Bootstrapping Realized Volatility*

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Abstract

In this paper, we propose bootstrap methods for statistics evaluated on high frequency data such as realized volatility. The bootstrap is as an alternative inference tool to the first-order asymptotic theory recently derived in the literature. We consider the i.i.d. bootstrap and the wild bootstrap (WB) and prove their first-order asymptotic validity. We then use Edgeworth expansions and Monte Carlo simulations to compare the accuracy of the bootstrap with the existing first-order feasible asymptotic theory. Our Edgeworth expansions show that the i.i.d. bootstrap provides a second-order asymptotic refinement when volatility is constant. Under stochastic volatility, the i.i.d. bootstrap is not able to match the cumulants through third order and therefore the i.i.d. bootstrap error has the same rate of convergence as the error implied by the standard normal approximation. Nevertheless, we show through simulations and using Edgeworth expansions that the i.i.d. bootstrap is still able to provide a smaller error than that of the standard normal approximation. For the possibly time-varying volatility case, the WB provides a second-order asymptotic refinement, provided we choose the external random variable used to construct the wild bootstrap pseudo data appropriately. Monte Carlo simulations suggest that both the i.i.d. bootstrap and the appropriately chosen wild bootstrap improve upon the first-order asymptotic theory in finite samples.

Keywords: Realized volatility, i.i.d. bootstrap, wild bootstrap, Edgeworth expansions.

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1 Introduction

The increasing availability of high frequency financial data has contributed to the popularity of realized volatility as a measure of volatility in empirical finance. Realized volatility is simple to compute (it is equal to the sum of squared high frequency returns) and it is a consistent estimator of integrated volatility under general nonparametric conditions (see e.g. Andersen, Bollerslev and Diebold (2002) for a survey of the properties of realized volatility).

Recently, a series of papers including Jacod (1994), Jacod and Protter (1998) and Barndorff-Nielsen and Shephard (henceforth BN-S) (2002) have developed an asymptotic theory for realized volatility-like measures. In particular, for a rather general stochastic volatility model, these authors establish a central limit theorem for realized volatility over a fixed interval of time, for instance a day, as the number of intraday returns increases to infinity. Similarly, BN-S (2003, 2004b) show that a CLT applies to empirical measures based on powers of intraday returns (realized power variation) and products of powers of absolute returns (e.g. bipower variation). More recently, BN-S (2005) provide a joint asymptotic distribution theory for the realized volatility and the realized bipower variation, and show how to use this distribution to test for the presence of jumps in asset prices.

In this paper, we propose bootstrap methods for statistics evaluated on high frequency data such as realized volatility. Our main motivation for using the bootstrap is to improve upon the existing asymptotic mixed normal approximations. The bootstrap can be particularly valuable in the context of high frequency data-based measures. Current practice is to use a moderate number of intraday returns, e.g. 30-minute returns, in computing realized volatility to avoid microstructure biases.[1] Sampling at long horizons may limit the value of the asymptotic approximations derived under the assumption of an infinite number of intraday returns. The Monte Carlo simulations in BN-S (2004a) show that the raw feasible asymptotic theory for realized volatility can be a poor guide to the finite sample distribution of the standardized realized volatility. BN-S (2004a) propose a logarithmic version of the raw statistic and show it has improved finite sample properties. In a different context, BN-S (2004d) use the Fisher-z transformation for realized correlation. However, analytical transformations may not be available for other applications, for instance for the realized regression parameters such as the realized beta. Similarly, Huang and Tauchen (2005) show that jump tests based on the (scaled) difference between realized volatility and bipower variation can have potential size problems for certain data generating processes when testing for jumps over a long time span.

We focus on realized volatility and ask whether we can improve upon the existing first-order asymptotic theory by relying on the bootstrap for inference on integrated volatility in the absence

[1]Recently, a number of papers has studied the impact of microstructure noise on realized volatility, including Aït-Sahalia, Mykland and Zhang (2005), Bandi and Russell (2004), Hansen and Lunde (2005), Zhang, Mykland and Aït-Sahalia (2004), Barndorff-Nielsen, Hansen, Lunde and Shephard (2004), and Zhang (2004). In particular, these papers propose alternative estimators of integrated volatility that are robust to microstructure noise and that do not coincide with realized volatility. Bootstrapping such measures is an interesting extension of our results, which we will consider elsewhere.

of microstructure noise. Since the effects of microstructure noise are more pronounced at very high frequencies, we expect the bootstrap to be a useful tool of inference based on realized volatility when sampling at moderate frequencies such as 30 minutes horizon, as is often done in practice. For instance, in their seminal paper, Andersen, Bollerslev, Diebold and Labys (2003) consider an empirical application based on 30 minutes intraday returns for three major spot exchange rates.

We propose and analyze two bootstrap methods for realized volatility: an i.i.d. bootstrap and a wild bootstrap. The i.i.d. bootstrap (cf. Efron, 1979) generates bootstrap pseudo intraday returns by resampling with replacement the original set of intraday returns. The wild bootstrap observations are generated by multiplying each original intraday return by an i.i.d. draw from a distribution that is completely independent of the original data. The wild bootstrap was introduced by Wu (1986), and further studied by Liu (1988) and Mammen (1993), in the context of cross-section linear regression models subject to unconditional heteroskedasticity in the error term. Both methods are well known in the bootstrap literature.[2] We are the first to the best of our knowledge to propose their application to realized volatility and to study their theoretical properties under a general stochastic volatility model. Zhang, Mykland and Aït-Sahalia (2004) and Zhang (2004) consider an application of the subsampling method to realized volatility under stochastic volatility. In particular, they use subsampling plus averaging to bias correct the realized volatility measure when microstructure noise is present. Our main goal here is to use the bootstrap to estimate the entire distribution (as opposed to just the bias) of realized volatility.

In a benchmark model in which the volatility is constant and therefore intraday returns are i.i.d., the i.i.d. bootstrap is the natural method of choice. In practice, volatility is highly persistent, especially over a daily horizon, implying that it is at least locally nearly constant. Hence we may expect the i.i.d. bootstrap to provide a good approximation even under stochastic volatility. Indeed, we show here that this approach remains valid under time-varying volatility if we center and scale the realized volatility measure appropriately.[3]

The wild bootstrap (WB) is an alternative approach that explicitly takes into account the conditional heteroskedasticity underlying stochastic volatility models. We show that this method is first-order asymptotically valid under conditions similar to BN-S, when the bootstrap statistic is appropriately centered and standardized.

A popular bootstrap for serially dependent data is the block bootstrap. In our context, intraday returns are (conditionally on the volatility path) independent, and this implies that blocking is not necessary for asymptotic refinements of the bootstrap. The issue here is heteroskedasticity and not serial correlation.

[2] Gonçalves and Kilian (2004) apply both methods in the context of autoregressions subject to conditional heteroskedasticity of unknown form.

[3] Recently, Gonçalves and Vogelsang (2004) show the validity of the i.i.d. bootstrap for t-tests based on heteroskedastic and autocorrelation consistent (HAC) variance estimators when data are serially dependent. There the i.i.d. bootstrap is applied in a naive fashion, without any centering or scaling correction.

We use Monte Carlo simulations and formal Edgeworth expansions to compare the accuracy of the bootstrap and the normal approximations at estimating confidence intervals for integrated volatility. Both one-sided and two-sided intervals are considered. One-term Edgeworth expansions show that the i.i.d. bootstrap provides a second-order asymptotic refinement when volatility is constant. This is as expected given that returns are i.i.d. under this simple model. Our simulations also suggest that the i.i.d. bootstrap outperforms the asymptotic normal approximation under more general stochastic volatility models. Based on our one-term Edgeworth expansions, we prove that although the rate of convergence of the i.i.d. bootstrap error is the same as that of the error of the normal approximation when volatility is stochastic, the absolute magnitude of the coefficients describing the i.i.d. bootstrap error is smaller than that of the coefficients correspoonding to the Edgeworth expansion for the original statistic (cf. Shao and Tu (1995, Section 3.3) and Davidson and Flachaire (2001) for a similar argument). This can explain the good finite sample behavior of the i.i.d. bootstrap for one-sided intervals in our simulations. One-term Edgeworth expansions for the WB statistic show that it provides a second-order asymptotic refinement when volatility is heterogeneous if we choose the external random variable used to construct the wild bootstrap observations appropriately. We propose an appropriate choice for this external random variable. Our Monte Carlo simulations show that the WB implemented with this choice outperforms the first-order asymptotic normal approximation. The comparison between this WB and the i.i.d. bootstrap favors the i.i.d. bootstrap, which is the preferred method in the context of our study.

Motivated by the good finite sample performance of the bootstrap for two-sided intervals, we also investigate the ability of the bootstrap to provide a third-order asymptotic refinement over the normal approximation. Our results show that although the i.i.d. and the WB bootstrap can provide second-order asymptotic refinements, third-order refinements are not possible. These theoretical predictions are not confirmed by our simulations, which show that both the i.i.d. and the WB outperform the normal approximation when estimating two-sided symmetric intervals for integrated volatility.

The remainder of this paper is organized as follows. In Section 2, we introduce the setup, review the existing first-order asymptotic theory and state regularity conditions. We also introduce the Monte Carlo design underlying all simulations in the paper and discuss the coverage probability results for the first-order asymptotic approach for nominal 95% one-sided and two-sided symmetric intervals. In Section 3, we introduce the bootstrap methods and establish their first-order asymptotic validity under the regularity conditions stated in Section 2. In Section 4 we discuss the second-order accuracy of the bootstrap whereas Section 5 deals with third-order accuracy results. These sections also contain a discussion of the Monte Carlo results for bootstrap one-sided and two-sided intervals, respectively. Section 6 concludes. In Appendix A we derive the asymptotic expansions for the cumulants of the original and bootstrap statistics. We also provide several auxiliary results. In Appendix B we collect all the proofs of the results appearing in Sections 3 through 5.

2 The first-order asymptotic approach

2.1 Setup

We consider the following continuous-time model for the log price process $\{\log S_t : t \geq 0\}$:

$$d \log S_t = \mu_t dt + v_t dW_t, \tag{1}$$

where W_t denotes a standard Brownian motion, μ_t denotes a drift term, and v_t a volatility term. For simplicity, we will assume that $\mu_t = 0$ for all t. The drift term is of order dt, which is smaller than the order $(dt)^{1/2}$ of the volatility term in (1) (see e.g. Andersen, Bollerslev and Diebold (2002) for a discussion of this result). Hence the drift term is negligible at high frequencies. Our model is thus given as

$$d \log S_t = v_t dW_t, \tag{2}$$

where $v_t > 0$ is in general a time-varying stochastic process. For the theoretical results, we assume the independence between the stochastic volatility process v_t and the Brownian motion W_t, i.e. we assume no leverage effects. Nevertheless our Monte Carlo study includes models with leverage and drift. A benchmark model useful for comparisons is the time-invariant diffusion model where $v_t = v$ for all $t > 0$. Given (2), the daily return for any day t is defined as

$$r_t \equiv \log S_t - \log S_{t-1} = \int_{t-1}^{t} v_u dW_u, \quad t = 1, 2, \ldots \ .$$

Since t is fixed in our analysis, we let $t = 1$ throughout without loss of generality. We can define intraday returns (for any given day) at horizon h as follows:

$$r_i \equiv \log S_{ih} - \log S_{(i-1)h} = \int_{(i-1)h}^{ih} v_u dW_u, \ \text{ for } i = 1, \ldots, 1/h,$$

with $1/h$ an integer. To simplify notation, we omit the dependence of intraday returns on the horizon h. When v is constant, intraday returns are i.i.d. $N\left(0, v^2 h\right)$, i.e. we have that

$$r_i = \int_{(i-1)h}^{ih} v_u dW_u = v\left(W_{ih} - W_{(i-1)h}\right) \equiv v u_i \sim \text{i.i.d. } N\left(0, v^2 h\right),$$

where $u_i \equiv W_{ih} - W_{(i-1)h} \sim$ i.i.d. $N(0, h)$ for $i = 1, \ldots, 1/h$. When volatility is time-varying and stochastic, intraday returns are (conditionally on the path of the volatility process v) independent but heteroskedastic, i.e. we can write $r_i = \sigma_i u_i$, where $\sigma_i^2 \equiv \int_{(i-1)h}^{ih} v_u^2 du$, and $u_i \sim$ i.i.d. $N(0,1)$. In this case, and conditionally on the path of volatility, $r_i \sim N\left(0, \sigma_i^2\right)$ for $i = 1, \ldots, 1/h$.

The parameter of interest is the integrated volatility over a day,

$$IV = \int_0^1 v_u^2 du,$$

which we assume to be finite. A simple estimator of the integrated volatility is the sum of squared intraday returns, known as realized volatility:

$$RV = \sum_{i=1}^{1/h} r_i^2.$$

This estimator is under certain assumptions (including absence of microstructure noise) a consistent estimator of IV when the number of intraday observations increases to infinity (i.e. if $h \to 0$). This result is theoretically justified by the theory of quadratic variation.

We introduce some notation. For any $q > 0$, define the realized q-th order power variation (cf. BN-S (2004b)) as

$$R_q = h^{-q/2+1} \sum_{i=1}^{1/h} |r_i|^q .$$

Note that for $q = 2$, $R_2 = RV$. Similarly, for any $q > 0$, define the integrated power volatility

$$\overline{\sigma^q} \equiv \int_0^1 v_u^q du.$$

Recently, Barndorff-Nielsen and Shephard (2004b, Theorem 1) show that $R_q \xrightarrow{P} \mu_q \overline{\sigma^q}$, where $\mu_q = E|Z|^q$, $Z \sim N(0,1)$, for a broad class of stochastic volatility models.

2.2 The existing theory

Our goal is to perform inference on the integrated volatility, e.g., we would like to form a confidence interval for IV. One approach is to rely on first-order asymptotic theory. This has been the standard approach in the realized volatility literature. We describe this approach here.

For the theory in this paper, we follow BN-S (2004b, 2005) and assume the following additional regularity condition on the stochastic volatility process.

Assumption (V) The volatility process v is (pathwise) càdlàg, bounded away from zero, and satisfies the following regularity condition:

$$\lim_{h \to 0} h^{1/2} \sum_{i=1}^{1/h} \left| v_{\eta_i}^r - v_{\xi_i}^r \right| = 0,$$

for some $r > 0$ (equivalently for every $r > 0$) and for any η_i and ξ_i such that $0 \leq \xi_1 \leq \eta_1 \leq h \leq \xi_2 \leq \eta_2 \leq 2h \leq \cdots \leq \xi_{1/h} \leq \eta_{1/h} \leq 1$.

As Barndorff-Nielsen, Jacod and Shephard (2004) note in their Remark 1, the càdlàg assumption implies that all powers of v are locally integrable with respect to Lebesgue measure, so that in particular $\int_0^1 v_u^q du < \infty$ for any $q > 0$. Under Assumption (V), v can exhibit jumps, intra-day seasonality and long-memory. Processes for $\{\log S_t\}$ satisfying (2) and Assumption (V) are a special case of the continuous stochastic volatility semimartingales.

Assumption (V) is stronger than required for the consistency of R_q for $\mu_q \overline{\sigma^q}$. It implies in particular that $R_q = \mu_q \overline{\sigma^q} + o_P\left(\sqrt{h}\right)$. It is also stronger than required to prove the central limit theorem for realized volatility (see e.g. Jacod (1994), Jacod and Protter (1998)). Under Assumption (V), BN-S (2004b, Theorem 3) show that $\overline{\sigma^q_h} - \overline{\sigma^q} = o_P\left(\sqrt{h}\right)$, where $\overline{\sigma^q_h} \equiv h^{1-q/2} \sum_{i=1}^{1/h} \left(\sigma_i^2\right)^{q/2}$ and $\sigma_i^2 \equiv \int_{(i-1)h}^{ih} v_u^2 du < \infty$, a result on which we rely subsequently to establish our bootstrap results. This is why we adopt Assumption (V) here.

The existing approach for constructing confidence intervals for IV relies on a CLT result for realized volatility derived by Jacod (1994), Jacod and Protter (1998) and Barndorff-Nielsen and Shephard (2002). In particular, under appropriate conditions, as $h \to 0$,

$$\frac{\sqrt{h^{-1}}\left(RV - IV\right)}{\sqrt{V}} \to^d N\left(0,1\right), \tag{3}$$

where

$$V = 2\int_0^1 v_u^4 du.$$

The result given in (3) is not immediately useful in practice because the asymptotic variance V depends on the unobserved quantity $\int_0^1 v_u^4 du$. Barndorff-Nielsen and Shephard (see e.g. 2002, 2003, 2004b) show that

$$T_h \equiv \frac{\sqrt{h^{-1}}\left(RV - IV\right)}{\sqrt{\hat{V}}} \to^d N\left(0,1\right), \tag{4}$$

where

$$\hat{V} = \frac{2}{3} h^{-1} \sum_{i=1}^{1/h} r_i^4 \equiv \frac{2}{3} R_4$$

is a consistent estimator of V. By replacing V with $\hat{V}$, T_h becomes a feasible statistic. We follow Barndorff-Nielsen and Shephard and refer to this approach as the feasible (first-order) asymptotic theory approach.

2.3 Simulations results for the feasible first-order asymptotic approach

Next we assess by simulation the accuracy of the feasible asymptotic theory of BN-S when computing 95% confidence intervals for IV. Our results confirm the previous simulation evidence by BN-S (2002, 2004a). In particular, we find that this approach leads to important coverage probability distortions when returns are not sampled too frequently. This motivates the bootstrap as an alternative method of inference in this context.

Our Monte Carlo design is inspired by Andersen, Bollerslev and Meddahi (2005). In particular, we consider the following stochastic volatility model

$$d\log S_t = \mu dt + v_t \left[\rho_1 dW_{1t} + \rho_2 dW_{2t} + \sqrt{1-\rho_1^2-\rho_2^2} dW_{3t}\right],$$

where W_{1t}, W_{2t} and W_{3t} are three independent standard Brownian motions. Our baseline models fix $\mu = \rho_1 = \rho_2 = 0$, implying that $d\log S_t = v_t dW_{3t}$ and no drift nor leverage effects exist.

We consider three different models for v_t. The first model is the log-normal diffusion reported in Andersen, Benzoni and Lund (2002) where v_t is such that

$$d\log v_t^2 = -0.0136\left[0.8382 + \log v_t^2\right]dt + 0.1148dW_{1t}.$$

Our second model is the GARCH(1,1) diffusion studied by Andersen and Bollerslev (1998):

$$dv_t^2 = 0.035\left(0.636 - v_t^2\right)dt + 0.144v_t^2 dW_{1t}.$$

Finally, we consider the two-factor diffusion model analyzed by Chernov et al. (2003) (and recently studied in the context of the nonparametric jump test by Huang and Tauchen (2005)):

$$\begin{aligned} v_t &= \text{s-exp}\left(-1.2 + 0.04v_{1t}^2 + 1.5v_{2t}^2\right) \\ dv_{1t}^2 &= -0.00137v_{1t}^2 dt + dW_{1t} \\ dv_{2t}^2 &= -1.386v_{2t}^2 dt + \left(1 + 0.25v_{2t}^2\right)dW_{2t}. \end{aligned}$$

According to this model, the stochastic volatility factor v_{2t}^2 has a feedback term in the diffusion effect.[4] This diffusion model has continuous sample paths but can imply sample paths for the price process that look like jumps. Chernov et. al. (2003) find that it fits well the S&P500 returns.

Our baseline models assume no drift and no leverage effects and satisfy our regularity conditions. Tables 1 through 3 report results for these models. Although our theory does not apply to stochastic volatility models with drift and/or leverage effects, we include in the Monte Carlo simulation three models for which $\mu \neq 0$ and for which leverage effects exist. The results are reported in Tables 4 and 5. Following Andersen, Bollerslev and Meddahi (2005), for the one-factor log-normal and GARCH(1,1) diffusions we consider

$$d\log S_t = 0.0314dt + v_t\left[-0.576dW_{1t} + \sqrt{1 - 0.576^2}dW_{3t}\right],$$

whereas for the two-factor diffusion model we follow Huang and Tauchen (2005) and assume that

$$d\log S_t = 0.030dt + v_t\left[-0.30dW_{1t} - 0.30dW_{2t} + \sqrt{1 - 0.30^2 - 0.30^2}dW_{3t}\right].$$

We study the finite sample performance of one-sided and two-sided 95% level asymptotic theory-based intervals. While one-sided intervals of IV are not as common in the econometrics literature, Mykland (2000, 2002, 2003) shows that these intervals are important for hedging in the context of option pricing.

The lower one-sided $100\left(1-\alpha\right)\%$ level confidence interval for IV based on the feasible asymptotic theory of BN-S is given by:

$$IC_{Feas,1-\alpha}^{(1)} = \left(0, RV - z_\alpha\sqrt{h\hat{V}}\right),$$

[4] The function s-exp is the usual exponential function with a linear growth function splined in at high values of its argument: s-exp$(x) = \exp(x)$ if $x \leq x_0$ and s-exp$(x) = \frac{\exp(x_0)}{\sqrt{x_0}}\sqrt{x_0 - x_0^2 + x^2}$ if $x > x_0$, with $x_0 = \log(1.5)$.

where z_α is the α-level critical value of the standard normal distribution. When $\alpha = 0.05$, $z_{0.05} = -1.645$. The two-sided $100\,(1-\alpha)\,\%$ level interval for IV is given by:

$$IC^{(2)}_{Feas,1-\alpha} = \left(RV - z_{1-\alpha/2}\sqrt{h\hat{V}}, RV + z_{1-\alpha/2}\sqrt{h\hat{V}}\right),$$

where $z_{1-\alpha/2}$ is the 97.5% critical value of the standard normal distribution when $\alpha = 0.05$, i.e. $z_{0.975} = 1.96$. This interval is symmetric about RV because the normal distribution is symmetric.

As a way of improving upon their feasible asymptotic theory approach, BN-S (2002) suggest to use a logarithmic version of this result. For comparison purposes, we also report results for confidence intervals based on this logarithmic transformation of T_h. These are referred to as "log" (as opposed to "raw" for the intervals described above) and are of the following form:

$$\begin{aligned} IC^{(1)}_{\log-feas,1-\alpha} &= \left(-\infty, \log(RV) - z_\alpha\sqrt{\frac{h\hat{V}}{RV^2}}\right) \\ IC^{(2)}_{\log-feas,1-\alpha} &= \left(\log(RV) - z_{1-\alpha/2}\sqrt{\frac{h\hat{V}}{(RV)^2}}, \log(RV) + z_{1-\alpha/2}\sqrt{\frac{h\hat{V}}{(RV)^2}}\right), \end{aligned}$$

where z_α and $z_{1-\alpha/2}$ are defined as before.

We compute the actual coverage probabilities of all these confidence intervals for each of the stochastic volatility models described above. We report results across 10,000 replications for six different sample sizes: $1/h = 1152, 576, 288, 96, 48$ and 12, corresponding to "1.25-minute", "2.5-minute", "5-minute", "15-minute", "half-hour", "2-hour" returns. Tables 1 and 2 contain results for the baseline models, for one-sided and two-sided symmetric intervals, respectively. (These tables also include results for the bootstrap methods but those results will be discussed later in Sections 4 and 5.) Tables 4 and 5 contain results for one-sided and two-sided symmetric intervals for the models with drift and leverage, respectively.

