I One Factor case

$$y_{t+1} = \mu_t + u_{t+1}$$

$$\mu_{t+1} = \omega + \gamma \mu_t + v_{t+1}$$

$$E\left(\begin{matrix} u_{t+1} \\ v_{t+1} \end{matrix} \middle| I_t, \underline{\mu_t} \right) = 0$$

Public information at time t; includes $\underline{y_t}$

Let $\{x_t\}$ be a second order stationary process

$$y_{t+1:t+m} \equiv \sum_{i=1}^{m} y_{t+i}$$

On has the regression

$$y_{t+1:t+m} = c^{(m)} + x_t' \beta^{(m)} + \varepsilon_{t+m}^{(m)}$$

whose coefficient of determination is denoted

$$R^2 (y_{t+1:t+m}, x_t)$$

Proposition

① One step ahead forecasts:

$$\beta^{(1)} = \left[\mathrm{Var}(x_t)\right]^{-1} \mathrm{Cov}(x_t, y_{t+1})$$

$$R^2(y_{t+1}, x_t) = \frac{1}{\mathrm{Var}(y_{t+1})} \mathrm{Cov}(y_{t+1}, x_t) \left[\mathrm{Var}(x_t)\right]^{-1} \mathrm{Cov}(x_t, y_{t+1})$$

② Multi step ahead forecasts

$$\beta^{(m)} = \frac{1-\gamma^m}{1-\gamma} \beta^{(1)}$$

$$R^2(y_{t+1:t+m}, x_t) = R^2(y_{t+1}, x_t) \frac{1-\gamma^m}{m(1-\gamma)^2 + 2\rho(\gamma^m - m\gamma + m - 1)}$$

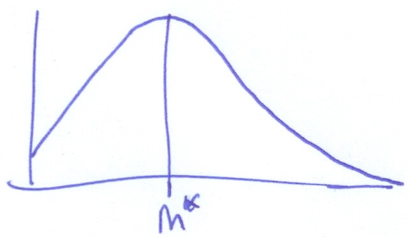
where $\rho =$ ACF(1) of y_t

③ Behavior when m varies:

$\gamma \geq 0$: $|\beta^{(m)}| \nearrow$ when $m \nearrow$ components by components

$\beta^{(m)}$ monotonic components by components

$R^2(y_{t+1:t+m}, x_t)$



or

flat if $R^2(y_{t+1}, x_t) = 0$

4. Asymptotic behavior

$$\beta^{(m)} \xrightarrow[m\to\infty]{} \frac{\beta^{(1)}}{1-\gamma}$$

$$R^2\left(y_{t+1:t+m}, x_t\right) \xrightarrow[m\to+\infty]{} 0$$

Proof

1. Definition or results in population
2. $\mathrm{cov}(y_{t+1:t+m}, x_t) = \mathrm{Cov}\left(E_t\, y_{t+1:t+m}, x_t\right)$

$$= \mathrm{cov}\left(E_t \sum_{i=1}^{m} \mu_{t-1+i}, x_t\right)$$

$$= \mathrm{cov}\left(\mu_t \sum_{i=0}^{m-1} \gamma^i, x_t\right) = \frac{1-\gamma^m}{1-\gamma}\,\mathrm{cov}(\mu_t, x_t)$$

$$= \frac{1-\gamma^m}{1-\gamma}\,\mathrm{cov}(y_{t+1}, x_t)$$

Hence $\beta^{(m)} = \left[\mathrm{Var}(x_t)\right]^{-1} \mathrm{cov}(x_t, y_{t+1:t+m})$

