Inspire…Educate…Transform.

Goal programming, NLP and quadratic programming

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October, 2015

Goal Programming

- Soft constraints or goals

- Let us look at a problem

DEFINING THE DECISION VARIABLES

- The decision facing the hotel owner is how many small, medium, and large conference rooms to include in the conference center expansion.
- These quantities are represented by X1, X2, and X3, respectively.

DEFINING THE GOALS

- The expansion should include
  - Goal 1: approximately 5 small conference rooms.
  - Goal 2: approximately 10 medium conference rooms.
  - Goal 3: approximately 15 large conference rooms.
  - Goal 4: approximately 25,000 square feet.
  - Goal 5: approximately $1,000,000

DEFINING THE GOAL CONSTRAINTS

$$X_1 + d_1^- - d_1^+ = 5 \quad \text{\} small rooms}$$

$$X_2 + d_2^- - d_2^+ = 10 \quad \text{\} medium rooms}$$

$$X_3 + d_3^- - d_3^+ = 15 \quad \text{\} large rooms}$$

$$\text{where } d_i^-, d_i^+ \geq 0 \text{ for all } i$$

- The RHS value of each goal constraint (the values 5, 10, and 15 in the previous constraints) is the **target value** for the goal because it represents the level of achievement that the decision maker wants to obtain for the goal.

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- The variables d_i- and d_i+ are called **deviational variables** because they represent the amount by which each goal deviates from its target value. The d_i represents the amount by which each goal's target value is *underachieved*, and the d_i+ represents the amount by which each goal's target value is *overachieved*.

How deviation variables work

- Suppose that we have a solution where X1 = 3, X2 = 13, and X3 = 15.

$$X_1 + d_1^- - d_1^+ = 5 \quad \} \text{ small rooms}$$

$$X_2 + d_2^- - d_2^+ = 10 \quad \} \text{ medium rooms}$$

$$X_3 + d_3^- - d_3^+ = 15 \quad \} \text{ large rooms}$$

$$\text{where } d_i^-, d_i^+ \geq 0 \text{ for all } i$$

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- To illustrate how deviational variables work, suppose that we have a solution where $X1 = 3$, $X2 = 13$, and $X3 = 15$.

- To satisfy the first goal constraint listed previously, its deviational variables would assume the values $d1^- = 2$ and $d1^+ = 0$ to reflect that the goal of having 5 small conference rooms is *underachieved* by 2.

- Similarly, to satisfy the second goal constraint, its deviational variables would assume the values $d2- = 0$ and $d2+ = 3$ to reflect that the goal of having 10 medium conference rooms is *overachieved* by 3.

- Finally, to satisfy the third goal constraint, its deviational variables would assume the values $d3- = 0$ and $d3+ = 0$, to reflect that the goal of having 15 medium conference rooms is *exactly* achieved.

$$400X_1 + 750X_2 + 1{,}050X_3 + d_4^- - d_4^+ = 25{,}000 \quad \} \text{ square footage}$$

$$18{,}000X_1 + 33{,}000X_2 + 45{,}150X_3 + d_5^- - d_5^+ = 1{,}000{,}000 \quad \} \text{ building cost}$$

What is the objective function

$$\text{MIN:} \qquad \sum_i (d_i^- + d_i^+)$$

7 rooms + 1,500 dollars = 1,507 units of what?

Option 2

$$\text{MIN:} \quad \sum_i \frac{1}{t_i}(d_i^- + d_i^+)$$

- suppose that we have a solution where first goal is underachieved by 1 room and the fifth goal is overachieved by $20,000 and all other goals are achieved exactly

$$\frac{1}{t_1}d_1^- + \frac{1}{t_5}d_5^+ = \frac{1}{5} \times 1 + \frac{1}{1{,}000{,}000} \times 20{,}000$$

- The percentage deviation objective can be used only if all the target values for all the goals are non-zero; otherwise a division by zero error will occur.

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- Is $20,000 1/10th as important as 1 room?

- Is $1.1 million as bad as $900K?

Minimize the weighted sum of the deviations: MIN: $\sum_i (w_i^- d_i^- + w_i^+ d_i^+)$

or

Minimize the weighted sum of the percentage deviations: MIN: $\sum_i \frac{1}{t_i}(w_i^- d_i^- + w_i^+ d_i^+)$

You need to follow an iterative procedure in which you try a particular set of weights, solve the problem, analyze the solution, and then refine the weights and solve the problem again.

To summarize, the LP model for our example GP problem is:

$$\text{MIN:} \quad \frac{w_1^-}{5}d_1^- + \frac{w_2^-}{10}d_2^- + \frac{w_3^-}{15}d_3^- + \frac{w_4^-}{25{,}000}d_4^- + \frac{w_4^+}{25{,}000}d_4^+ + \frac{w_5^+}{1{,}000{,}000}d_5^+$$

Subject to:

$$\begin{aligned}
X_1 + d_1^- - d_1^+ &= 5 && \text{\} small rooms} \\
X_2 + d_2^- - d_2^+ &= 10 && \text{\} medium rooms} \\
X_3 + d_3^- - d_3^+ &= 15 && \text{\} large rooms} \\
400X_1 + 750X_3 + 1{,}050X_3 + d_4^- - d_4^+ &= 25{,}000 && \text{\} square footage} \\
18{,}000X_1 + 33{,}000X_2 + 45{,}150X_3 + d_5^- - d_5^+ &= 1{,}000{,}000 && \text{\} building cost} \\
d_i^-, d_i^+ \geq 0 \text{ for all } i & && \text{\} non-negativity conditions} \\
X_i \geq 0 \text{ for all } i & && \text{\} non-negativity conditions} \\
X_i \text{ must be integers} & &&
\end{aligned}$$

Notice that this objective omits (or assigns weights of 0 to) the deviational variables about which the decision maker is indifferent. Thus, this objective would not penalize a solution where, for example, 7 small conference rooms were selected (and therefore $d_1^+ = 2$) because we assume that the decision maker would not view this as an undesirable deviation from the goal of having 5 small conference rooms.

Summary

1. Identify the decision variables in the problem.
2. Identify any hard constraints in the problem and formulate them in the usual way.
3. State the goals of the problem along with their target values.
4. Create constraints using the decision variables that would achieve the goals exactly.

GP Summary contd…

5. Transform the above constraints into goal constraints by including deviational variables.
6. Determine which deviational variables represent undesirable deviations from the goals.
7. Formulate an objective that penalizes the undesirable deviations.
8. Identify appropriate weights for the objective.
9. Solve the problem.
10. Inspect the solution to the problem. If the solution is unacceptable, return to step 8 and revise the weights as needed.

- Suppose we want to eliminate any solution that exceeds the target building cost by more than $50,000. We could build this requirement

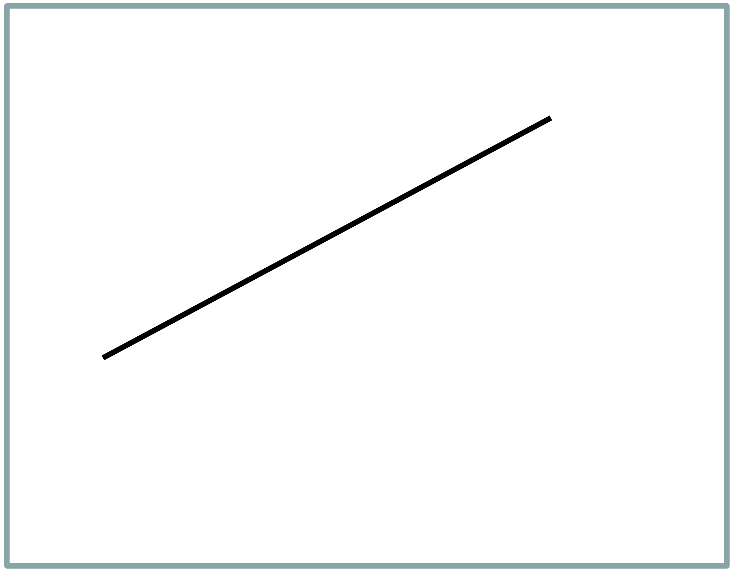
$$d5_{+} \leq 50{,}000$$

- In some GP problems, one or more goals might be viewed as being infinitely more important than the other goals. We could assign arbitrarily large weights to deviations from these goals

