LIMITS

1. NEIGHBOURHOOD OF ‘ $x = a$ ’

For some $h > 0$, sufficiently small, let the function $y = f(x)$ be defined in the interval $(a-h, a)$ then it is said that the function $y = f(x)$ is defined in the left-neighbourhood of $x = a$.

Similarly, if the function $y = f(x)$ be defined in the interval $(a, a+h)$ then it is said that the function $y = f(x)$ is defined in the right-neighbourhood of $x = a$.

If the function $y = f(x)$ be defined in left-neighbourhood of $x = a$ or right-neighbourhood of $x = a$, then it is said that the function $y = f(x)$ is defined in the neighbourhood of $x = a$.

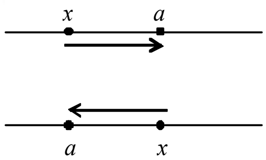
It must be noted here that the value ‘a’ itself may or may not be included in the domain which is actually not being considered in its neighbourhood.

2. MEANING OF ‘$x \to a$’

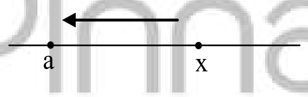
Let x be a variable and ‘a’ be a constant. If x assumes values nearer and nearer to ‘a’ but x is strictly smaller than ‘a’ then this statement is mathematically written as $x \to a^-$.

Similarly, $x \to a^+$, implies that x assumes values nearer and nearer to ‘a’ but x is strictly greater than ‘a’.

In general by ‘x tends to a’ we mean that
(i) $x \neq a$
(ii) x assumes values nearer and nearer to ‘a’ and
(iii) we are not specifying the manner in which x should approach to ‘a’.
x may approach to ‘a’ from left or right as shown in figure.

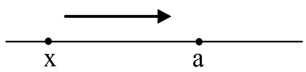


If ‘x’ approach to ‘a’ from any point on the right of $x = a$ in the real number line (i.e. the x axis) but it never crosses $x = a$, then it is written as $x \to a^+$.



$\Leftrightarrow \quad x \to a^+$

Similarly,



$\Leftrightarrow \quad x \to a^-$

3. INDETERMINATE FORMS

Some times we come across some functions which do not have definite value corresponding to some particular value of the variable.

For example, the function $f(x) = \dfrac{x^2-4}{x-2}$, converts into $\dfrac{0}{0}$ if $x = 2$ is substituted.

Hence, $f(2)$ cannot be determined. Such a form is called an ***Indeterminate Form***.

There are total 7 ***Indeterminate Forms*** given as

(1) $\frac{0}{0}$, (2) $\frac{\infty}{\infty}$, (3) $\infty - \infty$, (4) $0 \times \infty$, (5) 1^{∞}, (6) ∞^0, (7) 0^0.

Note : Here 0 and 1 are all approaching values, not the exact values.

EXAMPLE :

Which of the following are forming indeterminate form. Also indicate the form

(i)	$\frac{1}{x}$ as $x \to 0$	(ii)	$\frac{1-x}{1-x^2}$ as $x \to 1$
(iii)	$x \ell n\, x$ as $x \to 0$	(iv)	$\left(\frac{1}{x} - \frac{1}{x^2}\right)$ as $x \to 0$
(v)	$(\sin x)^x$ as $x \to 0$	(vi)	$(\ell n\, x)^x$ as $x \to 0$
(vii)	$(1+\sin x)^{\frac{1}{x}}$ as $x \to 0$	(viii)	$\frac{\sec x}{\tan x}$ as $x \to \frac{\pi}{2}$

Sol.

(i)	No	(ii)	$\frac{0}{0}$ form
(iii)	$0 \times \infty$ form	(iv)	$(\infty - \infty)$ form
(v)	$(0)^0$ form	(vi)	$(\infty)^0$ form
(vii)	$(1)^{\infty}$ form	(viii)	$\frac{\infty}{\infty}$ form

4. LIMIT OF A FUNCTION

DEFINITION 1

Let the function y = f(x) be defined in a certain neighbourhood of a point x = a . The function y = f(x) approaches the limit L ($y \to L$) as x approaches 'a' ($x \to a$). If for every positive number h, arbitrarily small, we are able to indicate k > 0, arbitrarily small, such that for all x, different from 'a' and satisfying the inequality.

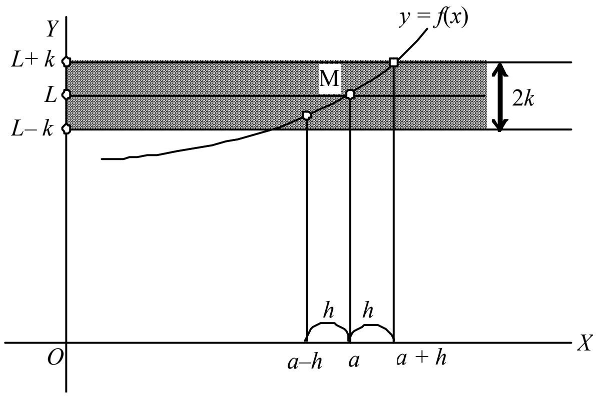
$| x - a | < h$

we have the inequality

$| f(x) - L | < k$

then $\lim_{x \to a} f(x) = L$

or $f(x) \to L$ as $x \to a$ or limiting value of f(x) is L as $x \to a$.



DEFINITION 2 :

Let y = f (x) be a function of x and the limiting value of y is required for $x \to a$, then we consider the values of the function at the points which are very near to 'a'.

If these values tend to a definite unique number L as x tends to 'a' (either from left or from right) then this unique number L is called the limit of f(x) at x = a and we write it as $\lim_{x\to a} f(x) = L$

EXAMPLES :

Ex.1 $\lim_{x\to 2} (x + 2)$

Sol. x + 2 being a polynomial in x, its limit as $x \to 2$ is given by $\lim_{x\to 2} (x + 2) = 2 + 2 = 4$

Ex.2 $\lim_{x\to 2} x(x - 1)$

Sol. x(x – 1) being a polynomial in x, its limit as $x \to 2$ is given by $\lim_{x\to 2} x(x - 1) = 2\,(2 - 1) = 2$

Ex.3 $\lim_{x\to 2} \frac{x^2+4}{x+2}$

Sol. $\lim_{x\to 2} \frac{x^2+4}{x+2} = \frac{(2)^2+4}{2+2} = 2$

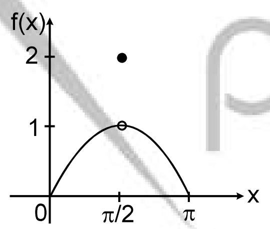
Ex.4 $\lim_{x\to 0} \cos(\sin x)$

Sol. $\lim_{x\to 0} \cos(\sin x) = \cos\left(\lim_{x\to 0} \sin x\right) = \cos 0 = 1$

Ex.5 If $f(x) = x^3 + 1$ then $\lim_{x\to 1} f(x) = f(1) = 2$.

Ex.6

Find $\lim_{x\to \pi/2} f(x)$

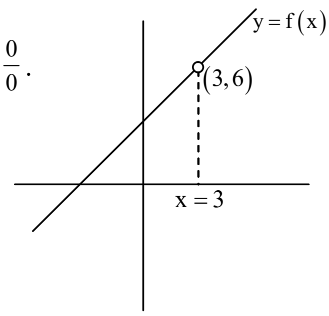


Sol. Here $\lim_{x\to \pi/2} f(x) = 1$

Ex.7 Find the limiting value of $f(x) = \frac{x^2-9}{x-3}$ as $x \to 3$.

Sol. At x = 3, $f(x) = \frac{x^2-9}{x-3}$ converts into an indeterminate form of $\frac{0}{0}$.

Now when x tends to 3 from left or from right, it can be easily observed from the graph that the value of f(x) tends to 6. Hence



$$\lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{(x-3)}$$

$$= \lim_{x\to 3} (x + 3) = 3 + 3 = 6$$

5. EXISTENCE OF LIMIT

The limit of a function f(x) at a point x = a exists and equals to L if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ = a finite value, L.

Here $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ are called left hand limit (L.H.L.) and right hand limit (R.H.L.) respectively.

Thus, if $\lim_{x\to a} f(x)$ exists then $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ = a finite value, L $\Rightarrow$ L.H.L. = R.H.L. = L

$\Rightarrow \lim_{x\to a} f(x) = L$

EXAMPLES :

Ex.1 The value of $\lim_{x\to 1} [x]$ is, where [ ] represents the greatest integer function.

(A) 1 (B) 2 (C) 4 (D) Does not exist

Sol. Left hand limit = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} [x] = 0$

and Right hand limit = $\lim_{x\to 1^+} f(x)$

$= \lim_{x\to 1^+} [x] = 1$

$\because \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

$\therefore$ limit does not exist.

Ex.2 If f(x) = $\begin{cases} \dfrac{1}{1+e^{-1/x}} & x \neq 0 \\ 0, & x = 0 \end{cases}$ then at x = 0

(A) right hand limit of f(x) exists but not left-hand limit
(B) left-hand limit of f(x) exists but not right- hand limit
(C) both limits exists but are not equal
(D) both limits exist and are equal

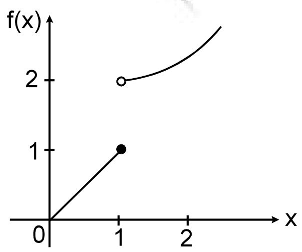
Sol. $f(0^-) = \lim_{h\to 0} \dfrac{1}{1+e^{1/h}} = \dfrac{1}{1+\infty} = 0$

$f(0^+) = \lim_{h\to 0} \dfrac{1}{1+e^{-1/h}} = \dfrac{1}{1+0} = 1$

$\therefore$ Both limits exist but are not equal.

Ex.3

Find $\lim_{x\to 1} f(x)$



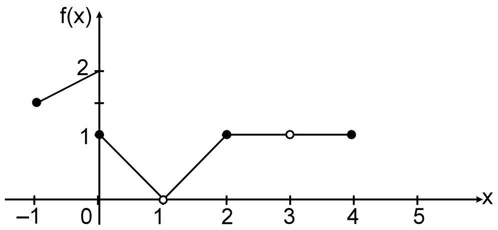
Sol.

