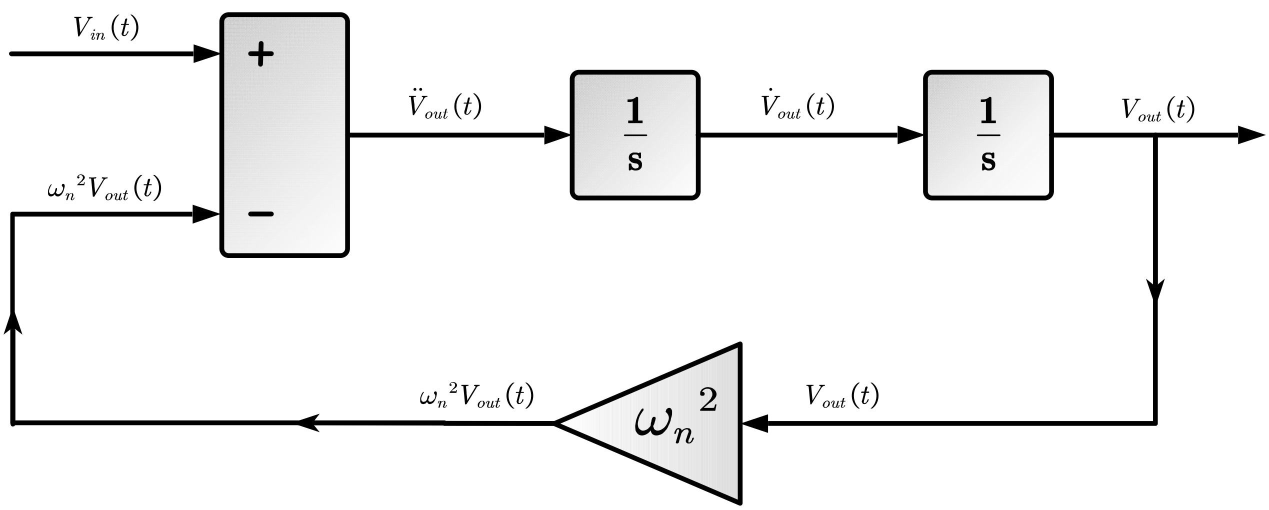
It is desired to build a pure sine wave oscillator from op-amps. Start with the equations of motion, which the block diagram can be derived from. The block diagram can later be turned into op-amps.

$$V_{in}(t) = \ddot{V}_{out}(t) + {\omega_n}^2 V_{out}(t)$$

$$\Downarrow$$

$$\ddot{V}_{out}(t) = V_{in}(t) - {\omega_n}^2 V_{out}(t)$$



A problem arises when the desired frequency ω_n , is too high.

$$\ddot{V}_{out}(t) = V_{in}(t) - {\omega_n}^2 V_{out}(t)$$

$$\Downarrow$$

$$V_{out}(t) = \frac{V_{in}(t) - \ddot{V}_{out}(t)}{{\omega_n}^2}$$

The higher the desired frequency , the larger (in magnitude) becomes, and the smaller turns out. Even at a frequency of 60Hz (≈377 rad/s), the difference in magnitude between and is already so massive. This would either mean that the op-amp that deals with $\ddot{V}_{out}(t)$ will saturate, or the output $V_{out}(t)$ would be in micro or nanovolts, too small to register. It is therefore desired to have the double derivative $\ddot{V}_{out}(t)$ and the output $V_{out}(t)$ at least within an order of magnitude within each other (ideally between 0 and 5V at most)

Let's change the input, increasing it by the scalar $K{\omega_n}^2$. This increase should balance out the large ${\omega_n}^2 V_{out}(t)$ term in the equation:

$$\ddot{V}_{out}(t) = V_{in}(t) - {\omega_n}^2 V_{out}(t) \quad \text{Will be changed to...}$$

$$\Downarrow$$

$$\ddot{V}_{out}(t) = K{\omega_n}^2 V_{in}(t) - {\omega_n}^2 V_{out}(t)$$

Meaning that $\ddot{V}_{out}(t)$ will not be so large compared to the actual output, $V_{out}(t)$

This scalar is chosen because the frequency desired is clearly accounted for in both terms $V_{in}(t)$ and $V_{out}(t)$. Now, the input voltage nearly matches the magnitude of ${\omega_n}^2 V_{out}(t)$ (due to being multiplied by ${\omega_n}^2$), while allowing for some customizability by changing the constant K.

The differential equation now becomes:

$$\ddot{V}_{out}(t) = K{\omega_n}^2 V_{in}(t) - {\omega_n}^2 V_{out}(t)$$

$$\Downarrow$$

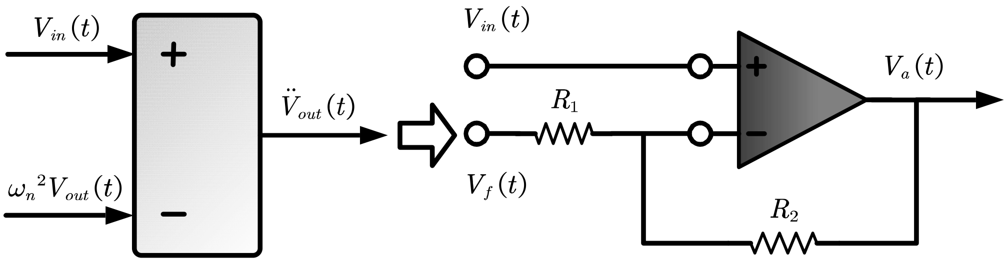
$$K{\omega_n}^2 V_{in}(t) = \ddot{V}_{out}(t) + {\omega_n}^2 V_{out}(t)$$

$$\Downarrow$$

$$V_{in}(t) = \frac{1}{K{\omega_n}^2}\ddot{V}_{out}(t) + \frac{1}{K}V_{out}(t)$$

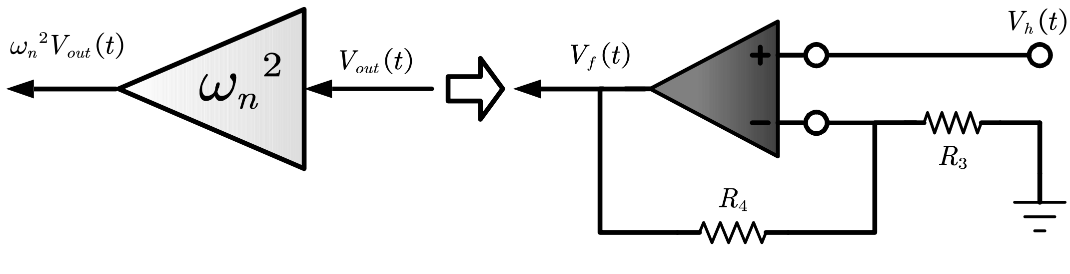
Remember, the none of the blocks on the diagram (which will be turned into op-amps) can exceed saturation. So simply multiplying outputs won't solve the problem. It is therefore difficult to see exactly where K goes on the block diagram. It might just be better to turn the existing block diagram into an op-amp circuit, and see where the math leads.

Turning summation block into differential op-amp (This setup ensures same polarity):



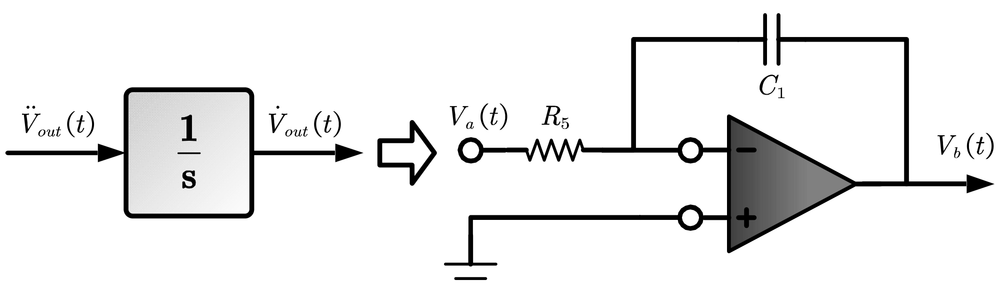
$$\frac{V_f(t)-V_-(t)}{R_1}=\frac{V_-(t)-V_a(t)}{R_2}$$
$$\Downarrow$$
$$\frac{V_f(t)-V_{in}(t)}{R_1}=\frac{V_{in}(t)-V_a(t)}{R_2}$$
$$\Downarrow$$
$$V_a(t)=\left(\frac{R_1+R_2}{R_1}\right)V_{in}(t)-\frac{R_2}{R_1}V_f(t)$$

Turning gain block into gain op-amp:

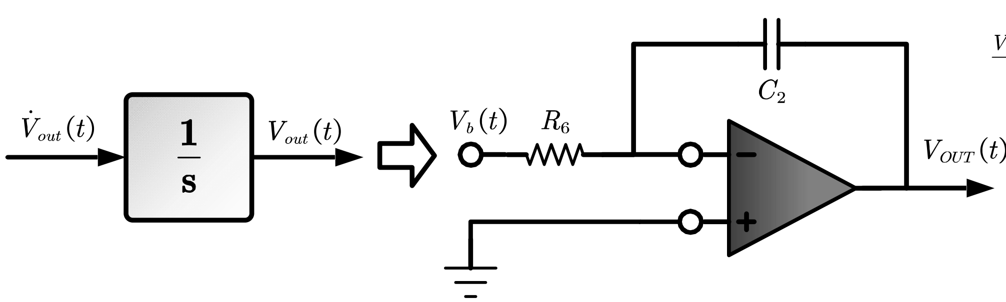


$$\frac{0-V_-(t)}{R_3}=\frac{V_-(t)-V_f(t)}{R_4}$$
$$\Downarrow$$
$$\frac{-V_h(t)}{R_3}=\frac{V_h(t)-V_f(t)}{R_4}$$
$$\Downarrow$$
$$V_f(t)=\left(\frac{R_3+R_4}{R_3}\right)V_h(t)$$

Turning integrator blocks into integrator op-amps:

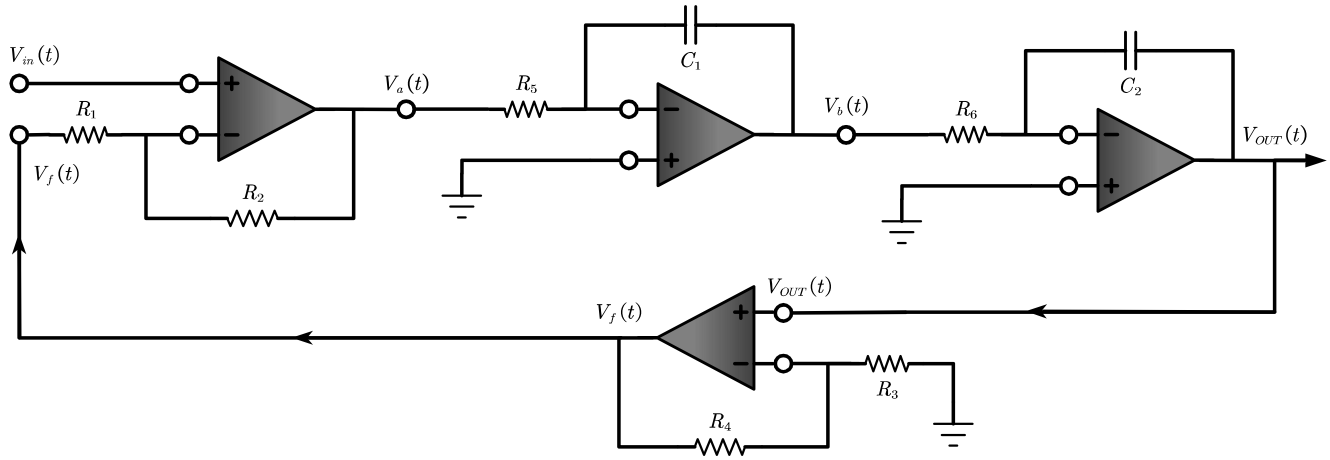


$$\frac{V_a(t)-V_-(t)}{R_5}=C_1\frac{d(V_-(t)-V_b(t))}{dt}$$
$$\Downarrow$$
$$\frac{V_a(t)-0}{R_5}=C_1\frac{d(0-V_b(t))}{dt}$$
$$\Downarrow$$
$$V_b(t)=\frac{-1}{R_5C_1}\int V_a(t)\,dt$$

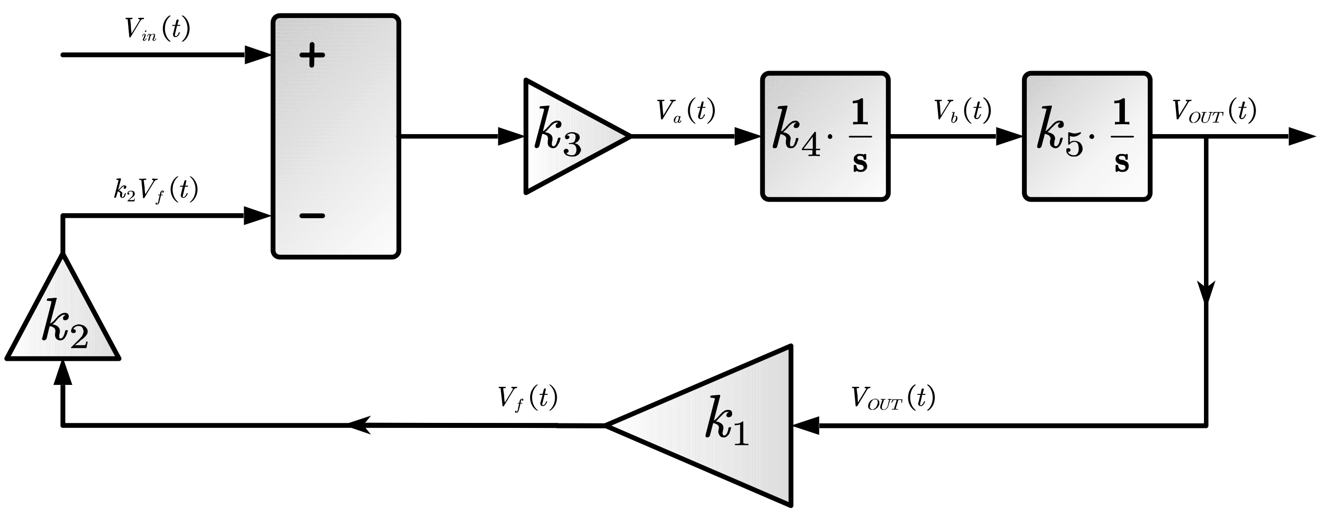


$$\frac{V_b(t)-V_-(t)}{R_6}=C_2\frac{d(V_-(t)-V_{OUT}(t))}{dt}$$
$$\Downarrow$$
$$\frac{V_b(t)-0}{R_6}=C_2\frac{d(0-V_{OUT}(t))}{dt}$$
$$\Downarrow$$
$$V_{OUT}(t)=\frac{-1}{R_6C_2}\int V_b(t)\,dt$$

Note, that the actual voltage outputs of each op-amp are labeled differently from the block diagram's outputs. This is because only a certain combination of resistors and capacitors will properly get those values, so for now, substitute voltages are put in place of the block diagram outputs. This includes the modified output $V_{OUT}(t)$, which is some scaled version of $V_{out}(t)$.



