YALE INTERNATIONAL CENTER FOR FINANCE

Yale ICF Working Paper No. 02-02
January 2, 2002

AN ANALYTIC SOLUTION FOR INTEREST RATE SWAP SPREADS

Mark Grinblatt

Anderson School at UCLA

Yale International Center for Finance

AN ANALYTIC SOLUTION FOR INTEREST RATE SWAP SPREADS

by

Mark Grinblatt*

*Professor, Anderson School at UCLA. Also, Research Associate, NBER and Fellow, Yale's International Center for Finance
Los Angeles, CA 90095

This Draft: January 2, 2002

The author is are grateful to the UCLA Academic Senate for financial support and to Francis Longstaff, Walter Torous, Wolfgang Buehler, Suresh Sundaresan, William Perraudin, Adam Olive, Yacine Ait-Sahalia, Arthur Warga, the editor Sheridan Titman, an anonymous referee, and seminar participants at the UCLA Fixed Income Conference, the NBER, and the CEPR Summer Symposium in Financial Markets for helpful comments and discussions. He also wishes to acknowledge the special contribution of Bing Han, whose assistance was invaluable in completing this paper.

Abstract

AN ANALYTIC SOLUTION FOR INTEREST RATE SWAP SPREADS

This paper argues that liquidity differences between government securities and short term Eurodollar borrowings account for interest rate swap spreads. It then models the convenience of liquidity as a linear function of two mean-reverting state variables and values it. The interest rate swap spread for a swap of particular maturity is the annuitized equivalent of this value. It has a closed form solution: a simple integral. Special cases examined include the Vasicek (1977) and Cox-Ingersoll-Ross (1985) one-factor term structure models. Numerical values for the parameters in both special cases illustrate that many realistic "swap spread term structures" can be replicated. Model parameters are estimated using weekly data on the "term structure of swap spreads from several countries. The model fits the data well.

Interest rate swaps, which are contracts to periodically exchange fixed for floating payments, are one of the most important financial instruments in the world. Aside from the sheer size of the swap market (about $60 trillion in notional amount outstanding on June 30, 2000 and $1.2 trillion in traded notional amount in the month of April 2001),[1] and their importance as hedging instruments, swaps offer data that are of great use to financial modelers. For example, many sophisticated banking houses use the "all-in-cost," which is the yield on the fixed side of the swap, to generate risk-free rates for their derivatives models.

The importance of swaps to the practice of finance has generated a modest amount of research on swap spread valuation. Sundaresan (1991), Longstaff and Schwartz (1995), Duffie and Huang (1996), Duffie and Singleton (1997),[2] and Jarrow and Yu (2001) model interest rate swap spreads as a default premium.[3] In the Sundaresan and Duffie-Singleton models, default risk arises from the possibility of default in the Eurocurrency (or LIBOR) market. The floating rate of most interest rate swaps is a short-maturity London Interbank Offered Rate (LIBOR), which, in their models, represents the yield on a risky financial instrument. The fixed payment is based on the yield of the most recently issued Treasury issue of the same maturity as the swap. If the two counterparties to the swap merely exchanged the Treasury yield for the LIBOR rate, the fixed payer would have essentially borrowed at the risk-free Treasury rate and invested at the risky LIBOR rate. If the two counterparties to the swap have symmetric (or no) default risk, this would be unfair to the payer of

---

[1]Sources, Swaps Monitor and Bank for International Settlements, respectively. Notional amount is roughly equivalent to open interest.

[2]Following up on the argument in an earlier draft of this paper, they also add a state variable for liquidity in their model of swap rates, rather than swap spreads.

[3]The swap spread is the difference between the yield of a recently issued government bond of identical maturity as the swap and the yield associated with the fixed rate of the swap.

the floating rate. Thus, to make the swap fair, the swap's fixed rate payment has to exceed the Treasury yield. Hence, the swap spread in Sundaresan and Duffie-Singleton models is always positive. However, the empirical evidence does not seem to be consistent with Eurocurrency default risk being a major factor in determining swap spreads.[4] In Longstaff and Schwartz (1995), there is no default premium built into the floating rate on which one side of the swap is based. Instead, swap spreads arise because of the possibility of counterparty default on the contract itself.[5] However, they show that for realistic parameters, the swap spreads that arise from counterparty default risk are small, on the order of one to two basis points.[6,7] Similar findings exist or are implicit in Duffie and Huang (1996) and Jarrow and Yu (2001), among others.

If default is not driving swap spreads, then what is? There is evidence that liquidity can have large price effects in the fixed income market.[8] In this paper, we model swap spreads as

---

[4]Litzenberger (1992), in his presidential address to the American Finance Association, argues that industry practice, which prices swaps as though they were not terribly credit sensitive, makes sense. Minton (1997) finds that swap spreads are unrelated to an aggregate default risk factor. Evans and Bales (1991) observe that swap spreads are not as cyclical as A-rated corporate spreads. Chen and Selender (1994) show that AA-AAA corporate spreads have some marginal explanatory power for swap spreads, but the explanatory power is extremely weak and only valid for long term swap spreads. Other credit spreads have no explanatory power in their paper. Also, the pattern of spreads between general collateral repurchase rates for U.S. Treasuries and T-bills, and between Eurodollar rates and the same repurchase rates in the period surrounding the Long-Term Capital Management debacle of Fall 1998 seems inconsistent with a default risk explanation.

[5]The correlation between a "default factor" and an interest rate factor can make the term structure of swap spreads upward sloping or downward sloping, depending on the sign of the correlation.

[6]Cooper and Mello (1991) find larger swap spreads generated by counterparty default risk, but only with parameters that generate swap defaults more frequently than they actually occur.

[7]Credit risk differences between counterparties will alter the bid-ask spread of the swap, but should have a negligible effect on a bank's reported mid-market swap spreads, which we study here.

[8]Boudoukh and Whitelaw (1993), for example, comment that "on-the-run" U.S. Treasury notes and bonds (which are the most recent issues of each Treasury maturity), have yields that are lower than those of comparable "off-the-run" Treasury securities by about 10 basis points. Warga (1992) argues that the duration-adjusted difference between on and off-the-run Treasuries is on the order of 50 basis points and that liquidity, but not taxes, are a potential explanation. In addition, Amihud and Mendelson (1991) show that the yields of U.S. Treasury bills are lower than those of otherwise identical government notes in their final

compensation for a liquidity-based convenience yield associated with government notes. This convenience yield is lost to an investor wishing to receive fixed rate payments, who--in lieu of purchasing a government note--enters into a swap to receive fixed payments.

The convenience yield is assumed to depend on two stochastic factors, one of which is related to short term interest rates. The assumed stochastic process for this convenience yield allows us to derive a closed form solution for the term structure of interest rate swap spreads.[9] Most stylized facts about swap spreads and about the curvature of the term structure of swap spreads can be captured with reasonable parameter values. For example, U-shaped and humped swap spread curves, with deviations on the order of 5 to 10 basis points, can be generated by the model, as well as the more common upward sloping and downward sloping curves.

The paper is organized as follows. Section 1 analyzes explanations for differences in the yields of two default-free or near default-free securities of the same maturity. It concludes that a convenience yield, tied to liquidity, is the only plausible explanation for this difference. Section 2 models swap spreads as the annuity payment that is equivalent to the present value of the liquidity-based convenience yield. Section 3 estimates the parameters of the model and assesses the model's performance. Section 4 concludes the paper.

---

coupon period by 70 to 110 basis points. Kamara (1994) argues that this difference can be driven by liquidity differences, but not taxes, assuming that demand for Treasury notes is perfectly elastic. Daves and Erhardt (1993) find that U.S. Treasury interest only and principal only strips that are otherwise identical differ in their prices. Grinblatt and Longstaff (2000) use regression to show that liquidity proxies affect the spread between synthetic Treasury bonds constructed from a portfolio of strips and the coupon-paying Treasury securities they replicate.

[9]Brennan (1991) values convenience yields for commodities using the standard contingent claims methodology. Despite some similarities, the stochastic processes for the convenience yields valued by Brennan are not the same as those valued here.

1. Differences Between Yields on Government Securities and other Risk-free Securities

Earlier, we noted that the floating rates in most interest rate swap agreements are Eurocurrency borrowing rates, which are the short-term rates at which creditworthy banks and corporations borrow. The spread in a swap represents a difference between the long-term fixed rate of the swap and a government borrowing rate of similar maturity. As we will show, this implies that interest rate swap spreads are tied to the spread between short-term borrowing rates of the most creditworthy corporations, (both financial and nonfinancial), and the borrowing rates of governments. For example, U.S. Dollar interest rate swap spreads are nothing more than the spreads between short-term borrowing rates in the Eurodollar market and an implicit short-term borrowing rate of the U.S. Treasury obtained by subtracting the swap spread from LIBOR. In this section, we argue that the former rate is a corporate "risk-free"[10] rate and the latter is a government risk-free rate and explore a variety of reasons for why the two risk-free rates might differ. The nature of these explanations enables us to model variables that can determine interest rate swap spreads.

A 6-month LIBOR loan is a loan to an AA or AAA-rated borrower.[11] Such a loan is essentially risk-free. And yet, this corporate risk-free rate or near risk-free rate is substantially larger than several other risk-free rates related to the government market as well as to the implicit short-term government borrowing rate referred to above. The size of the swap spread is almost always more than 25 basis points and is often more than 100 basis points. Differences of similar orders of

[10]We use the term "risk-free" interchangeably with the term "default-free." Obviously, the fixed income securities discussed here contain interest rate risk.

[11]Occasionally, LIBOR is the rate charged to an A-rated borrower. We refer to LIBOR as if it is a single rate, when in fact, different banks can in theory quote different LIBOR lending rates. In practice, the variation in quoted rates is small with the vast majority of banks having identical LIBOR rates. The rates at which simultaneous transactions take place exhibit even less variation. Hence, it is appropriate to treat LIBOR as if it were a single rate.

magnitude can exist for other short-term borrowing rates that are risk-free or nearly risk free. These include the spread between the rate on overcollateralized overnight loans that make use of government securities as collateral, known as repurchase rates, and the spreads between these repurchase rates and short-term government borrowing rates. Interestingly, such spreads vary, depending on the nature of the government security. Even the borrowing rates of the same government entity over the same horizon vary, as Amihud and Mendelson (1991) have documented.

Differences in the long-term rates charged to the most creditworthy borrowers may differ across borrower types; that is, short-term creditworthiness differs from long-term creditworthiness. For example, a creditworthy corporate borrower today may turn out to be a below-investment grade borrower in the distant future, while any government with the ability to print fiat money will always be able honor its nominal debt obligations in its own currency. However, large differences like this should not exist for short-maturity loans as credit quality deterioration over short horizons is extremely rare. (For example, there has never been a default of a LIBOR quality debtor.)

How then might we explain these differences between short-term risk-free rates or near risk-free rates across borrowers? The answer is critical because we will show that swap spreads are really spreads associated with the differences in short-term risk-free rates. Even though the swap maturity is long term, the swap spread represents a spread for a credit quality that is "refreshed" as AA or AAA every six months. Hence, we believe that what lies behind virtually all of this spread is a liquidity difference between the corporate and government borrowing instruments rather than a credit quality difference between these two types of borrowers.[12] However, this conclusion is

[12]This explains why after swap spreads widened in response to the Long-Term Capital Management crisis in September and October of 1998, they did not subsequently narrow once the consortium organized by the Federal Reserve stepped in, greatly reducing the credit risk in the financial system from the prospect of a default by Long Term Capital Management on its massive borrowing obligations.

reached only after ruling out several other non-risk-based explanations for these differences. Among these:

Taxation: The tax-favored status of interest in government securities (like state-income taxes for Treasuries at the state and municipal level) results in lower yields to the government securities.