For all DGP's, both one-sided and two-sided intervals tend to undercover. The degree of undercoverage is especially large for larger values of h, when sampling is not too frequent, and it is larger for one-sided than for two-sided intervals. For instance, if returns are sampled at every half-hour ($h = 1/48$), a 95% symmetric interval contains the true IV about 92% of the 10,000 replications for the log-normal and the GARCH diffusions. The corresponding one-sided 95% interval contains it about 90% of the time. For the two-factor diffusion, these rates decrease to 88% and 85%, respectively. This model implies overall lower coverage rates (hence larger coverage distortions) than the two other models, for all sample sizes and all confidence intervals. The simulations show that the results are robust to leverage and drift effects, as predicted by the theory of Jacod and Protter (1998) and BN-S (2004c). Finally, and confirming previous results by BN-S (2002, 2004a), although the logarithmic transformation helps reducing the coverage distortions, some distortions remain at the smaller sample sizes. For instance, for the two-factor diffusion model, the coverage rate of the two-sided 95% interval is 91% when sampling at half-hour horizon; for the one-sided interval, this rate is only equal to 88%.

3 The bootstrap

In this section we introduce the bootstrap methods and prove their first-order asymptotic validity under conditions similar to those used by Barndorff-Nielsen and Shephard.

3.1 The i.i.d. bootstrap

Consider the benchmark model in which volatility is constant, i.e. $v_t = v > 0$ for all t. In this case intraday returns at horizon h are i.i.d. $N\left(0, v^2 h\right)$, which suggests the use of an i.i.d. bootstrap. Although the i.i.d. bootstrap is motivated by this constant volatility model, we show here that it is asymptotically valid for general stochastic volatility models satisfying Assumption (V). This implies in particular that the i.i.d. bootstrap remains first-order asymptotically valid even when the volatility is not constant. Our Monte Carlo simulations in Sections 4 and 5 suggest that not only is the i.i.d. bootstrap valid but it outperforms the standard normal approximation. These sections also discuss the accuracy of the i.i.d. bootstrap approximation.

We denote the bootstrap intraday $h-$period returns as r_i^*. For the i.i.d. nonparametric bootstrap, we have that $r_i^* = r_{I_i}$, where $I_i \sim$ i.i.d. uniform on $\left\{1, \ldots, \frac{1}{h}\right\}$. This amounts to resampling with replacement the sample of $\frac{1}{h}$ intraday $h-$period returns. As usual in the bootstrap literature, we reserve the asterisk to denote bootstrap quantities. We let P^* denote the probability measure induced by the bootstrap, conditional on the original sample. Similarly, we let E^* (and Var^*) denote expectation (and variance) with respect to the bootstrap data, conditional on the original sample.[5]

The bootstrap realized volatility is the usual realized volatility, but evaluated on the bootstrap intraday returns:

$$RV^* = \sum_{i=1}^{1/h} r_i^{*2}.$$

It is easy to show that $E^*\left(RV^*\right) = RV$ and $V^* \equiv Var^*\left(\sqrt{h^{-1}} RV^*\right) = R_4 - RV^2$ (cf. Appendix A, Lemma A.5). We propose the following consistent estimator of the i.i.d. bootstrap variance V^*:

$$\hat{V}^* = h^{-1} \sum_{i=1}^{1/h} r_i^{*4} - \left(\sum_{i=1}^{1/h} r_i^{*2}\right)^2 \equiv R_4^* - RV^{*2}, \tag{5}$$

where for any $q > 0$ we let $R_q^* \equiv h^{-q/2+1} \sum_{i=1}^{1/h} \left|r_i^*\right|^q$. The i.i.d. bootstrap analogue of T_h is given by

$$T_h^* \equiv \frac{\sqrt{h^{-1}}\left(RV^* - RV\right)}{\sqrt{\hat{V}^*}}. \tag{6}$$

Note that although we center the bootstrap realized volatility around the sample realized volatility (since $E^*\left(RV^*\right) = RV$), the standard error that we propose to studentize the bootstrap statistic is

[5] Note that once we condition on the original intraday returns, adding the volatility path to the information set does not change the bootstrap probability measure. Thus, P^* can also be interpreted as being the probability measure induced by the bootstrap, conditional on the original sample *and* on the volatility path.

not of the same form as that used to studentize T_h. In particular, it is not given by $\frac{2}{3}h^{-1}\sum_{i=1}^{1/h} r_i^{*4}$, which would be the bootstrap analogue of $\hat{V}$. The naive estimator $\frac{2}{3}h^{-1}\sum_{i=1}^{1/h} r_i^{*4}$ is not consistent for V^* because it relies on a Gaussianity assumption that does not hold for the i.i.d. nonparametric bootstrap. In contrast, $\hat{V}^*$ given in (5) is a consistent estimator of V^*.

Theorem 3.1 *Consider DGP (2) and assume Assumption (V) holds. Let $\{r_i^* : i = 1, \ldots, 1/h\}$ denote an i.i.d. bootstrap sample of intraday returns. Then, as $h \to 0$,*

$$\sup_{x\in\mathbb{R}} \left|P^*\left(T_h^* \leq x\right) - \Phi\left(x\right)\right| \xrightarrow{P} 0, \tag{7}$$

where $\Phi(x) = P(Z \leq x)$, with $Z \sim N(0,1)$.

Theorem 3.1 establishes the first-order asymptotic validity of the i.i.d. bootstrap for general stochastic volatility models satisfying Assumption (V). In particular, (4) and (7) imply that as $h \to 0$

$$P^*\left(T_h^* \leq x\right) - P\left(T_h \leq x\right) = o_P(1),$$

uniformly in $x \in \mathbb{R}$. This result provides a theoretical justification for using the bootstrap distribution of T_h^* to estimate the quantiles of the distribution of T_h in the general context studied by BN-S. Sections 4 and 5 discuss the accuracy of this bootstrap approximation.

3.2 The wild bootstrap

As we argued previously, under stochastic volatility intraday returns are independent but heteroskedastic, conditional on the volatility path. This motivates our application of the WB in this context. Consider a sequence of i.i.d. external random variables η_i with moments given by $\mu_q^* = E^*|\eta_i|^q$, where $E^*(\cdot)$ denotes the expectation with respect to the distribution of η_i. The WB intraday returns are generated as $r_i^* = r_i\eta_i$, $i = 1, \ldots, 1/h$.

For applications we need to choose the distribution of η_i. As we will show here, this choice is not important for the first-order asymptotic validity of the WB as long as we carefully center and studentize the bootstrap realized volatility statistic. The choice of η_i implies a specific centering and studentization. Nevertheless, in order to prove an asymptotic refinement for the WB we need to choose the distribution of η_i appropriately. Section 4 proposes an appropriate choice for η_i.

Let RV^* denote the realized volatility evaluated on the WB pseudo data. Using the properties of the WB, we can show that $E^*(RV^*) = \mu_2^* RV$ and $V^* \equiv Var^*\left(\sqrt{h^{-1}}RV^*\right) = \left(\mu_4^* - \mu_2^{*2}\right)R_4$ (cf. Appendix A, Lemma A.2 and Remark 1). We propose the following consistent estimator of V^* :

$$\hat{V}^* = \left(\frac{\mu_4^* - \mu_2^{*2}}{\mu_4^*}\right)\left(h^{-1}\sum_{i=1}^{1/h} r_i^{*4}\right) \equiv \left(\frac{\mu_4^* - \mu_2^{*2}}{\mu_4^*}\right)R_4^*, \tag{8}$$

and define the WB studentized statistic T_h^* as

$$T_h^* = \frac{\sqrt{h^{-1}}\left(RV^* - \mu_2^* RV\right)}{\sqrt{\hat{V}^*}}. \tag{9}$$

Note that T_h^* is invariant to multiplication of η by a constant.

Suppose that we choose η_i such that $\mu_2^* = 1$ and $\mu_4^* = 3$, e.g. we let $\eta_i \sim N(0,1)$. Then

$$T_h^* = \frac{\sqrt{h^{-1}}\left(RV^* - RV\right)}{\sqrt{\hat{V}^*}}, \text{ with } \hat{V}^* = \frac{2}{3}R_4^*,$$

so that for this choice of η_i, the statistic T_h^* is of the same exact form as the original statistic T_h, with the bootstrap data replacing the original data. However, for other choices of η_i this is not necessarily the case. The bootstrap standard error and the centering of the bootstrap realized volatility depend on the particular choice of distribution for η_i through the moments μ_2^* and μ_4^*.

As long as we carefully center and studentize RV^* according to (8) and (9), the choice of η_i is not important for the first-order asymptotic validity of the WB, as the following theorem shows.

Theorem 3.2 *Consider DGP (2) and assume Assumption (V) holds. Let $\{r_i^* : i = 1, \ldots, 1/h\}$ denote a WB sample of intraday returns obtained with external random variables $\eta_i \sim$ i.i.d. such that $\mu_q^* = E^* |\eta_i|^q < \infty$ for $q = 2(2+\varepsilon)$ for some small $\varepsilon > 0$. Then, as $h \to 0$,*

$$\sup_{x \in \mathbb{R}} \left|P^*\left(T_h^* \leq x\right) - \Phi(x)\right| \xrightarrow{P} 0, \tag{10}$$

where $\Phi(x) = P(Z \leq x)$, with $Z \sim N(0,1)$, and T_h^* is the statistic defined in equations (8) and (9).

We note that although Theorems 3.1 and 3.2 explicitly rule out drift and leverage effects, the first-order asymptotic validity of our bootstrap methods can be easily extended to include these features. Indeed, as the proofs of these results reveal, we rely only on the convergence of R_q to $\mu_q \overline{\sigma^q}$, which holds under both drift and leverage effects (cf. Jacod and Protter (1998) and BN-S (2004c)). Similarly, following Mykland and Zhang (2001), we can extend our first-order results to non-equal spaced data. Here we abstract from these effects because their presence substantially complicates the higher-order accuracy of the bootstrap, which we will investigate next.

4 Second-order accuracy of the bootstrap

In this section we discuss second-order properties of the bootstrap. In particular, we investigate the ability of the bootstrap to provide an asymptotic refinement[6] through order $O(\sqrt{h})$ over the standard normal approximation when estimating the distribution function $P(T_h \leq x)$.

[6]We follow Horowitz (2001) and say that the bootstrap provides an "asymptotic refinement through order $O(h^r)$", for $r > 0$, when the bootstrap distribution of the statistic of interest is correct up to and including terms of order $O(h^r)$, with an estimation error of order $o(h^r)$. For $r = 1/2$, this amounts to matching the first term of an Edgeworth expansion (after the leading term given by the standard normal cdf), in which case the bootstrap is said to be second-order accurate.

Consider the following formal[7] one-term Edgeworth expansion for the distribution of T_h :

$$P\left(T_h \leq x\right) = \Phi\left(x\right) + \sqrt{h}\; q_1\left(x\right) \phi\left(x\right) + O\left(h\right), \tag{11}$$

uniformly over $x \in \mathbb{R}$, where $\Phi\left(x\right)$ is the standard normal cdf and $\phi\left(x\right)$ is the standard normal pdf. The function q_1 is a even function of x whose coefficients depend on the first three cumulants of T_h. In particular (see e.g. Hall, 1992, p. 48)

$$q_1\left(x\right) = -\left(\kappa_1 + \frac{1}{6}\kappa_3\left(x^2 - 1\right)\right), \tag{12}$$

where κ_1 and κ_3 are the leading terms of the first and third order cumulants of T_h.

Now consider the bootstrap. Let T_h^* denote a bootstrap version of T_h (either the i.i.d. or the WB). We can write a one-term Edgeworth expansion for the conditional distribution of T_h^* as follows:

$$P^*\left(T_h^* \leq x\right) = \Phi\left(x\right) + \sqrt{h}\; q_1^*\left(x\right) \phi\left(x\right) + O_P\left(h\right), \tag{13}$$

where q_1^* is an even polynomial in x, whose coefficients are now a function of the bootstrap cumulants of T_h^* (up to order three). In particular,

$$q_1^*\left(x\right) = -\left(\kappa_{1,h}^* + \frac{1}{6}\kappa_{3,h}^*\left(x^2 - 1\right)\right), \tag{14}$$

where $\kappa_{1,h}^*$ and $\kappa_{3,h}^*$ are the leading terms of the first and third order cumulants of T_h^*.

Given the cumulants expansions presented in Appendix A (cf. Theorems A.1 $-$ A.3) and the definitions (12) and (14), we can readily obtain expressions for q_1 and for q_1^*, for the i.i.d. bootstrap and the WB. The following proposition states these results.

Proposition 4.1 *Consider DGP (2). Suppose v is independent of W and in addition assume Assumption (V) holds. Then, conditional on v, as $h \to 0$, it follows that*

a) $q_1\left(x\right) = -\left(-\frac{A_1}{2} + \frac{1}{6}\left(B_1 - 3A_1\right)\left(x^2 - 1\right)\right)\frac{\overline{\sigma^6}}{\left(\overline{\sigma^4}\right)^{3/2}} = \frac{4\left(2x^2+1\right)}{6\sqrt{2}}\frac{\overline{\sigma^6}}{\left(\overline{\sigma^4}\right)^{3/2}}$, *with* $A_1 = B_1 = \frac{4}{\sqrt{2}}$.

b) *For the i.i.d. bootstrap,*

$$q_1^*\left(x\right) = -\left(-\frac{\tilde{A}_1}{2} + \frac{1}{6}\left(-2\tilde{A}_1\right)\left(x^2 - 1\right)\right) = \frac{1}{6}\left(2x^2+1\right)\frac{R_6 - 3R_4RV + 2RV^3}{\left(R_4 - RV^2\right)^{3/2}},$$

where $\tilde{A}_1 = \dfrac{R_6 - 3R_4RV + 2RV^3}{\left(R_4 - RV^2\right)^{3/2}}$.

[7] We will not provide a proof of the validity of the Edgeworth expansions we develop, which are in this sense only formal expansions. Proving the validity of our Edgeworth expansions would be a valuable contribution in itself, which we defer for future research. Here our focus is on using formal expansions to theoretically explain the superior finite sample properties of the bootstrap. Our approach follows Mammen (1993) and Davidson and Flachaire (2001), who also rely on formal Edgeworth expansions for studying the accuracy of the bootstrap in the context of linear regression models. Finally, all of our results are valid conditionally on the path of the stochastic process v.

c) *For the WB,*

$$q_1^*(x) = -\left(-\frac{A_1^*}{2} + \frac{1}{6}\left(B_1^* - 3A_1^*\right)\left(x^2 - 1\right)\right)\frac{R_6}{R_4^{3/2}},$$

where the constants A_1^* *and* B_1^* *are defined as*

$$A_1^* = \frac{\mu_6^* - \mu_2^*\mu_4^*}{\mu_4^*\left(\mu_4^* - \mu_2^{*2}\right)^{1/2}}, \quad \text{and} \quad B_1^* = \frac{\mu_6^* - 3\mu_2^*\mu_4^* + 2\mu_2^{*3}}{\left(\mu_4^* - \mu_2^{*2}\right)^{3/2}}.$$

Given (11), the standard normal approximation $\Phi(x)$ makes an error equal to

$$P\left(T_h \leq x\right) - \Phi(x) = \sqrt{h}\; q_1(x)\,\phi(x) + O(h), \tag{15}$$

uniformly in $x \in \mathbb{R}$, when estimating $P\left(T_h \leq x\right)$. The leading term is a function of $q_1(x)$ whose form is given in Proposition 4.1. a). When v is constant, $q_1(x)$ simplifies to $q_1(x) = \frac{1}{6}\frac{4}{\sqrt{2}}\left(2x^2+1\right)$. When v is stochastic, $q_1(x)$ is a function of the path of v through $\overline{\sigma^6}$ and $\overline{\sigma^4}$. In this case, (11) describes an asymptotic expansion of the distribution of T_h conditional on the volatility path. The leading term of the Edgeworth expansion is the standard normal approximation $\Phi(x)$. This is as expected, given that BN-S (2002) show that the first-order asymptotic distribution of T_h is the standard normal distribution. Here we provide a rate of convergence for the error committed by the first-order asymptotic approximation. In particular, (15) implies that the error of the standard normal approximation is of order $O\left(\sqrt{h}\right)$.

Given (13), the bootstrap error implicit in the bootstrap approximation of $P\left(T_h \leq x\right)$ (conditional on v) is given by

$$\begin{aligned} P^*\left(T_h^* \leq x\right) - P\left(T_h \leq x\right) &= \sqrt{h}\;\left(q_1^*(x) - q_1(x)\right)\phi(x) + O_P(h) \\ &= \sqrt{h}\left(\operatorname*{plim}_{h\to 0} q_1^*(x) - q_1(x)\right)\phi(x) + o_P\left(\sqrt{h}\right) \end{aligned} \tag{16}$$

uniformly in $x \in \mathbb{R}$. Thus, the bootstrap error in estimating the distribution function of T_h has a leading term of order $O\left(\sqrt{h}\right)$ equal to $\sqrt{h}\left(\text{plim}_{h\to 0}\, q_1^*(x) - q_1(x)\right)\phi(x)$. The ability of the bootstrap to improve upon the normal approximation depends on the magnitude of $\text{plim}_{h\to 0}\, q_1^*(x) - q_1(x)$, to order $O\left(\sqrt{h}\right)$. In particular, if the bootstrap is such that $\text{plim}_{h\to 0}\, q_1^*(x) - q_1(x) = 0$, then the bootstrap error is $o_P\left(\sqrt{h}\right)$, smaller than the $O\left(\sqrt{h}\right)$ normal error. That is, the bootstrap provides a second-order refinement. As we will see next, the bootstrap ability to match $q_1(x)$ with $q_1^*(x)$ depends on its ability to match the first three cumulants of T_h to order $O\left(\sqrt{h}\right)$.

4.1 The i.i.d. bootstrap error

The following result characterizes formally the i.i.d. bootstrap error.

Proposition 4.2 *Under the conditions of Proposition 4.1, conditionally on* v*, as* $h \to 0$*,*

a)

$$\operatorname*{plim}_{h\to 0} q_1^*(x) - q_1(x) = \frac{1}{6}\left(2x^2+1\right)\left[\frac{15\overline{\sigma^6} - 9\overline{\sigma^4}\ \overline{\sigma^2} + 2\left(\overline{\sigma^2}\right)^3}{\left(3\overline{\sigma^4} - \left(\overline{\sigma^2}\right)^2\right)^{3/2}} - \frac{4}{\sqrt{2}}\frac{\overline{\sigma^6}}{\left(\overline{\sigma^4}\right)^{3/2}}\right], \tag{17}$$

b) *When $v_t = v$ for all t, then*

$$\operatorname*{plim}_{h\to 0} q_1^*(x) - q_1(x) = 0. \tag{18}$$

c) *In the general case, we have that uniformly in $x \in \mathbb{R}$,*

$$\left|\operatorname*{plim}_{h\to 0} q_1^*(x) - q_1(x)\right| \leq |q_1(x)|\,. \tag{19}$$

An immediate consequence of (18) is that under constant volatility the error of the bootstrap approximation is of order $o_P\left(\sqrt{h}\right)$. This is of a smaller order of magnitude than the error of the standard normal approximation, which is of order $O\left(\sqrt{h}\right)$. Thus, the i.i.d. bootstrap provides an asymptotic refinement through order $O\left(\sqrt{h}\right)$ over the feasible asymptotic theory of BN-S under constant volatility.

When volatility is heterogeneous, $\operatorname{plim}_{h\to 0} q_1^*(x) - q_1(x) \neq 0$. Thus, the rate of convergence of the bootstrap error is in this case of order $O_P\left(\sqrt{h}\right)$, the same as that of the feasible asymptotic theory of BN-S. The i.i.d. bootstrap is not able to match the cumulants of the original statistic when volatility is time-varying and this explains why it does not provide an asymptotic refinement for the distribution of T_h (although it is asymptotically valid, as we showed in Section 3). This result is nevertheless at odds with our simulation evidence (to be discussed later) which shows that the i.i.d. bootstrap outperforms the normal approximation even when volatility is stochastic.

We propose the following explanation. To order $O\left(\sqrt{h}\right)$, the bootstrap error is determined by the difference $\sqrt{h}\left[\operatorname{plim}_{h\to 0} q_1^*(x) - q_1(x)\right]\phi(x)$. Similarly, the error of the first-order asymptotic normal approximation is determined by $\sqrt{h}q_1(x)\phi(x)$. (19) implies that the absolute magnitude of the i.i.d. bootstrap contribution of order $\sqrt{h}$ to the error in approximating the true sampling distribution of T_h is smaller than that of the standard normal approximation. Equivalently, the relative asymptotic error of the bootstrap, relative to the normal approximation can be approximated to order $O\left(\sqrt{h}\right)$ by the ratio

$$r_1(x) = \left|\frac{\operatorname{plim}_{h\to 0} q_1^*(x) - q_1(x)}{q_1(x)}\right|, \tag{20}$$

for any $x \in \mathbb{R}$. Part c) of Proposition 4.2 implies that $r_1(x) \leq 1$ uniformly in x and thus suggests that the bootstrap error is smaller (or never larger) than the error made by the normal approximation, to order $O\left(\sqrt{h}\right)$. The asymptotic relative error is one of several accuracy measures that one can use to compare the bootstrap with an alternative estimator such as the normal approximation when both estimators have the same rate of convergence. Shao and Tu (1995, Section 3.3) give a review of these alternative criteria. Davidson and Flachaire (2001) rely on a similar criterion to explain the superior performance of a certain wild bootstrap in the context of a cross-section linear regression model with unconditional heteroskedastic errors.

An important implication of Proposition 4.2 concerns the accuracy of the bootstrap critical values. Let q_α denote the true α-level critical value of T_h, i.e. $P(T_h \leq q_\alpha) = \alpha$. Similarly, let z_α be the α-level critical value of the normal distribution (i.e. $\Phi(z_\alpha) = \alpha$), and let q_α^* denote the corresponding α-level bootstrap quantile. Following Hall (1992, p. 92), and relying on the Cornish-Fisher expansions corresponding to the Edgeworth expansions (11) and (13), we have that for any $x \in \mathbb{R}$,

$$q_\alpha^* - q_\alpha = -\sqrt{h}\ \left(q_1^*(z_\alpha) - q_1(z_\alpha)\right) + O_P(h). \tag{21}$$

In contrast, the error made by the normal approximation is equal to

$$z_\alpha - q_\alpha = \sqrt{h}\ q_1(z_\alpha) + O(h). \tag{22}$$

Given (21) and (22), it follows that the relative error for i.i.d. bootstrap critical values, relative to the standard normal critical values, can be approximated to order $O\left(\sqrt{h}\right)$ by

$$\left|\frac{\operatorname{plim}_{h\to 0} q_1^*(z_\alpha) - q_1(z_\alpha)}{q_1(z_\alpha)}\right| = r_1(z_\alpha).$$

Thus, by Proposition 4.2 under time-varying volatility, $r_1(z_\alpha) \leq 1$, implying that the accuracy of the i.i.d. bootstrap critical value at level α cannot be worse than that of the standard normal approximation. Under constant volatility, $r_1(z_\alpha) = 0$ and the bootstrap critical value is second-order accurate.

