Edgeworth Corrections for Realized Volatility

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Abstract

The quality of the asymptotic normality of realized volatility can be poor if sampling does not occur at very high frequencies. In this paper we consider an alternative approximation to the finite sample distribution of realized volatility based on Edgeworth expansions. In particular, we show how confidence intervals for integrated volatility can be constructed using these Edgeworth expansions. The Monte Carlo study we conduct shows that the intervals based on the Edgeworth corrections have improved properties relatively to the conventional intervals based on the normal approximation. Contrary to the bootstrap, the Edgeworth approach is an analytical approach that is easily implemented, without requiring any resampling of one's data. A comparison between the bootstrap and the Edgeworth expansion shows that the bootstrap outperforms the Edgeworth corrected intervals. Thus, if we are willing to incur in the additional computational cost involved in computing bootstrap intervals, these are preferred over the Edgeworth intervals. Nevertheless, if we are not willing to incur in this additional cost, our results suggest that Edgeworth corrected intervals should replace the conventional intervals based on the first order normal approximation.

Keywords: Realized volatility, Edgeworth expansions, confidence intervals.

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1 Introduction

Realized volatility is now a standard measure of volatility in empirical finance. For moderate frequencies such as five or thirty minutes, realized volatility is an accurate measure of volatility. In addition, the limit theory provided by Jacod (1994), Jacod and Protter (1998) and Barndorff-Nielsen and Shephard (2002) shows that realized volatility has an asymptotic normal distribution under general conditions on the price and volatility processes (but excluding microstructure noise). This powerful result can be used for inference on volatility, e.g. for constructing confidence intervals for integrated volatility. Nevertheless, existing simulation results (see e.g. Barndorff-Nielsen and Shephard (2005), Gonçalves and Meddahi (2005) and Zhang et. al. (2005)) show that the asymptotic normality of realized volatility is not a good approximation for the moderate sample sizes often used in practice when computing realized volatility. To overcome this limitation, the logarithmic transformation of realized volatility is often used as an alternative to the raw version of realized volatility (see e.g. Andersen, Bollerslev, Diebold and Labys (2001), Andersen, Bollerslev, Diebold and Ebens (2001), and Barndorff-Nielsen and Shephard (2002, 2005)). Recently, Gonçalves and Meddahi (2006) propose a class of analytical transformations of realized volatility based on the Box-Cox transformation, which includes the log transformation and the raw realized volatility as special cases. Their results show that the log transformation is not the optimal choice from the viewpoint of eliminating skewness in finite samples. An alternative tool of inference for realized volatility based estimators is the bootstrap, which is considered in Gonçalves and Meddahi (2005). In particular, they propose several bootstrap methods for realized volatility and show that the bootstrap outperforms the normal approximation in small samples.

In this paper we explore an alternative method of inference for realized volatility. Specifically, we consider analytical corrections for realized volatility based on Edgeworth expansions. Edgeworth expansions correct the asymptotic normal approximation by including explicit corrections for skewness and kurtosis. These corrections can be quite important in small samples, where skewness and excess kurtosis are often present. In the bootstrap literature, Edgeworth expansions are the main tool to theoretically explain why the bootstrap outperforms the normal approximation. Using this approach, Gonçalves and Meddahi (2005) derived Edgeworth expansions for studentized statistics based on realized volatility and used them to assess the accuracy of the bootstrap in comparison to the normal approximation. Here we rely explicitly on the analytical Edgeworth expansions derived by Gonçalves and Meddahi (2005) to correct the normal approximation for realized volatility. In particular, we propose confidence intervals for integrated volatility that incorporate an analytical correction for skewness and kurtosis. Contrary to the bootstrap approach, the confidence intervals based on the Edgeworth expansions do not require any simulation and are therefore less computationally demanding than the bootstrap intervals.

Recently, Zhang et. al. (2005) also propose Edgeworth expansions for realized volatility estimators as a means to improve upon the first order asymptotic theory. Contrary to Gonçalves and Meddahi (2005), who abstract from the presence of microstructure noise and only consider realized volatility, Zhang et. al. (2005) allow for microstructure noise and therefore study a variety of realized volatility like estimators (including realized volatility but also other microstructure noise robust estimators). Whereas the Edgeworth expansions derived by Zhang et. al. (2005) apply only to normalized statistics based on the true variance of realized volatility, the expansions derived by Gonçalves and Meddadi (2005) apply also to studentized statistics (where the variance is replaced by a consistent estimator). Because the variance of realized volatility is usually unknown, studentized statistics are the statistics used in practice. For the special case of normalized statistics, the results of Zhang et. al. (2005) extend those of Gonçalves and Meddahi (2005) to allow for microstructure noise. Here we focus on the Edgeworth expansions for studentized statistics derived by Gonçalves and Meddahi (2005) and evaluate the coverage probability of confidence intervals corrected by these Edgeworth expansions. For comparison purposes, we also include in our study the infeasible, normalized statistics. Our results show that it is important to use the appropriate Edgeworth expansions when computing confidence intervals for integrated volatility.

We conduct extensive Monte Carlo simulations to evaluate the finite sample performance of the Edgeworth corrected confidence intervals. We consider two types of intervals: intervals based on infeasible Edgeworth expansions that depend on the true cumulants of the realized volatility statistic, and intervals based on a feasible version of the Edgeworth expansions, in which the true cumulants are replaced by consistent estimators. These are the empirical Edgeworth corrected intervals. For comparison purposes, we also include the traditional intervals based on the asymptotic normal distribution and the i.i.d. bootstrap interval proposed by Gonçalves and Meddahi (2005). Our simulation results show that the Edgeworth expansion corrected intervals outperform the conventional asymptotic theory based intervals. The infeasible intervals tend to perform extremely well, with coverage probabilities that are close to the desired 95% nominal level, especially for one-sided intervals. The feasible intervals based on the empirical Edgeworth expansions yield level distortions that are larger than the infeasible intervals. This is as expected given that cumulants are typically hard to estimate in finite samples. The empirical Edgeworth expansions corrected intervals are nevertheless superior to the conventional intervals based on first order asymptotic theory. A comparison between the Edgeworth approach and the bootstrap approach shows that the bootstrap outperforms the Edgeworth corrections. This finding is in agreement with the results found in the bootstrap literature, where the bootstrap is often found to be superior to analytical corrections based on empirical Edgeworth expansions (see e.g. Hardle, Kreiss and Horowitz (2003)). Our results suggest that the additional computational burden imposed by the bootstrap pays off in terms of accuracy. Nevertheless, if one wants to avoid this additional cost and not use the bootstrap, the Edgeworth corrected intervals should clearly replace the conventional intervals.