Collateral Rules: Government securities are often usable as collateral for margin investments (e.g. futures accounts), while corporate securities are not. Thus, if cash in a margin account earns a lower return than government securities do, there is some advantage to government securities as collateral.

Regulation: The marginal investor in government securities may have some kind of regulatory advantage, such as a different capital or reserve requirement for holding Treasuries rather than corporates (as, for example, implemented by the Basle accord), or a binding investment constraint (like no more than 30% of fixed income investments may be in the corporate sector).

Market Segmentation: The government and corporate fixed income markets may be segmented. For example, international demand for government securities relative to corporate securities may overprice the government securities relative to the corporates. While some portion of the investing public may recognize this, they cannot arbitrage the difference away because of market frictions like transaction costs.

Our evaluation of these four alternatives is based on stylized facts from the U.S., where differences in these borrowing rates are well documented. For the U.S., the tax equilibrium argument is implausible because the state tax advantage does not apply to broker-dealers, tax exempt investors like pension funds, or international investors who would then arbitrage away these differences. It is unlikely that the marginal corporate-Treasury investor is a wealthy individual in New York or California. In addition, the extant empirical evidence on the Treasury market argues that differences in the tax treatment of some Treasury securities are an unlikely source of pricing anomalies.[13]

The margin explanation fails because the difference between the interest earned on cash and collateral in margin accounts is large only for small investors. In addition, many of these margin

[13]See, for example, Warga (1992) and Kamara (1994).

accounts do not permit longer-term government securities as collateral. Yet, the yield difference between Treasuries and risk-free corporates (computed from the interest rate swap curve) does not generally decline with maturity. Finally, the fraction of Treasury securities used as collateral in this manner is miniscule compared to the amount outstanding. The yield difference between corporates and all Treasuries is unlikely to be generated by a benefit taken for only a few Treasury securities.

The regulatory constraint argument cannot explain why many unconstrained investment funds and mutual funds still invest in Treasuries. Thus, either a large segment of the market is irrational or (more likely), the constrained investors are not the marginal ones.

In addition to these inconsistencies, the first three explanations cannot account for the much smaller spread between corporates and "off-the-run" Treasury securities. For example, off-the-run Treasuries are treated the same as on-the-run Treasuries for the purpose of computing regulatory capital. Moreover, with a larger yield, off-the-runs have larger state tax advantages and advantages as collateral for margin accounts than the "on-the-run" (most recently issued) Treasuries.

As with the regulatory constraint argument, market segmentation requires a degree of irrationality that is not acceptable in most financial modeling. For example, international investors may favor U.S. Treasury securities over U.S. corporate fixed income securities (for a variety of reasons that we will not explore here). In this case, all domestic investors who are free to invest in both the corporate and Treasury markets should only be investing in corporates. Yet, we observe many of the savviest domestic investors holding both corporates and Treasuries.

Having ruled out the most plausible alternatives, we contend that a liquidity-based benefit to the holder of Treasury securities is the only reasonable explanation for the corporate-government risk-free yield difference. This is not so much a matter of lower bid-ask spreads and immediacy for purchases and sales of Treasuries (although this could alter the yield on Treasuries as well). Rather, it is a liquidity-related benefit driven by the needs of savvy investors. Because of the large volume of daily transactions taking place in the Treasury market, Treasury securities are the preferred vehicle for hedging interest rate risk associated with interest rate sensitive positions. A corporate bond buyer who acquires a bond because of a belief that the bond price reflects an overestimate of its default risk will often short a Treasury note of similar maturity to reduce the interest rate risk of the position. Those willing to lend out the Treasury note to such an investor in the Treasury securities repurchase (or "repo") market typically receive a loan at an abnormally low interest rate. Similar phenomena take place, to differing degrees, in the government securities markets of other major economies.

In this sense, there is a daily cash flow to holding Treasury notes if one is big enough and sophisticated enough to participate in the repo market. The cash flow is a premium paid to avoid the closing out of a short position, a phenomenon that Duffie (1996) modeled.[14] In this sense, the term liquidity advantage is something of a misnomer, in that when the likelihood of a short squeeze is high, (and thus, the Treasury note is in some sense less liquid), the convenience yield from lending out the Treasury security is greatest.

This distinction is important in that it points out a difference between a liquidity premium and a liquidity-based convenience yield. Up to a point, the value placed on a government security's liquidity advantage over a comparable corporate security may even be inversely related to the size of the government security's liquidity advantage. This would not be possible if the difference in yields was solely due to a liquidity premium. For example, when the U.S. Treasury decided to halt

[14] On rare occasions, this interest rate may even be negative.

the issuance of 30-year bonds, the relative liquidity advantage of 30-year Treasuries decreased. However, anyone who wanted to short a 30-year bond had to borrow these bonds. This drives up the value of each unit of additional liquidity derived from the 30-year Treasuries, possibly increasing the overall convenience yield of the 30-year Treasuries. However, this can only occur up to a point. If it becomes extraordinarily difficult to borrow 30-year Treasuries for these purposes, if equally liquid or more liquid substitutes arise, or if there is a persistent expectation that these short positions will be squeezed, it may be unwise to borrow these bonds in the first place. At that point, they lose some or all of their liquidity-based convenience yield.

Thus, to have a liquidity-based convenience yield, a security must have an ex-ante liquidity advantage to begin with. Securities that are ex-ante illiquid (e.g., most corporate bonds) would never be shorted in the first place as interest rate hedging vehicles. Because the likelihood of short squeezes with such illiquid securities is so high, they are inframarginal, and thus never carry the liquidity-based convenience yield of the Treasuries.

Below, we model an exogenously given stochastic process[15] that generates this liquidity-based "convenience yield" of government securities to derive an analytic formula for interest rate swap spreads in LIBOR-based swaps. For simplicity, we treat all of the government securities in the model as having the same convenience yield. In the conclusion to the paper, we discuss why we might want to relax that asssumption.

---

[15]A formal model of the general equilibrium behind this process, while obviously desirable, is beyond the scope of this paper. A sketch of the features of such a model would include: (i) asymmetric information about the default risk (or fair market credit spread) of corporate fixed income securities, (ii) an inability to short corporate securities, perhaps because of the likelihood of frequent "short squeezes," (iii) an insufficient supply of highly liquid Treasury notes and bonds in the repo market to meet the interest rate hedging needs of investors buying undervalued corporate securities, (iv) a general repo rate that is below LIBOR in equilibrium, which, along with special repo rates due to occasional liquidity crises in some Treasury issues, quantifies the convenience yield of Treasuries, and (v) equilibrium between the Treasury-repo market and the interest rate swap market.

2. The Model

Consider two riskless financial investments that last for T years:

Investment L: $1 invested in a zero coupon bond that has maturity T/N and an interest rate of LIBOR, which is rolled over every T/N years,[16] combined with a zero net present value interest rate swap with a notional amount of $1 that pays a floating amount equal to (T/N maturity) LIBOR and receives a fixed amount every T/N years.[17]

Investment G: $1 invested in the "on the run" note--a fixed rate government note issued at date 0 that pays coupons every T/N years and matures at year T.[18] For simplicity, we assume that the government note is trading at par at date 0.[19]

Investments L and G are, in effect, investments of equal amount in par fixed rate bonds[20] with identical principal but different coupons. For investment L, the floating rate interest payments cancel with floating payments from the swap, and all principal from the LIBOR investments is rolled over until one is left with $1 principal paid at year T. Hence, one is left with the fixed-rate payments from the swap and the principal from the terminal cash flow in the sequence of rolled over short-term LIBOR investments.

Investment L would be equivalent to a LIBOR bond except for the fact that a LIBOR bond of

---

[16]It is not essential that this investment be rolled over. Since no additional out-of-pocket cash is required to roll the investment over, the rolled over investment has the same present value as a short term LIBOR investment that terminates at T/N.

[17]For foreign swap spreads, we would denote the numeraire in the foreign currency.

[18]In reality, this date of issue will usually be slightly prior to date 0 and the note will be the most recently issued treasury security with a maturity between T-1/2 years and T+1/2 years at its issue date. For simplicity, we assume away this complication.

[19]In modeling swap spreads, we assume that there is an on-the-run note for the exact maturity date of every swap. In practice, there are a limited number of on-the-run notes and swap spreads of all maturities are quoted relative to interpolated yields from these on-the-run notes.

[20]The term "bond" here generically refers to the stream of cash flows generated from the purchase of a fixed income security.

maturity T has default risk. This is because a AA or AAA investor at an initial date may not be a AA or AAA investor two or three years later (as the 2001 experience of Enron would seem to indicate), but the bond contract will still be in place. By contrast, a sequence of short term LIBOR investments has the AA or AAA credit rating of the borrower "refreshed" periodically. In addition, while the swap is not risk-free per se because there is counterparty default risk, its pricing will be virtually equivalent to a default-free swap if the counterparties have symmetric credit risk.[21] We noted earlier that Longstaff and Schwartz (1995), among others, modeling this counterparty default risk, show that for most reasonable scenarios, there is a negligible deviation in swap pricing with default risk from the pricing of a default-free swap. Thus, the yield on investment L, which is also the all-in-cost of the swap, will be smaller than the yield of a LIBOR bond of comparable maturity.[22]

Our strategy is to derive the interest rate swap spread by comparing investments L and G and analyzing how they are affected by the liquidity-based convenience yield. The model used assumes that the zero coupon rates implied by L investments of various maturities should be used to compute present values of riskless cash flows. Hence, zero coupon rates derived from coupon-paying government notes of different maturities are inappropriate for obtaining the present values of risk-free cash flows that have no liquidity-based convenience yield.

2.1 The General Case

The derivation of the term structure of interest rate swap spreads employs the following notation:

[21]There may be a shortening of duration as a consequence of symmetric counterparty default risk, but since it is a small effect, we will treat default as if it does not exist. In addition, the cross-collateralization and mark-to-market dissolution of many swaps (see Minton (1997)) eliminates even this small effect.

[22]Sun, Sundaresan, and Wang (1993) have verified this empirically.

$r(t)$	=	stochastic short term interest rate factor at date t.
$q(t)$	=	vector of other state variables that determine the term structure of interest rates, which, without loss of generality, we take to have differentials that are orthogonal to the other state variables
$y(t)$	=	instantaneous convenience yield from holding Investment G (government note)
	=	$\beta r(t) + x(t)$
$L(r,q,t)$	=	date t fair market value of investment L (a portfolio consisting of the interest rate swap and the position in short term LIBOR that is periodically rolled over).
$G(r,q,x,t)$	=	date t fair market value of investment G (the government note).
$P(r_0,q_0,t)$	=	date 0 fair market value of a riskless zero coupon bond (with no convenience yield) paying \$1 at date t when the initial state variables have values $r(0) = r_0$, $q(0)= q_0$.

The convenience yield, y(t), is stochastic and is assumed to be independent of the maturity of Investment G. We assume that one can translate this yield into an instantaneous net cash flow by lending out investment G for an instant. We model the convenience yield as a linear function of r and another state variable, x. It makes economic sense to think of these variables as following a mean reverting stochastic process. From a purely mathematical perspective, however, some of our results assume that r and x can follow a more general stochastic process, while other results, such as those based on the AR1 and square root processes, require the process to be mean reverting and have more structure.