The magnitude of $r_1(x)$ is a useful measure of the accuracy of the bootstrap relatively to the accuracy of the normal approximation when both estimators have the same rate of convergence. Under stochastic volatility, this ratio is a function of the volatility path and can be quantified for a given stochastic volatility model by simulation. Figure 1 and Table 6 contain results for the baseline models considered in Section 2. The results suggest that this ratio is very small and close to zero for two of the three models considered (namely for the log-normal and GARCH(1,1) diffusions), and slightly larger for a two-factor diffusion model. This finding suggests that the bootstrap critical values are more accurate than the normal-based critical values even under stochastic volatility. This is consistent with the good performance of the i.i.d. bootstrap for these models for one-sided confidence intervals, as evidenced by the Monte Carlo results in Section 4.3.

4.2 The wild bootstrap error

The following result characterizes the WB error in estimating $P(T_h \leq x)$, to order $O\left(\sqrt{h}\right)$.

Proposition 4.3 *Under the assumptions of Proposition 4.1, conditionally on v,*

$$\operatorname*{plim}_{h\to 0} q_1^*(x) - q_1(x) = -\left[\left(\operatorname*{plim}_{h\to 0} \kappa_{1,h}^* - \kappa_1\right) + \frac{1}{6}\left(\operatorname*{plim}_{h\to 0} \kappa_{3,h}^* - \kappa_3\right)\left(x^2-1\right)\right],$$

where

$$\begin{aligned}
\operatorname*{plim}_{h\to 0} \kappa_{1,h}^* - \kappa_1 &= -\frac{1}{2}\frac{\overline{\sigma^6}}{\left(\overline{\sigma^4}\right)^{3/2}}\left(\frac{5}{\sqrt{3}}A_1^* - A_1\right)\\
\operatorname*{plim}_{h\to 0} \kappa_{3,h}^* - \kappa_3 &= \frac{\overline{\sigma^6}}{\left(\overline{\sigma^4}\right)^{3/2}}\left[\left(\frac{5}{\sqrt{3}}B_1^* - B_1\right) - 3\left(\frac{5}{\sqrt{3}}A_1^* - A_1\right)\right]
\end{aligned}$$

with $A_1 = B_1 = \frac{4}{\sqrt{2}}$, and where A_1^* and B_1^* are as defined in Proposition 4.1.

This result shows that the choice of η_i (which dictates the value of the constants A_1^* and B_1^* through μ_q^* for $q = 2, 4, 6$) influences the magnitude of the WB error. For instance, if we choose[8] $\eta_i \sim N(0,1)$, then $A_1^* = A_1 = B_1 = B_1^*$. This implies that $\operatorname{plim}_{h\to 0} \kappa_{1,h}^* - \kappa_1 = \left(\frac{5}{\sqrt{3}} - 1\right)\kappa_1 \neq 0$ and $\operatorname{plim}_{h\to 0} \kappa_{3,h}^* - \kappa_3 = \left(\frac{5}{\sqrt{3}} - 1\right)\kappa_3 \neq 0$. Thus, if $\eta_i \sim N(0,1)$, it follows that

$$\operatorname*{plim}_{h\to 0} q_1^*(x) - q_1(x) = \left(\frac{5}{\sqrt{3}} - 1\right) q_1(x) \approx 1.89 q_1(x),$$

showing that this choice of η_i does not deliver an asymptotic refinement over the normal approximation. It also shows that in absolute terms the contribution of the term $O\left(\sqrt{h}\right)$ to the bootstrap error is almost twice as large as the contribution of $q_1(x)$ that is associated with the error made by the normal approximation. We conclude that $\eta_i \sim N(0,1)$ is not a good choice for the WB. This is confirmed by our Monte Carlo simulations in the next section.

Our next result provides conditions on the external random variable η_i that ensure $\operatorname{plim}_{h\to 0} q_1^*(x) - q_1(x) = 0$, implying that the WB yields an asymptotic refinement through order $O\left(\sqrt{h}\right)$ over the normal approximation.

Proposition 4.4 *Suppose η_i is i.i.d. with moments $\mu_q^* = E^*|\eta_i|^q$ for $q = 2, 4$ and 6 such that*

$$\begin{aligned}
\frac{\mu_6^* - \mu_2^*\mu_4^*}{\mu_4^*\left(\mu_4^* - \mu_2^{*2}\right)^{1/2}} &= \frac{\sqrt{3}}{5}\frac{4}{\sqrt{2}}\\
\frac{\mu_6^* - 3\mu_2^*\mu_4^* + 2\mu_2^{*3}}{\left(\mu_4^* - \mu_2^{*2}\right)^{3/2}} &= \frac{\sqrt{3}}{5}\frac{4}{\sqrt{2}}.
\end{aligned}$$

Then under the assumptions of Proposition 4.1, conditionally on v, as $h \to 0$,

$$P^*\left(T_h^* \le x\right) - P\left(T_h \le x\right) = o_P\left(\sqrt{h}\right),$$

[8] Given that returns are (conditionally on v) normally distributed, choosing $\eta_i \sim N(0,1)$ could be a natural choice. Moreover, this is a first-order asymptotically valid choice that implies a WB statistic T_h^* whose form is exactly that of T_h but with the bootstrap data replacing the original data, as we argued above.

uniformly in $x \in \mathbb{R}$, where T_h^* is the statistic defined in equations (8) and (9).

The first equation in Proposition 4.4 is a rewriting of $A_1^* = \frac{\sqrt{3}}{5} A_1$ as a function of μ_2^*, μ_4^* and μ_6^*, whereas the second equation is equal to $B_1^* = \frac{\sqrt{3}}{5} B_1$. According to this result, any choice of η_i with moments μ_2^*, μ_4^* and μ_6^* satisfying these two conditions delivers an asymptotic refinement of the WB. There is an infinite number of solutions. In particular, we can show that for any $\gamma \neq 0$, the solution is of the form $\mu_2^* = \gamma^2$, $\mu_4^* = \frac{31}{25}\gamma^4$ and $\mu_6^* = \frac{31}{25}\frac{37}{25}\gamma^6$. Since the value of T_h^* is invariant to the choice of γ, we can choose $\gamma = 1$ without loss of generality, implying $\mu_2^* = 1$ (which ensures the WB realized volatility is an unbiased estimator of realized volatility), $\mu_4^* = \frac{31}{25} = 1.24$, and $\mu_6^* = \frac{31}{25}\frac{37}{25} = 1.8352$. Next, we propose a two point distribution for η_i that matches these three moments and thus implies a second-order asymptotic refinement for the WB.

Corollary 4.1 *Let η_i be i.i.d. such that*

$$\eta_i = \begin{cases} \frac{1}{5}\sqrt{31 + \sqrt{186}} \approx 1.33 & \text{with prob } p = \frac{1}{2} - \frac{3}{\sqrt{186}} \approx 0.28 \\ -\frac{1}{5}\sqrt{31 - \sqrt{186}} \approx -0.83 & \text{with prob } 1 - p. \end{cases}$$

Let

$$T_h^* = \frac{\sqrt{h^{-1}}\left(RV^* - RV\right)}{\sqrt{\hat{V}^*}}, \quad \text{with} \quad \hat{V}^* = \frac{6}{31}\left(h^{-1}\sum_{i=1}^{1/h} r_i^{*4}\right).$$

Under the assumptions of Proposition 4.1, conditionally on v, as $h \to 0$,

$$P^*\left(T_h^* \leq x\right) - P\left(T_h \leq x\right) = o_P\left(\sqrt{h}\right),$$

uniformly in $x \in \mathbb{R}$.

4.3 Simulations for one-sided confidence intervals

The theoretical results in the two previous subsections suggest that both the i.i.d. bootstrap and the WB with an appropriate choice of η_i are more accurate than the normal approximation for estimating the distribution function of T_h. Estimators of $P\left(T_h \leq x\right)$ are useful to compute critical values for one-sided confidence intervals for IV. In this section we evaluate the finite sample accuracy of the i.i.d. bootstrap and the WB in terms of the coverage probabilities of one-sided confidence intervals. A lower one-sided 95% bootstrap confidence interval for IV is given by

$$IC_{0.95}^{*(1)} = \left(0, RV - q_{0.05}^* \sqrt{h\hat{V}}\right),$$

where q_α^* is the α-quantile of the bootstrap distribution of T_h^*. We consider three different bootstrap methods for computing $q_{0.05}^*$: the i.i.d. bootstrap and two WB methods, one based on $\eta_i \sim N(0,1)$ and another based on the two-point distribution for which asymptotic refinements are to be expected. Notice that the bootstrap statistics T_h^* on which $q_{0.05}^*$ are based differ according to the bootstrap method in question. In particular, except for the WB based on the normal distribution, T_h^* is not of the same form as T_h.

We also report results for confidence intervals based on a logarithmic version of the statistic T_h, following BN-S. These are referred to as "log" and are of the following form:

$$IC_{\log,0.95}^{*(1)} = \left(-\infty, \log\left(RV\right) - q_{0.05}^{*}\sqrt{\frac{h\hat{V}}{\left(RV\right)^{2}}}\right),$$

where $q_{0.05}^*$ denotes the 5% percentile of the bootstrap distribution of the logarithmic version of each T_h^*. For the i.i.d. bootstrap, this is equal to

$$\frac{\sqrt{h^{-1}}\left(\log RV^{*} - \log RV\right)}{\sqrt{\frac{\hat{V}^{*}}{\left(RV^{*}\right)^{2}}}},$$

with $\hat{V}^* = R_4^* - RV^{*2}$. For the WB, it is equal to

$$\frac{\sqrt{h^{-1}}\left(\log RV^{*} - \log \mu_2^{*} RV\right)}{\sqrt{\frac{\hat{V}^{*}}{\left(RV^{*}\right)^{2}}}},$$

where $\hat{V}^* = \left(\frac{\mu_4^* - \mu_2^{*2}}{\mu_4^*}\right) R_4^*$. We note that the theory in this paper only provides the first-order asymptotic validity of the bootstrap "log" intervals (based on an application of the delta method, given Theorems 3.1 and 3.2). Our Edgeworth expansions do not apply to the log versions of T_h^*. Thus, we cannot use these expansions to make any predictions on the second-order correctness of the bootstrap "log" intervals. We include these only for comparison purposes with the feasible asymptotic theory of BN-S based on the logarithmic version of the statistic T_h.

Table 1 contains results for the baseline models. Table 4 refers to the models with drift and leverage. The bootstrap methods rely on 999 bootstrap replications for each of the 10,000 Monte Carlo replications. A comparison of the two tables shows that the results are very similar with and without leverage and drift. In all cases, the bootstrap intervals tend to undercover, with the exception of the WB intervals based on $\eta_i \sim N(0,1)$. However, the degree of undercoverage is larger for the feasible asymptotic-theory based intervals than for the bootstrap methods; it is larger the smaller the sample size (i.e. the larger is h), and it is larger for the "raw" version of the intervals than for the "log" version. As already noted for the feasible asymptotic approach, the bootstrap does generally worst for the two-factor diffusion model of Chernov et. al. (2003). Nevertheless, the i.i.d. bootstrap does remarkably well across all models, despite the fact that the volatility is stochastic and hence time-varying. It essentially eliminates the coverage distortions associated with the BN-S intervals for small values of $1/h$ for the log-normal and the GARCH(1,1) diffusions. The coverage probability of the i.i.d. bootstrap intervals deteriorates for the two-factor model, but it remains very competitive relatively to the other methods. The WB intervals based on the normal distribution tend to overcover across all models, with the degree of overcoverage being smaller for larger values of the sample size. The WB based on the two-point distribution tends to undercover, but significantly less than the feasible asymptotic theory-based intervals of BN-S. Except for the smaller sample sizes ($h = 1/12$ and $h = 1/48$) the WB based on the two-point distribution is competitive with the i.i.d. bootstrap.

5 Third-order accuracy of the bootstrap

Here we discuss the ability of the bootstrap to provide third-order asymptotic refinements. In particular, we develop Edgeworth expansions through order $O(h)$ and use these to evaluate the accuracy of the bootstrap for estimating the two-sided symmetric distribution function $P(|T_h| \leq x)$. This quantity is of interest for two-sided symmetric confidence intervals.

Following our analysis in Section 4, we consider now a two-term Edgeworth expansion for the distribution of T_h :

$$P(T_h \leq x) = \Phi(x) + \sqrt{h}\; q_1(x)\,\phi(x) + h\; q_2(x) + o(h), \tag{23}$$

for any $x \in \mathbb{R}$. The function q_1 is defined in (12). The function q_2 is defined as (cf. Hall, 1992, p. 48):

$$q_2(x) = -\left\{\frac{1}{2}\left(\kappa_2 + \kappa_1^2\right) He_1(x) + \frac{1}{24}\left(\kappa_4 + 4\kappa_1\kappa_3\right) He_3(x) + \frac{1}{72}\kappa_3^2 He_5(x)\right\}, \tag{24}$$

where for each j, He_j are Hermite polynomials (i.e. $He_1(x) = x$, $He_3(x) = x\left(x^2 - 3\right)$, and $He_5(x) = x\left(x^4 - 10x^2 + 15\right)$), and κ_j are the leading terms of the cumulants of T_h of order j. See Appendix A for the cumulants expansions of T_h and their corresponding leading terms for $j = 1, \ldots, 4$.

Similarly, for any $x \in \mathbb{R}$,

$$P^*(T_h^* \leq x) = \Phi(x) + \sqrt{h}\; q_1^*(x)\,\phi(x) + h\; q_2^*(x) + o_P(h), \tag{25}$$

where q_1^* and q_2^* are even and odd polynomials in x, respectively, whose coefficients are now a function of the bootstrap cumulants of T_h^* (up to order four). In particular,

$$q_2^*(x) = -\left\{\frac{1}{2}\left(\kappa_{2,h}^* + \kappa_{1,h}^{*2}\right) He_1(x) + \frac{1}{24}\left(\kappa_{4,h}^* + 4\kappa_{1,h}^*\kappa_{3,h}^*\right) He_3(x) + \frac{1}{72}\kappa_{3,h}^{*2} He_5(x)\right\}, \tag{26}$$

where $\kappa_{j,h}^*$ are the bootstrap cumulants.

Given the cumulants expansions presented in Appendix A and the definitions (24) and (26), we can readily derive expressions for q_2 and q_2^*, similarly to Proposition 4.1 for q_1 and q_1^*. To conserve space, we do not state these results formally here.

It follows from (23) and the symmetry properties of Φ, q_1, and q_2, that for any $x > 0$,

$$P(|T_h| \leq x) = 2\Phi(x) - 1 + 2h\; q_2(x)\,\phi(x) + o(h).$$

Thus the error in estimating $P(|T_h| \leq x)$ made by the normal approximation is of order $O(h)$:

$$P(|T_h| \leq x) - (2\Phi(x) - 1) = 2h\ q_2(x)\,\phi(x) + o(h). \tag{27}$$

Given (25) and the symmetry properties of q_1^* and q_2^*, the bootstrap estimator of $P(|T_h| \leq x)$ is given by

$$P^*(|T_h^*| \leq x) = 2\Phi(x) - 1 + 2h\ q_2^*(x)\,\phi(x) + o_P(h),$$

for any $x > 0$. Thus the bootstrap error in estimating a two-sided distribution is equal to

$$\begin{aligned} P^*(|T_h^*| \leq x) - P(|T_h| \leq x) &= 2h\ [q_2^*(x) - q_2(x)]\,\phi(x) + o_P(h) \\ &= 2h\left[\operatorname*{plim}_{h\to 0} q_2^*(x) - q_2(x)\right]\phi(x) + o_P(h). \end{aligned} \tag{28}$$

Because $\text{plim}_{h\to 0}\, q_2^*(x) - q_2(x)$ depends on the first four cumulants of T_h and T_h^*, the ability of the bootstrap to provide a third-order asymptotic refinement depends on its ability to consistently estimate the first four cumulants of T_h through order $O(h)$. Corollary A.1 in Appendix A gives $\text{plim}_{h\to 0}\, \kappa_{j,h}^* - \kappa_j$ for $j = 1, ..., 4$ for the i.i.d. bootstrap. Proposition 4.3 gives $\text{plim}_{h\to 0}\, \kappa_{j,h}^* - \kappa_j$ for $j = 1, 3$ for the WB whereas Corollary A.2 in Appendix A gives the corresponding results for $j = 2, 4$.

As discussed in Section 4.1, the i.i.d. bootstrap provides a second-order asymptotic refinement under constant volatility because it consistently estimates the first three cumulants of T_h through order $O\left(\sqrt{h}\right)$. By Corollary A.1, we can show that the i.i.d. bootstrap is unable to consistently estimate the second and fourth order cumulants through order $O(h)$. This is true even under constant volatility. Hence, even though the i.i.d. bootstrap provides a second-order refinement when volatility is constant, it does not provide a third-order refinement. We find this a surprising result. If volatility is constant, returns are i.i.d. and we would expect the i.i.d. bootstrap to be higher-order correct. Proposition 4.2 shows that this statement is true to second-order. It is nevertheless not true to third-order. We conjecture that one way of inducing the third-order accuracy of the i.i.d. bootstrap under constant volatility is to transform the original statistic T_h so as to more closely match the behavior of T_h^* (see Andrews (2004) for a similar approach). We will explore this possibility in future research. Our focus here is on bootstrapping the statistic T_h, which is the statistic originally proposed in the realized volatility literature by BN-S (2002).

The higher-order properties of the first four WB cumulants depend on the moments μ_q^* of the distribution of η_i, for $q = 2, 4, 6$ and 8. As we explain next, there is no choice of η_i that matches all four cumulants simultaneously. As discussed in Section 4.2, to match the first and third order cumulants we need to choose η_i with moments $\mu_2^* = \gamma^2$, $\mu_4^* = \frac{31}{25}\gamma^4$, and $\mu_6^* = \frac{31}{25}\frac{37}{25}\gamma^6$. Since the WB statistic is invariant to the choice of γ, we set $\gamma = 1$. We are left with two equations ($\text{plim}_{h\to 0}\, \kappa_{j,h}^* - \kappa_j$ for $j = 2, 4$) and one free parameter μ_8^*. Thus, there is no choice of η_i for which the WB can consistently estimate all four cumulants to order $O(h)$, and consequently the WB cannot provide a third-order asymptotic refinement. Nevertheless any choice of η_i satisfying Proposition 4.4 delivers second-order refinements.

One example is the two-point distribution proposed in Corollary 4.1, for which $\mu_8^* = 3.0137$. This choice of η_i only matches the first and third-order cumulants. We can show that in order to match κ_j for $j = 1, 2, 3$, we need $\mu_2^* = 1$, $\mu_4^* = \frac{31}{25}$, $\mu_6^* = \frac{31}{25}\frac{37}{25}$ and $\mu_8^* = \left(\frac{31}{25}\right)^2 \left(\frac{1}{25}\right) \left(\frac{1739}{35}\right) = 3.056.$[9] Any WB with η_i satisfying these moment restrictions implies second-order refinements. Because it also matches the second cumulant through order $O(h)$, any such choice could potentially perform better than our two-point choice of η_i in Corollary 4.1.

Next we evaluate the errors made by the i.i.d. bootstrap and the WB relatively to the error made by the normal approximation for estimating $P(|T_h| \leq x)$. Given that the two estimators converge at the same rate, we rely on the asymptotic relative error of the bootstrap as the criterion of comparison. This error is to order $O(h)$ equal to the ratio

$$r_2(x) = \left| \frac{\operatorname{plim}_{h \to 0} q_2^*(x) - q_2(x)}{q_2(x)} \right|, \tag{29}$$

where $x > 0$. If $r_2(x)$ is inferior to one, the bootstrap is better than the normal approximation in the sense that the absolute error implied by the bootstrap estimator is smaller than the error of the normal approximation, to order $O(h)$. Thus, the ratio $r_2(x)$ is a measure of the relative accuracy of the bootstrap when estimating two-sided symmetric distribution functions. In particular, $r_2\left(z_{1-\alpha/2}\right)$, with $z_{1-\alpha/2}$ the $(1-\alpha/2)$ critical value of Φ, has implications for the accuracy of the bootstrap critical values in nominal $100(1-\alpha)\%$ two-sided symmetric intervals.

In the general stochastic volatility case, $r_2(x)$ is a function of x and of v. When v is constant, it becomes only a function of x. Figure 2 plots $r_2(x)$ against x when v is constant. Four methods are considered: the i.i.d. bootstrap, the WB based on $\eta_i \sim N(0,1)$, the WB based on η_i chosen according to Corollary 4.1, and a third WB whose moments μ_q^* match the second cumulant, in addition to the first and third cumulants. Figure 2 shows that $\sup_x r_2(x) < 1$ for the i.i.d. bootstrap. Thus, although the i.i.d. bootstrap does not provide a third-order refinement even when volatility is constant, under the asymptotic relative error criterion it is better than the normal approximation. In particular, the value of r_2 at $x = z_{0.975} = 1.96$ is equal to 0.36, suggesting that the i.i.d. bootstrap critical value is more accurate than the normal-based critical value. Instead, Figure 2 shows that $r_2(x)$ can be larger or smaller than one for the WB methods depending on x. An exception is the WB based on $N(0,1)$, for which $r_2(x)$ is always very large, well above one, for any value of x. For the other two WB methods, $r_2(x)$ is smaller than one for all values of x sufficiently large. In particular, $r_2(x)$ is very small for $x = z_{1-\alpha/2} = 1.96$ with $\alpha = 0.05$, which suggests that in the constant volatility case, the WB methods that match cumulants up to the third order yield two-sided 95% critical values that are more accurate than the corresponding standard normal critical value equal to 1.96. We could evaluate $r_2(x)$ by simulation when v is stochastic, as we did for $r_1(x)$. The analysis is more complicated here because $r_2(x)$ depends on x. Some preliminary results suggest that $r_2(x)$ can be smaller or larger

[9] To match κ_j for $j = 1, 2, 4$, we would need $\mu_8^* = 1.225$. This value is not compatible with the other values of μ_2^*,μ_4^* and μ_6^*, because of Jensen's inequality. So, there is no distribution which can match κ_j for $j = 1, 2, 4$ simultaneously.

than one depending on the value of x. We will not pursue this analysis any further here.[10]

To conclude this section, we compare the bootstrap with the feasible asymptotic theory of BN-S when computing 95% two-sided confidence intervals for IV. We consider symmetric and equal-tailed intervals. The 95% level symmetric bootstrap confidence intervals for IV are of the form,

$$RV \pm p^*_{0.95}\sqrt{h\hat{V}},$$

where $p^*_{0.95}$ is the 95% percentile of the bootstrap distribution of $|T^*_h|$, i.e. instead of using the standard normal distribution to compute the critical value 1.96 we use the bootstrap. The 95% level equal-tailed bootstrap intervals are of the form

$$\left(RV - \sqrt{h\hat{V}}q^*_{0.975}, RV - \sqrt{h\hat{V}}q^*_{0.025}\right),$$

where q^*_α is the α-th percentile of the bootstrap distribution of T^*_h. We compute bootstrap critical values with the i.i.d. bootstrap and the two WB methods as in Section 4.3. Log versions of these bootstrap intervals are also considered.

Tables 2 and 5 contain results for the symmetric intervals, for the baseline models and for the models with drift and leverage, respectively. The results for symmetric intervals are in many ways similar to those mentioned for the one-sided intervals. Overall, the i.i.d. and the WB based on the two-point distribution outperform the normal approximation. This holds despite the fact that these bootstrap methods do not theoretically provide an asymptotic refinement for two-sided symmetric confidence intervals. The i.i.d. bootstrap is the preferred method in this case, followed by the WB based on the proposed two-point distribution.