Putting in place these substituted voltages in a new block diagram:



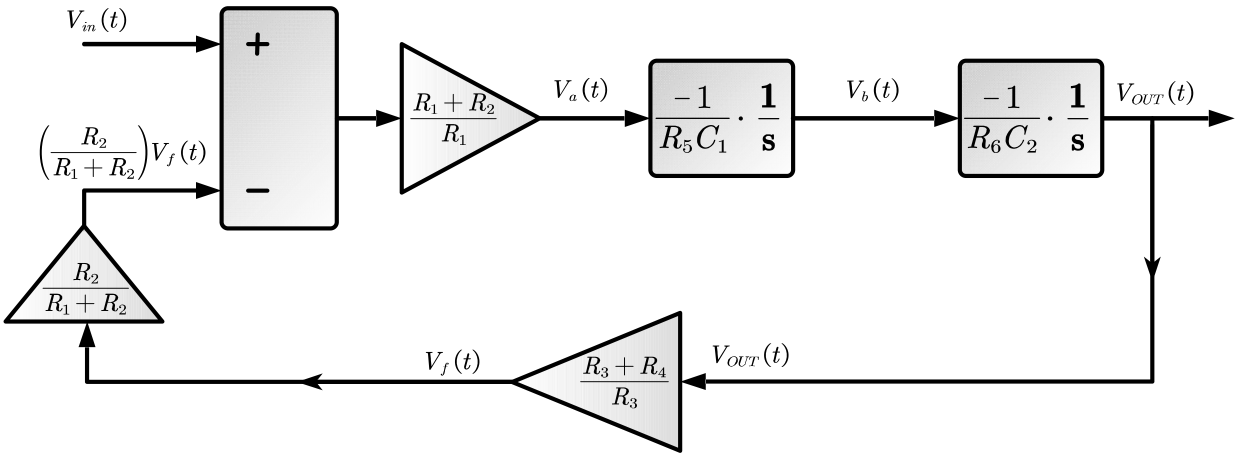
The equations on the previous page will be used to help plug in for the constants. Before that, a quick detour:

From $V_a(t) = \left(\frac{R_1+R_2}{R_1}\right)V_{in}(t) - \frac{R_2}{R_1}V_f(t)$, want to leave $V_{in}(t)$ with a multiplier of 1, so that the user does not have to put a gain on the input. Factoring out $\left(\frac{R_1+R_2}{R_1}\right)$ results in:

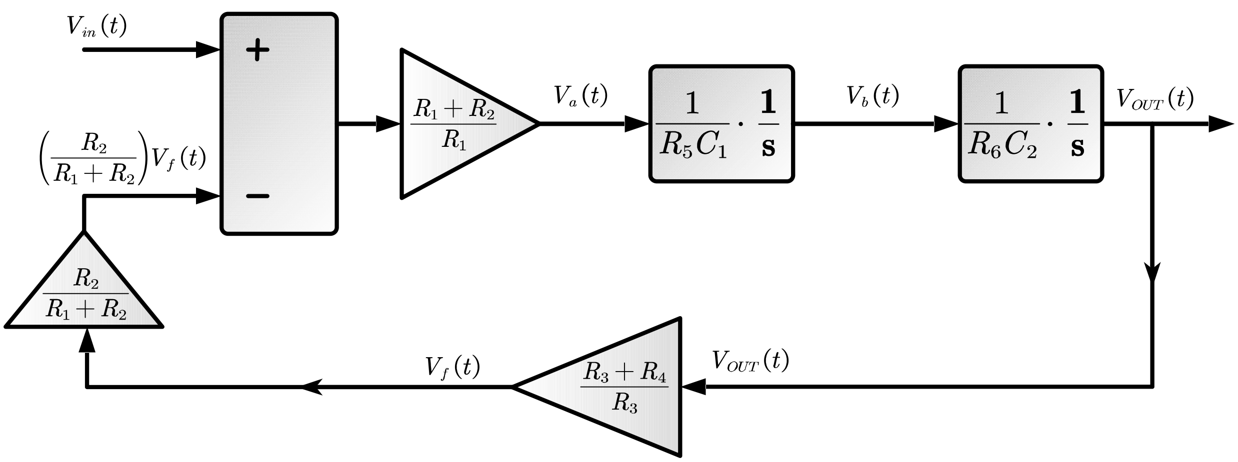
$$V_a(t) = \left(\frac{R_1+R_2}{R_1}\right)\left(V_{in}(t) - \left(\frac{R_1}{R_1+R_2}\right)\left(\frac{R_2}{R_1}\right)V_f(t)\right)$$

$$V_a(t) = \left(\frac{R_1+R_2}{R_1}\right)\left(V_{in}(t) - \left(\frac{R_2}{R_1+R_2}\right)V_f(t)\right)$$

It can be seen that k_3 the gain for the entire sum is $\left(\frac{R_1+R_2}{R_1}\right)$, and k_2, the gain for $V_f(t)$, is $\left(\frac{R_2}{R_1+R_2}\right)$. The other constants have been solved for two pages ago.



Note, that the integrators output negative. However, because two integrators are in series, then the -1s cascade, and cancel out. Therefore, the original math is correct when it didn't account for negatives. To make this easier, it is best to get rid of the -1s in the block diagram.



Although this rearranged block diagram isn't very useful, it does show that for example, the gain block below, ${\omega_n}^2$ isn't just comprised of a combination of R_3 and R_4 but is influenced by all other components in the circuit, and it isn't exactly clear.

To arrive at the final wave form produced, and to find how the natural frequency is generated, the calculation is conducted on the next page.

From:

- $V_a(t) = \left(\frac{R_1+R_2}{R_1}\right) V_{in}(t) - \frac{R_2}{R_1} V_f(t)$
- $V_b(t) = \frac{-1}{R_5 C_1} \int V_a(t)\, dt$
- $V_f(t) = \left(\frac{R_3+R_4}{R_3}\right) V_{OUT}(t)$
- $V_{OUT}(t) = \frac{-1}{R_6 C_2} \int V_b(t)\, dt$

Laplacing all equations, and assuming 0 initial conditions...

- $\mathcal{L}(V_a(t)) = \mathcal{L}\left(\left(\frac{R_1+R_2}{R_1}\right) V_{in}(t) - \frac{R_2}{R_1} V_f(t)\right) \Rightarrow V_a(s) = \left(\frac{R_1+R_2}{R_1}\right) V_{in}(s) - \frac{R_2}{R_1} V_f(s)$
- $\mathcal{L}(V_f(t)) = \mathcal{L}\left(\left(\frac{R_3+R_4}{R_3}\right) V_{OUT}(t)\right) \Rightarrow V_f(s) = \left(\frac{R_3+R_4}{R_3}\right) V_{OUT}(s)$
- $\mathcal{L}(V_b(t)) = \mathcal{L}\left(\frac{-1}{R_5 C_1} \int V_a(t)\, dt\right) \Rightarrow V_b(s) = \frac{-1}{R_5 C_1 s} V_a(s)$
- $\mathcal{L}(V_{OUT}(t)) = \mathcal{L}\left(\frac{-1}{R_6 C_2} \int V_b(t)\, dt\right) \Rightarrow V_{OUT}(s) = \frac{-1}{R_6 C_2 s} V_b(s)$

Solving for the transfer function $\frac{V_{OUT}(t)}{V_{in}(t)}$

- $V_a(s) = \left(\frac{R_1+R_2}{R_1}\right) V_{in}(s) - \frac{R_2}{R_1} V_f(s)$
- $V_f(s) = \left(\frac{R_3+R_4}{R_3}\right) V_{OUT}(s)$

$$\Rightarrow \; V_a(s) = \left(\frac{R_1+R_2}{R_1}\right) V_{in}(s) - \left(\frac{R_2R_3+R_2R_4}{R_1R_3}\right) V_{OUT}(s)$$

- $V_a(s) = \left(\frac{R_1+R_2}{R_1}\right) V_{in}(s) - \left(\frac{R_2R_3+R_2R_4}{R_1R_3}\right) V_{OUT}(s)$
- $V_b(s) = \frac{-1}{R_5 C_1 s} V_a(s)$

$$\Rightarrow \; V_b(s) = -\left(\frac{R_1+R_2}{R_1R_5C_1s}\right) V_{in}(s) + \left(\frac{R_2R_3+R_2R_4}{R_1R_3R_5C_1s}\right) V_{OUT}(s)$$

- $V_b(s) = -\left(\frac{R_1+R_2}{R_1R_5C_1s}\right) V_{in}(s) + \left(\frac{R_2R_3+R_2R_4}{R_1R_3R_5C_1s}\right) V_{OUT}(s)$
- $V_{OUT}(s) = \frac{-1}{R_6 C_2 s} V_b(s)$

$$\Rightarrow \; V_{OUT}(s) = \left(\frac{R_1+R_2}{R_1R_5R_6C_1C_2s^2}\right) V_{in}(s) - \left(\frac{R_2R_3+R_2R_4}{R_1R_3R_5R_6C_1C_2s^2}\right) V_{OUT}(s)$$

$$\Downarrow$$

$$V_{OUT}(s) + \left(\frac{R_2R_3+R_2R_4}{R_1R_3R_5R_6C_1C_2s^2}\right) V_{OUT}(s) = \left(\frac{R_1+R_2}{R_1R_5R_6C_1C_2s^2}\right) V_{in}(s)$$

$$\left(\frac{R_1R_3R_5R_6C_1C_2s^2 + R_2R_3 + R_2R_4}{R_1R_3R_5R_6C_1C_2s^2}\right) V_{OUT}(s) = \left(\frac{R_1+R_2}{R_1R_5R_6C_1C_2s^2}\right) V_{in}(s)$$

Continued on next page...

Continued...

$$\frac{V_{OUT}\left(t\right)}{V_{in}\left(t\right)}=\left(\frac{R_1 R_3 R_5 R_6 C_1 C_2 s^2}{C_1\,C_2\,R_1\,R_3\,R_5\,R_6\,s^2+R_2\,R_3+R_2\,R_4}\right)\left(\frac{R_1+R_2}{R_1 R_5 R_6 C_1 C_2 s^2}\right)$$

Now, have the transfer function:

$$\frac{V_{OUT}\left(t\right)}{V_{in}\left(t\right)}=\frac{R_1\,R_3+R_2\,R_3}{C_1\,C_2\,R_1\,R_3\,R_5\,R_6\,s^2+R_2\,R_3+R_2\,R_4}$$

Now solving for the response, and sizing components.
$V_{in}(t)$ is a step input, $V_{in}\left(t\right)=V_{set}\cdot 1(t)$, so $V_{in}(s)=V_{set}\cdot\frac{1}{s}$

$$V_{OUT}\left(s\right)=V_{set}\cdot\frac{R_1\,R_3+R_2\,R_3}{s(C_1\,C_2\,R_1\,R_3\,R_5\,R_6\,s^2+R_2\,R_3+R_2\,R_4)}$$

$$V_{OUT}\left(s\right)=V_{set}\left(\frac{R_1\,R_3+R_2\,R_3}{s\,(R_2\,R_3+R_2\,R_4)}-\frac{C_1\,C_2\,R_1\,R_3\,R_5\,R_6\,(R_1\,R_3+R_2\,R_3)s}{(R_2\,R_3+R_2\,R_4)\,(C_1\,C_2\,R_1\,R_3\,R_5\,R_6\,s^2+R_2\,R_3+R_2\,R_4)}\right)$$

$$V_{OUT}\left(s\right)=V_{set}\left(\frac{R_1\,R_3+R_2\,R_3}{R_2\,R_3+R_2\,R_4}\right)\frac{1}{s}-V_{set}\left(\frac{R_1\,R_3+R_2\,R_3}{R_2\,R_3+R_2\,R_4}\right)\frac{C_1\,C_2\,R_1\,R_3\,R_5\,R_6\,s}{C_1\,C_2\,R_1\,R_3\,R_5\,R_6\,s^2+R_2\,R_3+R_2\,R_4}$$

$$V_{OUT}\left(s\right)=V_{set}\left(\frac{R_1\,R_3+R_2\,R_3}{R_2\,R_3+R_2\,R_4}\right)\frac{1}{s}-V_{set}\left(\frac{R_1\,R_3+R_2\,R_3}{C_1\,C_2\,R_1\,R_3\,R_5\,R_6(R_2\,R_3+R_2\,R_4)}\right)\frac{C_1\,C_2\,R_1\,R_3\,R_5\,R_6\,s}{s^2+\left(\frac{R_2\,R_3+R_2\,R_4}{C_1\,C_2\,R_1\,R_3\,R_5\,R_6}\right)}$$

$$V_{OUT}\left(s\right)=V_{set}\left(\frac{R_1\,R_3+R_2\,R_3}{R_2\,R_3+R_2\,R_4}\right)\frac{1}{s}-V_{set}\left(\frac{C_1\,C_2\,R_1\,R_3\,R_5\,R_6(R_1\,R_3+R_2\,R_3)}{C_1\,C_2\,R_1\,R_3\,R_5\,R_6(R_2\,R_3+R_2\,R_4)}\right)\frac{s}{s^2+\left(\frac{R_2\,R_3+R_2\,R_4}{C_1\,C_2\,R_1\,R_3\,R_5\,R_6}\right)}$$

$$V_{OUT}\left(s\right)=V_{set}\left(\frac{R_1\,R_3+R_2\,R_3}{R_2\,R_3+R_2\,R_4}\right)\frac{1}{s}-V_{set}\left(\frac{R_1\,R_3+R_2\,R_3}{R_2\,R_3+R_2\,R_4}\right)\frac{s}{s^2+\left(\frac{R_2\,R_3+R_2\,R_4}{C_1\,C_2\,R_1\,R_3\,R_5\,R_6}\right)}$$

Can see that $\omega_n^2=\frac{R_2\,R_3+R_2\,R_4}{C_1\,C_2\,R_1\,R_3\,R_5\,R_6}$, So then, $\omega_n=\sqrt{\frac{R_2\,R_3+R_2\,R_4}{C_1\,C_2\,R_1\,R_3\,R_5\,R_6}}$

$$V_{OUT}\left(s\right)=\left(V_{set}\cdot\frac{R_1\,R_3+R_2\,R_3}{R_2\,R_3+R_2\,R_4}\right)\frac{1}{s}-\left(V_{set}\cdot\frac{R_1\,R_3+R_2\,R_3}{R_2\,R_3+R_2\,R_4}\right)\frac{s}{s^2+\omega_n^2}$$

Set $K=\frac{R_1\,R_3+R_2\,R_3}{R_2\,R_3+R_2\,R_4}$

$$V_{OUT}\left(s\right)=\left(KV_{set}\right)\frac{1}{s}-\left(KV_{set}\right)\frac{s}{s^2+\omega_n^2}$$

$$\mathcal{L}^{-1}\left(V_{OUT}\left(s\right)\right)=\ \mathcal{L}^{-1}\left(\left(KV_{set}\right)\frac{1}{s}-\left(KV_{set}\right)\frac{s}{s^2+\omega_n^2}\right)$$

$$V_{OUT}(t)=KV_{set}-KV_{set}cos(\omega_n t)$$

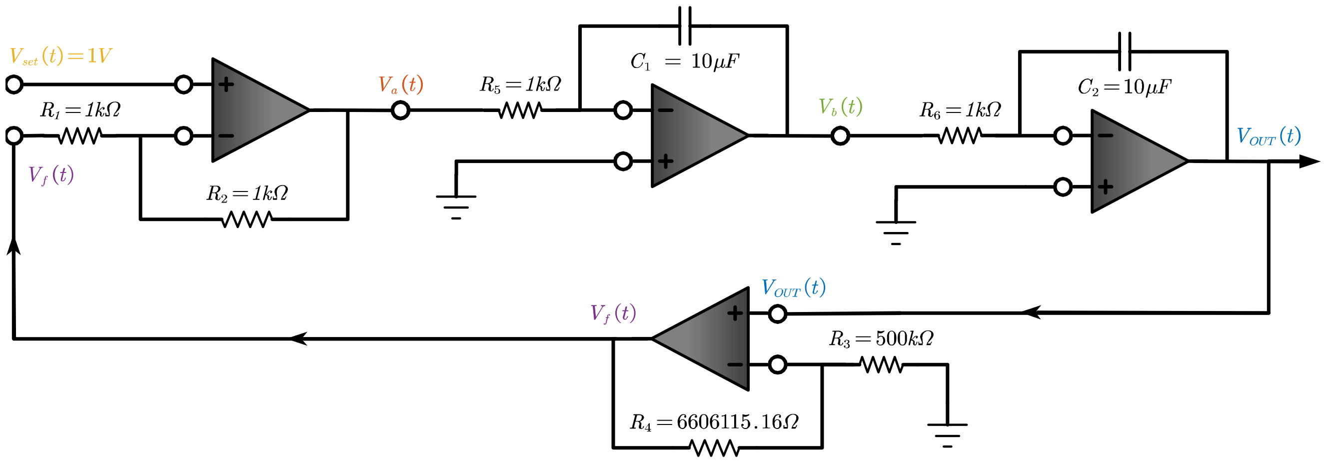
Expanding out...

$$V_{OUT}(t)=V_{set}\left(\frac{R_1\,R_3+R_2\,R_3}{R_2\,R_3+R_2\,R_4}\right)-V_{set}\left(\frac{R_1\,R_3+R_2\,R_3}{R_2\,R_3+R_2\,R_4}\right)cos\left(\left(\sqrt{\frac{R_2\,R_3+R_2\,R_4}{C_1\,C_2\,R_1\,R_3\,R_5\,R_6}}\right)t\right)$$

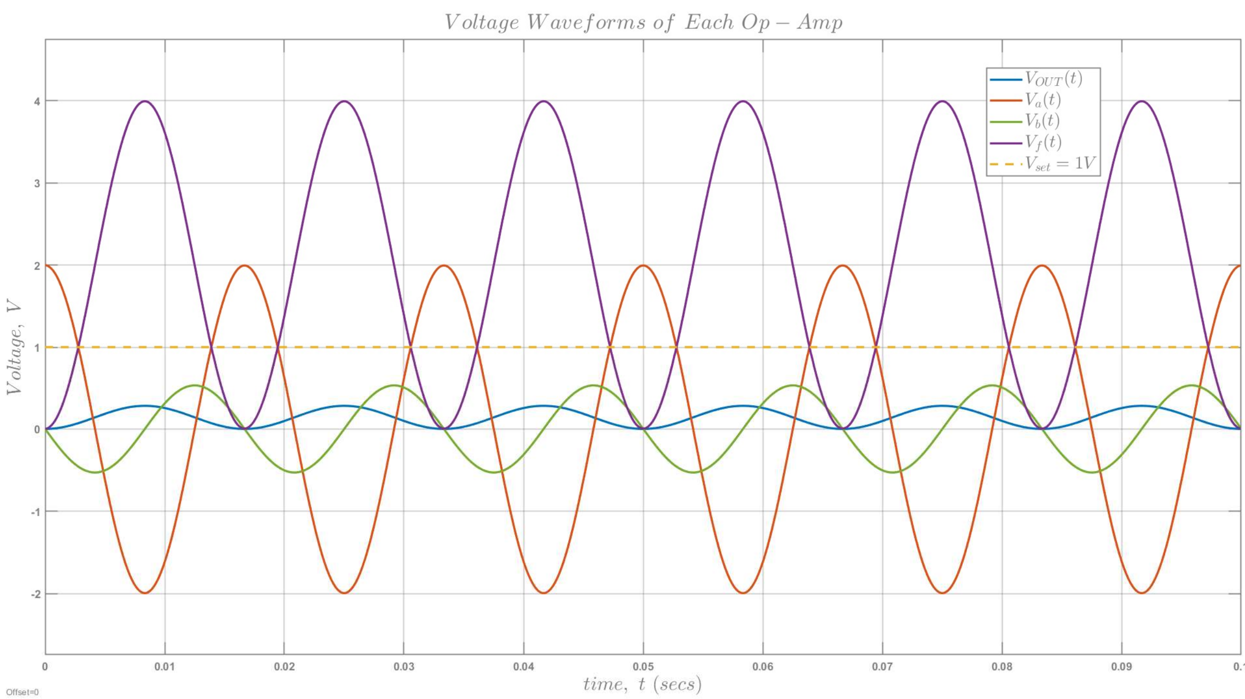
For $\omega_n\ =\ 60Hz\approx\ 376.991118\frac{rad}{s^2}$,