There are two reasons for allowing x(t) to be stochastic. First, holding the value of a given liquidity advantage constant, it is possible that the liquidity advantage of investment G over investment L may change as government securities become relatively more or less liquid. These changes may depend on factors like the amount of government notes outstanding, a change in the projected fiscal deficit, or the time since issue of the government security.[23] However, it is also possible that the convenience yield changes because the marginal preferences for liquidity change stochastically even when the relative liquidity advantage of government securities remains constant. In this case, a given liquidity advantage, measured in liquidity units, translates into a different amount of convenience yield units, since the latter is measured in monetary units.

The latter interpretation of x(t) may explain why certain risk proxies (like the yield spread between lower rated and higher rated corporate bonds) are often correlated with interest rate swap spreads. It is not that swaps, per se, have become riskier or less risky, but that the liquidity preference of the marginal investor has changed, depending on the estimates placed on a "credit crunch," or a run on their bank.

To derive the fair market interest rate swap spread, we need only compare the cash flows of investments L and G. The cash flows of investment L are the cash flows of a fixed rate bond. By definition, the coupons of this bond exceed those of Investment G by the fixed swap spread (see Figure 1). Thus, a short position in investment L and a long position in investment G results in a constant net cash outflow every T/N years. This constant net cash outflow is the swap spread, and it pays for the stream y, the liquidity-based convenience yield gained by holding investment G. However, if two counterparties entered into a contract to merely exchange the cash flows of the two investments with each other, the payer of investment L's cash flows would pay the swap spread every T/N years and not receive the liquidity-based convenience yield.

It follows that the present value of the swap spread cash flows is equivalent to the present value of the liquidity-generated convenience yield of investment G. We employ the risk-neutral

[23] For a description of how and why liquidity changes as a Treasury security ages, see Sarig and Warga

martingale valuation methodology to compute the present value of this convenience yield. At each point in time, we consider every pair consisting of a path for r and a convenience yield outcome y = βr + x. We discount y by the sequence of risk-free rates along the path for r, weighting each discounted y in a y path by the martingale probability measure for that pair generated by the stochastic processes for r and x.[24] The weighted sum is the present value of the liquidity-based benefit at that point in time. Integrating over all points in time from date 0 to the maturity date gives the present value of the liquidity-based convenience yield. We state and prove this formally as follows:

Proposition 1: Assume
(1) State variables x, r, and the vector q follow diffusion processes

$$dr = \kappa(r,q,t)dt + \sigma_r(x,r,q,t)dz \tag{1}$$

$$dx = \theta(x,r,q,t)dt + \sigma_x(x,r,q,t)dw \tag{2}$$

$$dq_i = a_i(r,q,t)dt + \sigma_i(r,q,t)du_i, \tag{3}$$

where du_i is independent of everything and $E(dzdw)=\rho(x,r,q,t)$.

(2) The interest rate swap spread is the cash flow from a portfolio that is long \$1 in investment L (described above) and short \$1 in investment G (described above)

(3) The instantaneous liquidity benefit of investment G, y dt, satisfies

$$y = \beta r + x,$$

(4) trading is continuous

(5) markets are frictionless

(5) counterparty default risk has no effect on swap pricing

(6) there is no arbitrage

Then, the fair market annualized swap spread is the annualized annuity equivalent of the present value of the liquidity-based convenience yield,

---

(1989).

[24]In the literature on derivatives, this is a standard methodology used to solve these kinds of problems.

Swap Spread(T) =(N/T) PV(liquidity-based benefit)/Σ_i $P(r_0,q_0,iT/N)$, where

$$PV(liquidity-based\ benefit) = \int_0^T E\left(e^{-\int_0^t \tilde{r}(\tau)d\tau}(\beta\tilde{r}(t)+\tilde{x}(t)) \mid r_0, x_0\right)dt$$

$$= \beta(1 - P(r_0,q_0,T)) + \int_0^T [P(r_0,q_0,t)E(\tilde{x}(t) \mid x_0) + cov(e^{-\int_0^t \tilde{r}(\tau)d\tau}\tilde{x}(t) \mid r_0, x_0)]dt$$

and where expectations, E(), and covariances are taken with respect to the "risk neutral" martingale probability measures generated by modifying the stochastic processes for r and x.[25]

Proof: Applying Ito's Lemma to derive the stochastic differential equations for investments L and G, with respective values L(r,q,t) and G(r,q,x,t) we obtain

$$dL = L_t dt + L_r dr + \Sigma L_{qi} dq_i + \tfrac{1}{2}(L_{rr}\sigma_r^2 + \Sigma L_{qi,qi}\sigma_i^2)dt \tag{4}$$

$$dG = G_t dt + G_r dr + G_x dx + \tfrac{1}{2}(G_{rr}\sigma_r^2 + 2G_{rx}\rho\sigma_x\sigma_r + G_{xx}\sigma_x^2 + \Sigma G_{qi,qi}\sigma_i^2)dt, \tag{5}$$

It is well known that Equation 4, in combination with no arbitrage conditions for as many type L investments as state variables determining L, gives rise to the differential equation

$$L_t + L_r[\kappa(r,q,t) - \lambda(r,q,t))] + \Sigma L_{qi}[a_i(r,q,t) - \eta_i(r,q,t))] + \tfrac{1}{2}(L_{rr}\sigma_r^2 + \Sigma L_{qi,qi}\sigma_i^2) = Lr. \tag{6}$$

where $\lambda(r,q,t)$ and $\eta_i(r,q,t)$ are free parameters.

We now turn to Investment G. Consider a sufficiently large number of investments of this type, each with different maturities, along with a type L investment of any maturity. Since all of these investments depend on at most r, x, and q, the instantaneous returns of a properly weighted replicating portfolio of the G type investments is perfectly correlated with the L type investment. Equations (4) and (5) can be used to show that arbitrage is prevented if and only if

$$E(dG) = G(r - y + \lambda(r,q,t)G_r/G + \mu(r,q,x,t)G_x/G + \Sigma\eta_i(r,q,t)G_{qi}/G)]$$

where $\mu(r,q,x,t)$ is a set of free parameters, or equivalently,

$$G_t + G_r[\kappa(r,q,t) - \lambda(r,q,t))] + G_x[\theta(r,q,x.t) - \mu(r,q,x,t))] + \Sigma G_{qi}[a_i(r,q,t) - \eta_i(r,q,t))]$$
$$+\tfrac{1}{2}(G_{rr}\sigma_r^2 + 2G_{rx}\rho\sigma_x\sigma_r + G_{xx}\sigma_x^2 + \Sigma G_{qi,qi}\sigma_i^2) = G(r-y) \tag{7}$$

Equations (6) and (7) suggest that investments G and L are priced as if they were traded in a risk neutral financial market where the stochastic processes that generate r, q, and x are modified to be

$$dr = [\kappa(r,q,t) - \lambda(r,q,t)]dt + \sigma_r(x,r,q,t)dz \tag{8}$$

$$dx = [\theta(x,r,q,t) - \mu(r,q,x,t)]dt + \sigma_x(x,r,q,t)dw \tag{9}$$

$$dq_i = [a_i(x,r,q,t) - \eta_i(r,q,t]dt + \sigma_i(x,r,q,t)du_i, \tag{10}$$

[25]The term that multiplies β in the numerator represents the present value of the interest cash flows of a floating rate bond that pays instantaneous LIBOR. With a one dollar principal payment at the end, such a bond (can be shown with some trivial inductive logic) to have a value of one dollar. Without the principal payment, its value is one dollar less the present value of the principal payment. The annuitized value of this term can be shown to be equal to the all-in-cost of the swap.

In such a risk neutral market, investment L appreciates at the instantaneous rate rdt. Investment G appreciates at the rate (r-y)dt, which is less than the riskless rate, reflecting the additional cash benefit obtainable by lending out investment G in the repo market.[26]

Any stream of cash flows that depend on x, r, q, and time has a value that can be replicated by a portfolio of investments L and G. Following a standard result from the derivatives literature, we can value such cash flows by the expectation of their risk-free discounted values, where expectations are computed with respect to the risk-neutral processes (8), (9), and (10). The interest rate swap spread is such a cash flow stream, and thus, trivially, has the present value expressed in the proposition.

Q.E.D.

The valuation formula becomes a closed form integral if we make r and x independent, and if the risk neutral expectation of x can be computed. One interesting class of such cases occurs when the modified risk neutral process for x is the mean-reverting AR1 process

$$dx = \theta^*(X^* - x)dt + \sigma_x(x,r,q,t)dw, \tag{11}$$

where θ^* and X^* are constants. Here, we can substitute

$$E(x(t)) = e^{-\theta^* t}x_0 + (1 - e^{-\theta^* t})X^* \tag{12}$$

into the Proposition 1's swap spread formula and set the covariance in the formula to zero. Alternatively, if the stochastic processes permit analytic computation of both the risk neutral expectation and covariance, we also obtain an analytic solution.

In the next two subsections, we discuss two specific cases that fall into these two classes.

2.2. The Cox-Ingersoll-Ross (CIR) Special Case

In this subsection, we explore a Cox-Ingersoll-Ross (1985) version of the model that generates

[26] We need not worry about coupons here. At ex-coupon dates, both investments L and G have prices that drop by the amount of the coupon, which is known with certainty at that time. In essence, the expected returns of investment L and G at the ex-coupon date, are still rdt and (r-y)dt at each date if we account for these cash flows.

an analytic solution. Here, r and x are assumed to follow a jointly independent mean-reverting square root process, i.e.,

$$dr = \kappa(r^* - r) + \sigma_r \sqrt{r}\, dz$$
$$dx = \theta(x^* - x) + \sigma_x \sqrt{x}\, dw,$$

where E(dzdw)=0dt. One can deduce from the technology and utility assumptions of this model that the interest rate risk premium parameter, $\lambda(r,t) = \lambda r$ and the risk premium for volatility in x, $\mu(x,r,t) = \mu x$. In addition, zero coupon bond prices are generated by a single factor, r.

For the CIR special case, the interest rate swap spread for a swap of maturity T with N equally spaced payments is given by

$$Swap\ Spread(T) = \frac{\beta(1 - P(r_0,T)) + \int_0^T P(r_0,t)\left[X^* + e^{-\theta^* t}(x_0 - X^*)\right]dt}{T/N \sum_{i=1}^{N} P(r_0, iT/N)}, \tag{13}$$

where

$P(r_0,t) =$ $A(t)\exp(-B(t)r_0)$, the date 0 value of \$1 paid at t, with A(t) and B(t) given by the CIR term structure model

$X^* = \theta x^*/(\theta+\mu)$

$\theta^* = \theta + \mu$

The risk neutral expectation operator used to derive this formula thus has probabilities generated by the modified stochastic processes

$$dr = (\kappa+\lambda)(\kappa r^*/(\kappa+\lambda) - r)dt + \sigma_r \sqrt{r}\, dz \tag{14}$$
$$dx = (\theta+\mu)(\theta x^*/(\theta+\mu) - x)dt + \sigma_x \sqrt{x}\, dw. \tag{15}$$

2.3. The Vasicek Special Case

In this section, we explore a Vasicek (1977) version of the model. This case assumes that r(t) and x(t) follow a bivariate Ornstein-Uhlenbeck (AR1) process:

$$dr = \kappa(r^* - r)dt + \sigma_r dz \qquad (16)$$

$$dx = \theta(x^* - x)dt + \sigma_x dw, \text{ where} \qquad (17)$$

$$dw = \rho dz + \sqrt{1 - \rho^2}\, du,$$

and where z and u follow independent standard Wiener processes. It also assumes that zero coupon bond prices are generated by a single factor, r, and that the risk premia for return volatility generated by z and w, are constant, i.e., $\lambda(r,t) = \lambda$ and $\mu(x,r,t) = \mu$. Note that with this process, we can relax the independence assumption of the CIR version of the model, and allow dz and dw to be correlated.