The structure of the paper is as follows. In Section 2, we present the setup and review the first order asymptotic theory for realized volatility. We also give the Edgeworth expansions for realized volatility derived in Gonçalves and Meddahi (2005). In Section 3, we show how to use these expansions to construct improved confidence intervals for integrated volatility. We give results for both the normalized and the studentized statistics, and for one-sided and two-sided intervals. In Section 4, we discuss the Monte Carlo simulation results. Section 5 concludes.

2 Setup and review of existing results

2.1 Setup

We assume the log price process $\{\log S_t : t \geq 0\}$ follows the continuous-time model

$$d \log S_t = v_t dW_t, \tag{1}$$

where W_t denotes a standard Brownian motion and v_t a volatility term. By assumption the drift term is zero and W_t and v_t are independent, thus excluding leverage and drift effects.

Intraday returns at a given horizon h are denoted r_i and are defined as follows:

$$r_i \equiv \log S_{ih} - \log S_{(i-1)h} = \int_{(i-1)h}^{ih} v_u dW_u, \text{ for } i = 1, \ldots, 1/h,$$

with $1/h$ an integer.

The parameter of interest is the integrated volatility over a day,

$$IV = \int_0^1 v_u^2 du,$$

where we have normalized the daily horizon to be the interval $(0, 1)$. The realized volatility estimator is defined as

$$RV = \sum_{i=1}^{1/h} r_i^2.$$

Following the notation of Gonçalves and Meddahi (2005), we let the integrated power volatility be denoted by

$$\overline{\sigma^q} \equiv \int_0^1 v_u^q du$$

for any $q > 0$. Its empirical analogue is the realized q-th order power variation, defined as

$$R_q = h^{-q/2+1} \sum_{i=1}^{1/h} |r_i|^q.$$

Under certain regularity conditions (see Barndorff-Nielsen and Shephard (2004)), $R_q \xrightarrow{P} \mu_q \overline{\sigma^q}$, where $\mu_q = E|Z|^q$, $Z \sim N(0, 1)$. The theory that follows involves the values of μ_4, μ_6 and μ_8, which are 3, 15 and 105, respectively.

2.2 Asymptotic normality of realized volatility

As $h \to 0$, Barndorff-Nielsen and Shephard (2002) (see also Jacod and Protter (1998), Jacod (1994)) show that

$$S_h \equiv \frac{\sqrt{h^{-1}}\left(RV - IV\right)}{\sqrt{V}} \to^d N\left(0,1\right), \tag{2}$$

where $V = 2\overline{\sigma^4}$ is the asymptotic variance of $\sqrt{h^{-1}}RV$. The statistic S_h is a normalized statistic. Since V is usually unknown because it depends on the volatility path through $\overline{\sigma^4}$, S_h is infeasible in practice. Barndorff-Nielsen and Shephard (2002) suggest replacing V with $\hat{V} = \frac{2}{3}R_4$, a consistent estimator of V. They prove that the studentized statistic T_h has also an asymptotic normal distribution:

$$T_h \equiv \frac{\sqrt{h^{-1}}\left(RV - IV\right)}{\sqrt{\hat{V}}} \to^d N\left(0,1\right). \tag{3}$$

The asymptotic normal distribution can be used to construct confidence intervals for IV. Alternatively, we can resort to Edgeworth expansions to improve upon the asymptotic normal approximation.

2.3 Edgeworth expansions for realized volatility

In this section we describe the Edgeworth expansions for S_h and T_h derived by Gonçalves and Meddahi (2005). These expansions depend on the cumulants of the statistic of interest.

Let $\kappa_j\left(S_h\right)$ denote the j^{th} order cumulant of S_h and let $\tilde{\kappa}_j$ denote the leading term of $\kappa_j\left(S_h\right)$ through order h. The normalized statistic is by construction centered at zero and it has unit variance (conditional on the volatility path). Hence

$$\begin{aligned} \kappa_1\left(S_h\right) &= 0, \text{ and} \\ \kappa_2\left(S_h\right) &= 1 \end{aligned}$$

which implies that $\tilde{\kappa}_1 = 0$ and $\tilde{\kappa}_2 = 1$. Based on the results of Gonçalves and Meddahi (2005), we can show that

$$\kappa_3\left(S_h\right) = \sqrt{h}\tilde{\kappa}_3 \tag{4}$$

$$\kappa_4\left(S_h\right) = h\tilde{\kappa}_4, \tag{5}$$

where

$$\tilde{\kappa}_3 = B_1 \frac{\overline{\sigma^6}}{\left(\overline{\sigma^4}\right)^{3/2}} \quad \text{and} \quad \tilde{\kappa}_4 = B_2 \frac{\overline{\sigma^8}}{\left(\overline{\sigma^4}\right)^2}, \tag{6}$$

with $B_1 = \frac{4}{\sqrt{2}}$ and $B_2 = 12$. Thus, although the first order asymptotic normal distribution of S_h has no skewness and no excess kurtosis (as $h \to 0$, $\kappa_3\left(S_h\right)$ and $\kappa_4\left(S_h\right)$ vanish), these are non zero at an higher order.

