

# Box-Cox Transforms for Realized Volatility

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## Abstract

The log transformation of realized volatility is often preferred to the raw version of realized volatility because of its superior finite sample properties. One of the possible explanations for this finding is the fact the skewness of the log transformed statistic is smaller than that of the raw statistic. Simulation evidence presented here shows that this is the case. It also shows that the log transform does not completely eliminate skewness in finite samples. This suggests that there may exist other nonlinear transformations that are more effective at reducing the finite sample skewness.

The main goal of this paper is to study the accuracy of a new class of transformations for realized volatility based on the Box-Cox transformation. This transformation is indexed by a parameter  $\beta$  and contains as special cases the log (when  $\beta = 0$ ) and the raw (when  $\beta = 1$ ) versions of realized volatility. Based on the theory of Edgeworth expansions, we study the accuracy of the Box-Cox transforms across different values of  $\beta$ . We derive an optimal value of  $\beta$  that approximately eliminates skewness. We then show that the corresponding Box-Cox transformed statistic outperforms other choices of  $\beta$ , including  $\beta = 0$  (the log transformation). We provide extensive Monte Carlo simulation results to compare the finite sample properties of different Box-Cox transforms. Across the models considered in this paper, one of our conclusions is that  $\beta = -1$  (i.e. relying on the inverse of realized volatility) is the best choice if we want to control the coverage probability of 95% level confidence intervals for integrated volatility.

*Keywords:* Realized volatility, Box-Cox transformation, Edgeworth expansions.

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# 1 Introduction

The logarithmic transformation of realized volatility (the sum of squared intraday returns) is known to have better finite sample properties than realized volatility and for this reason it is often used in empirical applications involving realized volatility. This transformation was first used by Andersen, Bollerslev, Diebold and Labys (2001) to induce normality for realized volatility measures for high frequency exchange rates returns. Andersen, Bollerslev, Diebold and Ebens (2001) applied it in the context of high frequency stock returns. Barndorff-Nielsen and Shephard (henceforth BN-S) (2002) studied the first order asymptotic properties of the log transformation, whereas BN-S (2005) showed by simulation that the finite sample distribution of the log transformation was closer to the asymptotic standard normal distribution than the finite sample distribution of the non-transformed (or raw) version of realized volatility.

The purpose of this paper is twofold. First, we compare the accuracy of the first order asymptotic theory for realized volatility and for log realized volatility by analytical methods. Our approach relies on the Edgeworth expansions<sup>1</sup> of the distribution of realized volatility and of its log transformation recently developed by Gonçalves and Meddahi (2006). In particular, we show that the first higher order term in the expansion of the distribution of the log transformation is smaller (in absolute value) than the corresponding higher order term in the expansion for realized volatility. This provides a theoretical explanation for the superior finite sample performance of the log transformation found previously by Monte Carlo simulation (see BN-S (2005) and Gonçalves and Meddahi (2006)).

Second, we study the accuracy of a class of analytical transformations of realized volatility based on the Box-Cox transformation. This transformation is indexed by a parameter  $\beta$  and includes as special cases the logarithmic transformation (when  $\beta = 0$ ) and the linear (or non-transformed) version of the statistic of interest (when  $\beta = 1$ ). Gonçalves and Meddahi (2006) studied the bootstrap for the Box-Cox transformed realized volatility statistics. In doing so, they developed Edgeworth expansions of the distribution of the Box-Cox transformation for realized volatility. Here we use these expansions to compare the accuracy of the Box-Cox transformation for different values of  $\beta$ . Our results suggest that the log transformation can be improved upon by choosing values of  $\beta$  other than zero. We derive an optimal value of  $\beta$  for which the skewness of the higher order expansion of the distribution associated with this transformation is equal to zero and show that its finite sample properties are often superior to those associated with other Box-Cox transformations (including the log). In practice, the transformation based on the optimal value of  $\beta$  is infeasible because it depends on the volatility process. We therefore propose a consistent estimator of the optimal value of  $\beta$  and study its finite sample properties. Our results suggest that estimation of  $\beta$  can induce significant distortions in finite samples. In practice, a fixed value of  $\beta$  equal to  $-1$  proves to be a better choice if the target is to

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<sup>1</sup>The idea of using Edgeworth expansions to compare the accuracy of the asymptotic distribution for alternative statistics of interest has been used before in other contexts. For instance, Phillips and Park (1988) use Edgeworth expansions to investigate alternative forms of the Wald test for nonlinear restrictions.

obtain 95% level confidence intervals for integrated volatility whose actual coverage is close to the desired level of 95%.

Recently, Chen and Deo (2004) propose a power transformation for statistics that are positive linear combinations of positive independent random variables. In particular, they provide a method to choose the optimal power that approximately eliminates skewness in finite samples. There are two main differences between this paper and Chen and Deo (2004). First, the setup of Chen and Deo (2004) applies only to linear combinations of independent and identically distributed random variables. Instead, our setup explicitly allows for conditional heteroskedasticity of unknown form (we focus on a general class of stochastic volatility models). Conditionally on the volatility path, the data are independent but heteroskedastic. Consequently, the optimal power transformations that we derive are a function of the volatility path and are infeasible in practice. In order to render our transformations feasible, we propose consistent estimators for the optimal power transformations. Second, Chen and Deo's (2004) results only apply to statistics whose asymptotic variance is known and does not need to be estimated (for realized volatility-based statistics, these correspond to infeasible statistics because in practice we do not know the asymptotic variance of realized volatility). As we will show, Chen and Deo's (2004) optimal power coincides with the optimal Box-Cox transformation for the infeasible statistic that we derive here. In addition, we also derive the optimal Box-Cox transform for feasible statistics based on realized volatility. These statistics are studentized statistics which replace the asymptotic variance with a consistent estimator. Our results show that the optimal Box-Cox transformation that approximately eliminates skewness for the feasible transformed statistic is different from the optimal Box-Cox transformation that applies to the infeasible statistic (and therefore different from the optimal power derived by Chen and Deo (2004)).

Barndorff-Nielsen and Shephard (2002) briefly discuss some nonlinear transformations for realized volatility as an alternative to the log transformation, but no higher order theory is presented.

The rest of this paper is organized as follows. In Section 2 we introduce the setup and study by simulation the finite sample properties of the log transformation for realized volatility in comparison with the raw realized volatility. In Section 3 we study the higher order properties of the Box-Cox transformation. We complement our theoretical analysis with Monte Carlo simulation results. Section 3.1 considers infeasible statistics, for which the asymptotic variance is assumed known. In Section 3.2 we study the Box-Cox transformed feasible statistics, which replace the unknown asymptotic variance with a consistent estimator. Section 4 concludes. Appendix A contains details on the Monte Carlo design and on how to construct confidence intervals for integrated volatility based on Box-Cox transformed realized volatility statistics. Appendix B contains the proofs of our theoretical results.

## 2 Setup and Motivation

The goal of this section is to explore by simulation some possible explanations for the improved accuracy of the normal distribution when applied to the log transformation. As we will see, the skewness of the log transformed realized volatility is substantially smaller than that of the non-transformed statistic and this can explain why the log transform performs better in finite samples. Our results also show that the log transform does not completely eliminate the skewness in finite samples. This finding suggests that there may exist other nonlinear transformations that more efficiently reduce the finite sample skewness. This motivates our study of the Box-Cox transformation in the next section.

We consider the same setup as in Gonçalves and Meddahi (2006). More specifically, the log price process  $\{\log S_t : t \geq 0\}$  is assumed to follow the continuous-time model

$$d \log S_t = v_t dW_t, \quad (1)$$

where  $W_t$  denotes a standard Brownian motion and  $v_t$  a volatility term. We assume the drift term is zero and we suppose the independence between  $W_t$  and  $v_t$ , thus excluding leverage and drift effects.

Intraday returns at a given horizon  $h$  are denoted  $r_i$  and are defined as follows:

$$r_i \equiv \log S_{ih} - \log S_{(i-1)h} = \int_{(i-1)h}^{ih} v_u dW_u, \text{ for } i = 1, \dots, 1/h,$$

with  $1/h$  an integer.

The parameter of interest is the integrated volatility over a day,

$$IV = \int_0^1 v_u^2 du,$$

where we have normalized the daily horizon to be the interval  $(0, 1)$ . The realized volatility estimator is defined as

$$RV = \sum_{i=1}^{1/h} r_i^2.$$

Following the notation of Gonçalves and Meddahi (2006), we let the integrated power volatility be denoted by

$$\overline{\sigma^q} \equiv \int_0^1 v_u^q du$$

for any  $q > 0$ . Note that  $\overline{\sigma^2} \equiv \int_0^1 v_u^2 du = IV$ . The realized  $q$ -th order power variation is defined as

$$R_q = h^{-q/2+1} \sum_{i=1}^{1/h} |r_i|^q.$$

Under certain regularity conditions (see BN-S (2004)),  $R_q \xrightarrow{P} \mu_q \overline{\sigma^q}$ , where  $\mu_q = E|Z|^q$ ,  $Z \sim N(0, 1)$ .

As  $h \rightarrow 0$ , it also follows that (BN-S (2002), Jacod and Protter (1998), Jacod (1994))

$$\frac{\sqrt{h^{-1}}(RV - IV)}{\sqrt{V}} \rightarrow^d N(0, 1), \quad (2)$$

where  $V = 2\sigma^4$  is the asymptotic variance of  $\sqrt{h^{-1}}RV$ . Similarly, as first proved by BN-S (2002),

$$\frac{\sqrt{h^{-1}}(RV - IV)}{\sqrt{\hat{V}}} \rightarrow^d N(0, 1), \quad (3)$$

where  $\hat{V} = \frac{2}{3}R_4$  is a consistent estimator of  $V$ .

The log versions of (2) and (3) are given by

$$\frac{\sqrt{h^{-1}}(\log(RV) - \log(IV))}{\sqrt{\frac{V}{(IV)^2}}} \rightarrow^d N(0, 1) \quad (4)$$

and

$$\frac{\sqrt{h^{-1}}(\log(RV) - \log(IV))}{\sqrt{\frac{\hat{V}}{(RV)^2}}} \rightarrow^d N(0, 1), \quad (5)$$

respectively.

We compare the log transformed statistics with the raw statistics in a Monte Carlo study. The design is the same as in Gonçalves and Meddahi (2006). In particular, we consider two stochastic volatility models. The first model is the GARCH(1,1) diffusion studied by Andersen and Bollerslev (1998) and the second model is a two-factor diffusion model analyzed by Chernov et. al. (2003) and recently studied in the context of jump tests by Huang and Tauchen (2005). We assume no drift and no leverage effects. 10,000 Monte Carlo replications are used throughout. Appendix B contains more details on the Monte Carlo design.

We consider two types of statistics: infeasible statistics, such as (2) and (4), for which the asymptotic variance depends on the true volatility process, and feasible statistics, such as (3) and (5), which are based on a consistent estimator of the asymptotic variance. Although the infeasible statistics are useless in practice, it is instructive to include them in our study for two reasons. First, the contrast between the infeasible and the feasible statistics is important to help us understand the finite sample performance of the feasible statistics (in particular, estimation of the asymptotic variance matters in finite samples and it deteriorates the properties of the feasible statistic). The second reason is that, as we will see in the next section, the optimal Box-Cox transformation is different when applied to the infeasible or to the feasible statistics. By including the infeasible statistic in the Monte Carlo study we can evaluate the performance of these two choices for both types of statistics.

Tables 3 and 4 contain the results for the infeasible realized volatility-based statistics for the GARCH(1,1) and the two-factor diffusion models, respectively. Tables 5 and 6 contain the corresponding results for the feasible versions of these statistics. In these tables we find results for several

Box-Cox transformed statistics, but for the moment we only discuss the results for the raw statistics (which correspond to  $\beta = 1$ ) and for the log transformation ( $\beta = 0$ ). For each sample size ( $h^{-1} = 12, 48, 288$  and  $1152$ ), we computed the finite sample summary statistics for each version of the RV-based statistic, namely the mean, the standard error, the skewness<sup>2</sup> and the excess kurtosis, across the 100,000 Monte Carlo replications. We also computed the coverage probability of two-sided symmetric and lower one-sided 95% level intervals for integrated volatility. See Appendix A for more details on how to build confidence intervals for integrated volatility based on the infeasible and feasible statistics.

Starting with the infeasible statistics, Tables 3 and 4 suggest the bias is smallest (and close to zero) for the raw version. This is as expected. In contrast, the log transformation introduces a negative bias, especially at the smallest sample sizes. The raw statistics have right skewed distributions, especially for the small sample sizes and for the two-factor diffusion. The degree of asymmetry is smaller for the log version than for the raw version of the statistics. This transformation clearly reduces the amount of finite sample skewness and is particularly effective in doing so for the two-factor diffusion. The log transformation is also able to reduce excess kurtosis. The actual coverage rates of two-sided symmetric 95% level confidence intervals based on the raw statistics are close to the desired level for all sample sizes. Interestingly, the one-sided intervals are too conservative whereas the log based intervals tend to undercover (this is probably due to the negative bias). Overall, Tables A1 and A2 indicate that although the raw statistics have some finite sample skewness and excess kurtosis, their finite sample performance in terms of coverage rates for two-sided symmetric intervals is good. The log transformation reduces the skewness and excess kurtosis, but it introduces a negative bias and this can hurt the coverage rates.

We now turn to Tables 5 and 6, which contain the results for the feasible statistics. The main difference with respect to Tables 3 and 4 is that there is a clear deterioration of the finite sample performance of the raw statistics for all sample sizes and models (although more so for the two-factor diffusion than for the GARCH(1,1) diffusion). The raw statistics are negatively biased in finite samples. This bias is introduced by the estimation of the asymptotic variance. Moreover, the amount of skewness and excess kurtosis can be quite substantial, especially for the small sample sizes and for the two-factor diffusion. The log transformation is effective in reducing the magnitude of the bias, skewness and excess kurtosis as well as in producing less variability for the two models. The coverage rates are therefore closer to the desired level of 95% than those of the raw statistics. However, significant coverage distortions remain for the log transformation, especially when  $h^{-1} = 12$  and  $48$ .

Given that the normal distribution is characterized by a zero mean, unit variance and zero skewness and zero excess kurtosis, a statistic that more closely matches these moments can be expected to be

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<sup>2</sup>The tables report the finite sample third central moment of the statistic of interest and not the skewness as it is usually calculated (i.e. we do not scale this quantity by the third power of the standard error). Equivalently, we report the finite sample third cumulants. The main reason for doing so is that we want to compare these empirical measures with the Edgeworth correction predictions for the third order cumulant of the statistics of interest.

better approximated by the normal distribution than otherwise. Our results suggest that this is the case for the log transformation. In the next section, we will provide a theoretical result that confirms this simulation finding.

Our results also suggest that the log transformation is not completely effective in eliminating the bias and the skewness in finite samples, which may explain why there are still some distortions in coverage probabilities at the smaller sample sizes. This is why we consider in the next section the more general Box-Cox transformation.

### 3 The Box-Cox Transformation

The Box-Cox transformation for realized volatility is defined as

$$g(RV; \beta) = \begin{cases} \frac{RV^\beta - 1}{\beta} & \text{when } \beta \neq 0 \\ \log(RV) & \text{when } \beta = 0. \end{cases} \quad (6)$$

It contains the log transformation for realized volatility (when  $\beta = 0$ ) and the raw statistic (when  $\beta = 1$ ) as special cases. Our main goal in this section is to study the accuracy of this transformation for several values of  $\beta$ .

The infeasible Box-Cox transformed statistic is given by

$$S_\beta = \frac{\sqrt{h^{-1}}(g(RV; \beta) - g(IV; \beta))}{g'(IV; \beta)\sqrt{V}} \equiv \frac{\sqrt{h^{-1}}(g(RV; \beta) - g(IV; \beta))}{\sqrt{V_\beta}},$$

where

$$V_\beta = (g'(IV; \beta))^2 V$$

is the asymptotic variance of  $\sqrt{h^{-1}}g(RV; \beta)$ . Here and throughout the paper  $g'(x; \beta)$  denotes the derivative of  $g$  with respect to  $x$ . Similarly,  $g''(x; \beta)$  denotes the second derivative of  $g$  with respect to  $x$ . By an application of the delta method, we have that  $S_\beta \xrightarrow{d} N(0, 1)$ . When  $\beta = 0$ ,  $S_\beta$  corresponds to (4). When  $\beta \neq 0$ ,

$$S_\beta = \frac{\sqrt{h^{-1}}(RV^\beta - IV^\beta)}{\sqrt{V_\beta}} \xrightarrow{d} N(0, 1),$$

where  $V_\beta = \beta^2 (IV)^{2(\beta-1)} V$ .  $\beta = 1$  corresponds to the raw version (2).