The interest rate swap spread for a swap of maturity T with N equally spaced payments is then given by

$$SwapSpread(T) = \frac{\beta(1 - P(r_0,T)) + \int_0^T P(r_0,t)\left[X^* + e^{-\theta t}(x_0 - X^*) - \frac{\rho\sigma_r\sigma_x}{\kappa}A(t)\right]dt}{T/N\sum_{i=1}^{N} P(r_0, iT/N)},$$

$$\text{where} \qquad (18)$$

$$A(t) = \frac{1 - e^{-\theta t}}{\theta} - \frac{1 - e^{-(\theta+\kappa)t}}{\theta+\kappa},$$

$P(r_0,t)$	=	$A(t)\exp(-B(t)r_0)$, the date 0 value of \$1 paid at t, with A(t) and B(t) given by the Vasicek term structure model
X^*	=	$x^* - \mu/\theta$.

The risk neutral expectation operator used to derive this formula thus has probabilities generated by the modified stochastic processes

$$dr = \kappa(r^* - \lambda/\kappa - r)dt + \sigma_r dz \text{ and} \qquad (19)$$
$$dx = \theta(x^* - \mu/\theta - x)dt + \sigma_x dw. \qquad (20)$$

The Appendix shows that the risk neutral covariance

$$cov\left(e^{-\int_0^t r(\tau)d\tau}, x(t)\right) = -P(r_0,t)\frac{\rho\sigma_r\sigma_x}{\kappa}\left[\frac{1 - e^{-\theta t}}{\theta} - \frac{1 - e^{-(\theta+\kappa)t}}{\theta+\kappa}\right]$$

The expression in the large brackets in equation (18) is the certainty equivalent of the x portion of the liquidity-based convenience yield of the government note. If $\rho = 0$, this certainty equivalent is the expected value of x, where the expectation is computed with the risk-neutral probability measure generated by the stochastic processes described in equations (19) and (20). However, if ρ is non-zero, realizations of particular liquidity benefits are associated with different probability measures for the interest rate path. The certainty equivalent must be adjusted to account for this.

2.4. Numerical Values for Swap Spreads

In this section, we explore a number of parametrizations of the model that illustrate a variety of swap spread curves that can be generated by the model. These will enable us to develop intuition about the effects of various model parameters. The integration necessary for the results reported in this section was approximated by a monthly summation.[27] The swap spread curves from ten parametrizations of the Vasicek version of the model and the associated term structure of interest rates are given in Table 1. Each parametrization corresponds to a column in the table.[28] Spreads for years 1-5, 7, and 10 are reported. Graphs of this table are provided in Figure 2.

[27] The approximation typically, but not always, generates a slight downward bias in the swap spread model value. The distortion is no more than two basis points and usually less than one basis point. The effect on the slope of the term structure is an order of magnitude smaller.

[28] Although theory does not require β to be nonnegative, all of our parametrizations have this feature. This prevents swap spreads from becoming negative. It also is consistent with our later empirical findings. The positive β may simply be due to the fact that expected inflation changes the price numeraire for everything, including liquidity benefits. In addition, as real interest rates rise, uninformed noise investors may naively move from the equity markets to the bond markets, creating larger opportunities for informed "arbitrageurs." Arbitrageurs, who hedge interest rate risk by shorting government securities, may find that as arbitrage activities increase, their added hedging needs may increase the value of the liquidity possessed by government securities.

Table 1: Ten Parametrizations of the Interest Rate Swap Model, Resulting Swap Spreads (in basis points (bp)), and Vasicek Term Structure (below swap spreads in %)

Params	1	2	3	4	5	6	7	8	9	10
R* (%)	6	6	6	6	6	6	10	6	6	4
X* (bp)	70	70	0	0	80	40	-25	-25	100	-150
r_0 (%)	6	6	6	6	6	6	6	10	14	12
x_0 (bp)	70	70	0	0	40	80	-25	-25	30	-400
K	.2	.2	.2	.2	.2	.2	.2	.2	.4	.2
Θ	.2	.2	.2	.2	.2	.2	.2	.2	.12	.4
σ_r (%)	2	2	2	2	2	2	2	2	2	2
σ_x (%)	1	1	1	1	1	1	1	1	1	1
ρ	0	.8	0	.5	0	0	0	0	0	0
β	0	0	.1	.1	0	0	.1	.1	.05	.4
	Swap Spreads and Term Structure									
1 yr spread: yld %:	71 5.99	71 5.99	61 5.99	61 5.99	45 5.99	77 5.99	39 6.37	73 9.62	100 12.59	102 11.25
2 yr spread: yld (%):	71 5.98	70 5.98	61 5.98	60 5.98	48 5.98	74 5.98	43 6.68	69 9.28	98 11.49	108 10.57
3 yr spread: yld (%):	71 5.96	69 5.96	61 5.96	60 5.96	51 5.96	71 5.96	45 6.95	67 8.97	97 10.63	110 9.98
4 yr spread: yld (%):	71 5.94	68 5.94	60 5.94	59 5.94	53 5.94	68 5.94	47 7.19	64 8.69	97 9.95	109 9.45
5 yr spread: yld (%):	71 5.92	68 5.92	60 5.92	58 5.92	55 5.92	66 5.92	49 7.39	62 8.44	97 9.41	107 8.97
7 yr spread: yld (%)	71 5.87	66 5.87	60 5.87	57 5.87	58 5.87	63 5.87	52 7.72	58 8.02	98 8.62	100 8.18
10 yr spread: yld (%)	71 5.81	64 5.81	59 5.81	55 5.81	62 5.81	59 5.81	55 8.08	54 7.54	100 7.88	88 7.27
Description:	flat	ρ effect	β effect	βρ effect	x up slope	x dn slope	r up slope	r dn slope	u shape	hump

Parametrization 1 illustrates a flat swap spread term structure. The swap spread generated is a constant seventy-one basis points. This swap spread curve is generated by parameters where x_0 and r_0, the initial values of the state variables, are at their risk-neutral long run equilibrium levels, X* (not necessarily x*), and R*. The associated term structure of interest rates in this case is slightly downward sloping because bond prices are convex functions of interest rates. In this case, altering κ and θ have no effect on the swap spread curve. The larger is ρ, the more downward sloping the swap spread term structure. A negative ρ generates an upward sloping term structure. With $\rho = 0$, σ_r and σ_x have no effect on the term structure of swap spreads.

Parametrization 2 illustrates a slightly downward sloping swap spread term structure, driven by a large positive correlation between changes in the liquidity-based convenience yield and changes in the interest rate factor. Essentially, the paths that lead to large liquidity-based convenience yields in the future are discounted at higher (lower) rates if ρ is positive (negative) and those paths that have low convenience yields have low discount rates. Here, in contrast to Parametrization 1, σ_r and σ_x have an effect. The larger either is, the larger is the covariance between changes in the two state variables and the more downward sloping is the swap spread term structure. However, the effect is rather slight. Even at the unreasonably large correlation of .8, the spread declines by 7 basis points as the maturity moves from one year to ten years.

Parametrization 3 illustrates the effect of a positive β, which is the coefficient on interest rates that, in part, determines the level of the convenience yield. The more positive is β, the larger is the swap spread. However, if, because of bond convexity, the term structure of interest rates is downward sloping, as is the case here, the swap spread term structure will also be downward sloping. Here the resulting slope is slight, with only two basis points separating the spread in a ten year swap from a one year swap. However, a larger β, a smaller κ, and a larger σ_r can magnify the downward slope. Altering θ or σ_x has no effect in this scenario, (assuming $X^* = x^* - \mu/\theta$ is held constant).

Parametrization 4 illustrates what happens if we combine a positive β with a positive ρ. There is, once again, a downward slope to the swap spread term structure, but the one year-ten year spread difference is now six basis points rather than two basis points. An increase in θ or κ makes the negative slope less steep, while an increase in either of the two standard deviations exacerbates the steepness of the slope.

Parametrizations 5 and 6 illustrate what happens if X^* deviates from x_0. If, as in Parametrization 5, $X^* > x_0$, the swap spread term structure is upward sloping. If the opposite is true, as in Parametrization 6, it is downward sloping. An increase in the liquidity mean reversion parameter θ increases the steepness of the slope, whether it is positive or negative. However, the interest rate mean reversion parameter, κ, and volatility σ_r, have no effect (assuming ρ=0). An increase in ρ makes the positive slope in Parametrization 5 less steep and the negative slope in Parametrization 6 more steep. A change in R^* essentially has no effect on the swap spreads. An increase in r_0 has a slight effect, both raising short maturity swap spreads and lowering long term swap spreads a bit.

Parametrizations 7 and 8 illustrate what happens if R^* deviates from r_0. These parametrizations have a different term structure of interest rates than the first six. For a deviation between R^* and r_0 to have an effect on the slope of the term structure of swap spreads, β must be non-zero. Here, it is assumed to be positive. Parametrization 7 assumes that $R^* > r_0$, which, for positive beta, generates an upward sloping swap spread term structure.

Parametrization 8 assumes $R^* < r_0$ and generates a downward sloping swap spread term structure. An increase in ρ or σ_r, because of the convexity effect and the positive β, makes the upward sloping term structure less upward sloping and the downward sloping term structure more downward sloping. σ_x and ρ, by contrast, have no effect (unless ρ is non-zero). An increase in κ exacerbates the steepness of the upward and downward sloping curves. Consistent with Parametrizations 5 and 6, an increase in X* makes a positively sloped swap spread curve steeper and a negatively sloped one less steep. The opposite is true for an increase in x_0.

Parametrizations 9 and 10 illustrate odd-shaped swap spread term structures. These examples demonstrate the flexibility of the model to generate patterns of swap spreads that may be rare, but which have been observed on occasion. Parametrization 9 has a U-shaped swap spread term structure. Parametrization 10 illustrates a hump in the swap spread term structure, with the peak spread occurring for three year swaps. Both are generated by downward sloping term structures of interest rates, a positive beta, and an x_0 that is below X*. The difference between the two shapes is driven largely by the ratio of the two mean reversion parameters. In the U-shaped case, interest rates revert much faster than the liquidity-based convenience yield. In the hump case, the reverse is true.

Table 2, using seven parametrizations, explores the effect of the CIR square root processes on the interest rate swap spread. The parametrization numbers at the top of the table correspond to those in Table 1. For comparison purposes, we use parameter values that are somewhat comparable (in a risk-neutral pricing environment) to those used in Table 1. This is not possible for parametrizations 3 and 4, which have a non-zero ρ, nor for parametrization 10, which has extreme negative values of x_0 and X^*, which are not permitted in a square root process. For the two parametrizations, 7 and 8, that have small, identical, negative values of x_0 and X^*, we use their absolute values, .0025. We employ the same values for parametrization 2, which had zero values for x_0 and X^*. The remaining four parametrizations, 1, 5, 6, and 9, essentially use the same parameters as found in Table 1, with one caveat: In all seven parametrizations, the two volatility parameters are not directly comparable between the Ornstein-Uhlenbeck model and the square root model. To make the comparison as close as possible, we convert the Ornstein-Uhlenbeck σ_r and σ_x into square root volatilities by dividing each respectively by the square roots of R* and X*, where $R^* = \kappa r^*/(\kappa + \lambda)$ and $X^* = \theta x^*/(\theta+\mu)$.

