T_{\beta,h}^*$

Notation

Let $\varepsilon_i^* = y_{1i}^* - \hat{\beta}_{12}y_{2i}^* = \hat{\varepsilon}_{I_i}$, with I_i a uniform draw from $\{1, \ldots, 1/h\}$, and let $\hat{\varepsilon}_i^* = y_{1i}^* - \hat{\beta}_{12}^* y_{2i}^*$ be the bootstrap OLS residual. We can write

$$T_{\beta,h}^* \equiv \frac{\sqrt{h^{-1}}(\hat{\beta}_{12}^* - \hat{\beta}_{12})}{\sqrt{\left(\sum_{i=1}^{1/h} y_{2i}^{*2}\right)^{-2}\hat{B}_{1h}^*}} = \frac{\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*}{\sqrt{\hat{B}_{1h}^*}} = S_h^*\left(1 + \sqrt{h}\left(\frac{\sqrt{h^{-1}}\left(\hat{B}_{1h}^* - \hat{B}_{1h}\right)}{\hat{B}_{1h}}\right)\right)^{-1/2}, \quad (9)$$

where

$$S_{\beta,h}^* = \frac{\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*}{\sqrt{\hat{B}_{1h}}}, \quad (10)$$

with $\hat{B}_{1h} = h^{-1}\sum_{i=1}^{1/h} y_{2i}^2\hat{\varepsilon}_i^2$, and $\hat{B}_{1h}^* = h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*2}\hat{\varepsilon}_i^{*2}$.

Let $\hat{A}_{0h} = h^{-1}\sum_{i=1}^{1/h} y_{2i}^3\hat{\varepsilon}_i$, $\hat{A}_{1h} = h^{-2}\sum_{i=1}^{1/h}(y_{2i}\hat{\varepsilon}_i)^3$, and $\tilde{B}_{1h}^* = h^{-1}\sum_{i=1}^{1/h}(y_{2i}^*\varepsilon_i^*)^2$. Note that

$$\tilde{B}_{1h}^{*}-\hat{B}_{1h}=h^{-1}\sum_{i=1}^{1/h}\left(y_{2i}^{*2}\varepsilon_i^{*2}-h^2B_{1h}^2\right), \tag{11}$$

where $E^*\left(y_{2i}^{*2}\varepsilon_i^{*2}\right)=h^2\hat{B}_{1h}$, so that $y_{2i}^{*2}\varepsilon_i^{*2}-h^2B_{1h}^2$ is i.i.d. with mean zero, conditional on the original sample.

Auxiliary lemmas

Lemma B.8 *Suppose (1) and (2) hold with* $\alpha\equiv 0$ *and* W *independent of* Σ. *Then, (a1)* $E^*\left(y_{2i}^{*}\varepsilon_i^{*}\right)=0$; *(a2)* $E^*\left(\left(y_{2i}^{*}\varepsilon_i^{*}\right)^2\right)=h^2\hat{B}_{1h}$; *(a3)* $E^*\left(\left(y_{2i}^{*}\varepsilon_i^{*}\right)^3\right)=h^3\hat{A}_{1h}$; *(a4)* $E^*\left(\left(y_{2i}^{*}\varepsilon_i^{*}\right)^q\right)=O_P\left(h^q\right)$ *for any* $q\geq 2$; *(a5)* $E^*\left(\sum_{i=1}^{1/h}y_{2i}^{*}\varepsilon_i^{*}\right)=0$; *(a6)* $E^*\left(\sum_{i=1}^{1/h}y_{2i}^{*}\varepsilon_i^{*}\right)^2=h\hat{B}_{1h}$; *(a7)* $E^*\left(\sum_{i=1}^{1/h}y_{2i}^{*}\varepsilon_i^{*}\right)^3=h^2\hat{A}_{1h}$; *(a8)* $E^*\left(\sum_{i=1}^{1/h}y_{2i}^{*}\varepsilon_i^{*}\right)^4=3h^2\left(\hat{B}_{1h}\right)^2+O_P(h^3)$, *as* $h\to 0$; *(a9)* $E^*\left(\sum_{i=1}^{1/h}y_{2i}^{*}\varepsilon_i^{*}\sum_{i=1}^{1/h}\left(y_{2i}^{*2}\varepsilon_i^{*2}-h^2B_{1h}^2\right)\right)=h^2\hat{A}_{1h}$; *(a10)* $E^*\left(\left(\sum_{i=1}^{1/h}y_{2i}^{*}\varepsilon_i^{*}\right)^2\sum_{i=1}^{1/h}\left(y_{2i}^{*2}\varepsilon_i^{*2}-h^2B_{1h}^2\right)\right)=O_P\left(h^3\right)$, *as* $h\to 0$;
(a11) $E^*\left(\left(\sum_{i=1}^{1/h}y_{2i}^{*}\varepsilon_i^{*}\right)^3\sum_{i=1}^{1/h}\left(y_{2i}^{*2}\varepsilon_i^{*2}-h^2B_{1h}^2\right)\right)=3h^3\hat{B}_{1h}\hat{A}_{1h}+O_P\left(h^4\right)$, *as* $h\to 0$.