For the feasible statistic, let $\kappa_j\left(T_h\right)$ denote the j^{th} order cumulant of T_h and let κ_j denote the leading term of $\kappa_j(T_h)$ through order h. Gonçalves and Meddahi (2005) show that conditional on v

$$\begin{aligned}
\kappa_1(T_h) &= \sqrt{h}\kappa_1, \\
\kappa_2(T_h) &= 1 + h\kappa_2, \\
\kappa_3(T_h) &= \sqrt{h}\kappa_3, \quad \text{and} \\
\kappa_4(T_h) &= h\kappa_4,
\end{aligned}$$

where

$$\begin{aligned}
\kappa_1 &= -\frac{A_1}{2}\frac{\overline{\sigma^6}}{\left(\overline{\sigma^4}\right)^{3/2}}, \\
\kappa_2 &= (C_1 - A_2)\frac{\overline{\sigma^8}}{\left(\overline{\sigma^4}\right)^2} + \frac{7}{4}A_1^2\frac{\left(\overline{\sigma^6}\right)^2}{\left(\overline{\sigma^4}\right)^3}, \\
\kappa_3 &= (B_1 - 3A_1)\frac{\overline{\sigma^6}}{\left(\overline{\sigma^4}\right)^{3/2}}, \quad \text{and} \\
\kappa_4 &= (B_2 + 3C_1 - 6A_2)\frac{\overline{\sigma^8}}{\left(\overline{\sigma^4}\right)^2} + \left(18A_1^2 - 6A_1B_1\right)\frac{\left(\overline{\sigma^6}\right)^2}{\left(\overline{\sigma^4}\right)^3},
\end{aligned}$$

where $A_1 = \frac{4}{\sqrt{2}}$, $A_2 = 12$, $B_1 = \frac{4}{\sqrt{2}}$, $B_2 = 12$ and $C_1 = \frac{32}{3}$.

The normalized and the studentized statistics have different higher order cumulants through order $O(h)$. In particular, the studentized statistic has a higher order bias equal to $\sqrt{h}\kappa_1$ due to the estimation of V. In contrast, the bias of S_h is zero by construction.

We now give the Edgeworth expansions for the distributions of S_h and T_h. Let

$$\begin{aligned}
He_1(x) &= x \\
He_3(x) &= x\left(x^2 - 3\right) \\
He_5(x) &= x\left(x^4 - 10x^2 + 15\right),
\end{aligned}$$

denote the first, third and fifth order Hermite polynomials.

The second order Edgeworth expansion for S_h is given by

$$P(S_h \leq x) = \Phi(x) + \sqrt{h}p_1(x)\phi(x) + hp_2(x)\phi(x) + o(h), \tag{7}$$

where

$$p_1(x) = -\frac{1}{6}\tilde{\kappa}_3\left(x^2 - 1\right) \tag{8}$$

$$p_2(x) = -\frac{1}{24}\tilde{\kappa}_4 He_3(x) - \frac{1}{72}\tilde{\kappa}_3^2 He_5(x). \tag{9}$$

The second order Edgeworth expansion for T_h is given by

$$P\left(T_h \leq x\right) = \Phi\left(x\right) + \sqrt{h} q_1\left(x\right) \phi\left(x\right) + h q_2\left(x\right) \phi\left(x\right) + o\left(h\right), \tag{10}$$

where

$$q_1\left(x\right) = -\left(\kappa_1 + \frac{1}{6}\kappa_3\left(x^2 - 1\right)\right) \tag{11}$$

$$q_2\left(x\right) = -\left\{\frac{1}{2}\left(\kappa_2 + \kappa_1^2\right) He_1\left(x\right) + \frac{1}{24}\left(\kappa_4 + 4\kappa_1\kappa_3\right) He_3\left(x\right) + \frac{1}{72}\kappa_3^2 He_5\left(x\right)\right\}. \tag{12}$$

The Edgeworth expansions in (7) and (10) are different. Both expansions contain correction terms for skewness and kurtosis but these differ depending on whether the statistic is normalized or studentized.

3 Edgeworth corrected intervals for realized volatility

The goal of this section is to explain how one can use the Edgeworth expansions presented above to construct confidence intervals for IV with improved coverage probability. Our discussion follows Hall (1992). We first consider one-sided intervals, which are easier to describe. While one-sided intervals of IV are not as common in the econometrics literature, Mykland (2000, 2002, 2003) shows that these intervals are important for hedging in the context of option pricing.. Two-sided symmetric intervals will follow next. In our discussion, we focus on intervals for IV based on the studentized statistic T_h. Similar arguments hold for the normalized statistic S_h. Therefore, we only present the final expressions for the intervals based on S_h, omitting the details that explain why these are expected to outperform the conventional intervals based on the normal approximation. For concreteness and because this is what we implement in the Monte Carlo simulations, we focus on 95% level confidence intervals throughout.

3.1 One-sided intervals

The conventional 95% level one-sided confidence interval based on the asymptotic normality result in (3) is of the following form:

$$\mathcal{I}^{AT-T,1} = \left(-\infty, RV - \sqrt{h\hat{V}} z_{0.05}\right),$$

where $z_{0.05} = -1.645$ is the 5% percentile of the normal distribution. This interval has coverage probability equal to

$$\begin{aligned} P\left(IV \in \mathcal{I}^{AT-T,1}\right) &= P\left(T_h \geq z_{0.05}\right) = 1 - P\left(T_h < z_{0.05}\right) \\ &= 1 - \left[\Phi\left(z_{0.05}\right) + \sqrt{h} q_1\left(z_{0.05}\right) \phi\left(z_{0.05}\right) + o\left(\sqrt{h}\right)\right] \\ &= 0.95 - \sqrt{h} q_1\left(z_{0.05}\right) \phi\left(z_{0.05}\right) + o\left(\sqrt{h}\right), \end{aligned}$$

given the Edgeworth expansion in (10). The error in coverage probability of $\mathcal{I}^{AT-T,1}$ is thus of order $O\left(\sqrt{h}\right)$.