CALCULUS AND COORDINATE GEOMETRY REFRESHER

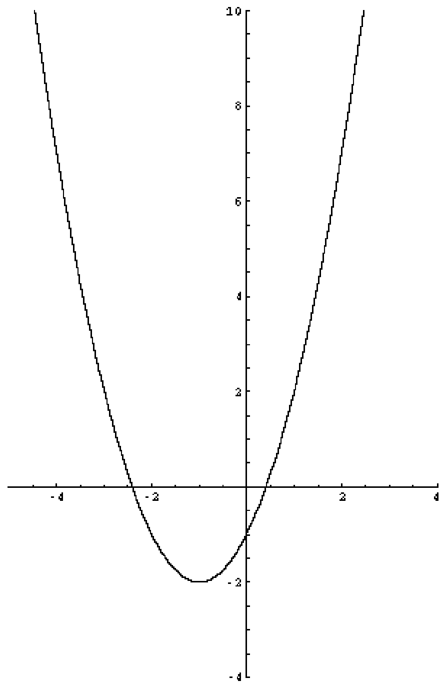
A linear function

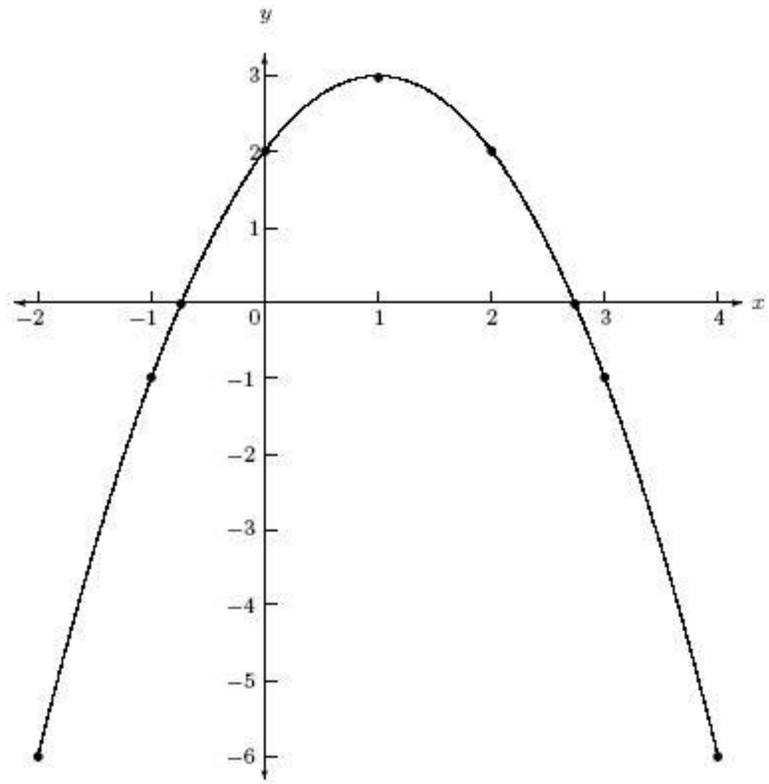
- $y=mx+c$
- $ax+by+c=0$



Quadratic equation

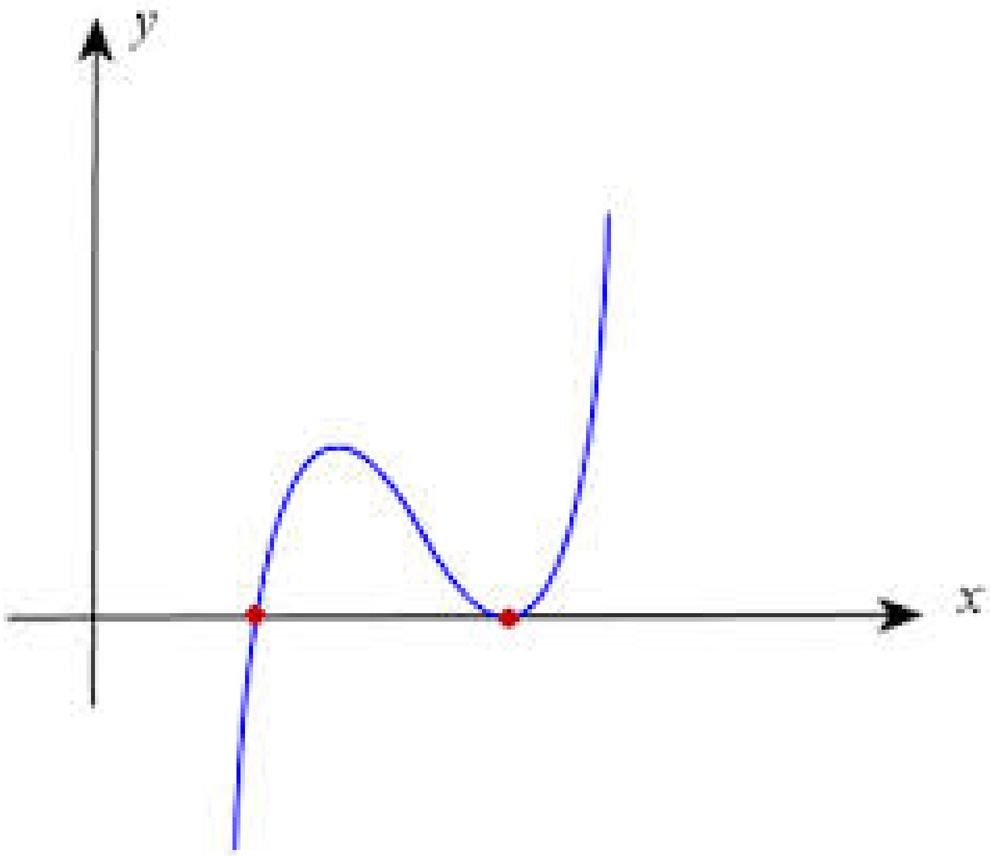
- $ax^2 + bx + c$



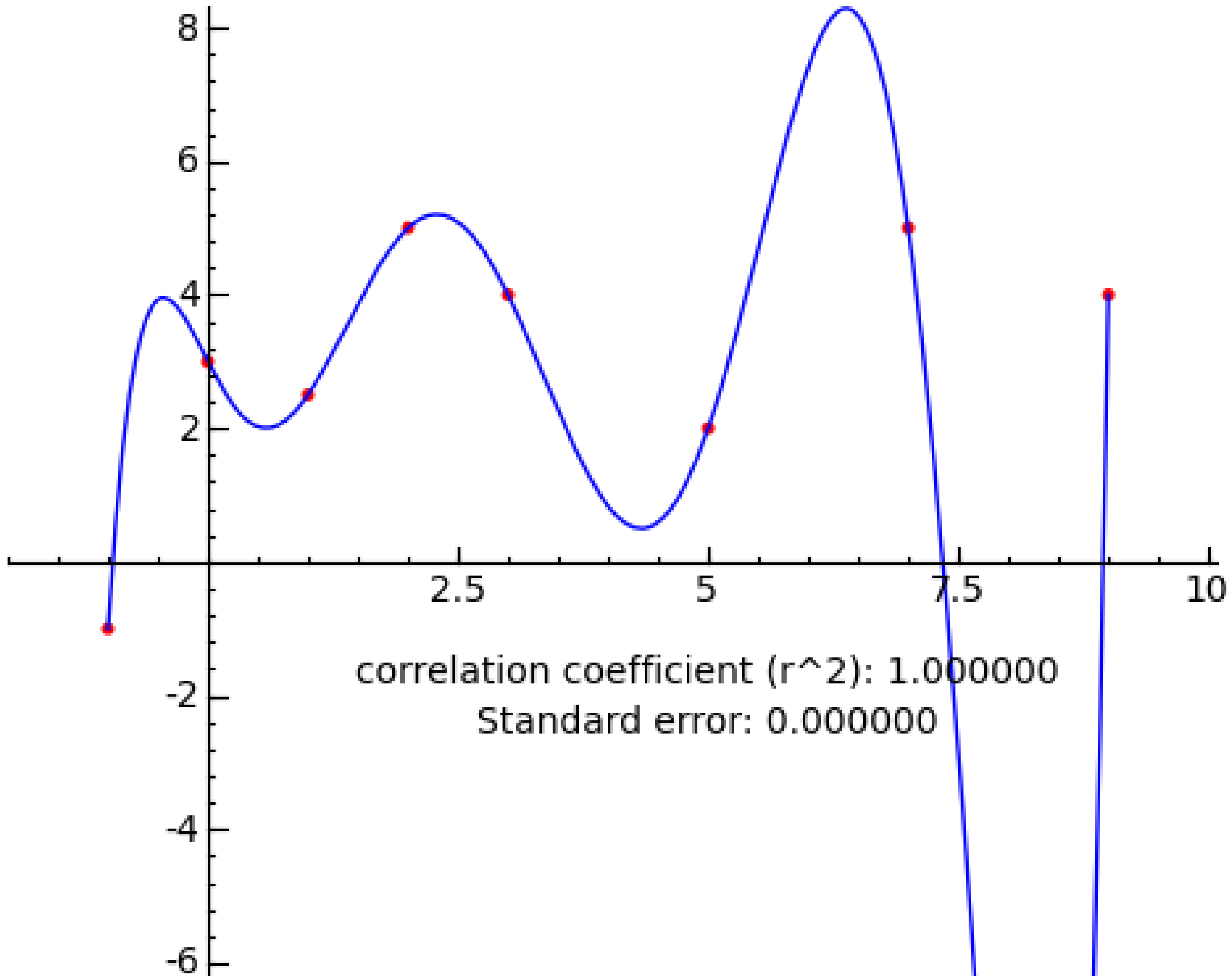