Lemma B.9 *Suppose (1) and (2) hold with* $\alpha\equiv 0$ *and* W *independent of* Σ. *Then, (a1)* $E^*\left(S_{\beta,h}^{*}\right)=0$; *(a2)* $E^*\left(S_{\beta,h}^{*2}\right)=1$; *(a3)* $E^*\left(S_{\beta,h}^{*3}\right)=\sqrt{h}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}}$; *(a4)* $E^*\left(S_{\beta,h}^{*4}\right)=3+O_P(h)$, *as* $h\to 0$;
(a5) $E^*\left(S_{\beta,h}^{*}\sqrt{h^{-1}}(\tilde{B}_{1h}^{*}-\hat{B}_{1h})\right)=\frac{\hat{A}_{1h}}{\sqrt{\hat{B}_{1h}}}$; *(a6)* $E^*\left(S_{\beta,h}^{*2}\sqrt{h^{-1}}\left(\tilde{B}_{1h}^{*}-\hat{B}_{1h}\right)\right)=O_P\left(\sqrt{h}\right)$ *as* $h\to 0$; *and*
(a7) $E^*\left(S_{\beta,h}^{*3}\sqrt{h^{-1}}\left(\tilde{B}_{1h}^{*}-\hat{B}_{1h}\right)\right)=3\frac{\hat{A}_{1h}}{\sqrt{\hat{B}_{1h}}}+O_P(h)$, *as* $h\to 0$.

Lemma B.10 *Suppose (1) and (2) hold with* $\alpha\equiv 0$ *and* W *independent of* Σ. *Then,*

$$\frac{\sqrt{h^{-1}}\left(\hat{B}_{1h}^{*}-\hat{B}_{1h}\right)}{\hat{B}_{1h}}=\frac{\sqrt{h^{-1}}\left(\tilde{B}_{1h}^{*}-\hat{B}_{1h}\right)}{\hat{B}_{1h}}-\frac{2\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_2}}S_{\beta,h}^{*}+O_{P^*}(\sqrt{h}),$$

in probability.

Lemma B.11 *Suppose (1) and (2) hold with* $\alpha\equiv 0$ *and* W *independent of* Σ. *Then,* $\kappa_1^*(T_{\beta,h}^{*})=\sqrt{h}\left(-\frac{\hat{A}_{1h}}{2\hat{B}_{1h}^{3/2}}+\frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}}\hat{\Gamma}_2}\right)\equiv\sqrt{h}\kappa_{1,h}^{*}$, *and* $\kappa_3^*(T_{\beta,h}^{*})=\sqrt{h}\left(-\frac{2\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}}+\frac{6\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}}\hat{\Gamma}_2}\right)+O_P(h)\equiv\sqrt{h}\kappa_{3,h}^{*}+O_P(h)$, *as* $h\to 0$.

Proof of Lemma B.8. For (a1), note that $E^*\left(y_{2i}^{*}\varepsilon_i^{*}\right)=h\sum_{i=1}^{1/h}y_{2i}\hat{\varepsilon}_i=0$ from the OLS first order condition that defines $\hat{\beta}_{12}$. The remaining results follow from the properties of the i.i.d bootstrap (in particular, the independence between $y_{2i}^{*}\varepsilon_i^{*}$ and $y_{2j}^{*}\varepsilon_j^{*}$ for $i\neq j$) and the definitions of $\hat{A}_{1h}$ and $\hat{B}_{1h}$. For instance, for (a2), $E^*\left(y_{2i}^{*2}\varepsilon_i^{*2}\right)=h\sum_{i=1}^{1/h}y_{2i}^2\hat{\varepsilon}_i^2=h^2\hat{B}_{1h}$, given the definition of $\hat{B}_{1h}$.
Proof of Lemma B.9. We apply Lemma B.8. For instance, for (a1) $E^*\left(S_{\beta,h}^{*}\right)=\frac{h^{-1/2}}{\hat{B}_1^{1/2}}E^*\left(\sum_{i=1}^{1/h}y_{2i}^{*}\varepsilon_i^{*}\right)=0$, given Lemma B.8 (a5). (a2) through (a4) follow similarly, using Lemma B.8, parts (a6) through (a8), respectively. For (a5)-(a7), use (a9)-(a11) of Lemma B.8 and the fact that $S_{\beta,h}^{*}$ and $\tilde{B}_{1h}^{*}-\hat{B}_{1h}$ are given by (10) and (11), respectively.