The statistic  $S_\beta$  is infeasible because  $V_\beta$  depends on the volatility path, which is unknown in practice. Therefore, we need to replace  $V_\beta$  with a consistent estimator given by

$$\hat{V}_\beta = (g'(RV; \beta))^2 \hat{V}.$$

The feasible statistic is given by

$$T_\beta = \frac{\sqrt{h^{-1}}(g(RV; \beta) - g(IV; \beta))}{\sqrt{\hat{V}_\beta}} = \begin{cases} \frac{\sqrt{h^{-1}}(RV^\beta - IV^\beta)}{\beta RV^{\beta-1} \sqrt{\hat{V}}} & \text{when } \beta \neq 0 \\ \frac{\sqrt{h^{-1}}(\log(RV) - \log(IV))}{\sqrt{\frac{\hat{V}}{RV^2}}} & \text{when } \beta = 0, \end{cases}$$

and because  $\hat{V}_\beta$  is a consistent estimator of  $V_\beta$  as  $h \rightarrow 0$  it is also the case that  $T_\beta \rightarrow^d N(0, 1)$ .

### 3.1 Infeasible statistic

In this section we study the accuracy of the Box-Cox transformation for the infeasible statistic  $S_\beta$ . Our approach will be based on a second order Edgeworth expansion of the distribution of  $S_\beta$ . This expansion depends on the first three cumulants of  $S_\beta$  which we denote by  $\kappa_i(S_\beta)$  for  $i = 1, 2, 3$ . The following result (from Gonçalves and Meddahi (2006)) provides asymptotic expansions for  $\kappa_i(S_\beta)$ ,  $i = 1, 2, 3$ .

**Proposition 3.1** *Under Gonçalves and Meddahi's (2006) regularity conditions, conditional on  $v$ , as  $h \rightarrow 0$ ,*

a)

$$\kappa_1(S_\beta) = \sqrt{h} \left( \frac{1}{2} \frac{g''(IV; \beta)}{g'(IV; \beta)} \sqrt{V} \right) + O(h).$$

b)

$$\kappa_2(S_\beta) = 1 + O(h).$$

c)

$$\kappa_3(S_\beta) = \sqrt{h} \left( \frac{4}{\sqrt{2}} \frac{\overline{\sigma^6}}{(\overline{\sigma^4})^{3/2}} + 3 \frac{g''(IV; \beta)}{g'(IV; \beta)} \sqrt{V} \right) + O(h).$$

Appendix B contains the proof of this and all other theoretical results in the paper.

Proposition 3.1 shows that the choice of  $\beta$  has an impact through order  $O(\sqrt{h})$  on the first and third order cumulants of the statistic  $S_\beta$ . In particular, note that

$$\frac{g''(IV; \beta)}{g'(IV; \beta)} = (\beta - 1) IV^{-1}.$$

Thus,  $\beta = 1$  is the optimal choice if eliminating bias is the goal, i.e. the raw statistic performs best with respect to bias. This theoretical result confirms the Monte Carlo simulation results discussed previously. The log transformation implies a negative bias for  $S_\beta$ , which is also in agreement with our simulation results. In general, applying a nonlinear transformation to  $RV$  introduces a bias term. The bias is positive if  $\beta > 1$  and it is negative if  $\beta < 1$ .



The third order cumulant is also affected by the choice of  $\beta$  through the ratio  $\frac{g''(IV;\beta)}{g'(IV;\beta)}$ . Because the asymptotic normal approximation is associated with a zero third order cumulant (or zero skewness), choosing  $\beta$  so as to make  $\kappa_3(S_\beta)$  approximately equal to zero can be beneficial.

**Proposition 3.2** *Conditionally on  $v$ , as  $h \rightarrow 0$ , up to order  $O(\sqrt{h})$ ,*

a) *Choosing  $\beta$  equal to 1 implies  $\kappa_1(S_\beta) = 0$ .*

b) *Choosing  $\beta$  equal to*

$$\beta^* = 1 - \frac{2}{3} \frac{\overline{\sigma^2} \overline{\sigma^6}}{(\overline{\sigma^4})^2} \leq \frac{1}{3}$$

*implies  $\kappa_3(S_\beta) = 0$ .*

Proposition 3.2 shows that different values of  $\beta$  are needed to make bias and skewness equal to zero. In particular, the optimal value of  $\beta$  from the viewpoint of skewness is random and depends on the volatility path. If the volatility is constant, then  $\beta^* = \frac{1}{3}$ . If volatility is stochastic,  $\beta^* < \frac{1}{3}$ . Recently, Chen and Deo (2004) proposed an optimal power transformation for random variables that are positive linear combinations of positive independent random variables and showed that this transformation induces normality in small samples. Their optimal transformation is equal to ours when applied to realized volatility for the infeasible statistic. Nevertheless, their results do not cover feasible statistics. As we will see later, the optimal power transform  $\beta^*$  derived for the infeasible statistic  $S_\beta$  is not applicable to the feasible statistic  $T_\beta$ .

In practice we cannot compute  $\beta^*$  because it depends on the volatility path. It is nevertheless possible to consistently estimate  $\beta^*$  by replacing  $\overline{\sigma^q}$  with  $R_q/\mu_q$  (see BN-S (2004)). This yields the following estimator of  $\beta^*$ :

$$\hat{\beta}^* = 1 - \frac{2}{3} \frac{RV \ (R_6/\mu_6)}{(R_4/\mu_4)^2} = 1 - \frac{2}{3} \frac{RV \ (R_6/15)}{(R_4/3)^2},$$

given that  $\mu_4 = 3$  and  $\mu_6 = 15$ .

The right-hand-side panels of Tables 7 and 8 contain some summary statistics for  $\beta^*$  and  $\hat{\beta}^*$  for different sample sizes, for the GARCH(1,1) diffusion and for the two-factor diffusion, respectively. We complement these results by providing the kernel densities for  $\beta^*$  and  $\hat{\beta}^*$  in Figures 1 and 2 (right panels), for the GARCH(1,1) and for the two-factor diffusion models, respectively. (The left-hand-side panels of these tables and figures also contain results for the optimal Box-Cox transforms for the feasible statistics, to be introduced and discussed later)

For the GARCH(1,1) diffusion, Table 7 shows that on average  $\beta^*$  equals 0.331 and its distribution is very concentrated around this value, with a minimum of 0.310 and a maximum of 0.332. Thus, for the GARCH(1,1) diffusion the optimal value of the Box-Cox transformation in terms of skewness is on average very close to  $1/3$ , which is the optimal value of  $\beta$  for the constant volatility case. This is not

very surprising because the GARCH(1,1) diffusion implies very persistent sample paths for volatility. Estimation of  $\beta^*$  is difficult. Table 7 and Figure 1 show that  $\hat{\beta}^*$  is very (positively) biased for finite samples. For instance, for  $h^{-1} = 12$ , the distribution of  $\hat{\beta}^*$  is centered at 0.482, with a minimum value of 0.323. Its range does not intersect the range of  $\beta^*$ ! The bias becomes less and less pronounced as we increase the sample size, as expected.

The results in Table 8 and Figure 2 suggest that the distribution of  $\beta^*$  for the two-factor diffusion is quite different from that of  $\beta^*$  for the GARCH(1,1) diffusion. On average,  $\beta^*$  equals 0.057, but its range is now much larger, going from a minimum of  $-1.031$  to a maximum of  $0.297$ . As with the GARCH(1,1) diffusion, for small samples  $\hat{\beta}^*$  is severely biased towards positive values.

We complete this section by studying the accuracy of normal approximation for the Box-Cox transformation for different values of  $\beta$ . We rely on a second order Edgeworth expansion of the distribution of  $S_\beta$  given by (see e.g. Hall, 1992)

$$P(S_\beta \leq x) = \Phi(x) + \sqrt{h} p_\beta(x) \phi(x) + o(h),$$

where  $\Phi(x)$  and  $\phi(x)$  denote the cumulative distribution function and the density function of a standard normal random variable, respectively. The correction term  $p_\beta(x)$  is defined as

$$p_\beta(x) = - \left( \bar{\kappa}_{1,\beta} + \frac{1}{6} \bar{\kappa}_{3,\beta} (x^2 - 1) \right),$$

where

$$\bar{\kappa}_{1,\beta} = \frac{1}{2} \frac{g''(IV; \beta)}{g'(IV; \beta)} \sqrt{V}$$

and

$$\bar{\kappa}_{3,\beta} = \frac{4}{\sqrt{2}} \frac{\overline{\sigma^6}}{(\overline{\sigma^4})^{3/2}} + 3 \frac{g''(IV; \beta)}{g'(IV; \beta)} \sqrt{V}$$

are the coefficients of the leading terms of  $\kappa_1(S_\beta)$  and  $\kappa_3(S_\beta)$ , respectively.

Given this expansion, the error incurred by the normal approximation in estimating the distribution of  $S_\beta$  for a given choice of  $\beta$  is given by

$$\sup_{x \in \mathbb{R}} |P(S_\beta \leq x) - \Phi(x)| = \sqrt{h} \sup_{x \in \mathbb{R}} |p_\beta(x) \phi(x)| + o(h).$$

Thus,  $\sup_{x \in \mathbb{R}} |p_\beta(x) \phi(x)|$  is the contribution of order  $O(\sqrt{h})$  to the normal error for the transformation indexed by  $\beta$ .

The following table shows the magnitude of  $\bar{\kappa}_{1,\beta}$  and  $\bar{\kappa}_{3,\beta}$  for several values of  $\beta$  when  $v$  is constant.

**Table 1. Coefficients of the correction term  $p_\beta(x)$  when  $v$  is constant**

$\beta$	1	0	-1	-1/3	1/3
$\bar{\kappa}_{1,\beta}$	0	$-\frac{1}{2}\sqrt{2}$	$-\sqrt{2}$	$-\frac{2}{3}\sqrt{2}$	$-\frac{1}{3}\sqrt{2}$
$\bar{\kappa}_{3,\beta}$	$2\sqrt{2}$	$-\sqrt{2}$	$-4\sqrt{2}$	$-2\sqrt{2}$	0

The results in Table 1 show that for the case of constant volatility, the raw statistic is dominated by the log transformation from the viewpoint of skewness but not from the viewpoint of bias (as already discussed above). The log transformation clearly dominates the choices of  $\beta = -1/3$  and  $\beta = -1$ . Choosing  $\beta = 1/3$  (the optimal value according to skewness when  $v$  is constant) induces a bias smaller than the log transformation, so in this case  $\beta = 0$  is dominated by the optimal choice  $\beta^*$ . The comparison between  $\beta = 1$  and  $\beta = \beta^*$  does not provide a clear ranking because  $\beta = 1$  eliminates bias but not skewness whereas  $\beta = \beta^*$  eliminates skewness but not bias.

For the GARCH(1,1) and the two-factor diffusion models, Tables 3 and 4 compare the finite sample value of the first and third order cumulants (in the tables, these are under the name of ‘Mean’ and ‘Skewness’, respectively) of the Box-Cox transformed statistics with the predictions of the asymptotic expansions for the first and third order cumulants given in 3.1 (reported as ‘EE Mean’ and ‘EE Skewness’, respectively). In particular, ‘EE mean’ is the average value of  $\sqrt{h}\bar{\kappa}_{1,\beta}$  across the 100,000 Monte Carlo replications. ‘EE Skewness’ is the average value of  $\sqrt{h}\bar{\kappa}_{3,\beta}$ . For the GARCH(1,1) diffusion, this comparison reveals generally close agreement between the two sets of results, which suggests that our expansions are good approximations to the true finite sample cumulants for the GARCH(1,1) model. The only exception is for  $\beta = -1$ , where the third order cumulant in finite samples is quite different from the Edgeworth expansion prediction, most especially when  $h^{-1} = 12$ . The same remark applies for the two-factor diffusion model. For this model the finite sample quality of the asymptotic expansions for the third order cumulants is inferior to the quality of these expansions for the GARCH(1,1) diffusion, at least for the smaller sample sizes. For instance, for  $h^{-1} = 12$ , the average sample value of  $\sqrt{h}\bar{\kappa}_{3,\beta}$  when  $\beta = \beta^*$  is zero (as predicted by our theory), but the finite sample value of the third order cumulant of  $S_{\beta^*}$  is  $-0.118$ . It reduces to  $-0.032$  when  $h^{-1} = 48$  and to  $0.004$  when  $h^{-1} = 288$ . For the larger sample sizes, the sample values of the first and third order cumulants agree with those predicted by the asymptotic expansions for the two-factor diffusion model.

The following result compares the accuracy of the normal approximation for several choices of  $\beta$  for the general case where volatility is not necessarily constant.

**Proposition 3.3** *Conditionally on  $v$ , as  $h \rightarrow 0$ , we have that for any  $\beta_1$  and  $\beta_2$  such that  $\beta_2 < \beta_1 < \beta^* \leq 1/3$  and for any  $x \neq 0$ ,*

$$|p_{\beta^*}(x)| < |p_{\beta_1}(x)| < |p_{\beta_2}(x)|,$$

and  $p_{\beta^*}(0) = p_{\beta_1}(0) = p_{\beta_2}(0)$ .

Proposition 3.3 shows that the log transform ( $\beta = 0$ ) is dominated by the optimal Box-Cox transform when  $v$  is constant ( $\beta^* = \frac{1}{3}$ ). When  $v$  is stochastic, the ranking is unclear as it depends on whether  $\beta^* < 0$  and  $\beta^* > 0$ . If  $\beta^*$  is always nonnegative, Proposition 3.3 applies and we can conclude that  $\beta^*$  is an improvement over the log transform. Our simulations show that for this is the case for the GARCH(1,1) diffusion, where  $\beta^* > 0$  across the 10,000 Monte Carlo simulations. Thus, for this

model we may conclude that the Box-Cox transform dominates the log transform in that the error of the normal approximation is uniformly (in  $x$ ) smaller for the Box-Cox transform when  $\beta$  is chosen according to  $\beta^*$ .

For the two-factor diffusion although the average  $\beta^*$  is positive, it can take on negative values. In this case, we can show that the ranking between the log transform and the Box-Cox transform using  $\beta^*$  is not uniform in  $x$ , i.e. none of the transformations dominates uniformly the other.

To evaluate the finite sample quality of the normal approximation for the different transformations, we present in Figures 3 and 4 the QQ plots for the infeasible Box-Cox transformed statistics for the GARCH(1,1) diffusion. Figure 3 contains the QQ plots for  $\beta = 1$ ,  $\beta = 0$  and  $\beta = -1$ . Figure 4 gives results for  $\beta = 1/3$ ,  $\beta = \beta^*$  and  $\beta = \hat{\beta}^*$ . The corresponding results for the two-factor diffusion model appear in Figures 5 and 6, respectively.

We can summarize the findings from Figures 3 and 4 as follows. The normal distribution is not a good approximation for the raw statistic ( $\beta = 1$ ) when the sample size is small. The log transformation ( $\beta = 0$ ) dominates the raw statistic, but some distortions remain for  $h^{-1} = 12$  and 48. The optimal Box-Cox transformation based on  $\beta^*$  induces a closer to normal QQ plot than the log transformation. The QQ plots of  $\beta = 1/3$  and  $\beta = \beta^*$  are almost identical. This is not surprising because  $\beta^*$  is very concentrated around  $\beta = 1/3$ , as discussed previously. The QQ plot for  $\beta = \hat{\beta}^*$  shows some distortions when  $h^{-1} = 12$ . Thus, estimation of  $\beta^*$  induces finite sample distortions.  $\beta = -1$  is clearly a bad choice from the viewpoint of closeness to normality.