Cubic

- $ax^3 + bx^2 + cx + d$



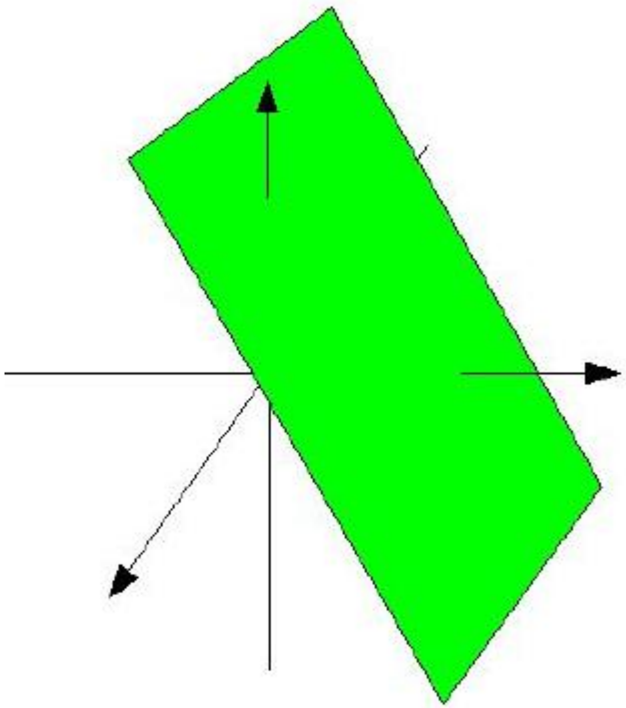
8th degree


8
6
4
2
-2
-4
-6
2.5
5
7.5
10
correlation coefficient (r^2): 1.000000
Standard error: 0.000000

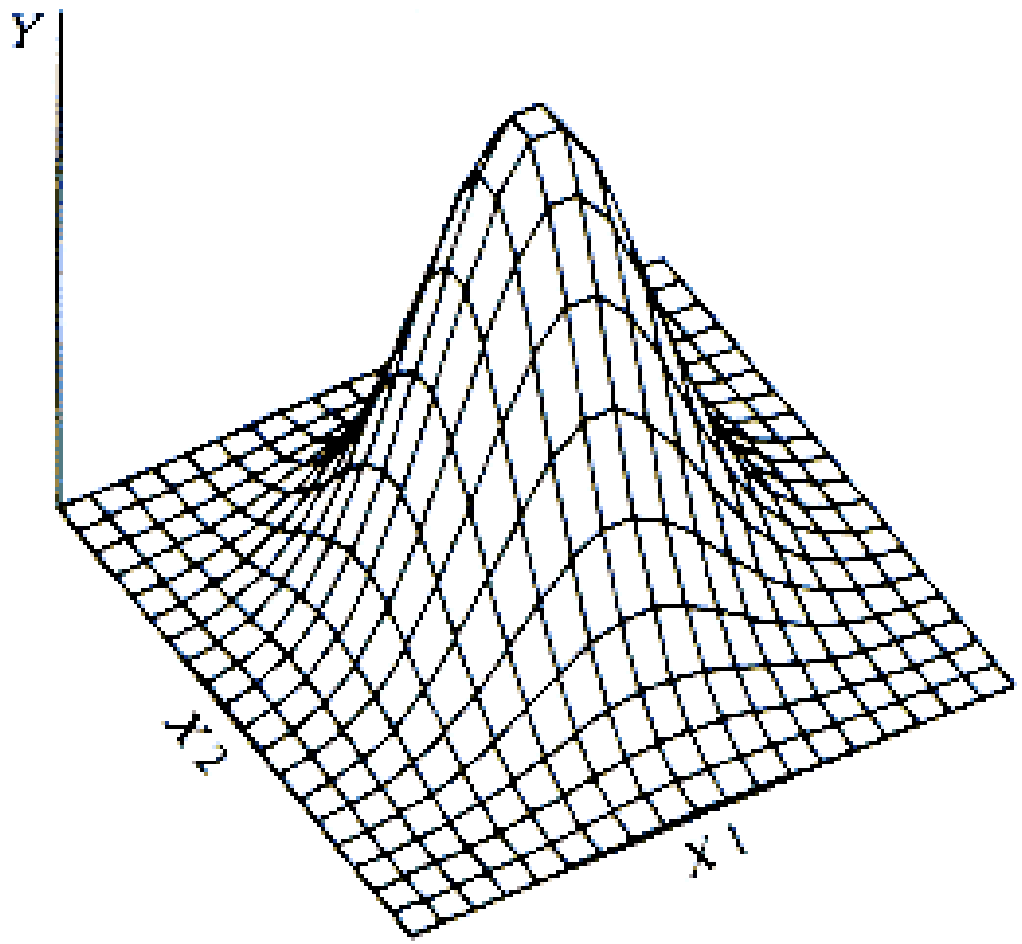
By increasing the polynomial, you make the curve more flexible!!!