$R_1\ =\ 1k\Omega$, $R_2\ =\ 1k\Omega$, $R_3\ =\ 500K\Omega$, $R_4\ =\ 6606115.16\Omega$, $R_5\ =\ 1k\Omega$, $R_6\ =\ 1k\Omega$, $C_1\ =\ 10\mu F$, $C_2\ =\ 10\mu F$

To produce a 60Hz sine wave with a 1V step input, the circuit becomes:



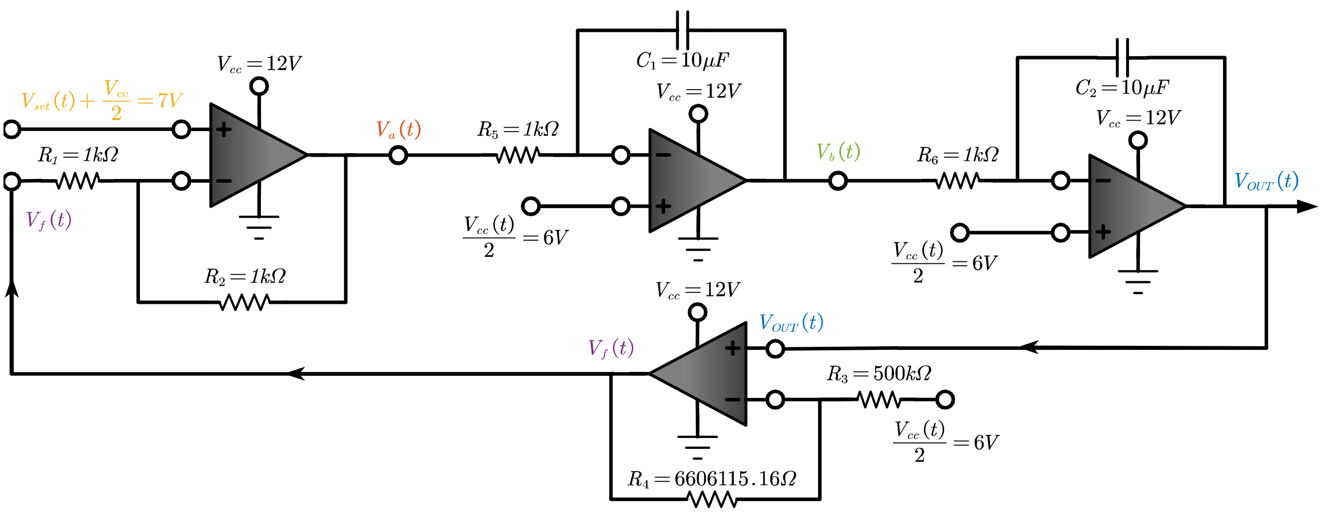
The voltages graphed:



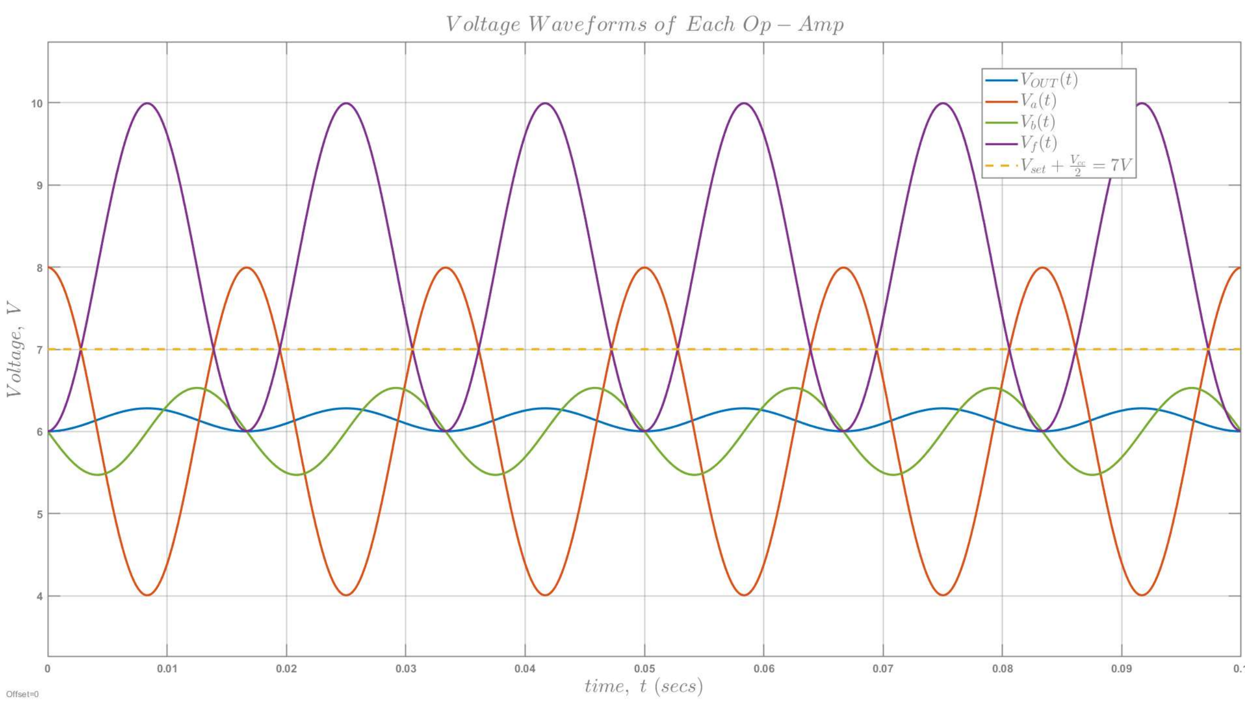
$V_{OUT}(t)$, and all its derivatives are plotted. It can be seen how much larger each successive derivative gets. It can also be seen that the frequency scaled $V_{OUT}(t)$, or $V_f(t)$, has the largest peak of all the voltages. $V_{OUT}(t)$ is the smallest, measuring in hundreds of mV, but still large enough to register on a meter or for other ICs.

If the op-amp circuit's integrators didn't have R and C values that scaled them down by small constants, then only R_3 and R_4 would make up the $V_f(t)$, then the output of $V_f(t)$ would spike past saturation.

Because there is no negative rail, each op-amp's negative side would be connected to ground. In order to keep the math the same, then each previous ground in the Op-Amp circuit must be half way between the supplied Positive voltage, V_{cc} and ground, meaning that they all become $\frac{V_{cc}}{2}$ This also means that V_{set} must be boosted up by the same voltage. Because it is such a common voltage, V_{cc} will be 12V.

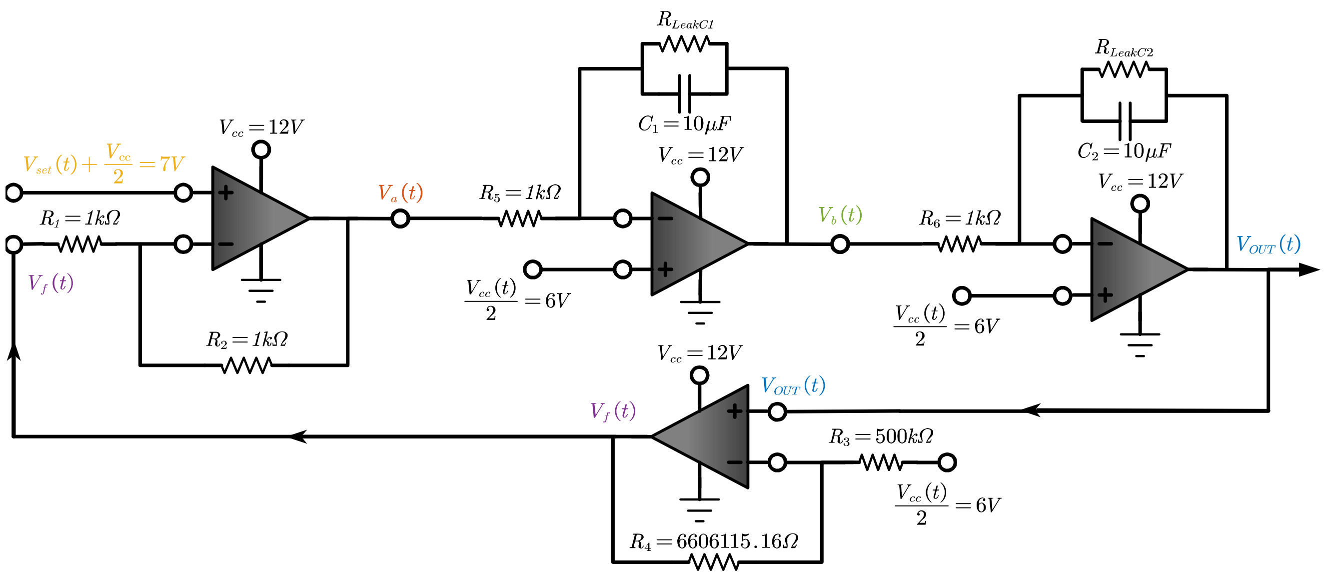


The resulting waveforms are then just offset by $\frac{V_{cc}}{2}$. or 6V.

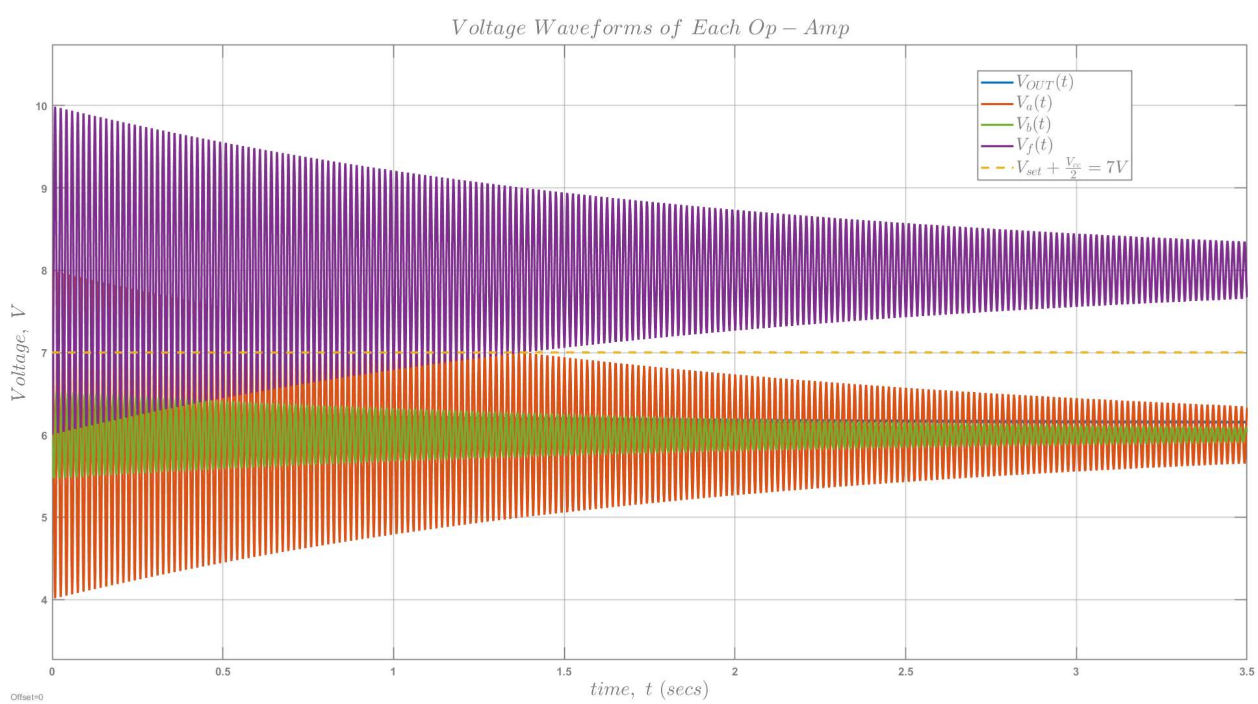


It can be seen that even $V_f(t)$ stays within the bounds of +12V and 0V, and there are no negative voltages. This means that no op-amp ever hits saturation at either end.

Of course, components are not ideal, especially the capacitors. The capacitors will come with leakage resistance, which while large (in the MΩ or GΩ range), will mean that over time, the waveforms will exponentially decay, only being useful as a sinewave generator for short durations. Long term uses, as say, an inverter, are not possible, since after days or weeks, the output will still have the same frequency, but the amplitude of the power supplied will drastically decrease.



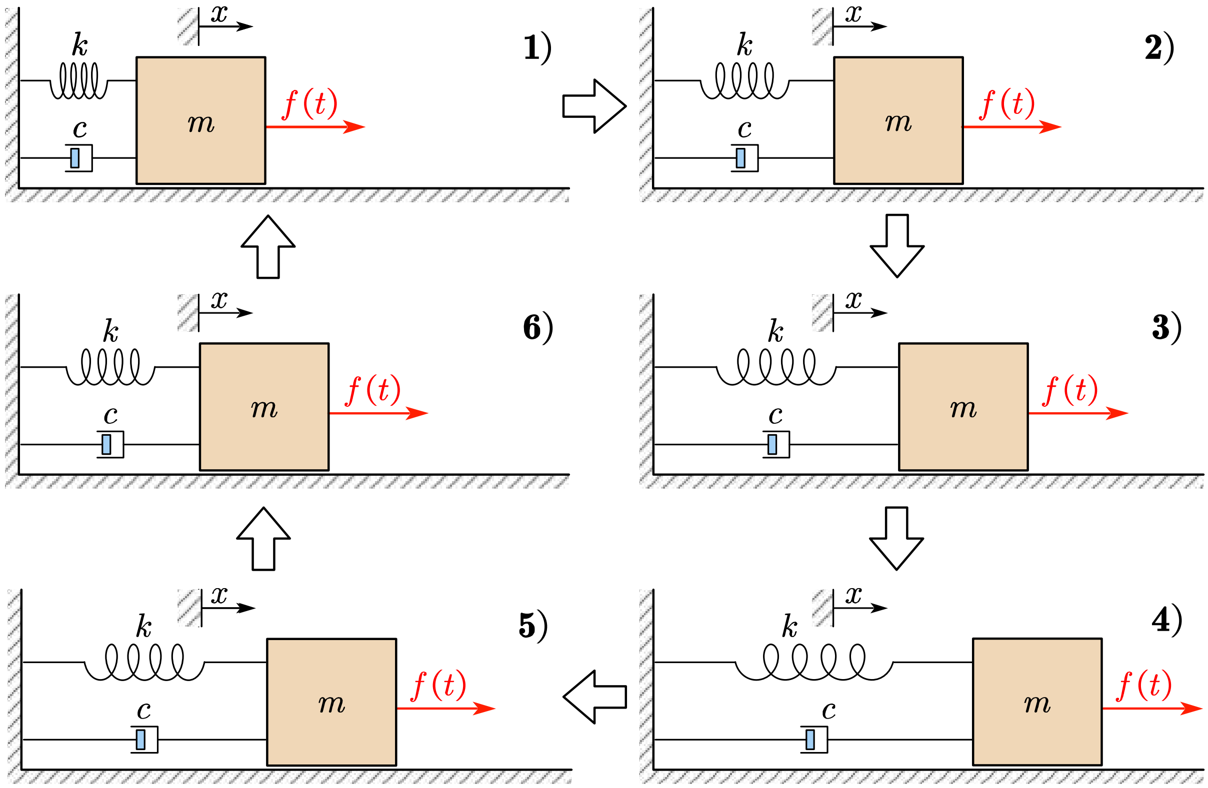
Below is an exagurated plot of the exponential decay when leakage resistors are taken into account.