For the two-factor diffusion, the QQ plots in Figures 5 and 6 suggest that the log improves upon the raw version of realized volatility. Nevertheless, the comparison between the QQ plot for the log with that for  $\beta^*$  shows that when  $h^{-1} = 12$ ,  $\beta^*$  does not uniformly dominate the log transform (there are some (small) distortions in the left hand side of the QQ plot). The differences disappear for  $h^{-1} = 48$  and 288. As for the GARCH(1,1) diffusion, estimation of  $\beta^*$  introduces some finite sample distortions and  $\beta = -1$  clearly underperforms relatively to other choices of  $\beta$ .

Previous simulation results (for instance, those discussed here, but also those of BN-S (2005) and Gonçalves and Meddahi (2006)) suggest that the log transformation improves the accuracy of first-order asymptotic theory based intervals. Nevertheless, any of the Box-Cox transformations for realized volatility can be used to build a confidence interval for  $IV$  (see Appendix A for more details on this). So, to end this section we investigate by simulation the coverage probabilities of 95% level confidence intervals for  $IV$  across several values of  $\beta$ . Results for the GARCH(1,1) diffusion are presented in Figure 11 (one-sided intervals) and Figure 12 (two-sided symmetric intervals) whereas Figures 13 (one-sided) and 14 (two-sided) contain the corresponding results for the two-factor diffusion. These figures contain two panels: the top panel refers to the infeasible statistics (which we discuss next) and the bottom panel refers to the feasible statistics (which we discuss in the next section). For each sample size, we computed the actual coverage probabilities of confidence intervals for  $IV$  based on Box-Cox transformed realized volatility for each value of  $\beta$  in a grid lying in the range  $[-2, 2]$ . Starting with the

GARCH(1,1) diffusion, Figures 11 and 12 suggest that the log transformation clearly improves upon the raw statistic for one-sided intervals but not for two-sided intervals (confirming previous results). They also suggest that other values of  $\beta$  different from zero may improve upon the log transformation. For instance, for one-sided intervals and for all sample sizes,  $\beta = 1/3$  (or slightly above) clearly improves coverage with respect to choosing  $\beta = 0$ . To complement these results, note that Table 3 shows that for  $h^{-1} = 12$  the coverage rates are 98.40 for  $\beta = 1$ , 91.00 for  $\beta = 0$ , 93.51 for  $\beta = 1/3$ , 93.50 for  $\beta = \beta^*$  and 94.67 for  $\beta = \hat{\beta}^*$ . For the two-sided symmetric intervals, coverage rates are equal to 95.66 for  $\beta = 1$ , 93.60 for  $\beta = 0$ , 94.82 for  $\beta = 1/3$ , 94.81 for  $\beta = \beta^*$  and 95.25 for  $\beta = \hat{\beta}^*$ , when  $h^{-1} = 12$ . Figure 12 shows that for this sample size  $\beta = 0$  is dominated by  $\beta$  in the range 0 to 1 (or slightly above).

For the two-factor diffusion, choosing  $\beta$  between 0 and  $1/3$  induces coverage rates closer to 95% for one sided intervals (see top panel of Figure 13). For two-sided intervals,  $\beta = 0$  seems to perform very well, with values of  $\beta$  larger than 0 but smaller than  $4/3$  producing too conservative intervals.  $\beta = 4/3$  does as well as  $\beta = 0$ . Table 4 shows that in general  $\beta = 0$  is dominated by  $\beta^*$  for one-sided intervals but not for two-sided intervals. Intervals based on  $\hat{\beta}^*$  tend to overcover for  $h^{-1} = 12$ .

### 3.2 Feasible statistic

In this section we study the Box-Cox transformed feasible statistics  $T_\beta$ . Our goal is to compare the higher order properties of  $T_\beta$  for different choices of  $\beta$  in order to study the accuracy of the normal approximation across values of  $\beta$ . We first analyze the properties of the first and third order cumulants of  $T_\beta$ . These are the main ingredients in the second-order Edgeworth expansions of the distribution of  $T_\beta$ .

We let  $\kappa_i(T_\beta)$  for  $i = 1, 2, 3$ , denote the first three cumulants of  $T_\beta$ . The following result (from Gonçalves and Meddahi (2006)) provides asymptotic expansions for  $\kappa_i(T_\beta)$ ,  $i = 1, 2, 3$ .

**Proposition 3.4** *Conditional on  $v$ , as  $h \rightarrow 0$ ,*

a)

$$\kappa_1(T_\beta) = \sqrt{h} \left[ -\frac{1}{2} \left( \frac{4}{\sqrt{2}} \frac{\overline{\sigma^6}}{(\overline{\sigma^4})^{3/2}} + \frac{g''(IV; \beta)}{g'(IV; \beta)} \sqrt{V} \right) \right] + O(h).$$

b)

$$\kappa_2(T_\beta) = 1 + O(h).$$

c)

$$\kappa_3(T_\beta) = \sqrt{h} \left( -\frac{8}{\sqrt{2}} \frac{\overline{\sigma^6}}{(\overline{\sigma^4})^{3/2}} - 3 \frac{g''(IV; \beta)}{g'(IV; \beta)} \sqrt{V} \right) + O(h).$$

Proposition 3.4 shows that the choice of  $\beta$  influences the higher order properties (through  $O(\sqrt{h})$ ) of the first and third order cumulants of  $T_\beta$ . Note that unlike in the infeasible case, the raw statistic ( $\beta = 1$ ) has a negative bias equal to  $-\frac{4}{2\sqrt{2}} \frac{\overline{\sigma^6}}{(\overline{\sigma^4})^{3/2}}$ . Thus, applying a nonlinear transformation to the feasible statistic has the potential to reduce bias *and* skewness. The following result states the optimal choices of  $\beta$  for approximately eliminating the bias and the skewness of  $T_\beta$ .

**Proposition 3.5** *Conditionally on  $v$ , as  $h \rightarrow 0$ , up to order  $O(\sqrt{h})$ ,*

a) *Choosing  $\beta$  equal to*

$$\beta_* = 1 - 2 \frac{\overline{\sigma^2} \overline{\sigma^6}}{(\overline{\sigma^4})^2} \leq -1$$

*implies  $\kappa_1(T_\beta) = 0$ .*

b) *Choosing  $\beta$  equal to*

$$\beta^{**} = 1 - \frac{4}{3} \frac{\overline{\sigma^2} \overline{\sigma^6}}{(\overline{\sigma^4})^2} \leq -\frac{1}{3}$$

*implies  $\kappa_3(T_\beta) = 0$ .*

The optimal choices of  $\beta$  are random and depend on the volatility path. In general, different values of  $\beta$  are required to eliminate bias and skewness. Under constant volatility, we get that  $\beta_* = -1$  and  $\beta^{**} = -\frac{1}{3}$ .

For the GARCH(1,1) diffusion, Table 7 shows that the average value of  $\beta^{**}$  is equal to  $-0.338$  and thus very close to  $-1/3$ , the optimal value of  $\beta^{**}$  when volatility is constant. The distribution of  $\beta^{**}$  is also very concentrated around the mean value, with a range going from  $-0.381$  up to  $-0.334$ . Similarly to the infeasible statistics, these results are not very surprising given that the GARCH(1,1) diffusion is highly persistent. For the two-factor diffusion, Table 8 shows that on average the value of  $\beta^{**}$  equals  $-0.886$ , with a distribution that is very dispersed (the minimum value is  $-3.061$  and the maximum value is  $-0.407$ ). Thus, in this case the value of  $\beta^{**}$  is quite different from the constant volatility value of  $-1/3$ . This is also not very surprising because the two-factor diffusion model is known to generate sample paths for prices and volatility that are very rugged, often close to those generated by a jump diffusion model. The comparison between the optimal  $\beta^{**}$  for the feasible statistics with the optimal  $\beta^*$  for the infeasible statistics shows clearly that the choice of  $\beta$  that minimizes skewness depends on whether we know the true asymptotic variance or we need to estimate it.

As discussed previously for  $\beta^*$ ,  $\beta^{**}$  is unknown in practice because it depends on  $\overline{\sigma^q}$  for  $q = 2, 4, 6$ . We can nevertheless estimate  $\beta^{**}$  consistently with the following estimator:

$$\hat{\beta}^{**} = 1 - \frac{4}{3} \frac{RV \ (R_6/\mu_6)}{(R_4/\mu_4)^2} = 1 - \frac{4}{3} \frac{RV \ (R_6/15)}{(R_4/3)^2}.$$

The left-hand-side panels of Figures 1 and 2 give the finite sample distributions of  $\hat{\beta}^{**}$ . Tables 7 and 8 (left-hand-side panels) give the corresponding summary statistics. Overall, these results suggest that estimating  $\beta^{**}$  is problematic in finite samples. The estimates are severely upward biased for the smallest sample sizes. The bias only becomes under control for  $h^{-1} = 1152$ , although for this sample size the variance is also larger than for the smaller sample sizes.

The log transformation does not correspond to the optimal Box-Cox transformation from the viewpoint of bias and skewness. To end this section we investigate to what extent this implies that the normal approximation is less accurate when applied to the log statistic as compared to other values of  $\beta$  (including the optimal choices).

The second order Edgeworth expansion of the distribution of  $T_\beta$  is given by

$$P(T_\beta \leq x) = \Phi(x) + \sqrt{h}q_\beta(x)\phi(x) + o(h),$$

where

$$q_\beta(x) = -\left(\kappa_{1,\beta} + \frac{1}{6}\kappa_{3,\beta}(x^2 - 1)\right),$$

where  $\kappa_{1,\beta}$  and  $\kappa_{3,\beta}$  denote the coefficients of the leading terms (of order  $O(\sqrt{h})$ ) of the first and third order cumulants of  $T_\beta$  given in Proposition 3.4.

The following table shows the magnitude of  $\kappa_{1,\beta}$  and  $\kappa_{3,\beta}$  for several values of  $\beta$  when  $v$  is constant.

**Table 2. Coefficients of the correction term  $q_\beta(x)$  when  $v$  is constant**

$\beta$	1	0	-1	-1/3	1/3
$\kappa_{1,\beta}$	$-\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	0	$-\frac{1}{3}\sqrt{2}$	$-\frac{2}{3}\sqrt{2}$
$\kappa_{3,\beta}$	$-4\sqrt{2}$	$-\sqrt{2}$	$2\sqrt{2}$	0	$-2\sqrt{2}$

When the volatility is constant, Table 2 shows that the coefficients of the raw statistic ( $\beta = 1$ ) are both larger (in absolute value) than those of the other transformations. In particular, the log transformation dominates  $\beta = 1$  both in terms of skewness and bias. Choosing  $\beta = \beta^{**} = -1/3$  eliminates skewness, as expected. Its bias is nonzero but it is smaller than the bias of both the log and the raw versions of realized volatility.  $\beta = -1$  eliminates bias (as expected in the constant volatility case), but induces a larger amount of skewness compared to the log transform or the choice of  $\beta = -1/3$ . This skewness is nevertheless smaller in absolute value than the skewness implicit in  $\beta = 1$ .

Similarly to the infeasible statistics, we can compare the scaled values of  $\kappa_{1,\beta}$  and  $\kappa_{3,\beta}$  (i.e.  $\sqrt{h}\kappa_{i,\beta}$  for  $i = 1, 3$ ) with the finite sample bias and skewness of the Box-Cox statistics given in Tables 5 and 6. The main feature of notice is that the cumulant expansions for the feasible statistics are in general less accurate than those for the infeasible statistics, especially when  $h^{-1} = 12$ . This is not surprising and it only confirms that the quality of the asymptotic theory (even to higher-order) is poorer for the

feasible statistics than for the infeasible statistics. Nevertheless, for the GARCH(1,1) diffusion, the first order cumulant expansion is quite accurate, even for the small sample sizes. The expansion for the third order cumulant is less accurate than for the first order cumulant, especially when  $h^{-1} = 12$ . It becomes reasonably accurate across the different values of  $\beta$  for the remaining values of  $h^{-1}$ . The distortions are larger for the two-factor diffusion model than the GARCH(1,1) diffusion. For instance, when  $h^{-1} = 12$ , the sample third order cumulant of the raw statistic is equal to  $-78.445$  whereas the cumulant expansion gives an average value of  $-3.365$ . For  $h^{-1} = 48$ , the values reduce to  $-1.609$  and  $-1.683$ , respectively. Thus, the quality of the third order cumulant expansion greatly improves with this increase in the sample size when  $\beta = 1$ . For  $\beta = 0$ , not only is the amount of skewness much smaller in small samples but it is also much more in line with our cumulant expansion. Choosing  $\beta = \beta^*$  reduces the amount of skewness relatively to the other transformations (an exception is when  $h^{-1} = 12$ ), but does not reduce it to zero, as predicted by our theory.

The following result compares the accuracy of the normal approximation for the Box-Cox transformation across several values of  $\beta$ . The result is general and does not assume that volatility is constant.

**Proposition 3.6** *Conditionally on  $v$ , as  $h \rightarrow 0$ , we have that for any  $\beta_1$  and  $\beta_2$  such that  $\beta^{**} < \beta_1 < \beta_2$  and for any  $x \neq 0$ ,*

$$|q_{\beta^{**}}(x)| < |q_{\beta_1}(x)| < |q_{\beta_2}(x)|,$$

and  $q_{\beta^{**}}(0) = q_{\beta_1}(0) = q_{\beta_2}(0)$ .

Proposition 3.6 proves that eliminating skewness helps increase the accuracy of the normal approximation. In particular, the error of the normal approximation (up to order  $O(\sqrt{h})$ ) is smaller for  $\beta^{**} < \beta_1$ , including the log transformation ( $\beta = 0$ ) and the raw statistic ( $\beta = 1$ ). Proposition 3.6 also proves that the log transformation is an improvement over the raw statistic since  $|q_{\beta_1}(x)| < |q_{\beta_2}(x)|$  when  $\beta_1 = 0$  and  $\beta_2 = 1$ , and  $\beta_1 > \beta^{**}$ .

The QQ plots for the feasible statistics are presented in Figures 7 and 8 for the GARCH(1,1) diffusion, and in Figures 9 and 10 for the two-factor diffusion. For the GARCH(1,1) model, the QQ plots confirm the ranking suggested by Proposition 3.6. These plots show that the best transformation is given by the optimal Box-Cox transform with  $\beta = \beta^{**}$ , followed by the log transform with  $\beta = 0$ , which is better than  $\beta = 1/3$ , which in turn dominates the raw transform with  $\beta = 1$ . The feasible transformation based on  $\hat{\beta}^{**}$  induces more finite sample distortions than the infeasible transformation, especially for the smaller sample sizes. The QQ plots for  $\hat{\beta}^{**}$  are close to those for the log. Choosing  $\beta = -1$  is better than choosing  $\beta = 1$ , but the QQ plots for  $\beta = -1$  show more distortions when compared to the other Box-Cox transforms, especially when  $h^{-1} = 12$ .

The two-factor diffusion model is associated with larger distortions than the GARCH(1,1) diffusion. This is evident when we compare the scales of the corresponding QQ plots. The log is clearly preferred



to the choosing  $\beta = 1/3$ , which in turn dominates  $\beta = 1$ . The QQ plots for  $\beta = 0$  and  $\beta = \beta^{**}$  are the reverse image of each other, with  $\beta = 0$  dominating  $\beta = \beta^{**}$  on the right-hand side of the distribution and being dominated by  $\beta = \beta^{**}$  on the left-hand side of the distribution. Although Proposition 3.6 does not cover the case where  $\beta = -1$ , the QQ plots suggest that this transformation is dominates the log transformation on the left-hand side but does worse at the right-hand side of the distribution. Estimation of  $\beta^{**}$  creates larger deviations from the normal approximation than in the GARCH(1,1) case.