The Monte Carlo results in Tables 2 and 5 show that the log versions of the original and bootstrap statistics outperform their raw versions. This suggests that asymmetry is more important for the raw statistics than for their log versions in finite samples. Therefore we also compute equal-tailed bootstrap intervals.[11] Table 3 contains the results. A comparison of Tables 2 and 3 shows that equal-tailed intervals tend to outperform symmetric intervals. The main conclusion from Table 3 is that the coverage rates of these intervals are very similar for both the raw and the log versions and therefore there is no additional gain from using the log transformation. This suggests that the bootstrap is an important inference tool in other contexts where the log transformation may be not applicable. For instance, BN-S (2004d) use the Fisher-z transformation in the context of realized correlation. Such transformation is not available for the realized regression parameters, e.g. realized beta.

[10] Although Edgeworth expansions are the main theoretical tool for proving bootstrap asymptotic refinements, it has already been pointed out in the bootstrap literature (see e.g. Härdle, Horowitz and Kreiss (2003)) that Edgeworth expansions can be imperfect guides to the relative accuracy of the bootstrap methods. The same comment appears to apply here to the asymptotic relative bootstrap error criterion for two-sided distribution functions.

[11] Note however that the standard arguments for asymptotic refinements based on comparing convergence rates suggest that symmetric intervals are more accurate than equal-tailed intervals when one matches q_1 and q_2.

6 Conclusions

In this paper we propose two bootstrap methods for realized-volatility based statistics. One is the i.i.d. bootstrap and the other is the wild bootstrap. We show that these methods are first-order asymptotically valid under quite general conditions, similar to those used recently by BN-S in a series of papers. In particular, they are valid under stochastic volatility. Next, we study the higher-order accuracy of these bootstrap methods in comparison to the standard normal approximation using Edgeworth expansions and Monte Carlo simulations.

The simulation evidence in this paper suggests that percentile-t bootstrap confidence intervals for IV (specifically, the i.i.d. bootstrap and a particular WB method which we propose in the paper) are more accurate in finite samples than the intervals based on the feasible first-order asymptotic theory. This superior performance of the bootstrap holds for both one-sided and two-sided (symmetric and equal-tailed) intervals.

The standard arguments based on Edgeworth expansions show that the i.i.d. bootstrap offers a second-order asymptotic refinement when volatility is constant but not otherwise. When volatility is heterogeneous, we compare the i.i.d. bootstrap and the normal approximations using the asymptotic relative bootstrap error. This criterion has been proposed in the statistics literature to compare the bootstrap with the normal approximation when both estimators have the same convergence rate. It shows that the i.i.d. bootstrap outperforms the normal approximation when estimating the distribution function of the RV statistic when volatility is time-varying. Second-order asymptotic refinements for the WB can be obtained in the general setup allowing for stochastic volatility provided we choose the external random variable appropriately. We provide an optimal choice of this random variable. The Monte Carlo results for one-sided intervals are consistent with these theoretical predictions.

Although both the i.i.d. and the WB can achieve second-order asymptotic refinements, we show that none of these methods can deliver refinements through third-order. Nevertheless our simulations show that the finite sample performance of two-sided bootstrap intervals is superior to that of the corresponding first-order asymptotic theory intervals.

Our focus here is on the bootstrap for realized volatility. Establishing the (first- and higher-order) validity of the bootstrap for this simple statistic is an important step towards establishing its (first- and higher-order) validity for more complicated statistics based on high-frequency data. For instance, an interesting application of the bootstrap is to realized beta, where the Monte Carlo results of BN-S (2004d) show that there are important distortions in finite samples. Another interesting application are the nonparametric jump tests studied by BN-S (2005), Huang and Tauchen (2005) and Andersen, Bollerslev and Diebold (2004). Similarly, we can apply the bootstrap for inference on integrated volatility in the presence of microstructure noise, relying on more robust measures of volatility. These extensions are the subject of ongoing research.

Table 1. Coverage rates of nominal 95% one-sided intervals for IV

Baseline volatility models: no leverage and no drift

			Bootstrap		*Wild Bootstrap*			
	CLT		*i.i.d.*		$\eta_i \sim N(0,1)$		$\eta_i \sim$ *2 point*	
h	*raw*	*log*	*raw*	*log*	*raw*	*log*	*raw*	*log*
Log-normal diffusion								
1/12	82.68	88.86	93.23	93.57	98.49	98.07	87.50	90.34
1/48	89.70	92.80	94.66	94.73	98.31	97.73	93.91	95.20
1/96	91.28	93.20	94.47	94.56	98.12	97.78	94.28	95.26
1/288	93.08	94.29	95.03	95.05	97.39	97.04	95.08	95.55
1/576	93.30	94.03	94.81	94.82	96.83	96.40	94.60	94.97
1/1152	94.00	94.59	95.07	95.06	96.56	96.26	95.00	95.22
GARCH(1,1) diffusion								
1/12	82.69	88.83	93.27	93.48	98.51	98.07	87.50	90.27
1/48	89.74	92.74	94.63	94.74	98.32	97.73	93.87	95.20
1/96	91.27	93.19	94.52	94.59	98.15	97.77	94.31	95.25
1/288	93.03	94.33	95.10	95.12	97.40	97.03	95.04	95.55
1/576	93.31	94.09	94.78	94.79	96.82	96.43	94.64	94.98
1/1152	94.01	94.56	95.02	95.00	96.51	96.22	95.04	95.21
Two-factor diffusion								
1/12	75.69	82.41	89.70	90.35	96.52	96.12	78.94	82.76
1/48	84.52	88.48	92.66	92.64	96.92	96.49	89.71	91.70
1/96	87.48	90.39	93.79	93.71	97.26	96.85	92.03	93.50
1/288	90.27	92.12	94.28	94.25	97.32	96.94	93.49	94.35
1/576	92.26	93.55	94.92	94.88	97.32	96.97	94.55	95.06
1/1152	93.20	94.04	95.02	94.99	96.93	96.60	94.95	95.30

Note: 10,000 replications, with 999 bootstrap replications each.

Table 2. Coverage rates of nominal 95% symmetric intervals for IV

Baseline volatility models: no leverage and no drift

			Bootstrap		*Wild Bootstrap*			
	CLT		*i.i.d.*		$\eta_i \sim N(0,1)$		$\eta_i \sim$ *2 point*	
h	*raw*	*log*	*raw*	*log*	*raw*	*log*	*raw*	*log*
Log-normal diffusion								
1/12	86.07	90.40	93.72	95.86	98.49	97.95	87.49	88.37
1/48	92.32	93.62	94.86	95.47	98.31	97.44	93.84	94.69
1/96	93.25	94.23	94.95	95.28	98.09	97.22	94.50	94.92
1/288	94.55	94.71	95.27	95.09	97.04	96.40	95.16	95.12
1/576	94.56	94.74	94.98	95.13	96.21	95.81	94.69	94.91
1/1152	94.83	94.88	94.98	95.06	95.63	95.41	94.85	94.84
GARCH(1,1) diffusion								
1/12	86.08	90.40	93.75	95.86	98.51	97.96	87.49	88.30
1/48	92.32	93.64	94.87	95.46	98.32	97.42	93.83	94.66
1/96	93.21	94.22	95.00	95.27	98.12	97.22	94.44	94.93
1/288	94.57	94.70	95.18	95.11	97.05	96.38	95.17	95.13
1/576	94.52	94.79	94.99	95.15	96.24	95.81	94.72	94.87
1/1152	94.81	94.85	94.97	94.99	95.69	95.43	94.88	94.86
Two-factor diffusion								
1/12	78.94	85.90	90.13	93.32	96.52	96.14	78.92	80.25
1/48	87.95	90.85	92.83	93.97	96.92	96.50	89.79	90.95
1/96	90.58	92.51	94.00	94.78	97.26	96.74	92.16	93.19
1/288	92.83	93.59	94.59	94.88	97.25	96.78	93.98	94.27
1/576	94.52	94.70	95.48	95.59	97.29	96.89	95.15	95.14
1/1152	94.64	94.77	95.20	95.11	96.52	96.08	94.89	94.92

Table 3. Coverage rates of nominal 95% equal-tailed intervals for IV

Baseline volatility models: no leverage and no drift

			Bootstrap		*Wild Bootstrap*			
	CLT		*i.i.d.*		$\eta_i \sim N(0,1)$		$\eta_i \sim$ *2 point*	
h	*raw*	*log*	*raw*	*log*	*raw*	*log*	*raw*	*log*
Log-normal diffusion								
1/12	86.07	90.40	95.94	95.89	94.33	96.34	86.65	87.92
1/48	92.32	93.62	95.57	95.37	94.17	95.78	94.08	94.23
1/96	93.25	94.23	95.36	95.33	94.48	95.59	94.64	94.77
1/288	94.55	94.71	95.13	95.07	94.72	95.25	94.86	94.83
1/576	94.56	94.74	95.09	95.08	94.86	95.18	94.77	94.86
1/1152	94.83	94.88	95.14	95.17	94.62	94.96	94.92	94.99
GARCH(1,1) diffusion								
1/12	86.08	90.40	95.91	95.88	94.32	96.36	86.56	87.85
1/48	92.32	93.64	95.54	95.43	94.14	95.79	94.07	94.24
1/96	93.21	94.22	95.37	95.33	94.46	95.64	94.60	94.74
1/288	94.57	94.70	95.11	95.12	94.70	95.24	94.85	94.80
1/576	94.52	94.79	95.11	95.07	94.86	95.20	94.71	94.84
1/1152	94.81	94.85	95.13	95.13	94.63	94.98	94.96	94.96
Two-factor diffusion								
1/12	78.94	85.90	93.79	93.89	94.31	95.86	78.69	80.32
1/48	87.95	90.85	94.38	94.32	93.51	95.64	90.57	91.20
1/96	90.58	92.51	94.73	94.75	93.77	95.43	92.67	92.94
1/288	92.83	93.59	94.77	94.85	93.97	95.25	93.96	94.14
1/576	94.52	94.70	95.22	95.30	94.49	95.37	94.72	94.71
1/1152	94.64	94.77	94.82	94.90	94.27	95.04	94.80	94.88

Table 4. Coverage rates of nominal 95% one-sided intervals for *IV*

Volatility models with leverage and constant drift

			Bootstrap		*Wild Bootstrap*			
	CLT		*i.i.d.*		$\eta_i \sim N(0,1)$		$\eta_i \sim$ *2 point*	
h	*raw*	*log*	*raw*	*log*	*raw*	*log*	*raw*	*log*
Log-normal diffusion								
1/12	82.47	88.44	92.98	93.35	98.37	98.03	87.25	90.05
1/48	89.82	92.76	94.67	94.84	98.63	98.03	94.05	95.20
1/96	91.23	93.48	94.73	94.76	98.28	97.86	94.44	95.32
1/288	92.80	94.26	95.05	95.05	97.37	96.88	94.98	95.47
1/576	93.63	94.53	94.99	95.02	97.01	96.55	94.97	95.28
1/1152	94.22	94.76	95.17	95.15	96.68	96.28	95.19	95.38
GARCH(1,1) diffusion								
1/12	82.40	88.40	93.00	93.32	98.36	98.04	87.21	89.99
1/48	89.81	92.72	94.70	94.79	98.57	98.01	94.01	95.17
1/96	91.28	93.43	94.69	94.73	98.27	97.87	94.48	95.30
1/288	92.84	94.25	94.98	95.00	97.37	96.87	94.95	95.46
1/576	93.69	94.53	94.98	94.97	96.99	96.58	95.01	95.26
1/1152	94.28	94.77	95.16	95.16	96.70	96.27	95.13	95.39
Two-factor diffusion								
1/12	75.79	83.09	90.44	90.67	96.75	96.34	79.57	82.97
1/48	84.16	88.51	92.69	92.76	97.05	96.60	89.68	91.73
1/96	87.04	90.07	93.20	93.24	97.04	96.62	91.71	93.10
1/288	90.75	92.39	94.56	94.57	97.34	97.04	93.76	94.69
1/576	92.20	93.50	94.91	94.88	97.36	97.09	94.63	95.12
1/1152	93.01	93.98	95.13	95.08	96.79	96.54	94.82	95.17

Table 5. Coverage rates of nominal 95% symmetric intervals for IV

Volatility models with leverage and constant drift

			Bootstrap		*Wild Bootstrap*			
	CLT		*i.i.d.*		$\eta_i \sim N(0,1)$		$\eta_i \sim$ *2 point*	
h	*raw*	*log*	*raw*	*log*	*raw*	*log*	*raw*	*log*
Log-normal diffusion								
1/12	85.86	90.80	93.92	95.90	98.38	97.92	87.60	88.90
1/48	92.60	93.76	95.24	95.68	98.70	97.86	94.20	94.74
1/96	93.66	94.34	95.10	95.44	98.22	97.42	94.56	95.22
1/288	94.58	94.68	95.34	95.18	96.90	96.32	95.18	94.98
1/576	94.40	94.62	94.68	94.68	96.06	95.58	94.64	94.84
1/1152	94.98	94.78	95.04	94.92	95.82	95.42	95.10	94.82
GARCH(1,1) diffusion								
1/12	85.72	90.48	93.69	95.70	98.36	97.93	87.22	88.29
1/48	92.35	93.65	94.97	95.55	98.57	97.70	93.92	94.66
1/96	93.44	94.23	94.99	95.41	98.25	97.29	94.50	95.11
1/288	94.41	94.56	95.15	95.09	96.84	96.19	94.94	94.80
1/576	94.62	94.95	94.94	95.10	96.29	95.88	94.91	95.15
1/1152	95.04	95.10	95.13	95.16	96.05	95.59	95.13	95.15
Two-factor diffusion								
1/12	79.52	86.09	90.87	93.50	96.75	96.34	79.55	80.40
1/48	87.81	90.76	92.89	94.08	97.05	96.57	89.69	90.82
1/96	90.31	92.04	93.57	94.43	97.04	96.51	91.99	92.79
1/288	93.14	93.76	94.81	94.99	97.30	96.68	94.08	94.36
1/576	94.25	94.46	95.15	95.24	97.20	96.67	94.86	95.00
1/1152	94.27	94.47	94.81	94.88	96.33	95.84	94.56	94.74

Table 6. Descriptive statistics for the ratio $r_1(x)$ for the i.i.d. bootstrap

Baseline volatility models: no leverage and no drift

	Log-normal	GARCH(1,1)	Two-factor
Mean	0.00160	0.00247	0.08885
Minimum	$1.4e-004$	$1.8e-004$	0.01389
25^{th} percentile	0.00069	0.00109	0.07006
Median	0.00116	0.00180	0.08883
75^{th} percentile	0.00204	0.00314	0.10680
Maximum	0.01692	0.02436	0.21943

Note: 10,000 replications.

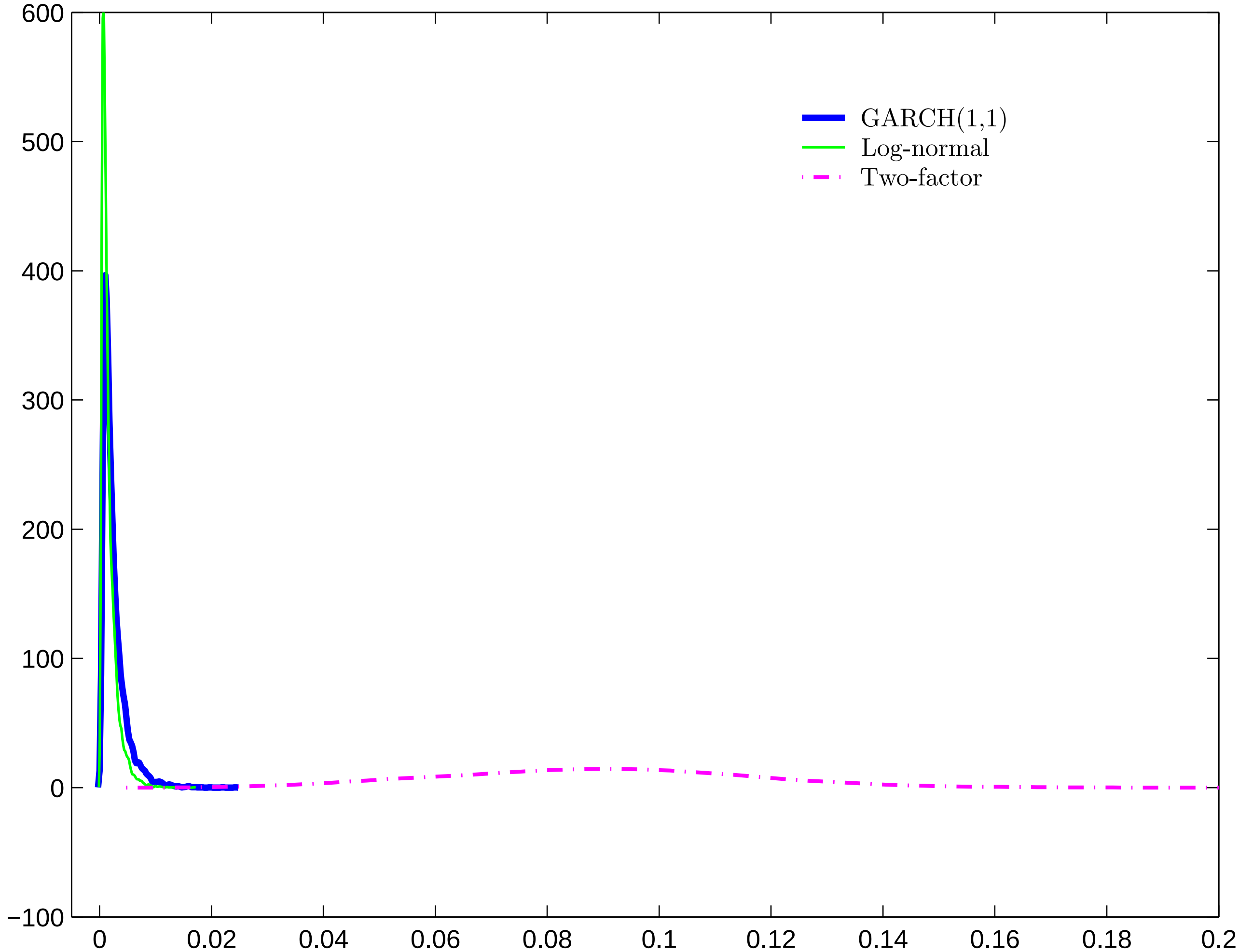

Figure 1: **Kernel density estimate of $r_1(x)$ for the i.i.d. bootstrap in the baseline models**

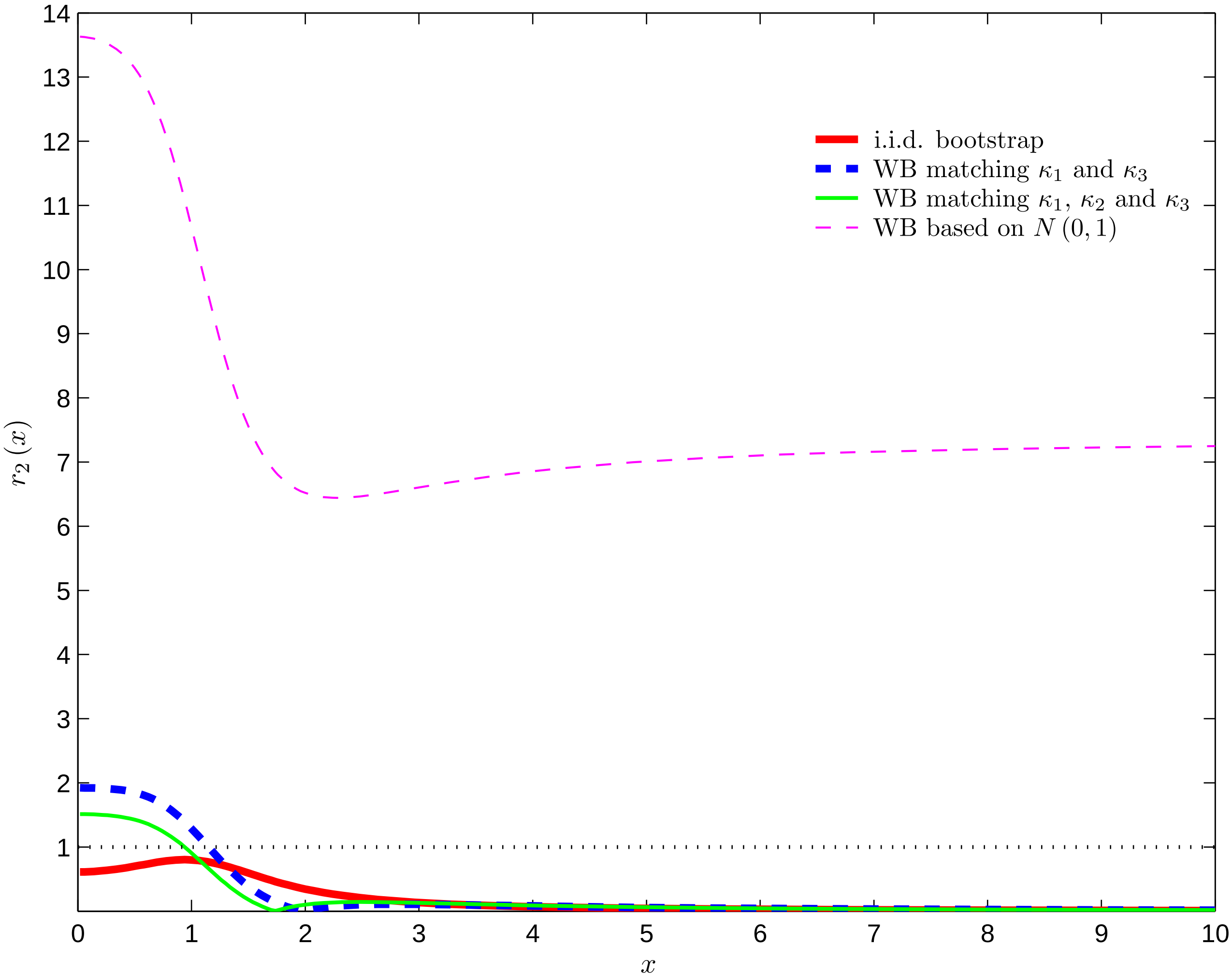

Figure 2: **The function $r_2(x)$ when v is constant**

Appendix A – Cumulants expansions and Lemmas

This Appendix is organized as follows. First, we provide expansions for the cumulants of T_h and their bootstrap analogues (cf. Theorems A.1–A.3 and Corollaries A.1 and A.2). These results are used to obtain the formal Edgeworth expansions through order $O(h)$ presented in the main text. Then, we state Lemmas A.1–A.7 useful for the proofs of these and other results in the paper. Next, we prove Theorems A.1–A.3. Finally, we prove Lemmas A.1–A.7.

