NATURAL FREQUENCY OF A BEAM

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Abstract

A beam's natural frequency was found using both theoretical textbook calculations, as well as experimentally. At the end, the results from the theoretical were compared to the experimental to see how close of a fit the theoretical and experimental models were

Introduction

When studying its mechanical vibrations, a beam must be modeled as a distributed mass, and therefore, has infinite natural frequencies. What is most interesting is the strongest of these, known as the fundamental frequency, and assuming negligible damping, would lead to catastrophic failure if vibrated to this resonant (or almost resonant) frequency. It is therefore important to find this out when using beams as structural members, or shafts as power transmission elements for that matter.

Method

In order to vibrate the beam, a motor must be attached to the center of it. This motor and assembly (which allows it to be attached to the beam) adds an additional mass to the system, however. Therefore, when the beam vibrates, what is being observed is actually the natural frequency of the entire system (of the beam and motor and assembly together), rather than the natural frequency of just the beam itself. It would be difficult to back out the natural frequency of the beam through experimentation with only this information, therefore, the motor assembly can also be fitted with additional weights. These additional weights would cause different natural frequencies of the system, and these data points can then be used to extrapolate the natural frequency of the beam alone. In the experiment, the natural frequency of the system was observed by measuring the motor RPM with a tachometer, and by having an accelerometer measuring the beam vibration, and having a Fast Fourier Transform (FFT) find the strongest frequency observed.

Theoretical Beam Natural Frequency

The natural frequency of a beam is given by[1]:

$$\omega_{n,beam} = \beta^2 \sqrt{\frac{EI}{\rho A}}$$

where E is the Young's modulus of the beam, I is the second moment of area, ρ is the density of material, A is the cross section of the beam, and where β is given from the characteristic equation: $sin(\beta L) = 0$, where L is the length of the beam. Solving for β, it is found that: $\beta = 0, \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \frac{4\pi}{L}$... The fundamental frequency is the first non-zero value of β, so $\beta = \frac{\pi}{L}$

Plugging into the equation for the natural frequency of the beam:

$$\omega_{n,beam} = \left(\frac{\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho A}} = \pi^2 \sqrt{\frac{EI}{L^4 \rho A}} \ , \ \text{where } L\rho A = m_{beam} \ , \ \omega_{n,beam} = \pi^2 \sqrt{\frac{EI}{L^3 m_{beam}}}$$

Because only one frequency is observed (the fundamental), the beam no longer needs to be modeled as a distributed mass. Since most natural frequencies are calculated as the square root of a quotient of elasticity and mass, an equivalent beam stiffness, $k_{equivalent}$, and equivalent concentrated mass of the beam, $m_{beamConcentrated}$, can be found.

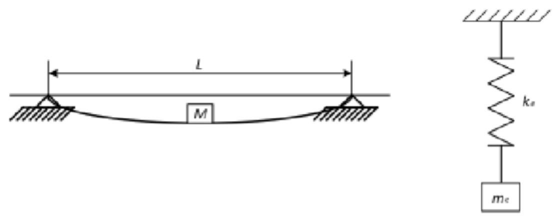


Figure 1: Turning a beam into a concentrated mass and equivalent stiffness

$$\omega_{n,beam} = \sqrt{\frac{k_{equivalent}}{m_{beamConcentrated}}} \text{, plugging in derivation from previous page:}$$

$$\omega_{n,beam} = \sqrt{\frac{k_{equivalent}}{m_{beamConcentrated}}} = \pi^2 \sqrt{\frac{EI}{L^3 m_{beam}}} = \sqrt{\frac{\pi^4 EI}{L^3 m_{beam}}}$$

and since[1] $k_{equivalent} = \frac{48EI}{L^3}$, then $m_{beamConcentrated} \approx 0.4857 m_{beam}$