Consider now the following (infeasible) Edgeworth expansion corrected confidence interval for IV,

$$\mathcal{I}_{\text{inf}}^{EE-T,1}=\left(-\infty,RV-\sqrt{h\hat{V}}z_{0.05}+h\sqrt{\hat{V}}q_1\left(z_{0.05}\right)\right).$$

This interval contains a skewness correction term equal to $h\sqrt{\hat{V}}q_1\left(z_{0.05}\right)$, where $q_1\left(x\right)$ is defined as in (11). The coverage probability of $\mathcal{I}_{\text{inf}}^{EE-T,1}$ is given by

$$P\left(IV\in\mathcal{I}_{\text{inf}}^{EE-T,1}\right)=P\left(T_h\geq z_{0.05}-\sqrt{h}q_1\left(z_{0.05}\right)\right)=1-P\left(T_h<z_{0.05}-\sqrt{h}q_1\left(z_{0.05}\right)\right) \tag{13}$$

Using arguments detailed in Hall (1992, pp. 119-120), we can show that

$$\begin{aligned}P\left(T_h<z_{0.05}-\sqrt{h}q_1\left(z_{0.05}\right)\right) &= \Phi\left(z_{0.05}-\sqrt{h}q_1\left(z_{0.05}\right)\right)\\ &\quad+\sqrt{h}q_1\left(z_{0.05}-\sqrt{h}q_1\left(z_{0.05}\right)\right)\phi\left(z_{0.05}-\sqrt{h}q_1\left(z_{0.05}\right)\right)+o\left(\sqrt{h}\right)\\ &= \Phi\left(z_{0.05}\right)+O\left(h\right).\end{aligned}$$

Thus, from (13), we have that

$$P\left(IV\in\mathcal{I}_{\text{inf}}^{EE-T,1}\right)=1-\Phi\left(z_{0.05}\right)+O\left(h\right)=0.95+O\left(h\right),$$

implying that the error in coverage probability associated with $\mathcal{I}_{\text{inf}}^{EE-T,1}$ is equal to

$$P\left(IV\in\mathcal{I}_{\text{inf}}^{EE-T,1}\right)-0.95=O\left(h\right),$$

smaller than the $O\left(\sqrt{h}\right)$ error in coverage probability of $\mathcal{I}^{AT-T,1}$.

The interval $\mathcal{I}_{\text{inf}}^{EE-T,1}$ is infeasible because the skewness correction term depends on the population cumulants κ_1 and κ_3 entering the function $q_1\left(x\right)$. The following empirical Edgeworth corrected confidence interval

$$\mathcal{I}_{feas}^{EE-T,1}=\left(-\infty,RV-\sqrt{h\hat{V}}z_{0.05}+h\sqrt{\hat{V}}\hat{q}_1\left(z_{0.05}\right)\right)$$

is a feasible version of $\mathcal{I}_{\text{inf}}^{EE-T,1}$ where the skewness correction term is now equal to $h\sqrt{\hat{V}}\hat{q}_1\left(z_{0.05}\right)$, with $\hat{q}_1\left(x\right)$ defined in (11) except that κ_1 and κ_3, the leading terms of the first and third order cumulants of T_h, are replaced by consistent estimators. In particular, the results of Barndorff-Nielsen and Shephard (2004) imply that the following estimators are consistent estimators of κ_1 and κ_3:

$$\begin{aligned}\widehat{\kappa}_1 &= -\frac{A_1}{2}\frac{R_6/\mu_6}{\left(R_4/\mu_4\right)^{3/2}}=-\frac{4}{2\sqrt{2}}\frac{R_6/15}{\left(R_4/3\right)^{3/2}},\\ \widehat{\kappa}_3 &= \left(B_1-3A_1\right)\frac{R_6/\mu_6}{\left(R_4/\mu_4\right)^{3/2}}=-2\frac{4}{\sqrt{2}}\frac{R_6/15}{\left(R_4/3\right)^{3/2}},\end{aligned}$$

given that $A_1=B_1=\frac{4}{\sqrt{2}}$ and $\mu_4=3$ and $\mu_6=15$. Using arguments similar to those described in Hall (1992, p. 119), we can show that the coverage probability error of the empirical Edgeworth corrected confidence interval is no larger than the order $o\left(\sqrt{h}\right)$, given the consistency of $\hat{\kappa}_j$ for κ_j. This error is smaller than the error implicit in the normal approximation intervals, which is equal to $O\left(\sqrt{h}\right)$.

Similar results hold for intervals based on the normalized statistic. In particular, let

$$\begin{aligned}\mathcal{I}^{AT-S,1} &= \left(-\infty, RV - \sqrt{hV}z_{0.05}\right), \text{ and} \\ \mathcal{I}_{\inf}^{EE-S,1} &= \left(-\infty, RV - \sqrt{hV}z_{0.05} + h\sqrt{V}p_1\left(z_{0.05}\right)\right)\end{aligned}$$

denote the conventional asymptotic theory-based interval and the Edgeworth expansion corrected confidence interval for IV based on the normalized statistic S_h (p_1 is as defined in (8)). By arguments similar to those described above, we can show that the error in coverage probabilities are $O\left(\sqrt{h}\right)$ for $\mathcal{I}^{AT-S,1}$ and $O\left(h\right)$ for $\mathcal{I}_{\inf}^{EE-S,1}$. As we remarked before, these intervals are usually infeasible because they depend on V as well as on other higher order moments that are unknown in practice. Suppose we replace all population moments by consistent estimators. In particular, for the Edgeworth expansion based interval, suppose we replace V with $\hat{V}$ and $p_1\left(x\right)$ with $\hat{p}_1\left(x\right)$, where $\hat{p}_1\left(x\right)$ is of the same form as in (8) except that $\tilde{\kappa}_3$ is replaced with

$$\widehat{\tilde{\kappa}}_3 = B_1 \frac{\overline{\sigma^6}}{\left(\overline{\sigma^4}\right)^{3/2}} = \frac{4}{\sqrt{2}}\frac{R_6/15}{\left(R_4/3\right)^{3/2}}, \tag{14}$$

a consistent estimator of $\tilde{\kappa}_3$. This yields a feasible interval of the form

$$\mathcal{I}_{feas}^{EE-S,1} = \left(-\infty, RV - \sqrt{h\hat{V}}z_{0.05} + h\sqrt{\hat{V}}\hat{p}_1\left(z_{0.05}\right)\right).$$

We can show that the error in coverage probability of this interval is of order $O\left(\sqrt{h}\right)$, the same order of magnitude as the error in coverage probability of the conventional interval based on the normal approximation. The reason why this interval does not provide an improvement over the conventional interval is that it relies on the Edgeworth expansion for the normalized statistic, which does not take into account the estimation of V (in particular it uses p_1 instead of q_1).