We introduce some notation. Let $\sigma_i^2 \equiv \int_{(i-1)h}^{ih} v_u^2 du < \infty$, and for any $q > 0$ define $\overline{\sigma_h^q} \equiv h^{1-q/2} \sum_{i=1}^{1/h} \left(\sigma_i^2\right)^{q/2} \equiv h^{1-q/2} \sum_{i=1}^{1/h} \sigma_i^q$, where $\sigma_i^q \equiv \left(\sigma_i^2\right)^{q/2}$. Note that for $q = 2$, $IV = \overline{\sigma_h^2} = \int_0^1 v_u^2 du \equiv \overline{\sigma^2}$, but in general $\overline{\sigma_h^q} \neq \overline{\sigma^q} \equiv \int_0^1 v_u^q du$, as defined in the main text. Let $\mu_q = E\left|Z\right|^q$, where $Z \sim N(0,1)$ and $q > 0$ and note that $\mu_2 = 1$, $\mu_4 = 3$, $\mu_6 = 15$ and $\mu_8 = 105$. Since $\mu_2 = 1$, we can write $IV = \mu_2 IV$, which will be convenient for proving the results for the WB. Define

$$S_h \equiv \frac{\sqrt{h^{-1}}\left(RV - \mu_2 IV\right)}{\sqrt{V}} \quad \text{and} \quad U_h \equiv \frac{\sqrt{h^{-1}}\left(\hat{V} - V\right)}{V}.$$

We can write

$$T_h = S_h \left(\frac{\hat{V}}{V}\right)^{-1/2} = S_h \left(1 + \sqrt{h} U_h\right)^{-1/2}.$$

Note also that

$$RV - \mu_2 IV = \sum_{i=1}^{1/h} \left(r_i^2 - \mu_2 \sigma_i^2\right) \quad \text{and} \quad \hat{V} - V = \frac{\mu_4 - \mu_2^2}{\mu_4} h^{-1} \sum_{i=1}^{1/h} \left(r_i^4 - \mu_4 \sigma_i^4\right),$$

where for any $q > 0$, $\left|r_i\right|^q - \mu_q \sigma_i^q$ are (conditionally on v) independent with zero mean since $r_i = \sigma_i u_i$, where $u_i \sim$ i.i.d. $N(0,1)$.

For any of the two bootstrap schemes, define the bootstrap statistics

$$S_h^* \equiv \frac{\sqrt{h^{-1}}\left(RV^* - E^*\left(RV^*\right)\right)}{\sqrt{V^*}}, \quad U_h^* \equiv \frac{\sqrt{h^{-1}}\left(\hat{V}^* - V^*\right)}{V^*}$$

where $V^* = Var^*\left(h^{-1/2} RV^*\right)$. By construction, $E^*\left(S_h^*\right) = 0$ and $Var^*\left(S_h^*\right) = 1$. Let $\hat{V}^*$ be a consistent estimator of V^*. Then the studentized statistic T_h^* can be written as

$$T_h^* = S_h^* \left(\frac{\hat{V}^*}{V^*}\right)^{-1/2} = S_h^* \left(1 + \sqrt{h} U_h^*\right)^{-1/2}.$$

In particular, for the i.i.d. bootstrap, $V^* = R_4 - RV^2$ and $\hat{V}^* = R_4^* - RV^{*2}$. For the WB, $V^* = \left(\mu_4^* - \mu_2^{*2}\right) R_4$ and $\hat{V}^* = \left(\frac{\mu_4^* - \mu_2^{*2}}{\mu_4^*}\right) R_4^*$. Recall that $R_q^* = h^{1-q/2} \sum_{i=1}^{1/h} \left|r_i^*\right|^q$.

Cumulants expansions

Theorem A.1 (Cumulants of T_h) *Consider DGP (2). Suppose v is independent of W and in addition assume Assumption (V) holds. Then for any $q > 0$, $\overline{\sigma_h^q} - \overline{\sigma^q} = o_P\left(\sqrt{h}\right)$, and conditionally on*

v, as $h \to 0$,

$$\kappa_1 (T_h) = \sqrt{h} \underbrace{\left(-\frac{A_1}{2} \frac{\overline{\sigma^6}}{\left(\overline{\sigma^4}\right)^{3/2}} \right)}_{\equiv \kappa_1} + o(h),$$

$$\kappa_2 (T_h) = 1 + h \underbrace{\left((C_1 - A_2) \frac{\overline{\sigma^8}}{\left(\overline{\sigma^4}\right)^2} + \frac{7}{4} A_1^2 \frac{\left(\overline{\sigma^6}\right)^2}{\left(\overline{\sigma^4}\right)^3} \right)}_{\equiv \kappa_2} + o(h),$$

$$\kappa_3 (T_h) = \sqrt{h} \underbrace{\left((B_1 - 3A_1) \frac{\overline{\sigma^6}}{\left(\overline{\sigma^4}\right)^{3/2}} \right)}_{\equiv \kappa_3} + o(h),$$

$$\kappa_4 (T_h) = h \underbrace{\left((B_2 + 3C_1 - 6A_2) \frac{\overline{\sigma^8}}{\left(\overline{\sigma^4}\right)^2} + \left(18A_1^2 - 6A_1 B_1\right) \frac{\left(\overline{\sigma^6}\right)^2}{\left(\overline{\sigma^4}\right)^3} \right)}_{\equiv \kappa_4} + o(h),$$

where, letting $\mu_q = E\,|Z|^q$, $Z \sim N(0,1)$, *and noticing in particular that* $\mu_2 = 1$, $\mu_4 = 3$, $\mu_6 = 15$ *and* $\mu_8 = 105$,

$$\begin{aligned}
A_1 &= \frac{\mu_6 - \mu_2 \mu_4}{\mu_4 \left(\mu_4 - \mu_2^2\right)^{1/2}} = \frac{4}{\sqrt{2}}, \quad A_2 = \frac{\mu_8 - \mu_4^2 - 2\mu_2\mu_6 + 2\mu_2^2\mu_4}{\mu_4\left(\mu_4 - \mu_2^2\right)} = 12, \\
B_1 &= \frac{\mu_6 - 3\mu_2\mu_4 + 2\mu_2^3}{\left(\mu_4 - \mu_2^2\right)^{3/2}} = \frac{4}{\sqrt{2}}, \quad B_2 = \frac{\mu_8 - 4\mu_2\mu_6 + 12\mu_2^2\mu_4 - 6\mu_2^4 - 3\mu_4^2}{\left(\mu_4 - \mu_2^2\right)^2} = 12, \\
C_1 &= \frac{\mu_8 - \mu_4^2}{\mu_4^2} = \frac{32}{3}.
\end{aligned}$$

Theorem A.2 (Cumulants of the i.i.d. bootstrap) *Let* $r_i^* \sim$ *i.i.d. from* $\{r_i : i = 1, \ldots, h\}$. *Under the same conditions as Theorem A.1, as* $h \to 0$,

$$\kappa_1^* (T_h^*) = \sqrt{h} \underbrace{\left(-\frac{\tilde{A}_1}{2} \right)}_{\equiv \kappa_{1,h}^*} + o_P(h),$$

$$\kappa_2^* (T_h^*) = 1 + h \underbrace{\left[\tilde{C} - \tilde{A}_2 - \frac{1}{4}\tilde{A}_1^2 \right]}_{\equiv \kappa_{2,h}^*} + o_P(h),$$

$$\kappa_3^* (T_h^*) = \sqrt{h} \underbrace{\left(-2\tilde{A}_1 \right)}_{\equiv \kappa_{3,h}^*} + o_P(h),$$

$$\kappa_4^* (T_h^*) = h \underbrace{\left[\left(\tilde{B}_2 - 2\tilde{D} + 3\tilde{E} \right) - 6\left(\tilde{C} - \tilde{A}_2 \right) - 4\tilde{A}_1^2 \right]}_{\equiv \kappa_{4,h}^*} + o_P(h),$$

where

$$
\begin{aligned}
\tilde{A}_1 &= \frac{R_6 - 3R_4 RV + 2RV^3}{\left(R_4 - RV^2\right)^{3/2}}, \quad \tilde{A}_2 = \frac{R_8 - 4R_4^2 - 4R_6 RV + 14R_4 RV^2 - 7RV^4}{\left(R_4 - RV^2\right)^2}, \\
\tilde{B}_2 &= \frac{R_8 - 4R_6 RV + 12R_4 RV^2 - 6RV^4 - 3R_4^2}{\left(R_4 - RV^2\right)^2}, \\
\tilde{C} &= \frac{R_8 - R_4^2}{\left(R_4 - RV^2\right)^2} + \frac{2\left(R_6 - R_4 RV\right)^2}{\left(R_4 - RV^2\right)^3} - \frac{12\left(R_6 - R_4 RV\right)(RV)}{\left(R_4 - RV^2\right)^2} + \frac{12RV^2}{R_4 - RV^2}, \\
\tilde{D} &= \frac{4\left(R_6 - 3R_4 RV + 2RV^3\right)\left(R_6 - R_4 RV\right)}{\left(R_4 - RV^2\right)^3} + \frac{6\left(R_8 - R_4^2 - 2R_6 RV + 2R_4 RV^2\right)}{\left(R_4 - RV^2\right)^2} \\
&\quad -15 - \frac{20RV\left(R_6 - 3R_4 RV + 2RV^3\right)}{\left(R_4 - RV^2\right)^2}, \\
\tilde{E} &= \frac{3\left(R_8 - R_4^2\right)}{\left(R_4 - RV^2\right)^2} + \frac{12\left(R_6 - R_4 RV\right)^2}{\left(R_4 - RV^2\right)^3} - \frac{60\left(R_6 - R_4 RV\right)(RV)}{\left(R_4 - RV^2\right)^2} + \frac{60\,(RV)^2}{R_4 - RV^2}.
\end{aligned}
$$

The following corollary follows from Theorems A.1 and A.2, noting in particular that $\operatorname{plim}_{h\to 0} R_q = \mu_q \overline{\sigma^q}$ for each $q > 0$ under our assumptions (cf. BN-S, 2004b, Theorem 1).

Corollary A.1 (Probability limits of i.i.d. bootstrap cumulants) *Under the assumptions of Theorem A.2, conditionally on v,*

$$
\begin{aligned}
\operatorname*{plim}_{h\to 0} \kappa_{1,h}^* - \kappa_1 &= -\frac{1}{2}\left(\frac{15\overline{\sigma^6} - 9\overline{\sigma^2}\ \overline{\sigma^4} + 2\left(\overline{\sigma^2}\right)^3}{\left(3\overline{\sigma^4} - \left(\overline{\sigma^2}\right)^2\right)^{3/2}} - \frac{4}{\sqrt{2}} \frac{\overline{\sigma^6}}{\left(\overline{\sigma^4}\right)^{3/2}} \right) \\
\operatorname*{plim}_{h\to 0} \kappa_{2,h}^* - \kappa_2 &= \operatorname*{plim}_{h\to 0}\left(\tilde{C}_h - \tilde{A}_{2,h} - \frac{\tilde{A}_{1,h}^2}{4}\right) + \frac{4}{3}\frac{\overline{\sigma^8}}{\left(\overline{\sigma^4}\right)^2} - 14\frac{\left(\overline{\sigma^6}\right)^2}{\left(\overline{\sigma^4}\right)^3} \\
\operatorname*{plim}_{h\to 0} \kappa_{3,h}^* - \kappa_3 &= -2\left(\frac{15\overline{\sigma^6} - 9\overline{\sigma^2}\ \overline{\sigma^4} + 2\left(\overline{\sigma^2}\right)^3}{\left(3\overline{\sigma^4} - \left(\overline{\sigma^2}\right)^2\right)^{3/2}} - \frac{4}{\sqrt{2}} \frac{\overline{\sigma^6}}{\left(\overline{\sigma^4}\right)^{3/2}} \right) \\
\operatorname*{plim}_{h\to 0} \kappa_{4,h}^* - \kappa_4 &= \operatorname*{plim}_{h\to 0}\left(\left(\tilde{B}_{2,h} - 2\tilde{D}_h + 3\tilde{E}_h\right) - 6\left(\tilde{C}_h - \tilde{A}_{2,h}\right) - 4\tilde{A}_{1,h}^2\right) + 28\frac{\overline{\sigma^8}}{\left(\overline{\sigma^4}\right)^2} - 96\frac{\left(\overline{\sigma^6}\right)^2}{\left(\overline{\sigma^4}\right)^3},
\end{aligned}
$$

where $\operatorname{plim}_{h\to 0}\left(\tilde{C}_h - \tilde{A}_{2,h} - \frac{\tilde{A}_{1,h}^2}{4}\right)$ *and* $\operatorname{plim}_{h\to 0}\left(\left(\tilde{B}_{2,h} - 2\tilde{D}_h + 3\tilde{E}_h\right) - 6\left(\tilde{C}_h - \tilde{A}_{2,h}\right) - 4\tilde{A}_{1,h}^2\right)$ *can be obtained from Theorem A.2 by noting that* $\operatorname{plim}_{h\to 0} R_q = \mu_q \overline{\sigma^q}$ *for each* $q > 0$.

Theorem A.3 (Cumulants of the WB) *Let $r_i^* = r_i \eta_i$, where $\eta_i \sim$ i.i.d. from a distribution independent of $\{r_i\}$ such that $\mu_q^* = E^* \left|\eta_i\right|^q$ for any $q > 0$. Then, under the same conditions as Theorem*

A.1, as $h \to 0$,

$$\kappa_1^*(T_h^*) = \underbrace{\sqrt{h}\left(-\frac{A_1^*}{2}\frac{R_6}{(R_4)^{3/2}}\right)}_{\equiv \kappa_{1,h}^*} + o_P(h),$$

$$\kappa_2^*(T_h^*) = 1 + \underbrace{h\left((C_1^* - A_2^*)\frac{R_8}{(R_4)^2} + \frac{7}{4}A_1^{*2}\frac{(R_6)^2}{(R_4)^3}\right)}_{\equiv \kappa_{2,h}^*} + o_P(h),$$

$$\kappa_3^*(T_h^*) = \underbrace{\sqrt{h}\left((B_1^* - 3A_1^*)\frac{R_6}{(R_4)^{3/2}}\right)}_{\equiv \kappa_{3,h}^*} + o_P(h),$$

$$\kappa_4^*(T_h^*) = \underbrace{h\left((B_2^* + 3C_1^* - 6A_2^*)\frac{R_8}{(R_4)^2} + \left(18A_1^{*2} - 6A_1^*B_1^*\right)\frac{(R_6)^2}{(R_4)^3}\right)}_{\equiv \kappa_{4,h}^*} + o_P(h),$$

where

$$A_1^* = \frac{\mu_6^* - \mu_2^*\mu_4^*}{\mu_4^*\left(\mu_4^* - \mu_2^{*2}\right)^{1/2}}, \quad A_2^* = \frac{\mu_8^* - \mu_4^{*2} - 2\mu_2^*\mu_6^* + 2\mu_2^{*2}\mu_4^*}{\mu_4^*\left(\mu_4^* - \mu_2^{*2}\right)},$$

$$B_1^* = \frac{\mu_6^* - 3\mu_2^*\mu_4^* + 2\mu_2^{*3}}{\left(\mu_4^* - \mu_2^{*2}\right)^{3/2}}, \quad B_2^* = \frac{\mu_8^* - 4\mu_2^*\mu_6^* + 12\mu_2^{*2}\mu_4^* - 6\mu_2^{*4} - 3\mu_4^{*2}}{\left(\mu_4^* - \mu_2^{*2}\right)^2},$$

$$C_1^* = \frac{\mu_8^* - \mu_4^{*2}}{\mu_4^{*2}}.$$

Corollary A.2 complements Proposition 4.3 (in Section 4.2) by providing results for the second and fourth order cumulants of the WB. Both results follow from Theorems A.1 and A.3 by noting that $\operatorname{plim}_{h\to 0} R_q = \mu_q \overline{\sigma^q}$ for each $q > 0$ under our assumptions (cf. BN-S, 2004b, Theorem 1).

Corollary A.2 (Probability limits of WB cumulants) *Under the assumptions of Theorem A.3, conditionally on* v,

$$\operatorname*{plim}_{h\to 0} \kappa_{2,h}^* - \kappa_2 = \frac{\overline{\sigma^8}}{\left(\overline{\sigma^4}\right)^2}\left[\frac{35}{3}(C_1^* - A_2^*) - (C_1 - A_2)\right] + \frac{7}{4}\frac{\left(\overline{\sigma^6}\right)^2}{\left(\overline{\sigma^4}\right)^3}\left(\frac{25}{3}A_1^{*2} - A_1^2\right)$$

$$\operatorname*{plim}_{h\to 0} \kappa_{4,h}^* - \kappa_4 = \frac{\overline{\sigma^8}}{\left(\overline{\sigma^4}\right)^2}\left[\frac{35}{3}(B_2^* + 3C_1^* - 6A_2^*) - (B_2 + 3C_1 - 6A_2)\right]$$
$$+ \frac{\left(\overline{\sigma^6}\right)^2}{\left(\overline{\sigma^4}\right)^3}\left[\frac{25}{3}\left(18A_1^{*2} - 6A_1^*B_1^*\right) - \left(18A_1^2 - 6A_1B_1\right)\right]$$

with $A_1 = B_1 = \frac{4}{\sqrt{2}}$, $A_2 = B_2 = 12$, *and* $C_1 = \frac{32}{3}$, *and where* $A_1^*, A_2^*, B_1^*, B_2^*$ *and* C_1^* *are defined in Theorem A.3.*

Auxiliary Lemmas

We rely on the following Lemmas for the proofs of Theorems A.1, A.2 and A.3. Throughout this Appendix we will use the following notation: $\sum_{i\neq j\neq\cdots\neq k}$ denotes a sum where all indices differ, e.g. $\sum_{i\neq j\neq k}\equiv\sum_{i\neq j,i\neq k,j\neq k}$.

Lemma A.1 *Let q, p and s be positive even integers. It follows that*

a1) $\sum_{i\neq j}^{1/h}\sigma_i^q\sigma_j^p=h^{-2+\frac{q+p}{2}}\left(\overline{\sigma_h^q}\;\overline{\sigma_h^p}-h\;\overline{\sigma_h^{q+p}}\right).$

a2)

$$\sum_{i\neq j\neq l}^{1/h}\sigma_i^q\sigma_j^p\sigma_l^s = h^{-3+\frac{q+p+s}{2}}\left(\overline{\sigma_h^q}\right)\left(\overline{\sigma_h^p}\right)\left(\overline{\sigma_h^s}\right)\overline{\sigma_h^4}-h^{-2+\frac{q+p+s}{2}}\left(\overline{\sigma_h^{q+p}}\;\overline{\sigma_h^s}+\overline{\sigma_h^{q+s}}\;\overline{\sigma_h^p}+\overline{\sigma_h^q}\;\overline{\sigma_h^{p+s}}\right)$$
$$+2h^{-1+\frac{q+p+s}{2}}\;\overline{\sigma_h^{q+p+s}}.$$

Lemma A.2 *Under the same conditions as Theorem A.1, as $h\to 0$,*

a1) $E\left|r_i\right|^q=\mu_q\sigma_i^q.$

a2) $V\equiv Var\left(h^{-1/2}RV\right)=\left(\mu_4-\mu_2^2\right)\overline{\sigma_h^4}.$

a3) $E\left[\left(RV-\mu_2IV\right)^3\right]=h^2\left(\mu_6-3\mu_2\mu_4+2\mu_2^3\right)\overline{\sigma_h^6}.$

a4) $E\left[\left(RV-\mu_2IV\right)^4\right]=3h^2\left(\mu_4-\mu_2^2\right)^2\left(\overline{\sigma_h^4}\right)^2+h^3\left(\mu_8-4\mu_2\mu_6+12\mu_2^2\mu_4-6\mu_2^4-3\mu_4^2\right)\overline{\sigma_h^8}.$

a5) $E\left[\left(RV-\mu_2IV\right)\left(\hat{V}-V\right)\right]=h\frac{\left(\mu_4-\mu_2^2\right)\left(\mu_6-\mu_2\mu_4\right)}{\mu_4}\overline{\sigma_h^6}.$

a6) $E\left[\left(RV-\mu_2IV\right)^2\left(\hat{V}-V\right)\right]=h^2\frac{\mu_4-\mu_2^2}{\mu_4}\left(\mu_8-\mu_4^2-2\mu_2\mu_6+2\mu_2^2\mu_4\right)\overline{\sigma_h^8}.$

a7) $E\left[\left(RV-\mu_2IV\right)^3\left(\hat{V}-V\right)\right]=3h^2\frac{\left(\mu_4-\mu_2^2\right)^2\left(\mu_6-\mu_2\mu_4\right)}{\mu_4}\overline{\sigma_h^4}\;\overline{\sigma_h^6}+O\left(h^3\right).$

a8) $E\left[\left(RV-\mu_2IV\right)^4\left(\hat{V}-V\right)\right]=h^3\frac{\mu_4-\mu_2^2}{\mu_4}\left[\begin{array}{c}4\left(\mu_6-3\mu_2\mu_4+2\mu_2^3\right)\left(\mu_6-\mu_2\mu_4\right)\left(\overline{\sigma_h^6}\right)^2\\+6\left(\mu_8-\mu_4^2-2\mu_2\mu_6+2\mu_2^2\mu_4\right)\left(\mu_4-\mu_2^2\right)\overline{\sigma_h^4}\;\overline{\sigma_h^8}\end{array}\right]+$ $O\left(h^4\right).$

a9) $E\left[\left(RV-\mu_2IV\right)\left(\hat{V}-V\right)^2\right]=\frac{\left(\mu_4-\mu_2^2\right)^2}{\mu_4^2}\left(\mu_{10}-2\mu_4\mu_6-\mu_2\mu_8+2\mu_2\mu_4^2\right)h^2\overline{\sigma_h^{10}}=O\left(h^2\right).$

a10) $E\left[\left(RV-\mu_2IV\right)^2\left(\hat{V}-V\right)^2\right]=h^2\frac{\left(\mu_4-\mu_2^2\right)^2}{\mu_4^2}\left(\left(\mu_4-\mu_2^2\right)\left(\mu_8-\mu_4^2\right)\overline{\sigma_h^4}\;\overline{\sigma_h^8}+2\left(\mu_6-\mu_2\mu_4\right)^2\left(\overline{\sigma_h^6}\right)^2\right)+$ $O\left(h^3\right).$

a11) $E\left[\left(RV-\mu_2IV\right)^3\left(\hat{V}-V\right)^2\right]=O\left(h^3\right)+O\left(h^4\right).$

a12) $E\left[(RV-\mu_2 IV)^4\left(\hat{V}-V\right)^2\right]=h^3\frac{\left(\mu_4-\mu_2^2\right)^2}{\mu_4^2}\left[\begin{array}{c}3\left(\mu_4-\mu_2^2\right)^2\left(\mu_8-\mu_4^2\right)\left(\overline{\sigma_h^4}\right)^2\overline{\sigma_h^8}\\+12\left(\mu_4-\mu_2^2\right)\left(\mu_6-\mu_2\mu_4\right)^2\left(\overline{\sigma_h^6}\right)^2\overline{\sigma_h^4}\end{array}\right]+O\left(h^4\right).$

Lemma A.3 *Under the same conditions as Theorem A.1, as $h\to 0$,*

a1) $E\left(S_h\right)=0.$

a2) $E\left(S_h^2\right)=1.$

a3) $E\left(S_h^3\right)=\sqrt{h}\left(B_1\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right).$

a4) $E\left(S_h^4\right)=3+h\left(B_2\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}\right).$

a5) $E\left(S_h U_h\right)=A_1\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}.$

a6) $E\left(S_h^2 U_h\right)=\sqrt{h}\left(A_2\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}\right).$

a7) $E\left(S_h^3 U_h\right)=A_3\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}+O\left(h\right).$

a8) $E\left(S_h^4 U_h\right)=\sqrt{h}\left[D_1\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+D_2\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right]+O\left(h^{3/2}\right).$

a9) $E\left(S_h U_h^2\right)=O\left(h^{1/2}\right).$

a10) $E\left(S_h^2 U_h^2\right)=\left[C_1\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+C_2\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right]+O\left(h\right).$

a11) $E\left(S_h^3 U_h^2\right)=O\left(h^{1/2}\right).$

a12) $E\left(S_h^4 U_h^2\right)=\left[E_1\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+E_2\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right]+O\left(h\right).$

The constants A_1, A_2, B_1, B_2, and C_1 are defined as in Theorem A.1, and $A_3=3A_1$, $C_2=2A_1^2$, $D_1=6A_2$, $D_2=4A_1B_1$, $E_1=3C_1$, and $E_2=12A_1^2$.