Left hand limit = 1 Right hand limit = 2

Hence $\lim_{x\to 1} f(x)$ does not exist.

Ex.4 From the adjoint graph of y = f(x), find

(i) $\lim_{x\to 0} f(x)$ (ii) $\lim_{x\to 1} f(x)$ (iii) $\lim_{x\to 2} f(x)$ (iv) $\lim_{x\to 3} f(x)$ (v) $\lim_{x\to 4} f(x)$



Sol. (i) Here L.H.L. = 2 and R.H.L. = 1

$\therefore$ $\lim\limits_{x\to 0} f(x)$ does not exist

because left hand limit $\neq$ right hand limit.

(ii) $\lim\limits_{x\to 1} f(x) = 0$ (iii) $\lim\limits_{x\to 2} f(x) = 1$

(iv) $\lim\limits_{x\to 3} f(x) = 1$ (v) $\lim\limits_{x\to 4} f(x) = 1$

6. **THEOREMS ON LIMITS**

Let $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both individually exist and are finite, then

(1) $\lim\limits_{x\to a} [k\, f(x)] = k \lim\limits_{x\to a} f(x)$, where k is a constant

(2) $\lim\limits_{x\to a} [f(x) + g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$

(3) $\lim\limits_{x\to a} [f(x) - g(x)] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$

(4) $\lim\limits_{x\to a} [f(x)\cdot g(x)] = \lim\limits_{x\to a} f(x)\cdot \lim\limits_{x\to a} g(x)$

(5) $\lim\limits_{x\to a} [f(x)/g(x)] = [\lim\limits_{x\to a} f(x)] / [\lim\limits_{x\to a} g(x)]$ provided that $\lim\limits_{x\to a} g(x) \neq 0$

(6) $\lim\limits_{x\to a} f[g(x)] = f[\lim\limits_{x\to a} g(x)]$, provided that $\lim\limits_{x\to a} g(x)$ = a finite value k and $\lim\limits_{x\to k} f(x)$ is also finite.

(7) $\lim\limits_{x\to a} [f(x) + k] = \lim\limits_{x\to a} f(x) + k$

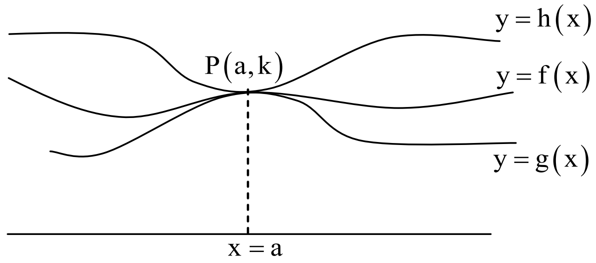
(8) $\lim\limits_{x\to a} [f(x)]^{g(x)} = \{\lim\limits_{x\to a} f(x)\}^{\lim\limits_{x\to a} g(x)}$

(9) $\lim\limits_{x\to +\infty} f(x) \Leftrightarrow \lim\limits_{x\to 0^+} f(1/x)$ and $\lim\limits_{x\to -\infty} f(x) \Leftrightarrow \lim\limits_{x\to 0^-} f(1/x)$ **(Important)**

(10) **Sandwitch theorem or Squeeze Play theorem :**

If $g(x) \le f(x) \le h(x)$ in the neighbourhood of x = a such that $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} h(x) = k$, a finite quantity,

then $\lim\limits_{x\to a} f(x) = k$



7. SOME STANDARD LIMITS

(1) $\lim\limits_{x\to 0}\frac{\sin x}{x} = \lim\limits_{x\to 0}\frac{x}{\sin x} = \lim\limits_{x\to 0}\frac{\sin^{-1}x}{x} = \lim\limits_{x\to 0}\frac{x}{\sin^{-1}x} = 1$

Also, $\left[\lim\limits_{x\to 0}\frac{\sin x}{x}\right]=\left[\lim\limits_{x\to 0}\frac{x}{\sin^{-1}x}\right]=0$ and $\left[\lim\limits_{x\to 0}\frac{x}{\sin x}\right]=\left[\lim\limits_{x\to 0}\frac{\sin^{-1}x}{x}\right]=1$

(where [ ] denotes the greatest integer function)

as $0<\frac{\sin x}{x}<1$ and $0<\frac{x}{\sin^{-1}x}<1$

(2) $\lim\limits_{x\to 0}\cos x = \lim\limits_{x\to 0}\left(\frac{1}{\cos x}\right)=1$

(3) $\lim\limits_{x\to 0}\frac{\tan x}{x} = \lim\limits_{x\to 0}\frac{x}{\tan x} = \lim\limits_{x\to 0}\frac{\tan^{-1}x}{x} = \lim\limits_{x\to 0}\frac{x}{\tan^{-1}x} = 1$

(4) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim\limits_{x\to 0}(1+x)^{1/x} = e$

(5) If $\lim\limits_{x\to a}f(x)\to 1$ and $\lim\limits_{x\to a}g(x)\to\infty$ then $\lim\limits_{x\to a}\{f(x)\}^{g(x)} = e^{\lim\limits_{x\to a}(f(x)-1)\cdot g(x)}$

(6) $\lim\limits_{x\to 0}\frac{a^x-1}{x} = \log_e a\ (a>0)$ and $\lim\limits_{x\to 0}\frac{e^x-1}{x} = 1$

(7) $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a} = n\cdot a^{n-1}$ and $\lim\limits_{x\to 0}\frac{(1+x)^n-1}{x} = n$

(8) $\lim\limits_{x\to 0}\frac{\ell n(1+x)}{x} = 1$ and $\lim\limits_{x\to 0}\frac{\log_a(1+x)}{x} = \log_a e$

(9) $\lim\limits_{x\to\infty}a^x = \begin{cases} 0 & \text{if } 0\le a<1 \\ 1 & \text{if } a=1 \\ \infty & \text{if } a>1 \\ \text{D.N.E.} & \text{if } a<0 \end{cases}$

(10) $\lim\limits_{x\to\infty}\frac{a_0x^m+a_1x^{m-1}+\dots\dots\dots+a_m}{b_0x^n+b_1x^{n-1}+\dots\dots\dots+b_n} = \begin{cases} \frac{a_0}{b_0} & \text{if } m=n \\ 0 & \text{if } m<n \\ \infty & \text{if } m>n\ \&\ a_0\cdot b_0>0 \\ -\infty & \text{if } m>n\ \&\ a_0\cdot b_0<0 \end{cases}$

8. IMPORTANT EXPANSIONS

(1) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\dots\dots\dots\ x\in R$

(2) $a^x = 1+\frac{x\,ln\,a}{1!}+\frac{x^2\,ln^2a}{2!}+\frac{x^3ln^3a}{3!}+\dots\dots\dots a>0$

(3) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... \text{for } -1 < x \le 1$

(4) $\sin x = \frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(5) $\cos x = \frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \; x \in (-\pi, \pi)$

(6) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ... \; x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(7) $\sin^{-1} x = x + \frac{1^2.x^3}{3!} + \frac{1^2.3^2.x^5}{5!} + \frac{1^2.3^2.5^2\,x^7}{7!} + ... = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + ...$

(8) $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x = \frac{\pi}{2} - \left(x + \frac{1^2.x^3}{3!} + \frac{1^2.3^2.x^5}{5!} + \frac{1^2.3^2.5^2\,x^7}{7!} + ...\right)$

$$= \frac{\pi}{2} - \left(x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + ...\right)$$

(9) $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

(10) $\sec^{-1}x = 1 + \frac{x^2}{2!} + \frac{5^2.x^4}{4!} + \frac{61x^6}{6!} + ...$

(11) $\sin h\, x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$

(12) $\cos h\, x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$

(13) $\tan h\, x = x - \frac{x^3}{3} + \frac{2x^5}{15} - ......$

(14) **The Binomial Theorem :**

$(1+x)^n = 1 + n\cdot x + \frac{n\cdot(n-1)}{2!}\cdot x^2 + \frac{n(n-1)(n-2)}{3!}\cdot x^3 + ...$ (for rational values of n only)

(15) $(1+x)^{1/x} = e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 + ...\right)$

9. STANDARD APPROACHES TO EVALUATE THE LIMIT OF A FUNCTION

1. **Substitution**
2. **Factorization**
3. **Rationalization or Double Rationalisation**
4. **Use of Binomial Theorem and other expansions**
5. **By Law of love**

   Here 'love' stands for zero. This law states that in the problems involving indeterminate form $\frac{\infty}{\infty}$, we should try to get 0 (zero) in numerator or ∞ in denominator by simple algebric operations.

