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Abstract

Swap spreads, the interest rate differentials between the fixed rates on fixed-for-floating swap contracts and the yields-to-maturity on maturity-matched government bonds, define a market for one of the most actively transacted securities in the global fixed-income arena. A large universe of fixed-income securities including corporate bonds and mortgaged-back securities use interest rate swap spreads as a key benchmark for pricing and hedging. Swap spreads have received renewed attention since the Fall of 1998 when their volatile movements contributed in a significant way to the financial turmoil that led the US Fed to cut short-term interest rates by 75 basis points. In this paper we present new insights on how to analyze term structure of interest swap spreads. Specifically, we focus on the determinants of swap spreads and show how quantities such as the spread of short-term LIBOR over GC-repo rates, the liquidity premium commended by government bonds, and the risk premium required for holding long-term bonds/swaps jointly determine term structures of swap spreads.

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1 Introduction

Swap spreads, the interest rate differentials between the fixed rates on fixed-for-floating swap contracts and the yields-to-maturity on maturity-matched government bonds (priced at par), define a market for one of the most actively transacted securities in the global fixed-income arena. A large universe of fixed-income securities including corporate bonds and mortgaged-back securities use interest rate swap spreads as a key benchmark for pricing and hedging. Swap spreads have received renewed attention since the Fall of 1998 when their volatile movements contributed in a significant way to the financial turmoil that led the US Fed to cut short-term interest rates by 75 basis points.

In this paper we present new insights on how to analyze term structures of swap spreads. Our objective is to set up an analytical framework so as to better explain some of the extraordinary movements in interest rate swaps or swap spreads observed in recent years. Specifically, we focus on the determinants of swap spreads, and show how quantities such as *short-term financing spreads* (i.e., LIBOR rates over GC-repo rates), the *liquidity premium* commended by government bonds, and the *risk premium* required for holding long-term government bonds or swaps jointly determine term structures of swap spreads.

When an investor enters a swap agreement as a fixed receiver in a plain-vanilla fixed-for-floating swap, the investor is promised to receive from the counterparty a series of semi-annual fixed payments in exchange for paying the counterparty a series of semi-annual floating payments. While the fixed payments are determined at the outset of the swap agreement, the floating payments are to be determined at later times, based on the six-month LIBOR rates prevailing at the beginning of each payment period. Given the apparent simple structure of swaps, it is evident that swap spreads can only be originated from the following three sources:

- the credit worthiness of the counter-parties involved;
- the floating leg (LIBOR) and its spread over the GC-repo;
- the yield curve shape or term structure of default-free interest rates.

The credit worthiness of counterparties has traditionally been taken as the primary factor affecting the fair market swap rates or swap spreads. Indeed, an overwhelming number of academic studies have all assumed that the main driving force behind the interest rate differentials between swaps and government bonds is due to the risk of counterparty default on its swap obligation, see Bollier and Sorensen (1994), Cooper and Mello (1991), Duffie and Huang (1996), and Litzenberger (1992), and etc. While this assumption was reasonable when the swap market was at its early stage of development, the current industry practice of swap agreements assumes that both counterparties enter a netting and collateral agreement, known as the *Master Swap Agreement*. Under this agreement, both parties net out all of the exiting swap positions and impose collateral against each other based on daily net mark to market values of all open positions. The current industry practice has essentially removed (in a significant way if not completely) the risk of default by either counterparty so that, for all practical purposes, swaps shall be valued without the consideration of counterparty risk. In a world in which counterparty default risk can be totally ignored, an investor receiving fixed in a swap agreement is equivalent to holding a long position in a default-free government bond while at the same time financing the long position by paying a six-month LIBOR interest rate.

If the risk of counterparty default is excluded from the list of potential sources for the interest rate differentials between swap rates and government bond yields, then the implicit credit risk associated with the floating leg in a swap contract becomes our next target for explaining swap spreads. Indeed, in determining the fair market swap rates relative to the equivalent government bond yields, we consider a competing investment strategy: buy a government bond with the same maturity as the swap while financing the purchase of the government bond via a repurchase agreement (repo), say at a standard term repo rate for general collateral (GC). Since the credit worthiness of the banks involved in resetting LIBOR rates ranges from AA to A (or worse), the six-month LIBOR rates have traditionally been quoted at a positive spread over the six-month GC term repo rates. Consequently, it is natural to expect that investors who receive fixed in a fixed-for-floating swap be awarded with a fixed rate that is slightly higher than the yield-to-maturity on an otherwise comparable government bond (priced at par). In this manner, investors are compensated for the extra financing cost that they have to bear due to the credit premium inherited in the six-month LIBOR rates. In other words, the fair market swap spreads are existed as a compensation for the short-term financing spreads between the six-month LIBOR rates and the six-month term GC-repo rates.

Notwithstanding our fair value theory based on short-term financing spreads, casual empirical observations suggest that short-term financing spreads have traditionally been well-behaved, fluctuating narrowly around a mean of 15-25 basis points, even during the period of financial crisis in 1998. In other words, the volatility we observed in recent swap market cannot possibly be triggered by the volatility of short-term financing spreads. This leads us to pursue other economic factors affecting term structures of swap spreads, namely the *liquidity premium* and the *risk premium* for holding bonds relative to holding swaps.

In this paper we present a multi-factor term structure framework to analyze the determinants of swap spreads. Our model assumes that all swaps are traded *default-free*. We first show in a formal setting that swap rates and their spreads over the default-free government bond yields are fully determined by the dynamics of the short-term financing spreads. The fair market swap rates are set in such a way that the present value of the (default-free) fixed payments equalizes the present value of the (default-free) floating payments. Under the assumption that the short-term financing spreads are independent of the current term structure of default-free interest rates, the fair market swap spreads can be shown to equal the present value of the short-term financing spreads properly amortized over the swap maturity.

Next, we take a closer look at the relationship between the term structure of swap rates and the term structure of government bond yields. Specifically, we emphasize that the slope and the curvature of the government curve plays an important role in determining the slope and the curvature of the swap curve or the swap spread curve, and that this relationship is closely related to the notion of liquidity premium and risk premium. More formally, term structures of interest rates are governed by three factors: level, slope and curvature, see Litterman and Scheinkman (1993). The level of default-free interest rates is set by the central bank or the monetary policy makers in accordance with the state of the economy. The slope and the curvature of the term structure of default-free interest rates are affected by a) the outlook of future interest rates, b) the liquidity premium commended by government bonds, and c) the risk premium that investors demand for holding long-term bonds. We note that

- The overall level of default-free interest rates shall have a direct impact on the overall level of swap rates but not necessary on the overall level of swap spreads;[1]

- Expectations of interest rate movements in the near and distant future affect the slope of the bond yield curve in the short and intermediate sectors. But, such effect is equally applicable to swaps. Thus, the net effect on the slope of swap spreads in the short and intermediate sector may be trivial. In other words, the slope of swap rates shouldn't be more or less affected by the expectations of future interest rates than the slope of government bond yields does;

- A large population of fixed-income money managers are mandated to invest a significant proportion of their assets in government bonds. Investors' demand for government bonds creates liquidity in the market for government bonds. In contrast, fixed-income money managers are typically forbidden from using swaps as an investment tool. The natural uses of swaps

[1] While some earlier papers have found a weak negative correlation between the level of interest rates and the swap spreads, such relationship has not been shown to be consistent over time.

are corporations or dealers involving in long-term financing or hedging. The special liquidity premium that government bonds command may cause the bond yield curve to be less steeper than it would otherwise be. The liquidity deferential in bonds and swaps may cause the swap curve to be steeper than the bond curve, which implies a steeper term structure of swap spreads. The liquidity premium for on-the-run bonds provides a further evidence that swap spreads may be widened or narrowed as the liquidity premium associated with the on-the-run bonds increases or decreases.

- Investors demand risk premium as compensation for holding long-term government bonds. This tends to affect the slope of the yield curve in the long-end or the overall curvature of the yield curve. Similarly, investors in the swaps market also demand risk premium for holding long-term swaps. Depending upon the supply and demand for long-dated swaps, the risk premium demanded by swap investors may be significantly different from those of bond investors. This may cause the slope of swap spreads in the long end to be either upward sloping or downward sloping, resulting an upward sloping or a humped spread curve.

To summarize, the liquidity premium and the risk premium associated with bonds and swaps play an important role in determining the slope and the curvature of the term structure of swap spreads. We formalize our intuitive discussions using a multi-factor/arbitrage-free term structure model.

A number of papers have done work on swap spreads that are closely related to this paper. Nielsen and Ronn (1996) contain a model with a one-factor instantaneous spread process that values swaps as the appropriately discounted expected value of the instantaneous TED spread (or financing spreads as called in this paper). Collin-Dufresne and Solnik (1999) also argue that swaps shall be valued as default-free and use the concept to compare the LIBRO curve to the swap curve. Grinblatt (1999) introduces a concept of convenience yield for holding government bonds due to its liquidity advantage and models swap spreads as the present value of a flow of convenience yields. His approach is consistent with our liquidity premium argument. However, it ignores the value of short-term financing spreads as compensation for the extra cost that swap investors have to bear.

The majority of research work on swap spreads focus on proper modeling and measuring of default risk so that interest rate swaps can be valued with default risk being fairly accounted for. There are basically two classes of credit models that fall into this line of research. The first class of models takes a *structural* approach, modeling default events as the first-passage time some economic variables fall below or hit certain pre-specified triggering level, see Merton (1974), Black and Cox (1976), Cooper and Melllo (1991), Hull and White (1991), Longstaff and Schwartz (1995), Leland (1994), and Leland and Toft (1995). Under this approach, defaults are endogenously triggered by the value of the underlying assets or firms, and are usually predictable. The second class of models takes a *reduce-form* approach, modeling default events as being triggered by an exogeneously specified jump process, see Das and Tufano (1996), Duffie and Huang (1996), and Duffie and Singleton (1999), Jarrow and Turnbull (1996), Jarrow, Lando, and Turnbull (1997), Madan and Unal (1994). Under the reduced-form approach, default events are typically unpredictable.

In addition to theoretical work, there is a strand of empirical literature on the determinants of swap spreads. While these papers have shown some statistical relationships between swap spreads and level of interest rates or other economic variables. These relationships have not been found consistently over time, see Brown, Harlow, and Smith (1994), Chen and Selender (1994), Evans and Bales (1991), Minton (1993), Sun, Sundaresan and Wang (1993).

The rest of the paper is organized as follows. Section 2 provides some institutional materials on swap markets. Section 3 sets up the model and shows that the fair market swap spreads equal the present value of the short-term financing spreads properly amortized over the swap maturity. Section 4 provides models of term structures of swap rates using the 1-factor, 2-factor and 3-factor setting. We conclude the paper in Section 5.

2 Institutions and Recent Market Experiences

In this section we present institutional information on swap markets. Specifically, we explain the process by which short-term LIBOR rates are determined, and illustrate how short-term LIBOR rates may influence long-term swap rates or swap spreads. We then present historical evidence of swap spreads for three major currencies, USD, EUR and JPY. Finally, we present recent experiences of swap markets, including events such as the explosion of Japan Premium in 1997, the financial crisis in 1998, the Y2K liquidity problem in 1999, and more recently, the US Treasury buy-back in 2000.

2.1 Swap Contracts, LIBOR Fixing and Financing Spreads

Since the inception of swap contracts in 1981, markets for generic interest rate swaps, cross currency interest rate swaps, and related swap options (swaptions) have experienced tremendous growth in the past twenty years. Currently, swap contracts are popular for all major currencies, although USD, EUR and JPY represent the three most demanded currencies with swapping needs. The total notional amount of outstanding swaps is estimated around 6 trillion USD, a size that is comparable to the size of global government bond market. Indeed, for many of the overseas markets where either the size of government bonds is too small or the liquidity is not readily available for trading government bonds, swap contracts are more popular than government bonds. However, there is a major difference between owning a swap and owning a bond. When a bond position is sold, it leaves the trading book, whereas market participants tend to close out a swap position by entering a new offsetting swap contract. As a result, the notional amount of swaps outstanding may be significantly exaggerated.

When an investor enters a swap agreement as a fixed receiver in a generic fixed-for-floating swap, the investor is promised to receive from the counterparty a series of semi-annual fixed payments in exchange for paying the counterparty a series of semi-annual floating payments. While the fixed payments are determined at the outset of the swap agreement, the floating payments are to be determined at later points in time, based on the six-month LIBOR rates prevailing at the beginning of each payment period. We have argued in the introduction that the spread between the six-month LIBOR rates and GC repo rates (or simply the short-term financing spread) represents an important factor in determining the term structure of swap spreads. Loosely speaking, the larger the short-term financing spreads, the larger the overall level of swap spreads. The term structure of swap spreads is closely related to market expectations of future short-term financing spreads.

The six-month LIBOR rate used for settling the floating leg of swap payments is based on a composite rate compiled by the British Banking Association (BBA) each day at 11:00am London time. The composite rate is calculated based on quotes provided by a basket of reference banks selected by BBA. A total of 16 banks are polled for their quotes of deposit rates from 1 week to 12 month. For each maturity, the highest four quotes and the lowest four quotes are dropped and the average of the rest of the eight banks is used as the official LIBOR fixing. The following table provides an example of LIBOR rates posted for the three major currencies, namely USD, JPY and EUR, as of May 8, 2000.

	US	EUR	JAPAN
1w	6.1809	4.0556	0.0662
1m	6.4612	4.1393	0.0812
2m	6.5725	4.2237	0.0937
3m	6.7000	4.3100	0.1037
6m	6.9137	4.4750	0.1437
12m	7.3300	4.7581	0.2637

For these three currencies, LIBOR rates are calculated based on the following set of reference banks:

- **USD**: BTM, Barclays, Citibank, CR Swiss, NetWest, BoA, Abbey, Norin BK Chase, Deutsche, Fuji, HSBC, Lloyds, West LB, RB Scot, RaboBank

- **EUR**: BTM, Barclays, Citibank, CR Swiss, NatWest, Chase, Lloyds, Halifax, BoA, Deutsche, MGT, HSBC, RBC, So Gen, UBS, West LB

- **JPY**: BTM, Sumitomo, Fuji, DKB, BOA, UBS, HSBC, NatWest, Chase, IBJ, Deutsche, Barclays, RaboBank, Norin, West LB, MGT, Sanwa

The deposit rates quoted by each reference bank incorporate both the credit premium (over the default-free short-term interest rate) that investors may demand for as well as the potential rate hikes or cuts by the central bank over the term under consideration. By the way of its design, the six-month LIBOR rate, chosen as the benchmark index of the floating leg of a generic swap contract, serves as a market indicator of credit worthiness of the banking sector. Several points are worth emphasizing here.

1. The credit quality among reference banks varies greatly, ranging from banks with AAA rating to banks with BBB rating.

2. There is a subset of brand-name banks serving as a part of the reference banks for all three major currencies;

3. In the past, banks with deteriorating credit quality in the LIBOR basket have been replaced by better banks. Thus, there is a tendency for LIBOR to maintain the overall credit quality of the basket at a consistent and stable level.

4. Since the credit quality for the basket banks is the average of the LIBOR rates offered by different banks, there is some diversification effect as well.