Remark 1 *The WB analogues of Lemmas A.2 and A.3 can be obtained from these same Lemmas by making two changes. First, replace the moments μ_q of the $N(0,1)$ distribution defining the constants A_1 through E_2 with the WB moments $\mu_q^* = E^* |\eta_i|^q$. This yields the WB constants A_1^* through E_2^*. Second, replace the power integrated volatilities $\overline{\sigma^q}$ with the power realized volatilities R_q for any $q > 0$. Thus, for instance, the WB analogue of Lemma A.3.a3) is $E^*\left(S_h^{*3}\right) = \sqrt{h}\left(B_1^* \frac{R_6}{(R_4)^{3/2}}\right)$, where $B_1^* = \frac{\mu_6^* - 3\mu_2^*\mu_4^* + 2\mu_2^{*3}}{\left(\mu_4^* - \mu_2^{*2}\right)^{3/2}}$.*

Lemma A.7 below is the i.i.d. bootstrap analog of Lemma A.3. Lemmas A.4−A.6 are auxiliary in proving Lemma A.7.

Lemma A.4 *Let $r_i^* \sim$ i.i.d. from $\{r_i : i = 1, \ldots, h\}$. Under the same conditions as Theorem A.1, for any $q > 0$ and for any $i = 1, \ldots, 1/h$,*

a1) $E^*\left(|r_i^*|^q\right) = h^{q/2} R_q$ *and* $E^*\left(R_q^*\right) = R_q = O_P(1)$.

a2) $E^*\left[\left(r_i^{*2} - hRV\right)^2\right] = h^2\left(R_4 - RV^2\right)$.

a3) $E^*\left[\left(r_i^{*2} - hRV\right)^3\right] = h^3\left(R_6 - 3R_4RV + 2RV^3\right)$.

a4) $E^*\left[\left(r_i^{*2} - hRV\right)^4\right] = h^4\left(R_8 - 4R_6RV + 6R_4RV^2 - 3RV^4\right)$.

a5) $E^*\left[\left(r_i^{*2} - hRV\right)^5\right] = h^5\left(R_{10} - 5R_8RV + 10R_6RV^2 - 10R_4RV^3 + 4RV^5\right)$.

a6) $E^*\left[\left(r_i^{*2} - hRV\right)^6\right] = h^6\left(R_{12} - 6R_{10}RV + 15R_8RV^2 - 20R_6RV^3 + 15R_4RV^4 - 5RV^6\right)$.

a7) $E^*\left[\left(r_i^{*2} - hRV\right)^q\right] = O_P\left(h^q\right)$, *for any* $q \geq 7$.

a8) $E^*\left[\left(r_i^{*4} - h^2R_4\right)^2\right] = h^4\left(R_8 - R_4^2\right)$.

a9) $E^*\left[\left(r_i^{*2} - hRV\right)\left(r_i^{*4} - h^2R_4\right)\right] = h^3\left(R_6 - R_4RV\right)$.

a10) $E^*\left[\left(r_i^{*2} - hRV\right)^2\left(r_i^{*4} - h^2R_4\right)\right] = h^4\left(R_8 - R_4^2 - 2R_6RV + 2R_4RV^2\right)$.

a11) $E^*\left[\left(r_i^{*2} - hRV\right)^3\left(r_i^{*4} - h^2R_4\right)\right] = h^5\left(R_{10} - R_4R_6 - 3R_8RV + 3R_4^2RV + 3R_6RV^2 - 3R_4RV^3\right)$.

a12) $E^*\left[\left(r_i^{*2} - hRV\right)^4\left(r_i^{*4} - h^2R_4\right)\right] = h^6\left(\begin{array}{c} R_{12} - R_4R_8 - 4R_{10}RV + 4R_4R_6RV + 6R_8RV^2 \\ -6R_4^2RV^2 - 4R_6RV^3 + 4R_4RV^4 \end{array}\right)$.

a13) $E^*\left(\left(r_i^{*2} - hRV\right)\left(r_i^{*4} - h^2R_4\right)^2\right) = h^5\left(R_{10} - 2R_4R_6 - R_8RV + 2R_4^2RV\right)$.

a14) *For any* $q, p > 0$, $E^*\left[\left(r_i^{*2} - hRV\right)^q\left(r_i^{*4} - h^2R_4\right)^p\right] = O_P\left(h^{q+2p}\right)$.

Lemma A.5 *Let $r_i^* \sim$ i.i.d. from $\{r_i : i = 1, \ldots, h\}$. Under the same conditions as Theorem A.1, for any $q > 0$,*

a1) $V^* \equiv Var^*\left(h^{-1/2}RV^*\right) = R_4 - RV^2$.

a2) $\hat{V}^* - V^* = R_4^* - R_4 - \left[\left(RV^* - RV\right)^2 + 2RV\left(RV^* - RV\right)\right]$.

a3) $E^*\left[(RV^*-RV)^3\right] = h^2\left(R_6 - 3R_4RV + 2RV^3\right).$

a4) $E^*\left[(RV^*-RV)^4\right] = h^2\left[3\left(R_4 - RV^2\right)^2\right] + h^3\left(R_8 - 4R_6RV + 12R_4RV^2 - 6RV^4 - 3R_4^2\right).$

a5) $E^*\left[(RV^*-RV)^5\right] = h^3\left[10\left(R_6 - 3R_4RV + 2RV^3\right)\left(R_4 - RV^2\right)\right] + O_P\left(h^4\right).$

a6) $E^*\left[(RV^*-RV)^6\right] = h^3\left[15\left(R_4 - RV^2\right)^3\right] + O_P\left(h^4\right).$

a7) $E^*\left[(RV^*-RV)^q\right] = O_P\left(h^4\right)$ *for* $q = 7, 8.$

a8) $E^*\left[(RV^*-RV)\left(R_4^*-R_4\right)\right] = h\left(R_6 - R_4RV\right).$

a9) $E^*\left[(RV^*-RV)^2\left(R_4^*-R_4\right)\right] = h^2\left(R_8 - R_4^2 - 2R_6RV + 2R_4RV^2\right).$

a10) $E^*\left[(RV^*-RV)^3\left(R_4^*-R_4\right)\right] = 3h^2\left(R_6 - R_4RV\right)\left(R_4 - RV^2\right) + O_P\left(h^3\right).$

a11) $E^*\left[(RV^*-RV)^4\left(R_4^*-R_4\right)\right] = h^3\left[\begin{array}{c} 4\left(R_6 - 3R_4RV + 2RV^3\right)\left(R_6 - R_4RV\right) \\ +6\left(R_4 - RV^2\right)\left(R_8 - R_4^2 - 2R_6RV + 2R_4RV^2\right) \end{array}\right] + O_P\left(h^4\right).$

a12) $E^*\left[(RV^*-RV)^5\left(R_4^*-R_4\right)\right] = h^3\left[15\left(R_4 - RV^2\right)^2\left(R_6 - R_4RV\right)\right] + O_P\left(h^4\right).$

a13) $E^*\left[(RV^*-RV)^6\left(R_4^*-R_4\right)\right] = O_P\left(h^4\right).$

a14) $E^*\left[(RV^*-RV)\left(R_4^*-R_4\right)^2\right] = h^2\left(R_{10} - 2R_4R_6 - R_8RV + 2R_4^2RV\right).$

a15) $E^*\left[(RV^*-RV)^2\left(R_4^*-R_4\right)^2\right] = h^2\left[\left(R_4 - RV^2\right)\left(R_8 - R_4^2\right) + 2\left(R_6 - R_4RV\right)^2\right] + O_P\left(h^3\right).$

a16) $E^*\left[(RV^*-RV)^3\left(R_4^*-R_4\right)^2\right] = O_P\left(h^3\right).$

a17) $E^*\left[(RV^*-RV)^4\left(R_4^*-R_4\right)^2\right] = h^3\left[\begin{array}{c} 3\left(R_4 - RV^2\right)^2\left(R_8 - R_4^2\right) \\ +12\left(R_6 - R_4RV\right)^2\left(R_4 - RV^2\right) \end{array}\right] + O_P\left(h^4\right).$

Lemma A.6 *Let* $r_i^* \sim$ *i.i.d. from* $\{r_i : i = 1, \ldots, h\}$*. Under the same conditions as Theorem A.1,*

a1) $E^*\left[(RV^*-RV)\left(\hat{V}^* - V^*\right)\right] = h\left(R_6 - 3R_4RV + 2RV^3\right) + O_P\left(h^2\right).$

a2) $E^*\left[(RV^*-RV)^2\left(\hat{V}^* - V^*\right)\right] = h^2\left[\begin{array}{c} \left(R_8 - R_4^2 - 2R_6RV + 2R_4RV^2\right) - 3\left(R_4 - RV^2\right)^2 \\ -2RV\left(R_6 - 3R_4RV + 2RV^3\right) \end{array}\right] + O_P\left(h^3\right).$

a3) $E^*\left[(RV^*-RV)^3\left(\hat{V}^* - V^*\right)\right] = h^2\left[3\left(R_4 - RV^2\right)\left(R_6 - 3R_4RV + 2RV^3\right)\right] + O_P\left(h^3\right).$

a4)

$$
\begin{aligned}
E^*\left[(RV^*-RV)^4\left(\hat{V}^* - V^*\right)\right] &= h^3\left[\begin{array}{c} 4\left(R_6 - 3R_4RV + 2RV^3\right)\left(R_6 - R_4RV\right) \\ +6\left(R_4 - RV^2\right)\left(R_8 - R_4^2 - 2R_6RV + 2R_4RV^2\right) \end{array}\right] \\
&\quad -h^3\left[15\left(R_4 - RV^2\right)^3\right] \\
&\quad -h^3\left[20RV\left(R_6 - 3R_4RV + 2RV^3\right)\left(R_4 - RV^2\right)\right] + O_P\left(h^4\right).
\end{aligned}
$$

a5) $E^*\left[\left(RV^*-RV\right)\left(\hat{V}^*-V^*\right)^2\right]=O_P\left(h^2\right).$

a6) $E^*\left[\left(RV^*-RV\right)^2\left(\hat{V}^*-V^*\right)^2\right]=h^2\left[\begin{array}{c}\left(R_4-RV^2\right)\left(R_8-R_4^2\right)+2\left(R_6-R_4RV\right)^2\\ -12\left(R_6-R_4RV\right)\left(R_4-RV^2\right)\left(RV\right)\\ +4\left(RV\right)^2\left[3\left(R_4-RV^2\right)^2\right]\end{array}\right]+O_P\left(h^3\right).$

a7) $E^*\left[\left(RV^*-RV\right)^3\left(\hat{V}^*-V^*\right)^2\right]=O_P\left(h^3\right).$

a8)

$$\begin{aligned}E^*\left[\left(RV^*-RV\right)^4\left(\hat{V}^*-V^*\right)^2\right] &= h^3\left[3\left(R_4-RV^2\right)^2\left(R_8-R_4^2\right)+12\left(R_6-R_4RV\right)^2\left(R_4-RV^2\right)\right]\\ &\quad -h^3\left[60\left(R_4-RV^2\right)^2\left(R_6-R_4RV\right)\left(RV\right)\right]\\ &\quad +h^3\left[60\left(R_4-RV^2\right)^3\left(RV\right)^2\right]+O_P\left(h^4\right).\end{aligned}$$

Lemma A.7 *Let $r_i^*\sim$ i.i.d. from $\{r_i : i=1,\ldots,h\}$. Under the same conditions as Theorem A.1,*

a1) $E^*\left(S_h^*\right)=0.$

a2) $E^*\left(S_h^{*2}\right)=1.$

a3) $E^*\left(S_h^{*3}\right)=\sqrt{h}\tilde{B}_1.$

a4) $E^*\left(S_h^{*4}\right)=3+h\tilde{B}_2.$

a5) $E^*\left(S_h^*U_h^*\right)=\tilde{A}_1+O_P\left(h\right).$

a6) $E^*\left(S_h^{*2}U_h^*\right)=\sqrt{h}\tilde{A}_2+O_P\left(h^{3/2}\right).$

a7) $E^*\left(S_h^{*3}U_h^*\right)=\tilde{A}_3+O_P\left(h\right).$

a8) $E^*\left(S_h^{*4}U_h^*\right)=\sqrt{h}\tilde{D}+O_P\left(h^{3/2}\right).$

a9) $E^*\left(S_h^*U_h^{*2}\right)=O_P\left(h^{1/2}\right).$

a10) $E^*\left(S_h^{*2}U_h^{*2}\right)=\tilde{C}+O_P\left(h\right).$

a11) $E^*\left(S_h^{*3}U_h^{*2}\right)=O_P\left(h^{1/2}\right).$

a12) $E^*\left(S_h^{*4}U_h^{*2}\right)=\tilde{E}+O_P\left(h\right).$

The bootstrap constants $\tilde{A}_1,\tilde{A}_2,\tilde{B}_2,\tilde{C},\tilde{D}$ and $\tilde{E}$ are as defined in Theorem A.2. $\tilde{A}_3$ and $\tilde{B}_1$ are such that $\tilde{A}_3=3\tilde{A}_1$ and $\tilde{B}_1=\tilde{A}_1$.

Proofs of Theorems A.1–A.3

Proof of Theorem A.1. The first four cumulants of T_h are given by (e.g., Hall, 1992, p. 42):

$$\begin{aligned}
\kappa_1(T_h) &= E(T_h), \kappa_2(T_h) = E\left(T_h^2\right) - [E(T_h)]^2, \\
\kappa_3(T_h) &= E\left(T_h^3\right) - 3E\left(T_h^2\right)E(T_h) + 2[E(T_h)]^3, \text{ and} \\
\kappa_4(T_h) &= E\left(T_h^4\right) - 4E\left(T_h^3\right)E(T_h) - 3\left[E\left(T_h^2\right)\right]^2 + 12E\left(T_h^2\right)[E(T_h)]^2 - 6[E(T_h)]^4.
\end{aligned}$$

Our goal is to identify the terms of order up to $O(h)$ in the asymptotic expansions of these four cumulants. We will first provide asymptotic expansions through order $O(h)$ for the first four moments of T_h. Note that for a given fixed value of k, a second-order Taylor expansion of $f(x) = (1+x)^{-k/2}$ around 0 yields $f(x) = 1 - \frac{k}{2}x + \frac{k}{4}\left(\frac{k}{2}+1\right)x^2 + O\left(x^3\right)$. Thus, provided $U_h = O_P(1)$, we have that for any fixed integer k,

$$T_h^k = S_h^k\left(1+\sqrt{h}U_h\right)^{-k/2} = S_h^k - \frac{k}{2}\sqrt{h}S_h^kU_h + \frac{k}{4}\left(\frac{k}{2}+1\right)hS_h^kU_h^2 + O\left(h^{3/2}\right) \equiv \tilde{T}_h^k + O\left(h^{3/2}\right).$$

For $k = 1, \ldots, 4$, the moments of $\tilde{T}_h^k$ are given by

$$E\left(\tilde{T}_h\right) = 0 - \sqrt{h}\frac{1}{2}E(S_hU_h) + \frac{3}{8}hE\left(S_hU_h^2\right) \tag{30}$$

$$E\left(\tilde{T}_h^2\right) = 1 - \sqrt{h}E\left(S_h^2U_h\right) + hE\left(S_h^2U_h^2\right) \tag{31}$$

$$E\left(\tilde{T}_h^3\right) = E\left(S_h^3\right) - \sqrt{h}\frac{3}{2}E\left(S_h^3U_h\right) + \frac{15}{8}hE\left(S_h^3U_h^2\right) \tag{32}$$

$$E\left(\tilde{T}_h^4\right) = E\left(S_h^4\right) - 2\sqrt{h}E\left(S_h^4U_h\right) + 3hE\left(S_h^4U_h^2\right), \tag{33}$$

where we have used the fact that $E(S_h) = 0$ and $E\left(S_h^2\right) = 1$ by construction. By Lemma A.3, we have that

$$\begin{aligned}
E\left(\tilde{T}_h\right) &= \sqrt{h}\left(-\frac{1}{2}A_1\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right) + \frac{3}{8}h\left(O\left(h^{1/2}\right)\right) = \sqrt{h}\left(-\frac{1}{2}A_1\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right) + O\left(h^{3/2}\right), \\
E\left(\tilde{T}_h^2\right) &= 1 - \sqrt{h}\left(\sqrt{h}\left(A_2\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}\right)\right) + h\left(C_1\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2} + C_2\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3} + O(h)\right) \\
&= 1 + h\left[(C_1 - A_2)\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2} + C_2\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right] + O\left(h^2\right), \\
E\left(\tilde{T}_h^3\right) &= \sqrt{h}\left(B_1\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right) - \sqrt{h}\frac{3}{2}\left(A_3\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}} + O(h)\right) + \frac{15}{8}h\left(O\left(h^{1/2}\right)\right) \\
&= \sqrt{h}\left[\left(B_1 - \frac{3}{2}A_3\right)\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right] + O\left(h^{3/2}\right),
\end{aligned}$$

$$\begin{aligned}
E\left(\tilde{T}_h^4\right) &= 3+h\left(B_2\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}\right)-2\sqrt{h}\left(\sqrt{h}\left[D_1\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+D_2\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right]+O\left(h^{3/2}\right)\right)\\
&\quad +3h\left(E_1\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+E_2\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}+O\left(h\right)\right)\\
&= 3+h\left(\left(B_2-2D_1+3E_1\right)\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+\left(3E_2-2D_2\right)\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right)+O\left(h^2\right).
\end{aligned}$$

Thus $\kappa_1\left(\tilde{T}_h\right)=\sqrt{h}\left(-\frac{A_1}{2}\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right)+O\left(h^{3/2}\right)$. Since under Assumption (V), BN-S (2004b) show that $\overline{\sigma_h^q}-\overline{\sigma^q}=o\left(h^{1/2}\right)$, we can write $\kappa_1\left(T_h\right)=\sqrt{h}\left(-\frac{A_1}{2}\frac{\overline{\sigma^6}}{\left(\overline{\sigma^4}\right)^{3/2}}\right)+o\left(h\right)\equiv\sqrt{h}\kappa_1+o\left(h\right)$, proving the first result. Next,

$$\begin{aligned}
\kappa_2\left(\tilde{T}_h\right) &= E\left(\tilde{T}_h^2\right)-\left[E\left(\tilde{T}_h\right)\right]^2=1+h\left[\left(C_1-A_2\right)\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+C_2\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right]-h\left(-\frac{1}{2}A_1\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right)^2+O\left(h^2\right)\\
&= 1+h\underbrace{\left[\left(C_1-A_2\right)\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+\left(C_2-\frac{1}{4}A_1^2\right)\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right]}_{\equiv\kappa_{2,h}}+O\left(h^2\right),
\end{aligned}$$

and $\kappa_2\left(T_h\right)=1+h\kappa_{2,h}+O\left(h^{3/2}\right)$. Notice that since $C_2=2A_1^2$, we can write $\kappa_{2,h}=\left(C_1-A_2\right)\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+\frac{7}{8}C_2\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}$. Since $\overline{\sigma_h^q}-\overline{\sigma^q}=o\left(h^{1/2}\right)$ under Assumption (V), $\kappa_{2,h}=\kappa_2+o\left(\sqrt{h}\right)$, proving the second result. Next,

$$\begin{aligned}
\kappa_3\left(\tilde{T}_h\right) &= E\left(\tilde{T}_h^3\right)-3E\left(\tilde{T}_h^2\right)E\left(\tilde{T}_h\right)+2\left[E\left(\tilde{T}_h\right)\right]^3=\sqrt{h}\left[\left(B_1-\frac{3}{2}A_3\right)\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right]\\
&\quad -3\left(1+h\left[\left(C_1-A_2\right)\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+C_2\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right]\right)\left(\sqrt{h}\left(-\frac{1}{2}A_1\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right)\right)\\
&\quad +2\left(\sqrt{h}\left(-\frac{1}{2}A_1\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right)\right)^3=\sqrt{h}\underbrace{\left[\left(B_1-\frac{3}{2}A_3+\frac{3}{2}A_1\right)\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right]}_{=\kappa_{3,h}}+O\left(h^{3/2}\right).
\end{aligned}$$

Since $A_3=3A_1$, we can write $\kappa_{3,h}=\left(B_1-A_3\right)\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}$, and under Assumption (V), $\kappa_{3,h}=\kappa_3+$

$o\left(\sqrt{h}\right)$, proving the third result. Finally, for $\kappa_4\left(T_h\right)$, we have that

$$\begin{aligned}
\kappa_4\left(\tilde{T}_h\right) &= 3+h\left(\left(B_2-2D_1+3E_1\right)\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+\left(3E_2-2D_2\right)\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right)+O\left(h^2\right) \\
&\quad -4\left[\sqrt{h}\left(\left(B_1-\frac{3}{2}A_3\right)\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right)\sqrt{h}\left(-\frac{1}{2}A_1\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right)\right] \\
&\quad -3\left[1+h\left(\left(C_1-A_2\right)\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+C_2\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right)\right]^2 \\
&\quad +12\left(1+h\left(\left(C_1-A_2\right)\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+C_2\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right)\right)h\left(-\frac{1}{2}A_1\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right)^2 \\
&\quad -6\left(\sqrt{h}\left(-\frac{1}{2}A_1\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right)\right)^4+O\left(h^{3/2}\right) \\
&= 3+h\left(\left(B_2-2D_1+3E_1\right)\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+\left(3E_2-2D_2\right)\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right)+h\left(2A_1\left(B_1-\frac{3}{2}A_3\right)\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right) \\
&\quad -3\left[1+2h\left(\left(C_1-A_2\right)\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+C_2\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}\right)\right]+12h\left(-\frac{1}{2}A_1\frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4}\right)^{3/2}}\right)^2+O\left(h^2\right) \\
&= h\left(\left(B_2-2D_1+3E_1\right)-6\left(C_1-A_2\right)\right)\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2} \\
&\quad +h\left(\left(3E_2-2D_2\right)+2A_1\left(B_1-\frac{3}{2}A_3\right)-6C_2+3A_1^2\right)\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}+O\left(h^2\right).
\end{aligned}$$

Thus,

$$\kappa_{4,h}=\left(\left(B_2-2D_1+3E_1\right)-6\left(C_1-A_2\right)\right)\frac{\overline{\sigma_h^8}}{\left(\overline{\sigma_h^4}\right)^2}+\left(\left(3E_2-2D_2\right)+2A_1\left(B_1-\frac{3}{2}A_3\right)-6C_2+3A_1^2\right)\frac{\left(\overline{\sigma_h^6}\right)^2}{\left(\overline{\sigma_h^4}\right)^3}.$$

The result follows by noting that $D_1=6A_2$ and $E_1=3C_1$, and by using Assumption (V) to write $\kappa_{4,h}=\kappa_4+o\left(\sqrt{h}\right)$.