6. **By application of standard limits**

7. **By using Sandwich Theorem**

8. **By using L' Hospital's Rule**
(This method will be discussed in the chapter of Method of Differentiation)

9. **By using Definite Integrals**
(This method will be discussed in the chapter of Definite Integrals)

SOLVED EXAMPLES :

1 <u>Substitution</u>

Ex.1 Evaluate : $\lim\limits_{x\to\infty}\dfrac{3x^2+5x+1}{7x^2+3x-1}$

Sol. Put $x = \dfrac{1}{y}$, $\lim\limits_{x\to\infty}\dfrac{3x^2+5x+1}{7x^2+3x-1} = \lim\limits_{y\to 0^+}\dfrac{y^2+5y+3}{y^2+3y+7} = \dfrac{3}{7}$

Ex.2 Evaluate : $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x^3}$

Sol. The given limit $l = \lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x^3}$, put x = 3y

$$= \lim_{x\to 0}\frac{e^{3y}-e^{-3y}-6y}{27y^3} = \lim_{x\to 0}\frac{\left(e^y-e^{-y}\right)^3+3\left(e^y-e^{-y}\right)-6y}{27y^3}$$

$$= \lim_{x\to 0}\frac{8}{27}\left(\frac{e^{2y}-1}{2y}\right)^3 + \lim_{x\to 0}\frac{1}{9}\left(e^y-e^{-y}-2y\right) \qquad \left(\text{Using } \lim_{y\to 0}\frac{e^{2y}-1}{2y}=1\right)$$

$$= \frac{8}{27}+\frac{1}{9}l = \frac{8l}{9} = \frac{8}{27} \Rightarrow l = \frac{1}{3}$$

Ex.3 Evaluate : $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}$

Sol. The given limit $l = \lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3} = \lim\limits_{x\to 0}\dfrac{3t-\sin 3t}{27t^3}$

$$= \lim_{x\to 0}\frac{3t-(3\sin t-4\sin^3 t)}{27t^3} = \lim_{x\to 0}\frac{3(t-\sin t)}{27t^3}+\lim_{x\to 0}\frac{4\sin^3 t}{27t^3} = \frac{1}{9}l+\frac{4}{27}$$

$$\Rightarrow \quad \frac{8l}{9}=\frac{4}{27} \qquad \Rightarrow \qquad l=\frac{1}{6}$$

Ex.4 Evaluate the limit : $\lim\limits_{x\to 1}\left(\dfrac{1}{\ell n\, x}-\dfrac{1}{x-1}\right)$

Sol. The given limit $\ell = \lim\limits_{x\to 1}\left(\dfrac{1}{\ell n\, x}-\dfrac{1}{x-1}\right)$

Put $x = \dfrac{1}{t}$, then $\ell = \lim\limits_{t\to 1}\left(\dfrac{1}{\ell n\left(\dfrac{1}{t}\right)}-\dfrac{1}{\dfrac{1}{t}-1}\right)$

$$= \lim_{t\to1}\left(\frac{-1}{\ell n\, t}-\frac{t}{1-t}\right)$$

$$= \lim_{t\to1}\left(\frac{-1}{\ell n\, t}+1+\frac{1}{t-1}\right)=1-\lim_{t\to1}\left(\frac{1}{\ell n\, t}-\frac{1}{t-1}\right)$$

$\therefore \quad \ell = 1-\ell$

$\therefore \quad 2\ell = 1$

$\Rightarrow \quad \ell = \frac{1}{2}$

2 <u>Factorization</u>

Ex.1 Evaluate : $\lim_{x\to1}\frac{x^2-3x+2}{x^2-1}$

Sol. Note that, for x = 1 both the numerator and the denominator of the given fraction vanish.

Hence, it converts into an indeterminate form of $\frac{0}{0}$.

Therefore we have $\lim_{x\to1}\frac{x^2-3x+2}{x^2-1} = \lim_{x\to1}\frac{(x-1)(x-2)}{(x-1)(x+1)} = \lim_{x\to1}\frac{x-2}{x+1} = -\frac{1}{2}$

Ex.2 Evaluate : $\lim_{x\to1}\frac{x^3-x^2\log x+\log x-1}{x^2-1}$

Sol. The given limit $= \lim_{x\to1}\frac{(x^3-1)-(x^2-1)\log x}{x^2-1}$

$$= \lim_{x\to1}\frac{(x-1)(x^2+x+1)-(x-1)(x+1)\log x}{(x-1)(x+1)}$$

$$= \lim_{x\to1}\frac{(x-1)\left[x^2+x+1-(x+1)\log x\right]}{(x-1)(x+1)}$$

$$= \frac{1^2+1+1-(1+1)\log 1}{(1+1)} = \frac{3}{2}$$

Ex.3 Evaluate : $\lim_{x\to2}\left[\frac{1}{x-2}-\frac{2(2x-3)}{x^3-3x^2+2x}\right]$

Sol. The given limit $= \lim_{x\to2}\left[\frac{1}{x-2}-\frac{2(2x-3)}{x(x-1)(x-2)}\right]$

$$= \lim_{x\to2}\left[\frac{x(x-1)-2(2x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to2}\left[\frac{x^2-5x+6}{x(x-1)(x-2)}\right] = \lim_{x\to2}\left[\frac{(x-2)(x-3)}{x(x-1)(x-2)}\right]$$

$$= \lim_{x\to2}\left[\frac{x-3}{x(x-1)}\right] = -\frac{1}{2}$$

3 Rationalization or Double Rationalisation

Ex.1 Evaluate : $\lim\limits_{x\to\pm\infty} x\left(\sqrt{x^2+k}-x\right), k>0$

Sol. The given limit $= \lim\limits_{x\to\pm\infty} x\left(\sqrt{x^2+k}-x\right)\dfrac{\left(\sqrt{x^2+k}+x\right)}{\left(\sqrt{x^2+k}+x\right)}$

$$= \lim_{x\to\pm\infty}\frac{x\left(x^2+k-x^2\right)}{\left(\sqrt{x^2+k}+x\right)} = \lim_{x\to\pm\infty}\frac{xk}{\left(|x|\sqrt{\left(1+\dfrac{k}{x^2}\right)}\right)+x}$$

Here we have to consider two cases

(i) When $x\to\infty$; $|x| = x$

then the given limit $= \lim\limits_{x\to\infty}\dfrac{xk}{x\sqrt{\left(1+\dfrac{k}{x^2}\right)}+x} = \lim\limits_{x\to\infty}\dfrac{xk}{x\left(\sqrt{\left(1+\dfrac{k}{x^2}\right)}+1\right)} = \dfrac{k}{2}$

(ii) When $x\to-\infty$; $|x| = -x$

then we have $\lim\limits_{x\to-\infty}\dfrac{xk}{-x\sqrt{\left(1+\dfrac{k}{x^2}\right)}+x}$

$$\lim_{x\to-\infty}\frac{xk}{x\left(-\sqrt{1+\dfrac{k}{x^2}}+1\right)} \to \frac{k}{-1^+ +1} \to \frac{k}{0^-} \to -\infty$$

Ex.2 Evaluate : $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x+x^2}-1}{x}$

Sol. By rationalization of numerator

$$= \lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{x}\cdot\frac{\sqrt{1+x+x^2}+1}{\sqrt{1+x+x^2}+1}$$

$$= \lim_{x\to 0}\frac{1+x+x^2-1}{x\left(\sqrt{1+x+x^2}+1\right)}$$

$$= \lim_{x\to 0}\frac{1+x}{\sqrt{1+x+x^2}+1} = \frac{1}{2}$$

Ex.3 Evaluate : $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$

Sol. The given limit taken the form $\dfrac{0}{0}$ when $x\to 0$. Rationalising the numerator, we get

$$= \lim_{x\to 0}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\times\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

$$= \lim_{x\to 0}\left[\frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}\right]$$

$$= \lim_{x\to 0}\left[\frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}\right]$$

$$= \lim_{x\to 0}\left[\frac{2}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{2}{2} = 1$$

Ex.4 $\lim_{x\to 1}\frac{(2x-3)\left(\sqrt{x}-1\right)}{2x^2+x-3}$

Sol. The given limit $= \lim_{x\to 1}\left[\frac{(2x-3)\left(\sqrt{x}-1\right)}{(2x+3)(x-1)}\right]$

$$= \lim_{x\to 1}\left[\frac{(2x-3)\left(\sqrt{x}-1\right)}{(2x+3)\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}\right]$$

$$= \lim_{x\to 1}\left[\frac{2x-3}{(2x+3)\left(\sqrt{x}+1\right)}\right]$$

$$= \frac{-1}{(5)(2)} = \frac{-1}{10}$$

4 <u>Use of Binomial Theorem and other expansions</u>

Ex.1 f(x) is integral of $\frac{2\sin x-\sin 2x}{x^3}$, $x\neq 0$ then, find $\lim_{x\to 0} f'(x)$

Sol. $\lim_{x\to 0} f'(x) = \lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}$

$$= \lim_{x\to 0}\frac{2\left(x-\frac{x^3}{3!}........\right)-\left(2x-\frac{8x^3}{3!}\right)}{x^3}$$

$$= \frac{8-2}{3!} = 1$$

Ex.2 Evaluate : $\lim_{x\to 0}\frac{2x-\sin^{-1}x}{2x+\tan^{-1}x}$

Sol. $\lim_{x\to 0}\frac{2x-\left(x+\frac{x^3}{6}+...\right)}{2x+\left(x-\frac{x^3}{3}+...\right)}=\frac{1}{3}$

Ex.3 Evaluate : $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$

Sol. $\lim\limits_{x \to 0} \dfrac{e^x - 1 - x}{x^2}$

$$= \lim_{x \to 0} \frac{\left(1 + x + \dfrac{x^2}{2!} ....\right) - 1 - x}{x^2} = \frac{1}{2}$$

Ex.4 Evaluate : $\lim\limits_{x \to 0} \dfrac{(7+x)^{1/3} - 2}{x - 1}$

Sol. Put x = 1 + h,

$$\lim_{h \to 0} \frac{(8+h)^{1/3} - 2}{h}$$

$$\lim_{h \to 0} \frac{2 \,.\left(1 + \dfrac{h}{8}\right)^{1/3} - 2}{h}$$

$$= \lim_{h \to 0} \frac{2\left\{1 + \dfrac{1}{3} \,.\, \dfrac{h}{8} + \dfrac{\dfrac{1}{3}\left(\dfrac{1}{3} - 1\right)\left(\dfrac{h}{8}\right)^2}{1.\,2} + ... - 1\right\}}{h}$$ {using binomial theorem}

$$= \lim_{h \to 0} 2 \times \frac{1}{24} = \frac{1}{12}$$

5 <u>By Law of love</u>

Ex.1 Evaluate : $\lim\limits_{x \to \infty} \dfrac{x - 2}{2x - 3}$

Sol. $\lim\limits_{x \to \infty} \dfrac{x - 2}{2x - 3}$

$$\lim_{x \to \infty} \frac{1 - 2/x}{2 - 3/x} = \frac{1}{2}.$$

Ex.2 Evaluate : $\lim\limits_{x \to \infty} \dfrac{x^2 - 4x + 5}{3x^2 - x^3 + 2}$

Sol. $\lim\limits_{x \to \infty} \dfrac{x^2 - 4x + 5}{3x^2 - x^3 + 2}$

$$= \lim_{x \to \infty} \frac{\dfrac{1}{x} - \dfrac{4}{x^2} + \dfrac{5}{x^3}}{\dfrac{3}{x} - 1 + \dfrac{2}{x^3}} = 0$$