3.2 Two-sided intervals

The conventional 95% level two-sided symmetric confidence interval based on the asymptotic normality result in (3) is of the following form:

$$\mathcal{I}^{AT-T,2} = \left(RV - \sqrt{h\hat{V}}z_{0.975}, RV + \sqrt{h\hat{V}}z_{0.975}\right),$$

where $z_{0.975} = 1.96$ is the 97.5% percentile of the normal distribution. This interval has coverage probability equal to

$$
\begin{aligned}
P\left(IV \in \mathcal{I}^{AT-T,2}\right) &= P\left(\left|T_h\right| \leq z_{0.975}\right) \\
&= 2\Phi\left(z_{0.975}\right) - 1 + 2hq_2\left(z_{0.975}\right)\phi\left(z_{0.975}\right) + o\left(h\right) \\
&= 0.95 + 2hq_2\left(z_{0.975}\right)\phi\left(z_{0.975}\right) + o\left(h\right),
\end{aligned}
$$

given the Edgeworth expansion in (10) and the symmetry properties of Φ, q_1 and q_2. The error in coverage probability of $\mathcal{I}^{AT-T,2}$ is thus of order $O\left(h\right)$.

We can obtain an improved symmetric confidence interval for IV by relying on the Edgeworth expansion in (10). In particular, consider the following interval

$$
\mathcal{I}_{\inf}^{EE-T,2} = \left(RV - \sqrt{h\hat{V}}z_{0.975} + h^{3/2}\sqrt{\hat{V}}q_2\left(z_{0.975}\right), RV + \sqrt{h\hat{V}}z_{0.975} - h^{3/2}\sqrt{\hat{V}}q_2\left(z_{0.975}\right)\right).
$$

This interval contains a skewness and kurtosis correction term equal to $h^{3/2}\sqrt{\hat{V}}q_2\left(z_{0.975}\right)$, where $q_2\left(x\right)$ is defined as in (12). By arguments similar to those used above, we can show that the coverage probability of $\mathcal{I}_{\inf}^{EE-T,2}$ is given by

$$
\begin{aligned}
P\left(IV \in \mathcal{I}_{\inf}^{EE-T,2}\right) &= P\left(\left|T_h\right| \leq z_{0.975} - h^{3/2}q_2\left(z_{0.975}\right)\right) \\
&= 0.95 + O\left(h^2\right),
\end{aligned}
$$

implying that the error in coverage probability associated with $\mathcal{I}_{\inf}^{EE-T,2}$ is equal to

$$
P\left(IV \in \mathcal{I}_{\inf}^{EE-T,2}\right) - 0.95 = O\left(h^2\right),
$$

smaller than the $O\left(h\right)$ error in coverage probability of $\mathcal{I}^{AT-T,2}$.

The interval $\mathcal{I}_{\inf}^{EE-T,2}$ is infeasible because it depends on $q_2\left(x\right)$, which in turn depends on the first four cumulants κ_j, for $j = 1, \ldots, 4$, of T_h. To obtain a feasible interval, we replace these cumulants by consistent estimators. In particular, $\widehat{\kappa}_1$ and $\widehat{\kappa}_3$ are defined as before and, in addition, we let

$$
\begin{aligned}
\widehat{\kappa}_2 &= \left(C_1 - A_2\right)\frac{R_8/\mu_8}{\left(R_4/\mu_4\right)^2} + \frac{7}{4}A_1^2\frac{\left(R_6/\mu_6\right)^2}{\left(R_4/\mu_4\right)^3}, \quad \text{and} \\
\widehat{\kappa}_4 &= \left(B_2 + 3C_1 - 6A_2\right)\frac{R_8/\mu_8}{\left(R_4/\mu_4\right)^2} + \left(18A_1^2 - 6A_1B_1\right)\frac{\left(R_6/\mu_6\right)^2}{\left(R_4/\mu_4\right)^3},
\end{aligned}
$$

where the constants A_1, A_2, B_1, B_2 and C_1 are as defined above, and $\mu_4 = 3$, $\mu_6 = 15$ and $\mu_8 = 105$.

We thus obtain the following feasible (or empirical) Edgeworth expansion corrected symmetric interval for IV:

$$
I_{feas}^{EE-T,2} = \left(RV - \sqrt{h\hat{V}}z_{0.975} + h^{3/2}\sqrt{\hat{V}}\hat{q}_2\left(z_{0.975}\right), RV + \sqrt{h\hat{V}}z_{0.975} - h^{3/2}\sqrt{\hat{V}}\hat{q}_2\left(z_{0.975}\right)\right).
$$

This interval has coverage probability error of order smaller than $O\left(h^2\right)$ given the consistency of $\hat{\kappa}_j$

for κ_j.

For comparison purposes, we also consider symmetric intervals for IV based on the normalized statistic S_h. These are defined as follows:

$$\begin{aligned}
\mathcal{I}^{AT-S,2} &= \left(RV - \sqrt{hV}z_{0.975}, RV + \sqrt{hV}z_{0.975}\right), \\
\mathcal{I}^{EE-S,2}_{\text{inf}} &= \left(RV - \sqrt{hV}z_{0.975} + h^{3/2}\sqrt{V}p_2\left(z_{0.975}\right), RV + \sqrt{hV}z_{0.025} - h^{3/2}\sqrt{V}p_2\left(z_{0.975}\right)\right), \text{ and} \\
\mathcal{I}^{EE-S,2}_{feas} &= \left(RV - \sqrt{h\hat{V}}z_{0.975} + h^{3/2}\sqrt{\hat{V}}\hat{p}_2\left(z_{0.975}\right), RV + \sqrt{h\hat{V}}z_{0.975} - h^{3/2}\sqrt{\hat{V}}\hat{p}_2\left(z_{0.975}\right)\right).
\end{aligned}$$

The error in coverage probabilities of the asymptotic theory-based interval is equal to $O\left(h\right)$, the same as that of the feasible Edgeworth corrected interval. In particular, the feasible version of $\mathcal{I}^{EE-S,2}_{\text{inf}}$ does not yield an asymptotic refinement over the conventional interval because it relies on the unappropriate Edgeworth expansion (i.e. it replaces V with $\hat{V}$ but uses p_2 instead of q_2). The infeasible interval promises an improvement, with a coverage probability error equal to $O\left(h^2\right)$.