Proof of Theorem A.2. We follow the proof of Theorem A.1 and use Lemma A.7 instead of Lemma A.3. The cumulants expansions follow by noting that $\tilde{A}_3=3\tilde{A}_1$ and $\tilde{B}_1=\tilde{A}_1$. More specifically, for $k=1,\ldots,4$, define $\tilde{T}_h^{*k}$ similarly to $\tilde{T}_h^k$ and note that $T_h^{*k}=\tilde{T}_h^{*k}+O_P\left(h^{3/2}\right)$. Then use Lemma A.7 to obtain $E^*\left(\tilde{T}_h^{*k}\right)$ by the bootstrap analogues of (30)–(33). This yields $\kappa_1^*\left(\tilde{T}_h\right)=\sqrt{h}\left(-\frac{1}{2}\tilde{A}_1\right)+$

$O_P\left(h^{3/2}\right)$. Similarly,

$$\begin{aligned}
\kappa_2^*\left(\tilde{T}_h^*\right) &= E^*\left(\tilde{T}_h^{*2}\right) - \left[E^*\left(\tilde{T}_h^*\right)\right]^2 = 1 + h\left(\tilde{C} - \tilde{A}_2\right) + O_P\left(h^2\right) - h\left(-\frac{1}{2}\tilde{A}_1\right)^2 + O_P\left(h^2\right) \\
&= 1 + h\underbrace{\left[\tilde{C} - \tilde{A}_2 - \frac{1}{4}\tilde{A}_1^2\right]}_{\equiv\kappa_{2,h}^*} + O_P\left(h^2\right),
\end{aligned}$$

and $\kappa_2^*\left(T_h^*\right) = 1 + h\kappa_{2,h}^* + O_P\left(h^{3/2}\right)$. Next,

$$\begin{aligned}
\kappa_3^*\left(\tilde{T}_h^*\right) &= E^*\left(\tilde{T}_h^{*3}\right) - 3E^*\left(\tilde{T}_h^{*2}\right)E^*\left(\tilde{T}_h^*\right) + 2\left[E^*\left(\tilde{T}_h^*\right)\right]^3 \\
&= \sqrt{h}\left(\tilde{B}_1 - \frac{3}{2}\tilde{A}_3\right) - 3\left[1 + h\left(\tilde{C} - \tilde{A}_2\right)\right]\left(\sqrt{h}\left(-\frac{1}{2}\tilde{A}_1\right)\right) + 2\left(\sqrt{h}\left(-\frac{1}{2}\tilde{A}_1\right)\right)^3 \\
&= \sqrt{h}\underbrace{\left(\tilde{B}_1 - \frac{3}{2}\tilde{A}_3 + \frac{3}{2}\tilde{A}_1\right)}_{=\kappa_{3,h}^*} + O_P\left(h^{3/2}\right).
\end{aligned}$$

Since $\tilde{A}_3 = 3\tilde{A}_1$ and $\tilde{B}_1 = \tilde{A}_1$, we can write $\kappa_{3,h}^* = -2\tilde{A}_1$. Finally, for $\kappa_4^*\left(T_h^*\right)$, we have that

$$\begin{aligned}
\kappa_4^*\left(\tilde{T}_h^*\right) &= E^*\left(\tilde{T}_h^{*4}\right) - 4E^*\left(\tilde{T}_h^{*3}\right)E^*\left(\tilde{T}_h^*\right) - 3\left[E^*\left(\tilde{T}_h^{*2}\right)\right]^2 + 12E^*\left(\tilde{T}_h^{*2}\right)\left[E^*\left(\tilde{T}_h^*\right)\right]^2 - 6\left[E^*\left(\tilde{T}_h^*\right)\right]^4 \\
&= 3 + h\left[\left(\tilde{B}_2 - 2\tilde{D} + 3\tilde{E}\right) + 2\tilde{A}_1\left(\tilde{B}_1 - \frac{3}{2}\tilde{A}_3\right)\right] - 3\left[1 + 2h\left(\tilde{C} - \tilde{A}_2\right)\right] + 12h\left(\frac{1}{4}\tilde{A}_1^2\right) + O_P\left(h^2\right) \\
&= h\underbrace{\left[\left(\tilde{B}_2 - 2\tilde{D} + 3\tilde{E}\right) + 2\tilde{A}_1\left(\tilde{B}_1 - \frac{3}{2}\tilde{A}_3\right) - 6\left(\tilde{C} - \tilde{A}_2\right) + 3\tilde{A}_1^2\right]}_{\equiv\kappa_{4,h}^*} + O_P\left(h^2\right).
\end{aligned}$$

Since $\tilde{B}_1 = \tilde{A}_1$ and $\tilde{A}_3 = 3\tilde{A}_1$, $2\tilde{A}_1\left(\tilde{B}_1 - \frac{3}{2}\tilde{A}_3\right) = -7\tilde{A}_1^2$ and it follows that $\kappa_{4,h}^* = \left(\tilde{B}_2 - 2\tilde{D} + 3\tilde{E}\right) - 6\left(\tilde{C} - \tilde{A}_2\right) - 4\tilde{A}_1^2$, which concludes the proof.

Proof of Theorem A.3. See the proof of Theorem A.1 and Remark 1.

Proof of Lemmas A.1–A.7

Proof of Lemma A.1. For a1), note that

$$\begin{aligned}
\sum_{i\neq j}^{1/h}\sigma_i^q\sigma_j^p &= \left(\sum_{i=1}^{1/h}\sigma_i^q\right)\left(\sum_{j=1}^{1/h}\sigma_j^p\right) - \left(\sum_{i=1}^{1/h}\sigma_i^{q+p}\right) \\
&= h^{-1+\frac{q}{2}}\left(h^{1-\frac{q}{2}}\sum_{i=1}^{1/h}\sigma_i^q\right)h^{-1+\frac{p}{2}}\left(h^{1-\frac{p}{2}}\sum_{j=1}^{1/h}\sigma_j^p\right) - h^{-1+\frac{q+p}{2}}\left(h^{1-\frac{q+p}{2}}\sum_{i=1}^{1/h}\sigma_i^{q+p}\right) \\
&= h^{-2+\frac{q+p}{2}}\left(\overline{\sigma_h^q}\;\overline{\sigma_h^p} - h\;\overline{\sigma_h^{q+p}}\right).
\end{aligned}$$

For a2), note that

$$\sum_{i\neq j\neq k}\sigma_i^q\sigma_j^p\sigma_k^s=\left(\sum_{i=1}^{1/h}\sigma_i^q\right)\left(\sum_{j=1}^{1/h}\sigma_j^p\right)\left(\sum_{k=1}^{1/h}\sigma_k^s\right)-\sum_{i=1}^{1/h}\sigma_i^{q+p+s}-\sum_{i\neq j}\sigma_i^{q+p}\sigma_j^s-\sum_{i\neq j}\sigma_i^{q+s}\sigma_j^p-\sum_{i\neq j}\sigma_i^q\sigma_j^{p+s},$$

and then proceed as for a1).

Proof of Lemma A.2. a1) follows from $r_i=\sigma_i u_i$, where $u_i\sim$ i.i.d. $N(0,1)$. For a2), note that $RV=\sum_{i=1}^{1/h}r_i^2$, where r_i^2 is (conditional on v) independent with $Var\left(r_i^2\right)=E\left(r_i^4\right)-\left(E\left(r_i^2\right)\right)^2=\mu_4\sigma_i^4-\left(\mu_2\sigma_i^2\right)^2=\left(\mu_4-\mu_2^2\right)\sigma_i^4$, with $\sigma_i^4\equiv\left(\sigma_i^2\right)^2$. *Proof of a3):* Write

$$I_1\equiv E\left[(RV-\mu_2 IV)^3\right]=E\left[\sum_{i=1}^{1/h}\sum_{j=1}^{1/h}\sum_{k=1}^{1/h}\left(r_i^2-\mu_2\sigma_i^2\right)\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_k^2-\mu_2\sigma_k^2\right)\right].$$

The only non zero contribution to I_1 is when $i=j=k$, in which case we get $E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^3\right]=\left(\mu_6-3\mu_2\mu_4+2\mu_2^3\right)\sigma_i^6$ and $I_1=h^2\left(\mu_6-3\mu_2\mu_4+2\mu_2^3\right)\overline{\sigma_h^6}$, proving a3). *Proof of a4):* Using the independence and zero mean property of $\left\{r_i^2-\mu_2\sigma_i^2\right\}$, we have that

$$\begin{aligned}
E\left[(RV-\mu_2 IV)^4\right] &= \sum_{i=1}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^4\right]+3\sum_{i\neq j}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)^2\right]\\
&= I_1 h^3\overline{\sigma_h^8}+3\left(\mu_4-\mu_2^2\right)^2\left[h^2\left(\left(\overline{\sigma_h^4}\right)^2-h\left(\overline{\sigma_h^8}\right)\right)\right]\\
&= 3h^2\left(\mu_4-\mu_2^2\right)^2\left(\overline{\sigma_h^4}\right)^2+h^3 J_1\left(\overline{\sigma_h^8}\right),
\end{aligned}$$

given Lemma A.1, and where $J_1=E\left[\left(u_i^2-\mu_2\right)^4\right]=\mu_8-3\mu_2^4+6\mu_2^2\mu_4-4\mu_2\mu_6$. *Proof of a5):*

$$\begin{aligned}
E\left[(RV-\mu_2 IV)\left(\hat{V}-V\right)\right] &= \frac{\mu_4-\mu_2^2}{\mu_4}h^{-1}\sum_{i=1}^{1/h}E\left(r_i^6-r_i^2\mu_4\sigma_i^4-\mu_2\sigma_i^2 r_i^4+\mu_2\mu_4\sigma_i^6\right)\\
&= \frac{\mu_4-\mu_2^2}{\mu_4}h^{-1}h^2\left(\mu_6-\mu_2\mu_4\right)\overline{\sigma_h^6}=h\frac{\left(\mu_4-\mu_2^2\right)\left(\mu_6-\mu_2\mu_4\right)}{\mu_4}\overline{\sigma_h^6}.
\end{aligned}$$

Proof of a6):

$$\begin{aligned}
E\left((RV-\mu_2 IV)^2\left(\hat{V}-V\right)\right) &= \frac{\mu_4-\mu_2^2}{\mu_4}h^{-1}\sum_{i=1}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\left(r_i^4-\mu_4\sigma_i^4\right)\right]\\
&= h^2\frac{\left(\mu_4-\mu_2^2\right)\left(\mu_8-\mu_4^2-2\mu_2\mu_6+2\mu_2^2\mu_4\right)}{\mu_4}\overline{\sigma_h^8}.
\end{aligned}$$

Proof of a7): Write $E\left((RV-\mu_2 IV)^3\left(\hat{V}-V\right)\right)=\frac{\mu_4-\mu_2^2}{\mu_4}h^{-1}I_2$, where by the independence and mean zero property of $|r_i|^q - \mu_2 \sigma_i^q$,

$$
\begin{aligned}
I_2 &= \sum_{i=1}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^3\left(r_i^4-\mu_4\sigma_i^4\right)\right] + 3\sum_{i\neq j}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)\right] \\
&= M_1\sum_{i=1}^{1/h}\sigma_i^{10} + 3E\left[\left(u_i^2-\mu_2\right)^2\right]E\left[\left(u_j^2-\mu_2\right)\left(u_j^4-\mu_4\right)\right]\sum_{i\neq j}\sigma_i^4\sigma_j^6 \\
&= 3h^3\left(\mu_4-\mu_2^2\right)\left(\mu_6-\mu_2\mu_4\right)\overline{\sigma_h^4}\;\overline{\sigma_h^6} + O\left(h^4\right),
\end{aligned}
$$

given Lemma A.1, the fact that $\overline{\sigma_h^{10}} = O(1)$ under our assumptions, and where $M_1 = E\left[\left(u_i^2-\mu_2\right)^3\left(u_i^4-\mu_4\right)\right]$ is a constant, and $E\left[\left(u_i^2-\mu_2\right)^2\right] = \mu_4-\mu_2^2$ and $E\left[\left(u_j^2-\mu_2\right)\left(u_i^4-\mu_4\right)\right] = \mu_6-\mu_2\mu_4$.

Proof of a8): Write $E\left(\left(RV-\mu_2 IV\right)^4\left(\hat{V}-V\right)\right) = \frac{\mu_4-\mu_2^2}{\mu_4}h^{-1}I_3$, where by the independence and mean zero property of $|r_i|^q-\mu_2\sigma_i^q$, and Lemma A.1,

$$
\begin{aligned}
I_3 &= \sum_{i=1}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^4\left(r_i^4-\mu_4\sigma_i^4\right)\right] + 4\sum_{i\neq j}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^3\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)\right] \\
&\quad +6\sum_{i\neq j}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)^2\left(r_j^4-\mu_4\sigma_j^4\right)\right] \\
&= M_1h^5\overline{\sigma_h^{12}} + 4M_2\left(h^4\left(\overline{\sigma_h^6}\right)^2 - h^5\overline{\sigma_h^{12}}\right) + 6M_3\left(h^4\overline{\sigma_h^4}\;\overline{\sigma_h^8} - h^5\overline{\sigma_h^{12}}\right) \\
&= h^4\left[4M_2\left(\overline{\sigma_h^6}\right)^2 + 6M_3h^4\overline{\sigma_h^4}\;\overline{\sigma_h^8}\right] + O\left(h^5\right),
\end{aligned}
$$

where $M_1 \equiv E\left[\left(u_i^2-\mu_2\right)^4\left(u_i^4-\mu_4\right)\right]$, $M_2 \equiv E\left[\left(u_i^2-\mu_2\right)^3\right]E\left[\left(u_j^2-\mu_2\right)\left(u_j^4-\mu_4\right)\right]$ $= \left(\mu_6-3\mu_2\mu_4+2\mu_2^3\right)\left(\mu_6-\mu_2\mu_4\right)$ and $M_3 \equiv E\left[\left(u_i^2-\mu_2\right)^2\right]E\left[\left(u_i^2-\mu_2\right)^2\left(u_i^4-\mu_4\right)\right]$ $= \left(\mu_8-\mu_4^2-2\mu_2\mu_6+2\mu_2^2\mu_4\right)\left(\mu_4-\mu_2^2\right)$, and given the fact that $\overline{\sigma_h^q} = O(1)$ under our assumptions.

Proof of a9): Write $E\left(\left(RV-\mu_2 IV\right)\left(\hat{V}-V\right)^2\right) = \frac{\left(\mu_4-\mu_2^2\right)^2}{\mu_4^2}h^{-2}\sum_{i=1}^{1/h}E\left[\left(r_i^2-\mu_2\sigma_i^2\right)\left(r_i^4-\mu_4\sigma_i^4\right)^2\right] = O\left(h^2\right).$

Proof of a10): Write $E\left(\left(RV-\mu_2 IV\right)^2\left(\hat{V}-V\right)^2\right) = \frac{\left(\mu_4-\mu_2^2\right)^2}{\mu_4^2}h^{-2}I_4$, where by the independence and mean zero property of $|r_i|^q-\mu_2\sigma_i^q$,

$$
\begin{aligned}
I_4 &= \sum_{i=1}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\left(r_i^4-\mu_4\sigma_i^4\right)^2\right] + \sum_{i\neq j}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\right]E\left[\left(r_j^4-\mu_4\sigma_j^4\right)^2\right] \\
&\quad +2\sum_{i\neq j}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)\left(r_i^4-\mu_4\sigma_i^4\right)\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)\right] \\
&= D_1h^5\overline{\sigma_h^{12}} + D_2\left(h^4\overline{\sigma_h^4}\;\overline{\sigma_h^8} - h^5\overline{\sigma_h^{12}}\right) + 2D_3\left(h^4\left(\overline{\sigma_h^6}\right)^2 - h^5\overline{\sigma_h^{12}}\right) \\
&= h^4\left(D_2\overline{\sigma_h^4}\;\overline{\sigma_h^8} + 2D_3\left(\overline{\sigma_h^6}\right)^2\right) + O\left(h^5\right),
\end{aligned}
$$

given Lemma A.1, and where $D_1 = E\left[\left(u_i^2-\mu_2\right)^2\left(u_i^4-\mu_4\right)^2\right]$, $D_2 = E\left[\left(u_i^2-\mu_2\right)^2\right]E\left[\left(u_j^4-\mu_4\right)^2\right] = \left(\mu_4-\mu_2^2\right)\left(\mu_8-\mu_4^2\right)$ and $D_3 = \left[E\left(\left(u_i^2-\mu_2\right)\left(u_i^4-\mu_4\right)\right)\right]^2 = \left(\mu_6-\mu_2\mu_4\right)^2$.

Proof of a11): Write $E\left((RV-\mu_2 IV)^3\left(\hat{V}-V\right)^2\right) = \frac{\left(\mu_4-\mu_2^2\right)^2}{\mu_4^2}h^{-2}I_5$, with

$$
\begin{aligned}
I_5 &= \sum_{i=1}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^3\left(r_i^4-\mu_4\sigma_i^4\right)^2\right] + \sum_{i\neq j}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^3\right]E\left[\left(r_j^4-\mu_4\sigma_j^4\right)^2\right] \\
&\quad +3\sum_{i\neq j}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)^2\right] \\
&\quad +6\sum_{i\neq j}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\left(r_i^4-\mu_4\sigma_i^4\right)\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)\right] \\
&= K_1 h^6\overline{\sigma_h^{14}} + K_2\left(h^5\overline{\sigma_h^6}\ \overline{\sigma_h^8} - h^6\overline{\sigma_h^{14}}\right) + 3K_3\left(h^5\overline{\sigma_h^4}\ \overline{\sigma_h^{10}} - h^6\overline{\sigma_h^{14}}\right) + 6K_4\left(h^5\overline{\sigma_h^8}\ \overline{\sigma_h^6} - h^6\overline{\sigma_h^{14}}\right) \\
&= h^5\left(K_2\overline{\sigma_h^6}\ \overline{\sigma_h^8} + 3K_3\overline{\sigma_h^4}\ \overline{\sigma_h^{10}} + 6K_4\overline{\sigma_h^8}\ \overline{\sigma_h^6}\right) + h^6\left(K_1-K_2-3K_3-6K_4\right)\overline{\sigma_h^{14}},
\end{aligned}
$$

where we have used the independence and mean zero property of $|r_i|^q - \mu_2\sigma_i^q$, Lemma A.1, and where K_1 through K_4 are constants depending μ_q. Since $\overline{\sigma_h^q} = O(1)$, the result follows.

Proof of a12): Write $E\left((RV-\mu_2 IV)^4\left(\hat{V}-V\right)^2\right) = \frac{\left(\mu_4-\mu_2^2\right)^2}{\mu_4^2}h^{-2}I_6$, with

$$
\begin{aligned}
I_6 &= \sum_{i=1}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^4\left(r_i^4-\mu_4\sigma_i^4\right)^2\right] + \sum_{i\neq j}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^4\right]E\left[\left(r_j^4-\mu_4\sigma_j^4\right)^2\right] \\
&\quad +8\sum_{i\neq j}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^3\left(r_i^4-\mu_4\sigma_i^4\right)\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)\right] \\
&\quad +6\sum_{i\neq j}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\left(r_i^4-\mu_4\sigma_i^4\right)^2\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)^2\right] \\
&\quad +4\sum_{i\neq j}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^3\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)^2\right] \\
&\quad +6\sum_{i\neq j}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\left(r_i^4-\mu_4\sigma_i^4\right)\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)^2\left(r_j^4-\mu_4\sigma_j^4\right)\right] \\
&\quad +3\sum_{i\neq j\neq k}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)^2\right]E\left[\left(r_k^4-\mu_4\sigma_k^4\right)^2\right] \\
&\quad +12\sum_{i\neq j\neq k}^{1/h} E\left[\left(r_i^2-\mu_2\sigma_i^2\right)^2\right]E\left[\left(r_j^2-\mu_2\sigma_j^2\right)\left(r_j^4-\mu_4\sigma_j^4\right)\right]E\left[\left(r_k^2-\mu_2\sigma_k^2\right)\left(r_k^4-\mu_4\sigma_k^4\right)\right] \\
&= h^5\left[3J_7\left(\overline{\sigma_h^4}\right)^2\overline{\sigma_h^8} + 12J_8\left(\overline{\sigma_h^6}\right)^2\overline{\sigma_h^4}\right] + O\left(h^6\right) + O\left(h^7\right),
\end{aligned}
$$

given the independence and mean zero property of $|r_i|^q - \mu_2 \sigma_i^q$, Lemma A.1, and where $J_7 = \left(E\left[\left(u_i^2 - \mu_2 \right)^2 \right] \right)^2 E\left[\left(u_i^4 - \mu_4 \right)^2 \right] = \left(\mu_4 - \mu_2^2 \right)^2 \left(\mu_8 - \mu_4^2 \right)$ and $J_8 = E\left[\left(u_i^2 - \mu_2 \right)^2 \right] \left(E\left[\left(u_i^2 - \mu_2 \right) \left(u_i^4 - \mu_4 \right) \right] \right)^2 = \left(\mu_4 - \mu_2^2 \right) \left(\mu_6 - \mu_2 \mu_4 \right)^2.$

Proof of Lemma A.3. a1) and a2) follow by construction given S_h. The remaining results from the definition of S_h and Lemma A1.a3) through a13). For instance for a3), given Lemma A1.a3) and the definition of S_h,

$$E\left(S_h^3\right) = \frac{h^{-3/2}}{V^{3/2}} E\left((RV - \mu_2 IV)^3 \right) = \sqrt{h} \frac{\mu_6 - 3\mu_2\mu_4 + 2\mu_2^3}{\left(\mu_4 - \mu_2^2 \right)^{3/2}} \frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4} \right)^{3/2}} \equiv \sqrt{h} B_1 \frac{\overline{\sigma_h^6}}{\left(\overline{\sigma_h^4} \right)^{3/2}}.$$

Proof of Lemma A.4. Part a1) follows from the properties of the i.i.d. bootstrap. The remaining results follow from a1), given the binomial expansions. Note in particular that since $R_q = O_P(1)$, it follows that $E^*\left[\left(r_i^{*2} - hRV \right)^q \right] = O_P\left(h^q \right)$. For instance, for a2), $E^*\left[\left(r_i^{*2} - hRV \right)^2 \right] = E^*\left(r_i^{*4} - 2 r_i^{*2} hRV + (hRV)^2 \right) = h^2 \left(R_4 - RV^2 \right)$. The other results follows similarly.