Ex.3 Evaluate : $\lim\limits_{x \to -\infty} \dfrac{\sqrt{3x^2+2}}{x-2}$

Sol. $\lim\limits_{x \to -\infty} \dfrac{\sqrt{3x^2+2}}{x-2}$

Put $x = \dfrac{-1}{t}$ as $x \to -\infty$, $t \to 0^+$

$$= \lim_{t \to 0^+} \frac{\sqrt{3+2t^2} \cdot \dfrac{1}{\sqrt{t^2}}}{\dfrac{-1-2t}{t}} = \lim_{t \to 0^+} \frac{\sqrt{3+2t^2}}{-(1+2t)} \frac{t}{|t|}$$

$$= \frac{\sqrt{3}}{-1} = -\sqrt{3}.$$

Ex.4 Evaluate : $\lim\limits_{n \to \infty} \dfrac{(3(n+1))!}{(n+1)^3(3n)!}$

Sol. $$\lim_{n \to \infty} \frac{(3n+3)(3n+2)(3n+1)(3n)!}{(n+1)^3(3n)!} = \lim_{n \to \infty} \frac{27n^3\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{2}{3n}\right)\left(1+\dfrac{1}{3n}\right)}{n^3\left(1+\dfrac{1}{n}\right)^3} = 27$$

Ex.5 Let $S_n = 1 + 2 + 3 + \ldots\ldots + n$ and

$P_n = \dfrac{S_2}{S_2-1} \cdot \dfrac{S_3}{S_3-1} \cdot \dfrac{S_4}{S_4-1} \cdot \ldots\ldots \cdot \dfrac{S_n}{S_n-1}$ where $n \in N$ $(n \geq 2)$. Find $\lim\limits_{n \to \infty} P_n$.

Sol. $S_n = \dfrac{n(n+1)}{2}$ and $S_n - 1 = \dfrac{(n+2)(n-1)}{2}$

$$\therefore \quad \frac{S_n}{S_n-1} = \frac{n(n+1)}{2} \cdot \frac{2}{(n+2)(n-1)} \quad \Rightarrow \quad \frac{S_n}{S_n-1} = \left(\frac{n}{n-1}\right)\left(\frac{n+1}{n+2}\right)$$

$$P_n = \left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \ldots\ldots\ldots \cdot \frac{n}{n-1}\right)\left(\frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \ldots\ldots\ldots \cdot \frac{n+1}{n+2}\right)$$

$$P_n = \left(\frac{n}{1}\right)\left(\frac{3}{n+2}\right)$$

$$\Rightarrow \quad \lim_{n \to \infty} P_n = 3$$

Ex.6 Evaluate : $\lim\limits_{x \to \infty} x \sin \dfrac{1}{x}$

Sol. $\lim\limits_{x \to \infty} x \sin \dfrac{1}{x}$

$$= \lim_{x \to \infty} \frac{\sin(1/x)}{1/x} = 1 \qquad \left(\text{Using } \lim_{y \to 0} \frac{\sin y}{y} = 1\right)$$

IN-CHAPTER EXERCISE 1

1. The value of $\lim_{x \to 1} \left(\frac{1}{x-1} + \frac{1}{1-x} \right)$ is

(A) 0 (B) $\frac{1}{2}$ (C) $-\frac{1}{2}$ (D) 1

2. The value of $\lim_{x \to 2} \frac{|x-2|}{x-2}$ is

(A) 0 (B) 1 (C) -1 (D) does not exist

3. The value of $\lim_{x \to 0} \frac{1}{3 + 2^{\frac{1}{x}}}$ is

(A) 0 (B) 1/3 (C) 1/2 (D) does not exist

4. The value of $\lim_{x \to 0} \frac{1 + 2^{\frac{1}{x}}}{3 + 2^{\frac{1}{x}}}$ is

(A) 0 (B) 1/3 (C) 1/2 (D) does not exist

5. The value of $\lim_{x \to 3} \frac{x}{[x]}$; where [ ] is the greatest integer function, is

(A) 0 (B) 1 (C) -1 (D) does not exist

6. The value of $\lim_{x \to \infty} \frac{\sin x}{x}$ is

(A) 0 B) 1 (C) -1 (D) does not exist

7. The value of $\lim_{n \to \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + ... + \frac{n}{n^2} \right)$ is

(A) 0 (B) 1/3 (C) 1/2 (D) 1/4

8. The value of $\lim_{n \to \infty} \frac{\sum n^3}{n^4}$ is

(A) 0 (B) $\frac{1}{4}$ (C) $\frac{1}{3}$ (D) $\frac{1}{2}$

9. The value of $\lim_{n \to \infty} \frac{n!}{(n+1)! - n!}$ is

(A) 0 (B) 1 (C) -1 (D) does not exist

10. The value of $\lim_{x \to \infty} \frac{\sqrt{x^2+1} - \sqrt[3]{x^2+1}}{\sqrt[4]{x^4+1} - \sqrt[5]{x^4+1}}$ is

(A) 0 (B) $\frac{1}{2}$ (C) $-\frac{1}{2}$ (D) 1

11. The value of $\lim_{x \to \infty} \frac{30 + 4\sqrt{x} + (7\sqrt[3]{x})}{2 + \sqrt{4x-7} + \sqrt[3]{6x-2}}$ is

(A) 0 (B) 1 (C) 2 (D) $\frac{7}{\sqrt[3]{6}}$

12. The value of $\lim\limits_{x \to \infty} (x - \sqrt{x^2 + x})$ is

(A) 0 (B) 1/2 (C) –1/2 (D) 1

13. The value of $\lim\limits_{x \to \infty} \dfrac{x}{\sqrt{4x^2 + 1} - 1}$ is

(A) –1/4 (B) 1/4 (C) –1/2 (D) 1/2

14. If $\lim\limits_{x \to \infty} \left(\dfrac{x^2 + 1}{x + 1} - ax - b \right) = b$, where a, b are constants, then the value of $a + b$, is

(A) 0 (B) 1/2 (C) –1/2 (D) 1

15. The value of $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x} - \sqrt{1 + x^2}}{x}$ is

(A) 0 (B) $\dfrac{1}{2}$ (C) $-\dfrac{1}{2}$ (D) 1

16. The value of $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x + x^2} - 1}{x}$ is

(A) 0 (B) $\dfrac{1}{2}$ (C) $-\dfrac{1}{2}$ (D) 1

17. The value of $\lim\limits_{x \to 1^+} \dfrac{\sqrt{x^2 - 1} + \sqrt{x - 1}}{\sqrt{x^2 - 1}}$ is

(A) 1 (B) $1 - \dfrac{1}{\sqrt{2}}$ (C) $1 + \dfrac{1}{\sqrt{2}}$ (D) does not exist

18. The value of $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{\sqrt{3x - 2} - \sqrt{x + 2}}$ is

(A) 1 (B) 2 (C) 4 (D) 8

19. The value of $\lim\limits_{x \to \frac{\pi}{2}} (\sec x - \tan x)$ is

(A) 0 (B) 1 (C) 1/2 (D) –1

20. The value of $\lim\limits_{n \to \infty} \dfrac{3^{n+1} + 2^{n+2}}{5.3^n - 2^{n-1}}$ is

(A) $\dfrac{1}{5}$ (B) $\dfrac{2}{5}$ (C) $\dfrac{3}{5}$ (D) 1

ANSWER KEY

1.	A	2.	D	3.	D	4.	D	5.	D
6.	A	7.	C	8.	B	9.	A	10.	D
11.	C	12.	C	13.	D	14.	B	15.	B
16.	B	17.	C	18.	D	19.	A	20.	C

6 By application of standard limits

Ex.1 Evaluate : $\lim\limits_{x\to 2} \dfrac{x^3-8}{x^2-4}$

Sol. **Method I**

$$\frac{x^3-(2)^3}{x^2-(2)^2} = \frac{x^3-(2)^3}{x-2} \div \frac{x^2-(2)^2}{x-2}$$

$$\Rightarrow \quad \lim_{x\to 2}\frac{x^3-8}{x^2-4} = \lim_{x\to 2}\frac{x^3-(2)^3}{x-2} \div \lim_{x\to 2}\frac{x^2-(2)^2}{x-2}$$

$$= 3(2^2) \div 2(2^1) \qquad \left(\text{using } \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right)$$

$$= 12 \div 4 = 3$$

Method II

$$\lim_{x\to 2}\frac{x^3-8}{x^2-4} = \lim_{x\to 2}\frac{(x^2+2x+4)(x-2)}{(x+2)(x-2)} = \lim_{x\to 2}\frac{x^2+2x+4}{x+2}$$

$$= \frac{(2)^2+2(2)+4}{2+2} = \frac{12}{4} = 3$$

Ex.2 Find $\lim\limits_{x\to -1}\dfrac{1+x^{1/3}}{1+x^{1/5}}$

Sol. Limit $= \lim\limits_{x\to -1}\dfrac{x^{1/3}-(-1)}{x^{1/5}-(-1)} = \dfrac{5}{3}$

Note : $\lim\limits_{x\to -1}\dfrac{\left(x^{1/5}\right)^5-(-1)}{x^{1/5}-(-1)} = 5(-1)^4$

$$\lim_{x\to 1}\frac{(x^{1/3})^3-(-1)}{x^{1/3}-(-1)} = 3\,(-1)^2$$

Ex.3 Evaluate : $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$

Sol. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x} \Rightarrow \lim\limits_{x\to 0}\dfrac{\sin 2x}{2x}.2 = 2$

Ex.4 Evaluate : $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\sin 3x}$

Sol. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\sin 3x} = \lim\limits_{x\to 0}\left[\dfrac{\sin 2x}{2x}\cdot\dfrac{2x}{3x}\cdot\dfrac{3x}{\sin 3x}\right]$

$$= \left[\lim_{2x\to 0}\frac{\sin 2x}{2x}\right] . \frac{2}{3} . \left[\lim_{3x\to 0}\frac{3x}{\sin 3x}\right]$$

$$= 1 . \frac{2}{3} . \left[\lim_{3x\to 0}\frac{\sin 3x}{3x}\right] = \frac{2}{3} \times 1 = \frac{2}{3}$$