A linear function in high dimensions

- $ax+by+cz+d=0$

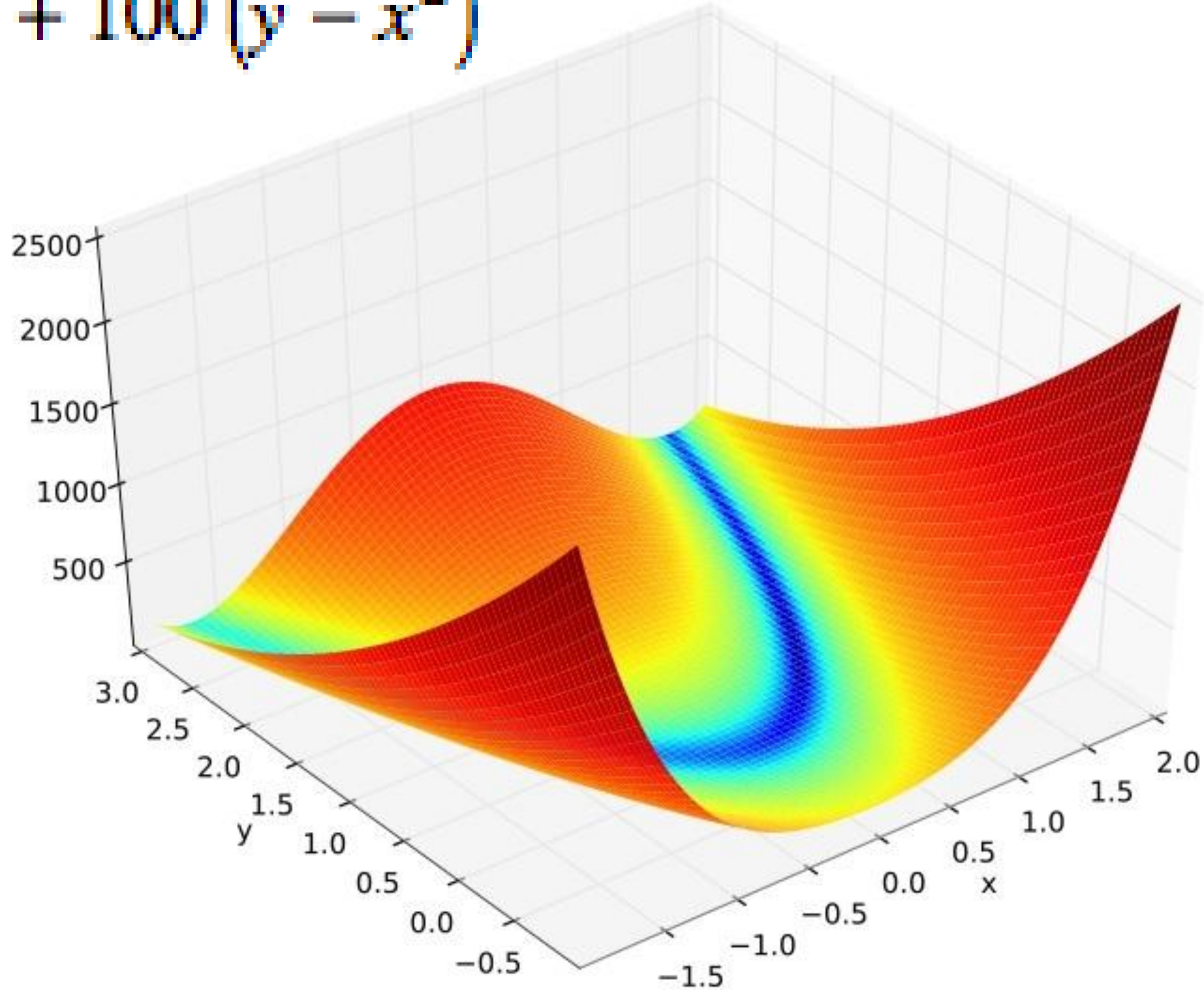


Quadratic in 2 dimensions

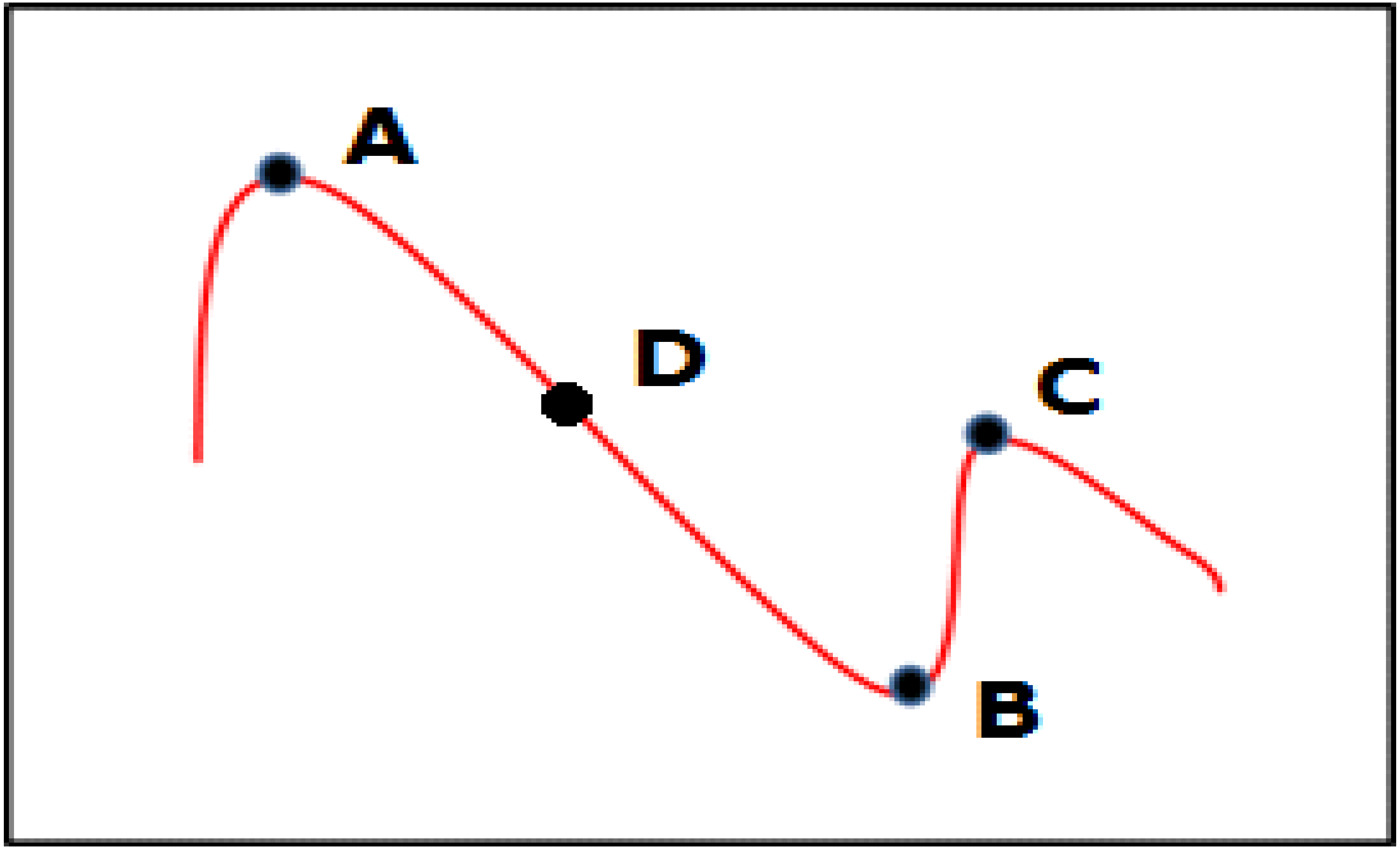


Rosen Brook Function

$$f(x, y) = (1 - x)^2 + 100\left(y - x^2\right)^2$$

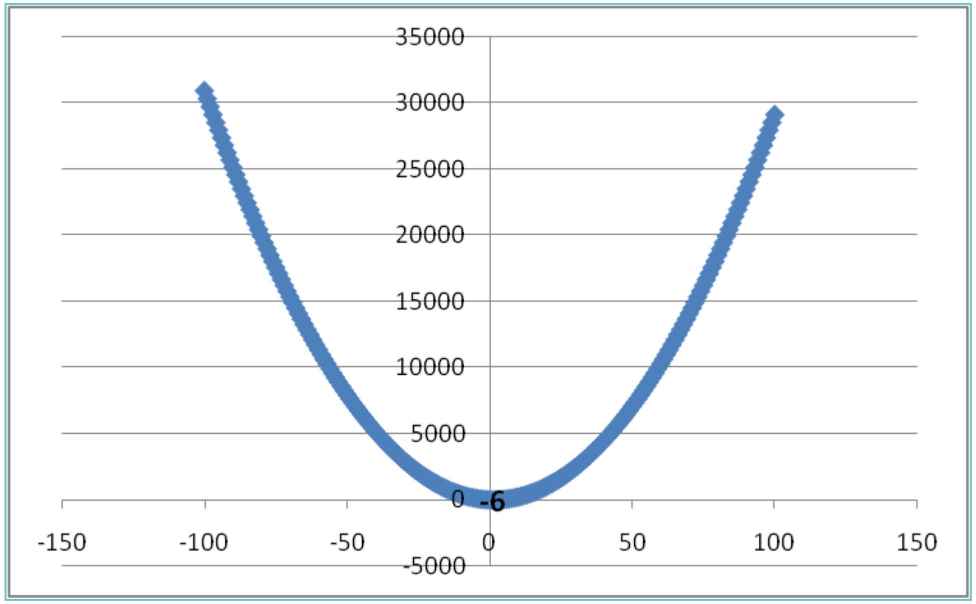


Non-linear optimization



Maxima Minima

- ***Find the critical points of $y=3x^2-9x$***



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Calculus approaches

$$Find\ the\ slope\ of\ y = 3x^2 - 9x$$

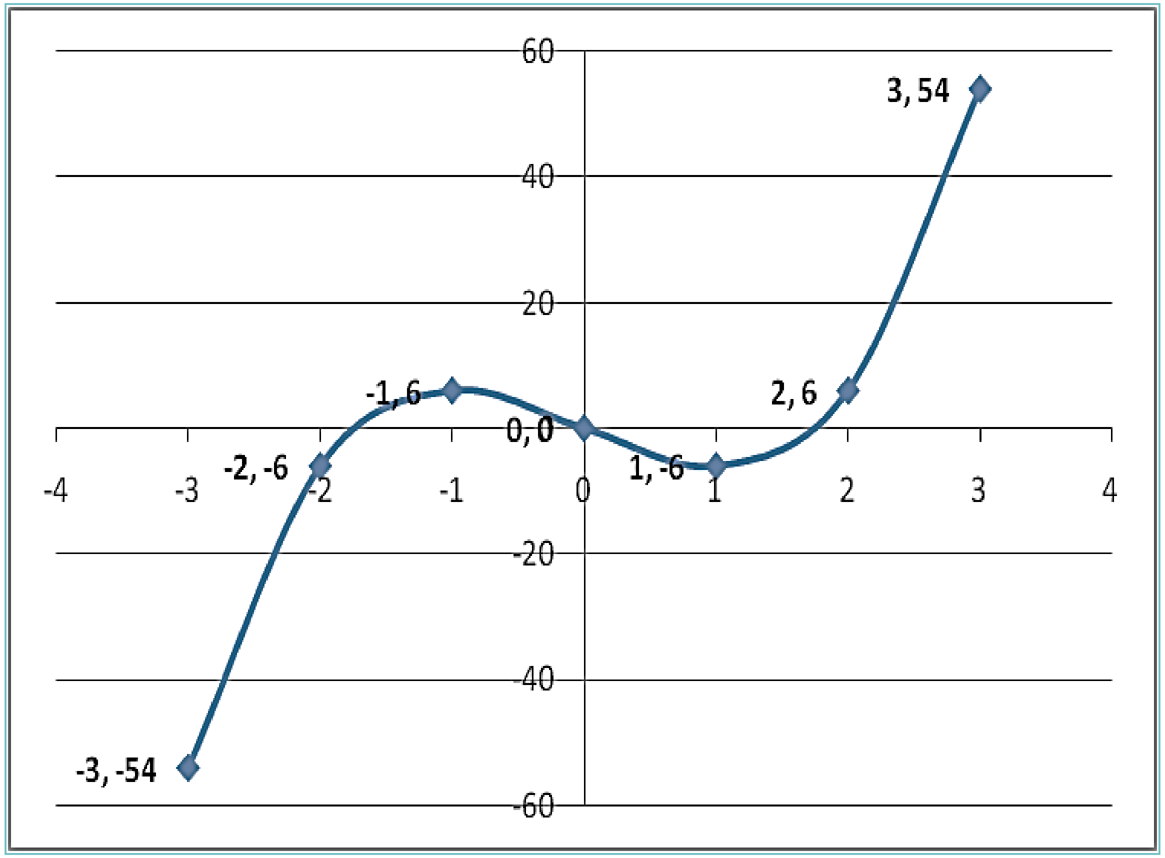
$$\frac{dy}{dx} = 6\mathrm{x} - 9; \text{Critical point is at x} = 1.5$$