In a perfect world in which the set of reference banks for all three major currencies are identical, short-term financing spreads (i.e., LIBOR over GC-repo) should be theoretically identical across different currencies. Consequently, swap spreads across different currencies ought to be theoretically identical, if the dynamics of short-term financing spreads is the only factor driving term structures of swap spreads. In such a world, variations in swap spreads may have to be explained by differences in investors' attitude towards risk or differences in liquidity premium and risk premium across difference countries.

Recent histories of short-term financing spreads (from 1996-2000) are reported in Figures 1(a), 1(b) and 1(c) for all three major currencies. The financing spread for US is measured as the difference between the 1-week dollar LIBOR rate and the Fed fund target rate. Note that Fed Fund target rate is usually used as the benchmark for pricing government bond repos. While the actual Fed fund rates were available, the Fed fund target rate represents a better proxy for GC-repos.[2] The financing spreads for Japan is calculated as the difference between the 1-week Yen

[2]For our purpose, the actual Fed fund rates are too volatile due to banks' bi-weekly refunding activity for meeting the reserve requirement.

LIBOR rate and the call rate, which is the Japanese equivalent of the Fed Fund rate and usually used as benchmark for pricing JGB repos. Finally, the financing spread for Germany is measured as the difference between the 1-week EURO LIBOR rate and the by-weekly repo target rate (which is currently set by the European Central Bank and previously set by the Bundesbank). All data are obtained from Bloomberg. Figures 1(a)-(c) show that short-term financing spreads have been well-behaved with a mean average of under 25 basis points. While there are a number of large spikes in these figures, the spikes appeared only at the term of the year, a well-understood phenomenon known as *the term effect*. Apart from those spikes, the short-term financing spread processes for all three currencies are not wildly volatile.

2.2 Historical Evidences

Historically, swap spreads have been volatile but reasonably well-behaved, exhibiting a strong degree of mean reversion. However, such historical relationship collapsed in 1998 in the aftermath of Russian default. Global swap spreads were blown out to a level that was never seen in recent years. Flight to quality and concern of systematic meltdown in the financial sector are forces behind the spread widening. While global swap spreads contracted in early 1999, they were blown out again in the second half of 1999 due to concerns over Y2K. Currently, US swap spreads once again reach their historical highs due to government budget surpluses and US Treasury Department's decision to buy back 30 billion dollars worth of treasury bonds in 2000.

In US, swap rates are quoted simply as spreads over the on-the-run treasuries. As a result, swap spreads data can be obtained with ease. The following table provides a summary statistics on 10yr USD swap spreads in recent years.[3] These data are obtained from Bloomberg, and are reported in basis points.

	1996	1997	1998	1999	2000
Mean	37	40	63	84	101
Std	5	6	15	10	19
High	45	54	90	113	129
Low	31	32	47	66	72

Swap rates in Germany and Japan are quoted simply in terms of their actual rates. As such, swap spreads need to be calculated based on the difference between the swap rates and yields on maturity matched par government bonds. The yields on government bonds priced at par are usually obtained by fitting a discount curve to the prices of German Bunds or Japanese Government Bonds (JGBs). The par rates implied by the fitted discount curve are considered as the yields on government bonds

[3]Swap spreads reported here are not adjusted for the on-the-run specialness. Had these spreads been adjusted for the specialness, we shall see a sizable decline in these spreads.

priced at par. The following table provides a summary statistics of 10-year Euro/D-Mark swap spreads (over the German Bunds). These data are obtained from Bloomberg, and are reported in basis points.

	1996	1997	1998	1999	2000
Mean	–	24	36	40	42
Std	–	5	11	5	4
High	–	29	62	51	50
Low	–	15	20	31	36

The same statistical summary of the 10-year JPY swap spreads (over the Japan Government Bonds) is reported in the following table.

	1996	1997	1998	1999	2000
Mean	19	26	39	35	32
Std	5	5	6	10	3
High	27	35	55	51	36
Low	4	18	25	11	29

We note that while global swap spreads widened almost simultaneously in the Fall of 1998, 10yr swap spreads in Japan collapsed to 11 basis points in early 1999 and have since resumed to its normal range of mid 30s. Swap spreads in Germany continue to be weakly correlated with US swap spreads, even though US swap spreads have maintained at their historical highs since 1998 while German swap spreads have resumed its normal range.

2.3 Global Swap Spreads Since 1998

We now take a closer look at term structures of swap spreads since the beginning of 1998. In the beginning of 1998, global swap spreads were at their normal level as shown in the following table:

	2yr	5yr	10yr	20/30yr
US	38	43	50	40
Germany	10	18	26	10
Japan	21	17	28	20

The term structure of swap spreads in US is typically humped, upward sloping from 0 to 10-year and downward sloping thereafter. The mean 10-year swap spreads level is around 40-50 basis points. Swap spreads in Germany and Japan are historically lower than their US counterpart. The term structure of swap spreads in Germany is also humped, while the term structure of swap spreads in Japan is decreasing (from 0-3yr)[4] and humped thereafter. Global swap spreads were blown out in the Fall of 1998 in the aftermath of Russian default. Flight to quality and concern of systematic meltdown in the financial sector are forces behind the spreads widening. By Sep 1998, they reached a high level as shown below:

[4]This phenomenon is due to the Japan Premium, see discussions below.

	2yr	5yr	10yr	20/30yr
US	73	78	97	94
Germany	17	30	54	27
Japan	23	33	55	55

We note that term structure of swap spreads in Germany and Japan became much steeper than it used to be.

While global swap spreads contracted in early 1999, they were blown out again in the second half of 1999 due to concerns over Y2K:

	2yr	5yr	10yr	20/30yr
US	62	86	102	108
Germany	16	17	46	23
Japan	27	33	60	32

Some of the real money investors refuse to lend out securities over the term of the century for fear of settlement risk. This triggered a serious short squeeze in the repo market. To alleviate liquidity concerns, central banks in various countries injected extra amount of liquidity into the system. In US, Fed enlarged the pool of securities eligible for repo transactions with the Fed. It also initiated liquidity options as an additional measure to secure Y2K funding. By historical standard, recent levels of swap spreads have been extraordinary wide. The last time swap spreads widened to the current level was in 1987 (stock market crash) and 1990 (S&L crisis). We note that this time the term structure of swap spreads in US became upward sloping.

Since the beginning of 2000, as US Treasury Department announced buying back 30 Billion dollars worth of US government bonds, swap spreads once again exploded to their highest level ever,

	2yr	5yr	10yr	20/30yr
US	74	94	129	140
Germany	17	24	37	35
Japan	20	22	30	20

The term structure of swap spreads in Germany as well Japan exhibits a humped shape, while the term structure of swap spreads in US is currently monotonically increasing, reflecting a strong demand for long-term bonds from real money investors. As a result, the risk premium demanded for holding long-term securities dropped significantly.

2.4 The Japan Premium

In late 1997, Japanese financial institutions suffered from one of the worst financial crises in recent history, as billions of dollars of non-performing loans were uncovered for virtually every Japanese bank. Several well-known financial institutions (e.g., Yamayichi Securities) went bankrupted. As a result, Japanese financial institutions faced credit crunch in the international capital market and had to pay a substantial premium in order to raise funds in the overseas market. This premium, popularly known as the *Japan Premium*, jumped as high as to 100 basis points. In other words, Japanese institutions would have to pay an extra 100 basis points relative to their foreign competitors in order to secure short-term borrowing.

The Japan Premium had a significant impact on the behavior of short-term LIBOR rates for Japanese Yen, and consequently, the behavior of short-term financing spreads. Since half of the reference banks for Yen LIBOR were based on Japanese names, 50% of the Japan Premium went into the Yen LIBOR. Moreover, the volatility of Japan Premium was also spilled over to the Yen LIBOR rates and short-term financing spreads. Specifically, higher Japan Premium led to higher YEN LIBOR rates, which in turn widened the financing spread between LIBOR and JGB repo rates. Ultimately, this created a pressure on Yen swap spreads. As we have shown early, the term structure of Yen swap spreads were downward sloping in the 0-3 year sector in later 1997 and early 1998. Figure 2 presents a recent history of the spreads between the 3-month TIBOR and the 3-month LIBOR, demonstrating the impact of the Japan Premium on LIBOR, TIBOR, and short-term financing spreads during that period.

2.5 A Short Summary

To summarize, we presented here recent experiences of swap spreads for three major currencies (USD, EUR and JPY). We illustrated the dynamic behavior of swap spreads in periods of normal economic environment and periods of financial crisis. We also showed a variety of term structures of swap spreads across different currencies under different economic environments (upward sloping (US), humped (US, Germany and Japan), and bi-modal (Japan)). The analytical framework to be presented in the next section will help us analyze the various term structures of swap spreads observed in the marketplace.

3 Arbitrage-Free Valuation of Swap Spreads

In this section we formally setup our model suitable for the construction of term structures of swap spreads. Taken as given is a probability space $(\Omega, P, \mathcal{F})$, where Ω denotes the states of nature, P the probability assessment for different states of nature, and $\mathcal{F}$ the information structure or filtration. Let r denote the stochastic process, defined on $(\Omega, P, \mathcal{F})$, for the instantaneous interest rate corresponding to the default-free government securities. In practice, this rate is best represented by the overnight interest rate charged on GC-repos (i.e., repurchase agreements for government bonds with general collateral). Let R be the stochastic process, also defined on $(\Omega, P, \mathcal{F})$, for the instantaneous interest rate corresponding to the LIBOR rates, which can be thought as the overnight deposit rate for banks with credit rating comparable to those in the reference banks of LIBOR fixing. Then,

$$\delta(t) = R(t) - r(t)$$

represents the instantaneous financing spread of LIBOR over GC repo. Given that banks are subject to the risk of default, it is expected that $\delta(t)$ is nonnegative.

Associated with the instantaneous interest rate processes r and R are the two discount curves that give the present value of one dollar to be paid at any future dates. Specifically, the discount curve corresponding to the government securities is given by

$$P_t(T) = \mathbf{E}^* \left[e^{-\int_t^{t+T} r(u)du} \right] \tag{1}$$

where $\mathbf{E}_t^*$ denotes an expectation under the equivalent martingale measure P^*, conditional on the information known at time t, and $P_t(T)$ denotes the price at time t of a discount bond that pays one unit at time $t+T$, see Harrison and Kreps (1979). Similarly, the discount curve associated with the LIBOR rates is given by[5]

$$Q_t(T) = \mathbf{E}_t^* \left[e^{-\int_t^{t+T} R(u)du} \right] = \mathbf{E}^* \left[e^{-\int_t^{t+T} (r(u)+\delta(u))du} \right] \tag{2}$$

The above formula is consistent with the reduce-form model of Duffie and Singleton (1999) for pricing defaultable securities. Indeed, if δ is interpreted as the product of hazard rate for default and fractional loss rate, then Q represents the price of a zero coupon bond subject to default risk. This interpretation is not necessary here for pricing swap rates, as all swaps considered in this paper are treated as default-free. However, we can rationalize the connection between the discount curve Q and the LIBOR rates (or LIBOR bonds). Note that under our setting δ is essentially observable from the financial market, which is extremely useful for model parameterization.

In the event that the spread process δ is independent of the default-free interest rate r,

$$Q_t(T) = P_t(T) \times \mathbf{E}_t^* \left[e^{-\int_t^{t+T} \delta(u)du} \right] = P_t(T) \times \Gamma_t(T)$$

where $\Gamma_t(T)$ is defined as

$$\Gamma_t(T) = \mathbf{E}_t^* \left[e^{-\int_t^{t+T} \delta(u)du} \right],$$

representing the credit adjustment associated with the LIBOR curve.

[5] At this point, we make a distinction between the discount curve corresponding to the LIBOR rates and the discount curve corresponding to the swap rates. This difference will be made clear below when we introduce the concept of par spreads and par swap spreads.

For purposes of understanding swap spreads, we take a look at various alternative forms of swap spreads that characterize more or less the same kind of credit differentials between LIBOR rates and government yields.

- (**Term Spreads**) Define the yields to maturity for the government and LIBOR curves as follows,

$$\begin{aligned} Y_T &= -\frac{1}{T}\ln P(T) \\ F_T &= -\frac{1}{T}\ln Q(T) \end{aligned}$$

  where $P(T) = P_0(T)$ and $Q(T) = Q_0(T)$. The *term spread* for maturity T is defined as the spread between the yields on these two zeros, $F_T - Y_T$. In the event that the instantaneous financing spread process δ is independent of the instantaneous default-free interest rate process r, we have

$$F_T - Y_T = -\frac{1}{T}\ln \mathbf{E}^*\left[e^{-\int_0^T \delta(u)du}\right] \tag{3}$$

  In this case, the term structure of term spreads is fully determined by the dynamic evolution of δ. For example, if δ is normally distributed, then

$$F_T - Y_T = \frac{1}{T}\mathbf{E}^*\left[\int_0^T \delta(u)du\right] - \frac{1}{2T}\mathbf{Var}\left[\int_0^T \delta(u)du\right]$$

  In particular, if $\delta(u) = \overline{\delta}$, a constant, then

$$F_T - Y_T = \overline{\delta}$$

  for every maturity T.

- (**Par Spreads**) An alternative way of looking at spreads between the government bond curve and the LIBOR curve is to compare the par rates implied by these two curves, known as *par spreads*. Specifically, we define par rates associated with the two zero curves as follows, for any maturity T,

$$\begin{aligned} Y_T &= \frac{2(1-P(T))}{\sum_{t=1}^{2T} P(t/2)} \\ F_T &= \frac{2(1-Q(T))}{\sum_{t=1}^{2T} Q(t/2)} \end{aligned}$$

  The par spread between government bonds and LIBOR rates is defined as $F_T - Y_T$. While we tied the discount factors Q to LIBOR rates in our definition, the industry practice is to construct Q based on market swap rates. Specifically, the discount factors Q are constructed in such a way that the par swap rates implied by Q fit market swap rates for all maturity. As such, par spreads defined here are called par swap spreads on the street when the discount factors Q are fitted to market swap rates, see Sundaresan (1991). We take a different approach here, as we will derive swap rates by assuming swaps are traded default-free.

- (**Spreads for Continuous Accruals**) In a simple one-period swap with continuous accrual, the floating-side receiver receives a lump sum payment at the maturity date based on a continuous compounding of the instantaneous LIBOR rates $e^{\int_0^T R(u)du}$, whereas the fixed-side receiver receives a lump sum payment at the maturity date based on a continuous compounding of a fixed rate $e^{F_T T}$. Under the assumption of no default, the fair fixed rate on such a swap can be determined by setting the present value of the floating side equal to the present value of the floating side:

$$\mathbf{E}^* \left[e^{-\int_0^T r(u)du} \times e^{\int_0^T R(u)du} \right] = e^{F_T T} P(T)$$

This implies that

$$F_T = Y_T + \frac{1}{T} \ln \mathbf{E}^* \left[e^{\int_0^T \delta(u)du} \right]$$

We note that we discount all cashflows at the default-free interest rate. The swap spreads for a continuous accrual, defined as the difference between the fixed swap rates and the yields on the corresponding default-free government bonds, is given by

$$F_T - Y_T = \frac{1}{T} \ln \mathbf{E}^* \left[e^{\int_0^T \delta(u)du} \right] \tag{4}$$

In the event that δ is normally distributed, then

$$F_T - Y_T = \frac{1}{T} \mathbf{E}^* \left[\int_0^T \delta(u)du \right] + \frac{1}{2T} \mathbf{Var} \left[\int_0^T \delta(u)du \right]$$

In particular, if $\delta(u) = \overline{\delta}$, a constant, then

$$F_T - Y_T = \overline{\delta}$$

for every maturity T.