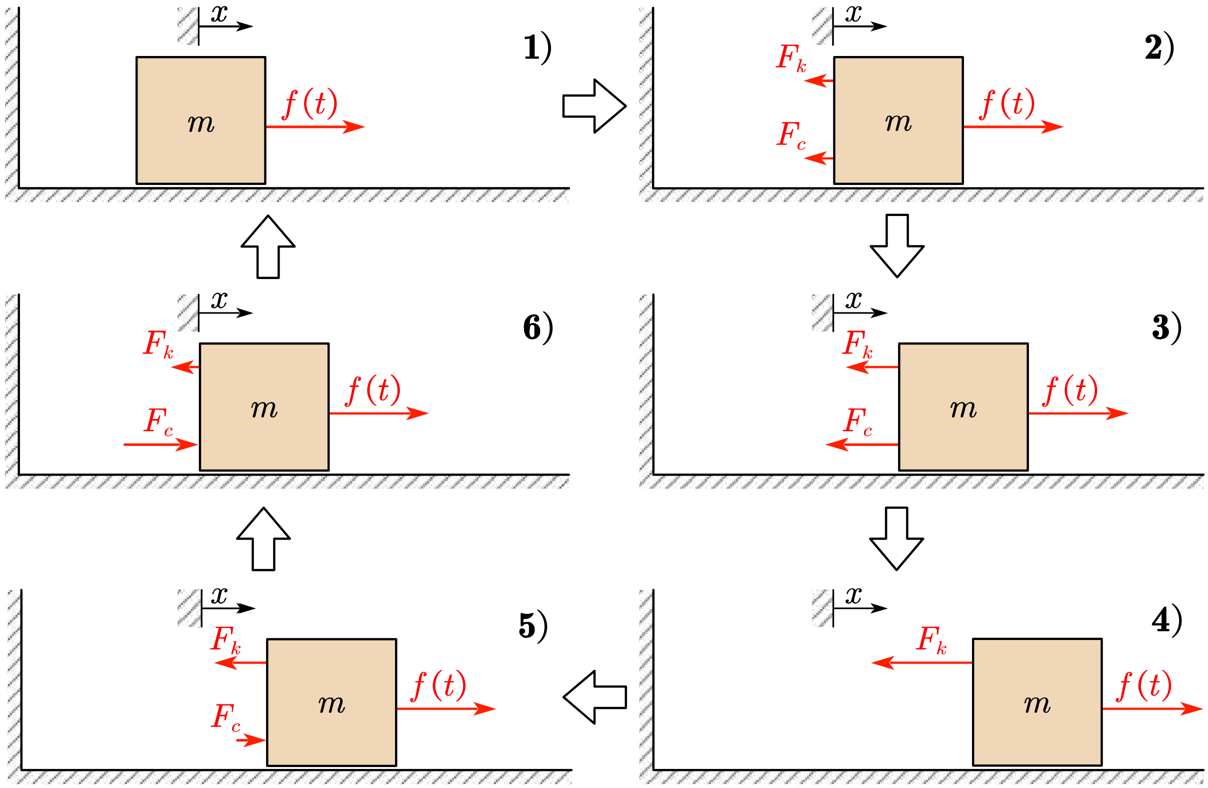
It can be seen why this decays. It attempted to turn a linear system into a marginally stable system that oscilates forever. However, if even these poles are slightly knocked either left or right, the system can easily turn unstable, or underdamped. Because the capacitors have a leakage resistance associated to it, a dampening term is introduced, and so the system becomes an underdamped oscilator.

So how do timers and enngines constantly reciprocate back and forth then? What separates an underdamped oscilator from an engine is the ability of the engine to shut of, and turn back on again, OR to redirect its source of force/pressure.

Shown below are the stages of an underdamped oscilation.

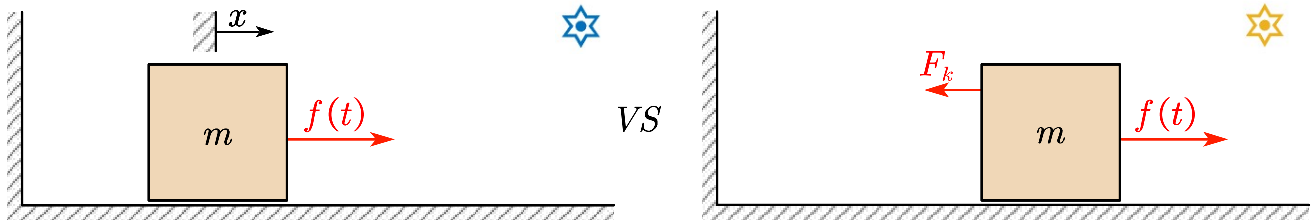


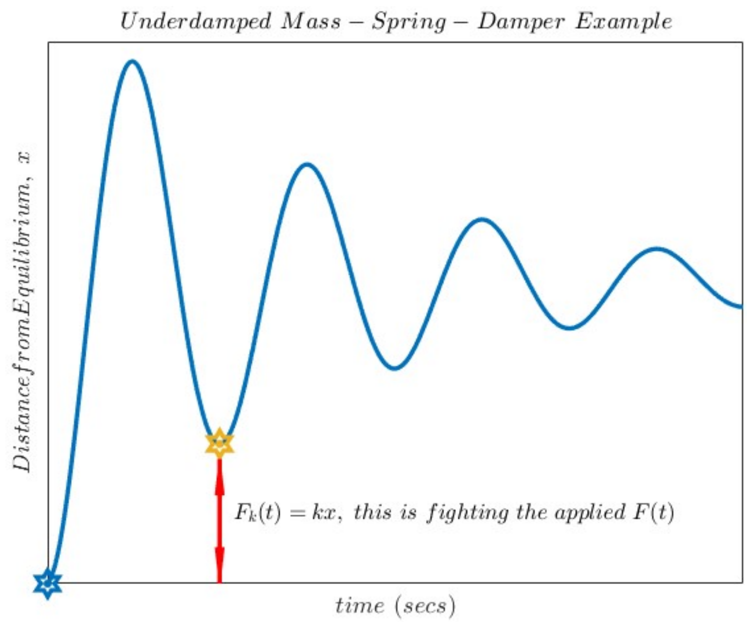
The stages will be redrawn, with forces shown instead of the components.



Note, that when the first cycle is completed, and goes back to step 1, it won't arrive at its original starting position. And neither will any of the other steps. This is due to the dampening force in the system. It will never be possible to get rid of drag (or friction, sometimes modeled as drag). This causes problems, because as the movement lessens in each cycle, eventually, the movement (practially) completely stops, and the oscilation motion, which is intended to drive machines, is lost.

It can be seen in the 5th and 6th stages that as the mass moves backwards, its backwards movement is being fought by the applied force, as well as the dampening force. The dampening force provides enough delay for the momentum to peter out before the applied force term again overpowers the spring force to start moving the block rightward. This is why it doesn't stop at its original equilibrium position. If there was no dampening term, then the momentum would continue to move the block backwards, until the spring force overpowers the applied force enough to accelerate and move the block in the other direction.

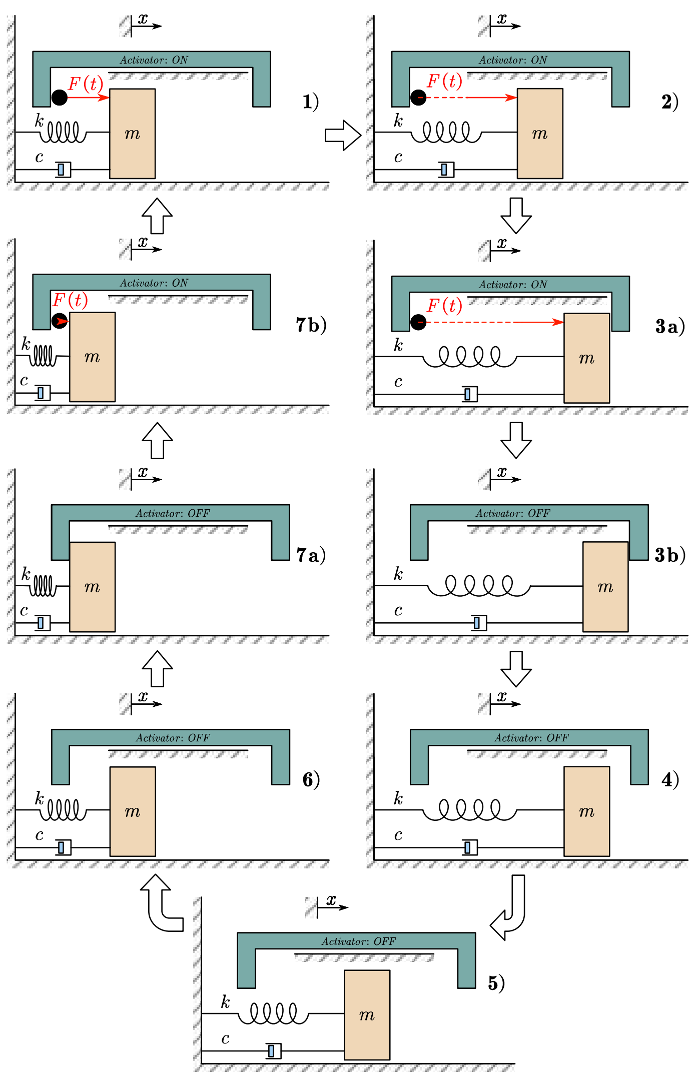




Let's compare the initial cycle start (blue star), with the start of the very next cycle (yellow star). It can be seen that in the blue star scenario, there is no displacement from equilibrium. This means that there is no spring force. Keep in mind F(t) is constantly being applied, even at the very start. This means, in the blue star scenario, for the first couple of milliseconds, F(t) can freely push against mass m, with either no, or negligible spring force. This allows F(t) to impart enough momentum for the mass to hit its highest peak, before oscilating down to the next cycle.

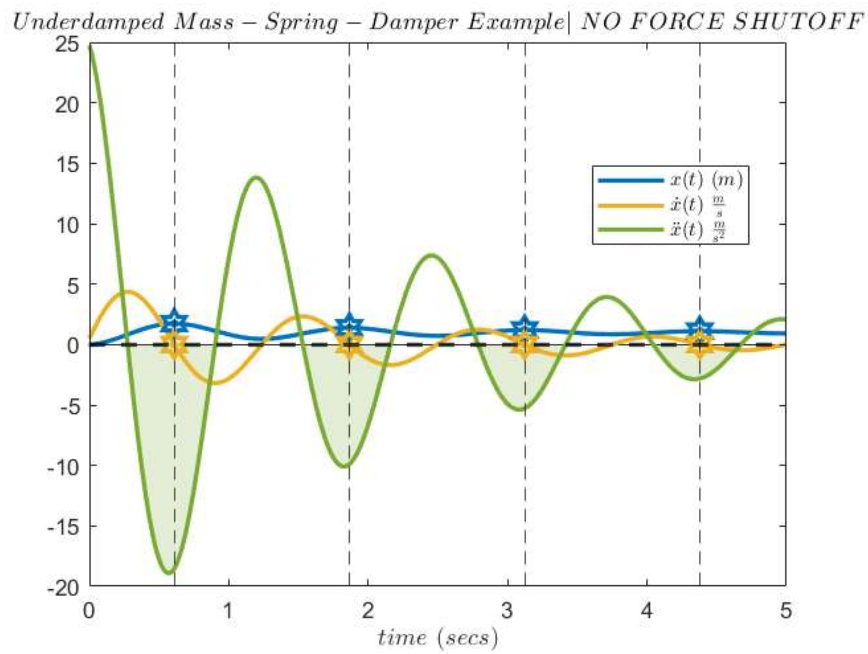
Compare this with the yellow star scenario. Because the dampening force in the previous (blue star) cycle gave enough delay for F(t) to overpower the displacement/movement caused by the spring force, the mass stopped at a more rightward position. This means, that from this new (instantaneous) resting position, F(t) isn't pushing mass m with no/negligible spring force fighting against it, since the spring starts stretched rather than at equlibrium. This means that in the same amount of time, F(t) is able to impart less momentum on mass m, meaning that it reaches a lower peak than in the blue star cycle. This delay and lessening of momentum plays out continuously through each cycle, until the oscilations (practically) stop.

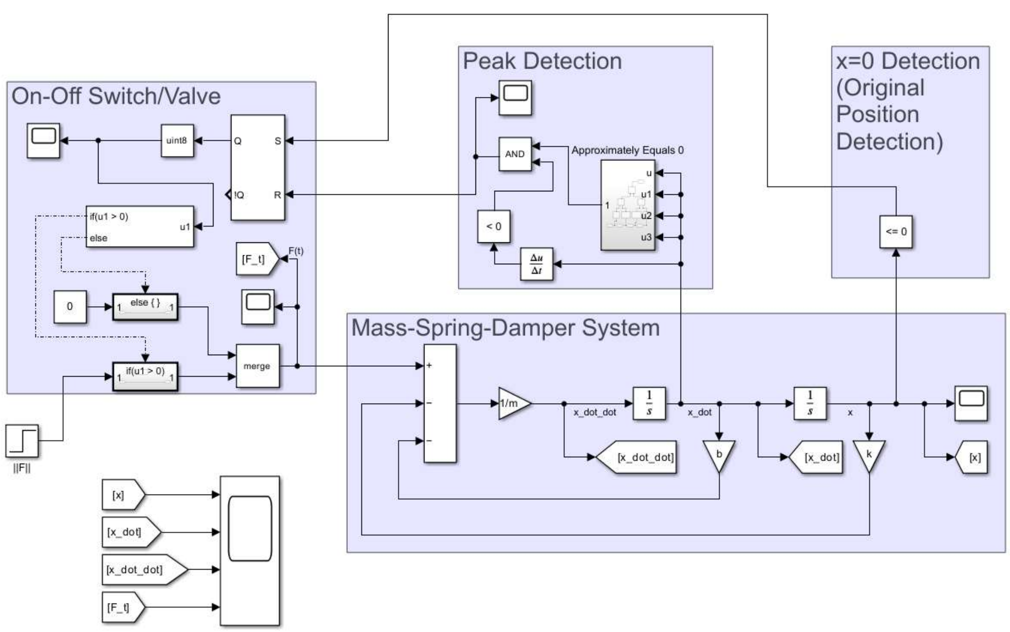
As mentioned before however, it is not possible to get rid of the dampening term. So instead of trying to minimize dampening, which will always be there and will stop the system eventually, we can try to turn off the force applied to the block as the block retracts. The goal is to get the block to stop/reach a position where the spring isn't stretched, before re-applying force F(t) so that F(t) has enough time to impart sufficient momentum to get mass m back to the previous peak.



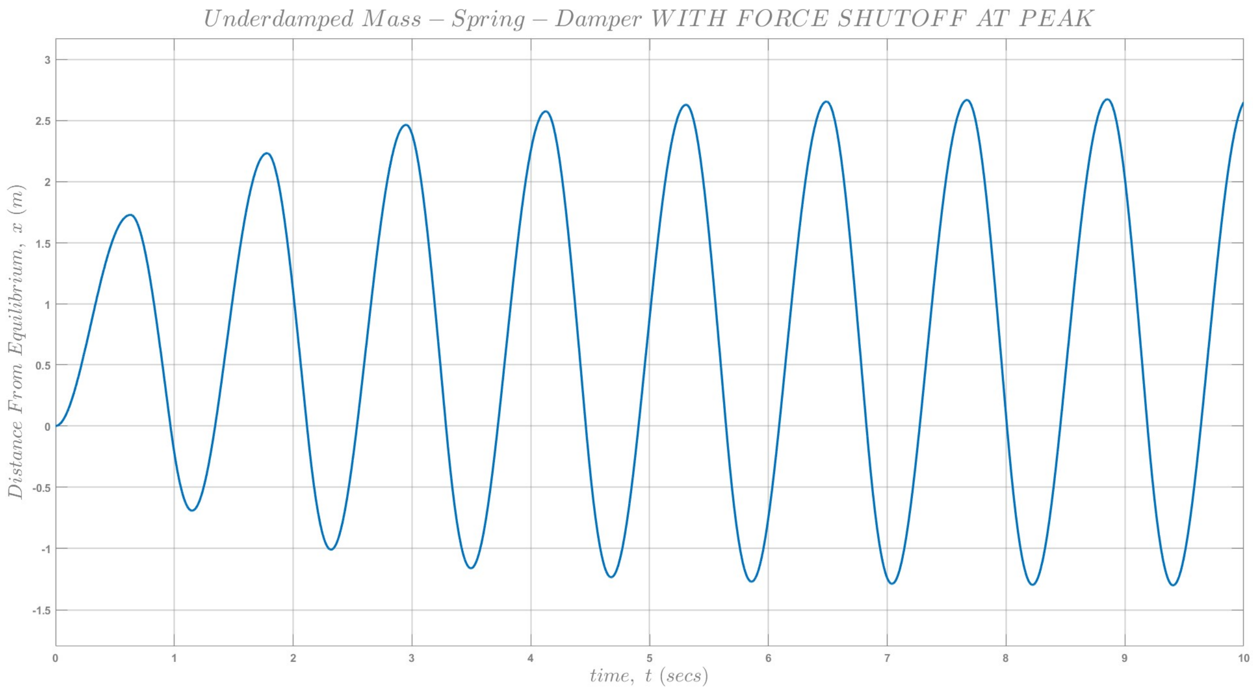
Seen to the left is a crude example of a machine that would achieve force shutoff, when the mass reaches some peak. The applied force, F(t) is modeled as air pressure that is shot through a hole, and pushes against mass m. Stages 1-3a are the normal stages of a mass-spring damper – the mass is applied a force, F(t), and it accelerates towards a peak, until the spring force starts to overcome both inertia and the applied force. However, once at the peak, the mass m nudges another block. As it is nudged right, it covers up the hole (it is perpendicular to the page, but pressure expands in all directions, so it is still able to push mass m right), and covers the hole. This shuts off the applied force, allowing mass m to retract all the way to or past the starting position. Once the mass moves far back enough, it nudges the block left, uncovering the hole. This allows the pressure to start acting against mass m again. And because F(t) is not fighting against spring force, due to it either not being stretched, or beacause it is compressed, force F(t) is able to move mass m to a peak not less than the original.