$$\frac{d^2y}{dx^2} = 6;\ \ As\ it\ is\ always\ positive, this\ is\ a\ minima$$

The minima is $3(1.5^2) - 9(1.5) = -6.75$

Maxima Minima

Find the critical points of $y=3x^3-9x$



Calculus

$$Find\ the\ slope\ of\ y = 3x^3 - 9x$$

$$\frac{dy}{dx} = 9x^2 - 9; \text{Critical point is at x} = \pm 1$$

$$\frac{d^2y}{dx^2} = 18x; \ \ Minima\ is\ at\ 1\ and\ maxima\ at\ -1$$

The minima is $-6\ and\ maxima\ is\ 6$

They are local minima and maxima

Multivariable calculus

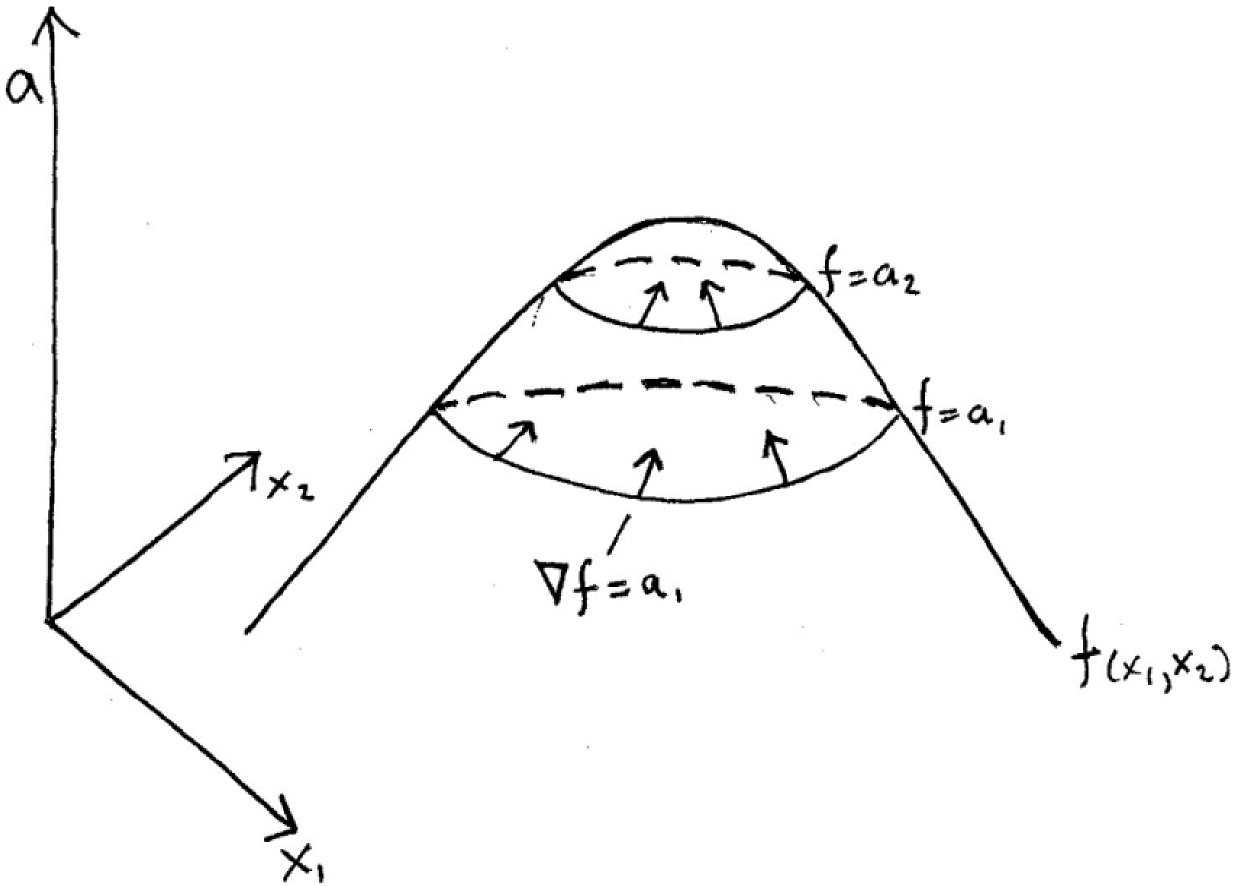
- Slope becomes Gradient: The first partial derivatives expressed as a vector is called the gradient. So, gradient of f is

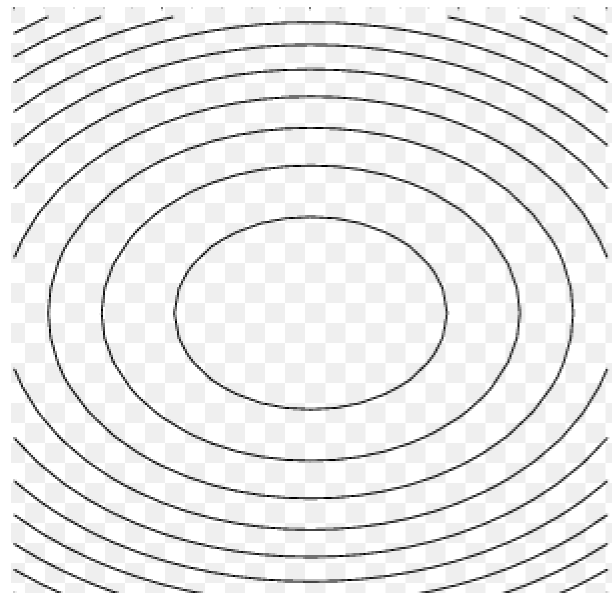
$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k \ldots$$

Gradient

- The **gradient** is a fancy word for derivative, or the rate of change of a function.
  - It’s a vector (a direction to move) that Points in the direction of greatest increase of a function.
  - It is zero at a local maximum or local minimum (because there is no single direction of increase)

Gradient and Level curves





Problem

- Find the gradient of

$$F(x,y,z) = x + y^2 + z^3 \qquad \nabla F(x,y,z) = (1, 2y, 3z^2)$$

If we want to find the direction to move to increase our function the fastest, we plug in our current coordinates (such as 3,4,5) into the equation and get:

$$direction = (1, 2(4), 3(5)^2) = (1, 8, 75)$$

Critical Point

- A **Critical Point** is any point where the gradient is zero. Take another example, say $f(x,y)=x^5+y^4-5x-32y$. Then,

$$\frac{\partial f}{\partial x} = 5x^4 - 5 = 0$$

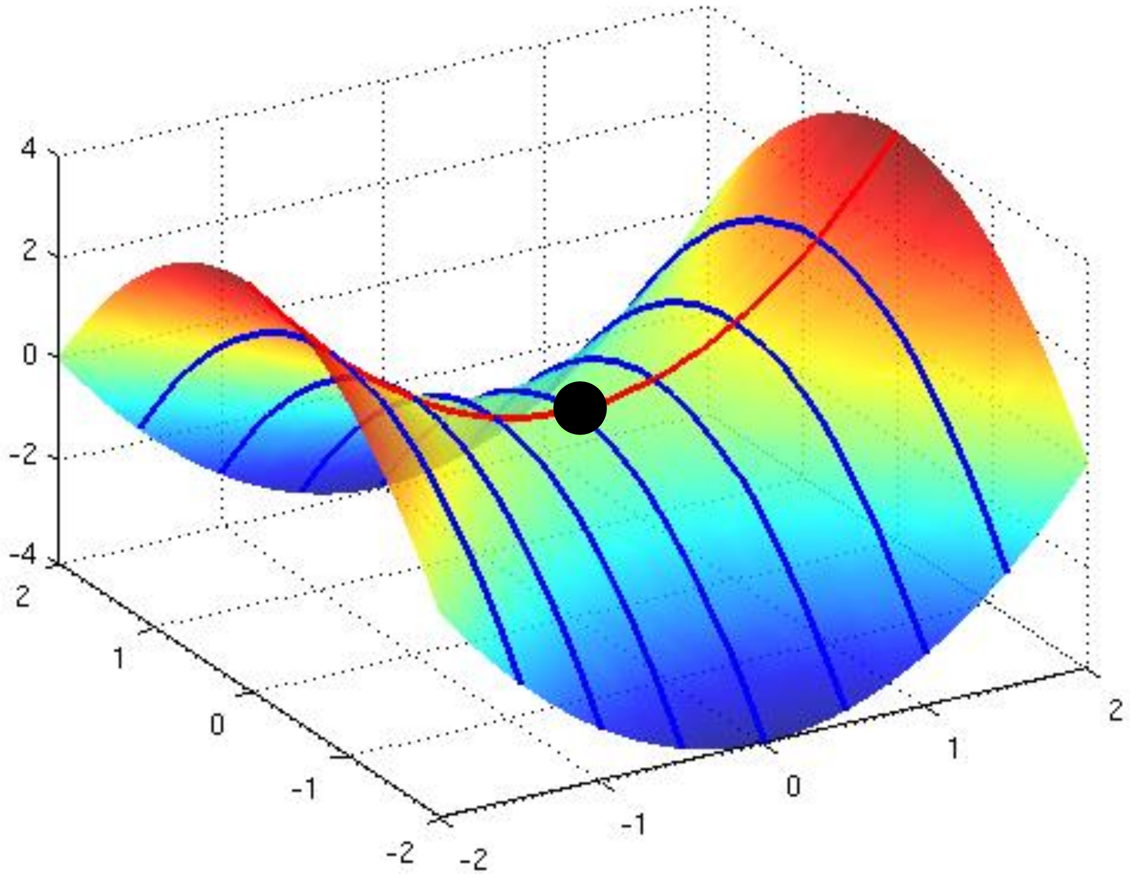
$$\frac{\partial f}{\partial y} = 4y^3 - 32 = 0$$

Solving for real critical points, we get (1,2) and (-1,2).