Proof of Lemma B.10. Noting that $\hat{\varepsilon}_i^* = \varepsilon_i^* - \left(\hat{\beta}_{12}^* - \hat{\beta}_{12}\right) y_{2i}^*$, we can write

$$\begin{aligned}
\hat{B}_{1h}^* &= h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*^2}\hat{\varepsilon}_i^{*^2} = h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*^2}(\varepsilon_i^* - \left(\hat{\beta}_{12}^* - \hat{\beta}_{12}\right) y_{2i}^*)^2 \\
&= h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*^2}\varepsilon_i^{*2} - 2\left(\hat{\beta}_{12}^* - \hat{\beta}_{12}\right) h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*^3}\varepsilon_i^* + \left(\hat{\beta}_{12}^* - \hat{\beta}_{12}\right)^2 h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*^4}.
\end{aligned}$$

Since $\hat{\beta}_{12}^* - \hat{\beta}_{12} = O_{P^*}\left(\sqrt{h}\right)$ and $h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*^4}$, in probability, the last term is $O_{P^*}(h)$, in probability. Next, note that $h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*^3}\varepsilon_i^* = \hat{A}_{0h} + O_{P^*}\left(\sqrt{h}\right)$, which together with $\hat{\beta}_{12}^* - \hat{\beta}_{12} = O_{P^*}\left(\sqrt{h}\right)$, implies that

$$\hat{B}_{1h}^* = h^{-1}\sum_{i=1}^{1/h} y_{2i}^{*^2}\varepsilon_i^{*2} - 2\left(\hat{\beta}_{12}^* - \hat{\beta}_{12}\right)\hat{A}_{0h} + O_{P^*}(h).$$

By definition, the first term is $\tilde{B}_{1h}^*$, and we can use $\hat{\beta}_{12}^* - \hat{\beta}_{12} = \frac{\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*}{\hat{\Gamma}_2^*}$ and the definition of $S_{\beta,h}^*$ to write

$$\begin{aligned}
\hat{B}_{1h}^* &= \tilde{B}_{1h}^* - 2\sqrt{h}\frac{\hat{A}_{0h}\sqrt{\hat{B}_{1h}}}{\hat{\Gamma}_2}\left(\frac{\sqrt{h^{-1}}\sum_{i=1}^{1/h} y_{2i}^*\varepsilon_i^*}{\sqrt{\hat{B}_1}}\right)\left(1 + \frac{\hat{\Gamma}_2^* - \hat{\Gamma}_2}{\hat{\Gamma}_2}\right)^{-1} + O_{P^*}(h), \\
&= \tilde{B}_{1h}^* - 2\sqrt{h}\frac{\hat{A}_{0h}\sqrt{\hat{B}_{1h}}}{\hat{\Gamma}_2}S_{\beta,h}^*\left(1 - \frac{\hat{\Gamma}_2^* - \hat{\Gamma}_2}{\hat{\Gamma}_2} + O_{P^*}(h)\right) + O_{P^*}(h), \\
&= \tilde{B}_{1h}^* - 2\sqrt{h}\frac{\hat{A}_{0h}\sqrt{\hat{B}_1}}{\hat{\Gamma}_2}S_{\beta,h}^* + \underbrace{2\sqrt{h}\frac{\hat{A}_{0h}\sqrt{\hat{B}_1}}{\hat{\Gamma}_2}S_{\beta,h}^*\left(\frac{\hat{\Gamma}_2^* - \hat{\Gamma}_2}{\hat{\Gamma}_2}\right)}_{O_{P^*}(h)} + O_{P^*}(h),
\end{aligned}$$

where we have used the fact that $S_{\beta,h}^* = O_{P^*}(1)$ and $\hat{\Gamma}_2^* - \hat{\Gamma}_2 = O_{P^*}\left(\sqrt{h}\right)$, in probability. Adding and subtracting appropriately gives the result.