To end this section, we compare the Box-Cox transformations for realized volatility in terms of coverage probabilities for both one-sided and two-sided symmetric intervals for  $IV$ . Our analysis is similar to that followed for the infeasible statistics, but now we concentrate on the bottom panels of Figures 11 through 14, which refer to the feasible statistics. Starting with the one-sided intervals for the GARCH(1,1) diffusion, Figure 11 shows that choosing  $\beta = 0$  improves upon  $\beta = 1$  for all sample sizes, but  $\beta = 0$  does not correspond to the best possible choice in terms of coverage accuracy. Indeed, any value of  $\beta$  between  $-1$  and  $0$  dominates  $\beta = 0$ , with the optimal choice lying somewhere close to  $-1$ . For one-sided intervals, Table 5 shows that for  $\beta = -1$  the coverage rate is equal to 96.07, 95.74, 95.38, 95.19 for  $h^{-1} = 12, 48, 288$  and  $1152$ , respectively. In contrast, the log transformation has coverage rates equal to 88.54, 92.36, 94.17 and 94.55. These rates are systematically worse than those for  $\beta = -1$  and they are also worse than those for  $\beta^{**}$  (which are equal to 90.94, 93.51, 94.59 and 94.77). Estimating  $\beta^{**}$  causes the coverage rates to be slightly below those of  $\beta^{**}$ , but  $\hat{\beta}^{**}$  is still preferred to  $\beta = 0$ . The results for the two-sided symmetric intervals are qualitatively similar to those for the one-sided intervals, with the main difference being that two-sided intervals are associated with smaller coverage distortions for almost all transformations and for all sample sizes. The exception is when  $\beta = -1$ , which is associated with a slight overcoverage for one-sided intervals and with undercoverage for two-sided intervals. Nevertheless, Figure 12 (bottom panel) shows that  $\beta = -1$  is the choice of  $\beta$  that produces more accurate two-sided intervals for  $IV$ . The difference in coverage probability with respect to other choices of  $\beta$  (including  $\beta = 0$ ) is especially important for  $h^{-1} = 12$ . The results for the two-factor diffusion model are qualitatively similar to those for the GARCH(1,1) diffusion. The main differences are that the two-factor diffusion has larger distortions for all methods and all sample sizes (the lines in Figures 13 and 14 shift to the left in comparison to Figures 11 and 12). Choosing the optimal value of  $\beta^{**}$  is an improvement over choosing  $\beta = 0$ , which dominates  $\beta = 1/3$ , which dominates  $\beta = 1$ . Choosing  $\beta = \hat{\beta}^{**}$  outperforms  $\beta = 0$ , but underperforms  $\beta = \beta^{**}$ . Figures 13 and 14 suggest that the optimal choice of  $\beta$  in terms of coverage probability control for one-sided and two-sided intervals is slightly above  $-1$  for the two-factor diffusion model. Overall, taking into account the two models, a choice of  $\beta = -1$  appears to do best if we want to produce intervals for  $IV$  based on the Box-Cox transform with good coverage rates. In particular, this choice dominates the optimal choice  $\beta^{**}$  which is based on eliminating the finite sample skewness. A careful look at Tables 5 and 6 suggests that although  $\beta = -1$  is in general associated with a larger skewness (which explains why

the QQ plots for  $\beta = 1$  look worse than those for the other transformations), its bias is smaller than that for  $\beta = \beta^{**}$ . Controlling bias appears as important (if not more) as controlling skewness if good coverage accuracy is the goal.

## 4 Conclusion

The log transformation is often preferred to the raw version of realized volatility because of its superior finite sample properties (including finite sample skewness). Although the finite sample improvements of the log transformation are well known in the literature, no theoretical explanation appears to be available so far. In this paper, we provide a possible explanation based on a comparison of Edgeworth expansions of the distribution of the log and the raw realized volatility statistics.

The fact that the log transformation improves upon the raw statistic does not imply that the log transformation is the best possible transformation available. The second contribution of this paper is to consider a broader class of analytical transformations (indexed by the parameter  $\beta$ ) that includes the log ( $\beta = 0$ ) and the raw ( $\beta = 1$ ) versions of realized volatility as special cases. This is the Box-Cox transformation. Based on higher order expansions of the cumulants of the Box-Cox transformed statistics, we first derive the optimal values of  $\beta$  that induce zero bias and zero skewness. The values of  $\beta$  that eliminate bias are generally different from those that eliminate skewness. These values are also different depending on whether we consider infeasible statistics (which are constructed under the assumption that their asymptotic variances are known) or on whether we consider feasible statistics (which are the statistics used in practice). Therefore in this paper we treat the two cases separately. For each case, we rely on Edgeworth expansions of the distribution of the Box-Cox transforms to study the accuracy of these transforms across different values of  $\beta$ . We also provide extensive Monte Carlo simulation to investigate how the different Box-Cox transforms behave in finite samples. Our general conclusion is that there exist other Box-Cox transformations that dominate the log transformation. In particular, the optimal but infeasible transformations derived in this paper generally outperform the log. Their feasible versions are less well behaved than their infeasible versions in finite samples and are comparable to the log transform. However, if coverage probability accuracy is the goal, choosing  $\beta = -1$  appears as the best choice across the two models studied in this paper.

**Table 3. Infeasible RV-based statistics, GARCH(1,1) diffusion**

$h^{-1}$		$\beta = 1$ (Raw)	$\beta = 0$ (Log)	$\beta = -1$	$\beta = 1/3$	$\beta^*$	$\hat{\beta}^*$
12	Mean	0.000	-0.210	-0.490	-0.136	-0.136	-0.105
	EE Mean	0.000	-0.204	-0.409	-0.136	-0.137	-0.106
	Skewness	0.824	-0.489	-8.170	0.009	0.006	0.186
	EE Skewness	0.821	-0.406	-1.663	0.003	0.000	0.185
	St. Error	0.999	1.042	1.471	0.998	0.998	0.989
	Ex. Kurtosis	1.064	0.387	14.419	-0.016	-0.015	0.030
	Cov. one-sided 95%	98.40	91.00	84.63	93.51	93.50	94.67
	Cov. two-sided 95%	95.66	93.60	87.91	94.82	94.81	95.25
	Mean	0.002	-0.101	-0.211	-0.066	-0.066	-0.059
	EE Mean	0.000	-0.102	-0.204	-0.068	-0.068	-0.061
48	Skewness	0.392	-0.224	-1.164	-0.010	-0.012	0.034
	EE Skewness	0.410	-0.203	-0.817	0.001	0.000	0.046
	St. Error	0.999	1.010	1.091	1.000	1.000	0.998
	Ex. Kurtosis	0.210	0.074	1.503	-0.026	-0.026	-0.023
	Cov. one-sided 95%	96.48	93.09	90.03	94.28	94.27	94.52
	Cov. two-sided 95%	95.27	94.69	92.90	95.06	95.06	95.10
	Mean	-0.001	-0.043	-0.085	-0.029	-0.029	-0.028
	EE Mean	0.000	-0.042	-0.083	-0.028	-0.028	-0.027
	Skewness	0.160	-0.091	-0.361	-0.006	-0.007	-0.002
	EE Skewness	0.168	-0.083	-0.333	0.001	0.000	0.005
288	Standard Error	0.999	1.001	1.014	0.999	0.999	0.999
	Excess Kurtosis	0.039	0.023	0.236	0.004	0.004	0.003
	Cov. one-sided 95%	95.55	94.30	93.06	94.73	94.72	94.75
	Cov. two-sided 95%	95.02	94.88	94.65	94.98	94.98	94.98
	Mean	0.003	-0.018	-0.039	-0.011	-0.011	-0.011
	EE Mean	0.000	-0.021	-0.042	-0.014	-0.014	-0.014
	Skewness	0.077	-0.048	-0.177	-0.006	-0.007	-0.006
	EE Skewness	0.084	-0.041	-0.167	0.000	0.000	0.001
	Standard Error	1.002	1.002	1.005	1.002	1.002	1.002
	Excess Kurtosis	-0.001	-0.003	0.051	-0.008	-0.008	-0.008
1152	Cov. one-sided 95%	95.22	94.62	93.99	94.81	94.81	94.81
	Cov. two-sided 95%	94.99	94.98	94.93	94.99	94.99	94.99

Notes: 100,000 Monte Carlo replications. EE Mean and EE skewness denote the Egdeworth corrections for the mean and skewness. The infeasible statistic is  $\sqrt{h^{-1}} (RV^\beta - IV^\beta) / \sqrt{V_\beta}$ , where  $V_\beta = \beta^2 IV^{2(\beta-1)} 2IQ$ .

**Table 4. Infeasible RV-based statistics, two-factor diffusion**

$h^{-1}$		$\beta = 1$ (Raw)	$\beta = 0$ (Log)	$\beta = -1$	$\beta = 1/3$	$\beta^*$	$\hat{\beta}^*$
12	Mean	0.000	-0.244	-0.603	-0.158	-0.230	-0.121
	EE Mean	0.000	-0.294	-0.588	-0.196	-0.280	-0.152
	Skewness	1.183	-0.206	-11.646	0.205	-0.118	0.384
	EE Skewness	1.683	-0.082	-1.846	0.507	0.000	0.771
	Standard Error	0.943	0.938	1.524	0.896	0.929	0.896
	Excess Kurtosis	3.381	0.168	26.234	0.108	0.099	0.474
	Cov. one-sided 95%	99.89	92.82	83.52	96.27	93.39	97.53
	Cov. two-sided 95%	95.88	95.60	87.03	97.16	95.88	97.25
	Mean	0.002	-0.135	-0.288	-0.089	-0.129	-0.083
	EE Mean	0.000	-0.147	-0.294	-0.098	-0.140	-0.092
48	Skewness	0.761	-0.073	-1.335	0.183	-0.032	0.219
	EE Skewness	0.841	-0.041	-0.923	0.253	0.000	0.289
	Standard Error	0.985	0.975	1.095	0.965	0.973	0.965
	Excess Kurtosis	1.134	0.026	1.980	0.081	0.024	0.123
	Cov. one-sided 95%	98.21	93.73	89.16	95.31	93.95	95.50
	Cov. two-sided 95%	95.67	95.25	92.38	95.78	95.38	95.81
	Mean	0.003	-0.056	-0.116	-0.036	-0.053	-0.043
	EE Mean	0.000	-0.060	-0.120	-0.040	-0.057	-0.047
	Skewness	0.349	-0.013	-0.393	0.105	0.004	0.064
	EE Skewness	0.343	-0.017	-0.377	0.103	0.000	0.061
288	Standard Error	0.998	0.994	1.013	0.993	0.994	0.991
	Excess Kurtosis	0.257	0.018	0.285	0.041	0.016	0.011
	Cov. one-sided 95%	96.30	94.50	92.69	95.12	94.58	94.97
	Cov. two-sided 95%	95.18	95.17	94.61	95.22	95.17	95.26
	Mean	0.008	-0.022	-0.052	-0.012	-0.021	-0.018
	EE Mean	0.000	-0.030	-0.060	-0.020	-0.029	-0.026
	Skewness	0.152	-0.029	-0.214	0.031	-0.020	-0.006
	EE Skewness	0.172	-0.008	-0.188	0.052	0.000	0.013
	Standard Error	0.999	0.999	1.004	0.998	0.999	0.997
	Excess Kurtosis	0.057	0.019	0.107	0.018	0.018	0.008
1152	Cov. one-sided 95%	95.64	94.76	93.92	95.05	94.79	94.89
	Cov. two-sided 95%	95.01	95.00	94.86	95.02	94.99	95.04

Notes: 100,000 Monte Carlo replications. EE Mean and EE skewness denote the Egdeworth corrections for the mean and skewness. The infeasible statistic is  $\sqrt{h^{-1}} (RV^\beta - IV^\beta) / \sqrt{V_\beta}$ , where  $V_\beta = \beta^2 IV^{2(\beta-1)} 2IQ$ .

**Table 5. Feasible RV-based statistics, GARCH(1,1) diffusion**

$h^{-1}$		$\beta = 1$ (Raw)	$\beta = 0$ (Log)	$\beta = -1$	$\beta = 1/3$	$\beta^{**}$	$\hat{\beta}^{**}$
12	Mean	-0.548	-0.235	0.000	-0.325	-0.151	-0.227
	EE Mean	-0.410	-0.206	-0.001	-0.274	-0.137	-0.198
	Skewness	-11.944	-0.755	1.198	-2.028	-0.004	-0.730
	EE Skewness	-1.641	-0.414	0.812	-0.823	0.000	-0.370
	Standard Error	1.658	1.175	1.125	1.269	1.124	1.172
	Excess Kurtosis	14.831	0.621	1.293	2.105	0.180	0.663
	Cov. one-sided 95%	82.55	88.54	96.07	86.32	90.94	88.83
	Cov. two-sided 95%	85.87	90.33	93.04	89.07	91.50	90.46
	Mean	-0.219	-0.105	0.001	-0.142	-0.069	-0.086
	EE Mean	-0.205	-0.103	-0.001	-0.137	-0.068	-0.084
48	Skewness	-1.335	-0.259	0.450	-0.540	-0.011	-0.144
	EE Skewness	-0.821	-0.207	0.406	-0.412	0.000	-0.093
	Standard Error	1.134	1.051	1.039	1.070	1.039	1.045
	Excess Kurtosis	1.638	0.178	0.335	0.448	0.077	0.131
	Cov. one-sided 95%	89.38	92.36	95.74	91.32	93.51	92.97
	Cov. two-sided 95%	92.05	93.64	94.29	93.24	93.97	93.79
	Mean	-0.086	-0.043	-0.001	-0.057	-0.029	-0.031
	EE Mean	-0.084	-0.042	0.000	-0.056	-0.028	-0.030
	Skewness	-0.374	-0.096	0.162	-0.185	-0.008	-0.022
	EE Skewness	-0.335	-0.085	0.166	-0.168	0.000	-0.010
288	Standard Error	1.022	1.009	1.007	1.012	1.007	1.007
	Excess Kurtosis	0.260	0.043	0.056	0.088	0.022	0.027
	Cov. one-sided 95%	92.93	94.17	95.38	93.76	94.59	94.53
	Cov. two-sided 95%	94.44	94.69	94.83	94.61	94.77	94.77
	Mean	-0.040	-0.019	0.002	-0.026	-0.012	-0.012
	EE Mean	-0.042	-0.021	0.000	-0.028	-0.014	-0.014
	Skewness	-0.179	-0.050	0.077	-0.092	-0.007	-0.009
	EE Skewness	-0.168	-0.042	0.083	-0.084	0.000	-0.001
	Standard Error	1.007	1.004	1.004	1.005	1.004	1.004
	Excess Kurtosis	0.059	0.004	0.006	0.016	-0.002	-0.001
1152	Cov. one-sided 95%	93.96	94.55	95.19	94.35	94.77	94.76
	Cov. two-sided 95%	94.86	94.91	94.96	94.90	94.92	94.93

Notes: 100,000 Monte Carlo replications. EE Mean and EE skewness denote the Egdeworth corrections for the mean and skewness. The feasible statistic is  $\sqrt{h^{-1}} (RV^\beta - IV^\beta) / \sqrt{\hat{V}_\beta}$ , where  $\hat{V}_\beta = \beta^2 RV^{2(\beta-1)} \frac{2}{3} R_4$ .

**Table 6. Feasible RV-based statistics, two-factor diffusion**

$h^{-1}$		$\beta = 1$ (Raw)	$\beta = 0$ (Log)	$\beta = -1$	$\beta = 1/3$	$\beta^{**}$	$\hat{\beta}^{**}$
12	Mean	-0.972	-0.428	-0.085	-0.572	-0.115	-0.417
	EE Mean	-0.841	-0.547	-0.253	-0.645	-0.280	-0.280
	Skewness	-78.445	-1.709	1.864	-5.231	2.001	-1.687
	EE Skewness	-3.365	-1.601	0.163	-2.189	0.000	-1.541
	Standard Error	2.475	1.348	1.236	1.540	1.247	1.339
	Excess Kurtosis	73.102	1.020	1.849	4.373	3.785	1.227
	Cov. one-sided 95%	75.74	82.86	93.31	80.17	92.12	83.27
	Cov. two-sided 95%	79.45	85.74	90.95	83.89	90.45	86.12
	Mean	-0.421	-0.235	-0.074	-0.293	-0.088	-0.196
	EE Mean	-0.421	-0.274	-0.127	-0.323	-0.140	-0.140
48	Skewness	-1.609	-0.504	0.385	-0.825	0.341	-0.329
	EE Skewness	-1.683	-0.801	0.082	-1.095	0.000	-0.578
	Standard Error	1.373	1.158	1.111	1.207	1.112	1.138
	Excess Kurtosis	5.249	0.586	0.444	1.382	0.595	0.395
	Cov. one-sided 95%	84.56	88.62	93.32	87.14	92.93	89.77
	Cov. two-sided 95%	88.01	90.85	92.41	90.07	92.34	91.31
	Mean	-0.166	-0.102	-0.040	-0.123	-0.046	-0.068
	EE Mean	-0.172	-0.112	-0.052	-0.132	-0.057	-0.057
	Skewness	-0.658	-0.307	0.109	-0.461	0.076	-0.095
	EE Skewness	-0.687	-0.327	0.033	-0.447	0.000	-0.121
288	Standard Error	1.078	1.043	1.032	1.052	1.032	1.039
	Excess Kurtosis	0.910	0.226	0.125	0.381	0.111	0.145
	Cov. one-sided 95%	90.84	92.57	94.30	92.01	94.13	93.46
	Cov. two-sided 95%	93.22	93.89	94.22	93.70	94.21	93.99
	Mean	-0.078	-0.047	-0.017	-0.057	-0.019	-0.024
	EE Mean	-0.086	-0.056	-0.026	-0.066	-0.029	-0.029
	Skewness	-0.391	-0.184	0.007	-0.251	-0.010	-0.049
	EE Skewness	-0.343	-0.163	0.017	-0.223	0.000	-0.027
	Standard Error	1.024	1.014	1.011	1.017	1.010	1.014
	Excess Kurtosis	0.311	0.112	0.050	0.163	0.046	0.072
1152	Cov. one-sided 95%	93.02	93.87	94.77	93.58	94.70	94.51
	Cov. two-sided 95%	94.49	94.65	94.70	94.57	94.70	94.61

Notes: 100,000 Monte Carlo replications. EE Mean and EE skewness denote the Egdeworth corrections for the mean and skewness. The feasible statistic is  $\sqrt{h^{-1}} (RV^\beta - IV^\beta) / \sqrt{\hat{V}_\beta}$ , where  $\hat{V}_\beta = \beta^2 RV^{2(\beta-1)} \frac{2}{3} R_4$ .