$$= \frac{1-\gamma^m}{1-\gamma}\,\mathrm{Var}(x_t)\,\mathrm{cov}(x_t, y_{t+1})$$

$$\boxed{\beta^{(m)} = \frac{1-\gamma^m}{1-\gamma}\,\beta^{(1)}}$$

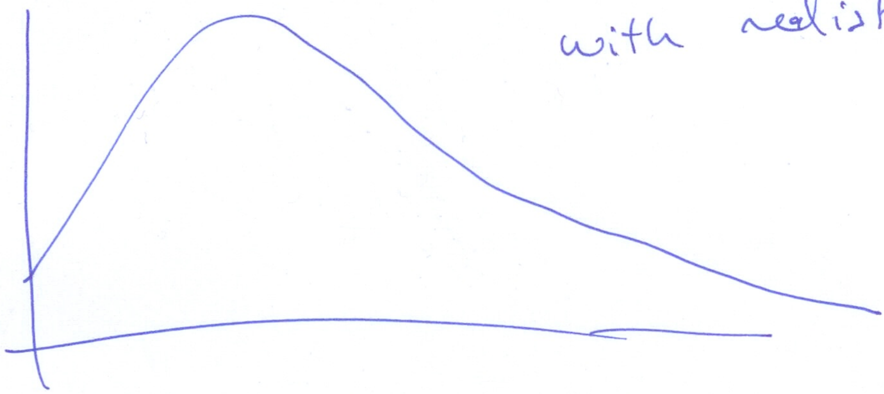
The result

$$R^2(y_{t+1:t+m}, x_t) = R^2(y_{t+1}, x_t) \frac{1-\gamma^m}{m(1-\gamma)^2 + 2\rho(\gamma^m - m\gamma + m - 1)}$$

is proved elsewhere (For instance first version of Campbell-Mehtahit).

3 likewise CM proved

$$g(m) = \frac{1-\gamma^m}{m(1-\gamma)^2 + 2\rho(\gamma^m - m\gamma + m - 1)}$$



with realistic numbers

$1-\gamma^m \nearrow$ when $m \nearrow$

$\Rightarrow |\beta^m| \nearrow$ components by components

4 $1-\gamma^m \xrightarrow[m\to\infty]{} 1 \Rightarrow \beta^{(m)} \longrightarrow \frac{1}{1-\gamma}\beta^{(1)}$

$R^2(y_{t+1:t+m}) \xrightarrow[m\to\infty]{} 0$ due to denominator.

II) Two factor case

$$y_{t+1} = \mu_t + u_{t+1}$$

$$\mu_{t+1} = \mu_{1,t+1} + \mu_{2,t+2}$$

$$\mu_{i,t+1} = \omega_i + \gamma_i \mu_{i,t} + v_{i,t+1}$$

$$E\left[\begin{pmatrix} u_{t+1} \\ v_{1,t+1} \\ v_{2,t+1} \end{pmatrix} \middle| I_t, \underline{\mu_{1,t}}, \underline{\mu_{2,t}} \right] = 0$$

$$y_{t+1:t+m} = c^{(m)} + x_t' \beta^{(m)} + \varepsilon_{t+m}^{(m)}$$

We can find constants $a_1(m, \gamma_1, \gamma_2)$ and $a_2(m, \gamma_1, \gamma_2)$ such that

$$\beta^{(m)} = a_1(m, \gamma_1, \gamma_2)\, \beta^{(1)} + a_2(m, \gamma_1, \gamma_2)\, \beta^{(2)}$$

Proof

$$\beta^{(1)} = \left[Var(x_t)\right]^{-1} cov(x_t, y_{t+1})$$

$$= \left[Var(x_t)\right]^{-1}\left[cov(x_t, \mu_{1,t}) + cov(x_t, \mu_{2,t})\right]$$

$$\beta^{(2)} = \left[Var(x_t)\right]^{-1}\left[cov(x_t, y_{t+1} + y_{t+2})\right]$$

$$= \left[Var(x_t)\right]^{-1}\left[cov(x_t, \mu_{1,t} + \mu_{2,t} + \mu_{1,t+1} + \mu_{2,t+1})\right]$$

$$= \left[Var(x_t)\right]^{-1}\left[cov(x_t, \mu_{1,t} + \mu_{2,t} + E_t(\quad))\right]$$

$$= \left[Var(x_t)\right]^{-1}\left[cov(x_t, \mu_{1,t} + \mu_{2,t} + \gamma_1 \mu_{1,t} + \gamma_2 \mu_{2,t})\right]$$

$$= \left[Var(x_t)\right]^{-1}\left[(1+\gamma_1) cov(x_t, \mu_{1,t}) + (1+\gamma_2) cov(x_t, \mu_{2,t})\right]$$

~~Lemma~~ Define $\lambda^{(m)} = \mathrm{cov}(x_t, y_{t+1:t+m})$

$$\lambda^{(1)} = \mathrm{cov}(x_t, \mu_{1,t}) + \mathrm{cov}(x_t, \mu_{2,t})$$

$$\lambda^{(2)} = (1+\gamma_1)\,\mathrm{cov}(x_t, \mu_{1,t}) + (1+\gamma_2)\,\mathrm{cov}(x_t, \mu_{2,t})$$

$$\lambda^{(m)} = \frac{1-\gamma_1^m}{1-\gamma_1}\,\mathrm{cov}(x_t, \mu_{1,t}) + \frac{1-\gamma_2^m}{1-\gamma_2}\,\mathrm{cov}(x_t, \mu_{2,t})$$