Experimental Beam Natural Frequency

As stated before, in order to vibrate the beam, a motor assembly must be attached to it. This however, means that when the beam vibrates, the entire system (beam and motor assembly) vibrates with it, and so the natural frequency of the beam itself is not seen. It is difficult to back out this natural frequency only knowing the mass of the motor and beam, so to gain more data points, masses in increments of 2 lbs were added for each run, and when the beam vibrated most violently, the RPM of the motor was taken with a tachometer, and the accelerometer had a Fast Fourier Transform (FFT) done on it to record the highest frequency in Hz. Although in this lab, the beam was a rectangle, and the natural frequency can be calculated as above, some beams/shafts can be strangely shaped, or have lightning patterns that make the elasticity and concentrated mass difficult to predict. This is where the experimental method can become useful.

From theory, it is known then, that[1]:

$$\omega_{n,system} = \sqrt{\frac{k_{equivalent}}{M + m_{beamConcentrated}}}$$, where the mass of the motor assembly and additional mass M, are added to the beam

Rearranging algebraically:

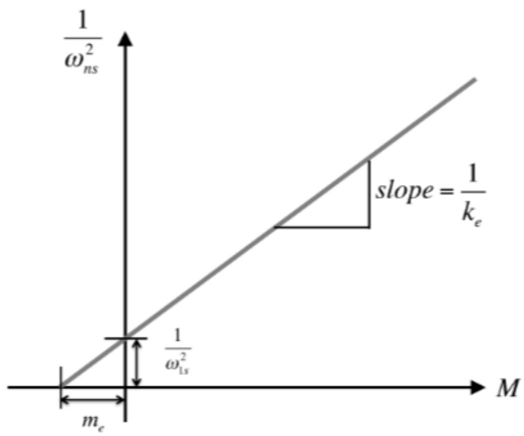
$$(\omega_{n,system})^2 = \frac{k_{equivalent}}{M + m_{beamConcentrated}} \Rightarrow \frac{1}{(\omega_{n,system})^2} = \frac{m_{beamConcentrated}}{k_{equivalent}} + \frac{M}{k_{equivalent}}$$

We can now define:

$$\frac{1}{(\omega_{n,beam})^2} = \frac{m_{beamConcentrated}}{k_{equivalent}} \Rightarrow \omega_{n,beam} = \sqrt{\frac{k_{equivalent}}{m_{beamConcentrated}}} \ \& \ \frac{1}{(\omega_{n,M})^2} = \frac{M}{k_{equivalent}} \Rightarrow \omega_{n,M} = \sqrt{\frac{k_{equivalent}}{M}}$$

Plotting $\frac{1}{(\omega_{n,system})^2}$ vs M, where when no additional mass is added, M is the mass of the motor assembly, a similar graph is expected.

$$\frac{1}{(\omega_{n,system})^2} = \frac{m_{beamConcentrated}}{k_{equivalent}} + \frac{M}{k_{equivalent}}$$



$Figure\ 2\text{:}\ Dunkerley's\ Plot$

- Since M is varying, can see that $\frac{1}{k_{equivalent}}$ is slope
- M=0 is NOT when no additional masses are present, it is when there is no motor assembly either (is extrapolated, since isn't possible to experiment like this)
  - When M=0, can see that becomes $\frac{1}{(\omega_{n,system})^2} = \frac{m_{beamConcentrated}}{k_{equivalent}}$, so is just natural frequency of beam alone = $\omega_{n,beam}$
- Extrapolating the line to negative additional mass, can see that to make natural frequency 0, additional mass must be the negative equivalent of the concentrated beam mass $M = \ - m_{beamConcentrated}$
  Through this extrapolation, can find out concentrated beam mass

Results/Discussion

The beam's dimensions were length of L = 32in, b = 1in (horizontal width) h = 0.5in (vertical height), mass of m = 4lbm 9.964ozm = 0.143517 slugs, and young's modulus of E = 30x10^6 psi. The motor assembly was placed in the center of the beam, and mad mass of M_noextra = 10lbm 4.34ozm = 0.318983 slugs, that allowed incremental masses of 2lbm = 0.062112 slugs to be placed onto it.