Ex.5 Evaluate : $\underset{x\to 0}{\text{Limit}}\ \dfrac{\tan x-\sin x}{x^3}$

Sol. $\underset{x\to 0}{\text{Limit}}\ \dfrac{\tan x-\sin x}{x^3}$

$$= \underset{x\to 0}{\text{Limit}} \frac{\tan x(1-\cos x)}{x^3} = \lim_{x\to 0}\frac{\tan x}{x}\cdot\frac{1-\cos x}{x^2} = \underset{x\to 0}{\text{Limit}} \frac{\tan x \,.\, 2\sin^2\frac{x}{2}}{x^3}$$

$$= \underset{x\to 0}{\text{Limit}} \frac{\tan x}{x} \cdot \frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}. \qquad \left(\textbf{Note that : } \lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}\right)$$

Ex.6 Evaluate : $\lim_{x\to 1^-}\frac{1-\sqrt{x}}{\left(\cos^{-1}x\right)^2}$

Sol. Put $\cos^{-1}x = y$ and $x \to 1^- \Rightarrow y \to 0$

The given limit $= \lim_{y\to 0}\frac{1-\sqrt{\cos y}}{y^2}$

now rationalizing numerator

$$= \lim_{y\to 0}\frac{(1-\cos y)}{y^2\left(1+\sqrt{\cos y}\right)}$$

$$= \lim_{y\to 0}\frac{1-\cos y}{y^2}\lim_{y\to 0}\frac{1}{1+\sqrt{\cos y}} = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$$

Ex.7 Evaluate : $\underset{x\to 0}{\text{Lim}}\frac{x^2\cos\frac{1}{x}}{\sin x}$

Sol. The given limit $= \underset{x\to 0}{\text{Lim}}\frac{x\cdot\cos(1/x)\cdot x}{\sin x} = \underset{x\to 0}{\text{Lim}}\; x\cos\frac{1}{x} = 0 \times$ (a finite quantity between -1 and 1) $= 0$

Ex.8 $\underset{x\to 0^+}{\text{Lim}}\frac{\cos^{-1}(1-x)}{\sqrt{x}}$

Sol. Put $\cos^{-1}(1-x) = \theta$, then as $x \to 0^+, \theta \to 0^+$

$$\underset{\theta\to 0^+}{\text{Lim}}\frac{\theta}{\sqrt{1-\cos\theta}} = \underset{\theta\to 0^+}{\text{Lim}}\frac{\theta}{\sqrt{2}\sin(\theta/2)} = \underset{\theta\to 0^+}{\text{Lim}}\frac{2(\theta/2)}{\sqrt{2}\sin(\theta/2)} = \sqrt{2}$$

Ex.9 Evaluate : $\lim_{x\to 0}\frac{1-\cos(1-\cos x)}{\sin^4 x}$

Sol. $\lim_{x\to 0}\frac{1-\cos\left(2\sin^2\frac{x}{2}\right)}{x^4} \Rightarrow \lim_{x\to 0}\frac{1-\cos(1-\cos x)}{(1-\cos x)^2}\cdot\frac{(1-\cos x)^2}{x^4} = \lim_{x\to 0}\frac{1}{2}\cdot\left(\frac{1-\cos x}{x^2}\right)^2 = \frac{1}{8}$

Ex.10 Evaluate : $\underset{x\to 0}{\text{Lim}}\frac{\cos\left(\frac{\pi}{2\cos x}\right)}{\sin(\sin x^2)}$

Sol. $l = \underset{x\to 0}{\text{Lim}}\frac{\cos\left(\frac{\pi}{2\cos x}\right)\cdot\left(\sin x^2\right)}{\sin x^2\cdot\sin(\sin x^2)} = \underset{x\to 0}{\text{Lim}}\frac{\cos\left(\frac{\pi}{2\cos x}\right)}{x^2}\cdot\underset{x\to 0}{\text{Lim}}\frac{x^2}{\sin x^2} = \underset{x\to 0}{\text{Lim}}\frac{\cos\left(\frac{\pi}{2\cos x}\right)}{x^2}$

$$l = \lim_{x\to0} \frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2\cos x}\right)}{\left(\frac{\pi}{2}-\frac{\pi}{2\cos x}\right)} \cdot \lim_{x\to0} \frac{\left(\frac{\pi}{2}-\frac{\pi}{2\cos x}\right)}{x^2} \qquad \left(\text{taking } \left(-\frac{\pi}{2}\right) \text{ common}\right)$$

$$= (1)\left(-\frac{\pi}{2}\right)\lim_{x\to0} \frac{(1-\cos x)}{\cos x}\cdot\frac{1}{x^2} = -\frac{\pi}{2}\cdot\frac{1}{2} = -\frac{\pi}{4} \text{ Ans.]}$$

Ex.11 Evaluate : $\lim\limits_{x\to3} \dfrac{e^x - e^3}{x-3}$

Sol. Put y = x – 3. So, as $x \to 3$, $y \to 0$. Thus

$$\lim_{x\to3} \frac{e^x-e^3}{x-3} = \lim_{y\to0} \frac{e^{3+y}-e^3}{y} = \lim_{y\to0} \frac{e^3 . e^y - e^3}{y} = e^3 \lim_{y\to0} \frac{e^y-1}{y} = e^3 . 1 = e^3$$

Ex.12 Evaluate : $\underset{x\to0}{\text{Limit}} \dfrac{e^{3x}-1}{x/2}$

Sol. $\underset{x\to0}{\text{Limit}} \dfrac{e^{3x}-1}{x/2} = \underset{x\to0}{\text{Limit}}\ 2 \times 3\ \dfrac{e^{3x}-1}{3x} = 6.$

Ex.13 Evaluate : $\lim\limits_{x\to0} \dfrac{x(e^x-1)}{1-\cos x}$

Sol. $\lim\limits_{x\to0} \dfrac{x(e^x-1)}{1-\cos x} = \lim\limits_{x\to0} \dfrac{x^2}{1-\cos x}\cdot\dfrac{e^x-1}{x} = \dfrac{1}{\left(\frac{1}{2}\right)}\cdot 1 = 2$

Ex.14 Evaluate : $\lim\limits_{x\to0} \dfrac{\sin(\pi\cos^2 x)}{x^2}$

Sol. $\lim\limits_{x\to0} \dfrac{\sin(\pi\cos^2 x)}{x^2} = \lim\limits_{x\to0} \dfrac{\sin\{\pi(1-\sin^2 x)\}}{x^2}$

$$= \lim_{x\to0} \frac{\sin(\pi - \pi\sin^2 x)}{x^2}$$

$$= \lim_{x\to0} \frac{\sin(\pi\sin^2 x)}{x^2}$$

$$= \lim_{x\to0} \left\{\frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}\times\frac{\pi}{1}\times\frac{\sin^2 x}{x^2}\right\}$$

$$= \lim_{x\to0} \frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}\times\pi\times\lim_{x\to0}\frac{\sin^2 x}{x^2} = 1 \times \pi \times 1 = \pi$$

Ex.15 Evaluate : $\lim\limits_{x\to\infty} \left(1+\dfrac{2}{x}\right)^x$

Sol. $\lim\limits_{x\to\infty} \left(1+\dfrac{2}{x}\right)^x = e^{\lim_{x\to\infty} \frac{2}{x}.x} = e^2.$

Ex.16 Evaluate : $\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{1/x}$

Sol. $\lim_{x\to 0}\left\{\tan\left(\frac{\pi}{4}+x\right)\right\}^{\frac{1}{x}} = \lim_{x\to 0}\left\{\dfrac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}\right\}^{\frac{1}{x}} = \lim_{x\to 0}\left\{\dfrac{1+\tan x}{1-\tan x}\right\}^{\frac{1}{x}} = \lim_{x\to 0}\left\{1+\dfrac{2\tan x}{1-\tan x}\right\}^{\frac{1}{x}}$

$$= e^{\lim_{x\to 0}\left\{\frac{2\tan x}{1-\tan x}\right\}\frac{1}{x}}$$

$$= e^{\lim_{x\to 0} 2\frac{\tan x}{x}.\frac{1}{1-\tan x}} = e^2$$

Ex.17 Evaluate : $\lim_{x\to e}\dfrac{\ln x - 1}{x - e}$

Sol. The given limit $= \lim_{x\to e}\dfrac{\ln\left(\frac{x}{e}\right)}{e\left(\frac{x}{e}-1\right)}$

Put $\frac{x}{e}-1 = y$, as $x\to e, y\to 1$

$\therefore$ The given limit $= \lim_{y\to 0}\dfrac{\ln(1+y)}{ey} = \dfrac{1}{e}$

7 **By using Sandwich Theorem**

Ex.1 Evaluate : $\lim_{x\to\infty}\dfrac{[x]}{x}$

Sol. $x - 1 < [x] \le x, \Rightarrow 1-\frac{1}{x} < \frac{[x]}{x} \le 1$ $\{\text{As } x\to\infty \ \therefore\ x > 0\}$

Now $\lim_{x\to\infty}\left(1-\frac{1}{x}\right) = 1.$

Therefore by Sandwich theorem $\lim_{x\to\infty}\dfrac{[x]}{x} = 1$

Ex.2 Evaluate : $\operatorname{Lim}_{x\to 0} x^3\cos\frac{2}{x}$

Sol. $-1 \le \cos\frac{2}{x} \le 1; -x^3 \le x^3\cos\frac{2}{x} \le x^3$ for $x > 0$ and $x^3 \le x^3\cos\frac{2}{x} \le -x^3$ for $x < 0$

in both the cases limit is zero

Ex.3 Evaluate : $\lim_{x\to\infty}\dfrac{5x^2-\sin 3x}{x^2+10}$

Sol. $\because -1 \le \sin 3x \le 1$

$\therefore$ $\dfrac{5x^2-1}{x^2+10} \le \dfrac{5x^2-\sin 3x}{x^2+10} \le \dfrac{5x^2+1}{x^2+10}$

$\therefore \quad \lim\limits_{x\to\infty}\frac{5x^2-1}{x^2+10} \le \lim\limits_{x\to\infty}\frac{5x^2-\sin 3x}{x^2+10} \le \lim\limits_{x\to\infty}\frac{5x^2+1}{x^2+10}$

Since $\lim\limits_{x\to\infty}\frac{5x^2-1}{x^2+10} = 5$ and $\lim\limits_{x\to\infty}\frac{5x^2+1}{x^2+10} = 5$. Hence $\lim\limits_{x\to\infty}\frac{5x^2-\sin 3x}{x^2+10} = 5$.