Table 2: Seven Parametrizations of the Interest Rate Swap Model, Resulting Swap Spreads (in basis points (bp)), and Cox-Ingersoll-Ross Term Structure (below swap spreads in %)

Params	1	2	5	6	7	8	9
R* (%)	6	6	6	6	10	6	6
X* (bp)	70	25	80	40	25	25	100
r_0 (%)	6	6	6	6	6	10	14
x_0 (bp)	70	25	40	80	25	25	30
κ	.2	.2	.2	.2	.2	.2	.4
θ	.2	.2	.2	.2	.2	.2	.12
σ_r (%) (CIR)	8.165	8.165	8.165	8.165	8.165	8.165	8.165
σ_x (%) (CIR)	11.952	11.952	11.952	11.952	11.952	11.952	11.952
ρ	0	0	0	0	0	0	0
β	0	.1	0	0	.1	.1	.05
	Swap Spreads and Term Structure						
1 yr spread:	71	86	45	77	90	124	100
yld %:	5.99	5.99	5.99	5.99	6.37	9.62	12.58
2 yr spread:	71	86	48	74	93	121	98
yld (%):	5.98	5.98	5.98	5.98	6.69	9.26	11.48
3 yr spread:	71	86	51	71	96	117	97
yld (%):	5.96	5.96	5.96	5.96	6.97	8.95	10.61
4 yr spread:	71	86	53	68	98	115	97
yld (%):	5.94	5.94	5.94	5.94	7.21	8.66	9.92

Table 2: Seven Parametrizations of the Interest Rate Swap Model, Resulting Swap Spreads (in basis points (bp)), and Cox-Ingersoll-Ross Term Structure (below swap spreads in %)

5 yr spread:	71	85	55	66	100	112	97
yld (%):	5.92	5.92	5.92	5.92	7.41	8.40	9.38
7 yr spread:	71	85	58	63	103	108	98
yld (%)	5.87	5.87	5.87	5.87	7.76	7.97	8.58
10 yr spread:	71	85	62	59	106	104	100
yld (%)	5.82	5.82	5.82	5.82	8.13	7.48	7.85
Description:	Flat	β effect	x up slope	x dn slope	r up slope	r dn slope	u shape

The results for the four parametrizations that are directly comparable--1, 5, 6, and 9--are remarkably similar to those found in Table 1. Those for which we needed to raise the values of x_0 and X* to positive numbers, specifically 2, 7, and 8, have term structures that have been shifted up, but which have essentially the same slope as those in Table 1. This suggests that the model is not terribly sensitive to the particular functional form of the stochastic process.

3. Estimation of Parameters of the Model

We analyze the model with data from four countries. For the U.S., U.K., and Canada, weekly data on Libor rates, interest swap rates, and swap spreads over matched maturity on-the-run Government Bonds (e.g., U.S. Treasuries for the U.S.) are obtained from Datastream. We analyze swaps of two, three, four, five and seven-year maturities, which are the most liquid swap markets. All rates are taken at the 17:30 London close.[29] For Japan, weekly data (Friday close) on Yen Libor rates, interest swap rates and swap spreads over the government bond yield curve are obtained from Bloomberg. The data are the midpoints of the best bid/ask rates from the latest

[29]An earlier draft of this paper studied weekly data from 1/1/88 to 2/28/92 on U.S. dollar fixed-LIBOR floating interest swap spreads for maturities of one through five years (inclusive) was obtained from Salomon Brothers, Inc. These data yielded similar results and insights to those presented here.

quoted rates collected by Bloomberg. For each security, we exclude weeks where any of the data for a particular maturity are missing. The U.S. and Canadian data are from June 1, 1993 to April 3, 2001 (410 weeks). The U.K. data are from September 8, 1998 to April 3, 2001 (135 weeks). The data from Japan are from January 15, 1999 to April 3, 2001 (108 weeks).

The payment frequency on the swap rates and spreads are semi-annual and the floating leg is 6 month LIBOR in the relevant currency. The LIBOR day count convention is actual/360. We adjust all rates to be semi-annually compounded rates for consistency.

In this section, we use this data to estimate the parameters of the general model assuming x and r are independent. Recall from the discussion at the end of section 2.1 that when x and r are independent and equation (11) describes the risk neutral stochastic process for x, the swap spread for a swap of maturity T is

$$Swap\ Spread(T) = \frac{\beta(1 - P(r_0, T)) + \int_0^T P(r_0, t)\left[X^* + e^{-\theta^* t}(x_0 - X^*)\right]dt}{T/N \sum_{i=1}^{N} P(r_0, iT/N)}, \tag{21}$$

This encompasses both the CIR and Vasicek versions of the model, although the zero prices $P(r_0, t)$ and the risk-neutral parameters, X* and θ* for a given X and θ would vary between the models. Rather than second guess the functional form of the term structure, we use the actual term structure to compute the zero coupon prices P(). Specifically, using data on 6 month and 1-year LIBOR, as well as the 2,3,4, 5, and 7-year all-in-cost, we employ a cubic spline to interpolate the all-in-cost for maturities 1.5, 2.5, 3.5, 4.5, 5.5, 6.0, and 6.5 years and then compute ten semi-annual zero coupon bond prices from the all-in-cost rates. We then apply a second cubic spline to the semi-annual zero coupon prices to obtain interpolated zero coupon prices for each interval of time necessary for the numerical integration.[30]

Equation (21), the swap spread valuation equation, can be viewed as a fairly simple regression that is nonlinear in only one parameter, θ^* and, given θ^*, for a time series of length τ, is linear in the coefficients $x_0(1)$, ..., $x_0(\tau)$, β, and X^*. There is no constant in the regression. To estimate parameters, we pool cross-section and time series and minimize the total sum of squares. For the U.S. data, with 410 time series observations on five maturities, we have 2050 pooled observations and 413 parameters to estimate.[31] To obtain parameters that minimize the total sum of squares, we first orthogonalize x and the term structure of interest rates (as represented by the LIBOR all-in-cost curve), by regressing the pooled cross-section and time series of swap spreads on the all-in-cost of the swap and a constant. The slope coefficient in this univariate regression is β. We then use the intercept plus the residuals from this regression as dependent variables in a nonlinear multiple regression.[32] We first guess a value for θ^* and then obtain OLS estimates of the remaining parameters, $x_0(1),. . .,x_0(\tau)$, and X^*.[33] Iteration on θ^* is guided by the Newton-Raphson algorithm. Table 3 summarizes the parameter estimates for the four countries. Panels A through E of Figure 3, each panel corresponding to a particular maturity, graph the actual and predicted values for the swap spreads in each of the four countries through time.

---

[30]This numerical integration was performed with the routine provided with MATLAB.

[31]In general, with T times series observations and five maturities, there are 5T data points used to estimate T+3 parameters.

[32]The intercept in the second regression is suppressed.

[33]Since these coefficient estimates are linear functions of dependent variables that are constructed to be orthogonal to interest rates, we would expect them to be reasonably orthogonal to the r's that generate the interest rates.

Table 3. Parameter Estimates and Standard Errors (in Parentheses) for the General Model

Parameters are obtained by minimizing the total sum of squared errors in Equation 21, using data from a pooled cross-section and time series. For a given θ*, OLS estimates of the remaining parameters are obtained in a two step regression--the first step a univariate regression of swap spreads on the all-in-cost and the second step a regression of residuals plus the constant from the first regression on the remaining variables in the swap spread equation. Iteration on θ* is guided by the Newton-Raphson algorithm. Standard errors for X* are obtained assuming that θ* is estimated without error and using the OLS formulae for standard errors on the remaining coefficients. Standard errors of the average x_0 are computed as the standard error of the mean of the reproduced time series.

	United States	Canada	Japan	United Kingdom
Beta	.053 (.006)	.001 (0.0018)	0.095 (0.0066)	0.098 (0.0104)
X* (basis points)	6.937 (3.128)	8.431 (4.63)	-4.822 (17.4243)	10.780 (2.1735)
Mean of x0 (basis points)	1.45 (22.53)	17.88 (8.67)	6.2774 (6.2052)	17.05 (13.04)
θ*	0.000	0.081	0.000	0.163
θ from slope of x0(t) on x0(t-1)	.229 (1.46)	2.485 (1.38)	9.082 (2.02)	6.554 (3.87)
ρ(predicted,actual) 2 yr	0.986	0.717	0.847	0.928
ρ(predicted,actual) 3 yr	0.994	0.920	0.947	0.950
ρ(predicted,actual) 4 yr	0.999	0.974	0.974	0.976
ρ(predicted,actual) 5 yr	0.995	0.916	0.947	0.971
ρ(predicted,actual) 7 yr	0.993	0.756	0.927	0.823

The U.S. and Japan have mean reversion parameters that are positive but are rounded to zero. This indicates that the component of the convenience yield that is unrelated to the level of interest rates is virtually indistinguishable from a random walk.[34] It is important to point out, however, that the model is fit only with swap spread data and can therefore only determine the "risk-neutral" parameters.

Is the model is reasonable? First, a glance at the graphs in Panels A through E of Figure 3 illustrates that the predicted swap spreads closely track the actual swap spreads. The correlations between actual and predicted swap spreads all are above 0.98 for the U.S. For other countries,

[34]An earlier draft of this paper, using data from Salomon Brothers from the late 80s and early 90s found this mean reversion to be more positive. Many hedge funds in the 90s, including some that are now defunct, had strategies that attempted to capitalize on mean reversion in the valuation impact of liquidity. If, indeed, the mean reversion in the liquidity parameter is negligible, then these strategies would have been doomed to failure.

where the data may be less reliable, they range from 0.72 to 0.98. None of the parameter estimates is of an unreasonable sign or magnitude. Finally, different ways of obtaining the same parameter estimates yield similar results. For example, no constraints are placed on the estimation of x_0's, (as would be the case, for example, with Kalman filter estimation) other than that they fit the swap spread data. However, if x follows an AR1 process, $\theta = \theta^*$. A comparison of the mean reversion parameter obtained from regressing the x_0's on their lag1 values with the mean reversion parameter that provides the best fit for the swap spreads indicates that, except for Japan, they are within two standard errors of one another.[35]

4. Conclusion

This paper is a first attempt to develop a model where liquidity considerations alone generate interest rate swap spreads. It may also be the first paper to value liquidity with the valuation techniques developed for derivatives. For certain parameter restrictions, closed form solutions can be found for a number of interest rate processes, including multifactor ones.

An alternative way to think about these issues is to note that we have implicitly added a factor to traditional models of the government yield curve. The model starts with the yield curve generated by the all-in-cost of the swap (also known as the LIBOR term structure). The government yield curve is then derived from the LIBOR term structure by annuitizing the present value of a stochastic liquidity factor (the swap spread) and subtracting it from the LIBOR term structure. Hence, a 1-factor model of the yield curve can be thought of, not as a 1-factor model of the government yield curve but as 1-factor model of the all-in-cost curve (the LIBOR yield curve). The government yield curve is then generated by considering the effect of the additional liquidity factor.