Proof of Lemma A.5. For a1), since r_i^* are i.i.d. from $\{r_i : i = 1, \ldots, 1/h\}$, it follows that

$$V^* = h^{-1} Var^* \left(\sum_{i=1}^{1/h} r_i^{*2} \right) = h^{-1} \sum_{i=1}^{1/h} Var^* \left(r_i^{*2} \right) = h^{-2} Var^* \left(r_1^{*2} \right).$$

But $Var^*\left(r_1^{*2} \right) = E^*\left(r_1^{*4} \right) - \left(E^*\left(r_1^{*2} \right) \right)^2 = h^2 R_4 - (hRV)^2$. Thus, $V^* = R_4 - RV^2$. Part a2) follows because $V^* = R_4 - RV^2$ and $\hat{V}^* = R_4^* - RV^{*2}$. For the remaining of the proof, note that $\sum_{i \neq j}^{1/h} 1 = h^{-2} - h^{-1}$, $\sum_{i \neq j \neq k} 1 = h^{-3} + 2h^{-1} - 3h^{-2}$, and $\sum_{i \neq j \neq k \neq m}^{1/h} 1 = h^{-4} - 6h^{-3} + 11h^{-2} - 6h^{-1}$. In addition, note that

$$RV^* - RV = \sum_{i=1}^{1/h} \left(r_i^{*2} - hRV \right) \quad \text{and} \quad R_4^* - R_4 = h^{-1} \sum_{i=1}^{1/h} \left(r_i^{*4} - h^2 R_4 \right),$$

where for any $q > 0$ $\left\{ |r_i^*|^q - h^{q/2} R_q \right\}$ are (conditionally on the sample) i.i.d. with zero mean, and $R_q = O_P(1)$. Using this independence property, we evaluate the bootstrap expectations of the sums of products and cross products of $|r_i^*|^q - h^{q/2} R_q$ by relying on Lemma A.4 to compute the appropriate bootstrap moments of products and cross products of $|r_i^*|^q - h^{q/2} R_q$. We proceed as in the proof of Lemma A.2 and use the multinomial expansions to compute the number of coefficients in each sum.

Proof of Lemma A.6. Using part a2) of Lemma A.5, for $q = 1, \ldots, 4$, we can write

$$\begin{aligned} E^*\left[(RV^* - RV)^q \left(\hat{V}^* - V^* \right) \right] &= E^*\left[(RV^* - RV)^q \left(R_4^* - R_4 \right) \right] - E^*\left[(RV^* - RV)^{2+q} \right] \\ &\quad - 2(RV) E^*\left[(RV^* - RV)^{1+q} \right] \\ &\equiv I_1^q - I_2^q - I_3^q. \end{aligned} \tag{34}$$

Similarly, for $q=1,\ldots,4$, note that

$$\begin{aligned} E^*\left[(RV^*-RV)^q\left(\hat{V}^*-V^*\right)^2\right] &= E^*\left[(RV^*-RV)^q\left(R_4^*-R_4\right)^2\right]-2E^*\left[(RV^*-RV)^{2+q}\left(R_4^*-R_4\right)\right] \\ &\quad -4\,(RV)\,E^*\left[(RV^*-RV)^{1+q}\left(R_4^*-R_4\right)\right]+E^*\left[(RV^*-RV)^{4+q}\right] \\ &\quad +4\,(RV)\,E^*\left[(RV^*-RV)^{3+q}\right]+4\,(RV)^2\,E^*\left[(RV^*-RV)^{2+q}\right] \end{aligned}$$

For a1), set $q=1$ in (34). We have that

$$\begin{aligned} I_1^1 &= E^*\left[(RV^*-RV)\left(R_4^*-R_4\right)\right]=h\left(R_6-R_4RV\right) \\ I_2^1 &= E^*\left[(RV^*-RV)^3\right]=h^2\left(R_6-3R_4RV+2RV^3\right) \\ I_3^1 &= 2\,(RV)\,E^*\left[(RV^*-RV)^2\right]=2\,(RV)\left[h\left(R_4-RV^2\right)\right], \end{aligned}$$

by Lemma A.5. a8), a3), a1), respectively. Thus

$$\begin{aligned} E^*\left[(RV^*-RV)\left(\hat{V}^*-V^*\right)\right] &= h\left[\left(R_6-R_4RV\right)-2RV\left(R_4-RV^2\right)\right]-h^2\left(R_6-3R_4RV+2RV^3\right) \\ &= h\left(R_6-3R_4RV+2RV^3\right)+O_P\left(h^2\right). \end{aligned}$$

The remaining results follow similarly.
Proof of Lemma A.7. Parts a1) and a2) follow by construction given S_h^*. The remaining parts follow as in the proof of Lemma A.3, given the definition of V^* in Lemma A.5. a1) and given Lemmas A.5 and A.6. For instance, for a3), given V^* and Lemma A.5.a3), the definition of S_h^* implies that

$$E^*\left(S_h^{*3}\right)=\frac{h^{-3/2}}{V^{*3/2}}E^*\left((RV^*-RV)^3\right)=\sqrt{h}\left(\frac{R_6-3R_4RV+2RV^3}{\left(R_4-RV^2\right)^{3/2}}\right)\equiv\sqrt{h}\tilde{B}_1$$

The other results follow similarly.

Appendix B - Proofs of results in Sections 3 and 4

Proof of Theorem 3.1. The proof contains two steps. Step 1: We show that the desired result is true for S_h^*. Step 2: We show that $\hat{V}^*\to^{P^*}V^*$ in prob-P. *Proof of Step 1.* We can write $S_h^*=\sum_{i=1}^{1/h}z_i^*$, where $z_i^*\equiv\frac{r_i^{*2}-E^*\left(r_i^{*2}\right)}{\sqrt{hV^*}}$ are (conditionally on the original sample) i.i.d. with $E^*\left(z_i^*\right)=0$ and $Var^*\left(z_i^*\right)=\frac{h^2V^*}{hV^*}=h$ such that $Var^*\left(\sum_{i=1}^{1/h}z_i^*\right)=h^{-1}h=1$. Thus, by Katz's (1963) Berry-Esseen bound, for some small $\varepsilon>0$ and some constant K,

$$\sup_{x\in\mathbb{R}}\left|P^*\left(\frac{\sum_{i=1}^{1/h}z_i^*}{\sqrt{Var^*\left(\sum_{i=1}^{1/h}z_i^*\right)}}\le x\right)-\Phi(x)\right|\le K\sum_{i=1}^{1/h}E^*\left|z_i^*\right|^{2+\varepsilon}. \tag{35}$$

We show that the RHS of (35) converges to zero in probability. We have that

$$\begin{aligned}
\sum_{i=1}^{1/h} E^* \left|z_i^*\right|^{2+\varepsilon} &= h^{-1} E^* \left|z_1^*\right|^{2+\varepsilon} = h^{-1} h^{-\frac{2+\varepsilon}{2}} \left|V^*\right|^{-\frac{2+\varepsilon}{2}} E^* \left(\left| r_i^{*2} - E^* \left|r_i^*\right|^2 \right|^{2+\varepsilon} \right) \\
&\leq 2 \left|V^*\right|^{-\frac{2+\varepsilon}{2}} h^{-1} h^{-\frac{2+\varepsilon}{2}} E^* \left|r_i^*\right|^{2(2+\varepsilon)} = 2 \left|V^*\right|^{-\frac{2+\varepsilon}{2}} h^{-1} h^{-\frac{2+\varepsilon}{2}} h^{2+\varepsilon} R_{2(2+\varepsilon)} \\
&= 2 \left|V^*\right|^{-\frac{2+\varepsilon}{2}} h^{\frac{\varepsilon}{2}} R_{2(2+\varepsilon)} = O_P\left(h^{\frac{\varepsilon}{2}}\right),
\end{aligned}$$

given that $V^* \xrightarrow{P} 3\overline{\sigma^4} - \left(\overline{\sigma^2}\right)^2 > 0$ and $R_{2(2+\varepsilon)} \xrightarrow{P} \mu_{2+\varepsilon} \overline{\sigma^{2(2+\varepsilon)}} = O(1)$ (along any path of v, given Assumption (V)). As $h \to 0$, $O_P\left(h^{\frac{\varepsilon}{2}}\right) = o_P(1)$. *Proof of Step 2.* We use Lemma A.5 to show that $Bias^*(\hat{V}^*) \xrightarrow{P} 0$ and $Var^*\left(\hat{V}^*\right) \xrightarrow{P} 0$.

Proof of Theorem 3.2. We proceed as in the proof of Theorem 3.1. *Proof of Step 1.* We can write $S_h^* = \sum_{i=1}^{1/h} x_i^*$, where $x_i^* = \frac{r_i^2\left(\eta_i^2 - \mu_2^*\right)}{\sqrt{hV^*}}$. Notice that x_i^* is an array of independent random variables with $E^*\left(x_i^*\right) = 0$ and $Var^*\left(x_i^*\right) = \frac{r_i^2}{hV^*} Var^*\left(\eta_i^2\right) = \frac{\left(\mu_4^* - \mu_2^{*2}\right) r_i^2}{hV^*}$ (so x_i^* is heteroskedastic). Thus, $Var^*\left(\sum_{i=1}^{1/h} x_i^*\right) = \sum_{i=1}^{1/h} Var^*\left(x_i^*\right) = \left(\mu_4^* - \mu_2^{*2}\right) \frac{\sum_{i=1}^{1/h} r_i^4}{hV^*} = \left(\mu_4^* - \mu_2^{*2}\right) \frac{R_4}{V^*} = 1$, given the definition of V^*. It suffices to verify Lyapunov's condition (35) using the properties of the wild bootstrap. In particular, we can show that

$$\sum_{i=1}^{1/h} \left|x_i^*\right|^{2+\varepsilon} = \left|V^*\right|^{-\frac{2+\varepsilon}{2}} h^{\frac{\varepsilon}{2}} R_{2(2+\varepsilon)} E^* \left|\eta_i^2 - \mu_2^*\right|^{2+\varepsilon} \leq K \left|V^*\right|^{-\frac{2+\varepsilon}{2}} h^{\frac{\varepsilon}{2}} R_{2(2+\varepsilon)} \left(\mu_{2(2+\varepsilon)}^* + \mu_2^{*2+\varepsilon}\right) = O_P\left(h^{\frac{\varepsilon}{2}}\right),$$

arguing as in the proof of Theorem 3.1. *Proof of Step 2.* We have that $E^*\left(\hat{V}^*\right) = \frac{\mu_4^* - \mu_2^{*2}}{\mu_4^*} E^*\left(R_4^*\right) = V^*$, given the definition of V^*. Thus, it suffices to show that $Var^*\left(\hat{V}^*\right) = o_P(1)$. We can show that $Var^*\left(\hat{V}^*\right) = h\left(\frac{\mu_4^* - \mu_2^{*2}}{\mu_4^*}\right)^2 \left(\mu_8^* - \mu_4^{*2}\right) R_8 = O_P(h) = o_P(1)$, since $h \to 0$.

Proof of Proposition 4.1. The results follow from the definition of $q_1(x)$ and $q_1^*(x)$ in (12) and (14), respectively, given the cumulants expansions in Theorems A.1, A.2 and A.3.

Proof of Proposition 4.2. a) follows from Corollary A.1; b) follows trivially when v is constant because $\left(\overline{\sigma^q}\right)^p = v^{qp}$ for any $q, p > 0$. We prove c) next. Define $C = \frac{4\overline{\sigma^6}}{\sqrt{2}(\overline{\sigma^4})^{3/2}}$ and $C^* = \frac{15\overline{\sigma^6} - 9\overline{\sigma^4}\ \overline{\sigma^2} + 2(\overline{\sigma^2})^3}{(3\overline{\sigma^4} - (\overline{\sigma^2})^2)^{3/2}}$, and note that $C > 0$. Proving c) is equivalent to proving $|C - C^*| \leq |C|$, which in turn is equivalent to proving $0 \leq C^* \leq 2C$. Next we show that $C^* \geq 0$. The Jensen's inequality implies that $\overline{\sigma^4} \geq (\overline{\sigma^2})^2$, and since $\overline{\sigma^4} > 0$, it follows that the denominator of C^* is positive. For the numerator of C^*, note we can write

$$15\overline{\sigma^6} - 9\overline{\sigma^4}\ \overline{\sigma^2} + 2(\overline{\sigma^2})^3 \geq 15\overline{\sigma^6} - 9(\overline{\sigma^4})^{3/2} + 2(\overline{\sigma^2})^3 \geq 9(\overline{(\sigma^4)^{3/2}} - (\overline{\sigma^4})^{3/2}) + 6\overline{\sigma^6} + 2(\overline{\sigma^2})^3,$$

using $-(\overline{\sigma^2})^2 \geq -\overline{\sigma^4}$. Since the function $\psi(x) = x^{3/2}$ for $x > 0$ is convex, we have that $\overline{(\sigma^4)^{3/2}} - (\overline{\sigma^4})^{3/2} \geq 0$, which implies $15\overline{\sigma^6} - 9\overline{\sigma^4}\ \overline{\sigma^2} + 2(\overline{\sigma^2})^3 \geq 6\overline{\sigma^6} + 2(\overline{\sigma^2})^3 > 0$, proving that the numerator of C^* is also positive. Next we prove $\frac{C^*}{C} \leq 2$. We can write

$$\frac{C^*}{C} = \frac{15\overline{\sigma^6} - 9\overline{\sigma^4}\ \overline{\sigma^2} + 2(\overline{\sigma^2})^3}{8\overline{\sigma^6}} \frac{2\sqrt{2}(\overline{\sigma^4})^{3/2}}{(3\overline{\sigma^4} - (\overline{\sigma^2})^2)^{3/2}} \equiv C_1 \times C_2.$$

We show that $C_1 \leq 2$ and $C_2 \leq 1$. First, note that

$$\frac{15\overline{\sigma^6} - 9\overline{\sigma^4}\ \overline{\sigma^2} + 2(\overline{\sigma^2})^3}{8\overline{\sigma^6}} \leq 2 \Longleftrightarrow 15\overline{\sigma^6} - 9\overline{\sigma^4}\ \overline{\sigma^2} + 2(\overline{\sigma^2})^3 \leq 16\overline{\sigma^6} \Longleftrightarrow 0 \leq \overline{\sigma^6} + 7\overline{\sigma^4}\ \overline{\sigma^2} + 2\overline{\sigma^2}\left(\overline{\sigma^4} - (\overline{\sigma^2})^2\right),$$

which proves the result since $\left(\overline{\sigma^4} - (\overline{\sigma^2})^2\right) \geq 0$ and $0 \leq \overline{\sigma^6} + 7\overline{\sigma^4}\ \overline{\sigma^2}$. Finally, we have that

$$\frac{2\sqrt{2}(\overline{\sigma^4})^{3/2}}{(3\overline{\sigma^4} - (\overline{\sigma^2})^2)^{3/2}} \leq 1 \Longleftrightarrow 8(\overline{\sigma^4})^3 \leq \left(3\overline{\sigma^4} - (\overline{\sigma^2})^2\right)^3 \Longleftrightarrow 8(\overline{\sigma^4})^3 \leq \left(2\overline{\sigma^4} + \left(\overline{\sigma^4} - (\overline{\sigma^2})^2\right)\right)^3,$$

which holds true since $\left(\overline{\sigma^4} - (\overline{\sigma^2})^2\right) \geq 0$.

Proof of Proposition 4.3. This follows from Theorem A.1 and A.3, given that $R_q \rightarrow \mu_q \overline{\sigma^q}$ in probability (conditional on v), for any $q > 0$, by BN-S (2004b, Theorem 1).

Proof of Proposition 4.4. The conditions on the bootstrap moments μ_2^*, μ_4^* and μ_6^* are a restatement of the equations $\frac{5}{\sqrt{3}}A_1^* = A_1$ and $\frac{5}{\sqrt{3}}B_1^* = B_1$. Thus, $\mathrm{plim}_{h\rightarrow 0}\, q_1^*(x) = q_1(x)$, implying the result.

Proof of Corollary 4.1. We seek η_i such that its moments are equal to $\mu_2^* = 1$, $\mu_4^* = \frac{31}{25}$, and $\mu_6^* = \frac{31}{25}\frac{37}{25}$. Let

$$\eta_i = \begin{cases} a_1 & \text{with prob } p \\ a_2 & \text{with prob } 1-p \end{cases}.$$

We determine a_1, a_2 and p such that $E\eta_i^2 = 1$, $E\eta_i^4 = \frac{31}{25}$ and $E\eta_i^6 = \frac{31}{25}\frac{37}{25}$. In particular, we can show that $a_1 = \frac{1}{5}\sqrt{31 + \sqrt{186}}$, $a_2 = -\frac{1}{5}\sqrt{31 - \sqrt{186}}$, and $p = \frac{1}{2} - \frac{3}{\sqrt{186}}$ solve the following system of equations

$$\begin{aligned} a_1^2 p + a_2^2(1-p) &= 1 \\ a_1^4 p + a_2^4(1-p) &= \frac{31}{25} \\ a_1^6 p + a_2^6(1-p) &= \frac{31}{25}\frac{37}{25}. \end{aligned}$$

References

[1] Aït-Sahalia, Y., Mykland, P.A. and L. Zhang, 2005. How often to sample a continuous-time process in the presence of market microstructure noise, *Review of Financial Studies* 18 351-416.

[2] Andersen, T.G., L. Benzoni and J. Lund, 2002. An Empirical Investigation of Continuous-Time Equity Return Models, *Journal of Finance*, 57, 1239-1284.

[3] Andersen, T.G. and T. Bollerslev, 1998. Answering the Skeptics: Yes, Standard Volatility Models Do Provide Accurate Forecasts, *International Economic Review*, 39, 885-905.

[4] Andersen, T.G., T. Bollerslev, and F.X. Diebold, 2002. Parametric and Nonparametric Measurements of Volatility, in Y. Aït-Sahalia and L.P. Hansen (Eds.), *Handbook of Financial Econometrics*, forthcoming.

[5] Andersen, T.G., T. Bollerslev, and F.X. Diebold, 2004. Some Like it Smooth and Some Like it Rough: Untangling Continuous and Jump Components in Measuring, Modeling and Forecasting Asset Return Volatility, manuscript, Duke University.

[6] Andersen, T.G., T. Bollerslev, F.X. Diebold and P. Labys, 2003. Modeling and Forecasting Realized Volatility, *Econometrica*, 71, 529-626.

[7] Andersen, T.G., T. Bollerslev, and N. Meddahi, 2005. Correcting the errors: volatility forecast evaluation using high-frequency data and realized volatilities. *Econometrica*, 73, 279 - 296.

[8] Andrews, D.K., 2004. The Block-block Bootstrap: Improved Asymptotic Refinements, *Econometrica*, 72, 673-700.

[9] Bandi, F.M. and J.R. Russell, 2004. Microstructure noise, realized volatility, and optimal sampling, manuscript, Graduate School of Business, University of Chicago.

[10] Barndorff-Nielsen, O. and N. Shephard, 2002. Econometric analysis of realized volatility and its use in estimating stochastic volatility models, *Journal of the Royal Statistical Society*, Series B, 64, 253-280.

[11] Barndorff-Nielsen, O. and N. Shephard, 2003. Realised power variation and stochastic volatility models, *Bernoulli*, volume 9, 2003, 243-265 and 1109-1111.

[12] Barndorff-Nielsen, O. and N. Shephard, 2004a. How accurate is the asymptotic approximation to the distribution of realised variance?, in Identification and Inference for Econometric Models, ed. by D.F. Andrews, J.L. Powel, and J.H. Stock. Cambridge University Press, forthcoming.

[13] Barndorff-Nielsen, O. and N. Shephard, 2004b. Power and bipower variation with stochastic volatility and jumps, *Journal of Financial Econometrics*, 2, 1-48.

[14] Barndorff-Nielsen, O. and N. Shephard, 2004c. A feasible central limit theory for realised volatility under leverage, manuscript, University of Oxford.

[15] Barndorff-Nielsen, O. and N. Shephard, 2004d. Econometric analysis of realised covariation: high frequency based covariance, regression and correlation in financial economics, *Econometrica*, 72, 885-925.

[16] Barndorff-Nielsen, O. and N. Shephard, 2005. Econometrics of testing for jumps in financial economics using bipower variation, *Journal of Financial Econometrics*, forthcoming.

[17] Barndorff-Nielsen, O., J. Jacod, and N. Shephard, 2004. Limit theorems for bipower variation in financial econometrics, manuscript. University of Oxford.

[18] Barndorff-Nielsen, O., P.R. Hansen, A. Lunde and N. Shephard, 2004. Regular and modified kernel-based estimators of integrated variance: the case with independent noise, manuscript, University of Oxford.

[19] Chernov, M., R. Gallant, E. Ghysels, and G. Tauchen, 2003. Alternative models for stock price dynamics, *Journal of Econometrics*, 116, 225-257.

[20] Davidson, R. and Flachaire, E., 2001. The wild bootstrap, tamed at last. Working Paper, Darp58, STICERD, London School of Economics.

[21] Efron,B., 1979. Bootstrap methods: another look at the jackknife. *Annals of Statistics* 7, 1-26.

[22] Gonçalves, S. and L. Kilian, 2004. Bootstrapping Autoregressions with Conditional Heteroskedasticity of Unknown Form. *Journal of Econometrics*, 2004, 123, 89-120.

[23] Gonçalves, S. and T. Vogelsang, 2004. Block Bootstrap Puzzles in HAC Robust Testing: The Sophistication of the Naive Bootstrap, manuscript, Université de Montréal.

[24] Hall, P., 1992. The bootstrap and Edgeworth expansion. Springer-Verlag, New York.

[25] Hansen, P.R. and A. Lunde, 2005. Realized variance and market microstructure, *Journal of Business and Economic Statistics*. The 2005 Invited Address, forthcoming.

[26] Härdle, W., Horowitz, J., Kreiss, J.-P., 2003. Bootstrap methods for time series. *International Statistical Review*, 71, 435-459, 2003

[27] Horowitz, J., 2001. The Bootstrap in Econometrics, in Handbook of Econometrics, Vol. 5, J.J. Heckman and E.E. Leamer, eds., Elsevier Science B.V., Ch. 52, pp. 3159-3228.

[28] Huang, X. and G. Tauchen, 2005. The relative contribution of jumps to total price variance, manuscript, Duke University.

[29] Jacod, J. (1994). Limit of random measures associated with the increments of a Brownian semimartingale. Preprint number 120, Laboratoire de Probabilitités, Université Pierre et Marie Curie, Paris.

[30] Jacod, J. and P. Protter (1998). Asymptotic error distributions for the Euler method for stochastic differential equations. *Annals of Probability* 26, 267-307.

[31] Katz, M.L., 1963. Note on the Berry-Esseen theorem. *Annals of Mathematical Statistics* 34, 1107-1108.

[32] Liu, R.Y., 1988. Bootstrap procedure under some non-i.i.d. models. *Annals of Statistics* 16, 1696-1708.

[33] Mammen, E., 1993. Bootstrap and wild bootstrap for high dimensional linear models. *Annals of Statistics* 21, 255-285.

[34] Mykland, P.A., 2000. Conservative delta hedging. *Annals of Applied Probability* 10, 664-683.

[35] Mykland, P.A., 2002. Options pricing bounds and statistical uncertainty, *Handbook of Financial Econometrics*, forthcoming.

[36] Mykland, P.A., 2003. Financial options and statistical prediction intervals. *Annals of Statistics* 31, 1413-1438.

[37] Mykland, P.A. and L. Zhang, 2001. ANOVA for diffusions. Technical report no. 507, Department of Statistics, University of Chicago.

[38] Shao, J., and Tu, D., 1995. *The jackknife and bootstrap*, Springer-Verlag, New York.

[39] Wu, C.F.J., 1986. Jackknife, bootstrap and other resampling methods in regression analysis. *Annals of Statistics* 14, 1261-1295.

[40] Zhang, L, P.A. Mykland, and Y. Ait-Sahalia, 2004. A tale of two time-scales: determining integrated volatility with noisy high frequency data, *Journal of the American Statistical Association*, forthcoming.

[41] Zhang, L., 2004. Efficient estimation of stochastic volatility using noisy observations: a multi-scale approach, manuscript, Carnegie Mellon University, Department of Statistics.