- (**Swap Spreads for Discrete Accruals**) In a discrete accrual, the floating-side receiver receives a lump-sum payment at the maturity date by the amount that is equivalent to rolling a six-month time deposit every six months:

$$Q_0(\tau)^{-1} Q_\tau(2\tau)^{-1} \cdots Q_{2T\tau-\tau}(2T\tau)^{-1}$$

where T is the maturity of the swap and $\tau = 1/2$. The fixed-side receiver receives a lump-sum payment in the same way as in the previous case. Under the assumption of no-default, the fair fixed swap rate for the discrete accrual can be determined by setting

$$\mathbf{E}^* \left[e^{-\int_0^T r(u)du} \prod_{t=0}^{2T-1} \frac{1}{\mathbf{E}_{t/2}[e^{-\int_{t/2}^{t/2+\tau} R(u)du}]} \right] = e^{F_T T} P(T)$$

or equivalently

$$\mathbf{E}^* \left[\prod_{t=0}^{2T-1} \frac{e^{-\int_{t/2}^{t/2+\tau} r(u)du}}{\mathbf{E}^*_{t/2}[e^{-\int_{t/2}^{t/2+\tau} R(u)du}]} \right] = e^{F_T T} P(T)$$

Thus, the spread of the swap rate for a continuous accrual relative to the corresponding default-free government yield is

$$F_T - Y_T = \frac{1}{T} \ln \mathbf{E}^* \left[\prod_{t=0}^{2T-1} \frac{e^{-\int_{t/2}^{t/2+\tau} r(u)du}}{\mathbf{E}^*_{t/2}[e^{-\int_{t/2}^{t/2+\tau} R(u)du}]} \right] \tag{5}$$

In the event that r is independent of δ, we can simplify the above expression as

$$F_T - Y_T = \frac{1}{T} \ln \mathbf{E}^* \left[\prod_{t=0}^{2T-1} \frac{1}{\mathbf{E}^*_{t/2}[e^{-\int_{t/2}^{t/2+\tau} \delta(u)du}]} \right]$$

where we have used the following equation

$$\mathbf{E}^* \left[\frac{e^{-\int_{t/2}^{t/2+\tau} r(u)du}}{\mathbf{E}^*_{t/2}[e^{-\int_{t/2}^{t/2+\tau} R(u)du}]} \right] = \mathbf{E}^* \left[\frac{1}{\mathbf{E}^*_{t/2}[e^{-\int_{t/2}^{t/2+\tau} \delta(u)du}]} \right]$$

In particular, if $\delta(u) = \overline{\delta}$, a constant, we have

$$F_T - Y_T = \overline{\delta}$$

for every maturity T.

- (**Par Swap Spreads**) We are now ready to define a par swap and its spread over the equivalent par bond. A (semi-annual) par swap is an exchange of a series of (semi-annual) floating payments for a series (semi-annual) fixed payments, where the floating legs are reset on a semi-annual basis. Specifically, the floating receiver receives, at the end of each payment period t ($t = 1, \cdots, 2T$), an amount equivalent to the six-month LIBOR interest, i.e.,

$$\frac{1}{Q_{t/2-\tau}(\tau)} - 1, \quad (\tau = \frac{1}{2})$$

whereas the fixed-side receiver receives a fixed amount of $\frac{F_T}{2}$. The notional of one dollar is exchanged at the maturity date T. Under the assumption of no-default, the fair market fixed rate for a par swap can be determined in such a way that the market value of all floating payments equals to the market value of all fixed payments:

$$\sum_{t=0}^{2T-1} \mathbf{E}^* \left[e^{-\int_0^{t/2+\tau} r(u)du} \times \left(\frac{1}{Q_{t/2}(\tau)} - 1 \right) \right] = \sum_{t=1}^{2T} \frac{F_T}{2} P(t\tau)$$

or equvalently,

$$\sum_{t=0}^{2T-1} \left(\mathbf{E}^* \left[e^{-\int_0^{t/2} r(u)du} \times \frac{e^{-\int_{t/2}^{t/2+\tau} r(u)du}}{\mathbf{E}_{t/2}[e^{-\int_{t/2}^{t/2+\tau} R(u)du}]} \right] - P(t/2+\tau) \right) = \frac{F_T}{2} \sum_{t=1}^{2T} P(t\tau) \qquad (6)$$

Let Y_T denote the yield of a government bond priced at par, then Y_T is Determined as

$$1 - P(T) = \frac{Y_T}{2} \sum_{t=1}^{2T} P(t\tau)$$

or equivalently,

$$\sum_{t=0}^{2T-1} \left(\mathbf{E}^* \left[e^{-\int_0^{t/2+\tau} r(u)du} \right] - P(t/2+\tau) \right) = \frac{Y_T}{2} \sum_{t=1}^{2T} P(t\tau) \qquad (7)$$

Subtracting (7) from (6), we obtain

$$\sum_{t=0}^{2T-1} \mathbf{E}^* \left[e^{-\int_0^{t/2} r(u)du} \times \left(\frac{\mathbf{E}_{t/2}[e^{-\int_{t/2}^{t/2+\tau} r(u)du}]}{\mathbf{E}_{t/2}[e^{-\int_{t/2}^{t/2+\tau} R(u)du}]} - 1 \right) \right] = \frac{F_T - Y_T}{2} \sum_{t=1}^{2T} P(t\tau)$$

The left-hand-side of the equation is the present value of LIBOR-repo differentials or short-term financing spreads. Indeed, when δ is independent of r, we can simplify the above expression as

$$\sum_{t=0}^{2T-1} \mathbf{E}^* \left[e^{-\int_0^{t/2} r(u)du} \times \left(\frac{1}{\mathbf{E}_{t/2}[e^{-\int_{t/2}^{t/2+\tau} \delta(u)du}]} - 1 \right) \right] = \frac{F_T - Y_T}{2} \sum_{t=1}^{2T} P(t\tau) \qquad (8)$$

In particular, if $\delta(u) = \overline{\delta}$, a constant, then

$$\sum_{t=0}^{2T-1} \mathbf{E}^* \left[e^{-\int_0^{t/2} r(u)du} \times \left(e^{\delta\tau} - 1 \right) \right] = \frac{F_T - Y_T}{2} \sum_{t=1}^{2T} P(t\tau)$$

In summary, par swap spreads can be interpreted as the present value of the stream of short-term financing spreads, properly amortized over the swap payment period.

We point out that although the various forms of swap spreads defined in this section are based on different assumptions and different structures, these term structures of swap spreads, i.e., swap spreads as a function of maturity, shall share more or less the same shape. In other words, if term spreads are upward sloping (downward sloping or humped), then par spreads as well as par swap spreads will also be upward sloping (downward sloping or humped).

We shall also point out that even though swaps are priced as default-free, there is strong correlation between corporate spreads and swap spreads. Loosely speaking, par spreads are essentially corporate spreads for those banks that are involved in the LIBOR resetting. As such, the term structure of swap spreads is highly correlated with the term structure of corporate spreads. In practice, corporate spreads are defined for a combination of financial and non-financial firms while the credit quality of the reference banks in the LIBOR basket tends to be controlled by BBA. Consequently, corporate spreads are usually priced wider than swap spreads (see Collin-Dufresne and Solnik (1999)) and are highly correlated with swap spreads.

4 Term Structures of Swap Spreads

In this section we construct term structures of swap spreads using the basic analytical framework outlined in the previous section. Specifically, we analyze the term structure of swap spreads in three different settings. First, we take a one-factor normal model or Vasicek's model for the default-free instantaneous interest rate as well as for the instantaneous financing spread. We demonstrate how the dynamics of financing spreads affects term structures of swap spreads. Next, we extend our one-factor model to several multi-factor models and show how the dynamics of instantaneous financing spreads gives rise to various shapes of term structures of swap spreads. As we extend our model into multi-factor settings, we place emphases on term structure factors that reflect the liquidity premium and the risk premium associated with government bonds and swaps and show how they play a role in determining term structures of swap spreads.

4.1 Swap Spreads in a Single-Factor Setting

We first analyze term structures of swap spreads under the assumption of a one-factor Vasicek's model.[6] Specifically, we assume that the default-free instantaneous interest rate process r (i.e., the overnight GC-repo rate) and the instantaneous financing spread process δ (i.e., the overnight LIBOR rate minus the overnight GC-repo rate) are determined by the following stochastic processes, see

[6] Alternatively, we may choose a one-factor Cox, Ingersoll and Ross (1985)'s model as our basic interest rate process. Our results shall not be materially different.

Vasicek (1977),

$$\begin{aligned} dr &= k_r(\overline{r} - r)dt + \sigma_r dw_r \\ d\delta &= k_\delta(\overline{\delta} - \delta)dt + \sigma_\delta dw_\delta \end{aligned}$$

where k_r and k_δ are (non-negative) constants for the mean-reversion of the two processes, σ_r and σ_δ (positive) constants for volatility, $\overline{r}$ and $\overline{\delta}$ (positive) constants for the long-run mean, w_r and w_s are two independent Brownian motions under the probability measure P.

For the purpose of pricing under no-arbitrage, we seek an equivalent martingale measure P^* for r and δ so that under P^*,

$$w_r^* = w_r - \lambda_r dt, \quad w_\delta^* = w_\delta - \lambda_\delta dt$$

become two independent Brownian motions, where λ_r and λ_δ, respectively, are the market price of risk or the risk premium corresponding to the risk associated with each factor. Both risk premium parameters are usually positive. We assume for the purpose of this paper that λ_r and λ_δ are constants. Under the equivalent martingale measure P^*, we can write the stochastic dynamics of r and δ as

$$\begin{aligned} dr &= k_r(r^* - r)dt + \sigma_r dw_r^* \\ d\delta &= k_\delta(\delta^* - \delta)dt + \sigma_\delta dw_\delta^* \end{aligned}$$

where

$$\begin{aligned} k_r r^* &= k_r\overline{r} + \lambda_r\sigma_r \\ k_\delta\delta^* &= k_\delta\overline{\delta} + \lambda_\delta\sigma_\delta \end{aligned}$$

Note that when $k_r > 0$, $r^* = \overline{r} + \lambda_r\sigma_r/k_r$. Thus, the long-run mean of r under P^* is adjusted upward by a constant term. Moreover, the larger the risk premium of the short rate factor, λ_r, the larger the long term mean r^*. For example, with $k_r = 0.5$, $\lambda_r = 0.2$ and $\sigma_r = 0.01$, the long-run mean of r is adjusted upward by about 40bp. Similar statement can be made for δ^*.

For ease of illustrations, we first consider the term structure of term spreads.

Proposition 1 *The term structure of term spreads under our 1-factor setting is given by*

$$F_T - Y_T = \psi(k_\delta T)\delta_0 + (1 - \psi(k_\delta T))\delta^* - \eta_T \tag{9}$$

where $\psi(x) \equiv \frac{1-e^{-x}}{x}$ *and* η_T *is a deterministic function of* T*, given by*

$$\eta_T = \frac{\sigma_\delta^2}{2Tk_\delta^2}\int_0^T (1 - e^{-k_\delta(T-u)})^2 du$$

Based on Proposition 1, the following list of observations can be made on the term structure of term spreads:

- The term structure of term spreads mean-reverts to their long run mean of δ^*;
- An increase in the initial short-term financing spreads increases the level of term spreads in the short-end, and thereby flattens the term structure of term spreads;
- An increase in the long-run mean of the short-term financing spread increases the level of term spreads in the long-end, and thereby steepens the term structure of term spreads;
- When the current short-term financing spread is less (or greater) than its long run mean, the term structure of term spreads is *strictly* upward (or downward) sloping;
- The larger the volatility of short-term financing spreads, the smaller the overall level of term spreads. However, the volatility effect is almost negligible. For example, if $\sigma_\delta = 0.01$, $k_\delta = 0.45$ and $T = 10$, the volatility contribution to the 10yr term spread is a mere 1.5bp.
- Finally, the larger the risk premium λ_δ for δ, the larger the long-run mean δ^* and the steeper the term spread curve.

Figure 3 plots term structures of term spreads, par spreads and (default-free) par swap spreads for the following set of parameters:

$$k_r = 0.5,\ \sigma_r = 0.01,\ r(0) = 0.06,\ \overline{r} = 0.065,\ \lambda_r = 0.15$$
$$k_\delta = 0.5,\ \sigma_r = 0.0025,\ \delta(0) = 0.0025,\ \overline{\delta} = 0.0050,\ \lambda_\delta = 0.075$$

Par spreads are calculated based on the discount factors used for term spreads. In calculating par swap spreads, we make use of formula (8), which implies that,

$$F_T - Y_T = \frac{2}{\sum_{t=1}^{2T} P(t\tau)} \sum_{t=0}^{2T-1} P(t\tau)\Delta(t\tau)$$

where

$$\Delta(t) = \mathbf{E}^* \left[\frac{1}{\mathbf{E}_t[e^{-\int_t^{t+\tau} \delta(u)du}]} - 1 \right]$$

Under our assumption, $\delta(t)$ is normally distributed which allows us to derive a close form expression for $\Delta(t)$:

$$\begin{aligned} \Delta(t) &= \mathbf{E}^* \left[e^{\tau\psi(k_\delta\tau)\delta(t)+\tau(1-\psi(k_\delta\tau))\delta^*-\gamma(\tau)} - 1 \right] \\ &= e^{\tau\psi(k_\delta\tau)\mathbf{E}^*[\delta(t)]+\frac{1}{2}\tau^2\psi(k_\delta\tau)^2\mathbf{Var}^*[\delta(t)]+\tau(1-\psi(k_\delta\tau))\delta^*-\gamma(\tau)} - 1 \end{aligned}$$

where

$$\begin{aligned}
\gamma(\tau) &= \frac{\sigma^2}{2k_\delta^2}\int_0^\tau (1-e^{-k_\delta(\tau-u)})^2 du \\
\mathbf{E}^*[\delta(t)] &= e^{-k_\delta t}\delta_0 + (1-e^{-k_\delta t})\delta^* \\
\mathbf{Var}^*[\delta(t)] &= \sigma_\delta^2 t\psi(2k_\delta t)
\end{aligned}$$

We note that in Figure 3 the term structure of term spreads and the term structure of par spreads share much of the same shape. This is not surprising, as it is consistent with the observation that the spot yield curve usually has the same shape as the par yield curve. Figure 3 also indicates that the term structure of par spreads is almost exactly identical to the term structure of par swap spreads. The difference between the two, for the parameters chosen here, is less than 0.5 basis points. This is comforting, since it suggests that the current industry practice of fitting the par rates implied from the LIBOR zero curve to market swap rates is practically equivalent to fitting the default-free par swap rates to market swap rates. Given the close relationship between term spreads and par spreads or par swap spreads, for the rest of the paper, we will focus primarily on the term structure of term spreads for our purposes of studying term structures of swap spreads.