Theoretical Results

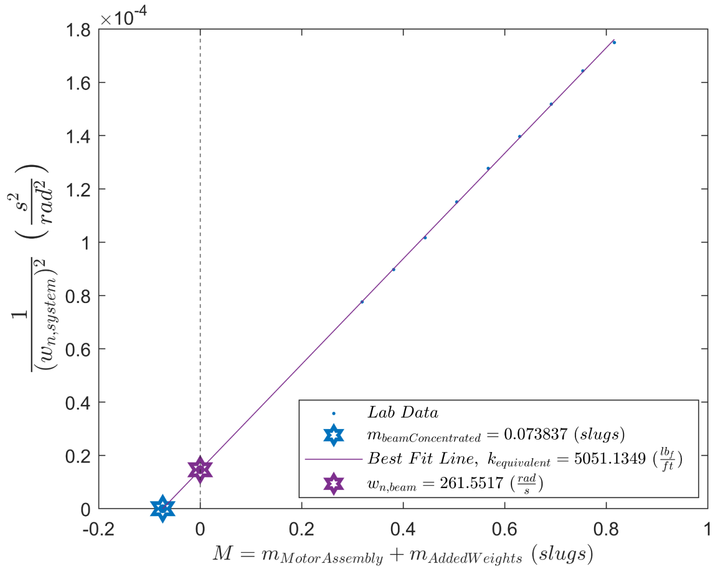
Calculating the second moment of area, $I = \frac{bh^3}{12} = \frac{1in * (0.5in)^3}{12} = 0.01042\ in^4 = 5.0234 \times 10^{-7}\ ft$. This results in a theoretical natural frequency, theoretical equivalent stiffness, and theoretical equivalent concentrated mass of:

$$\omega_{n,beam} = \sqrt{\frac{\pi^4 EI}{L^3 m_{beam}}} = 278.70106\frac{rad}{s}\ ,\ k_{equivalent} = \frac{48EI}{L^3} = 5493.16406\frac{lb_f}{ft}\ ,$$

$$m_{beamConcentrated} \approx 0.4857 m_{beam} \approx 0.06971\ slugs$$

Experimental Tachometer Results

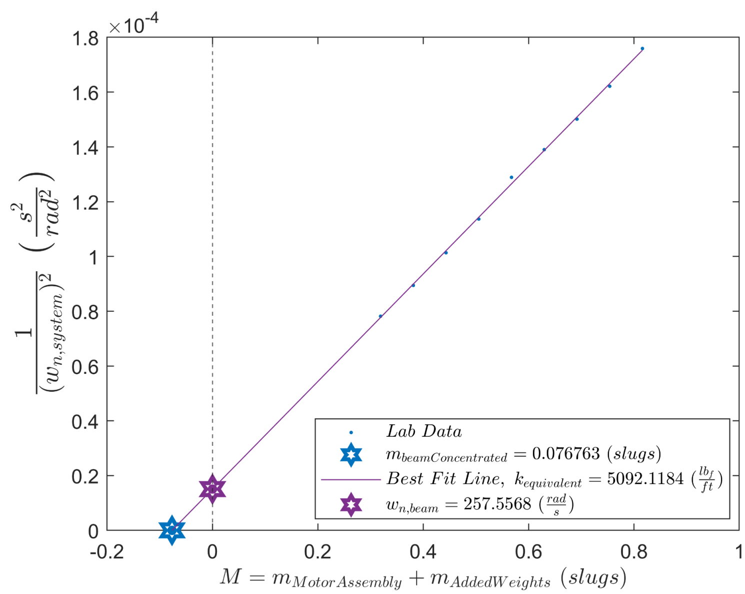
The tachometer readings were recorded in RPM. They were converted to rad/s for the data analysis. Plotting a dunkerley line, it is found that $k_{equivalent} = 5051.13494 \frac{lb_f}{ft}$, $m_{beamConcentrated} = 0.07384\ slugs$, and $\omega_{n,beam} = 261.55168 \frac{rad}{s}$.