**Table 7. Summary statistics for the infeasible and feasible versions of the optimal power**

*GARCH(1,1) diffusion*

$h^{-1}$	Statistics	Infeasible statistic		Feasible statistic	
		$\beta^*$	$\hat{\beta}^*$	$\beta^{**}$	$\hat{\beta}^{**}$
12	Mean	0.331	0.482	-0.338	-0.036
	St. Error	0.002	0.045	0.004	0.089
	Max	0.332	0.591	-0.334	0.182
	Min	0.310	0.323	-0.381	-0.354
48	Mean	0.331	0.407	-0.338	-0.187
	St. Error	0.002	0.069	0.004	0.138
	Max	0.333	0.542	-0.334	0.083
	Min	0.310	0.061	-0.381	-0.877
288	Mean	0.331	0.351	-0.338	-0.298
	St. Error	0.002	0.067	0.004	0.133
	Max	0.333	0.477	-0.334	-0.046
	Min	0.310	-0.409	-0.381	-1.817
1152	Mean	0.331	0.336	-0.338	-0.327
	St. Error	0.002	0.045	0.004	0.089
	Max	0.333	0.439	-0.334	-0.122
	Min	0.310	-0.344	-0.381	-1.688

Notes: 100,000 Monte Carlo replications.

**Table 8. Summary statistics for the infeasible and feasible versions of the optimal power**

*Two-factor diffusion*

$h^{-1}$	Statistics	Infeasible statistic		Feasible statistic	
		$\beta^*$	$\hat{\beta}^*$	$\beta^{**}$	$\hat{\beta}^{**}$
12	Mean	0.057	0.482	-0.886	-0.036
	St. Error	0.120	0.043	0.240	0.086
	Max	0.297	0.596	-0.407	0.192
	Min	-1.031	0.325	-3.061	-0.350
48	Mean	0.057	0.373	-0.886	-0.254
	St. Error	0.120	0.076	0.240	0.152
	Max	0.297	0.561	-0.407	0.122
	Min	-1.031	0.038	-3.061	-0.925
288	Mean	0.057	0.216	-0.886	-0.567
	St. Error	0.120	0.136	0.240	0.271
	Max	0.297	0.464	-0.407	-0.072
	Min	-1.031	-0.615	-3.061	-2.230
1152	Mean	0.057	0.126	-0.886	-0.748
	St. Error	0.120	0.164	0.240	0.327
	Max	0.297	0.412	-0.407	-0.177
	Min	-1.031	-1.458	-3.061	-3.916

Notes: 100,000 Monte Carlo replications.

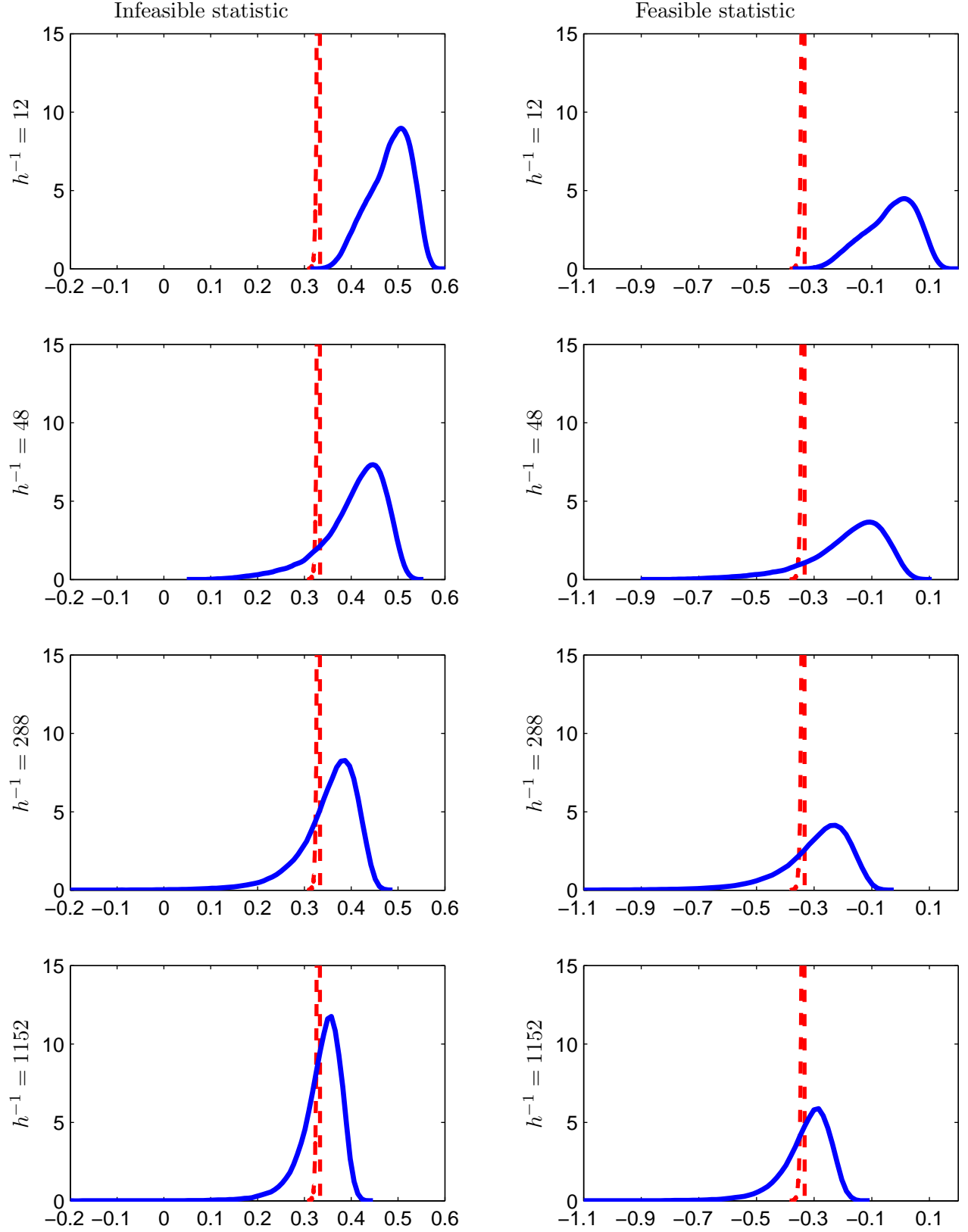


Figure 1: The left-hand-side panel shows the kernel densities of  $\beta^*$  and  $\hat{\beta}^*$  for the GARCH(1,1) diffusion. The solid line refers to  $\hat{\beta}^*$  and the dashed line refers to  $\hat{\beta}^*$ . The right-hand-side panel refers to the kernel densities of  $\beta^{**}$  and  $\hat{\beta}^{**}$ .



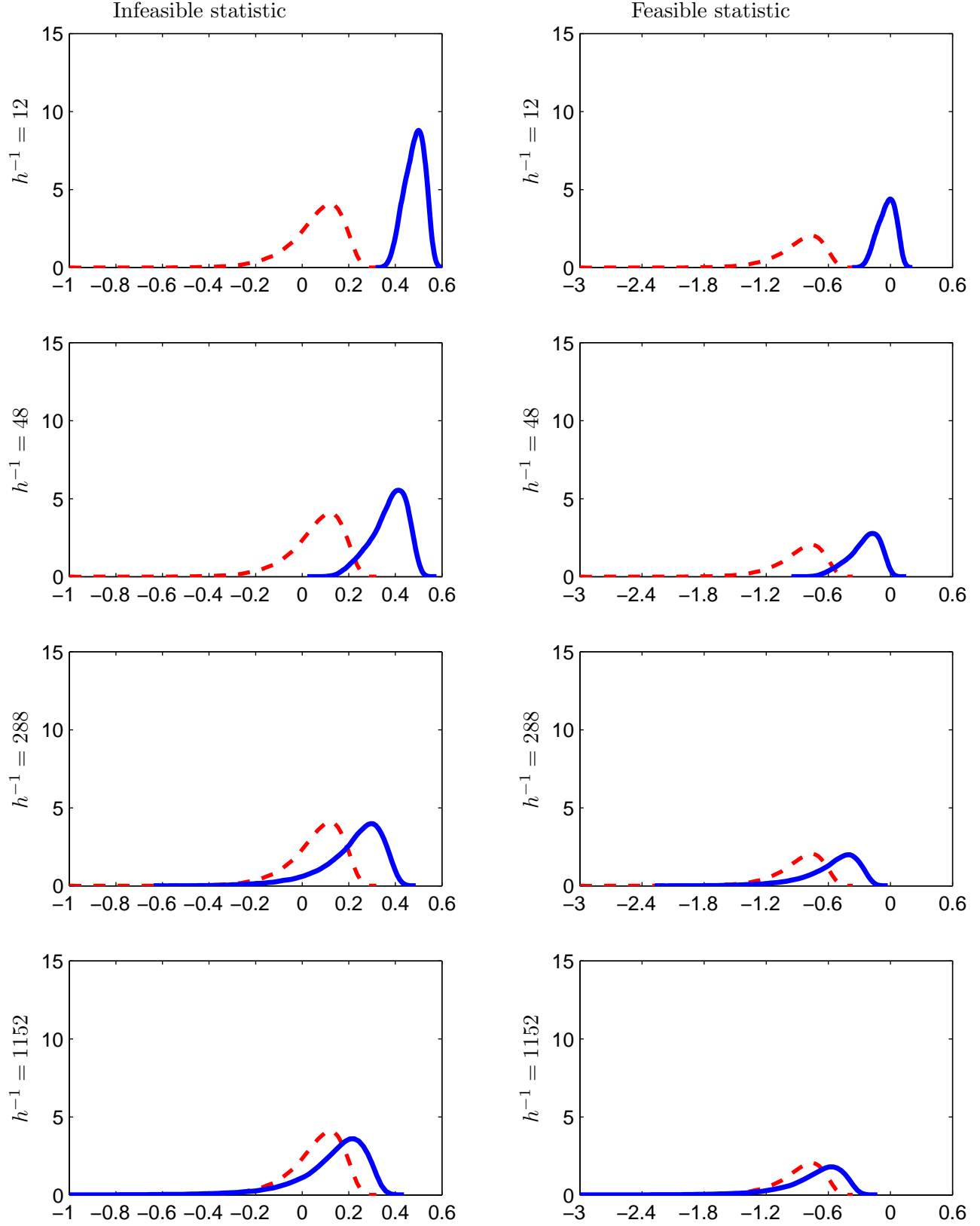


Figure 2: The left-hand-side panel shows the kernel densities of  $\beta^*$  and  $\hat{\beta}^*$  for the two-factor diffusion. The solid line refers to  $\hat{\beta}^*$  and the dashed line refers to  $\hat{\beta}^*$ . The right-hand-side panel refers to the kernel densities of  $\beta^{**}$  and  $\hat{\beta}^{**}$ .

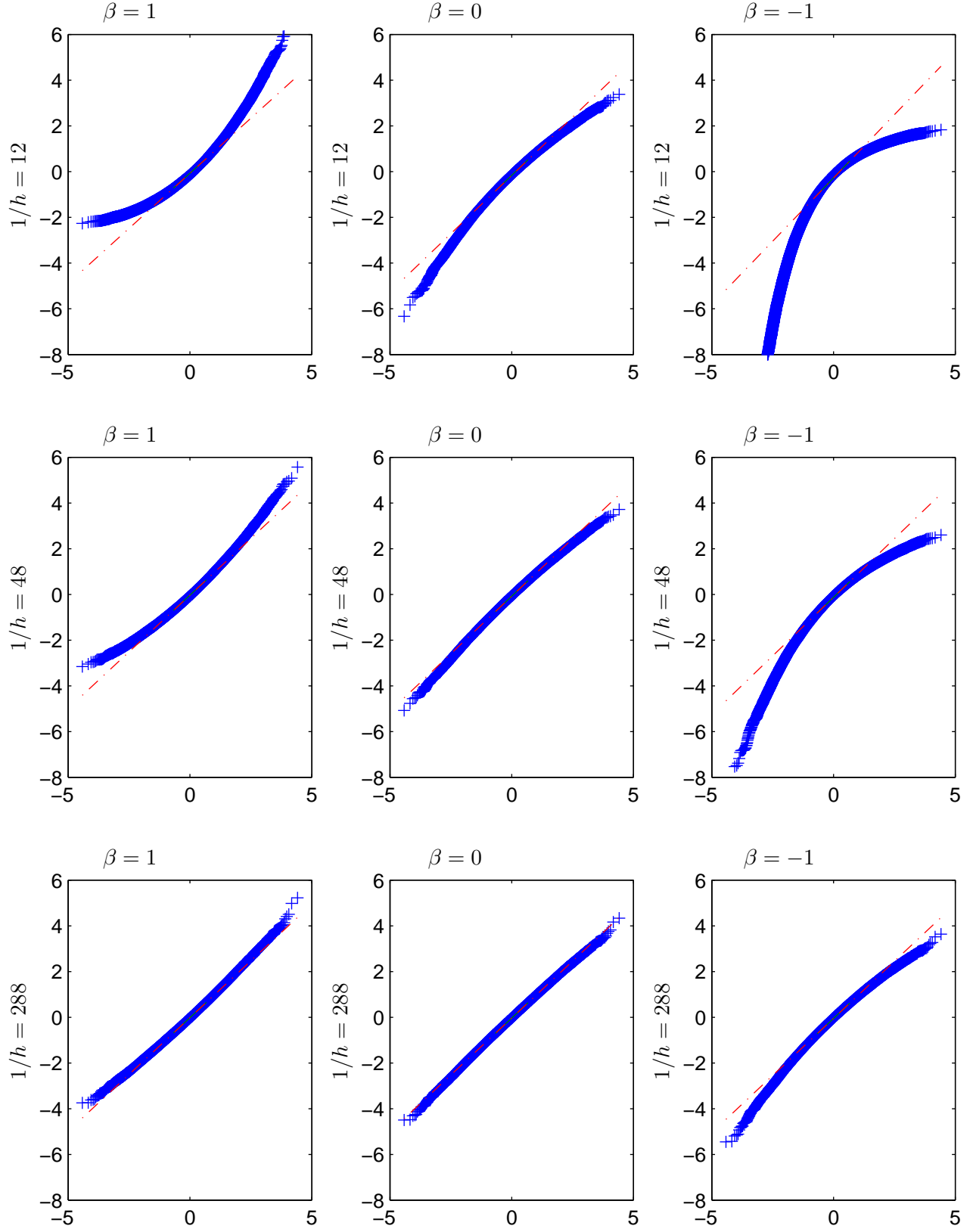


Figure 3: QQ-plots for infeasible statistics with  $\beta = 1, \beta = 0$  and  $\beta = -1$ , GARCH(1,1) diffusion.