It is useful to note that when the mean-reversion coefficient of the instantaneous financing spreads equals that of the default-free instantaneous interest rates (as in Figure 3), we can collapse the two state variables into a new state variable. This allows us to characterize the term structure of swap rates using the new state variable. Specifically, letting $R = r + \delta$, we have

$$dR = k_R(\overline{R} - R)dt + \sigma_R dw_R$$

where $k_R = k_r = k_\delta$, $\overline{R} = \overline{r} + \overline{\delta}$, $\sigma_R = \sqrt{\sigma_r^2 + \sigma_\delta^2}$, and $\sigma_R dw_R = \sigma_r dw_r + \sigma_\delta dw_\delta$. Under this setup, the instantaneous interest rate associated with swap rates can be equivalent modeled by a single-factor R, which has a starting value of $R_0 = r_0 + \delta_0$, a long-run mean of $\overline{R} = \overline{r} + \overline{\delta}$ under P and a long-run mean of $R^* = r^* + \delta^*$ under P^*, implying a risk premium of

$$\lambda_R = \frac{\lambda_r\sigma_r + \lambda_\delta\sigma_\delta}{\sqrt{\sigma_r^2 + \sigma_\delta^2}}$$

In Figure 3, given the parameters for processes r and δ, we have

$$k_R = 0.5,\ \sigma_R = 0.0103,\ R(0) = 0.0625,\ \overline{R} = 0.0700,\ \lambda_R = 0.1637$$

The usefulness of this approach will be demonstrated in the next two sub-sections when we introduce multi-factor models of swap spreads.

4.2 Swap Spreads in a 2-Factor Setting

One of the critical disadvantages of single-factor interest rate models is that these models can only generate term structures of interest rates and term structures of swap spreads that are either upward sloping or downward sloping. Yield curve shifts implied by single-factor models can only explain curve reshapings corresponding to either a steepening or a flattening move. It is empirically known that two or three independent factors are necessary in order to characterize the dynamic behavior of yield curve movements, e.g., Litterman and Scheinkman (1993). It is equally plausible that two or three factors are required to describe the stochastic behavior of swap spread movements.

In this section, we extend the single-factor swap spread model of the previous section into a 2-factor framework in which the instantaneous default-free interest rates as well as the instantaneous financing spreads are driven by two latent state variables. Specifically, we assume that the instantaneous short-rate is given by

$$r = x + y$$

where the two stochastic latent factors are described by two mean-reverting processes:

$$\begin{aligned} dx &= k_x(\overline{x} - x)dt + \sigma_x dw_x \\ dy &= k_y(\overline{y} - y)dt + \sigma_y dw_y \end{aligned}$$

Here, k_x, k_y, $\overline{x}$, $\overline{y}$, σ_x and σ_y are (non-negative) constants, and w_x and w_y are two independent Brownian motions under the probability measure P. With appropriately chosen parameters, we can relate factor x to the overall interest rate level or long-term interest rate and factor y to the slope factor or spread between the short- and long-term interest. The sum of x and y becomes the instantaneous short rate.[7] For the purpose of pricing, we again seek the equivalent martingale measure P^* so that under P^*,

$$w_x^* = w_x - \lambda_x t, \quad w_y^* = w_y - \lambda_y t$$

are two independent Brownian motions, and the dynamics of x and y under P^* become

$$\begin{aligned} dx &= k_x(x^* - x)dt + \sigma_x dw_x^* \\ dy &= k_y(y^* - y)dt + \sigma_y dw_y^* \end{aligned}$$

where

$$k_x x^* = k_x \overline{x} + \lambda_x \sigma_x, \quad k_y y^* = k_y \overline{y} + \lambda_y \sigma_y$$

[7] In this 2-factor setup, the factor which has a slower speed of mean-reversion is likely to become the level factor; whereas the factor with a faster speed of mean-reversion is likely to become the slope factor.

Similarly, we can extend our single-factor financing spread process into a two-factor setting such that the process for the instantaneous financing spreads is given by

$$\delta = \delta_x + \delta_y$$

where δ_x and δ_y are two (independent) mean-reverting processes (independent of w_x and w_y):

$$\begin{aligned} d\delta_x &= k_{\delta_x}(\overline{\delta}_x - \delta_x)dt + \sigma_{\delta_x} dw_{\delta_x} \\ d\delta_y &= k_{\delta_y}(\overline{\delta}_y - \delta_y)dt + \sigma_{\delta_y} dw_{\delta_y} \end{aligned}$$

with constants k_{δ_x}, k_{δ_y}, $\overline{\delta}_x$, $\overline{\delta}_y$, σ_{δ_x} and σ_{δ_y}. With properly chosen parameters, δ_x can be related to the level of swap spreads, while δ_y can be interpreted as the slope of swap spreads. Together, $\delta_x + \delta_y$ forms the implied swap spread in the short end or the instantaneous financing spread. We assume that under the equivalent martingale measure P^*,

$$w^*_{\delta_x} = w_{\delta_x} - \lambda_{\delta_x} t, \quad w^*_{\delta_y} = w_{\delta_y} - \lambda_{\delta_y} t$$

are two independent Brownian motions and the dynamics of δ_x and δ_y under P^* become

$$\begin{aligned} d\delta_x &= k_{\delta_x}(\delta^*_x - \delta_x)dt + \sigma_{\delta_x} dw^*_{\delta_x} \\ d\delta_y &= k_{\delta_y}(\delta^*_y - \delta_y)dt + \sigma_{\delta_y} dw^*_{\delta_y} \end{aligned}$$

where

$$k_{\delta_x}\delta^*_x = k_{\delta_x}\overline{\delta}_x + \lambda_{\delta_x}\sigma_{\delta_x}, \quad k_{\delta_y}\delta^*_y = k_{\delta_y}\overline{\delta}_y + \lambda_{\delta_y}\sigma_{\delta_y}$$

We present our main result on the term structure of swap spreads in the following proposition.

Proposition 2 *The term structure of term spreads under our 2-factor setting is given by*

$$F_T - Y_T = \psi(k_{\delta_x}T)\delta_x(0) + \psi(k_{\delta_y}T)\delta_y(0) + (1 - \psi(k_{\delta_x}T))\delta^*_x + (1 - \psi(k_{\delta_y}T))\delta^*_y - \eta_T \tag{10}$$

where η_T is a deterministic function of T, given by

$$\eta_T = \frac{\sigma^2_{\delta_x}}{2Tk^2_{\delta_x}} \int_0^T (1 - e^{-k_{\delta_x}(T-u)})^2 du + \frac{\sigma^2_{\delta_y}}{2Tk^2_{\delta_y}} \int_0^T (1 - e^{-k_{\delta_y}(T-u)})^2 du$$

We can draw a number of conclusions based on Proposition 2. To make our discussions more specific, we assume that $k_{\delta_x} << k_{\delta_y}$ and k_{δ_x} is close to zero so that δ_x is highly correlated with the level of swap spreads while δ_y is highly correlated with the slope of swap spreads.

- The term structure of term spreads mean-reverts to their long run mean of $\delta^*_x + \delta^*_y$;

- An increase in $\delta_y(0)$ (i.e., the slope factor) increases the term spreads in the short end, and thereby flatten the term structure of term spreads. Similarly, an increase in $\delta_x(0)$ has the effect of increasing the overall level of term spreads and slightly flatten the spread curve;
- An increase in δ_x^* increases the level of term spreads in the long-end, and thereby steepen the term structure of term spreads. Similarly, an increase in δ_y^* has the effect of increasing the overall level of term spreads and slightly steepen the spread curve;
- The term structure of term spreads can be *strictly* upward, downward sloping, or humped;

Figure 4 plots the term structure of term spreads for the following set of parameters:

$$k_x = 0.001,\ \sigma_x = 0.010,\ x(0) = 0.06337,\ \overline{x} = 0.06,\ \lambda_x = 0.15$$
$$k_y = 0.500,\ \sigma_y = 0.015,\ y(0) = -0.0043,\ \overline{y} = 0.0,\ \lambda_y = 0.0$$
$$k_{\delta_x} = 0.001,\ \sigma_{\delta_x} = 0.0050,\ \delta_x(0) = 0.0060,\ \overline{\delta}_x = 0.0050,\ \lambda_{\delta_x} = 0.075$$
$$k_{\delta_y} = 0.500,\ \sigma_{\delta_y} = 0.0075,\ \delta_y(0) = -0.0045,\ \overline{\delta}_y = 0.0,\quad \lambda_{\delta_y} = 0.0$$

In contrast to single factor models, the 2-factor model can accommodate a variety of curve shapes for the term structure of swap spreads including humps, i.e., upward sloping for the first part of the curve and downward sloping for the second part of the curve.

We point out that under the setting of Proposition 2, the term structure of default-free interest rates is given by

$$Y_T = \psi(k_xT)x(0) + \psi(k_yT)y(0) + (1 - \psi(k_xT))x^* + (1 - \psi(k_yT))y^* - \epsilon_T$$

where

$$\epsilon_T = \frac{\sigma_x^2}{2Tk_x^2}\int_0^T (1 - e^{-k_x(T-u)})^2 du + \frac{\sigma_y^2}{2Tk_y^2}\int_0^T (1 - e^{-k_y(T-u)})^2 du$$

If we are given market yields of default-free bonds and swaps for certain maturity sectors, e.g., 2-year and 10-year, we can fit the above formula to the default-free bond yields at 2-year and 10-year, thereby obtaining the realization of $x(0)$ and $y(0)$. Once we find the $x(0)$ and $y(0)$, we can fit the formula in Proposition 2 to the swap rates at 2-year and 10-year, thereby obtaining the realization of $\delta_x(0)$ and $\delta_y(0)$. The following table illustrates a fit of bond and swap yields, based on market data on April 28, 2000 and the same set of model parameters as in Figure 4:

	2-yr	10-yr	$x(0)/\delta_x(0)$	$y(0)/\delta_y(0)$	$x(0)+y(0)$
Bonds	0.06676	0.06212	0.05254	0.02034	0.07288
Swaps	0.07299	0.07381	0.06493	0.01007	0.07500
Spreads (BP)	62.3	116.9	123.9	-103.3	20.6

The starting values of the two bond factors are 5.254% and 2.034 %, respectively, implying an instantaneous short-rate of 7.288%. The starting values of the two swap spread factors are 123.9 bp and -103.3 bp, respectively, implying an instantaneous financing spread of 20.6 bp.

Similar to the single-factor case, we can explore the special case when $k_x = k_{\delta_x}$ and $k_y = k_{\delta_y}$. Define

$$\begin{aligned} X &= x + \delta_x \\ Y &= y + \delta_y \end{aligned}$$

We can verify that

$$\begin{aligned} dX &= k_X(\overline{X} - X)dt + \sigma_X dw_X \\ dY &= k_Y(\overline{Y} - Y)dt + \sigma_Y dw_Y \end{aligned}$$

where $k_X = k_x$, $k_Y = k_y$, $\overline{X} = \overline{x} + \overline{\delta}_x$, $\overline{Y} = \overline{y} + \overline{\delta}_y$, $\sigma_X = \sqrt{\sigma_x^2 + \sigma_{\delta_x}^2}$ and $\sigma_Y = \sqrt{\sigma_y^2 + \sigma_{\delta_y}^2}$. Similarly, under the equivalent martingale measure P^*, we have

$$\begin{aligned} dX &= k_X(X^* - x)dt + \sigma_X dw_X^* \\ dY &= k_Y(Y^* - Y)dt + \sigma_Y dw_Y^* \end{aligned}$$

where $X^* = x^* + \delta_x^*$ and $Y^* = y^* + \delta_y^*$. In summary, similar to our 2-factor model for the government curve, we obtain a 2-factor model for the swap curve with exactly the same stochastic structure as for bonds. For the parameters given in Figure 4, we have

$$\begin{aligned} &k_X = 0.001,\ \sigma_X = 0.0118,\ X(0) = 0.06940,\ \overline{X} = 0.065,\ \lambda_X = 0.1677 \\ &k_Y = 0.500,\ \sigma_Y = 0.01677,\ Y(0) = -0.00884,\ \overline{Y} = 0.0,\ \lambda_Y = 0.0 \end{aligned}$$

The parameters in Figure 4 were chosen so that swaps and bonds can be fitted separately. We note when the term structure of swap spreads is upward sloping, a typical fit to bond yields and swap rates usually gives rise to $\delta_x(0) = X(0) - x(0) > 0$ while $\delta_y(0) = Y(0) - y(0) < 0$. In other words, the implied level or long rate based on the swap curve is higher than that implied from the government curve, while the implied the slope factor based on the swap curve is steeper than that of the government curve.

We argue here that processes δ_x and δ_y have important economic interpretations. δ_x represents the spread differential between the overall level of swap curve and the overall level of government curve. As such it represents market view of the required compensation for swap holders paying the extra short-term financing spreads in the long run. The way that δ_x is determined is similar to how long-term interest rates are determined, as long-term interest rates represent market view of the required return for bond holders given their expectation of future short-term interest rates. In the same way that long-term interest rates are affected by the supply and the demand of government bonds, δ_x can also be affected by the supply and the demand of bonds vs swaps.

In contrast, δ_y can be interpreted as a process characterizing the liquidity differential between swaps and government bonds. We have argued in the introduction that the slope of the term structure of default-free interest rates is affected by two factors: 1) market expectation of future interest rates and 2) the liquidity premium associated with government bonds.

- Market expectations of interest rate movements in the near and distant future affect the slope of the yield curve in the short and intermediate sectors. But, such effect is applicable to swaps as well. The net effect on the slope differential between the swap curve and the government curve may be trivial. In other words, the slope of swap spreads shouldn't be materially affected by the expectations of future interest rates.

- It is known that a large population of fixed-income money managers are mandated to invest a significant proportion of their assets in government bonds. Globally, the 10-year sector has traditionally been the most demanded sector among all maturity sectors. Unbalanced demand for government bonds creates liquidity in markets for government bonds. In contrast, fixed-income money managers are typically forbidden from using swaps as an investment tool. The natural demand for swaps comes from corporations or dealers involving in long-term financing or hedging. The liquidity premium which government bonds command may cause the government yield curve to be flatter than it would otherwise be. The liquidity deferential in bonds and swaps may cause the swap curve to be steeper than the bond curve, which implies a steeper term structure of swap spreads. The liquidity premium or specialness associated with the on-the-run bonds vs the off-the-run bonds provides a further evidence that swap spreads may be widened or narrowed as the liquidity premium associated with the on-the-run bonds increases or decreases.

In summary, the slope differential factor δ_y provides a good proxy for bond liquidity premium. While we cannot explicitly model the liquidity differential between swaps and bonds, we provide some empirical evidences to support our viewpoint in the next section as we continue to extend our model to a 3-factor setting.

4.3 Swap Spreads in a 3-Factor Setting

In constructing term structures of swap spreads using the 2-factor model proposed in the previous section, we took the approach to fit bond yields and swap rates for maturities at 2-year and 10-year. Unfortunately, the instantaneous interest rates implied by fitting the 2-year and 10-year sectors may be ill-behaved and not consistent with the actual overnight rates observed in the market. Consequently, instantaneous finance spreads, upon which the term structure of swap spreads is constructed, may not be consistent with the actual financing spreads observed in the market. To resolve this problem, we extend our 2-factor model to include an additional factor which tracks the dynamic behavior of the instantaneous interest rates.