4 Monte Carlo Simulations

In this section, we conduct a Monte Carlo study to evaluate the finite sample performance of the Edgeworth expansion corrected intervals in comparison to the asymptotic theory-based intervals and the i.i.d. bootstrap of Gonçalves and Meddahi (2005). The design is the same as in Gonçalves and Meddahi (2005). In particular, we consider two stochastic volatility models. The first model is the GARCH(1,1) diffusion studied by Andersen and Bollerslev (1998) and the second model is a two-factor diffusion model analyzed by Chernov et. al. (2003) and recently studied in the context of jump tests by Huang and Tauchen (2005).

More specifically, we consider the following stochastic volatility model

$$d\log S_t = \mu dt + v_t\left[\rho_1 dW_{1t} + \rho_2 dW_{2t} + \sqrt{1-\rho_1^2-\rho_2^2}dW_{3t}\right],$$

where W_{1t}, W_{2t} and W_{3t} are three independent standard Brownian motions. We fix $\mu = \rho_1 = \rho_2 = 0$, i.e. we assume no drift and no leverage effects.

The first model for v_t is the GARCH(1,1) diffusion studied by Andersen and Bollerslev (1998):

$$dv_t^2 = 0.035\left(0.636 - v_t^2\right)dt + 0.144v_t^2 dW_{1t}.$$

Finally, we consider the two-factor diffusion model analyzed by Chernov et al. (2003):[1]

$$\begin{aligned}
v_t &= \text{s-exp}\left(-1.2 + 0.04v_{1t}^2 + 1.5v_{2t}^2\right) \\
dv_{1t}^2 &= -0.00137v_{1t}^2 dt + dW_{1t} \\
dv_{2t}^2 &= -1.386v_{2t}^2 dt + \left(1 + 0.25v_{2t}^2\right) dW_{2t}.
\end{aligned}$$

10,000 Monte Carlo replications are used throughout for six different sample sizes: $1/h = 1152, 576, 288, 96, 48$ and 12, corresponding to "1.25-minute", "2.5-minute", "5-minute", "15-minute", "half-hour", "2-hour" returns.

Table 1 contains the first four cumulants of S_h and T_h across different sample sizes. We use $\kappa_j(\cdot)$ (for $j = 1, 2, 3$ and 4) to denote the j^{th} order cumulant of the relevant statistic, e.g. $\kappa_1(\cdot)$ denotes either $\kappa_1(S_h)$, the first order cumulant of S_h, or $\kappa_1(T_h)$, the first order cumulant of T_h. For each statistic and for each sample size, we report three values for each cumulant. The column entitled "Finite Sample" contains the finite sample cumulants, averaged across the 10,000 simulations. For instance, if $M = 10,000$ denotes the number of Monte Carlo replications and $S_h^{(k,j)}$ denotes the value of S_h for the k^{th} replication and j^{th} observation, the finite sample value of $\kappa_1(S_h)$ is $\frac{1}{M}\sum_{k=1}^{M}\left(\frac{1}{h^{-1}}\sum_{j=1}^{h^{-1}} S_h^{(k,j)}\right)$. Similarly,

$$\kappa_2(S_h) = \frac{1}{M}\sum_{k=1}^{M}\left(\frac{1}{h^{-1}}\sum_{j=1}^{h^{-1}}\left(S_h^{(k,j)} - \frac{1}{h^{-1}}\sum_{j=1}^{h^{-1}} S_h^{(k,j)}\right)^2\right),$$

$$\kappa_3(S_h) = \frac{1}{M}\sum_{k=1}^{M}\left(\frac{1}{h^{-1}}\sum_{j=1}^{h^{-1}}\left(S_h^{(k,j)} - \frac{1}{h^{-1}}\sum_{j=1}^{h^{-1}} S_h^{(k,j)}\right)^3\right), \quad \text{and}$$

$$\kappa_4(S_h) = \frac{1}{M}\sum_{k=1}^{M}\left(\frac{1}{h^{-1}}\sum_{j=1}^{h^{-1}}\left(S_h^{(k,j)} - \frac{1}{h^{-1}}\sum_{j=1}^{h^{-1}} S_h^{(k,j)}\right)^4\right) - 3\kappa_2^2(S_h),$$

with $\kappa_2(S_h)$ the sample variance of S_h, average across the M Monte Carlo simulations. Similar formulas were used for T_h. Note in particular that $\kappa_3(\cdot)$ is the finite sample third central moment of the statistic of interest. This is related but not equal to the traditional skewness coefficient because it is not scaled by the $(3/2)^{th}$ power of the sample variance. Similarly, $\kappa_4(\cdot)$ is related but not equal to the excess kurtosis coefficient because it is not scaled by the 2^{nd} power of the sample variance. The column entitled "EE" contains the simulation values of the higher order cumulants given in Section 2.3, which enter the Edgeworth expansions presented in Section 3. For example, the third order "EE" cumulant for S_h is equal to the average across the 10,000 Monte Carlo replications of $\sqrt{h}\tilde{\kappa}_3$, as given in (4) with $\tilde{\kappa}_3$ defined in (6). The column "Est. EE" contains the estimated version of the cumulants given in "EE". In the previous example, it contains $\sqrt{h}\widehat{\tilde{\kappa}}_3$, where $\widehat{\tilde{\kappa}}_3$ is as defined in (14), averaged across the 10,000 Monte Carlo replications.

[1] The function s-exp is the usual exponential function with a linear growth function splined in at high values of its argument: s-exp$(x) = \exp(x)$ if $x \leq x_0$ and s-exp$(x) = \frac{\exp(x_0)}{\sqrt{x_0}}\sqrt{x_0 - x_0^2 + x^2}$ if $x > x_0$, with $x_0 = \log(1.5)$.