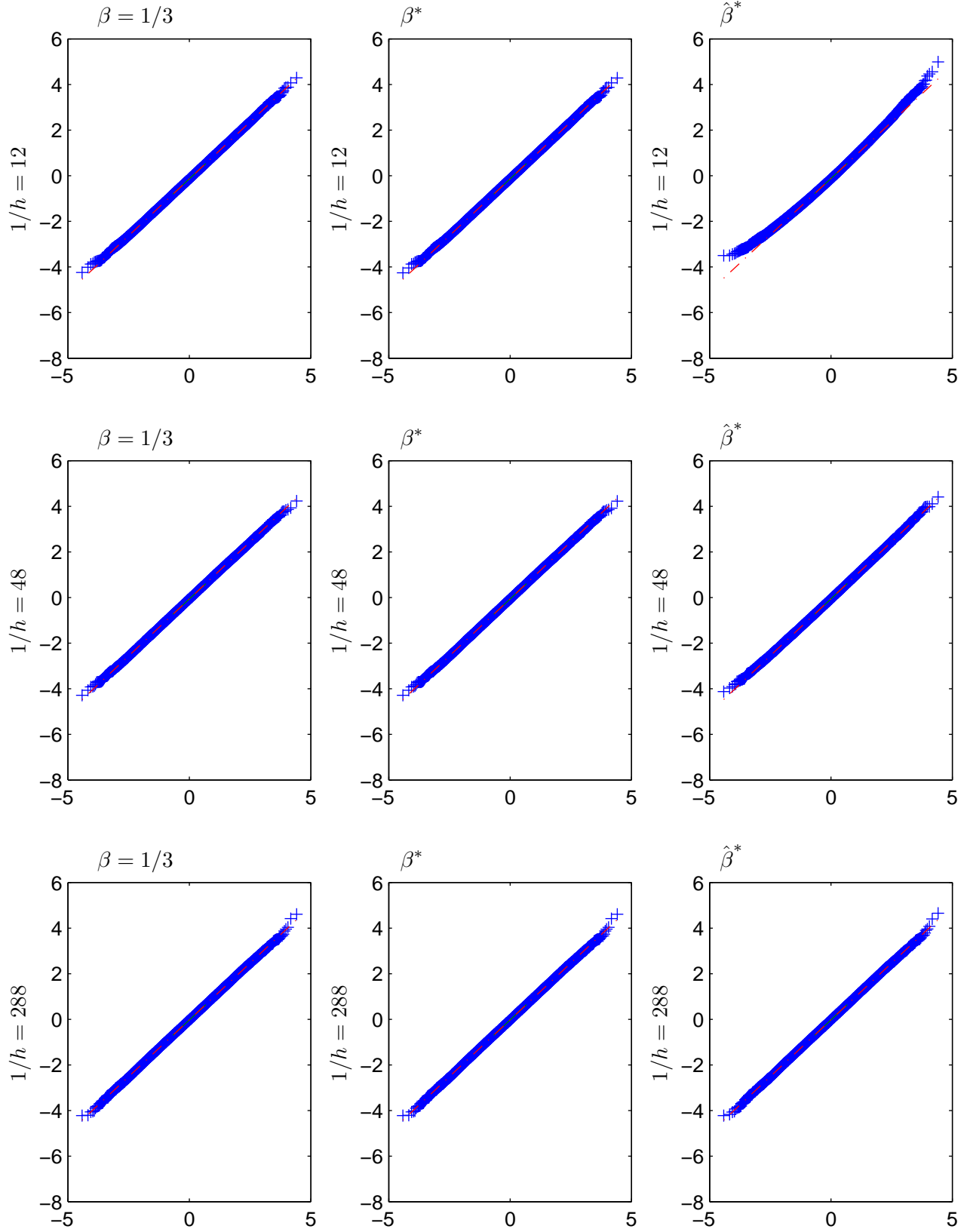


Figure 4: QQ-plots for infeasible statistics with  $\beta = 1/3$ ,  $\beta = \beta^*$  and  $\beta = \hat{\beta}^*$ , GARCH(1,1) diffusion.

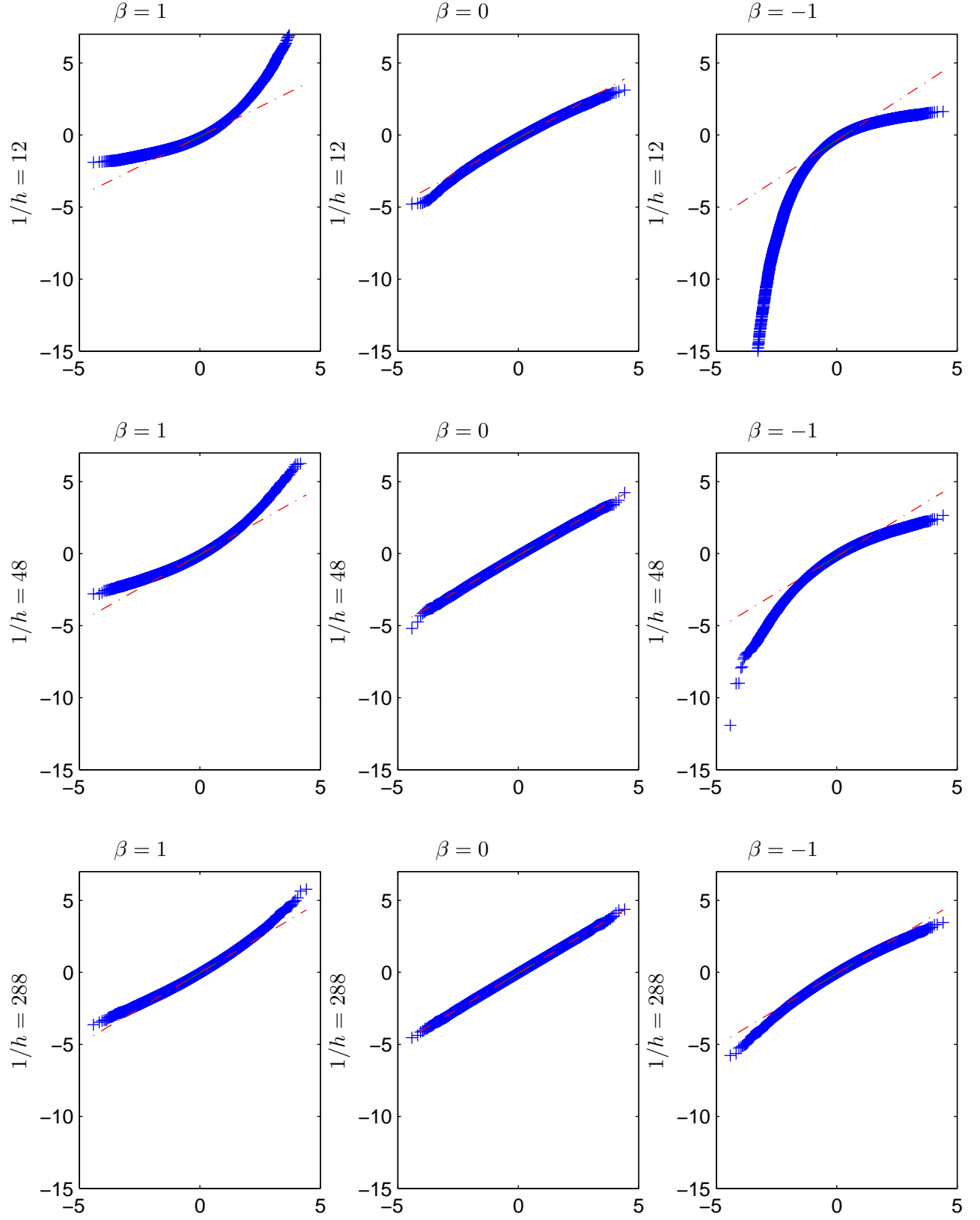


Figure 5: QQ-plots for infeasible statistics with  $\beta = 1, \beta = 0$  and  $\beta = -1$ , two-factor diffusion.

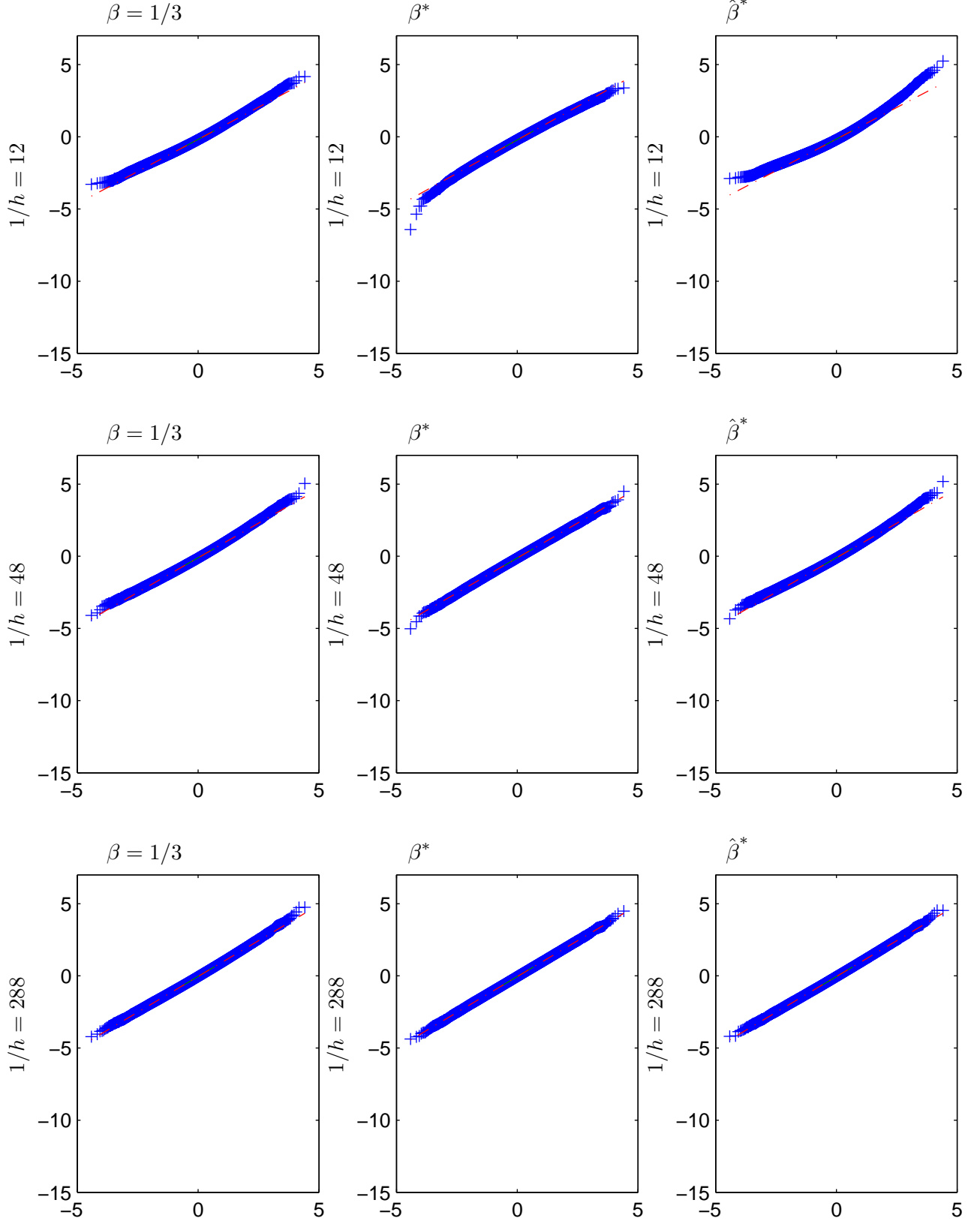


Figure 6: QQ-plots for infeasible statistics, with  $\beta = 1/3$ ,  $\beta = \beta^*$  and  $\beta = \hat{\beta}^*$ , two-factor diffusion.

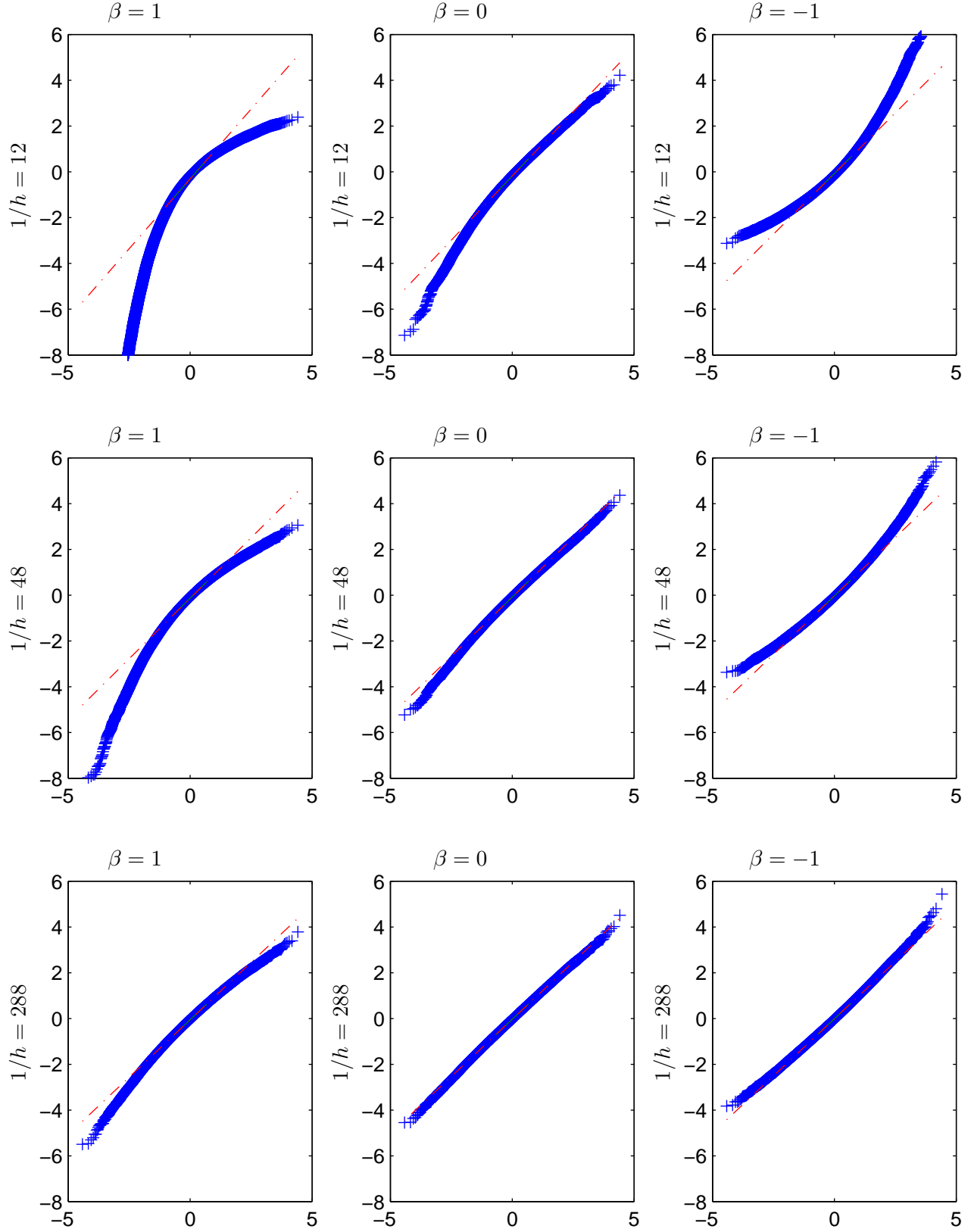


Figure 7: QQ-plots for feasible statistics with  $\beta = 1, \beta = 0$  and  $\beta = -1$ , GARCH(1,1) diffusion.