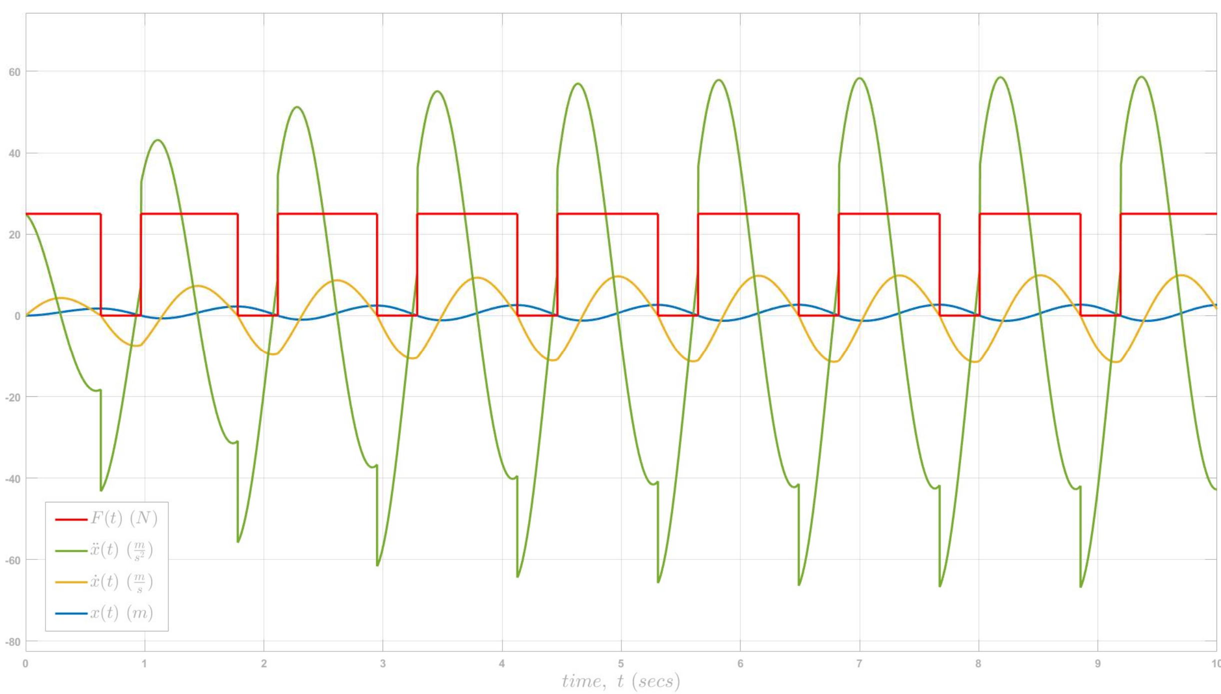
A simulink example of this is given here, which is a simple mass spring damper system at its core. The modification is done to detect either when mass m is at a peak, or at or left past it's equilibrium position. When the mass is in between the peak, and rightward past it's equilibrium position, force F(t) is applied to mass m. When mass m reaches its peak (detected when the velocity is about 0, and acceleration is negative), force F(t) is shutoff. F(t) is shutoff until mass m moves to, or left past its original position. Once that happens, F(t) is applied again, and the cycle repeats.





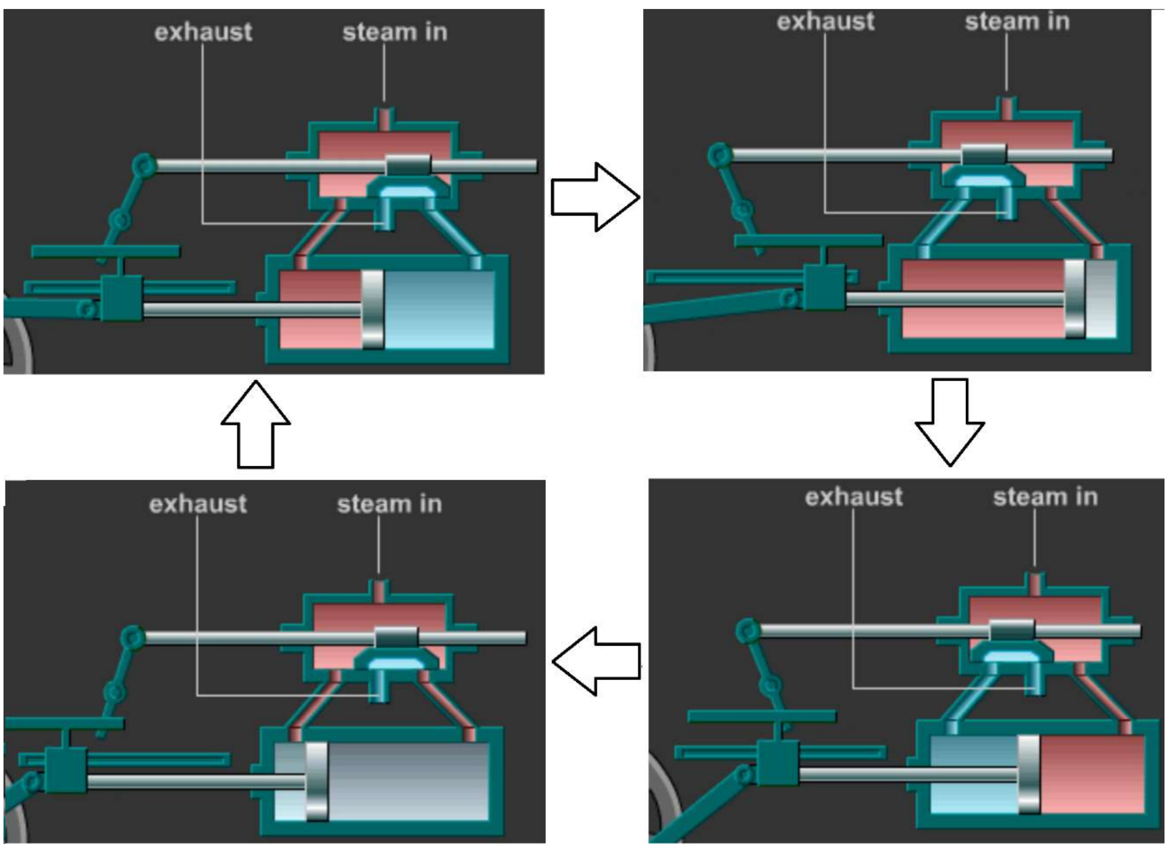
Shown below are the plots of position, velocity, and acceleration along with the force shutoffs. It can be seen that the mass-spring-damper no longer peters out and settles to a single value. Instead, the on-off applied force allows the system to constantly oscilate.





Every engine works on this exact principle. In ICE engines, the power stroke only happens at a specific position. In steam engines, the steam that drives the piston is controlled by a shuffler to redirect the steam. Even in constant heat engines like the Stirling engine, the hot air that drives the engine is shuffled and moved around, redirecting the pressure applied onto the piston.

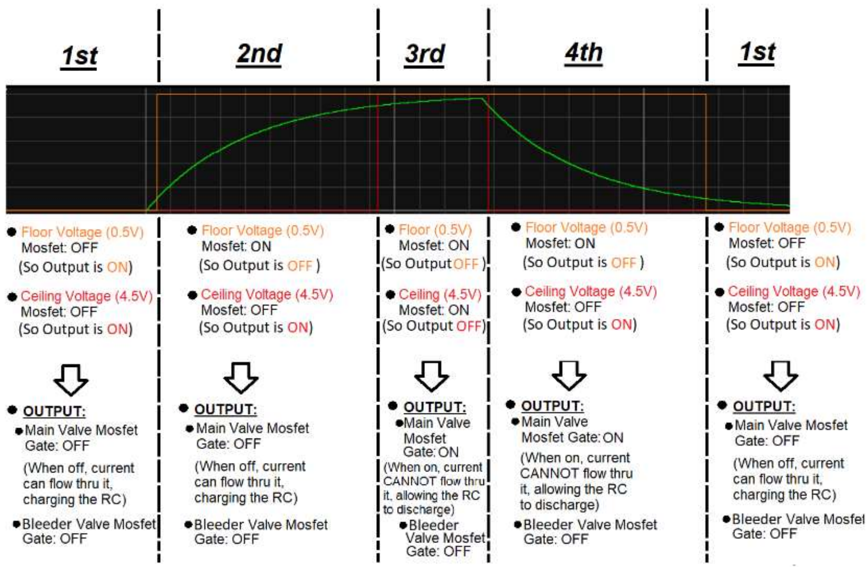
Shown below are four parts of the cycle of a steam engine. The hot air (red) pushes the piston to the cold side (blue). However, when the piston reaches its max extent (at either end), the shuffler piston (the one above the main piston) covers the hole where the steam previously filled the chamber, and redirects the steam to the other hole (the hole on the side where the piston ended up).

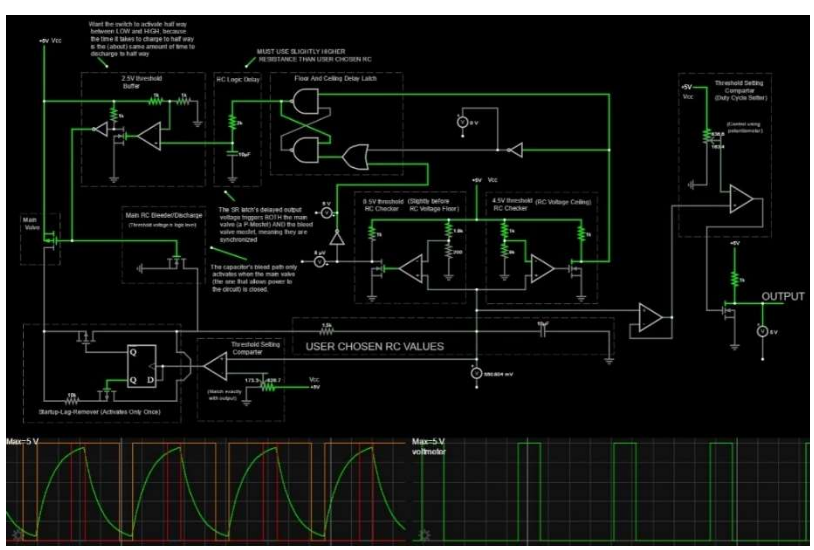


The steam piston of course has no elastic element. It is a mass-damper system, but the oscillations are caused by redirecting the source of force (steam) rather than just turning it on and shutting it off. The reason the mass-spring-damper system oscillated only with on-off control was because the spring drove the mass to its original position. The steam piston doesn't have an element that can reposition it back to where it started.

Of course, the oscillator doesn't have to be this complicated in electronics. In fact, the inertia (mass or inductance) can be entirely done away with. An oscillator can just be built with an RC circuit that turns itself off and on again. This was shown in the previous "Square Wave RC Oscillator" was built. A 555 timer works roughly the same way in astable mode.

In this circuit, an initial voltage V_cc is applied to the RC circuit. The RC circuit slowly charges and asymptotes towards V_cc. When the RC about reaches V_cc, the circuitry turns off V_cc, and 0V is applied to the RC circuit. This allows it to discharge. When the RC circuit about reaches 0V, the circuitry turns V_cc on again, and the cycle repeats. Even if the capacitor in the main RC circuit had leakage resistance, it would only affect the timing of the cycles, not dampen its amplitude overtime.





Now that a stable and reliable oscillation has been obtained, it is now time to back out a sine wave out of the square wave. This is because it is known that square waves can be turned into the sum of several sine waves.

$$V(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)\right)$$

$$a_0 = \frac{1}{T}\int_0^T V(t)dt$$

$$a_n = \frac{1}{T}\int_0^T V(t)\cos\left(\frac{2n\pi t}{T}\right)dt$$

$$b_n = \frac{1}{T}\int_0^T V(t)\sin\left(\frac{2n\pi t}{T}\right)dt$$

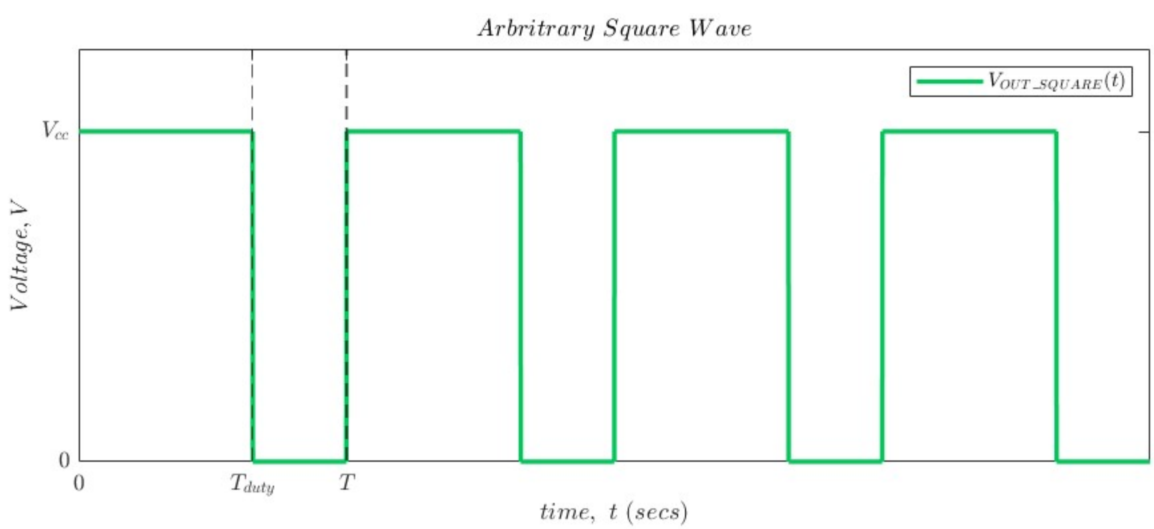
The Fourier transform of a square wave, with any duty cycle is:

$$V(t) = V_{cc}\frac{T_{duty}}{T} + \sum_{n=1}^{\infty}\left(\frac{V_{cc}}{T}\sin\left(\frac{2n\pi T_{duty}}{T}\right)\cos\left(\frac{2n\pi t}{T}\right) + \frac{V_{cc}}{n\pi}\left(1-\cos\left(\frac{2n\pi T_{duty}}{T}\right)\right)\sin\left(\frac{2n\pi t}{T}\right)\right)$$

$$a_0 = \frac{1}{T}\int_0^T V(t)dt = \frac{1}{2T}\left(\int_0^{T_{duty}} V_{cc}dt + \cancel{\int_{T_{duty}}^{T} 0dt}\right) = V_{cc}\frac{T_{duty}}{2T}$$

$$a_n = \frac{1}{T}\int_0^T V(t)\cos\left(\frac{n\pi t}{T}\right)dt = \frac{1}{T}\left(\int_0^{T_{duty}} V_{cc}\cos\left(\frac{2n\pi t}{T}\right)dt + \cancel{\int_{T_{duty}}^{T} 0dt}\right) = \frac{V_{cc}}{n\pi}\sin\left(\frac{2n\pi T_{duty}}{T}\right)$$

$$b_n = \frac{1}{T}\int_0^T V(t)\sin\left(\frac{n\pi t}{T}\right)dt = \frac{1}{T}\left(\int_0^{T_{duty}} V_{cc}\sin\left(\frac{2n\pi t}{T}\right)dt + \cancel{\int_{T_{duty}}^{T} 0dt}\right) = -\frac{V_{cc}}{n\pi}\cos\left(\frac{2n\pi T_{duty}}{T}\right) + \frac{V_{cc}}{n\pi}$$



To make this simpler, it is best to use a square wave with a duty cycle of 50%. This eliminates the cosine elements, thus making the analysis easier.

$$V(t) = V_{cc}\frac{T_{duty}}{T} + \sum_{n=1}^{\infty}\left(\frac{V_{cc}}{T}\sin\left(\frac{2n\pi T_{duty}}{T}\right)\cos\left(\frac{2n\pi t}{T}\right) + \frac{V_{cc}}{n\pi}\left(1 - \cos\left(\frac{2n\pi T_{duty}}{T}\right)\right)\sin\left(\frac{2n\pi t}{T}\right)\right)$$

Remember, $\frac{T_{duty}}{T} = \frac{1}{2}$, so…

$$V(t) = \frac{V_{cc}}{2} + \sum_{n=1}^{\infty}\left(\cancel{\frac{V_{cc}}{T}\sin(n\pi)\cos\left(\frac{2n\pi t}{T}\right)} + \frac{V_{cc}}{n\pi}(1 - \cos(n\pi))\sin\left(\frac{2n\pi t}{T}\right)\right)$$

$\sin(n\pi) = 0$ for every n, since n is an integer (every multiple of π results in 0)