Formally, we consider a 3-factor default-free term structure model under which the dynamics of the first two factors, economic latent variables x_1 and x_2, is given by

$$\begin{aligned} dx_1 &= k_1(\overline{x}_1 - x_1)dt + \sigma_1 dw_1 \\ dx_2 &= k_2(\overline{x}_2 - x_2)dt + \sigma_2 dw_2 \end{aligned}$$

where k_1, k_2, $\overline{x}_1$, $\overline{x}_2$, σ_1 and σ_2 are (non-negative) constant parameters defining mean-reversions, long-run means and volatilities, respectively, while w_1 and w_2 are two independent Brownian motions. These two factors are designed to mimic the same two factors as in our previous 2-factor setting. Thus, under suitable choices of the mean reversion parameters k_1 and k_2, the first factor x_1 can tract movements of long-term interest rates (i.e., the theoretical long rate factor), while the second factor x_2 can tract movements of yield curve slope (i.e., the theoretical slope factor). Together, $\Theta = x_1 + x_2$ defines the model implied theoretical short-rate, which can be interpreted as the *equilibrium short rate* necessary to accommodate the current state of the economy. For example, if the current state of the economy is such that it is heading towards recession, then the equilibrium short-rate may necessitate a level that is lower than the current market short-rate.

Alternatively, we interpret Θ as a *theoretical target rate*. Specifically, we model the evolution of the instantaneous default-free interest rate as a policy rule controlled by the central bank,

$$dr = k_r(\Theta - r)dt + \sigma_r dw_r$$

where k_r and σ_r are constants, and w_r is a Brownian motion independent of w_1 and w_2. That is, the instantaneous short-rate, controlled by the monetary policy maker, mean-reverts to the equilibrium target rate, subject to a random noise due possibly to transitory imbalance in the money market.[8] If the current short-term interest rate is below the equilibrium target rate, then

[8]See Piazessi (2000) for a similar specification with the central bank controlling short-term rates under a jump-diffusion framework.

there is a tendency that the central bank may hike the short-term interest rate to gradually catch up with the equilibrium short-rate. Similarly, if the current short-rate is higher than the equilibrium target rate, then it is more likely that the central bank will guide the short-rate gradually down towards its equilibrium target.

For purposes of pricing securities under the no-arbitrage condition, we assume there exists an equivalent martingale measure P^* under which,

$$w_i^* = w_i - \lambda_i t, \quad i = 1, 2, r$$

are converted into standard Brownian motions. In other words, under P^*, the dynamics of x_1, x_2 and r take the following form:

$$\begin{aligned} dx_1 &= [k_1(\overline{x}_1 - x_1) + \lambda_1\sigma_1]dt + \sigma_1 dw_1^* \\ dx_2 &= [k_2(\overline{x}_2 - x_2) + \lambda_2\sigma_2]dt + \sigma_2 dw_2^* \\ dr &= [k_r(\Theta - r) + \lambda_r\sigma_r]dt + \sigma_r dw_r^* \end{aligned}$$

Moreover, the no-arbitrage price at time t of a zero coupon bond maturing at time T can be determined as

$$P_t(T) = \mathbf{E}_t^* \left[e^{-\int_t^T r(s)ds} \right]$$

We need the following lemma for constructing term structures of government bond yields, swap rates and swap spreads.

Lemma 1 *Under the 3-factor setting specified above, the yield to maturity of default-free zero coupon bonds has the following closed form formula:*

$$Y_T = A_T r(0) + B_T x_1(0) + C_T x_2(0) + D_T \tag{11}$$

where

$$\begin{aligned} A_T &= \psi(k_r T) \\ B_T &= \frac{k_r}{k_r - k_1}(\psi(k_1 T) - \psi(k_r T)) \\ C_T &= \frac{k_r}{k_r - k_2}(\psi(k_2 T) - \psi(k_r T)) \\ D_T &= \lambda_r\sigma_r T H(k_r T) + \frac{k_r(k_1\overline{x}_1 + \lambda_1\sigma_1)T}{k_r - k_1}(H(k_1 T) - H(k_r T)) \\ &\quad + \frac{k_r(k_2\overline{x}_2 + \lambda_2\sigma_2)T}{k_r - k_2}(H(k_2 T) - H(k_r T)) - \epsilon_T \\ \epsilon_T &= \frac{\sigma_1^2 k_r^2}{2T(k_r - k_1)^2}\int_0^T (T-u)^2(\psi(k_1(T-u)) - \psi(k_r(T-u)))^2 du \end{aligned}$$

$$+\frac{\sigma_2^2 k_r^2}{2T(k_r-k_2)^2}\int_0^T (T-u)^2(\psi(k_2(T-u))-\psi(k_r(T-u)))^2 du$$
$$+\frac{\sigma_r^2}{2T}\int_0^T (T-u)^2\psi(k_r(T-u))^2 du$$

Lemma 1 provides a complete solution to the term structure of zero coupon bonds. Specifically, given initial values of x_1, x_2, and r, we can generate yields to maturity for zero coupon bonds of all maturities. Formula (11) suggests that yields to maturity are linearly related to the three interest rate factors. Under the assumption that $k_1 << k_2$ and k_1 is close to zero, we can make the following observations:

- An increase in $x_1(0)$ increases the overall level of interest rates (especially in the long-end) while having no material impact on the short-end of the curve;
- An increase in $x_2(0)$ flattens the yield curve; and
- An increase in $r(0)$ sharply increases the short-end of the curve while having no material impact on the long-end of the curve.

Our 3-factor model also suggests that yields to maturity are linearly related to the three risk premium parameters, i.e., λ_1, λ_2 and λ_r. Each risk premium plays the role of steepening the yield curve for each of its relevant maturity sector. In other words, each demands additional yield compensation for risk bearing. In fact, it is easily shown that that for any zero coupon bond, its excess return can be explained by

$$\mathbf{E}\left[\frac{dP}{P}\right]-r=\lambda_r(\sigma_r T A_T)+\lambda_1(\sigma_1 T B_T)+\lambda_2(\sigma_2 T C_T)$$

Based on the loadings of A_T, B_T and C_T, bonds with very long maturity are primarily priced by λ_1 while bonds with very short maturity are primarily priced by λ_r, and the intermediate sector is priced by λ_2.

Given our experience with 1- and 2-factor models, we specify the dynamics of short-term financing spreads δ using the same analytical structure. Specifically, under the probability measure P, we assume that δ is governed by the following dynamic system,

$$\begin{aligned} d\delta_1 &= k_{\delta_1}(\bar{\delta}_1-\delta_1)dt+\sigma_{\delta_1}dw_{\delta_1} \\ d\delta_2 &= k_{\delta_2}(\bar{\delta}_2-\delta_2)dt+\sigma_{\delta_2}dw_{\delta_2} \\ d\delta &= k_\delta(\delta_1+\delta_2-\delta)dt+\sigma_\delta dw_\delta \end{aligned}$$

With a suitable choice of parameters, δ_1 represents the overall level of swap spreads (level factor) while δ_2 represents the slope of swap spreads (slope factor). The sum of δ_1 and δ_2 is the model implied short-term financing spreads, i.e., theoretical financing spreads that are consistent with the term structure of swap spreads in the intermediate and long maturity sectors. Finally, δ itself corresponds to the instantaneous financing spread. The 3-factor structure imposed here allows us to capture the over-all shape of the swap spread curve including the short-end as well as the long-end. Moreover, we can study the dynamic behavior of the swap spread curve by tracking the dynamic behavior of each of the three swap spread factors. For the dynamics of δ under the equivalent martingale measure P^*, we assume that

$$\begin{aligned} d\delta_1 &= [k_{\delta_1}(\delta_1 - \delta_1) + \lambda_{\delta_1}\sigma_{\delta_1}]dt + \sigma_{\delta_1} dw^*_{\delta_1} \\ d\delta_2 &= [k_{\delta_2}(\delta_2 - \delta_2) + \lambda_{\delta_2}\sigma_{\delta_2}]dt + \sigma_{\delta_2} dw^*_{\delta_2} \\ d\delta &= [k_{\delta}(\delta_1 + \delta_2 - \delta) + \lambda_{\delta}\sigma_{\delta}]dt + \sigma_{\delta} dw^*_{\delta} \end{aligned}$$

where

$$w^*_{\delta_1} = w_{\delta_1} - \lambda_{\delta_1} t, \quad w^*_{\delta_2} = w_{\delta_2} - \lambda_{\delta_2} t, \quad w^*_{\delta} = w_{\delta} - \lambda_{\delta} t$$

are independent Brownian motions under P^*, and λ_{δ_1}, λ_{δ_2} and λ_{δ} are constant parameters for risk premiums. The following proposition constructs the term structure of term spreads using the formula derived in Lemma 1.

Proposition 3 *The term structure of term spreads under our 3-factor setting is given by*

$$F_T - Y_T = a_T\delta(0) + b_T\delta_1(0) + c_T\delta_2(0) + d_T \tag{12}$$

where a_T, b_T, c_T, and d_T are the same as A_T, B_T, C_T and D_T in Lemma 1 with all parameters for x_1, x_2 and r replaced by those of δ_1, δ_2 and δ.

We now conduct empirical analyses to study the implication of our 3-factor term structure model of swap spreads using historical data from the US market. Specifically, we construct term structures of government bond yields, term structures of swap rates, and term structures of swap spreads, based on the formulae derived in Lemma 1 and Proposition 3. TO do so, we first parameterize our model based on some of the existing empirical literature on estimating term structures of interest rates. We then fit our model to historical bond yields and swap rates, allowing us to study the dynamic behavior of term structures of swap spread.

For parameterization, we again make the convenient assumption that $k_1 = k_{\delta_1}$, $k_2 = k_{\delta_2}$ and $k_3 = k_\delta$, i.e., the mean-reversion parameters corresponding to the 3-term structure factors in the swap space are identical to those of the bond space. This assumption allows us to assume the same

3-factor structure for swap rates:

$$\begin{aligned} dX_1 &= k_1(\overline{X}_1 - X_1)dt + \sigma_{X_1} dw_{X_1} \\ dX_2 &= k_2(\overline{X}_2 - X_2)dt + \sigma_{X_2} dw_{X_2} \\ dR &= k_3(X_1 + X_2 - R)dt + \sigma_R dw_R \end{aligned}$$

where $\overline{X}_1$, $\overline{X_2}$, σ_{X_1}, σ_{X_2} and σ_R are related to those parameters defining the default-free interest rate process and the short-term financing spread process, i.e.,

$$\overline{X}_1 = \overline{x}_1 + \overline{\delta}_1, \quad \overline{X}_2 = \overline{x}_2 + \overline{\delta}_2$$

$$\sigma_{X_1} = \sqrt{\sigma_{x_1}^2 + \sigma_{\delta_1}^2}, \quad \sigma_{X_2} = \sqrt{\sigma_{x_2}^2 + \sigma_{\delta_2}^2}, \quad \sigma_R = \sqrt{\sigma_r^2 + \sigma_\delta^2}$$

Following the empirical work of Duffie and Singleton (1997), Babbs and Nowman (1999), and Mendev-Vives and Naik (2000), we set $k_1 = 0$ and $k_2 = 0.5$, i.e., x_1 follows a random walk while x_2 is mean-reverting with a half-life of 1.4 years. In all these three studies, the mean reversion of the first factor is near zero while the mean reversion of the second factor is around 0.5.[9] For k_3, which is a parameter characterizing the central bank's responsiveness, we assign a value of 1.5, implying a half-life of about 6 months.[10] Given that $k_1 = 0$, we can assume without loss of generality that $\overline{x}_1 = \overline{x}_2 = 0$ and $\overline{\delta}_1 = \overline{\delta}_2 = 0$ since all constant terms can be absorbed into the initial values of x_1 and δ_1. For specification of the volatility parameters, we again borrow the empirical estimates from Duffie and Singleton (1997), setting $\sigma_1 = 0.009$ and $\sigma_2 = 0.014$, which is consistent with their estimates of the instantaneous volatility (of the square-root processes evaluated at the means).[11] For σ_3, we assign a value of 0.0050 or 50 basis points, which is consistent with that of Mendev-Vives and Naik (2000). For the spread volatility parameters, we set $\sigma_{\delta_1} = 0.0025$ and $\sigma_{\delta_2} = 0.0035$ and $\sigma_\delta = 0.0025$. This implies that $\sigma_{X_1} = 0.009341$, $\sigma_{X_2} = 0.014431$, and $\sigma_R = 0.00559$. Finally, for the risk premium parameters, we assume that $\lambda_2 = \lambda_r = 0$ and $\lambda_{\delta_2} = \lambda_{\delta_3} = 0$.[12] In doing so, we can focus exclusively on the risk premium associated with holding long-term securities. To summarize, we assume the following set of parameters for our historical analysis:

$$k_1 = 0.0, \ \sigma_1 = 0.009, \ \overline{x} = 0.0,$$

[9]We note that the estimates from Duffie and Singleton (1997) are based on 2-factor square-root processes, while our x_1 and x_2 are 2-factor nromal processes.

[10]Meandev-Vives and Naik (2000) report a value of 0.82. We choose a higher numerical value based on our intuition that the central bank in US, under the leadership of Mr. Greenspan, is aggressive when it comes to setting monetary policy in accordance with macroeconomic fundamentals.

[11]Both Babbs and Norman (1999) and Mendev-Vives and Naik (2000) obtain much higher estimates for σ_1 and σ_2 due to the negative correlation between the two factors.

[12]The preliminary study by Meandev-Vives and Naik (2000) indicates that both λ_2 and λ_r are not equal to zero. We choose to set these parameters to zero as we are not confident about their true values.

$$k_2 = 0.5,\ \sigma_2 = 0.014,\ \overline{y} = 0.0,\ \lambda_2 = 0.0$$
$$k_r = 1.5,\ \sigma_r = 0.0050,\ \lambda_r = 0.0$$
$$k_{\delta_1} = 0.0,\ \sigma_{\delta_1} = 0.0025,\ \overline{\delta}_1 = 0.0,$$
$$k_{\delta_2} = 0.5,\ \sigma_{\delta_2} = 0.0035,\ \overline{\delta}_2 = 0.0,\ \lambda_{\delta_2} = 0.0,$$
$$k_{\delta} = 1.5,\ \sigma_{\delta} = 0.0025,\ \lambda_{\delta} = 0.0$$

To generate our model yield curves that fit the historical rates, we take as given the market yields for maturities at the short-end (the instantaneous short-rate), the 2-year sector, the 10-year sector and the long-end (the 30-year sector). While we have previously fitted x_1 and x_2 to rates at the 2-year and 10-year sectors in our 2-factor framework, we now expand our fitting of r to the short-end as well. In addition, we would like to take the 30-year market rate as a benchmark for assessing the market-required risk premium for holding long-term government bonds or swaps. To summarize, for each given market yield curve, we search for a risk premium parameter λ_1 (or λ_{X_1}) so that the model yield curve generated by Lemma 1 fits the market yield curve jointly at the short-end, 2-year, 10-year and 30-year sectors.