Ex.4 Evaluate : $\text{Limit}_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+2}+\frac{n}{n^2+3}+....+\frac{n}{n^2+n}\right)$

Sol. Let $f(n) = \frac{n}{n^2+1}+\frac{n}{n^2+2}+\frac{n}{n^2+3}+....+\frac{n}{n^2+n}$

note that f(n) has n terms which are decreasing

Suppose $h(n) = \left(\frac{n}{n^2+1}+\frac{n}{n^2+1}+\frac{n}{n^2+1}+....+\frac{n}{n^2+1}\right)$, n terms

$h(n) = \frac{n^2}{n^2+1}$ (obviously $f(n) < h(n)$)

and $g(n) = \left(\frac{n}{n^2+n}+\frac{n}{n^2+n}+\frac{n}{n^2+n}+....+\frac{n}{n^2+n}\right)$, n terms

$= \frac{n^2}{n^2+n}$ (obviously $g(n) < f(n)$)

Hence $g(n) < f(n) < h(n)$

Since $\text{Limit}_{n\to\infty}\ g(n) = 1 = \text{Limit}_{n\to\infty}\ h(n)$

Hence, using Sandwich Theorem $\text{Limit}_{n\to\infty}\ f(n) = 1$

Ex.5 If [x] denotes the integral part of x, then evaluate

$$\lim_{n\to\infty}\frac{[1^2x]+[2^2x]+...+[n^2x]}{n^3}$$

Sol. Let $S_n = [1^2x] + [2^2x] + . . . + [n^2x]$

$x - 1 < [x] \le x$

$\therefore \quad 1^2x - 1 < [1^2x] \le 1^2x$

$2^2x - 1 < [2^2x] \le 2^2x$

$3^2x - 1 < [3^2x] \le 3^2x$

. . . .

. . . .

$n^2x - 1 < [n^2x] \le n^2x$

$\therefore \quad (1^2 + 2^2 + . . . + n^2)\, x - n < S_n \le (1^2 + 2^2 + . . . + n^2)\, x$

$\Rightarrow \quad \frac{n(n+1)(2n+1)}{6n^3}x - \frac{n}{n^3} < \frac{S_n}{n^3} \le \lim\limits_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}.x$

$\Rightarrow \quad \lim\limits_{n\to\infty}\left\{\frac{n(n+1)(2n+1)}{6n^3}x - \frac{1}{n^2}\right\} < \lim\limits_{n\to\infty}\frac{S_n}{n^3} \le \lim\limits_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}.x$

$\because \quad \lim\limits_{n\to\infty}\left\{\frac{n(n+1)(2n+1)}{6n^3}x - \frac{1}{n^2}\right\} = \frac{x}{3}$ and $\lim\limits_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}.x = \frac{x}{3}$

Hence required limit i.e. $\lim\limits_{n\to\infty}\frac{S_n}{n^3} = \frac{x}{3}$

IN-CHAPTER EXERCISE 2

1. The value of $\lim\limits_{x \to 0} \dfrac{\sin 5x - \sin 3x}{\sin x}$ is

(A) 0 (B) 1 (C) 2 (D) does not exist

2. The value of $\lim\limits_{x \to 0} \dfrac{\tan 2x - x}{3x - \sin x}$ is

(A) 0 (B) 1/2 (C) –1/2 (D) 1

3. The value of $\lim\limits_{x \to 0} \dfrac{e^{\sin x} - 1}{x}$ is

(A) 0 (B) 1 (C) –1 (D) None of these

4. The value of $\lim\limits_{x \to 0} \dfrac{\sin(2+x) - \sin(2-x)}{x}$ is

(A) $\sin 2$ (B) $\cos 2$ (C) $2\sin 2$ (D) $2\cos 2$

5. The value of $\lim\limits_{x \to 0} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{\sin^{-1} x}$ is

(A) 0 (B) 1/2 (C) 1 (D) 2

6. The value of $\lim\limits_{x \to \frac{\pi}{4}} \dfrac{\cos x - \sin x}{\pi - 4x}$ is

(A) $\dfrac{1}{\sqrt{2}}$ (B) $-\dfrac{1}{\sqrt{2}}$ (C) $\dfrac{1}{2\sqrt{2}}$ (D) $-\dfrac{1}{2\sqrt{2}}$

7. The value of $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{1 + \cos 2x}{(\pi - 2x)^2}$ is

(A) 0 (B) 1 (C) 1/2 (D) –1/2

8. The value of $\lim\limits_{x \to 0} \dfrac{x\,\ell n(1+x)}{1 - \cos x}$ is

(A) 0 (B) 1 (C) 2 (D) does not exist

9. The value of $\lim\limits_{x \to 0} \dfrac{a^x - b^x}{x}$ is

(A) $\ell n\left(\dfrac{a}{b}\right)$ (B) $\ell n\left(\dfrac{b}{a}\right)$ (C) $\ell n\left(\dfrac{a}{b}\right)^2$ (D) $\ell n\left(\dfrac{b}{a}\right)^2$

10. The value of $\lim\limits_{x \to 1} \dfrac{2.3^x - 3.2^x}{x - 1}$ is

(A) $6\ell n\dfrac{3}{2}$ (B) $6\ell n\dfrac{2}{3}$ (C) $3\ell n\dfrac{3}{2}$ (D) $3\ell n\dfrac{2}{3}$

11. The value of $\lim\limits_{x \to 0} \dfrac{a^x + b^x + c^x - 3}{x}$ $(a, b, c > 0)$, is

(A) $\ell n\,(abc)$ (B) $\ell n\,(abc)^{-1}$ (C) $\ell n\,(abc)^{1/3}$ (D) $\ell n\,(abc)^{-1/3}$

12. The value of $\lim\limits_{x \to 1} \dfrac{x + x^2 + ... + x^n - n}{x - 1}$ is

(A) $\dfrac{n}{2}$ (B) $\dfrac{n+1}{2}$ (C) $\dfrac{n(n+1)}{2}$ (D) $\dfrac{n(n+1)}{4}$

13. The value of $\lim\limits_{x \to \frac{1}{\sqrt{2}}} \dfrac{x - \cos(\sin^{-1} x)}{1 - \tan(\sin^{-1} x)}$ is

(A) $1/\sqrt{2}$ (B) $-1/\sqrt{2}$ (C) 1 (D) does not exist

14. The value of $\lim\limits_{x \to \infty} x^2 \sin\left(\ell n\sqrt{\cos\dfrac{\pi}{x}}\right)$ is

(A) $-\dfrac{\pi^2}{2}$ (B) $-\dfrac{\pi^2}{4}$ (C) $\dfrac{\pi^2}{2}$ (D) $\dfrac{\pi^2}{4}$

15. The value of $\lim\limits_{x \to 0} (1 + \sin x)^{\cot x}$ is

(A) 1 (B) e (C) e^{-1} (D) $e^{-1/2}$

16. The value of $\lim\limits_{x \to 0} \left(\dfrac{\tan x}{x}\right)^{\frac{1}{x^2}}$ is

(A) 1 (B) e (C) $\sqrt{e}$ (D) $\sqrt[3]{e}$

17. The value of $\lim\limits_{x \to \infty} \left[\dfrac{2x^2 + 3}{2x^2 + 5}\right]^{8x^2 + 3}$ is

(A) e^{-2} (B) e^{-4} (C) e^{-8} (D) e^{-1}

18. If $\lim\limits_{x \to \infty} \left(\dfrac{x + c}{x - c}\right)^x = 4$, then the value of c is

(A) 1 (B) $\ell n 2$ (C) $-\ell n 2$ (D) $2\ell n 2$

19. The value of $\lim\limits_{x \to 0} \left(\dfrac{x - 1 + \cos x}{x}\right)^{\frac{1}{x}}$ is

(A) 1 (B) e (C) $\sqrt{e}$ (D) $1/\sqrt{e}$

20. The value of $\lim\limits_{n \to \infty} \dfrac{[x] + [2x] + ... + [nx]}{n^2}$, where [ ] is the step function (x is real), is

(A) 0 (B) x (C) x/2 (D) does not exist

ANSWER KEY

1.	C	2.	B	3.	B	4.	D	5.	C
6.	C	7.	C	8.	C	9.	A	10.	A
11.	A	12.	C	13.	B	14.	B	15.	B
16.	D	17.	C	18.	B	19.	D	20.	C

CONTINUITY

1. GENERAL INTRODUCTION

After conceiving the notion of limits, the next element which is taken into consideration is the ***continuity*** of the function.

Qualitatively, the graph of a function is said to be continuous at x = a if while travelling along the graph of the function and in crossing over the point at x = a either from left to right or from right to left, one does not have to lift his pen.

In case one has to lift his pen the graph of the function is said to have a break or discontinuity at x = a.

2. CONTINUITY AT A POINT

A function f(x) is said to be continuous at x = a, if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

i.e. LHL = RHL = value of the function at 'a' i.e. $\lim_{x\to a} f(x) = f(a)$.

if f(x) is not continuous at x = a, we say that f(x) is discontinuous at x = a.

It must be noted here that

(1) If a function f(x) is continuous at x = a then it implies existence of the limit at x = a, but not the converse.

(2) If a function f(x) is continuous at x = a then it implies that f is well defined at x = a, but not the converse.

(3) Continuity at x = a is meaningful to talk only if x = a is in the domain of f(x). Hence, to discuss the continuity of a function, one is allowed to move his pen on the graph of the function only in its domain.

By this concept, we can say that all the standard trigonometric, inverse trigonometric, exponential, logarithmic and polynomial functions are continuous in their standard domains.