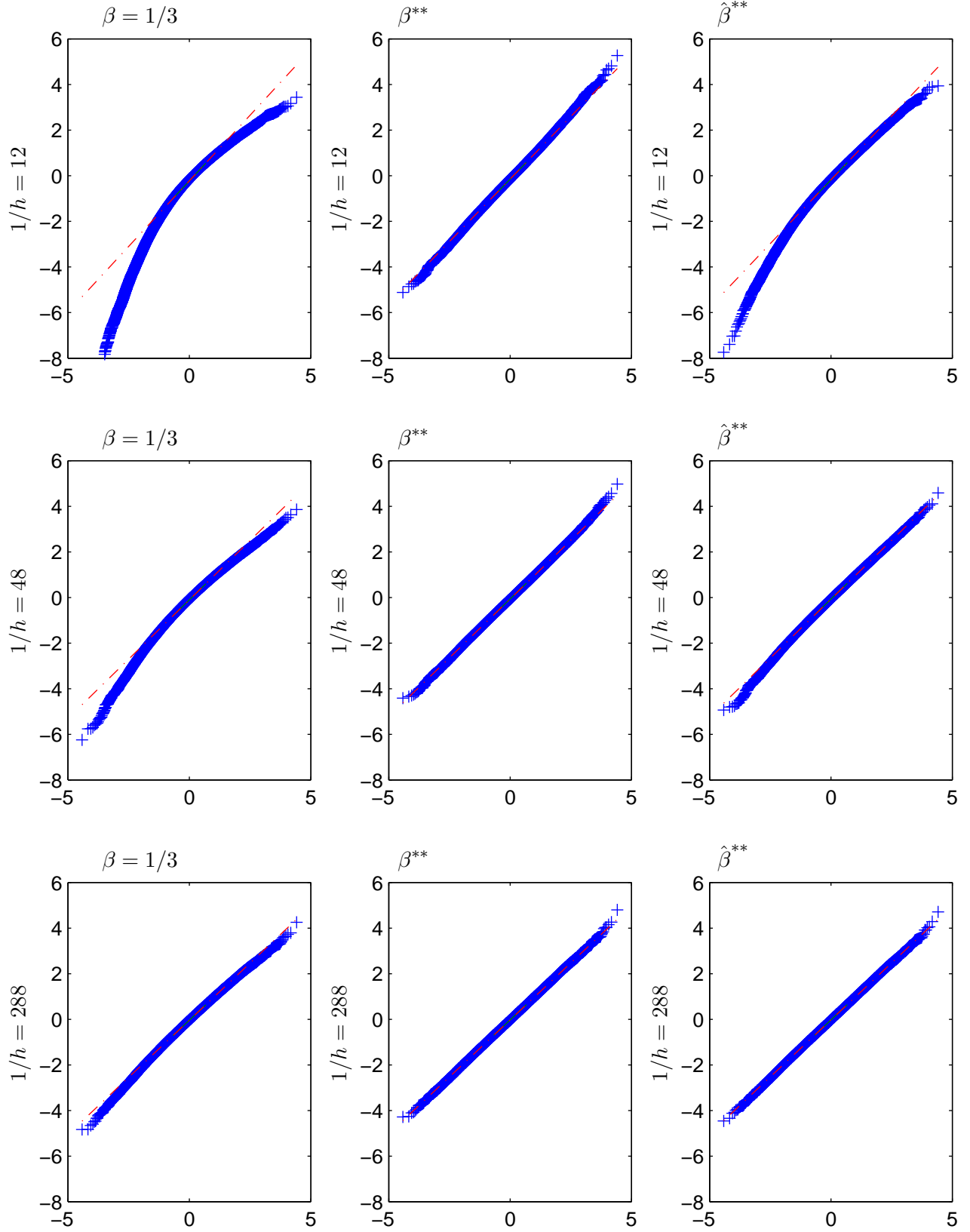


Figure 8: QQ-plots for feasible statistics with  $\beta = 1/3$ ,  $\beta = \beta^{**}$  and  $\beta = \hat{\beta}^{**}$ , GARCH(1,1) diffusion.

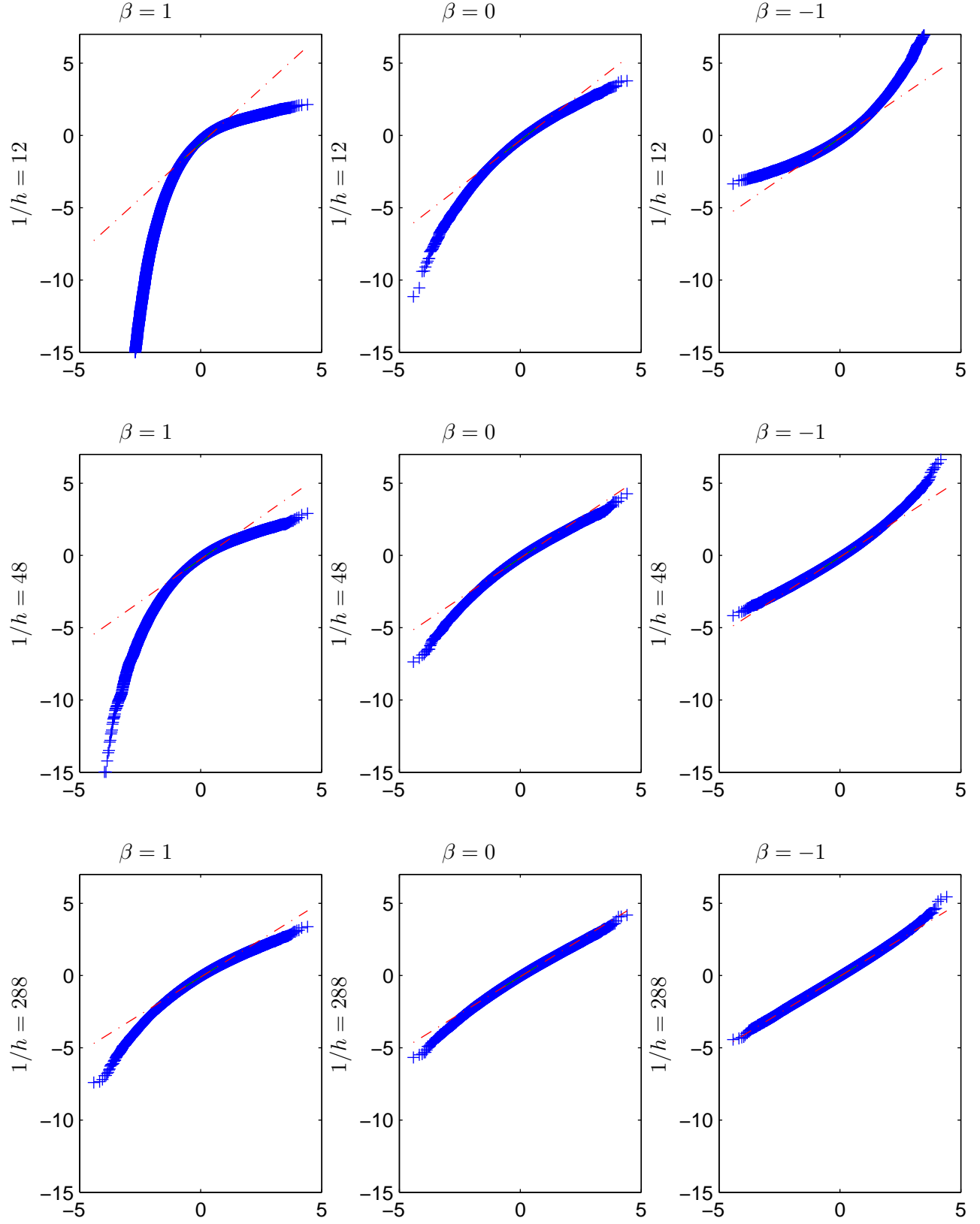


Figure 9: QQ-plots for feasible statistics with  $\beta = 1, \beta = 0$  and  $\beta = -1$ , two-factor diffusion.



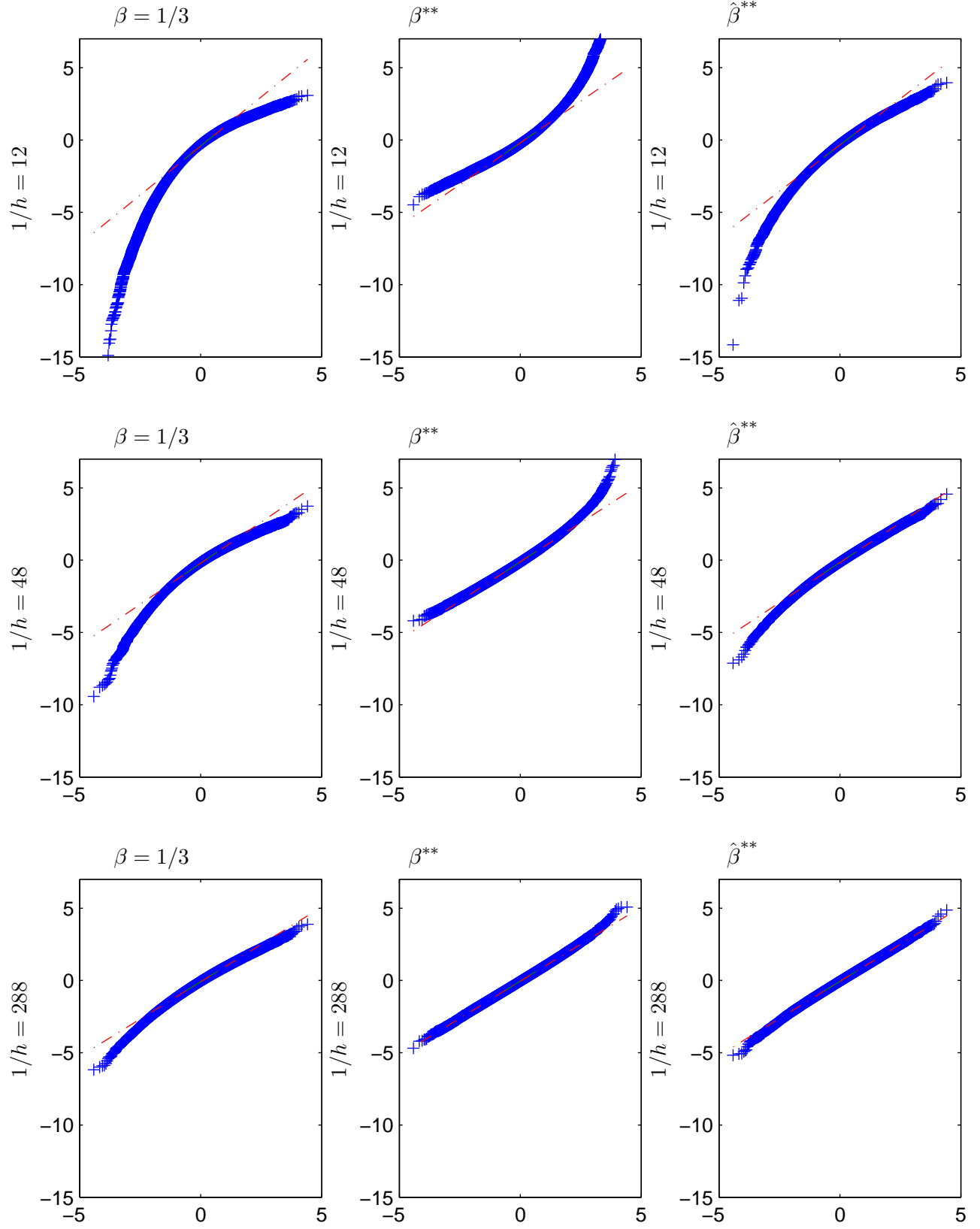


Figure 10: QQ-plots for feasible statistics with  $\beta = 1/3$ ,  $\beta = \beta^{**}$  and  $\beta = \hat{\beta}^{**}$ , two-factor diffusion.

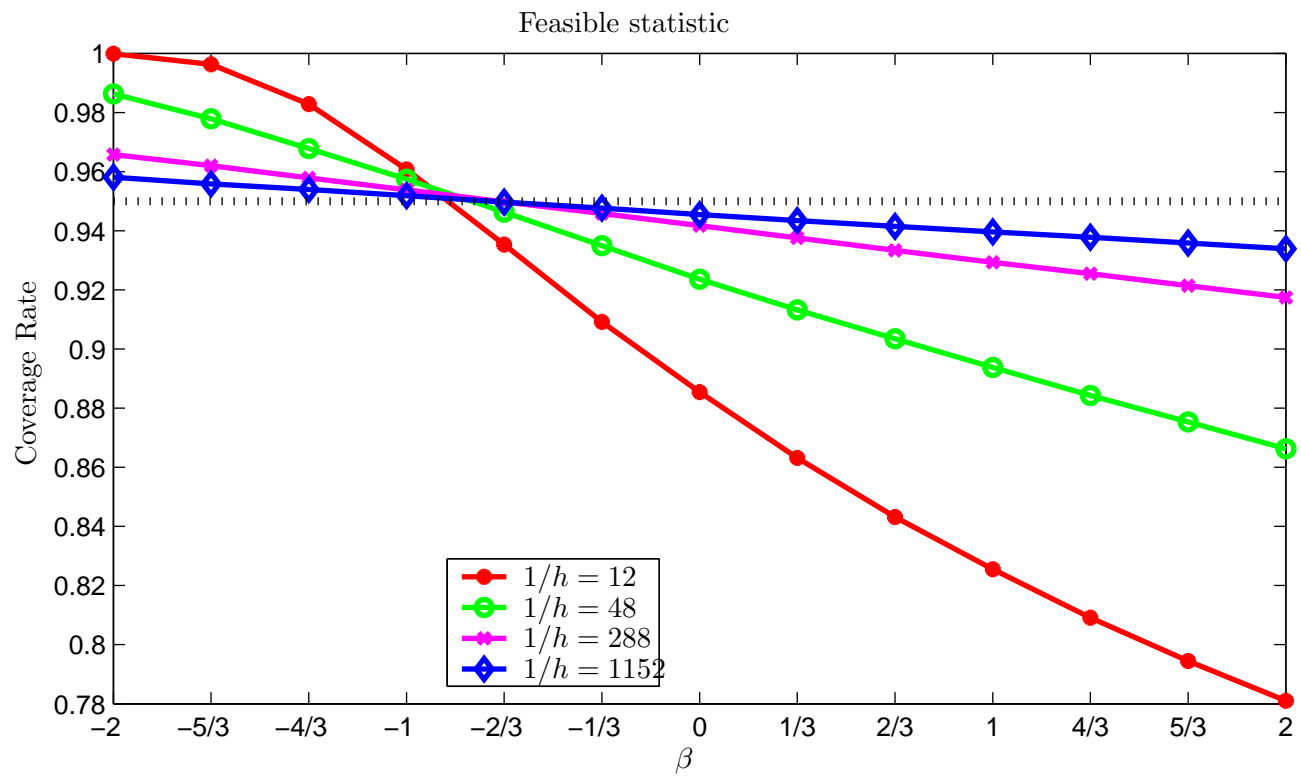
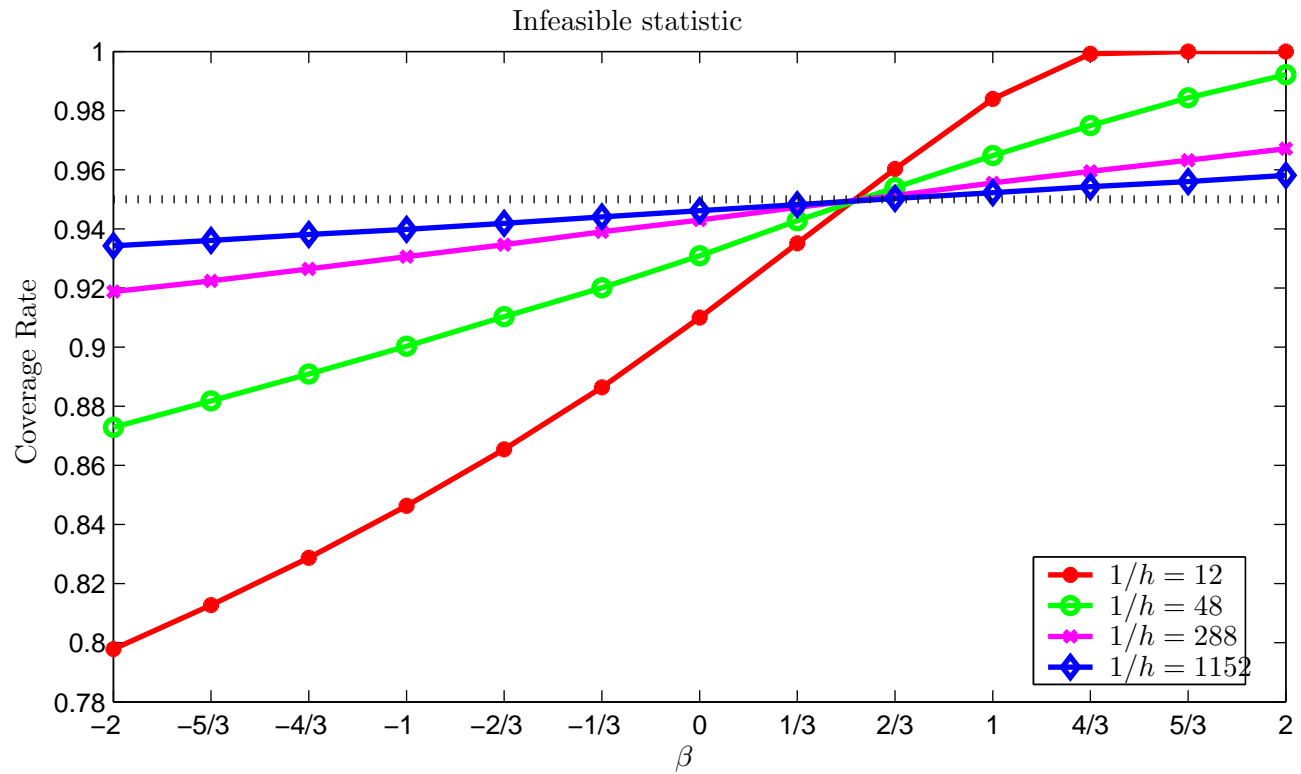


Figure 11: Coverage probabilities of one-sided confidence intervals across several values of  $\beta$ , GARCH(1,1) diffusion.