$Figure\ 3:\ Dunkerly\ Plot\ of\ Tachometer\ Data$

Experimental FFT Results

The FFT readings were recorded in Hz. They were converted to rad/s for the data analysis. Plotting a dunkerley line, it is found that $k_{equivalent} = 5092.11836 \frac{lb_f}{ft}$, $m_{beamConcentrated} = 0.07676\ slugs$, and $\omega_{n,beam} = 257.55682 \frac{rad}{s}$.



$Figure\ 3:\ Dunkerly\ Plot\ of\ FFT\ Data$

Percent Error Compared To The Theoretical

Where $Error = 100\% \times \left|\frac{Theoretical - Experimental}{Theoretical}\right|$

	$k_{equivalent}$	$m_{beamConcentrated}$	$\omega_{n,beam}$
Tachometer	8.0469%	5.9259%	6.1533%
FFT	7.3008%	10.124%	7.5867%

Table 1: Percent Error of Experimental Results Compared To Theoretical Calculations

Conclusion

It seems that due to the low percent error between the experimental results (both tachometer and FFT), and the theoretical calculations, that both Dunkerley's method and the theoretical stiffness and equivalent mass for a simply supported uniform beam are consistent with each other. This will prove useful when attempting to find the natural frequency, equivalent stiffness, and concentrated mass of more complex shaped beams and shafts, or with non-simply supported boundary conditions, when it would be too difficult or time consuming to do so theoretically.

References

[1]Inman, D.J. and Singh, R.C. (2014) *Engineering vibration*. Boston: Pearson. pg539