3. CONTINUITY IN AN INTERVAL

(1) A function f(x) is said to be continuous in an open interval (a, b) if it is continuous at each and every point of (a, b) i.e. y = [x] is continuous in (1, 2), where [ ] is greatest integer function

(2) A function f(x) is said to be continuous in a closed interval [a, b] if

(a) it is continuous in (a, b)

(b) value of the function at "b" is equal to left hand limit at "b" i.e., $f(b) = \lim_{x\to b^-} f(x)$

(c) value of the function at "a" is equal to right hand limit at "a" i.e., $f(a) = \lim_{x\to a^+} f(x)$

i.e. $y = \sin^{-1} x$ is continuous in $[-1, 1]$

EXAMPLES :

Ex.1 Discuss the continuity of the function [cos x] at $x = \frac{\pi}{2}$, where [.] denotes the greatest integer function.

Sol. L.H.L $= \lim\limits_{x \to \frac{\pi}{2}^-} [\cos x] = 0$

R.H.L $= \lim\limits_{x \to \frac{\pi}{2}^+} [\cos x] = -1$

$f\left(\frac{\pi}{2}\right) = \left[\cos\frac{\pi}{2}\right] = 0$

Clearly, L.H.L $\neq$ R.H.L

so, the function is discontinuous at $x = \frac{\pi}{2}$.

Ex.2 Check the continuity of the function $f(x) = [x^2] - [x]^2 \ \forall\, x \in R$ at the end points of the interval $[-1, 0]$, where [.] denotes the greatest integer function.

Sol. Continuity at $x = -1$

$f(-1) = [(-1)^2] - [-1]^2 = [1] - (-1)^2 = 1 - 1 = 0$

R.H.L $= \lim\limits_{x \to -1^+} \{[x^2] - [x]^2\} = 0 - 1 = -1$

so, $f(-1) \neq$ R.H.L

Continuity at $x = 0$

$f(0) = [(0)^2] - [0]^2 = 0 - 0 = 0$

LHL $= \lim\limits_{x \to 0^-} \{[x^2] - [x]^2\} = 0 - 1 = -1$

So, $f(0) \neq$ L.H.L , Hence the function is not continuous at the end points of the interval $[-1, 0]$

Ex.3 A function f is defined as follows: $f(x) = \begin{cases} 1 & , \text{when } -\infty < x < 0 \\ 1 + \sin x & , \text{when } 0 \le x < \frac{\pi}{2} \\ 2 + \left(x - \frac{\pi}{2}\right)^2 & , \text{when } \frac{\pi}{2} \le x < \infty \end{cases}$,

Discuss the continuity of f.

Sol. Continuity at $x = 0$

L.H.L at $x = 0$ $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (1) = 1$

R.H.L at $x = 0$ $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (1 + \sin x) = 1$

$f(0) = 1 + \sin 0 = 1$ = L.H.L = R.H.L = f(0) so f(x) is continuous at $x = 0$.

continuity at $x = \frac{\pi}{2}$

L.H.L at $x = \frac{\pi}{2} = \lim\limits_{x \to \frac{\pi}{2}^-} f(x) = \lim\limits_{x \to \frac{\pi}{2}^-} (1 + \sin x) = 1 + 1 = 2$

R.H.L at $x = \frac{\pi}{2} = \lim\limits_{x \to \frac{\pi}{2}^+} f(x) = 2 + \left(\frac{\pi}{2} - \frac{\pi}{2}\right)^2 = 2$

$f\left(\frac{\pi}{2}\right) = 2 + \left(\frac{\pi}{2} - \frac{\pi}{2}\right)^2 = 2$

$\therefore$ L.H.L = R.H.L = $f\left(\frac{\pi}{2}\right)$

so, f(x) is continuous at $x = \left(\frac{\pi}{2}\right)$

Hence, f(x) is continuous over the whole real number.

Ex.4 If $f(x) = \frac{\sin 2x + A\sin x + B\cos x}{x^3}$ $(x \neq 0)$ is continuous at x = 0. Find the values of A and B. Also find f(0).

Sol. As f(x) is continuous at x = 0,

$\therefore \lim\limits_{x \to 0} f(x) = f(0)$ and both f(0) and $\lim\limits_{x \to 0}$ f(x) are finite.

$\Rightarrow \quad f(0) = \lim\limits_{x \to 0} \frac{\sin 2x + A\sin x + B\cos x}{x^3}$

As denominator $\to 0$, when $x \to 0$.

Numerator should also $\to 0$, when $x \to 0$ which is possible only if

$\Rightarrow \quad \sin 2(0) + A\sin(0) + B\cos(0) = 0 \quad \Rightarrow \quad B = 0$

$\therefore \quad f(0) = \lim\limits_{x \to 0} \frac{\sin 2x + A\sin x}{x^3}$

$\Rightarrow \quad f(0) = \lim\limits_{x \to 0} \left(\frac{\sin x}{x}\right)\left(\frac{2\cos x + A}{x^2}\right) = \lim\limits_{x \to 0} \left(\frac{2\cos x + A}{x^2}\right)$

Again we can see that denominator $\to 0$ as $x \to 0$

$\therefore$ Numerator should also approach 0 as $x \to 0$

$\Rightarrow \quad 2 + A = 0 \Rightarrow A = -2$

$\Rightarrow \quad f(0) = \lim\limits_{x \to 0} \left(\frac{2\cos x - 2}{x^2}\right) = \lim\limits_{x \to 0}\left(\frac{-4\sin^2 x/2}{x^2}\right) = \lim\limits_{x \to 0}\left(\frac{\sin^2 x/2}{x^2/4}\right) = -1$

So, we get $A = -2$, $B = 0$ and $f(0) = -1$

Ex.5 $f(x) = \frac{\sqrt{2}\cos x - 1}{\cot x - 1}$ $\forall x \in \left(0, \frac{\pi}{2}\right)$ except at $x = \frac{\pi}{4}$. Define $f\left(\frac{\pi}{4}\right)$ so that f(x) may be continuous at $x = \frac{\pi}{4}$.

Sol. f(x) will be continuous at $x = \frac{\pi}{4}$, if $\lim\limits_{x \to \pi/4} f(x) = f\left(\frac{\pi}{4}\right)$

$\therefore \quad f\left(\frac{\pi}{4}\right) = \lim\limits_{x \to \pi/4} \frac{\sqrt{2}\cos x - 1}{\cot x - 1} = \lim\limits_{x \to \pi/4} \frac{(\sqrt{2}\cos x - 1)\sin x}{\cos x - \sin x}$

$= \lim\limits_{x \to \pi/4} \frac{(\sqrt{2}\cos x - 1)(\sqrt{2}\cos x + 1)(\cos x + \sin x)\ \sin x}{(\sqrt{2}\cos x + 1)(\cos x - \sin x)\ (\cos x + \sin x)}$

$$= \lim_{x\to\pi/4} \frac{(2\cos^2 x - 1)}{(\cos^2 x - \sin^2 x)} \frac{(\cos x + \sin x)\sin x}{(\sqrt{2}\cos x + 1)} = \lim_{x\to\pi/4} \frac{\sin x(\cos x + \sin x)}{\sqrt{2}\cos x + 1} = \frac{\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)}{\sqrt{2}.\frac{1}{\sqrt{2}} + 1} = \frac{1}{2}$$

Ex.6 Let $f(x) = \lim_{n\to\infty} \frac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}$. If $f(x)$ is continuous for all $x \in R$ then find the values of a and b.

Sol. $$f(x) = \begin{cases} ax^2 + bx & \text{for } -1 < x < 1 \\ \frac{a-b-1}{2} & x = -1 \\ \frac{a+b+1}{2} & x = 1 \\ \frac{1}{x} & \text{for } x > 1 \text{ or } x < -1 \end{cases}$$

for continuity at $x = 1$ we have $a + b = \frac{a+b+1}{2}$

hence, $a + b = 1$ ....(1)

for continuity at $x = -1$

$a - b = -1 \Rightarrow a - b = -1$ ....(2)

hence $a = 0$ and $b = 1$.

Ex.7 Let $f(x) = \begin{cases} \frac{(e^{2x}+1)-(x+1)(e^x+e^{-x})}{x(e^x-1)} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$

if $f(x)$ is continuous at $x = 0$ then k is equal to

(A) 1/2 (B) 1 (C) 3/2 (D) 2

Sol. $$l = \lim_{x\to 0} \frac{(e^{2x}+1)-(e^x+e^{-x})-x(e^x+e^{-x})}{x^2\left(\frac{e^x-1}{x}\right)}$$

multiply N^r and D^r by e^x

$$\lim_{x\to 0} \frac{e^x(e^{2x}+1)-(e^{2x}+1)-x(e^{2x}+1)}{x^2 \cdot e^x} = \lim_{x\to 0} (e^{2x}+1)\left[\frac{(e^x-1)-x}{x^2}\right] = 2 \cdot \frac{1}{2} = 1$$

Ex.8 Let $f(x) = \frac{\sqrt{x^2+kx+1}}{x^2-k}$. The interval(s) of all possible values of k for which f is continuous for every $x \in R$, is

(A) $(-\infty, -2]$ (B) $[-2, 0)$ (C) $R - (-2, 2)$ (D) $(-2, 2)$

Sol. $$f(x) = \frac{\sqrt{x^2+kx+1}}{x^2-k}$$

for f to be continuous $\forall x \in R$

$x^2 + kx + 1 \geq 0$ and $x^2 - k$ must not have any root i.e. $k < 0$

$\therefore$ $k^2 - 4 \leq 0$ and $k < 0$ ....(1)

$\Rightarrow$ $k \in [-2, 2]$ ....(2)

from (1) and (2)

$k \in [-2, 0)$

4. PROPERTIES OF CONTINUOUS FUNCTIONS

Let f(x) and g(x) are continuous functions at x = a. Then,

(1) cf(x) is continuous at x = a where c is any constant

(2) $f(x) \pm g(x)$ is continuous at x = a

(3) $f(x) \cdot g(x)$ is continuous at x = a

(4) f(x)/g(x) is continuous at x = a, provided that $g(a) \neq 0$

Following important points should be remembered :

(a) If f(x) is continuous & g(x) is discontinuous at x = a then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at x = a.

e.g. (i) $f(x) = x \ \& \ g(x) = \begin{bmatrix} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{bmatrix}$;

(ii) $f(x) = \cos\left(\frac{2x-1}{2}\right)\pi$ is continuous at x = 1 and g (x) = [x] is discontinuous at x = 1

but $f(x) \cdot g(x)$ is continuous at x = 1.