Figure 5(a) plots the time-series history of the three interest rate factors (i.e., the short-rate factor, the long-rate factor and the slope factor) and the implied risk premium parameter (associated with the long-rate factor), based on fitting the Federal Fund target rates, the on-the-run 2-year Treasury yields, the on-the-run 10-year Treasury yields and the on-the-run 30-year Treasury yields. The data set, which is sampled on a weekly basis and obtained from Bloomberg, covers a period from January 1998 to June 2000. Figure 5(b) plots the same time-series history of the three interest rate factors and the implied risk premium parameter, based on fitting the 1-month LIBOR rates, 2-year, 10-year and 30-year swap rates. Figure 5(c) plots the difference of the two time-series histories (swap factors minus bond factors), while Table 1 provides summary statistics for the same set of time-series histories.

Figure 5(c) and Table 1 show that the short-rate and long-rate factors corresponding to swap rates are usually higher than those same factors corresponding to government bond yields, with a mean difference of 19.4 and 101.7 basis points, respectively. The mean short-rate difference represents an average of short-term financing spreads, while the mean long-rate difference represents an average of long-term term spreads (or swap spreads). While the short-rate differences have been well behaved, fluctuating narrowly around its mean of 20 basis points (except for periods that are close the end of calendar year known as the term effect), the long-rate differences have been steadily rising from around 60-70 basis points in early 1998 to near 130-150 basis point by early 2000. Clearly, this is related to the fact that long-term swap spreads reached their historical wides in early 2000.

The slope factor corresponding to swap rates is always less than that of bonds, with a mean difference of -51.8 basis points. In other words, on average, the yield curve corresponding to bonds is flatter than that of swaps. The slope differential between the bond curve and swap curve can be interpreted as liquidity differential between swaps and bonds. The liquidity premium associated with government bonds makes the yield curve for bonds flatter than it would otherwise be. As shown in Figure 5(c), the slope differential between swap curves and bond curves reached its lows in the Fall of 1998, the Summer of 1999 and the Spring of 2000, all corresponding to periods of extreme liquidity shocks and exploding swap spreads.

The sum of the mean long-rate difference and the mean slope difference is about 50 basis points. This sum can be interpreted as the model implied short-term financing spread, which serves as the near-term target towards which the current short-term financing spread will move. The model implied financing spreads have been steadily rising during our sample period, from around 30-40 basis points in early 1998 to near 100 basis points in early 2000. In other words, currently, the US bond market is expecting that the short-term financing spread will become much higher in the future.

On average, the risk premium implied from long-dated swap rates is about 1.4% less than that of long-term bond yields. Figures 5(a) and 5(b) demonstrate that during the 1998 financial crisis, the risk premium implied from the long-term bond yields exploded to as high as 20.7%, while the risk premium implied from long-dated swaps also reached its high of 19%. The risk premium difference between the two curves dropped to -4.5% during that period. In other words, relative to long-dated swaps, the market was demanding a much higher risk premium for holding long-term government bonds. We note that the term structure of swap spreads had a humped shaped in the beginning of 1998 (upward sloping from 0 to 10-year and downward sloping from 10 to 30-year). This can happen when long-term bonds (such as those of 30-year's) become less attractive relative to intermediate-term bonds (such as those of 10-year's), or when the risk premium implied from the swap market is much less than the risk premium implied from the bond market. However, the term structure of swap spreads became strictly increasing in the Spring of 2000 after US Treasury Department announced its first-ever Treasury Buy-back program in recent history. The risk premium implied from bonds dropped from its high of 20% in 1998 to a low of 6-8% in the beginning of 2000. The risk premium demanded for holding long-term bonds dropped significantly due to the strong demand for long-term government bonds (especially the 20-year plus sector). In the mean time, the risk premium associated with long-dated swaps reached a level around 8-10%, and the risk premium differential between swaps and bonds peaked to 3.9%. The sign change in the risk premium differential is the main reason why the term structure of swap spreads has become strictly upward sloping since the beginning of 2000.

Figure 6 and Table 2 reproduce the same graph and table while using the off-the-run Treasury yields for the construction of bond curves. The off-the-run Treasury yields are constructed from bonds that are issued 2-cycles ahead of the current on-the-runs. This step is important, as the on-the-runs may carry substantial specialness that may influence our results.[13] As seen from Figure 6, this adjustment doesn't materially alter any of the conclusions we have drawn so far. The one important effect of this revision is that the average size of slope differential is reduced significantly even though it still reached the same level of low (-137 basis point) in the Fall of 1998.

Finally, to support our argument that the slope differential is closely related to the liquidity premium associated with government bonds, we plot in Figure 7 the time series of the slope differential (properly re-scaled) constructed from the off-the-run fit and the time series of the on-the-run and off-the-run yield spreads (i.e. a proxy for specialess) for Treasuries at the 10-year and 30-year sectors. Figure 7 demonstrates a correlation between the slope differential and the on/off-the-run spreads, implying a steeper swap spread curve when benchmark bonds are priced with a significant liquidity premium.[14]

5 Summary

In this paper we present a new approach of analyzing term structures of swap spreads using the notion of short-term financing spreads, liquidity premium and risk premium. Our approach departs from the traditional approach of attributing swap spreads to counterparty risk of default. We present a historical analysis of swap spreads using data from the US market. Using our 3-factor term structure model, we are able to explain some of the extreme movements in US swap markets in recent years.

[13]Ideally, we would like to get repo quotes to correct for the on-the-run specialness, but such quotes are very difficult to obtain. The 2-cycle off series shall be a good proxy for the off-the-run yields.

[14]In periods when the entire sector of 10-year or 30-year bonds become special, the on-the-run and off-the-run spreads may not be able to capture the liquidity premium associated with the sector. This was the case for the 10-year sector in the Fall of 1998. To partially overcome this problem, we include the on/off-the-run spreads for both 10-year and 30-year sectors.

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Table 1: Historical Yield Curve Factors (Based on the On-the-Run Yields)

Swaps	Short-Rate	Long-Rate	Slope	Risk Premium
Mean	5.6107	5.9784	0.0588	0.1350
Stdev	0.4688	0.6928	0.5574	0.0195
Maximum	6.6911	7.3476	1.1225	0.1890
Minimum	4.9101	4.5977	-1.8780	0.0796
Bonds (ON)	Short-Rate	Long-Rate	Slope	Risk Premium
Mean	5.4169	4.9617	0.5766	0.1494
Stdev	0.4617	0.6343	0.5603	0.0329
Maximum	6.5944	6.4560	2.0042	0.2067
Minimum	4.7985	3.4862	-1.729	0.0604
Diff	Short-Rate	Long-Rate	Slope	Risk Premium
Mean	0.1938	1.0167	-0.5180	-0.0144
Stdev	0.2172	0.2502	0.2781	0.0200
Maximum	1.0048	1.6128	-0.0017	0.0395
Minimum	-0.0770	0.5799	-1.3600	-0.0449

Table 2: Historical Yield Curve Factors (Based on the Off-the-Run Yields)

Swaps	Short-Rate	Long-Rate	Slope	Risk Premium
Mean	5.6107	5.9784	0.0588	0.1350
Stdev	0.4688	0.6928	0.5574	0.0195
Maximum	6.6911	7.3476	1.1225	0.1890
Minimum	4.9101	4.5977	-1.8780	0.0796
Bonds (OFF)	Short-Rate	Long-Rate	Slope	Risk Premium
Mean	5.4183	5.1965	0.2432	0.1444
Stdev	0.4715	0.6803	0.5551	0.0382
Maximum	6.5957	6.6266	1.7610	0.2220
Minimum	4.7998	3.6046	-1.8840	0.0473
Diff	Short-Rate	Long-Rate	Slope	Risk Premium
Mean	0.1924	0.7818	-0.1840	-0.0094
Stdev	0.2174	0.2786	0.3350	0.0249
Maximum	1.0038	1.4706	0.3324	0.0537
Minimum	-0.0780	0.3110	-1.3790	-0.0732

Figure 1(a): Short-Term Financing Spreads (US)

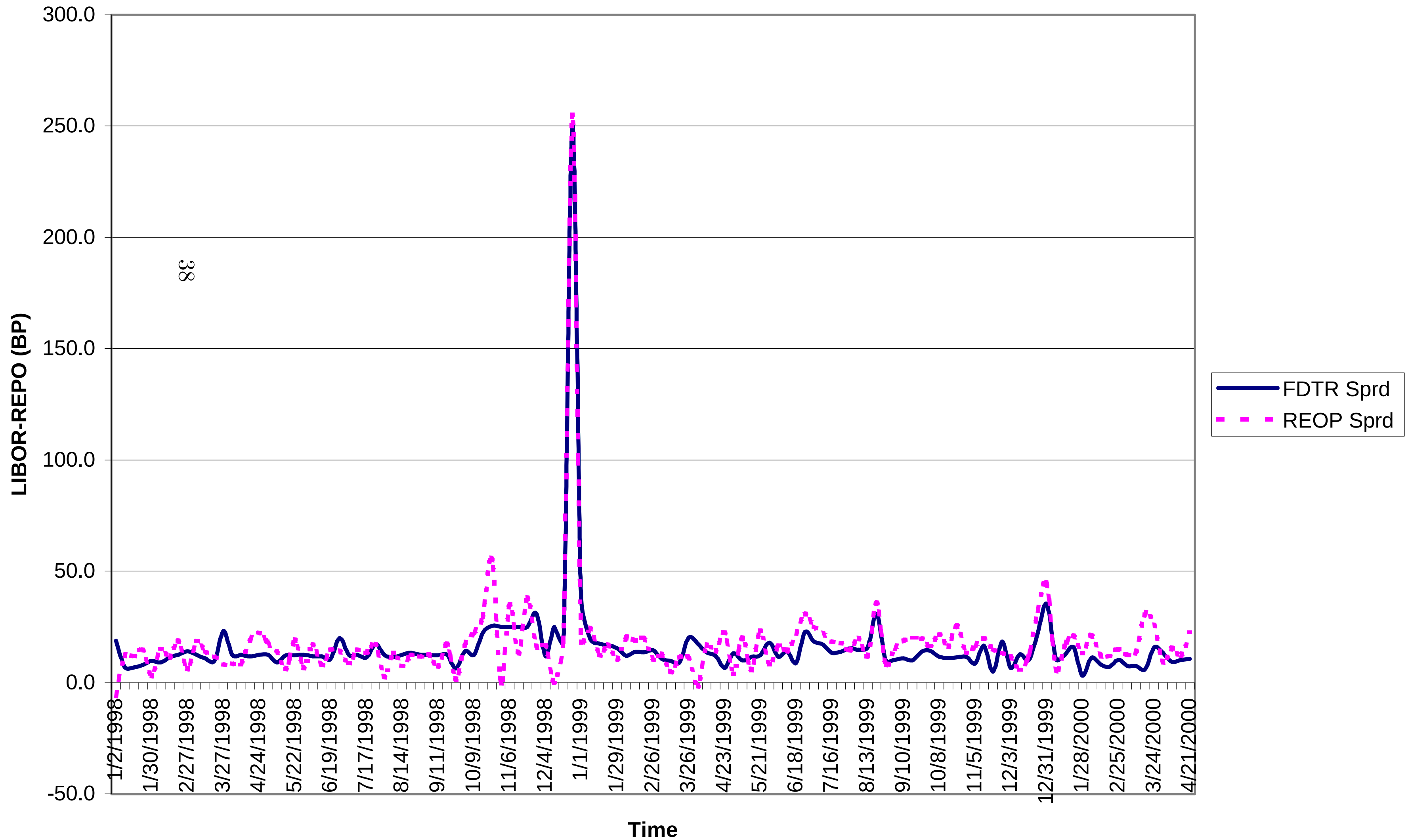

Figure 1(b): Short-Term Financing Spreads (EU)

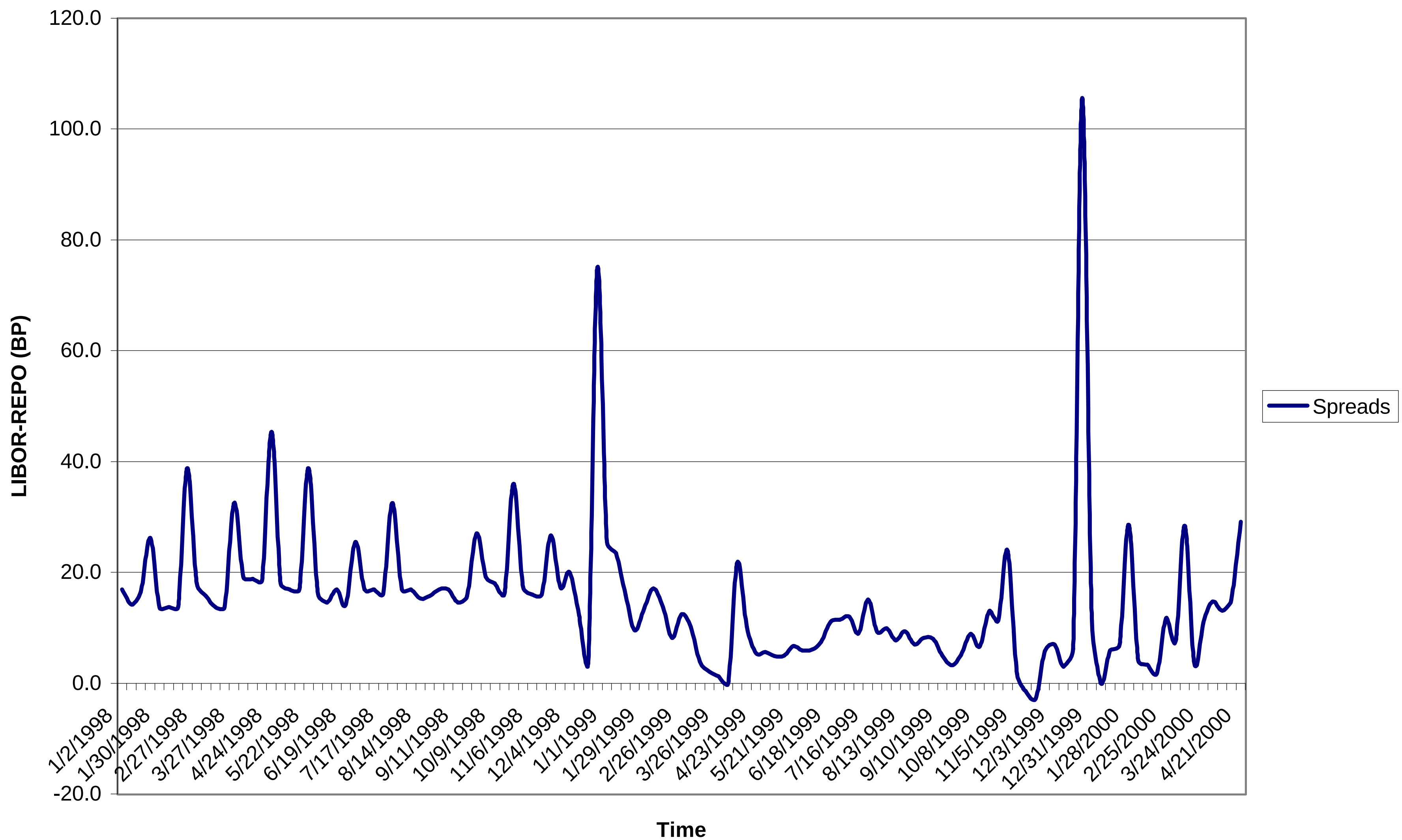

Figure 1(c): Short-Term Financing Spreads (JP)