Appendix

Matlab Code

```
clear;
clc;
close all;
resolution = 100;
%%Lab Data
L_beam = 32/12;%inches to ft
b_Beam = 1/12;%feet
h_Beam = 0.5/12;%feet
E_beam = 30*10^6 * 144;%psi to psf
I_beam = (b_Beam*h_Beam^3)/12; %ft^4
m_beam = (4 + 9.94/16)/32.2;%slugs
m_motorAssem = (10 + 4.34/16)/32.2;%slugs
range = 'A2:C10';
labData = table2array(readtable("Lab Data.xlsx",'Range',range));
%columns:
% Additional Weights | Tachometer Reading (of entire system)  | FFT (of entire
system)
labData(:,1) = labData(:,1)/32.2 + m_motorAssem;%converting lbm to slugs, and
adding the motor assembly mass
labData(:,2) = labData(:,2)*(pi/30);%converting rpm to rad/s
labData(:,3) = labData(:,3)*(2*pi);%converting Hz to rad/s
%%From Theoretical Calculations
%See derivation in report
m_beamConcentrated_theoretical = 0.4857*m_beam
k_equivalent_theoretical = 48*E_beam*I_beam/L_beam^3 %lbf/ft
```

```
%w_n_theoretical = sqrt(k_equivalent_theoretical/m_beamConcentrated) %rad/s
w_n_theoretical = (pi^2)*sqrt((E_beam*I_beam)/(m_beam*L_beam^3))
%%From Tachometer
[k_equivalent_tach,m_beamConcentrated_tach,w_n_tach] = ...
   dunkerleySolver([labData(:,1),labData(:,2)],resolution,"Tachometer")
%%From FFT
[k_equivalent_FFT,m_beamConcentrated_FFT,w_n_FFT] = ...
   dunkerleySolver([labData(:,1),labData(:,3)],resolution,"FFT")
%%Tabulating Results Data
results = [ [k_equivalent_theoretical;k_equivalent_tach;k_equivalent_FFT] ,
...

[m_beamConcentrated_theoretical;m_beamConcentrated_tach;m_beamConcentrated_FFT
] , ...
   [w_n_theoretical;w_n_tach;w_n_FFT]];
array2table(results, 'VariableNames', {'k_equivalent', 'm_concentratedBeam',
'w_nBeam'} , ...
   'RowNames',["Theoretical","Tachometer","FFT"])
percentError = [ [abs( (k_equivalent_tach -
k_equivalent_theoretical)/k_equivalent_theoretical) ; ...
   abs( (k_equivalent_FFT -
k_equivalent_theoretical)/k_equivalent_theoretical)] ...
   , [abs( (m_beamConcentrated_tach -
m_beamConcentrated_theoretical)/m_beamConcentrated_theoretical) ; ...
   abs( (m_beamConcentrated_FFT -
m_beamConcentrated_theoretical)/m_beamConcentrated_theoretical)] ...
   , [abs( (w_n_tach - w_n_theoretical)/w_n_theoretical) ; abs( (w_n_FFT -
w_n_theoretical)/w_n_theoretical)] ];
percentError = percentError * 100;
array2table(percentError, 'VariableNames', {'k_equivalent',
'm_concentratedBeam', 'w_nBeam'} , ...
   'RowNames',["Tachometer","FFT"])
function [k_equivalent,m_beamConcentrated,w_n] =
dunkerleySolver(dataIn,res,titleStuff)
   lineEqn = polyfit(dataIn(:,1), 1./(dataIn(:,2).^2),1);
   M = linspace(0,dataIn(end,1),res);%from 0lbs to 16lbs
   %solving for x intercept (will be negative), and is negative
m_beamConcentrated
   m_beamConcentrated = lineEqn(2)/lineEqn(1);
   M = linspace(-m_beamConcentrated,0,res) + M;
   %solving for spring constant
   k_equivalent = 1/lineEqn(1);
   %solving for natural frequncy of beam itself
   w_n = sqrt(k_equivalent/m_beamConcentrated);

   %plotting
   figure;
   pointPlot1 = plot(dataIn(:,1), 1./(dataIn(:,2).^2),"." );
   hold on;

plot(-m_beamConcentrated,0,".",'color',get(pointPlot1,'color'),'HandleVisibili
ty','off','MarkerSize',16);

plot(-m_beamConcentrated,0,"hexagram",'color',get(pointPlot1,'color'),'MarkerS
ize',12,'linewidth',2);
   hold on;
```

```
   pointPlot2 = plot(M,lineEqn(1)*M + lineEqn(2));
   hold on;

plot(0,lineEqn(2),".",'color',get(pointPlot2,'color'),'HandleVisibility','off'
,'MarkerSize',16);

plot(0,lineEqn(2),"hexagram",'color',get(pointPlot2,'color'),'MarkerSize',12,'
linewidth',2);
   hold on
   xline(0,"--")
   title(strcat("$Dunkerly \ Line \ Plot \ From \ ", titleStuff , " \ Data
$"),'Interpreter','latex')
   xlabel("$ M = m_{MotorAssembly} + m_{AddedWeights}  \ ( slugs )
$",'Interpreter','latex')
   ylabel("$ \frac{1}{(w_{n,system}) ^2} \ ( \frac{s^2}{rad^2} )
$",'Interpreter','latex',FontSize= 18)
   legendStuff = ["$Lab \ Data$" ,strcat("$ m_{beamConcentrated} = " ,
num2str(m_beamConcentrated) , " \  (slugs) $") ...
       strcat("$Best \  Fit \ Line, \ k_{equivalent} = " ,
num2str(k_equivalent) , " \ ( \frac{lb_f}{ft} ) $")...
       strcat("$ w_{n,beam} = " , num2str(w_n) , " \ ( \frac{rad}{s} ) $")];
   lgnd= legend(legendStuff,'Location','best');
   set(lgnd, 'Interpreter','latex')
   hold off;
   plot1 = strcat("Dunkerly Line Plots For", titleStuff ,  " data");
   print('-r600','-dpng',plot1);
end
```