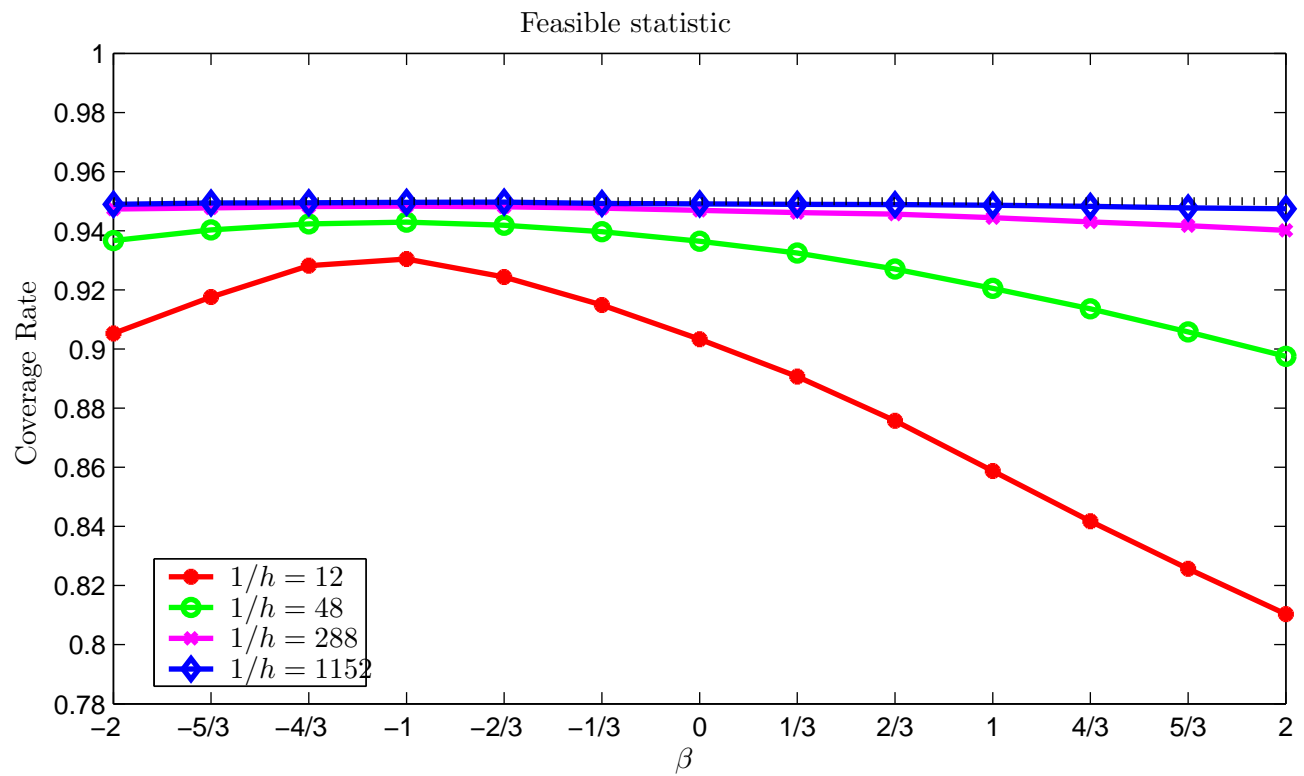
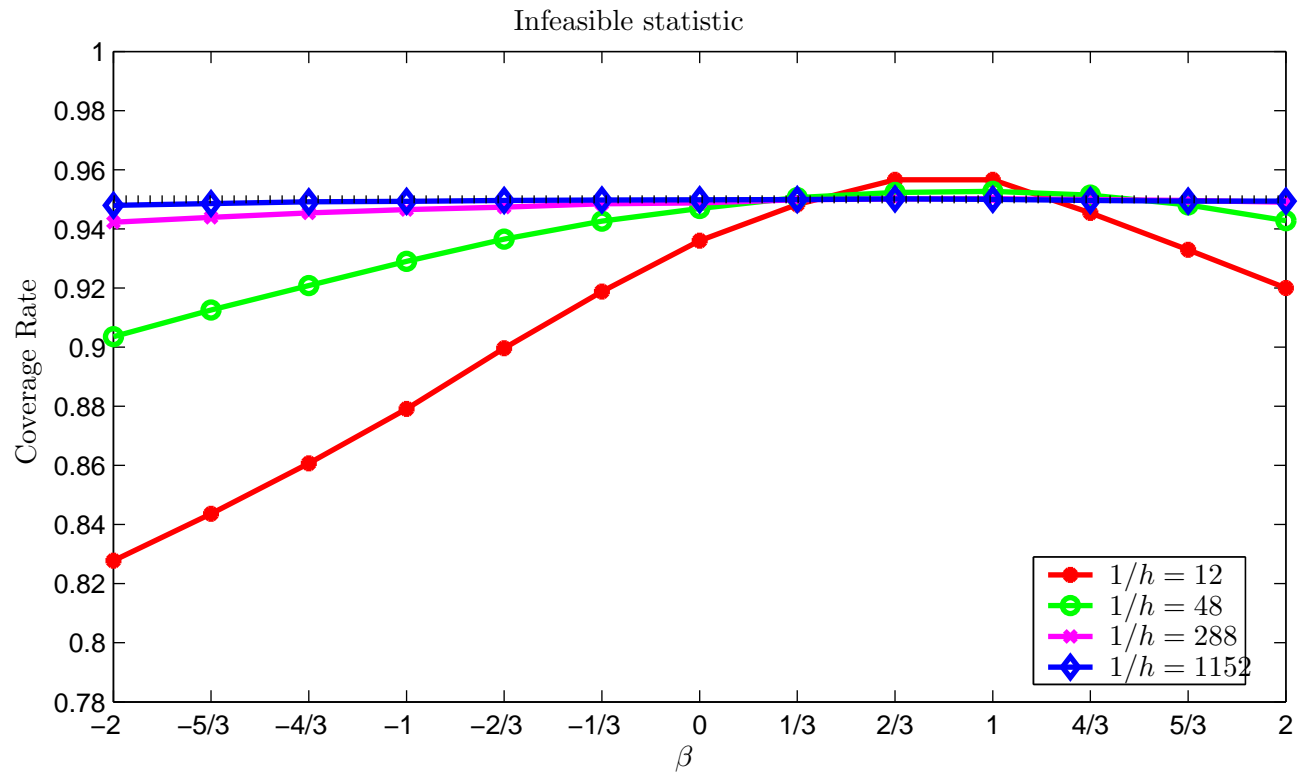


Figure 12: Coverage probabilities of two-sided confidence intervals across several values of  $\beta$ , GARCH(1,1) diffusion.

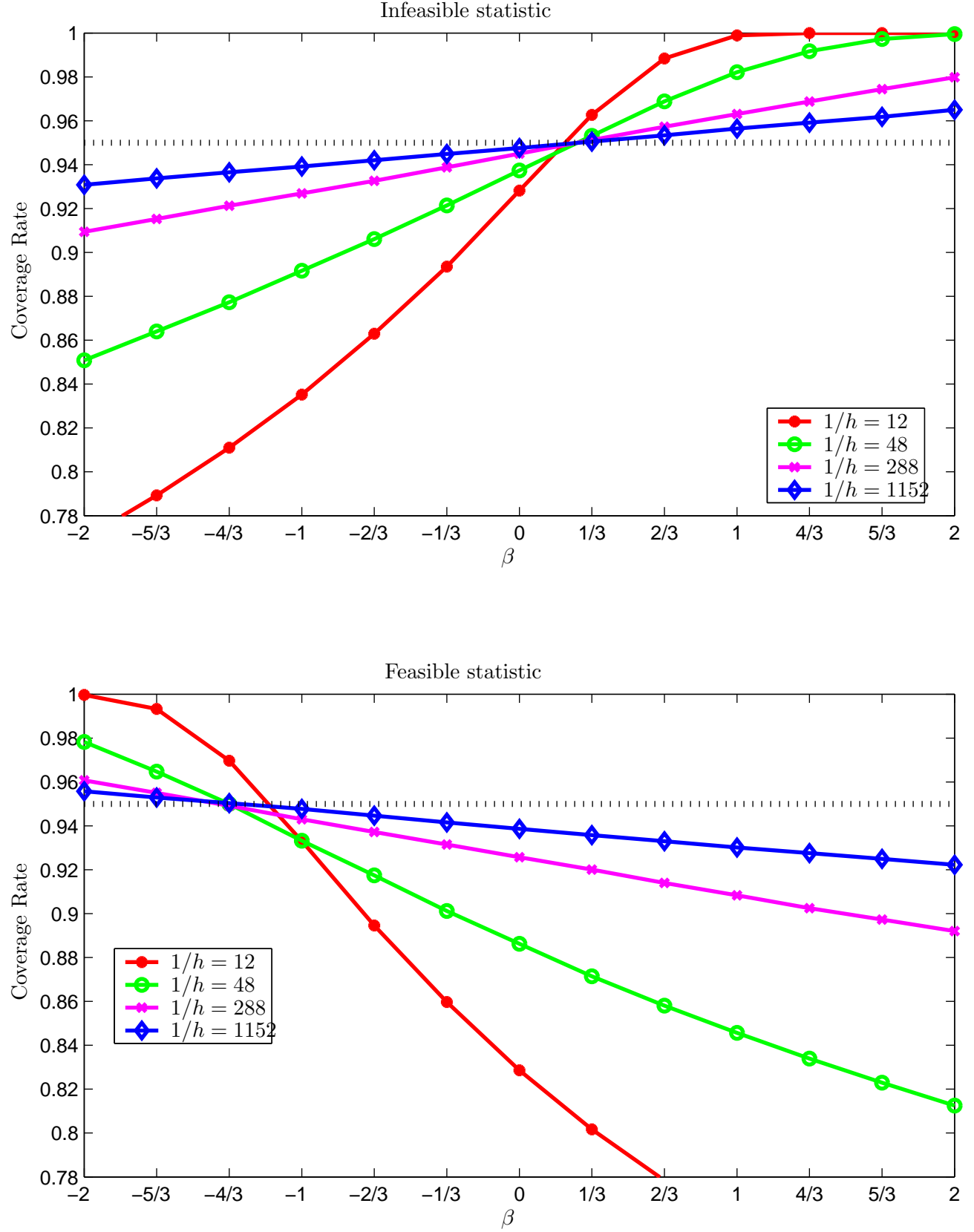


Figure 13: Coverage probabilities of one-sided confidence intervals across several values of  $\beta$ , two-factor diffusion.

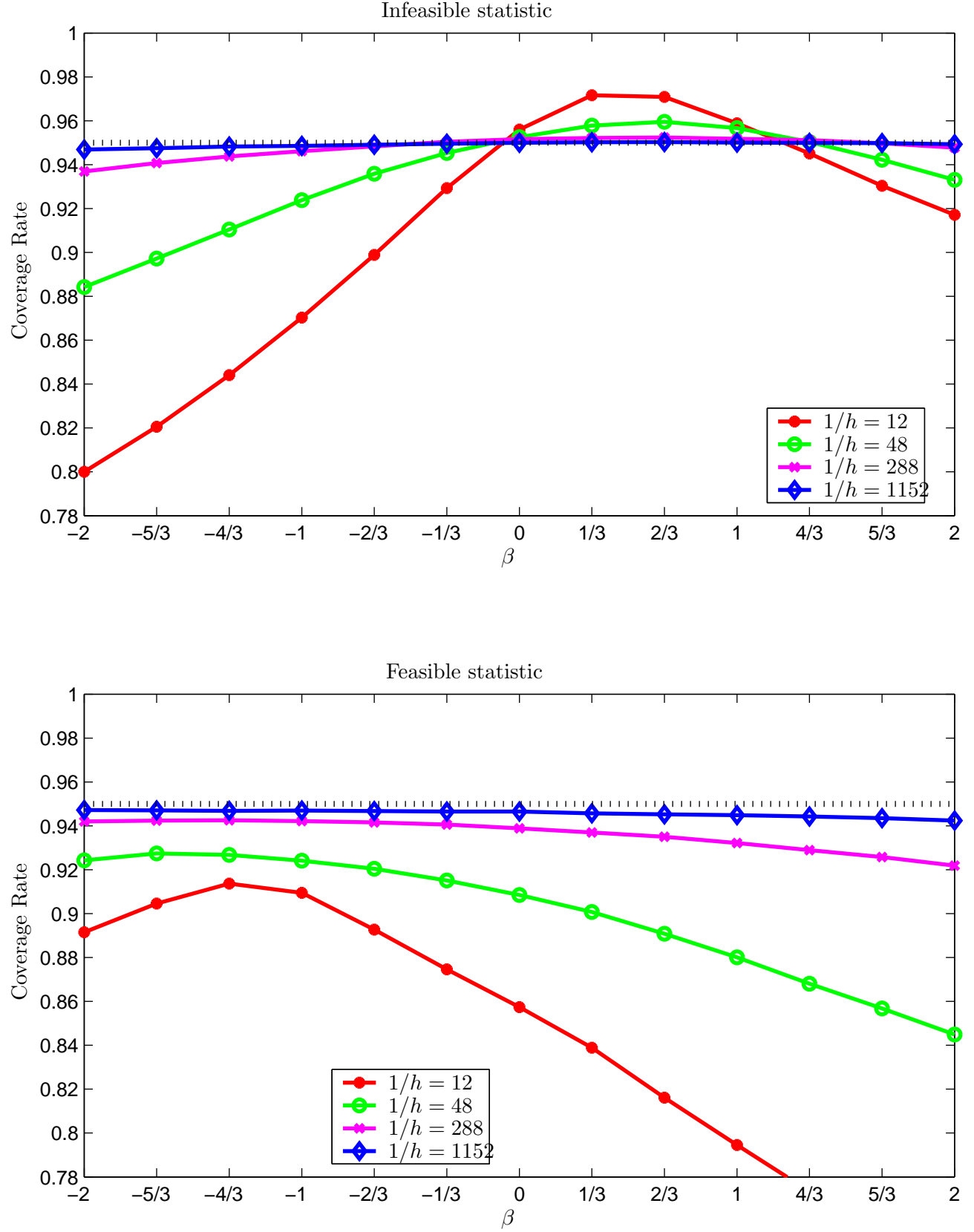


Figure 14: Coverage probabilities of two-sided confidence intervals across several values of  $\beta$ , two-factor diffusion.

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