In higher dimensions

- Critical point can be
  - A local maxima
  - A local minima or
  - A saddle point

Saddle point



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Process for finding Maxima, minima

- We define the Hessian Matrix for a n-variable function y= $f(x_1, x_2, \ldots x_n)$, as the n by n matrix whose (i,j)-th entry is the function of the second-order partial derivative

$$\frac{\partial^2 f}{\partial x_i \partial x_j}$$

Hessian for a 2 and 3D function

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

For more variables

- For a function of more variables, one must look at the eigenvalues of the Hessian matrix at the critical point.
  - If the Hessian is positive definite at x, then f attains a local minimum at x.
  - If the Hessian is negative definite at x, then f attains a local maximum at x.
  - If the Hessian has both positive and negative eigenvalues then x is a saddle point for f.
  - Otherwise the test is inconclusive.

Problem

- Find the maxima and minima of $f(x, y, z) = x^2 + 2y^3 + 3z^2 + 4x - 6y + 9$

- Gradient
  - $\frac{\partial f}{\partial x} = 2x + 4; x = -2$
  - $\frac{\partial f}{\partial y} = 6y^2$-6; y=1 or -1
  - $\frac{\partial f}{\partial z}$=6z; z=0

Problem

- $x^2 + 2y^3 + 3z^2 + 4x - 6y + 9$
- Critical points = (-2, -1, 0) and (-2, 1, 0)
- Hessian matrix
  - $\frac{\partial^2 f}{\partial x^2} = 2; \frac{\partial^2 f}{\partial y^2}$=12y; ; $\frac{\partial^2 f}{\partial z^2}$=6
  - $\frac{\partial^2 f}{\partial x \partial y}=\frac{\partial^2 f}{\partial x \partial z}=\frac{\partial^2 f}{\partial z \partial y}=0$

Hessian matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 12y & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

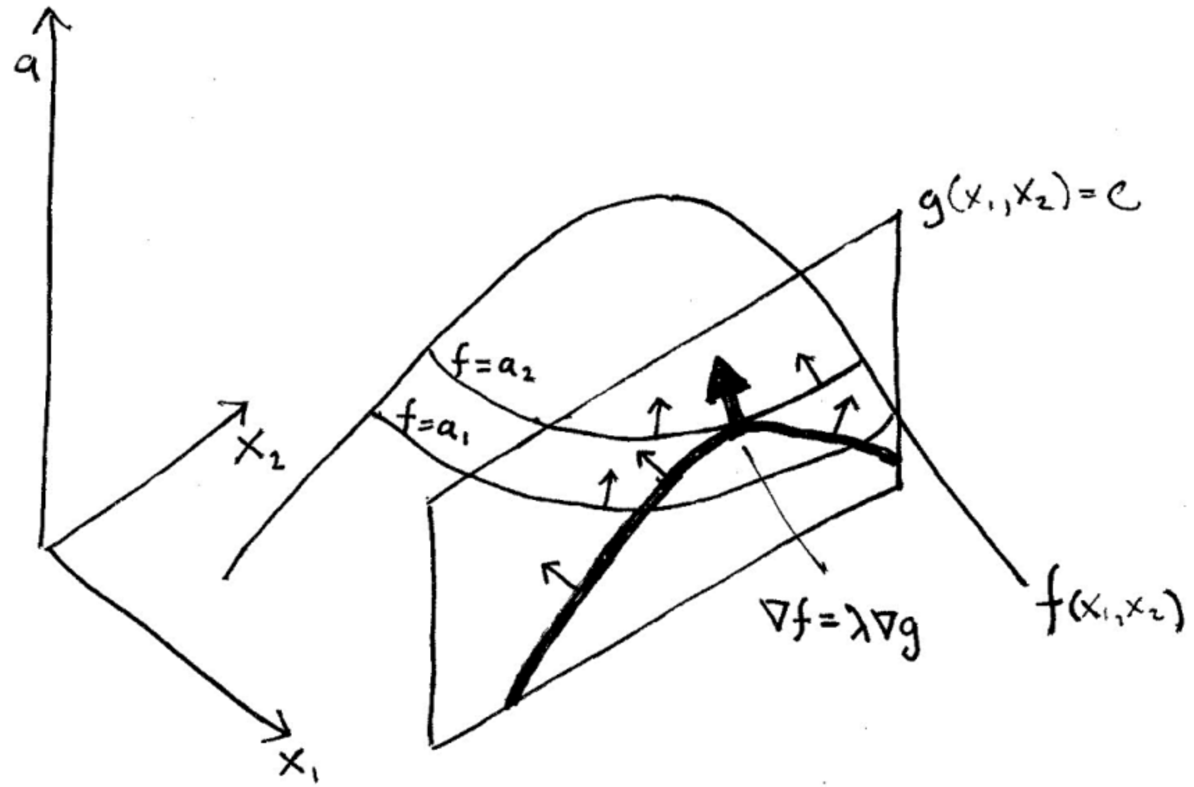
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

At (-2, -1, 0)

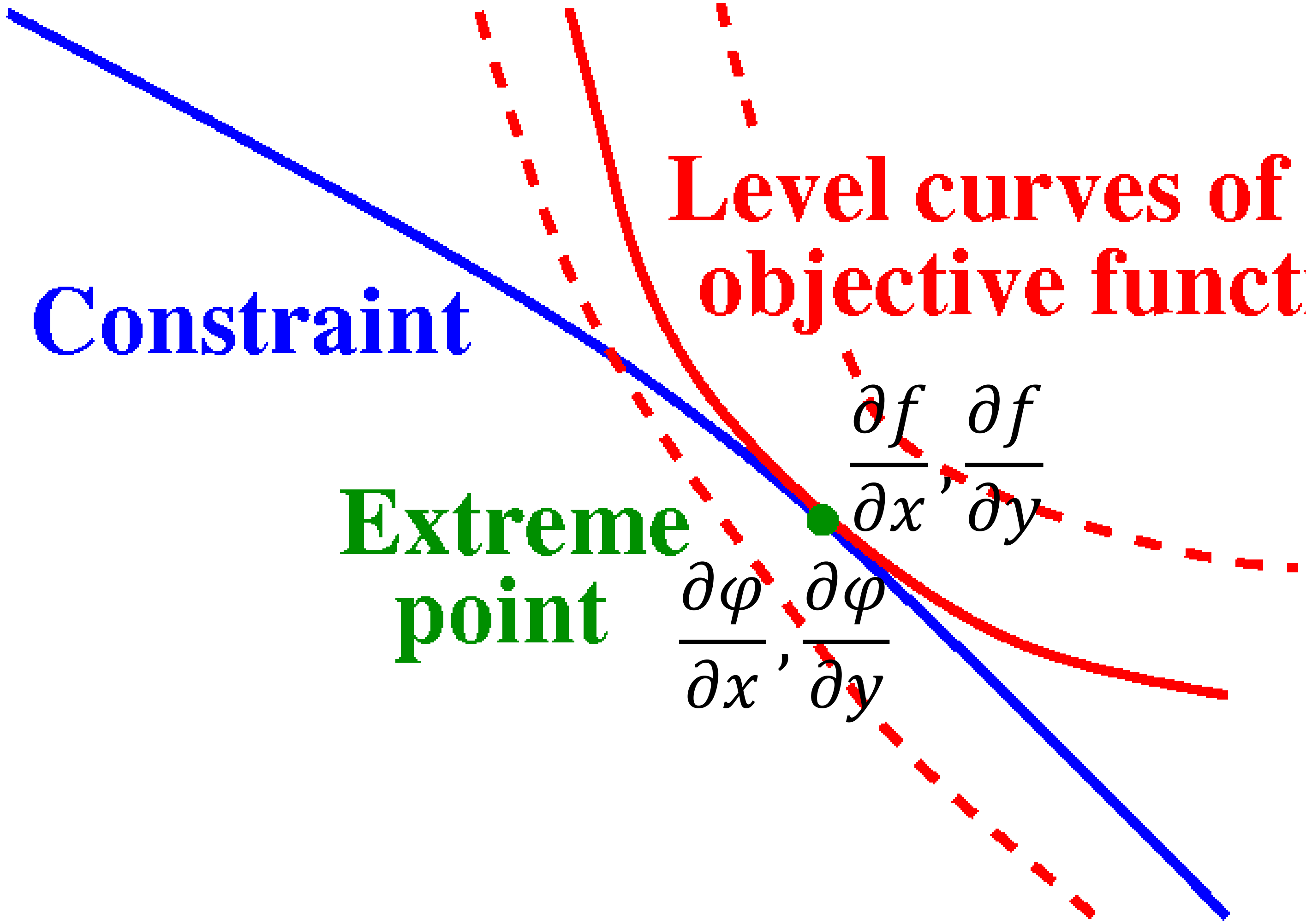
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

