Natural Deduction on Predicate Logic

All this work is based on the book *Logic and Structure* by Dirk Van Dalen.

Strategy to build correct derivations in Predicate Logic

From an university test:

Let $\Gamma = \{(\exists x)P_1(x), (\exists x)(\exists y)(\neg P_2(x,y)), (\forall x)(P_1(x) \rightarrow (\exists y)P_2(x,y))\}$

Build a derivation that proves the following:

$$\Gamma \cup \{(\forall x)(\forall y)(P_3(x,y) \leftrightarrow P_2(x,y))\} \vdash (\exists x)(\exists y)(\neg P_3(x,y))$$

First of all build the derivation without canceling any hypotheses:

$$
\begin{prooftree}
\AxiomC{$(\exists x)(\exists y)(\neg P_2(x,y))$}
\AxiomC{$(\exists y)(\neg P_2(x,y))$}
\AxiomC{$\neg P_2(x,y)$}
\AxiomC{$(\forall x)(\forall y)(P_3(x,y) \leftrightarrow P_2(x,y))$}
\RightLabel{$E_{\forall *6}$}
\UnaryInfC{$(\forall y)(P_3(x,y) \leftrightarrow P_2(x,y))$}
\RightLabel{$E_{\forall *5}$}
\UnaryInfC{$P_3(x,y) \leftrightarrow P_2(x,y)$}
\AxiomC{$P_3(x,y)$}
\RightLabel{$E_{\leftrightarrow 1}$}
\BinaryInfC{$P_2(x,y)$}
\RightLabel{$E_{\neg}$}
\BinaryInfC{$\bot$}
\LeftLabel{(3)}
\RightLabel{$I_{\neg}$}
\UnaryInfC{$\neg P_3(x,y)$}
\RightLabel{$I_{\exists *4}$}
\UnaryInfC{$(\exists y)(\neg P_3(x,y))$}
\RightLabel{$I_{\exists *3}$}
\UnaryInfC{$(\exists x)(\exists y)(\neg P_3(x,y))$}
\LeftLabel{(2)}
\RightLabel{$E_{\exists *2}$}
\BinaryInfC{$(\exists x)(\exists y)(\neg P_3(x,y))$}
\LeftLabel{(1)}
\RightLabel{$E_{\exists *1}$}
\BinaryInfC{$(\exists x)(\exists y)(\neg P_3(x,y))$}
\end{prooftree}
$$

During the construction the following hypotheses were generated:

1. $(\exists y)(\neg P_2(x,y))$
2. $\neg P_2(x,y)$
3. $P_3(x,y)$

Next, start checking every atomic step top-down. In each atomic step, if the rule needs proofs we write them down. If the rule generated hypotheses we cancel them. If you can give a correct proof for all rules that need it, the derivation is correct.

$E_{\forall *6}$:

- This rule doesn't generate hypotheses.
- ***6 justification:** x is free for x in any formula (in particular, in: $(\forall y)(P_3(x,y) \leftrightarrow P_2(x,y))$).

$E_{\forall *5}$:

- This rule doesn't generate hypotheses.
- ***5 justification:** y is free for y in any formula (in particular, in $(P_3(x,y) \leftrightarrow P_2(x,y))$).

$E_{\leftrightarrow 1}$:

- This rule doesn't generate hypotheses and doesn't need justification.

$E_{\neg 1}$:

- This rule doesn't generate hypotheses doesn't need justification.

$I_{\neg 1}$:

- This rule generates the hypotheses $(P_3(x,y))$, se let's cancel it.
- This rule doesn't need justification.