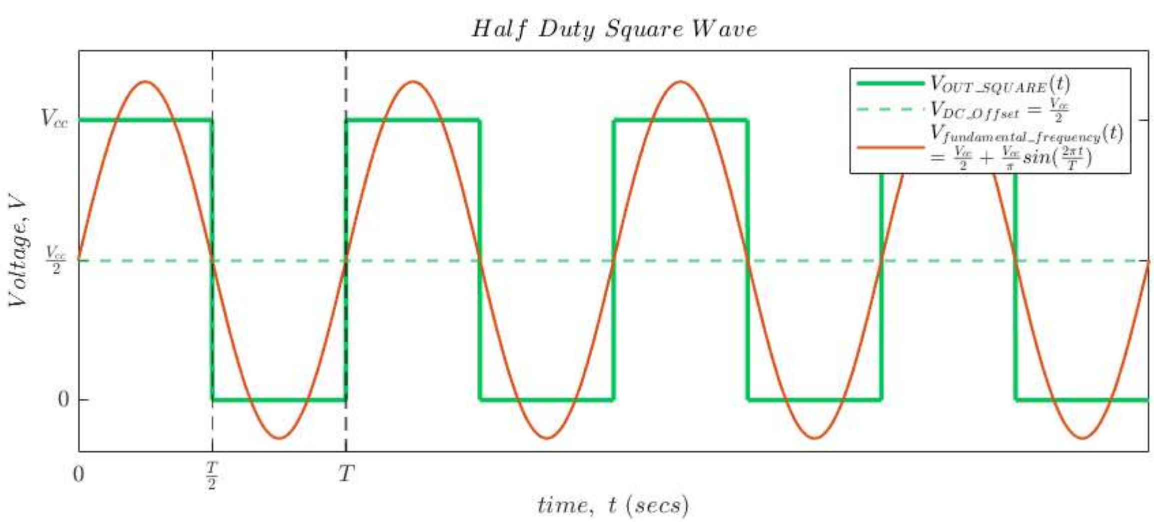
$$V(t) = \frac{V_{cc}}{2} + \sum_{n=1}^{\infty}\frac{V_{cc}}{n\pi}(1 - \cos(n\pi))\sin\left(\frac{2n\pi t}{T}\right)$$

$\left(1 - \cos(n\pi)\right) = 0$, only for n = 2,4,6,8… , so the formula reduces to:

$$V(t) = \frac{V_{cc}}{2} + \frac{V_{cc}}{(1)\pi}\sin\left(\frac{2(1)\pi t}{T}\right) + \frac{V_{cc}}{(3)\pi}\sin\left(\frac{2(3)\pi t}{T}\right) + \frac{V_{cc}}{(5)\pi}\sin\left(\frac{2(5)\pi t}{T}\right) + \cdots$$

The fundamental frequency happens at n = 1. Shown below are the half duty square wave along with the fundamental wave and DC offset. The fundamental sine wave has the same frequency as the square wave and is in phase with it. It has a DC offset of $\frac{V_{cc}}{2}$.

$$V_{fundamental\_frequency}(t) = \frac{V_{cc}}{2} + \frac{V_{cc}}{\pi}\sin\left(\frac{2\pi t}{T}\right)$$

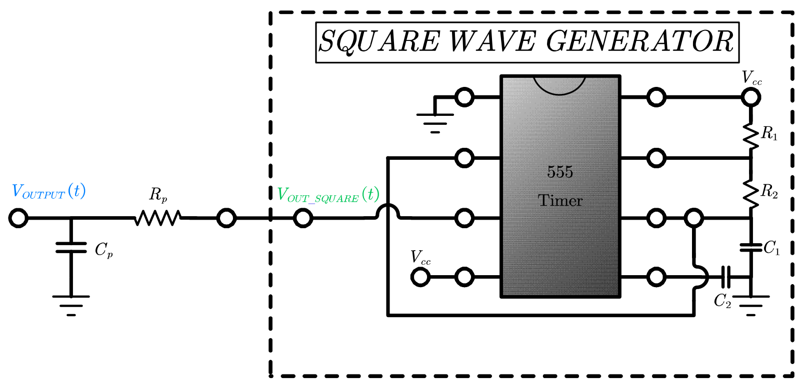


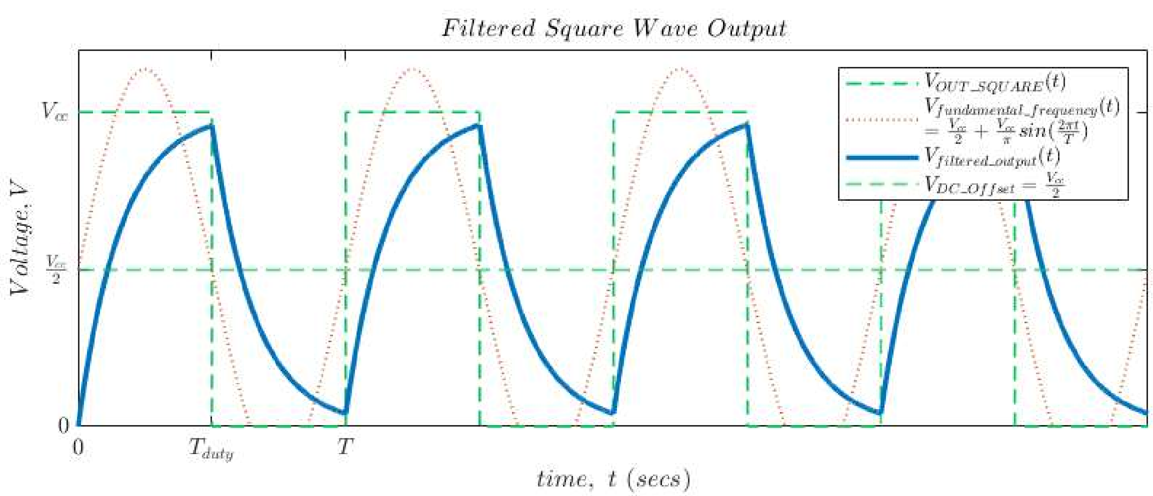
Keep in mind that period to frequency in Hz is $f = \frac{1}{T}$, and frequency in Hz to frequency in rad/s is $f = \frac{\omega_n}{2\pi}$

$$V_{fundamental\_frequency}(t) = \frac{V_{cc}}{2} + \frac{V_{cc}}{\pi}\sin\left(\frac{2\pi t}{T}\right) = \frac{V_{cc}}{2} + \frac{V_{cc}}{\pi}\sin(2\pi f_n t) \; = \frac{V_{cc}}{2} + \frac{V_{cc}}{\pi}sin(\omega_n t)$$

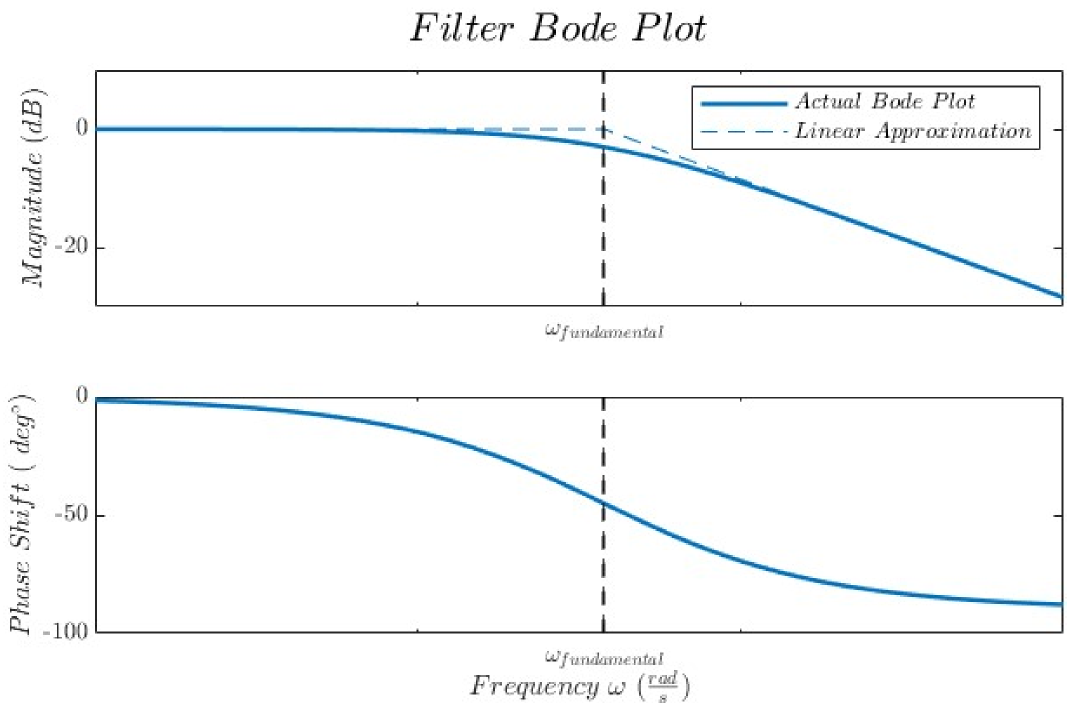
To obtain a desired AC frequency of 60hz, this must mean that the square wave of half duty cycle must also operate at 60Hz. 1 Hz = 1 cycle per second. $f = \frac{1}{T} \Rightarrow T = \frac{1}{f}$, and so the period of the square wave is 1/60 seconds.

Backing out this sine wave be done with RC filtering. However, only one RC filter stage is insufficient to back out this sine wave. The resulting waveform from a square wave passed through a single RC filter is shown below:



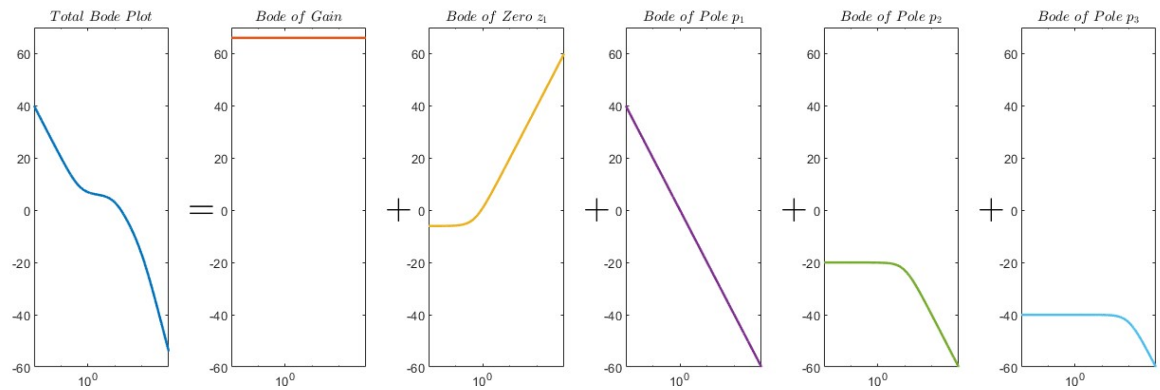


It can be clearly seen that one filter does not output a sine wave. In fact, it looks like a series of simple RC step responses, because it is. Imagine turning on and off a voltage to an RC circuit - it will just slowly charge the capacitor up until it plateaus towards the applied voltage, resulting in that shape. Another way to think about why this shape forms is to analyze the bode plot of the filter.

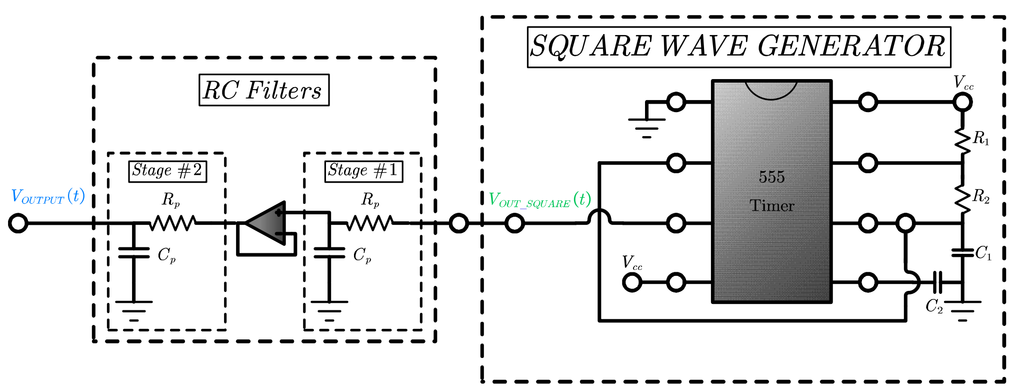


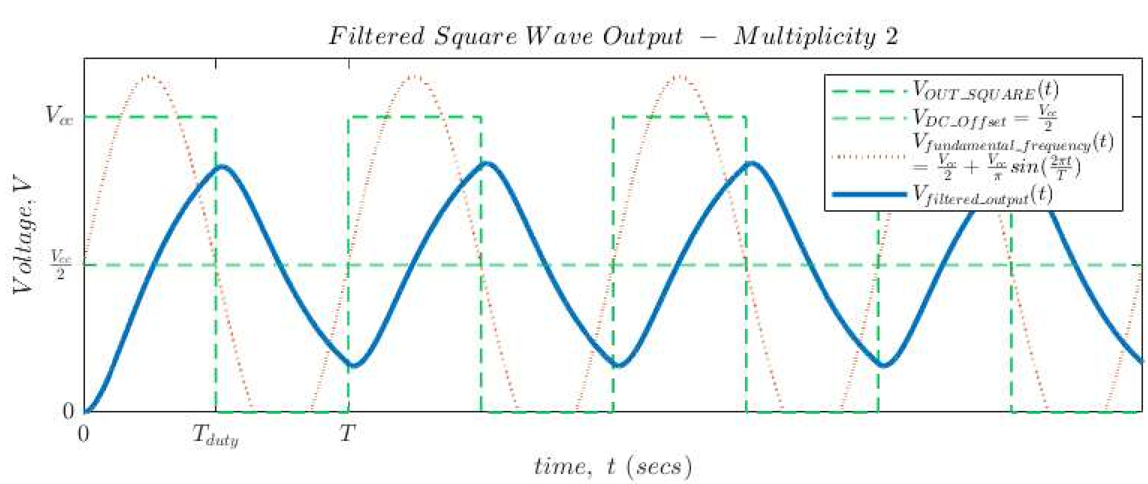
Here, the RC filter is set to the exact fundamental frequency of 60Hz (377rad/s), which is marked by the thick dashed black line. Take a look at the linear bode plot assumption (the blue dashed line, which is the ideal bode plot). Ideally, every frequency greater than the corner frequency should get filtered out of the signal. However, the linear approximation of the bode plot after the corner frequency does not immediately set the response to 0, because the magnitudes aren't that negative yet. This means that the next few frequencies up have a comparable strength to the fundamental sine wave – in this case, from a square wave, the next frequencies will be 1/3rd as strong, 1/5th as strong, and 1/7th as strong, and so on WITHOUT attenuation (which is how the square wave is formed). Because the sloped portion of the bode plot is still in the -20, -40, -60, -80 range, this means these next frequencies aren't completely attenuated to 0, and so while the square wave form is gone, these waves still add to the fundamental sine wave, creating the RC step response waveform.

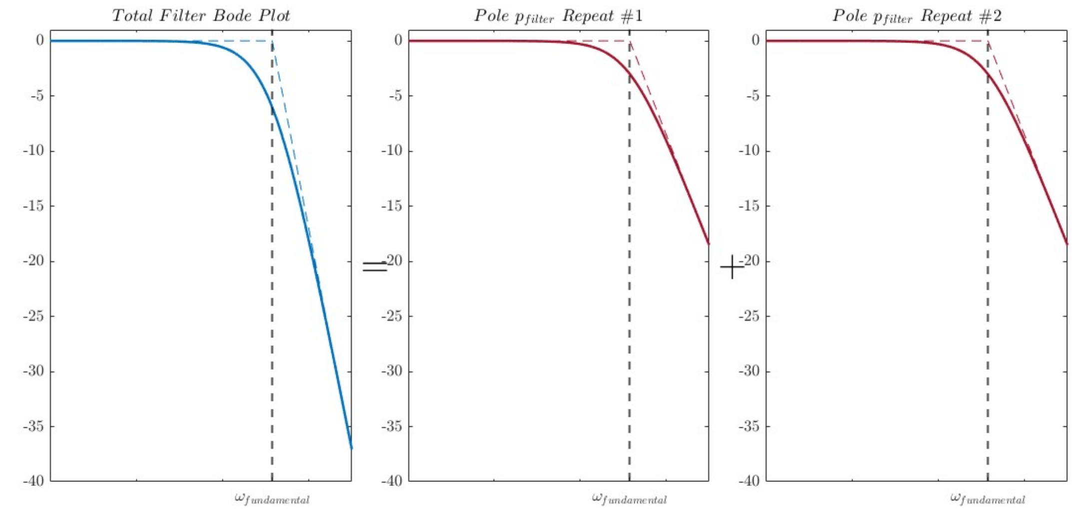
The solution here is to have repeat stages of the same exact RC filter (having a multiplicity of the same pole). To see why, it is important to take a detour to remember how bode plots work with multiple poles. The resultant bode plot is just the sum of the individual bode plots of each gain, pole, and zero.