The model can generate term structures for interest rate swap spreads that have a variety of shapes, including U-shaped and humped curves. Elementary inspections of the swap spreads generated by the estimated model indicate that it is adequate at explaining the existing empirical data on swap spreads.

Interestingly, the model implies that the risk-free rate derived from government securities may be inappropriate as implicit risk-free rates for models of the pricing of corporate securities. Empirical analyses of U.S. data have long suggested that the implicit risk-free (or zero beta) rates for popular risk return models of corporate securities, like the CAPM and APT, are substantially higher than the government risk-free rate derived from Treasury Bills. A number closer to the corporate risk free rate derived from LIBOR is also implicit in option prices. It is possible that the failings of asset pricing models in this arena arise from the inappropriate use of the government risk-free rate, which is lowered by the liquidity-based convenience yield, as parameters. A risk-free rate that is close to short-term LIBOR, which this model suggests is the appropriate one to use for the CAPM, APT, and Option Pricing Models, provides a far better empirical fit to historical data.

An issue that the model does not address is the effect of special repo. The model makes the assumption that the liquidity advantage of government notes of all maturities is the same. Realistically, differences in the liquidity-based convenience yield of the government notes, due to some issues being "more special" than others at certain times needs to be accounted for. However, a model where there are different liquidity-based state variables for different maturities is not testable without further restrictions. One alternative is to keep the model in its current form and treat

[35] This is an overly conservative statement since the comparison assumes that θ^* is estimated without error.

the effect of special repo as serially correlated measurement error in tests. This presents identification problems of its own, in that it may be impossible to determine if the serial correlation is driven by special repo or by some other misspecification of the model.

A potential limitation of the model is that the state variable for the liquidity-based convenience yield is observationally equivalent to a state variable for credit risk. We circumvented this issue by assuming that there is zero default risk of any type embedded in swap spreads. In practice, this assumption may be a bit extreme. We know for example that a handful of corporations with the very best credit rating borrow, short-term, at a few basis points below LIBOR. This would seem to indicate that there is a small amount of compensation for default risk built into LIBOR. The thrust of our argument, however, is that this is miniscule--at most on the order of 10% to 20% of the swap spread. By far, the critical determinant of the swap spread is an amortized present value of a liquidity-based convenience yield.

In principle, one could differentiate the miniscule default risk component of the swap spread from the liquidity-based component by modeling the stochastic processes for these components differently. However, there is little economic guidance for modeling a distinction between these components. Moreover, such arbitrary distinctions in the modeling alone will dictate whether empirical research finds that the swap spread is largely due to default risk or largely due to liquidity.

A better alternative is to test the model by supplementing the swap spread data with data on other rates or spreads known to have differing default or liquidity properties. In addition to special repo rates, as discussed above, general collateral repo rates, spreads between such rates and LIBOR, and spreads between short-term borrowing rates of the most creditworthy corporations and LIBOR may be useful. Such data were unavailable to us, but represent an interesting avenue for future research.

APPENDIX

Proof that

$$cov\left(e^{-\int_0^t r(\tau)d\tau}, x(t) \right) = -P(r_0,t)\frac{\rho\sigma_r\sigma_x}{\kappa}\left[\frac{1-e^{-\theta t}}{\theta} - \frac{1-e^{-(\theta+\kappa)t}}{\theta+\kappa}\right]$$

.

Define for all points in time τ

$\varphi = e^{\kappa\tau}r.$
$\psi = e^{\theta\tau}x.$

By Ito's Lemma

$d\varphi = \kappa e^{\kappa\tau}R^*dt + e^{\kappa\tau}\sigma_r dz$ and
$d\psi = \theta e^{\theta\tau}X^*dt + e^{\theta\tau}\sigma_x dw,$

By Stein's Lemma,

$$cov\left(e^{-\int_0^t r(\tau)d\tau}, x(t) \right) = -E\left(e^{-\int_0^t r(\tau)d\tau} \right) cov\left(\int_0^t r(\tau)d\tau, x(t)\right)$$

$$= -P(r_0,t)cov\left(\int_0^t e^{-\kappa\tau}\phi(\tau)d\tau,\ e^{-\theta t}\psi(t)\right)$$

$$= -P(r_0,t)E\left[\left(\int_0^t e^{-\kappa\tau}\int_0^\tau e^{\kappa s}\sigma_r dz(s)d\tau\right)\left(e^{-\theta t}\int_0^t e^{\theta\tau}\rho\sigma_x dz(\tau)\right)\right],$$

where s is a variable indexing time. Reversing the order of integration for the double (partially stochastic) integral above,

$$cov\left(e^{-\int_0^t r(\tau)d\tau}, x(t) \right) = -P(t)E[(\int_0^t e^{\kappa\tau}(\int_\tau^t e^{-\kappa\tau}\sigma_r d\tau)dz(\tau))(e^{-\theta t}\int_0^t e^{\theta\tau}\rho\sigma_x dz(\tau))]$$

$$= -P(t)e^{-\theta t}E[(\int_0^t e^{\kappa\tau}\frac{-\sigma_r}{\kappa}(e^{-\kappa t}-e^{-\kappa\tau})dz(\tau))(\int_0^t e^{\theta\tau}\rho\sigma_x dz(\tau))]$$

$$= -P(t)e^{-\theta t}\frac{-\rho\sigma_r\sigma_x}{\kappa}\int_0^t e^{(\kappa+\theta)\tau}(e^{-\kappa t}-e^{-\kappa\tau})d\tau\ ,$$

where the latter equality follows from the independence of the non-contemporaneous dz's and the Brownian motion assumption that $E(dz^2) = d\tau$. Completing the integration,

$$cov\left(e^{-\int_0^t r(\tau)d\tau}, x(t) \right) = -P(t)e^{-\theta t}\frac{-\rho\sigma_r\sigma_x}{\kappa}\left[\frac{e^{(\kappa+\theta)t}-1}{\kappa+\theta}e^{-\kappa t} - \frac{e^{\theta t}-1}{\theta}\right]$$

$$= -P(t)\frac{\rho\sigma_r\sigma_x}{\kappa}\left[\frac{1-e^{-\theta t}}{\theta} - \frac{1-e^{-(\theta+\kappa)t}}{\theta+\kappa}\right].$$

References

Amihud, Yacov and Haim Mendelson, 1991, "Liquidity, Maturity, and the Yields on U.S. treasury Securities," Journal of Finance, 46, 1411-1425.

Boudoukh, Jacob and Robert Whitelaw, 1993, "Liquidity as a Choice Variable: A lesson from the Japanese Government Bond Market," Review of Financial Studies, 6 (2), 266-292.

Brennan, Michael, 1991, "The Price of Convenience and the Valuation of Commodity Contingent Claims," in D. Lund and B. Øksendal (eds.), Stochastic Models and Option Values, Elsevier Science Publishers (North Holland).

Chen, Andrew and Arthur K. Selender, 1994, "Determination of Swap Spreads: An Empirical Analysis," Southern Methodist University Working Paper.

Cooper, Ian and Antonio Mello, 1991, "The Default Risk of Swaps," Journal of Finance 46, 597-620.

Cox, John, Jonathan Ingersoll, and Stephen Ross, 1985, "A Theory of the Term Structure of Interest Rates," Econometrica 53, 385-407.

Daves, Phillip and Michael Erhardt, 1993, "Liquidity, Reconstitution, and the Value of U.S. Treasury Strips," Journal of Finance, 48 (1), 315-330..

Duffie, Darrell, 1996, "Special Repo Rates," 1996, Journal of Finance, 51 (2), 493-526.

Duffie, Darrell and Ming Huang, 1996, “Swap rates and Credit Quality,” Journal of Finance, 51 (3), 921-949.

Duffie, Darrell and Kenneth Singleton, 1997, “An Econometric Model of the Term Structure of Interest-Rate Swap Yields,” Journal of Finance, 52 (4), 1287-1321.

Evans, E. and Giola Parente Bales, 1991, "What Drives Interest Rate Swap Spreads?" in Interest Rate Swaps, Carl Beidleman, ed. 280-303.

Grinblatt, Mark and Francis Longstaff, 2000, “Financial Innovation and the Role of Derivative Securities: An Empirical Analysis of the Treasury Strips Program,” Journal of Finance 55 (3), 1415-1436.

Jarrow, Robert and Fan Yu, 2001, “Counterparty Risk and the Pricing of Defaultable Securities,” Journal of Finance, 56 (5), 1765-1799.

Kamara, Avraham, 1994, "Liquidity, Taxes, and Short-Term Treasury Yields," Journal of Financial and Quantitative Analysis, 29(3), 403-417.

Litzenberger, Robert, 1992, "Swaps: Plain and Fanciful," The Journal of Finance, 597-620.

Longstaff, Francis and Eduardo Schwartz, 1995, "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt," Journal of Finance, 50 (3), 789-819.

Minton, Bernadette, 1997, "An Empirical Examination of Basic Valuation Models for Plain Vanilla U.S. Interest Rate Swaps," Journal of Financial Economics, 44 (2), 251-277.

Sarig, Oded and Arthur Warga, 1989, "Bond Price Data and Bond Market Liquidity," Journal of Financial and Quantitative Analysis, 24(3), 367-378.

Sun, Tong-Sheng, Suresh Sundaresan, and Ching Wang, 1993, "Interest Rate Swaps: An Empirical Investigation," Journal of Financial Economics, August, 77-99.

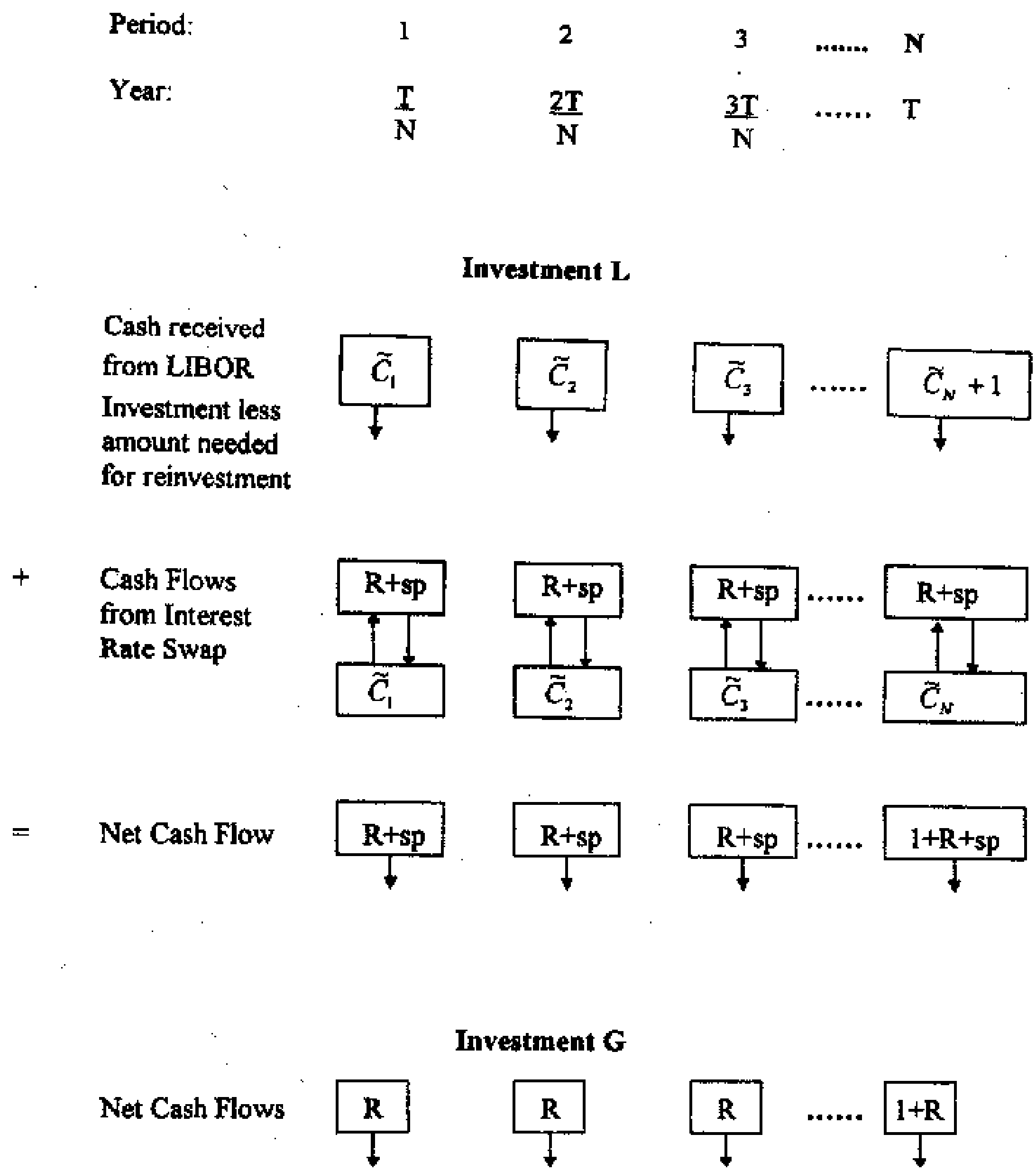
Sundaresan, Suresh, 1991, "Valuation of Swaps," in Recent Developments in International Banking and Finance, S. Khoury, ed. Amsterdam: North Holland (Vols. IV and V, 1991).