For the GARCH(1,1) diffusion, Table 1 shows that the normalized statistic S_h has negligible finite sample bias across all sample sizes and sample variance close to 1. Thus, the first and second higher order cumulants entering the Edgeworth expansions agree well with their finite sample counterparts in this case. The same is generally true for the third and fourth order cumulants. These can be significantly different from zero, the value predicted by the first order asymptotic theory. The close agreement between the first four finite sample cumulants of S_h and their higher order versions used in the (infeasible) Edgeworth expansions suggests that the latter can provide a good approximation to the finite sample distribution of S_h. As we will see next, this translates into good coverage probabilities for the infeasible Edgeworth corrected intervals based on S_h ($\mathcal{I}_{\text{inf}}^{EE-S}$ in Table 2 below). A comparison between the "EE" and "Est. EE" columns shows that we tend to underestimate the finite sample third and fourth order cumulants. The results for the studentized statistic T_h show that differently from S_h, there is a finite sample bias that is non negligible at the smaller sample sizes. However, this finite sample bias is well matched with the higher order cumulants (compare "Finite Sample" with "EE" for $\kappa_1(\cdot)$). This is also true for $\kappa_2(\cdot)$, the variance of T_h. The most striking difference between S_h and T_h is the fact that the third and fourth order finite sample cumulants of T_h are much larger (in absolute value) than those of S_h. This indicates that the finite sample distribution of T_h is strongly (negatively) skewed and it has large excess kurtosis. The comparison between the columns "Finite Sample" and "EE" shows that there is a significant difference between the finite sample values of $\kappa_3(\cdot)$ and $\kappa_4(\cdot)$ and their EE predictions, especially for the smaller sample sizes (this is particularly true for $h^{-1} = 12$). Estimating these cumulants implies a further distortion in finite samples.

The comparison between the results for the GARCH(1,1) and the two factor diffusion models yields the following conclusions. First, the finite sample distributions of S_h and T_h for the two-factor diffusion have more skewness and excess kurtosis than those for the GARCH(1,1) diffusion. This is especially true for T_h, where the third and fourth order cumulants can be very large at the smaller sample sizes. This result is not very surprising given that the two-factor diffusion model is characterized by very rugged sample paths, often comparable to those generated by a jump diffusion model. See Gonçalves and Meddahi (2006) for more results on the comparison between these two models from the viewpoint of skewness and kurtosis. Second, the distortions between the finite sample cumulants and their higher order theoretical analogues are also larger for the two-factor diffusion than for the GARCH(1,1) diffusion. Third, estimation of the higher order cumulants induces even larger distortions, especially for the third and fourth order cumulants, when the sample size is small. Nevertheless, the estimated third and fourth order cumulants are very different from zero, the value assumed by the asymptotic normal distribution. This is true even for the larger sample sizes, which suggests that the Edgeworth approach should yield better inference than the first order asymptotic approach. Based on these results, we can also expect the coverage probabilities for the intervals based on the Edgeworth corrections (as well as for the intervals based on the first order asymptotic theory) to be poorer for the two-factor diffusion model compared to the GARCH(1,1) diffusion. These predictions are confirmed by the results in Table 2, which we discuss next.

Table 2 contains the actual coverage probabilities of the confidence intervals described in Section 3. Specifically, we consider intervals based on the normalized statistic S_h and on the studentized statistic T_h. For each statistic and for both one-sided and two-sided symmetric intervals, three types of intervals are considered: first order asymptotic theory-based intervals, intervals based on the Edgeworth expansion of the statistic of interest, where the coefficients are the true population cumulants (these correspond to the infeasible Edgeworth expansion intervals), and intervals based on the empirical Edgeworth expansion, which replaces the true cumulants with consistent estimates. The intervals based on S_h are infeasible, except for the feasible version of the Edgeworth expansion of S_h, where all moments are estimated (this corresponds to the third column). These intervals are presented mainly for comparison purposes. For the intervals based on T_h, we also give the coverage probabilities of the i.i.d. bootstrap. The results are from Gonçalves and Meddahi (2005).

The coverage probabilities for the infeasible Edgeworth corrected intervals based on S_h are generally close to the desired 95% level for both diffusion models. One-sided intervals are particularly well behaved. This is as expected given that Table 1 showed good agreement between the finite sample cumulants and the higher order cumulants entering the infeasible Edgeworth expansions for S_h. The asymptotic theory-based one-sided intervals tend to overcover for the smaller sample sizes. The feasible version of the Edgeworth corrected intervals ($\mathcal{I}_{feas}^{EE-S}$) performs much worse, with coverage rates well below the 95% desired level. Although this interval is feasible in practice, it is the worst interval in the table. The main reason for this performance is the fact that $\mathcal{I}_{feas}^{EE-S}$ is based on the unappropriate Edgeworth expansion, as discussed in Section 3. It replaces V with $\hat{V}$ but relies on the correction terms derived under the assumption that V is known. It is therefore important to use the appropriate Edgeworth expansion when constructing confidence intervals for IV. The correct approach in this case is to use $\mathcal{I}_{feas}^{EE-T}$. The first-order asymptotic theory-based intervals and the infeasible Edgeworth expansion corrected intervals based on the studentized statistic T_h are typically more distorted than the corresponding intervals based on S_h. This is as expected given that the accuracy of the higher order cumulants entering the Edgeworth expansions is smaller for T_h than for S_h. The empirical Edgeworth corrected intervals have larger coverage distortions than the infeasible Edgeworth intervals (but significantly smaller than those of the conventional intervals). This is especially true for the two-factor diffusion model, where the third and fourth order finite sample cumulants are not well matched by the estimated cumulants. A comparison between the two last columns reveals that the i.i.d. bootstrap outperforms the Edgeworth corrected intervals based on the estimated cumulants. Thus, although more computationally intensive, the bootstrap approach does yield a further refinement over the Edgeworth approach. A comparison between the fourth and sixth columns reveals nevertheless that the empirical Edgeworth corrected intervals outperform the first-order asymptotic theory approach.

5 Conclusion

The main contribution of this paper has been to propose confidence intervals for integrated volatility based on correction terms for skewness and kurtosis derived from Edgeworth expansions. The traditional arguments based on Edgeworth expansions show that these intervals have coverage probability errors smaller than the errors underlying the first-order asymptotic theory. The results of our Monte Carlo study confirm these predictions. For two diffusion models, we show that the finite sample performance of the Edgeworth corrected intervals is better than the finite sample performance of the conventional intervals based on the normal approximation. We also show that it is important to rely on the appropriate Edgeworth expansion when computing the corrected confidence intervals. In particular, the Edgeworth expansion for the studentized statistic should be used in constructing Edgeworth corrected intervals based on this statistic. Simply replacing population moments by consistent estimators in an Edgeworth expansion derived for the normalized statistic (which assumes that the true variance is known) leads to intervals with poor coverage probabilities.