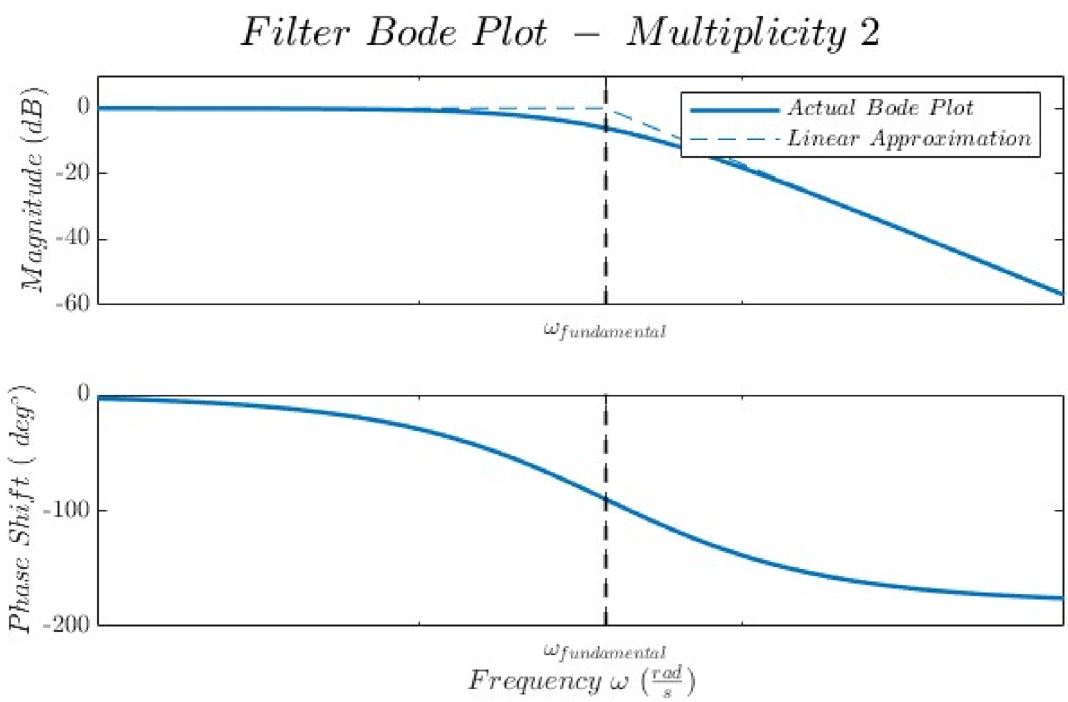


This means that if the same exact pole was added to the filter, the slope on the resultant bode plot would be double the slope of each individual filter. This attenuates the higher frequencies from about the [-20 to -30] range to [-30 to -60], creating an output closer to a sine wave (although it's closer to a triangle wave for 2 filters). Note the use of Op-Amp followers to feed the voltage into the next stage. These draw no current, which is why if the RC filters just fed into each other directly, then the system obtains extra poles, and not a multiplicity of the original.

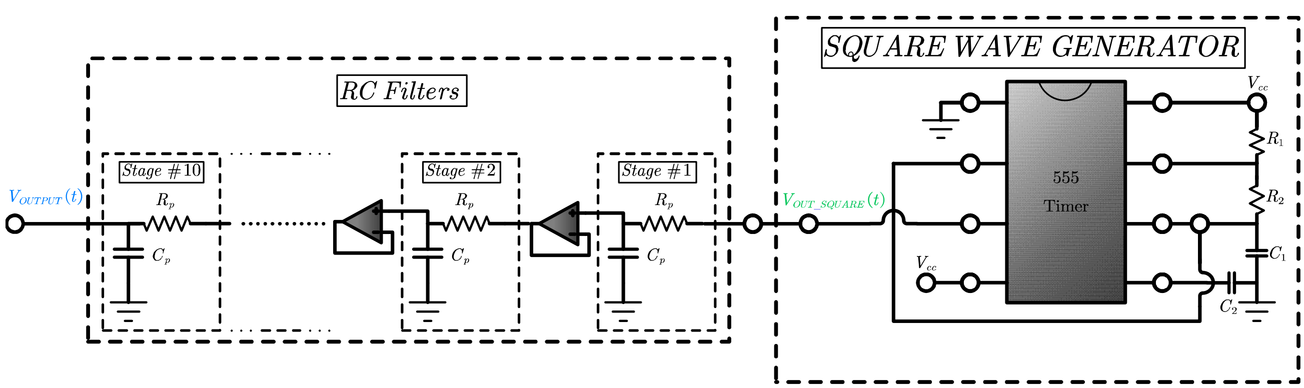


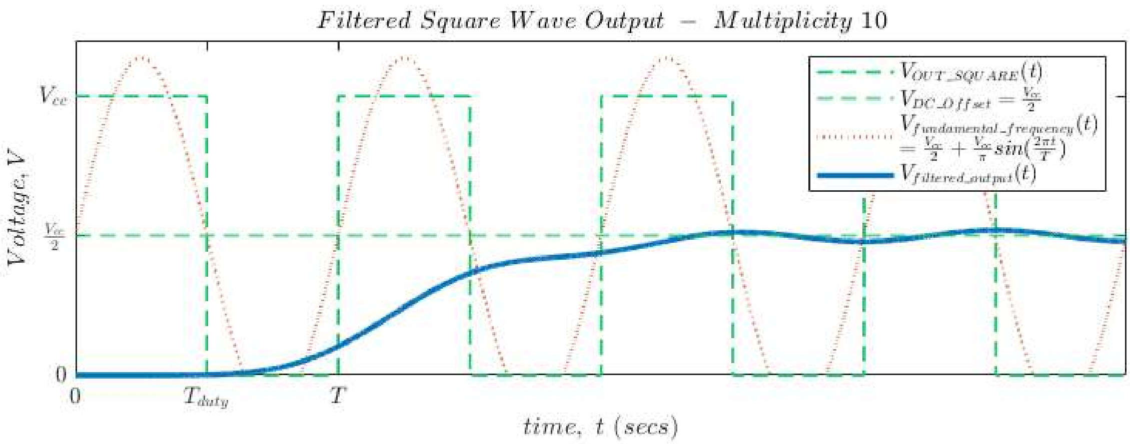




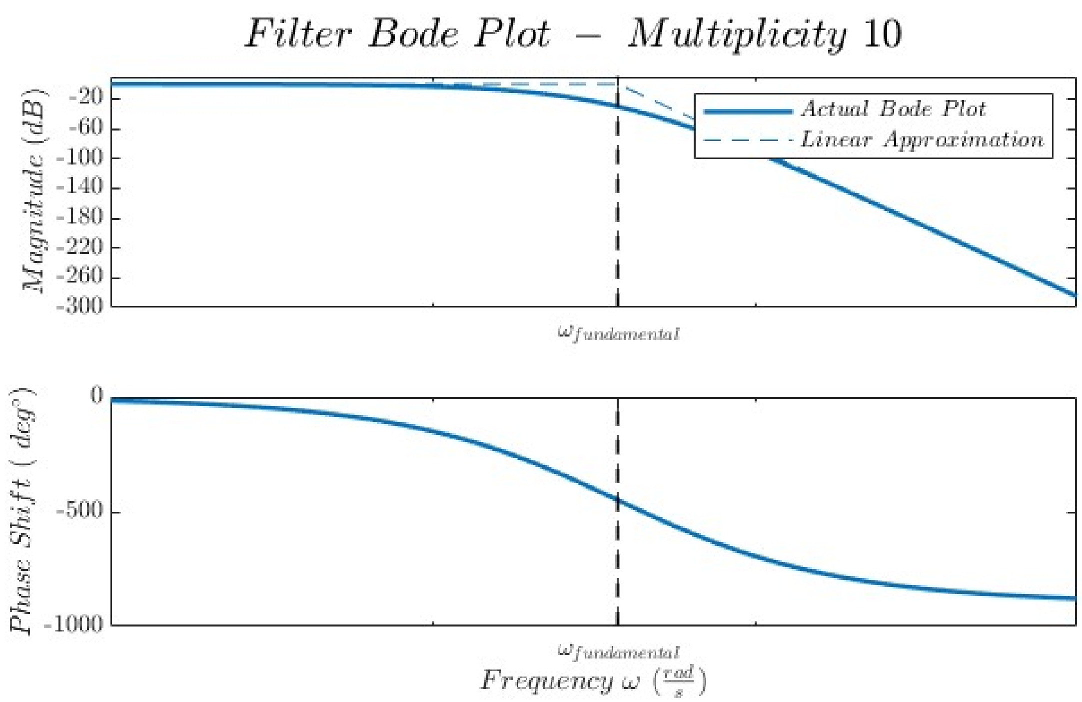


It stands to reason that as the same RC filter is repeated more and more, the output should result in a closer sine wave. Below is the result of 10 RC filters in series.





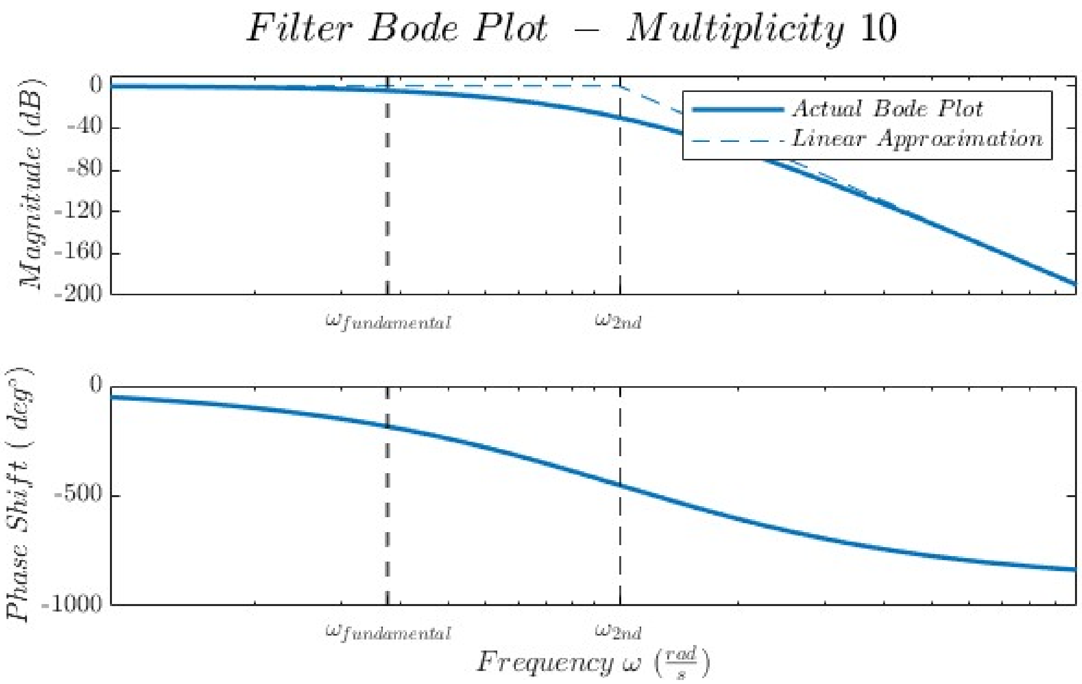
While the sine wave is much more visible, it is also very clear that it is much weaker than desired. Why does this happen? The sinewave is significantly weaker than desired because of the ideal, linear bode plot approximation. It can be seen below that although the linear approximation holds the magnitude of the fundamental frequency to be 0 (meaning the output is the same magnitude as the input), the actual bode plot has a curve where the corner should be. At one pole, the drop from the corner to the curve is 3dB. At 10 poles, this drop is 30dB. Because the formula for magnitude in dB is $M_{dB} = 20\log(\|Z(j\omega)\|)$, this represents a steep logarithmic drop in output voltage.



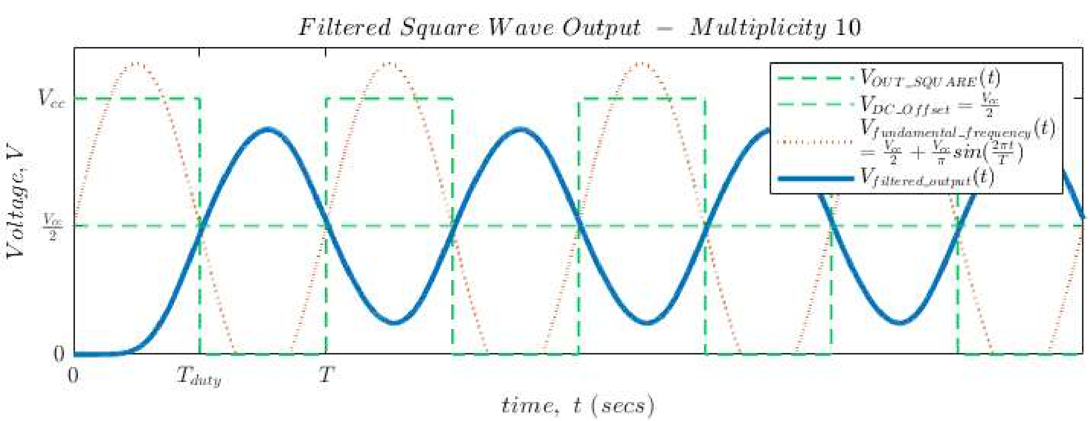
To compensate for the curve at the corner, we can instead filter for the next frequency above the fundamental frequency. Remembering that a square wave with a 50% duty cycle is made up of cosine waves:

$$V(t) = \frac{V_{cc}}{2} + \frac{V_{cc}}{(1)\pi}\sin\left(\frac{2(1)\pi t}{T}\right) + \frac{V_{cc}}{(3)\pi}\sin\left(\frac{2(3)\pi t}{T}\right) + \frac{V_{cc}}{(5)\pi}\sin\left(\frac{2(5)\pi t}{T}\right) + \cdots$$

The next frequency (the second sine term) is three times the frequency of the fundamental wave. So if $f_{fundamental} = 60Hz \approx 377\frac{rad}{s}$, then $f_{2nd} = 180Hz \approx 1131\frac{rad}{s}$. At a multiplicity of 10, when filtering at the 2nd frequency, the resulting bode plot is:



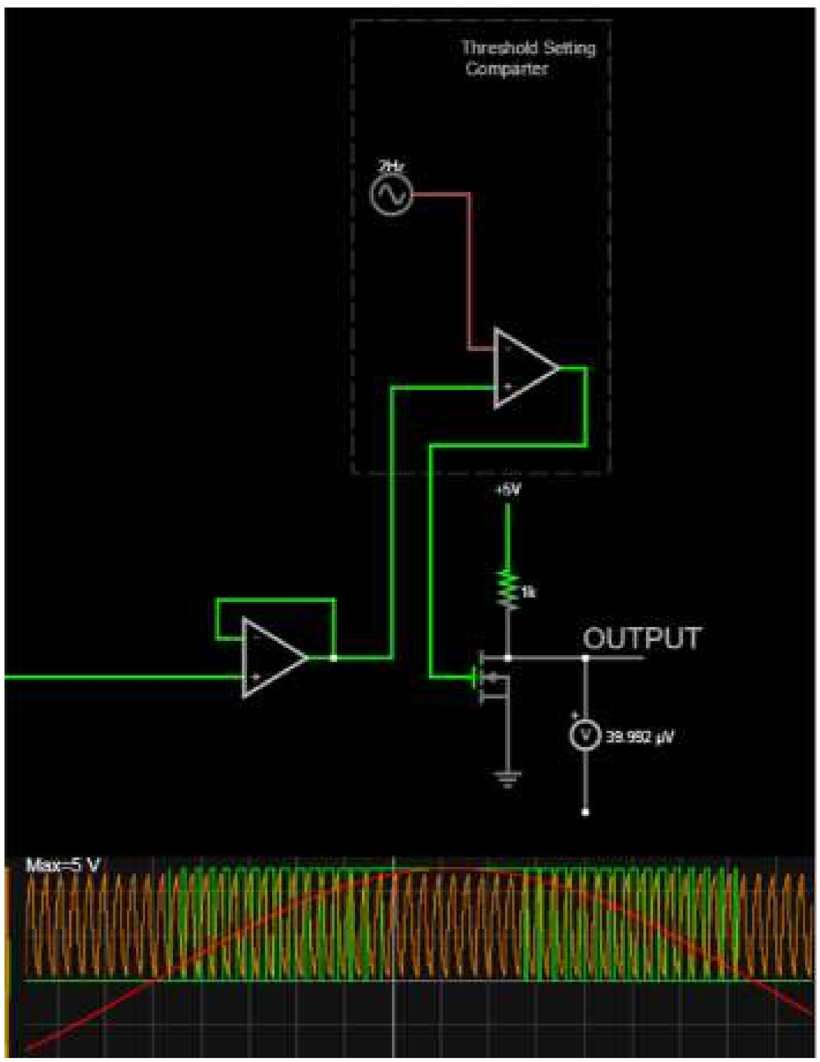
As seen, the fundamental frequencies are almost unaffected, having only about a 4.6dB drop, whereas the 2nd frequency has a 30dB drop. The 3rd, 4th, 5th, and other higher frequencies get even more attenuated. The resulting output wave is shown.



This result looks much closer to a sine wave. It even has less “rise time” compared to the 10 multiplicity filter at the fundamental frequency. The output sine wave is phase shifted however, and the magnitude doesn’t span V_{cc} to 0V. However, this is OK, since this can just be amplified to a desired voltage using an op-amp.