If one consider the system

$$\mathrm{cov}(x_t, \mu_{1,t}) + \mathrm{cov}(x_t, \mu_{2,t}) = \lambda^{(1)} = \mathrm{cov}(x_t, y_{t+1})$$

$$(1+\gamma_1)\,\mathrm{cov}(x_t, \mu_{1,t}) + (1+\gamma_2)\,\mathrm{cov}(x_t, \mu_{2,t}) = \lambda^{(2)} = \mathrm{cov}(x_t, y_{t+1} + y_{t+2})$$

The determinant of the system is

$$\det \begin{pmatrix} 1 & 1 \\ 1+\gamma_1 & 1+\gamma_2 \end{pmatrix} = \gamma_2 - \gamma_1 \neq 0$$

$\Rightarrow$ we can invert the system

i.e.

$$\boxed{\begin{aligned} \mathrm{cov}(x_t, \mu_{1,t}) &= \mathrm{cov}(x_t, y_{t+1})\,\frac{1+\gamma_2}{\gamma_2-\gamma_1} - \mathrm{cov}(x_t, y_{t+1:t+2})\,\frac{1}{\gamma_2-\gamma_1} \\ \mathrm{cov}(x_t, \mu_{2,t}) &= -\,\mathrm{cov}(x_t, y_{t+1})\,\frac{1+\gamma_1}{\gamma_2-\gamma_1} + \mathrm{cov}(x_t, y_{t+1:t+2})\,\frac{1}{\gamma_2-\gamma_1} \end{aligned}}$$

Consequently

$$\mathrm{cov}(x_t, y_{t+1:t+m}) = \mathrm{cov}(x_t, E_t\, y_{t+1:t+m})$$

$$= \mathrm{cov}\left(x_t, E_t \sum_{i=1}^{m} \mu_{1,t-1+i}\right) + \mathrm{cov}\left(x_t, E_t\left(\sum_{i=1}^{m} \mu_{2,t-1+i}\right)\right)$$

$$= \mathrm{cov}\left(x_t, \left(\sum_{i=0}^{m-1} \gamma_1^i\right) \mu_{1,t}\right) + \mathrm{cov}\left(x_t, \left(\sum_{i=0}^{m-1} \gamma_2^i\right) \mu_{2,t}\right)$$

$$= \frac{1-\gamma_1^m}{1-\gamma_1} \mathrm{cov}(x_t, \mu_{1,t}) + \frac{1-\gamma_2^m}{1-\gamma_2} \mathrm{cov}(x_t, \mu_{2,t})$$

$$= \left\{ \frac{1-\gamma_1^m}{1-\gamma_1} \frac{1+\gamma_2}{\gamma_2-\gamma_1} - \frac{1-\gamma_2^m}{1-\gamma_2} \frac{1+\gamma_1}{\gamma_2-\gamma_1} \right\} \mathrm{cov}(x_t, y_{t+1})$$

$$+ \left\{ -\frac{(1-\gamma_1^m)}{1-\gamma_1} \frac{1}{\gamma_2-\gamma_1} + \frac{1-\gamma_2^m}{1-\gamma_2} \frac{1}{\gamma_2-\gamma_1} \right\} \mathrm{cov}(x_t, y_{t+1:t+2})$$

$$= a_1(m,\gamma_1,\gamma_2)\, \mathrm{cov}(x_t, y_{t+1}) + a_2(m,\gamma_1,\gamma_2)\, \mathrm{cov}(x_t, y_{t+1:t+2})$$

with

$$a_1(m,\gamma_1,\gamma_2) = \frac{1}{(1-\gamma_1)(1-\gamma_2)(\gamma_2-\gamma_1)} \left[(1-\gamma_1^m)(1-\gamma_2^2) - (1-\gamma_2^m)(1-\gamma_1^2) \right]$$

$$a_2(m,\gamma_1,\gamma_2) = \frac{1}{(1-\gamma_1)(1-\gamma_2)(\gamma_2-\gamma_1)} \left[-(1-\gamma_1^m)(1-\gamma_2) + (1-\gamma_2^m)(1-\gamma_1) \right]$$