A comparison between the empirical Edgeworth corrected intervals for IV and the i.i.d. bootstrap proposed by Gonçalves and Meddahi (2005) shows that the later approach is superior. Therefore, if one is willing to incur in the additional computational cost involved in computing bootstrap intervals, these are preferred over the Edgeworth-based intervals. If one is not willing to incur in this additional cost, then our results suggest that Edgeworth-corrected confidence intervals should replace the conventional intervals based on the normal approximation.

Table 1. Finite Sample and Edgeworth expansion-based Cumulants of S_h and T_h

h^{-1}		Normalized Statistic S_h Finite Sample	EE	Est. EE	Studentized Statistic T_h Finite Sample	EE	Est. EE
		GARCH(1,1) diffusion					
12	$\kappa_1(\cdot)$	−0.024	0.000	0.000	−0.590	−0.410	−0.293
	$\kappa_2(\cdot)$	1.005	1.000	1.000	2.873	2.066	1.579
	$\kappa_3(\cdot)$	0.860	0.820	0.586	−14.032	−1.641	−1.172
	$\kappa_4(\cdot)$	1.136	1.014	0.422	171.46	5.716	3.309
48	$\kappa_1(\cdot)$	−0.019	0.000	0.000	−0.166	−0.145	−0.134
	$\kappa_2(\cdot)$	0.985	1.000	1.000	1.288	1.267	1.216
	$\kappa_3(\cdot)$	0.428	0.410	0.359	−1.398	−0.821	−0.717
	$\kappa_4(\cdot)$	0.419	0.253	0.173	3.400	1.429	1.208
288	$\kappa_1(\cdot)$	−0.004	0.000	0.000	−0.088	−0.084	−0.081
	$\kappa_2(\cdot)$	0.988	1.000	1.000	1.040	1.044	1.043
	$\kappa_3(\cdot)$	0.124	0.168	0.162	−0.412	−0.335	−0.324
	$\kappa_4(\cdot)$	0.017	0.042	0.038	0.448	0.238	0.233
1152	$\kappa_1(\cdot)$	0.027	0.000	0.000	−0.014	−0.042	−0.042
	$\kappa_2(\cdot)$	0.995	1.000	1.000	1.000	1.011	1.011
	$\kappa_3(\cdot)$	0.110	0.084	0.083	−0.141	−0.168	−0.166
	$\kappa_4(\cdot)$	−0.010	0.011	0.010	0.041	0.060	0.059
		Two-factor diffusion					
12	$\kappa_1(\cdot)$	−0.016	0.000	0.000	−1.005	−0.845	−0.335
	$\kappa_2(\cdot)$	0.870	1.000	1.000	6.107	5.779	1.763
	$\kappa_3(\cdot)$	1.155	1.689	0.670	−67.825	−3.378	−1.339
	$\kappa_4(\cdot)$	2.901	5.588	0.542	1735.303	23.987	4.381
48	$\kappa_1(\cdot)$	−0.016	0.000	0.000	−0.451	−0.422	−0.245
	$\kappa_2(\cdot)$	0.969	1.000	1.000	1.920	2.195	1.421
	$\kappa_3(\cdot)$	0.750	0.845	0.491	−4.102	−1.689	−0.981
	$\kappa_4(\cdot)$	0.824	1.397	0.322	17.854	5.997	2.379
288	$\kappa_1(\cdot)$	0.020	0.000	0.000	−0.146	−0.172	−0.140
	$\kappa_2(\cdot)$	0.984	1.000	1.000	1.148	1.199	1.139
	$\kappa_3(\cdot)$	0.292	0.345	0.279	−0.833	−0.690	−0.558
	$\kappa_4(\cdot)$	0.182	0.233	0.122	1.133	1.000	0.759
1152	$\kappa_1(\cdot)$	0.024	0.000	0.000	−0.063	−0.086	−0.079
	$\kappa_2(\cdot)$	1.002	1.000	1.000	1.044	1.050	1.044
	$\kappa_3(\cdot)$	0.169	0.172	0.159	−0.365	−0.345	−0.317
	$\kappa_4(\cdot)$	−0.004	0.058	0.044	0.184	0.250	0.236

Table 2. Coverage probabilities of nominal 95% confidence intervals for IV

	Normalized Statistic S_h			Studentized Statistic T_h			
h^{-1}	$\mathcal{I}^{AT-S}$	$\mathcal{I}^{EE-S}_{\inf}$	$\mathcal{I}^{EE-S}_{feas}$	$\mathcal{I}^{AT-T}$	$\mathcal{I}^{EE-T}_{\inf}$	$\mathcal{I}^{EE-T}_{feas}$	i.i.d Boot
				GARCH(1,1) diffusion			
One-Sided							
12	98.34	95.18	79.34	82.69	89.87	87.93	93.27
48	96.31	95.07	87.96	89.74	93.47	92.98	92.74
288	95.56	95.07	92.37	93.03	94.82	94.79	94.33
1152	95.89	95.61	94.40	94.01	95.57	95.58	94.56
Two-sided symmetric							
12	95.68	94.85	84.42	86.08	93.14	90.13	93.75
48	95.50	95.28	91.94	92.32	94.59	93.99	94.87
288	95.32	95.26	94.47	94.57	95.01	95.00	95.18
1152	95.14	95.14	95.30	94.81	95.41	95.40	94.97
				Two-factor diffusion			
One-sided							
12	99.90	95.42	72.67	75.69	89.34	82.18	93.27
48	98.28	95.52	81.90	84.52	92.45	88.95	94.63
288	96.36	95.26	90.27	90.27	94.64	93.65	95.10
1152	95.76	95.29	92.71	93.20	95.04	94.83	95.02
Two-sided symmetric							
12	96.08	93.48	78.08	78.94	96.88	84.98	90.13
48	95.71	95.00	86.85	87.95	95.44	90.50	92.83
288	95.32	95.17	93.48	92.83	95.37	94.66	94.59
1152	95.11	95.07	94.41	94.64	95.22	95.05	95.20

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