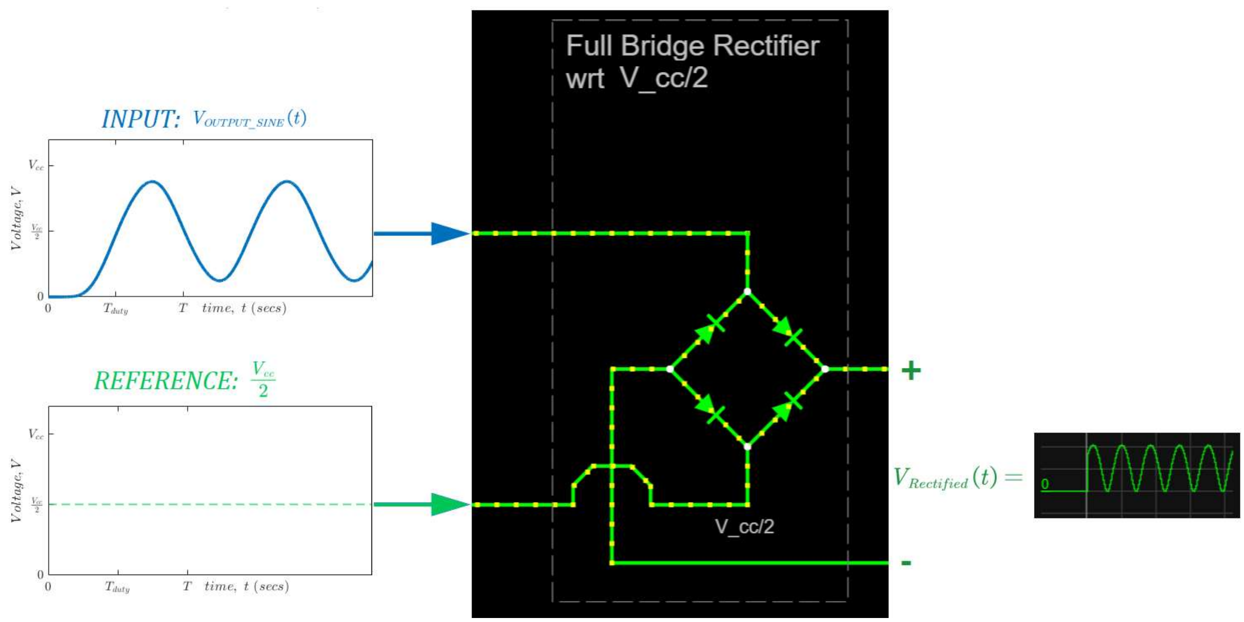
The next step from turning a weak sine wave voltage source into powerful AC source would be to either use amplification, or to feed this wave into another square wave timer to create a sine wave PWM.

The digital solution involves feeding this sine wave into the comparter of the square wave generator. An H-Bridge will be used to flip the polarity.

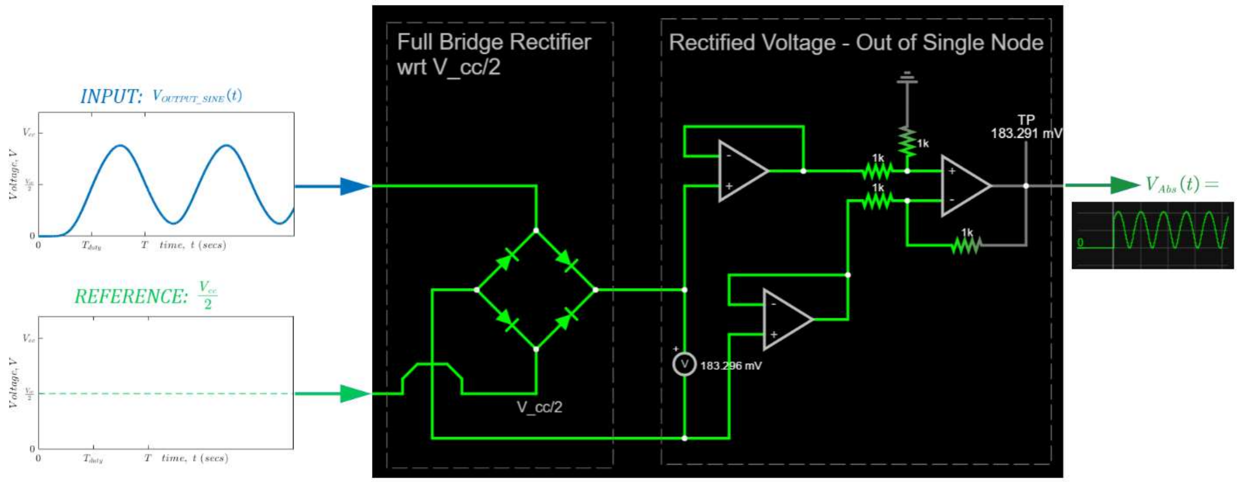


The analog solution will be explored here. The idea is to get both a positive and negative voltage output from the AC wave. Because there is no negative voltage rail, the way to supply negative voltage is to flip the leads connected to V_power and ground. This can be done with an H-bridge, that switches polarity depending on what voltage the sine wave is. There is also the issue of turning a positive sine wave with a positive offset into a voltage that is equal in waveform both states of the H-bridge. This is done by passing the wave through a full bridge rectifier, which will be explained before the H-Bridge.

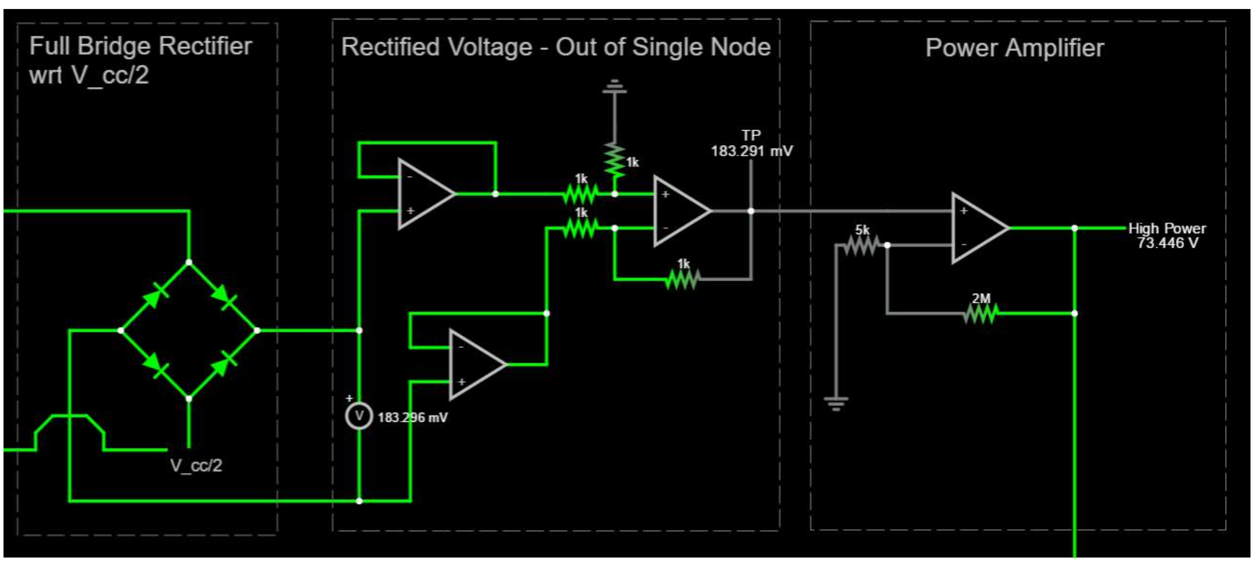
This modified full bridge rectifier essentially takes the absolute value of the wave with respect to the total offset of the wave. For the sinewave generated by passing a square wave through a series of RC filters, this is just $\frac{V_{cc}}{2}$. If the op-amp analog sine wave was used, then the sine wave's troughs are at $\frac{V_{cc}}{2}$, and therefore, it hovers at a voltage slightly over it. Solving the differential equation $V_{set} = \ddot{V}_{out}(t) + {\omega_n}^2 V_{out}(t)$ results in $V_{out}(t) = \frac{V_{set}}{\omega^2} + \frac{V_{set}}{\omega^2}\, cos(\omega t)$. Combined with the offset of $\frac{V_{cc}}{2}$ as virtual ground, the total offset of the wave created by the analog sine wave generator is $V_{offset} = \frac{V_{set}}{\omega^2} + \frac{V_{cc}}{2}$.



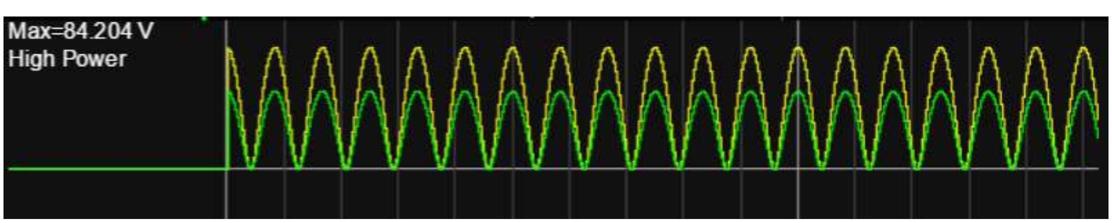
However, this rectified voltage only exists between the two leads of the right side. If one wanted to amplify this using a single node, additional circuitry would be needed.



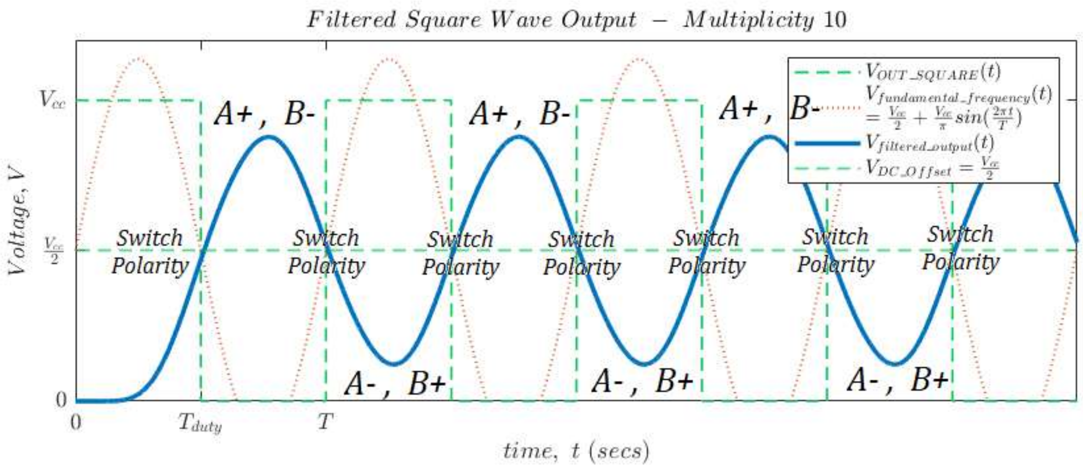
Here, the two output leads of the full bridge rectifier are connected to op-amp follower circuits, which exactly match the voltage at each lead without drawing current from them. This allows the math to avoid taking current into consideration. These followers then feed into a subtraction op-amp. Because the leads of the rectifier are either in the state of [off, Sine Voltage] of [-Sine Voltage, off], subtracting the two will always give a positive sine wave voltage. This can then be fed into a power amplifier.

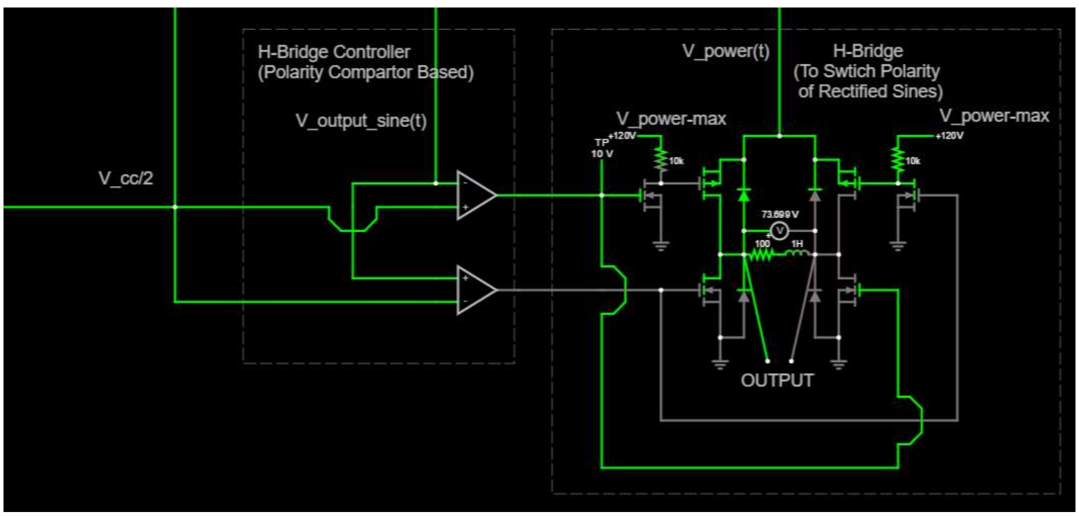


This amplified rectified wave, $\|V_{power}(t)\|$, can be fed into an H-bridge to switch polarity.

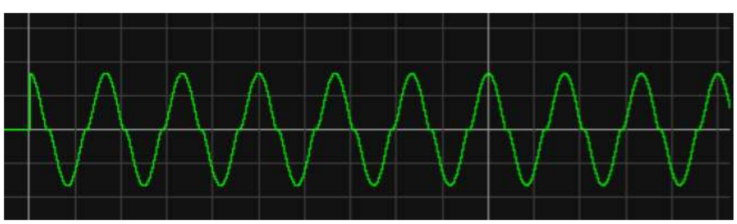


The H-Bridge should switch polarity based on when $V_{OUTPUT\_SINE}(t)$ crosses the total offset (which in the case of a filtered square wave, is $V_{offset} = \frac{V_{cc}}{2}$, and for the op-amp sine wave generator, is $V_{offset} = \frac{V_{set}}{\omega^2} + \frac{V_{cc}}{2}$).



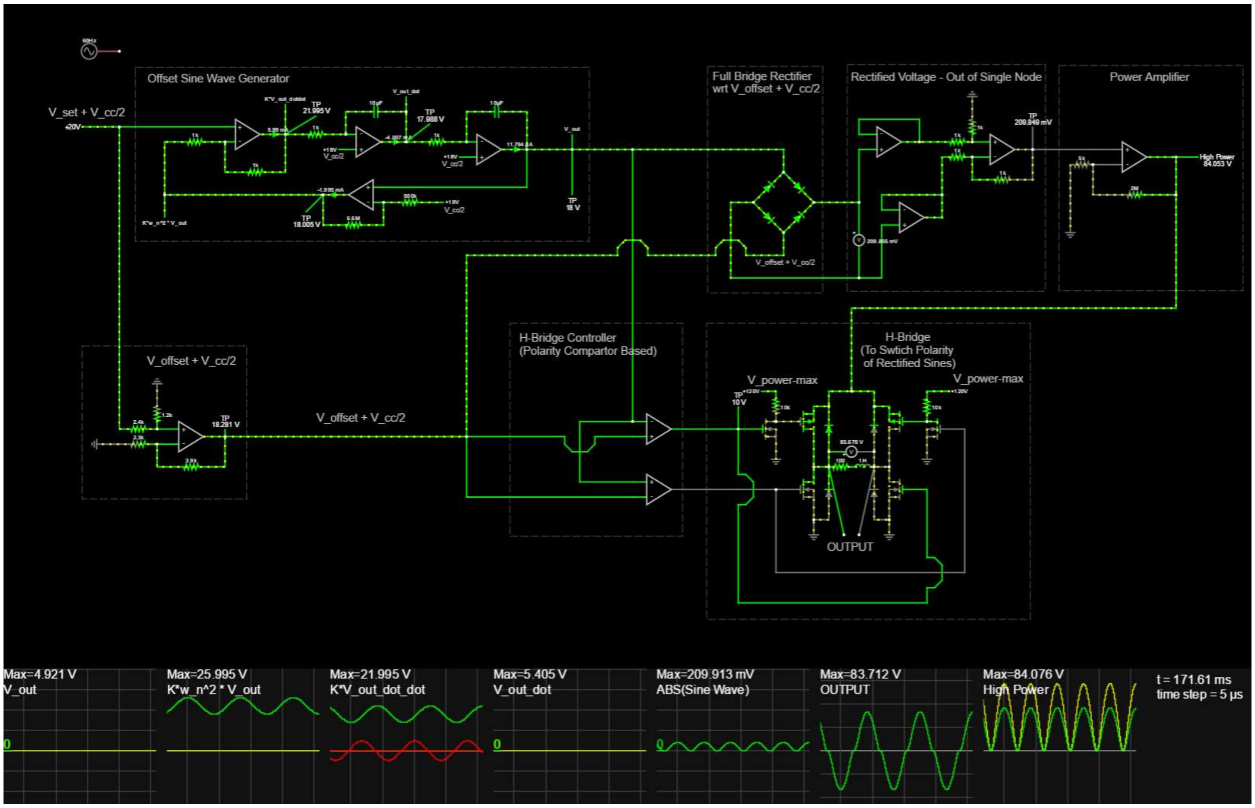


The resulting voltage across the applied load is then:



Note the distortion around the 0V line. This is due to the minimum voltage difference between the gate and drain for the MOSFETs to start conducting

The entire machine put together using analog sine wave generator:



The entire machine put together sing filtered square wave timer (Note, in the simulation the reference voltage on the rectifier isn't $V_{offset} = \frac{V_{cc}}{2}$, since the sine wave hovers a little under it for some reason. If you were building this, you could determine the reference voltage experimentally):