The derivation at this step is:

$$(1)\,\dfrac{(\exists x)(\exists y)(\neg P_2(x,y)) \qquad (2)\,\dfrac{(\exists y)(\neg P_2(x,y)) \qquad \dfrac{\dfrac{(3)\,\dfrac{\dfrac{\neg P_2(x,y) \qquad \dfrac{\dfrac{\dfrac{(\forall x)(\forall y)(P_3(x,y)\leftrightarrow P_2(x,y))}{(\forall y)(P_3(x,y)\leftrightarrow P_2(x,y))}E_{\forall *6}}{P_3(x,y)\leftrightarrow P_2(x,y)}E_{\forall *5} \qquad [P_3(x,y)]^3}{P_2(x,y)}E_{\leftrightarrow 1}}{\bot}E_{\neg}}{\neg P_3(x,y)}I_{\neg}}{(\exists y)(\neg P_3(x,y))}I_{\exists *4}}{(\exists x)(\exists y)(\neg P_3(x,y))}I_{\exists *3}}{(\exists x)(\exists y)(\neg P_3(x,y))}E_{\exists *2}}{(\exists x)(\exists y)(\neg P_3(x,y))}E_{\exists *1}$$

$I_{\exists *4}$:

- This rule doesn't generate hypotheses.
- ***4 justification:** y is free for y in any formula (in particular, in $\neg P_3(x,y)$).

$I_{\exists *3}$:

- This rule doesn't generate hypotheses.
- ***3 justification:** x is free for x in any formula (in particular, in $(\exists y)(\neg P_3(x,y))$).

$E_{\exists *2}$:

- This rule generates the hypotheses $\neg P_2(x,y)$, let's cancel it.
- ***2 justification:** $y \notin FV((\exists x)(\exists y)(\neg P_3(x,y)))$, $y \notin FV((\forall x)(\forall y)(P_3(x,y) \leftrightarrow P_2(x,y)))$.
  Note that there's no need to check if other hypotheses contain y free: ($P_3(x,y)$ ni en $\neg P_2(x,y)$) as in this step they are canceled.

The derivation at this step is:

$$(1)\,\dfrac{(\exists x)(\exists y)(\neg P_2(x,y)) \qquad (2)\,\dfrac{(\exists y)(\neg P_2(x,y)) \qquad \dfrac{\dfrac{(3)\,\dfrac{\dfrac{[\neg P_2(x,y)]^2 \qquad \dfrac{\dfrac{\dfrac{(\forall x)(\forall y)(P_3(x,y)\leftrightarrow P_2(x,y))}{(\forall y)(P_3(x,y)\leftrightarrow P_2(x,y))}E_{\forall *6}}{P_3(x,y)\leftrightarrow P_2(x,y)}E_{\forall *5} \qquad [P_3(x,y)]^3}{P_2(x,y)}E_{\leftrightarrow 1}}{\bot}E_{\neg}}{\neg P_3(x,y)}I_{\neg}}{(\exists y)(\neg P_3(x,y))}I_{\exists *4}}{(\exists x)(\exists y)(\neg P_3(x,y))}I_{\exists *3}}{(\exists x)(\exists y)(\neg P_3(x,y))}E_{\exists *2}}{(\exists x)(\exists y)(\neg P_3(x,y))}E_{\exists *1}$$

$E_{\exists *1}$:

- This rule generates the hypotheses: $(\exists y)(\neg P_2(x,y))$, let's cancel it.
- ***1 justification:** $x \notin FV((\exists x)(\exists y)(\neg P_3(x,y)))$, $x \notin FV((\forall x)(\forall y)(P_3(x,y) \leftrightarrow P_2(x,y)))$.
  Let's note that for this case also there's no need to check if any other hypotheses contain x free: ($P_3(x,y)$ ni en $\neg P_2(x,y)$) because they are all canceled.