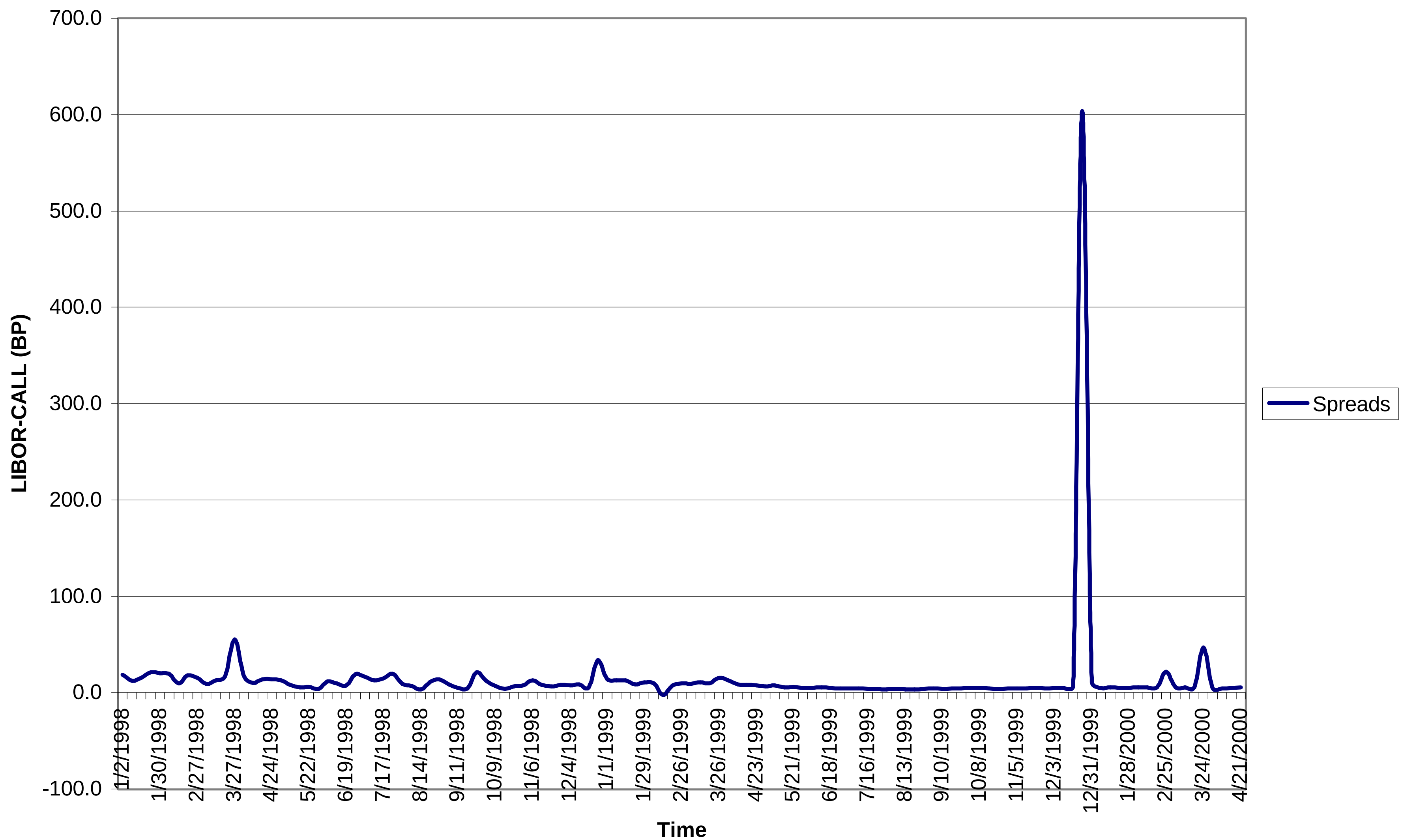

Figure 2: Japan Premiums

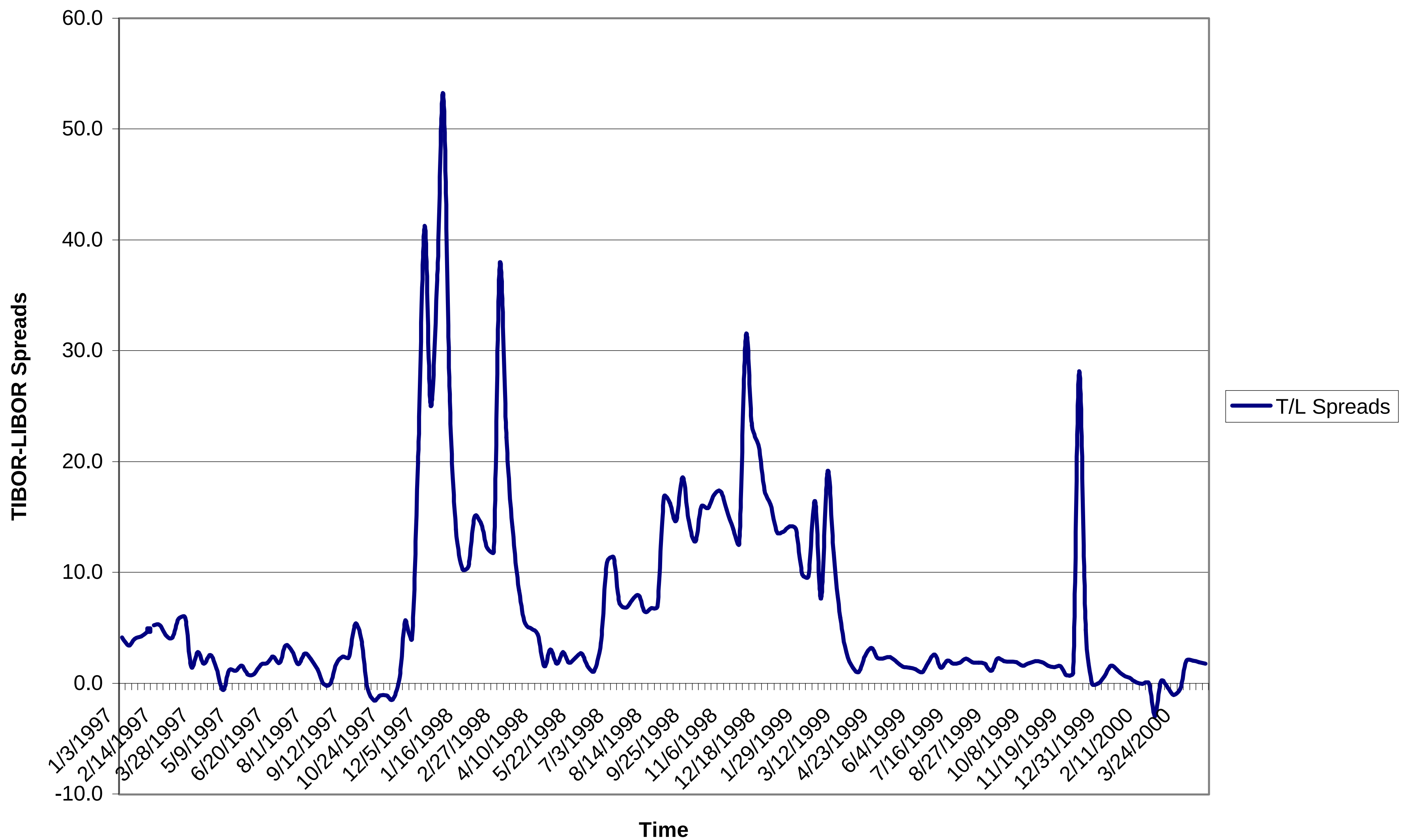

Figure 3: Term Structures of Swap Spreads (1 factor)

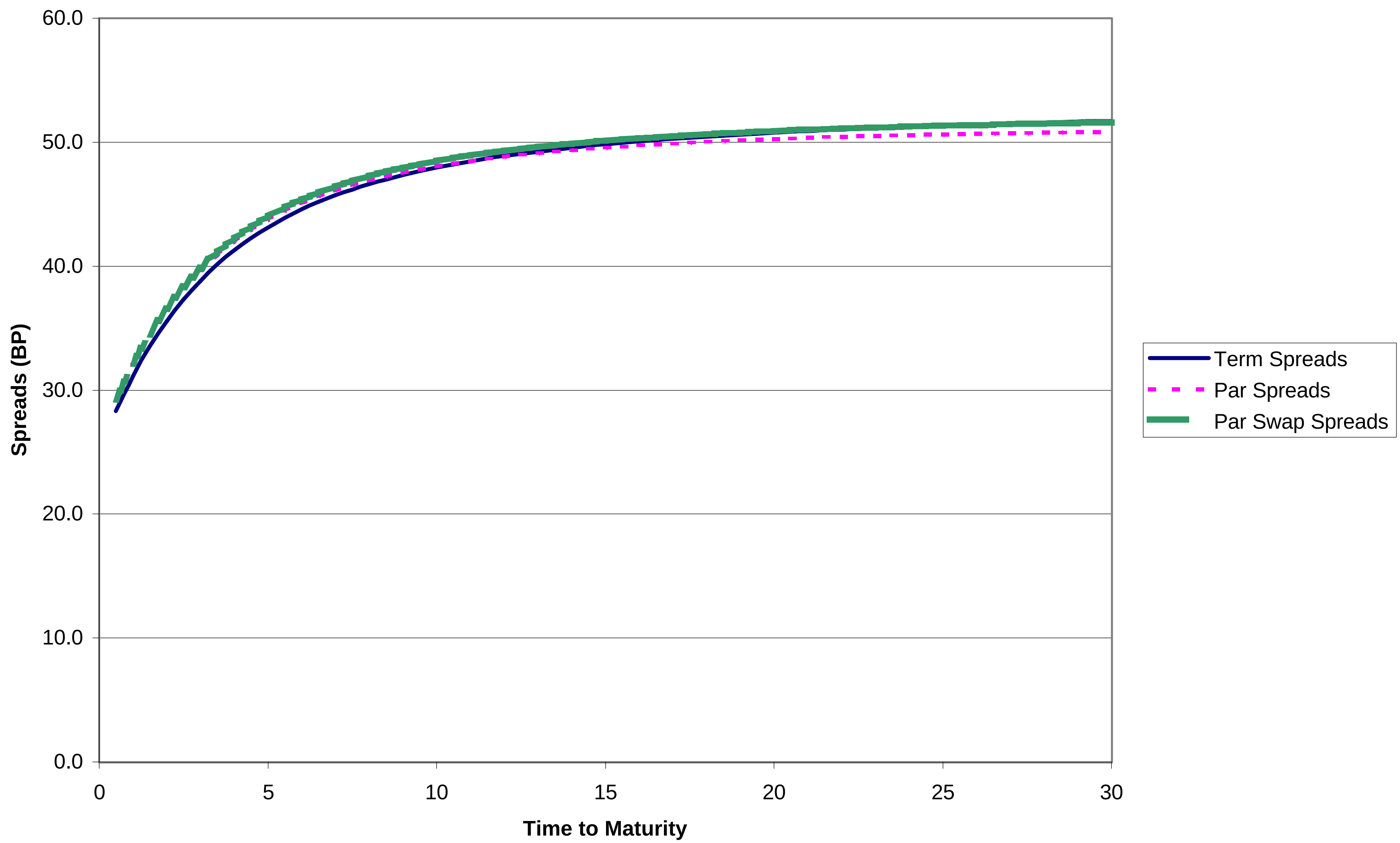

Figure 4: Term Structure of Term Spreads (2 factors)

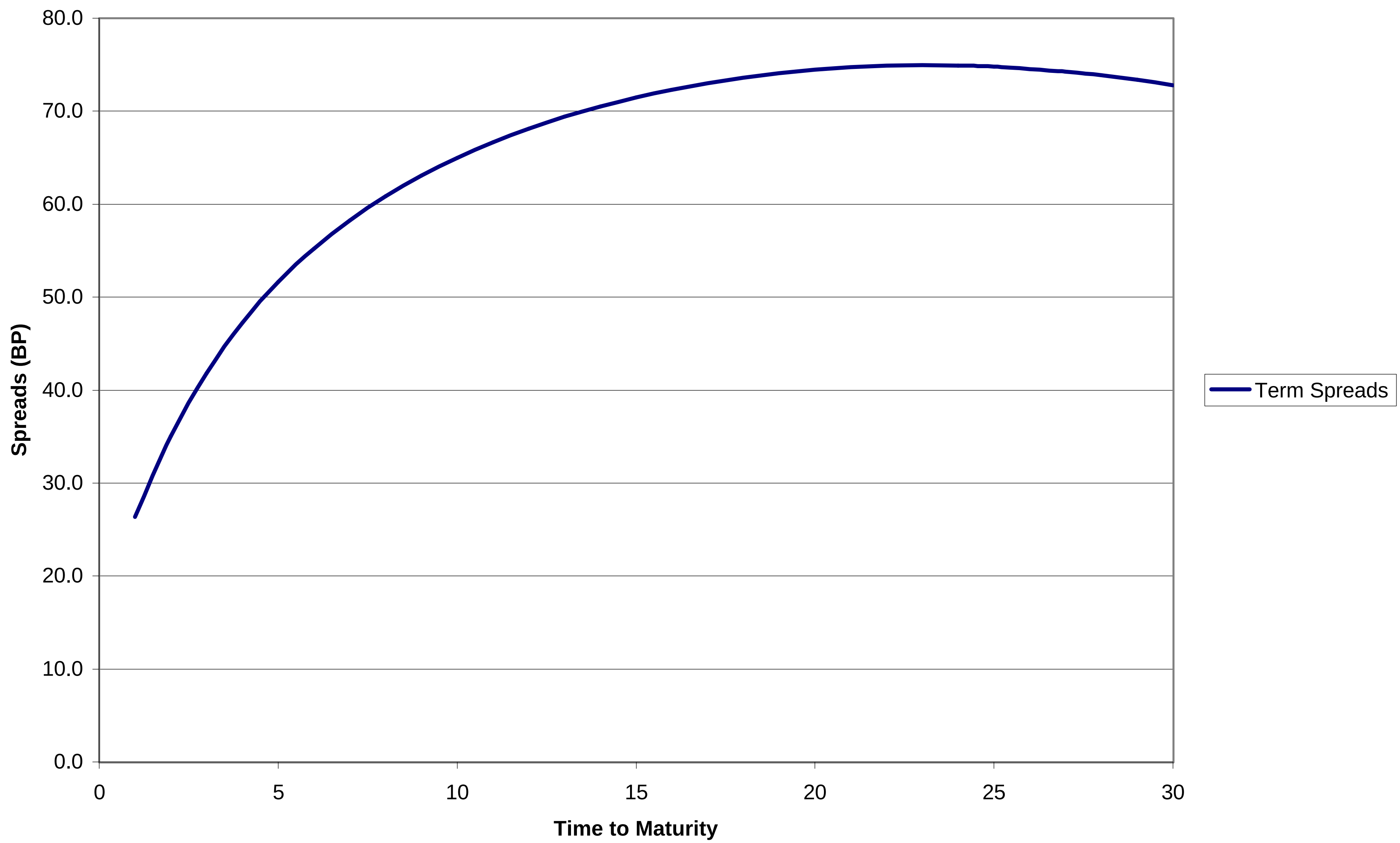

Figure 5(a): Term Structure Factors (Govt On-the-Runs)