Proof of Lemma B.11. By (9) and Lemma B.10,

$$T_{\beta,h}^* = S_{\beta,h}^*\left(1 + \sqrt{h}\left(\frac{\sqrt{h^{-1}}\left(\tilde{B}_{1h}^* - \hat{B}_{1h}\right)}{\hat{B}_{1h}} - \frac{2\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}}\hat{\Gamma}_2}S_{\beta,h}^* + O_{P^*}(\sqrt{h})\right)^{-1/2}\right).$$

Following the proof of Proposition 5.2.(a), for any fixed integer k, we have that

$$T_{\beta,h}^{*^k} = S_{\beta,h}^{*^k}\left(1 - \sqrt{h}\frac{k}{2}\frac{\sqrt{h^{-1}}}{\hat{B}_{1h}}\left(\tilde{B}_{1h}^* - \hat{B}_{1h}\right) + \sqrt{h}k\frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}}\hat{\Gamma}_2}S_{\beta,h}^*\right) + O_P(h) \equiv \tilde{T}_{\beta,h}^{*^k} + O_P(h).$$

For $k = 1, 2, 3$, the moments of $\tilde{T}_h^{*k}$ are given by

$$
\begin{aligned}
E^{*}(\tilde{T}^{*}_{\beta,h}) &= E^{*}(S^{*}_{\beta,h}) - \sqrt{h}\frac{1}{2}\frac{1}{\hat{B}_{1h}}E^{*}\left(S^{*}_{\beta,h}\sqrt{h^{-1}}\left(\tilde{B}^{*}_{1h}-\hat{B}_{1h}\right)\right) + \sqrt{h}\frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{2}}}E^{*}(S^{*^{2}}_{\beta,h}),\\
E^{*}(\tilde{T}^{*^{2}}_{\beta,h}) &= E^{*}(S^{*^{2}}_{\beta,h}) - \sqrt{h}\frac{1}{\hat{B}_{1h}}E^{*}\left(S^{*^{2}}_{\beta,h}\sqrt{h^{-1}}\left(\tilde{B}^{*}_{1h}-\hat{B}_{1h}\right)\right) + \sqrt{h}\frac{2\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{2}}}E^{*}(S^{*^{3}}_{\beta,h}),\\
E^{*}(\tilde{T}^{*^{3}}_{\beta,h}) &= E^{*}(S^{*^{3}}_{\beta,h}) - \sqrt{h}\frac{3}{2}\frac{1}{\hat{B}_{1h}}E^{*}\left(S^{*^{3}}_{\beta,h}\sqrt{h^{-1}}\left(\tilde{B}^{*}_{1h}-\hat{B}_{1h}\right)\right) + \sqrt{h}\frac{3\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{2}}}E^{*}(S^{*^{4}}_{\beta,h}).
\end{aligned}
$$

Lemma B.9 implies that

$$
\begin{aligned}
E^{*}(\tilde{T}^{*}_{\beta,h}) &= -\sqrt{h}\frac{1}{2}\frac{1}{\hat{B}_{1h}}\frac{\hat{A}_{1h}}{\sqrt{\hat{B}_{1h}}} + \sqrt{h}\frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{2}}} = \sqrt{h}\left(-\frac{1}{2}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} + \frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{2}}}\right)\\
E^{*}\left(\tilde{T}^{*}_{\beta,h}\right) &= 1 + O_{P}\left(h\right),\\
E^{*}(\tilde{T}^{*^{3}}_{\beta,h}) &= \sqrt{h}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} - \sqrt{h}\frac{9}{2}\frac{1}{\hat{B}_{1h}}\frac{\hat{A}_{1h}}{\sqrt{\hat{B}_{1h}}} + \sqrt{h}\frac{9\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{2}}} = \sqrt{h}\left(-\frac{7}{2}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} + 9\frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{2}}}\right).
\end{aligned}
$$

Thus $\kappa_{1}^{*}(T^{*}_{\beta,h}) = E^{*}(\tilde{T}^{*}_{\beta,h}) = \sqrt{h}\left(-\frac{1}{2}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} + \frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{2}}}\right) \equiv \sqrt{h}\kappa^{*}_{1,h}$, and

$$
\begin{aligned}
\kappa_{3}^{*}(T^{*}_{\beta,h}) &= E^{*}(\tilde{T}^{*^{3}}_{\beta,h}) - 3E^{*}(\tilde{T}^{*^{2}}_{\beta,h})E^{*}(\tilde{T}^{*}_{\beta,h}) + 2[E^{*}(\tilde{T}^{*}_{\beta,h})]^{3}\\
&= \sqrt{h}\left(-\frac{7}{2}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} + 9\frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{2}}}\right) - 3\sqrt{h}\left(-\frac{1}{2}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} + \frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{2}}}\right) + O_{P}\left(h\right)\\
&= \sqrt{h}\left(-2\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}} + 6\frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_{2}}}\right) + O_{P}\left(h\right) \equiv \sqrt{h}\kappa^{*}_{3,h} + O_{P}\left(h\right).
\end{aligned}
$$

Proof of Theorem 5.2(b).