The derivation at this step is:

$$(1)\,\dfrac{(\exists x)(\exists y)(\neg P_2(x,y)) \qquad (2)\,\dfrac{[(\exists y)(\neg P_2(x,y))]^1 \qquad \dfrac{\dfrac{(3)\,\dfrac{\dfrac{[\neg P_2(x,y)]]^2 \qquad \dfrac{\dfrac{\dfrac{(\forall x)(\forall y)(P_3(x,y) \leftrightarrow P_2(x,y))}{(\forall y)(P_3(x,y) \leftrightarrow P_2(x,y))}E_{\forall *6}}{P_3(x,y) \leftrightarrow P_2(x,y)}E_{\forall *5} \qquad [P_3(x,y)]^3}{P_2(x,y)}E_{\leftrightarrow 1}}{\bot}E_{\neg}}{\neg P_3(x,y)}I_{\neg}}{(\exists y)(\neg P_3(x,y))}I_{\exists *4}}{(\exists x)(\exists y)(\neg P_3(x,y))}I_{\exists *3}}{(\exists x)(\exists y)(\neg P_3(x,y))}E_{\exists *2}}{(\exists x)(\exists y)(\neg P_3(x,y))}E_{\exists *1}$$

And this, together with the justifications is the complete finished derivation.

1 How to use identity rules.

Special thanks to Prof. Luis Sierra (FIng UdelaR) for the corrections.

From an university test:

Build a derivation that proves:

$$(\forall x)P(x, f(x)), (\exists y)f(y) =' y \vdash (\exists z)P(f(z), z)$$

First of all, looking at the premises we can figure out that to reach the conclusion, using substition in the application of the rule $E_\forall$ won't lead us anywhere. For example, if we wanted to reach $(\exists z)P(f(z), z)$ with the premise $(\forall x)P(x, f(x))$, we should apply the rule $E_\forall$. If we wanted to obtain $f(z)$ instead of x, when substituting, we would get $P(f(z), f(f(z)))$. The only way to swawp the arguments of P is by using the rule $RI4'$.

Let's build the derivation:

All we can do with the premises we have is an existential elimination:

$$(1)\ \frac{(\exists y)f(y) =' y \qquad (\exists z)P(f(z), z)}{(\exists z)P(f(z), z)}\, E_{\exists *1}$$

Here, we obtained the hypotheses $f(y) =' y$. Checking the premises again we can see that with $(\forall x)P(x, f(x))$, by applying the rule $E_\forall$ and substituting adequately we obtain $P(y, f(y))$. If we manage to substitute in this formula in a way that we get $P(f(y), y)$, applying the rule $I_\exists$ and substituting adequately we can obtain $(\exists z)P(f(z), z)$. We can do this by using the identity rule $RI4'$. Doing this, the resulting derivation is:

$$(1)\ \cfrac{(\exists y)f(y) =' y \qquad \cfrac{\cfrac{\cfrac{(\forall x)P(f(x), x)}{P(f(y), y)} \qquad f(y) =' y \qquad \cfrac{f(y) =' y}{y =' f(y)}}{P(f(y), y)}}{(\exists z)P(f(z), z)}}{(\exists z)P(f(z), z)}\, E_{\exists *1}$$

By adding the rules in each step and using the strategy of the previous example, we have built the derivation:

***4**: y free for x in $P(f(x), x)$.
***3**: y free for z_1; $f(y)$ free for z_2 en $P(z_1, z_2)$.
$f(y)$ free for z_1; y free for z_2 en $P(z_1, z_2)$.
***2**: y free for z en $P(f(z), z)$.
***1**: $y \notin FV(\{(\exists z)P(f(z), z), (\forall x)P(f(x), x)\})$.

$$(1)\ \cfrac{(\exists y)f(y) =' y \qquad \cfrac{\cfrac{\cfrac{[f(y) =' y]^1}{y =' f(y)}\, RI2 \qquad [f(y) =' y]^1 \qquad \cfrac{(\forall x)P(x, f(x))}{P(z_1, z_2)[y/z_1, f(y)/z_2])}\, E_{\forall *4}}{P(z_1, z_2)[f(y)/z_1, y/z_2)]}\, RI4'_{*3}}{(\exists z)P(f(z), z)}\, I_{\exists *2}}{(\exists z)P(f(z), z)}\, E_{\exists *1}$$