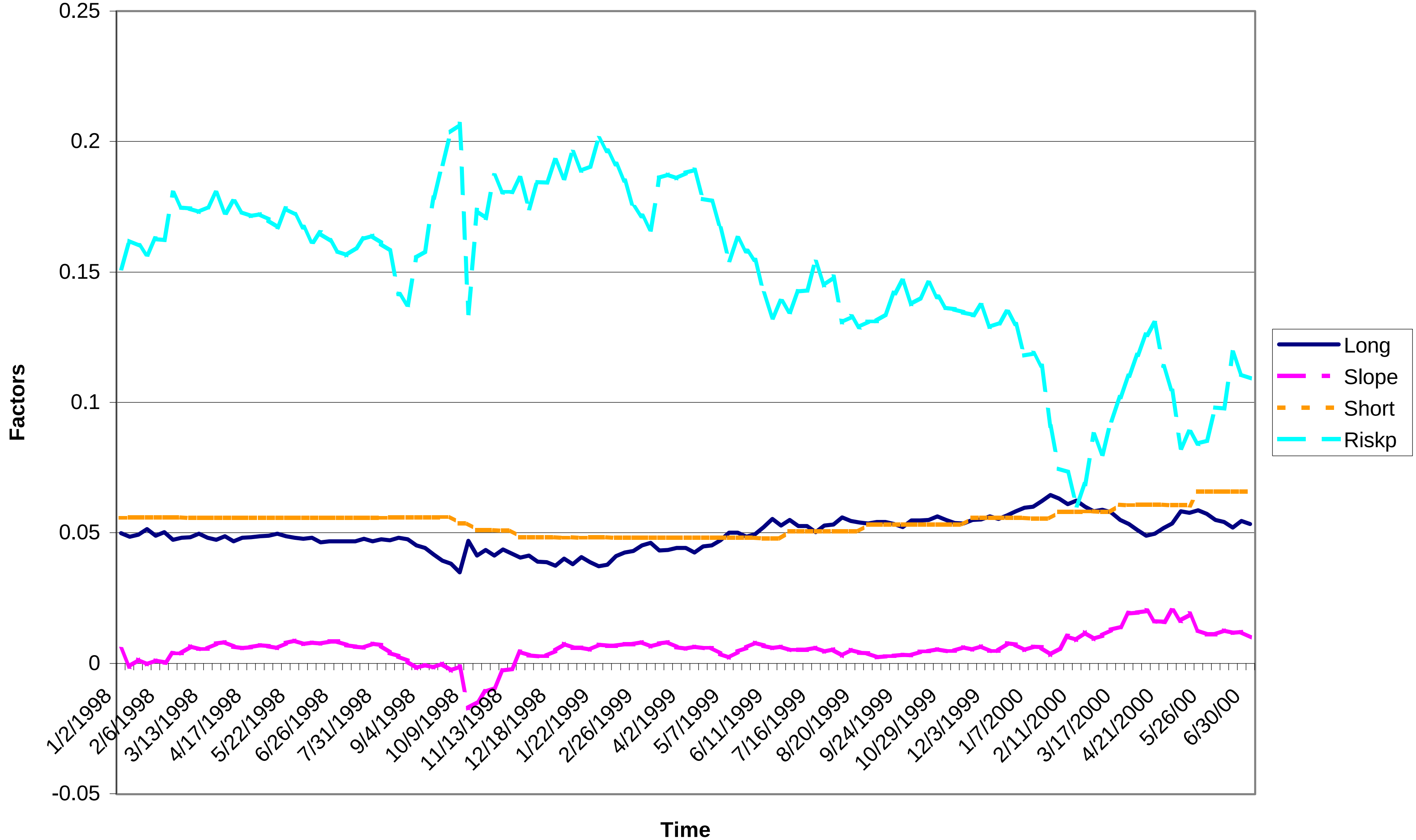

Figure 5 (b): Term Structure Factors (Swaps)

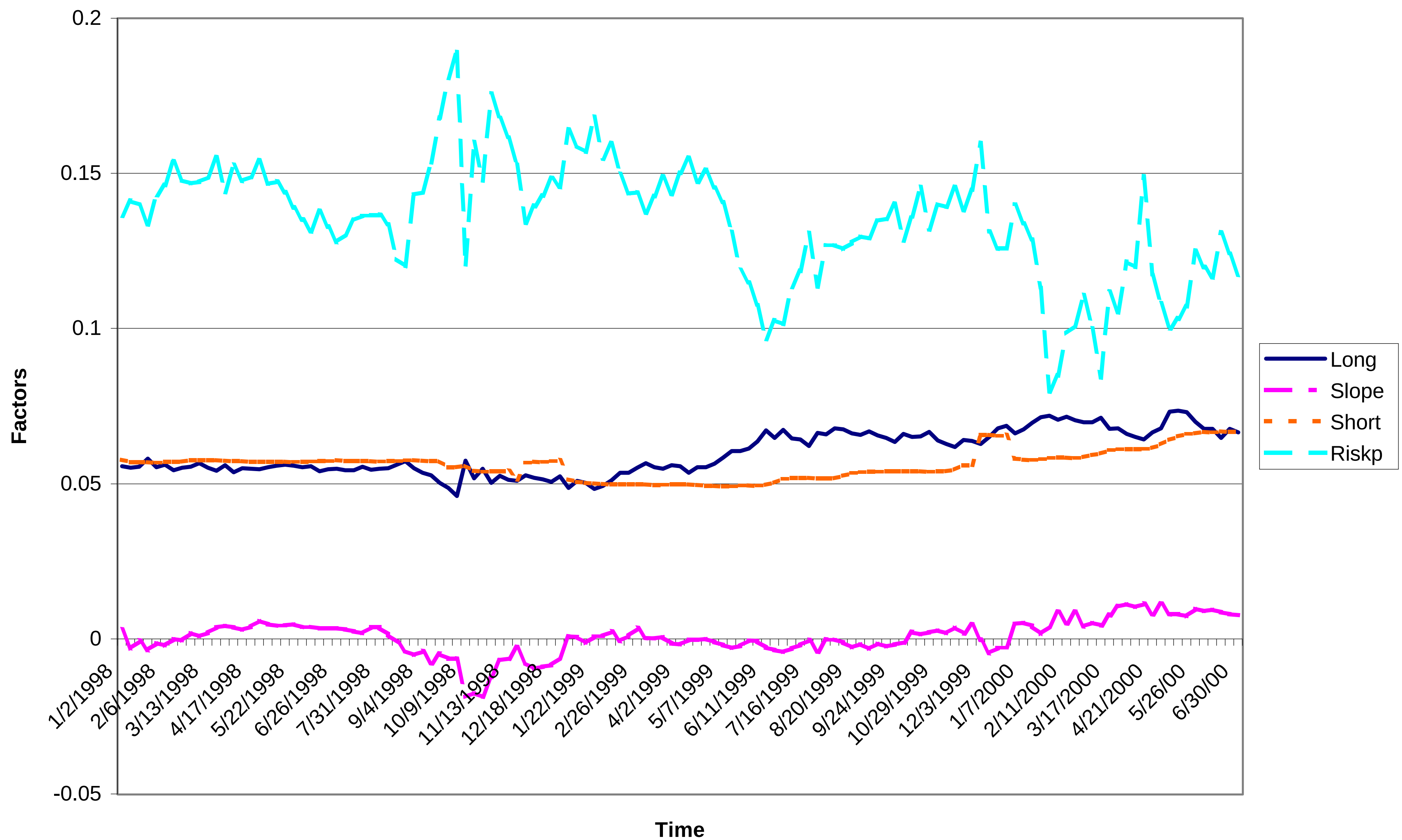

Figure 5(c): Swap Factors vs Gov (On) Factors

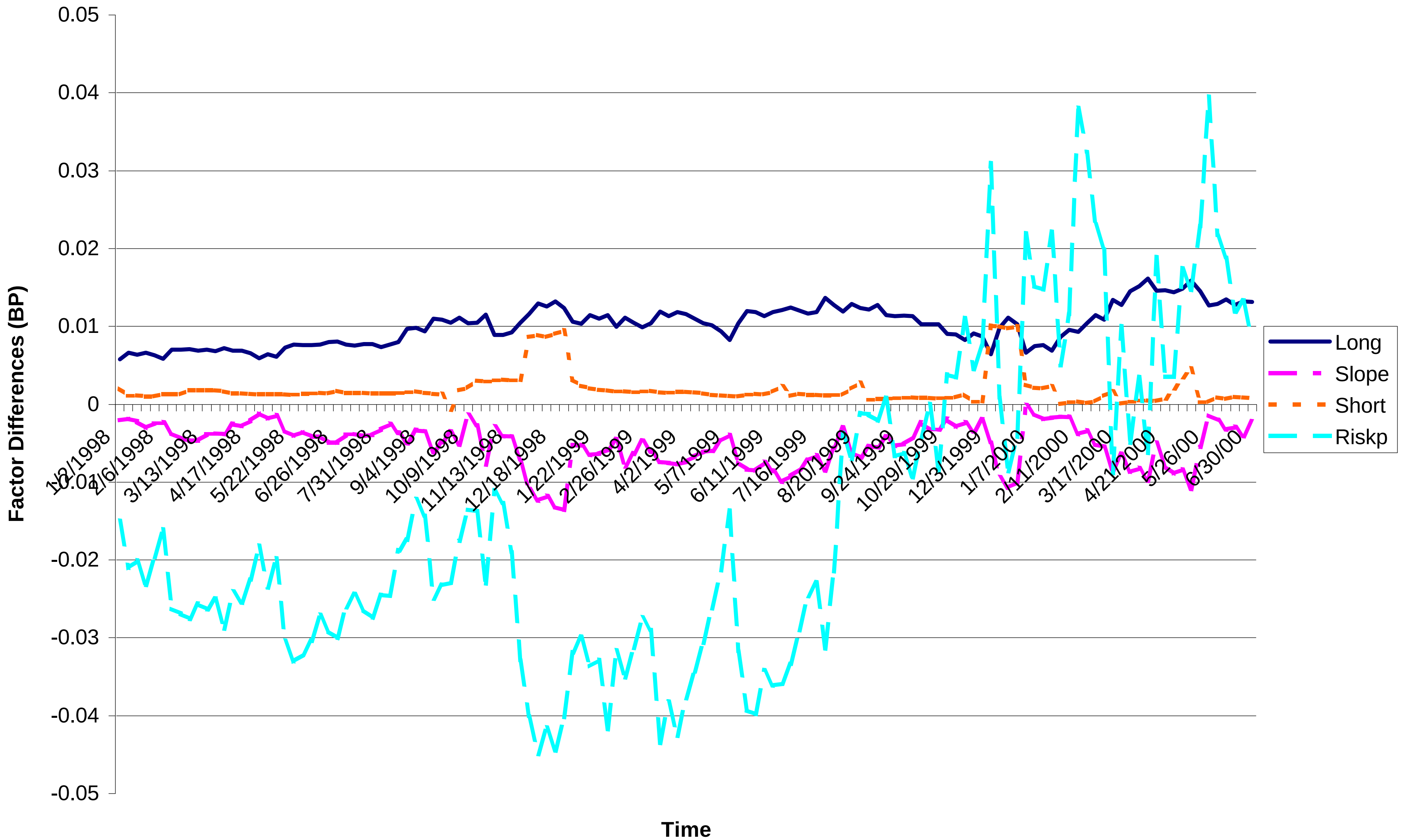

Figure 6: Swap Factors vs Gov (Off) Factors

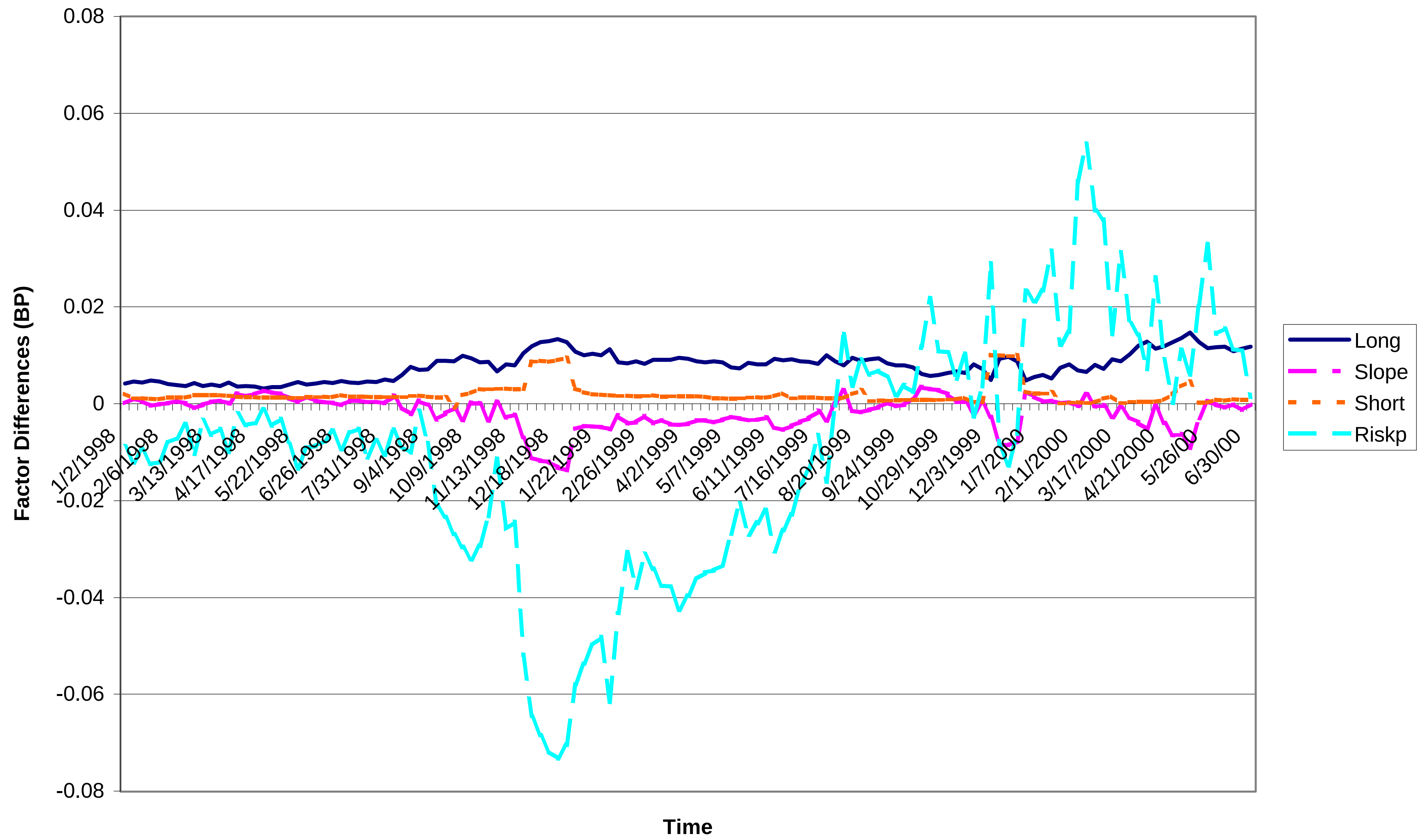

Figure 7: On/Off Spreads and Slope Difference