Therefore

$$\beta^{(m)} = \left[Var(x_t)\right]^{-1} cov(x_t, y_{t+1:t+m})$$

$$= \left[Var(x_t)\right]^{-1} \left[a_1(m, \gamma_1, \gamma_2) \, cov(x_t, y_{t+1}) + a_2(m, \gamma_1, \gamma_2) \, cov(x_t, y_{t+1:t+2})\right.$$

$$\boxed{\beta^{(m)} = a_1(m, \gamma_1, \gamma_2) \, \beta^{(1)} + a_2(m, \gamma_1, \gamma_2) \, \beta^{(2)}}$$

Asymptotic behavior of $\beta^{(m)}$:

$$a_1(m,\gamma_1,\gamma_2) \xrightarrow[m\to\infty]{} \frac{\gamma_1^2-\gamma_2^2}{(1-\gamma_1)(1-\gamma_2)(\gamma_2-\gamma_1)} = -\frac{\gamma_1+\gamma_2}{(1-\gamma_1)(1-\gamma_2)}$$

$$a_2(m,\gamma_1,\gamma_2) \xrightarrow[m\to\infty]{} \frac{\gamma_2-\gamma_1}{(1-\gamma_1)(1-\gamma_2)(\gamma_2-\gamma_1)} = \frac{1}{(1-\gamma_1)(1-\gamma_2)}.$$

Hence

$$\beta^{(m)} \xrightarrow[m\to\infty]{} -\frac{(\gamma_1+\gamma_2)}{(1-\gamma_1)(1-\gamma_2)}\beta^{(1)} + \frac{1}{(1-\gamma_1)(1-\gamma_2)}\beta^{(2)}$$

$$= \frac{\left[Var(x_t)\right]^{-1}}{(1-\gamma_1)(1-\gamma_2)}\left\{ \begin{array}{l} -(\gamma_1+\gamma_2)\left\{cov(x_t,\mu_{1,t}) + cov(x_t,\mu_{2,t})\right\} \\ +\ 1\left\{(1+\gamma_1)\,cov(x_t,\mu_{1,t}) + (1+\gamma_2)\,cov(x_t,\mu_{2,t})\right\} \end{array}\right\}$$

$$= \frac{\left[Var(x_t)\right]^{-1}}{(1-\gamma_1)(1-\gamma_2)}\left\{(1-\gamma_2)\,cov(x_t,\mu_{1,t}) + (1-\gamma_1)\,cov(x_t,\mu_{2,t})\right\}$$

$$= \cancel{\frac{1}{(1-\beta_1)}\left[\left(Var(x_t)\right)^{-1}\left(cov(x_t,\mu_{1,t})\right) + \left(Var(x_t)\right)^{-1}\left(cov(x_t,\mu_{2,t})\right)\right]}$$

$$= \frac{1}{(1-\gamma_1)}\left[Var(x_t)\right]^{-1} cov(x_t,\mu_{1,t}) + \frac{1}{1-\gamma_2}\left[Var(x_t)\right]^{-1} cov(x_t,\mu_{2,t})$$

Define $\lambda_1^{(m)}$ and $\lambda_2^{(m)}$ by

$\cancel{\mu_{1,t}}$: $$\mu_{1,t:t+m-1} = \sum_{i=1}^{m} \mu_{1,t-1+i}$$

$$\mu_{2,t:t+m-1} = \sum_{i=1}^{m} \mu_{2,t-1+i}$$

$$\mu_{1;t:t+m-1} = \omega_1^{(m)} + x_t'\lambda_1^{(m)} + \varepsilon^{(m)}_{1,t+m-1}$$

$$\mu_{2,t:t+m-1} = \omega_2^{(m)} + x_1'\lambda_2^{(m)} + \varepsilon^{(m)}_{2,t+m-1}$$

Then $$\beta^{(m)} = \lambda_1^{(m)} + \lambda_2^{(m)}$$

$$\lambda_1^{(m)} = \frac{1-\gamma_1^m}{1-\gamma_1}\lambda_1^{(1)}$$ (one factor result)

$$\lambda_2^{(m)} = \frac{1-\gamma_2^m}{1-\gamma_2}\lambda_2^{(1)}$$ (")

$$\boxed{\lim_{m\to\infty} \beta^{(m)} = \frac{\lambda_1^{(1)}}{1-\gamma_1} + \frac{\lambda_2^{(1)}}{1-\gamma_2}}$$