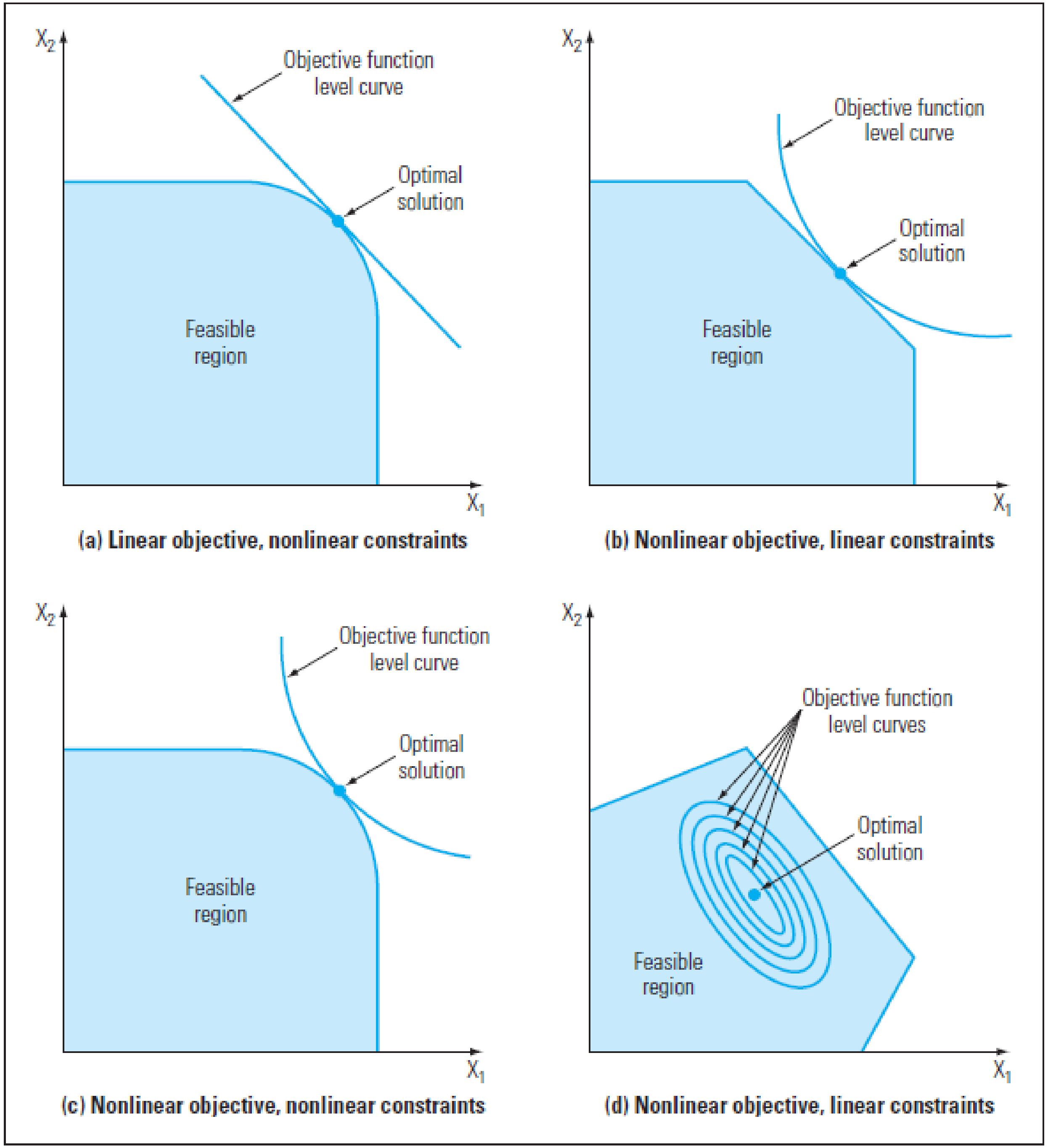
At (-2, 1, 0)

Constrained optimization



Levels curves and constraints




X2
Objective function level curve
Optimal solution
Feasible region
X1
(a) Linear objective, nonlinear constraints
X2
Objective function level curve
Optimal solution
Feasible region
X1
(b) Nonlinear objective, linear constraints
X2
Objective function level curve
Optimal solution
Feasible region
X1
(c) Nonlinear objective, nonlinear constraints
X2
Objective function level curves
Optimal solution
Feasible region
X1
(d) Nonlinear objective, linear constraints

Lagrange multipliers

$$\frac{\partial f}{\partial x} = K \frac{\partial \varphi}{\partial x} \qquad \frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0$$

$$\frac{\partial f}{\partial y} = K \frac{\partial \varphi}{\partial y} \qquad \frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0$$

λ *is a lagrange multiplier and you add one for each constraint*

Lagrange Multipliers

- $U=x^2 + y^2$
- Subject to $g= x^2+y^2+2x-2y+1 = 0$

- Lagrange equation

- $L=U+\lambda g = 0$

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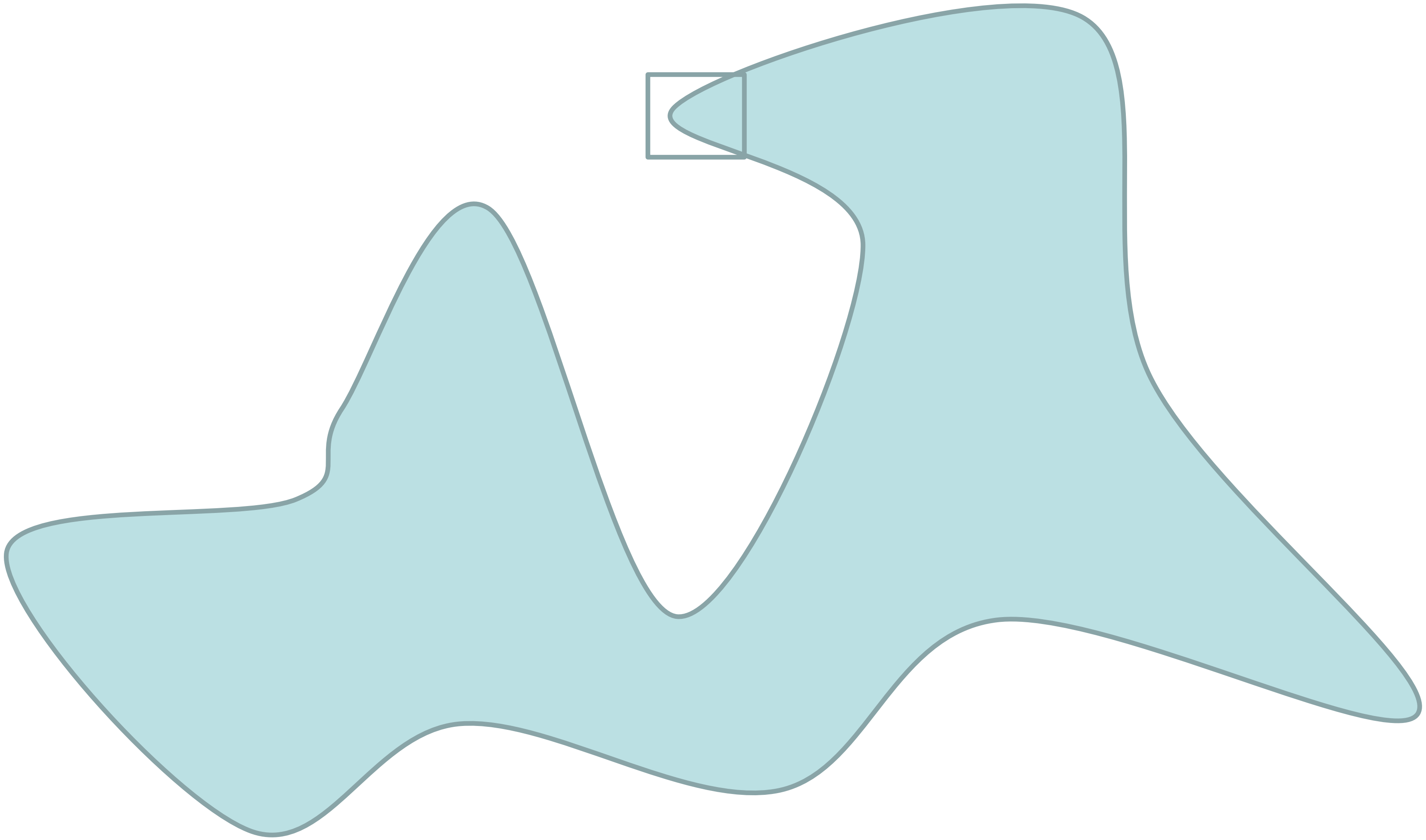
Method of Lagrange multipliers

- $\frac{\partial L}{\partial x} = 2x + \lambda(2x + 2) = 0;\ x+\lambda(x+1)=0$
- $\frac{\partial L}{\partial y} = 2y + \lambda(2y - 2) = 0;\ y+\lambda(y-1)=0$
- $\frac{\partial L}{\partial \lambda} = x^2+y^2+2x-2y+1 = 0$

- From 1 and 2, we get y=-x and substituting in 3, we get x and y values.