By Theorem 4 by BN-S (2004), and because $\hat{\beta}_{12} \xrightarrow{P} \beta_{12}$, we have that $\hat{B}_{1h} \xrightarrow{P} B^{*}$, and $\hat{A}_{0h} \xrightarrow{P} 3\int_{0}^{1}\left(\Sigma_{12}(u) - \beta_{12}\Sigma_{2}^{2}(u)\right)du = 3A_{0}$. Similarly, we can show that

$$
\hat{A}_{1h} = h^{-2}\sum_{i=1}^{1/h}\left(\varepsilon_{i}y_{2i}\right)^{3} + o_{P}(1) = h^{-2}\sum_{i=1}^{1/h}E\left(\left(\varepsilon_{i}y_{2i}\right)^{3}\right) + R_{h} + o_{P}\left(1\right),
$$

where $R_{h} = h^{-2}\sum_{i=1}^{1/h}\left(\varepsilon_{i}y_{2i}\right)^{3} - E\left(\left(\varepsilon_{i}y_{2i}\right)^{3}\right)$. $E\left(R_{h}\right) = 0$ and by straightforward calculations, $Var\left(h^{-2}\sum_{i=1}^{1/h}\left(\varepsilon_{i}y_{2i}\right)^{3}\right) = O\left(h\right) = o\left(1\right)$, which implies that $R_{h} = o_{P}\left(1\right)$. By tedious but simple algebra we can verify that

$$
h^{-2}\sum_{i=1}^{1/h}E\left(\left(\varepsilon_{i}y_{2i}\right)^{3}\right) = h^{-2}\sum_{i=1}^{1/h}\left(\begin{array}{c} 6\Gamma_{12i}^{3} + 9\Gamma_{1i}\Gamma_{12i}\Gamma_{2i} - 36\beta_{12}\Gamma_{12i}^{2}\Gamma_{2i}\\ -9\beta_{12}\Gamma_{1i}\Gamma_{2i}^{2} + 45\beta_{12}^{2}\Gamma_{12i}\Gamma_{2i}^{2} - 15\beta_{12}^{3}\Gamma_{2i}^{3}\end{array}\right).
$$

By Lemma B.3, this last expression converges to

$$
\int_{0}^{1}\left(\begin{array}{c} 6\Sigma_{12}^{3}\left(u\right) + 9\Sigma_{1}\left(u\right)\Sigma_{12}\left(u\right)\Sigma_{2}\left(u\right) - 36\beta_{12}\Sigma_{12}^{2}\left(u\right)\Sigma_{2}\left(u\right)\\ -9\beta_{12}\Sigma_{1}\left(u\right)\Sigma_{2}^{2}\left(u\right) + 45\beta_{12}^{2}\Sigma_{12}\left(u\right)\Sigma_{2}^{2}\left(u\right) - 15\beta_{12}^{3}\Sigma_{2}^{3}\left(u\right)\end{array}\right)du = \frac{3}{2}A_{1}^{*},
$$

proving that $\hat{A}_{1h} \to^{P} \frac{3}{2}A_{1}^{*}$. Thus, using Proposition B.11, we get that

$$p\lim \kappa^*_{1,h} = p\lim\left(-\frac{1}{2}\frac{\hat{A}_{1h}}{\hat{B}_{1h}^{3/2}}+\frac{\hat{A}_{0h}}{\sqrt{\hat{B}_{1h}\hat{\Gamma}_2}}\right) = -\frac{1}{2}\frac{\frac{3}{2}A_1^*}{B^{*3/2}}+\frac{3A_0}{\sqrt{B^*}\Gamma_2} = \frac{3}{4}\left(\frac{4A_0}{\sqrt{B^*}\Gamma_2}-\frac{A_1^*}{B^{*3/2}}\right) \equiv \frac{3}{4}\left(H_1^*-H_2^*\right).$$

Similarly,

$$p\lim \kappa^*_{3,h} = \left(-2\frac{\frac{3}{2}A_1^*}{B^{*3/2}}+6\frac{3A_0}{\sqrt{B^*}\Gamma_2}\right) = \left(\frac{3\times 3}{2}H_1^*-3H_2^*\right) = \frac{3}{2}\left(3H_1^*-2H_2^*\right).$$

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