Uhlenbeck, G.E. and L.S. Ornstein, 1930, "On the Theory of Brownian Motion," Physical Review 36, 823-841.

Vasicek, Oldrich, 1977, "An Equilibrium Characterization of the Term Structure," Journal of Financial Economics 5, 177-188.

Warga, Arthur, 1992, "Bond Returns, Liquidity, and Missing Data," Journal of Financial and Quantitative Analysis, 27(4), 605-617.

FIGURE 1



$\tilde{C}_i$ = LIBOR rate at beginning of period

R = Government yield for note of maturity T

sp = Swap spread for swap of maturity T

N = Number of coupons

FIGURE 2

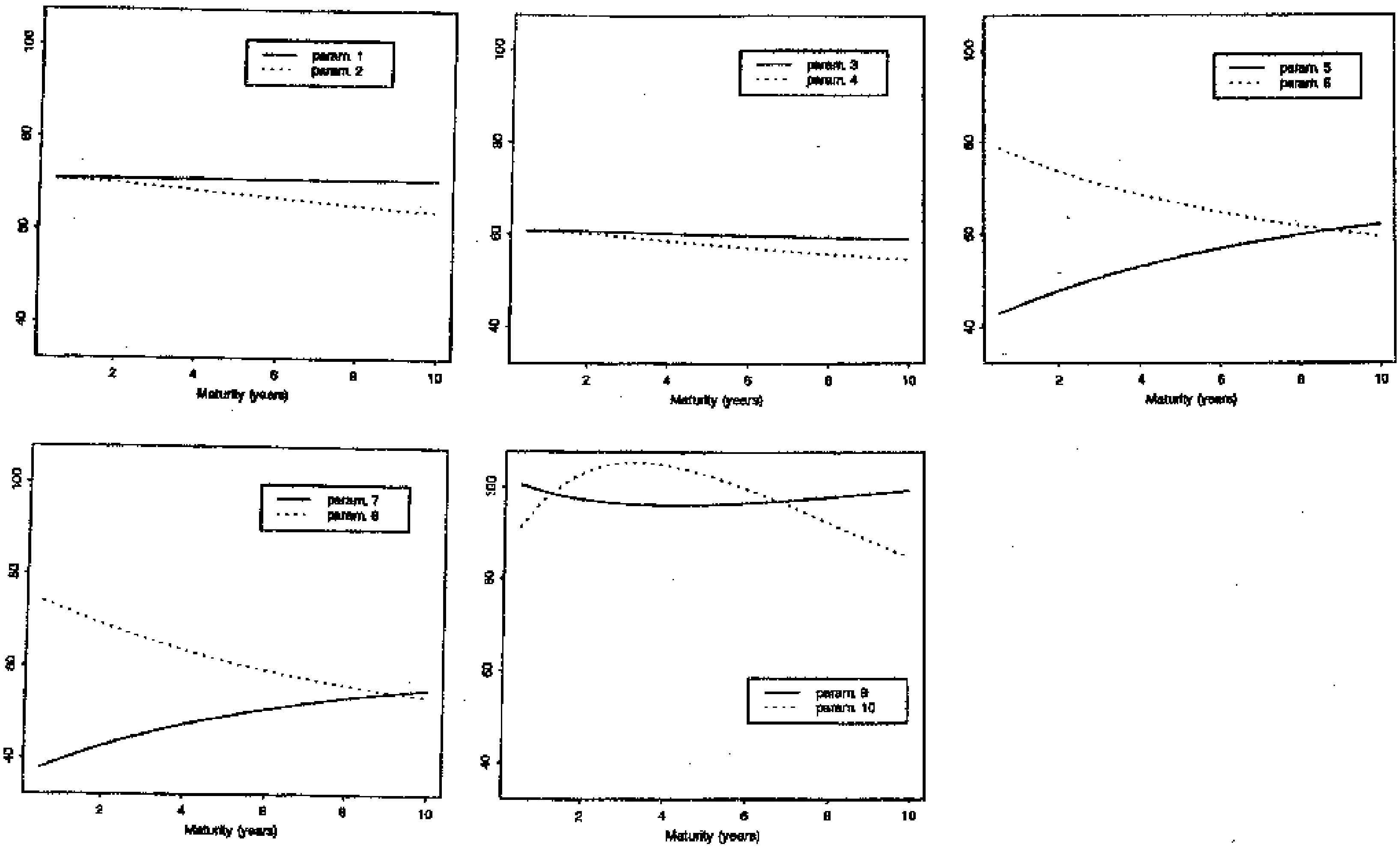
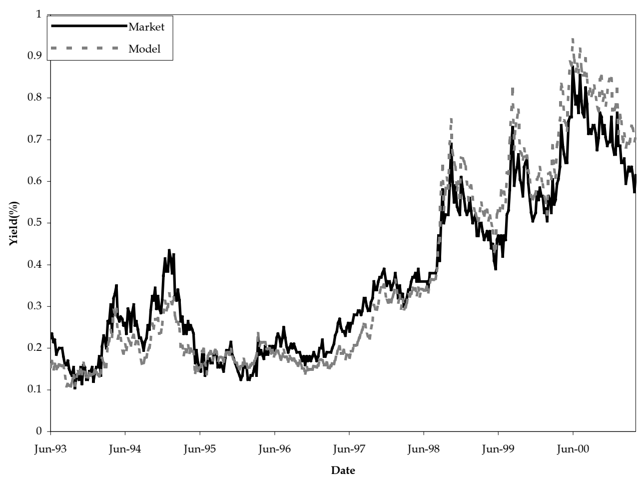
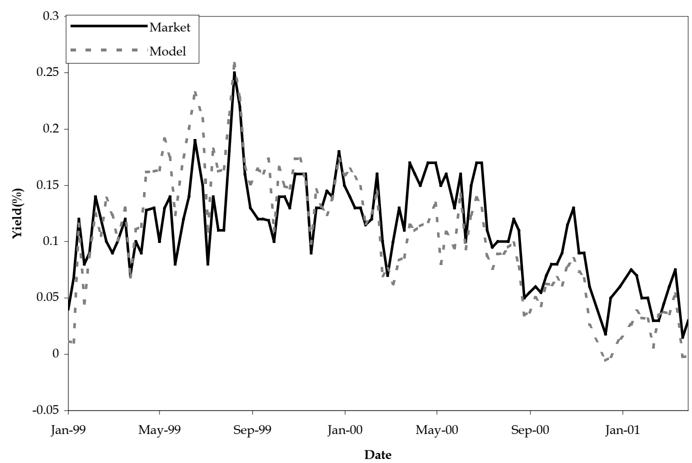

param. 1
param. 2
param. 3
param. 4
param. 5
param. 6
param. 7
param. 8
param. 9
param. 10
Maturity (years)
100
80
60
40
2
4
6
8
10

FIGURE 3: Panel A
Fitted and Actual Values for 2 Year Swap Spreads Over Time for 4 Countries

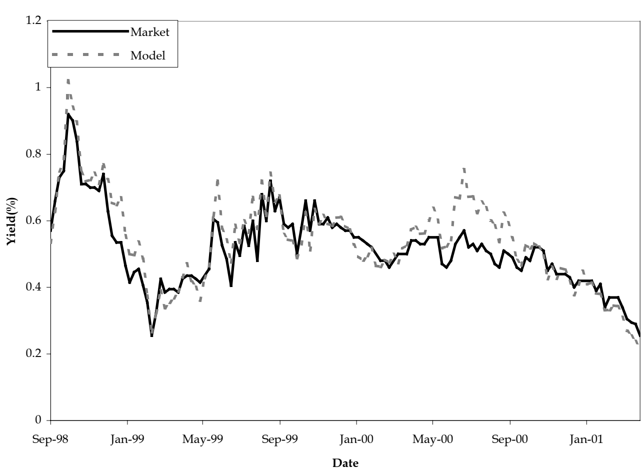
Two Year Swap Spread: United States



Two Year Swap Spread: Japan



Two Year Swap Spread: United Kingdom



Two Year Swap Spread: Canada

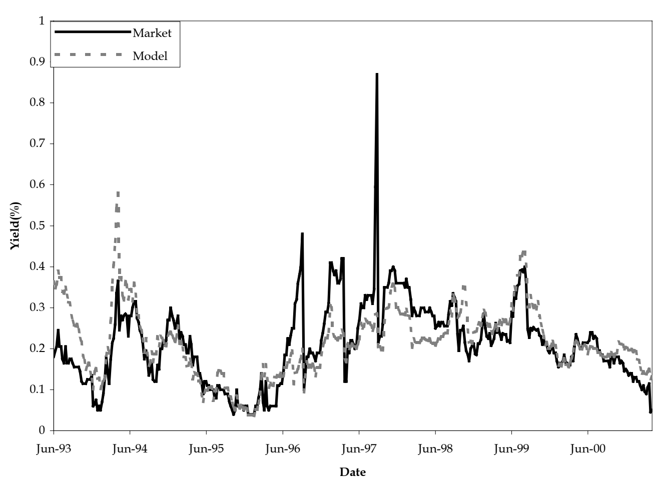


Figure 3
Panel B: Fitted and Actual Values for 3 Year Swap Spreads Over Time for 4 Countries