For extremely complex functions



Intuition behind NR

- Write the function as a quadratic polynomial f(x) in the vicinity of x0

$$= a(x - x_0)^2 + b(x - x_0) + c$$

- The first and second derivatives are

  $f'(x) = 2a(x - x_0) + b$

  $f''(x) = 2a$

- $f'(x_0) = b; f''(x_0) = 2a$

- We set the gradient to zero for critical points

$$0 = 2a(x_{crit} - x_0) + b$$

$$x_{crit} = x_0 - \frac{b}{2a} = x_0 - \frac{f'(x_0)}{f''(x_0)}$$

Gradient Descent

- Do not worry about hessian

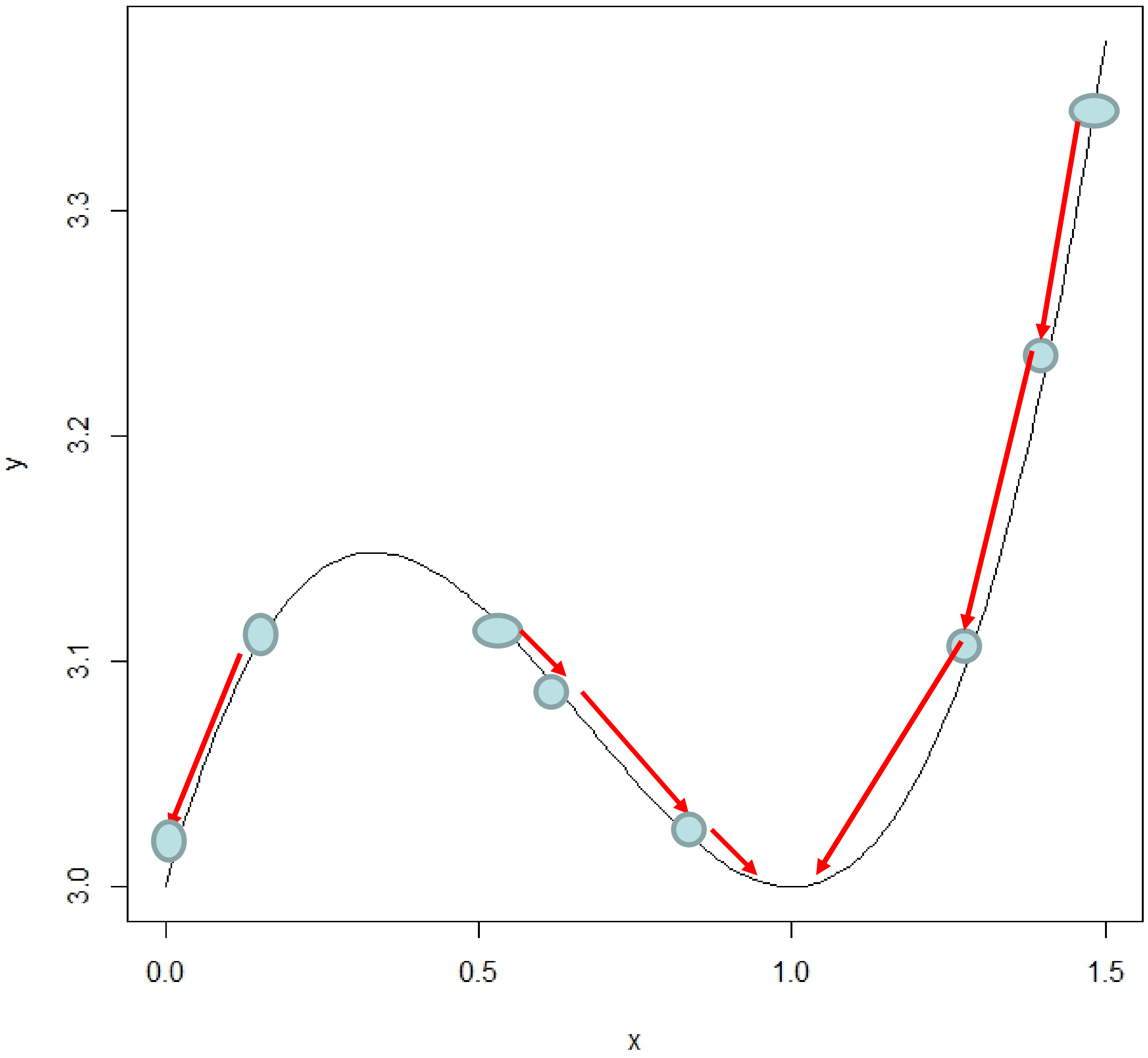
- Start randomly at a point. Move in the negative direction of gradient

$$x_{k+1} = x_k - \alpha \nabla f(x_k)$$

You need less curvature information that Newton's method.

http://www.onmyphd.com/?p=gradient.descent&ckattempt=1

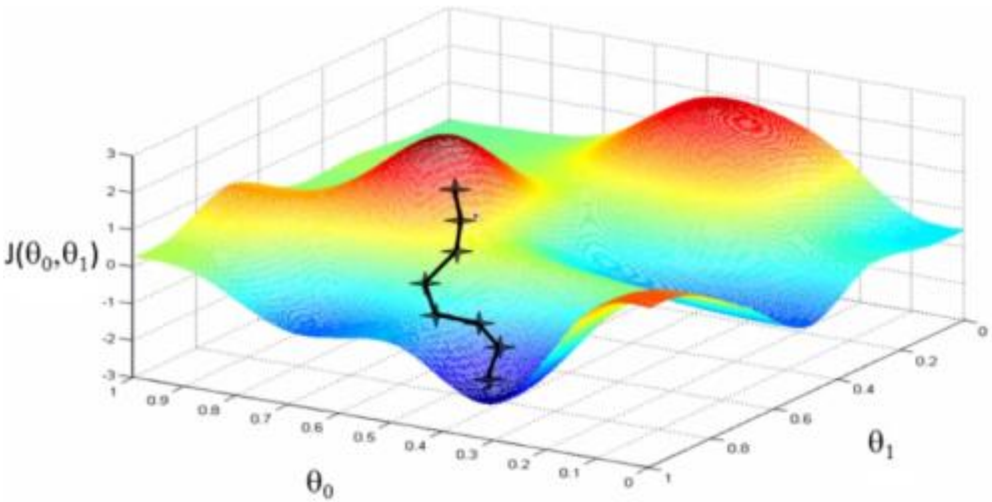
GD

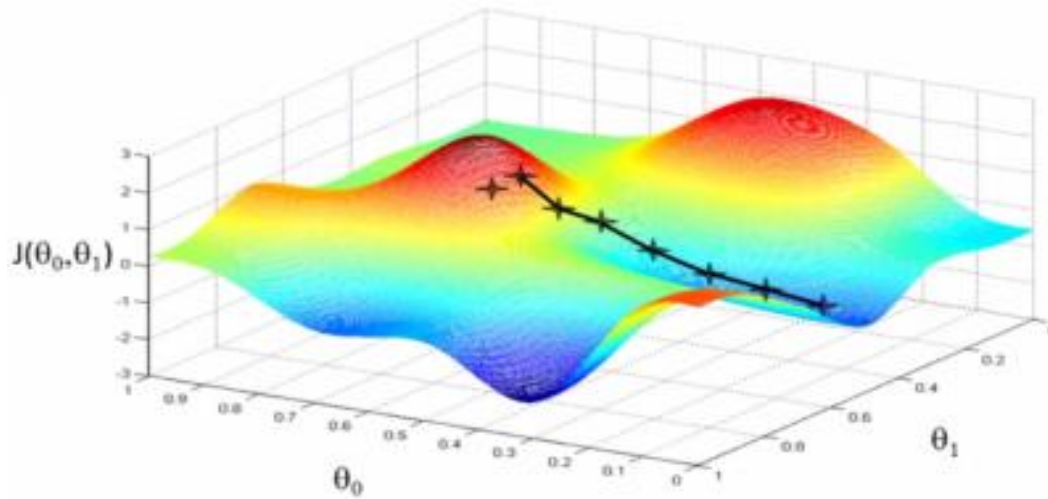


$$X_{k+1} = X_k - \propto \nabla f$$

Gradient descent also gets stuck local minima

Gradient descent: Numerical Solution





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Quadratic programming in R

$$\min(-d^T b + 1/2 b^T D b)$$

$$A^T b >= b_0$$

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