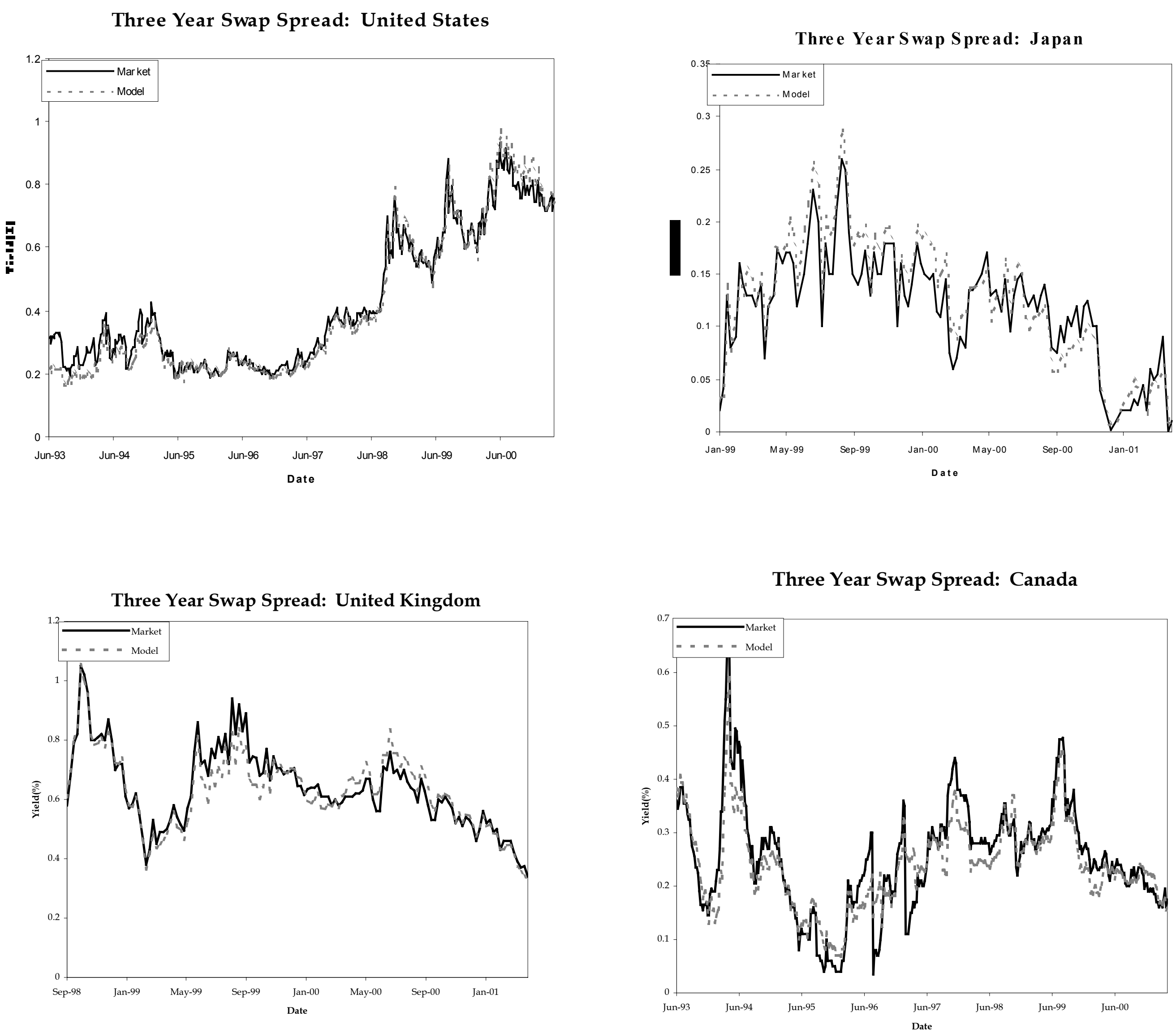


FIGURE 3

Panel C: Fitted and Actual Values for 4 Year Swap Spreads Over Time for 4 Countries

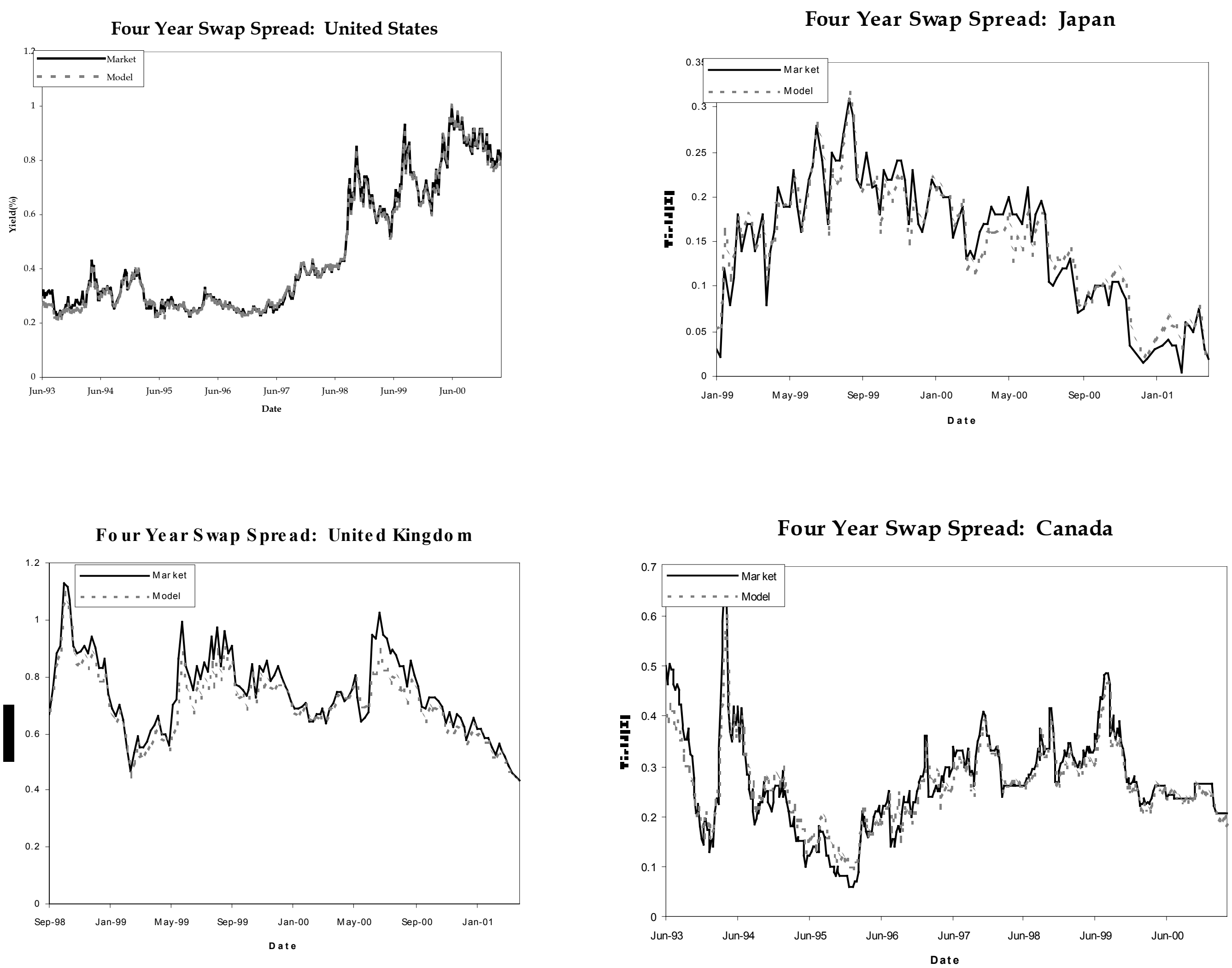


FIGURE 3
Panel D:Fitted and Actual Values for 5 Year Swap Spreads Over Time for 4 Countries

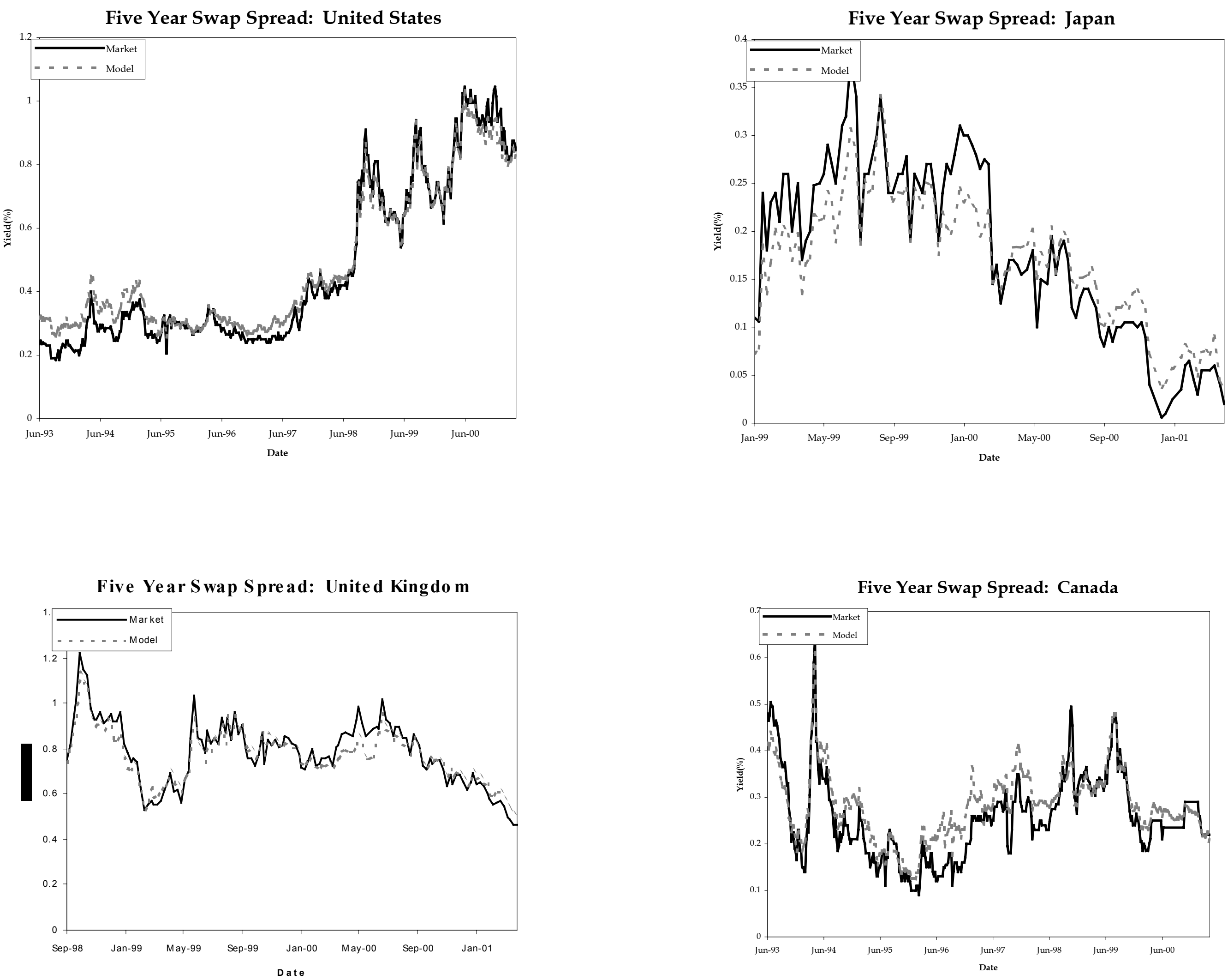


FIGURE 3

Panel E: Fitted and Actual Values for 7 Year Swap Spreads Over Time for 4 Countries