(b) If f(x) and g(x) both are discontinuous at x = a then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at x = a.

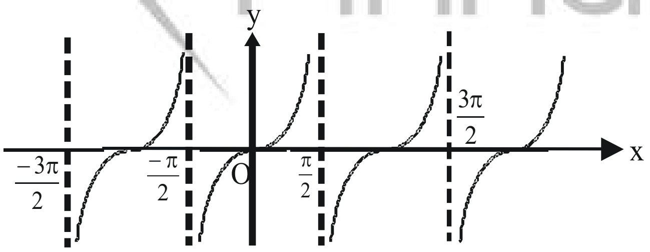
e.g. $f(x) = -g(x) = \begin{bmatrix} 1 & x \geq 0 \\ -1 & x < 0 \end{bmatrix}$

5. DISCONTINUITY

Discussion of discontinuity of a function y = f(x) at any point x = a is meaningful only if the graph of y = f(x) exists in any neighbourhood of x = a.

EXAMPLE :

(1) f(x) = tan x is discontinuous at $x = (2n+1)\frac{\pi}{2}, \ \forall \ n \in I$.



It should be noted here that $x = (2n+1)\frac{\pi}{2}$, where $n \in I$, is not in the domain of f(x) = tan x.

(2) The graph of $f(x) = \frac{1}{[x]}$ doesn't exist in [0,1)

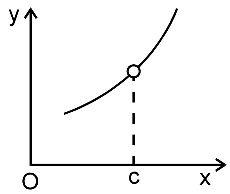
$\therefore$ The discussion of discontinuity of f(x) at any point in (0,1) is meaningless,
while the function is continuous at x = 1 and discontinuous at x = 0.

6. TYPES OF DISCONTINUITY

(1) Removable Discontinuity

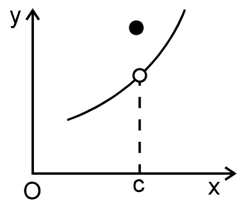
In case, $\lim\limits_{x \to c} f(x)$ exists but is not equal to f(c) then the function is said to have a removable discontinuity. In this case, we can redefine the function such that $\lim\limits_{x \to c} f(x) = f(c)$ & make it continuous at x = c.

Removable type of discontinuity can be further classified as :



(i) Missing Point Discontinuity :

Where $\lim\limits_{x \to a} f(x)$ exists finitely but f(a) is not defined.
e.g. f(x) = has a missing point discontinuity at x = 1.

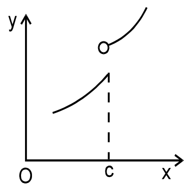


(ii) Isolated Point Discontinuity:

Where $\lim\limits_{x \to a} f(x)$ exists & f(a) also exists but;

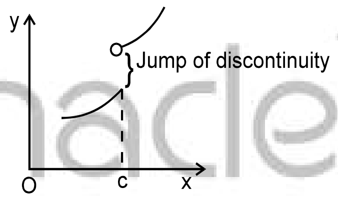
$\lim\limits_{x \to a} f(x) \neq f(a)$. e.g. $f(x) = \dfrac{x^2 - 16}{x - 4}$, $x \neq 4$ & f (4) = 9 has a break at x = 4.

(2) Irremovable Discontinuity



In case, $\lim\limits_{x \to c} f(x)$ does not exist then it is not possible to make the function continuous by redefining it. However, if both the limits (i.e. L.H. L. & R.H.L.) are finite, then it is said to be non–removable discontinuity of first kind otherwise it is non–removable discontinuity of second kind.

Jump of discontinuity :



In case of non-removable discontinuity of the first kind the non-negative difference between the value of the RHL at x = c & LHL at x = c is called, the Jump of discontinuity.

Hence, the jump of discontinuity = $|\text{RHL} - \text{LHL}|$

NOTE :

A function having a finite number of jumps in a given interval is called a Piece Wise Continuous or Sectionally Continuous function in this interval.
Examples are the greatest integer function [x] and the fractional part function {x},

Irremovable discontinuity can be further classified as :

(i) Finite discontinuity e.g. f(x) = x – [x] at all integral x.

(ii) Infinite discontinuity e.g. $f(x) = \dfrac{1}{x - 4}$ or $g(x) = \dfrac{1}{(x-4)^2}$ at x = 4.

(iii) Oscillatory discontinuity e.g. $f(x) = \sin\dfrac{1}{x}$ at x = 0.

In all these cases, the value of f(a) of the function at x = a (point of discontinuity) may or may not exist but $\lim\limits_{x \to a}$ does not exist.

EXAMPLES :

Ex.1 If $f(x)=\begin{cases} x & x<1 \\ x^2 & x>1 \end{cases}$, then check if f(x) is continuous at x = 1 or not. If not, then comment on the type of discontinuity.

Sol. $f(x)=\begin{cases} x & \forall \quad x<1 \\ x^2 & \forall \quad x>1 \end{cases}$

$\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^-} x = 1$ and $\lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^+} x^2 = 1$

$\therefore$ $\lim\limits_{x\to1^-} f(x) = \lim\limits_{x\to1^+} f(x) =$ finite and f(1) is not defined.

So f(x) is discontinuous at x = 1 and this discontinuity is removable missing point discontinuity.

Ex.2 f(x) = $\begin{cases} \cos^{-1}(\cot x) & \text{if} \quad x<\dfrac{\pi}{2} \\ \pi[x]-1 & \text{if} \quad x\ge\dfrac{\pi}{2} \end{cases}$, where [ ] represents greatest integer function.

Find the jump of discontinuity.

Sol. f(x) = $\begin{cases} \cos^{-1}(\cot x) & \text{if} \quad x<\dfrac{\pi}{2} \\ \pi[x]-1 & \text{if} \quad x\ge\dfrac{\pi}{2} \end{cases}$

$\lim\limits_{x\to\frac{\pi}{2}^-} f(x) = \lim\limits_{x\to\frac{\pi}{2}^-} \cos^{-1}(\cot x) = \cos^{-1}(0^+) = \cos^{-1} 0 = \dfrac{\pi}{2}$

$\lim\limits_{x\to\frac{\pi}{2}^+} f(x) = \lim\limits_{x\to\frac{\pi}{2}^+} \pi[x] - 1 = \pi - 1$

$\therefore$ jump of discontinuity = $\left|(\pi-1)-\dfrac{\pi}{2}\right| = \dfrac{\pi}{2} - 1$

7. CONTINUITY OF COMPOSITE FUNCTIONS

If f is continuous at x = c & g is continuous at x = f(c) then the composite g[f(x)] is continuous at x = c.

e.g. $f(x)=\dfrac{x\sin x}{x^2+2}$ & $g(x) = |x|$ are continuous at x = 0, hence the composite (gof) (x) $=\left|\dfrac{x\sin x}{x^2+2}\right|$ will also be continuous at x = 0.

EXAMPLE :

If f(x) = $\dfrac{x+1}{x-1}$ and g(x) = $\dfrac{1}{x-2}$, then discuss the continuity of f(x), g(x) and fog (x).

Sol. f(x) = $\dfrac{x+1}{x-1}$

f(x) is a rational function it must be continuous in its domain
and f is not defined at x = 1
$\therefore$ f is discontinuous at x = 1

$g(x) = \dfrac{1}{x-2}$

g(x) is also a rational function. It must be continuous in its domain and g is not defined at x = 2

$\therefore$ g is discontinuous at x = 2

Now fog (x) will be discontinuous at

(i) x = 2 (point of discontinuity of g(x))

(ii) g(x) = 1 (i.e. when g(x) = point of discontinuity of f(x))

if g(x) = 1

$\Rightarrow \quad \dfrac{1}{x-2} = 1 \quad \Rightarrow \quad x = 3$

$\therefore$ discontinuity of fog(x) should be checked at x = 2 and x = 3

at x = 2

$$fog\,(x) = \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1}$$

fog (2) is not defined

$$\lim_{x\to 2} fog\,(x) = \lim_{x\to 2} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \lim_{x\to 2} \frac{1+x-2}{1-x+2} = 1$$

$\therefore$ fog (x) is discontinuous at x = 2 and it is removable discontinuity.

At x = 3, fog (3) is not defined

Also, $\displaystyle\lim_{x\to 3^+} fog\,(x) = \lim_{x\to 3^+} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} \to \infty$ and $\displaystyle\lim_{x\to 3^-} fog\,(x) = \lim_{x\to 3^-} \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} \to -\infty$

$\therefore$ fog (x) is discontinuous at x = 3 and it has non removable discontinuity of second kind.

8. INTERMEDIATE VALUE THEOREM (I.V.T.)

A function f which is continuous in $[a, b]$ possesses the following properties:

(1) If f(a) & f(b) possess opposite signs, then there exists at least one solution of the equation f(x) = 0 in the open interval (a, b).

(2) If k is any real number between f(a) & f(b), then there exists at least one solution of the equation f(x) = k in the open inetrval (a, b).

EXAMPLES :

Ex.1 Given that a > b > c > d then prove that the equation (x – a) (x – c) + 2(x – b) (x – d) = 0 will have real and distinct roots.

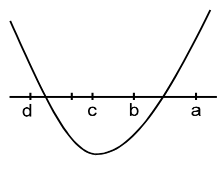
Sol. Let f(x) = (x – a) (x – c) + 2 (x – b) (x – d)

f(a) = (a – a) (a – c) + 2 (a – b) (a – d) = + ve

f(b) = (b – a) (b – c) + 0 = –ve

f(c) = 0 + 2 (c – b) (c – d) = –ve

f(d) = (d – a) (d – c) + 0 = +ve



hence (x – a) (x – c) + 2(x – b) (x – d) = 0

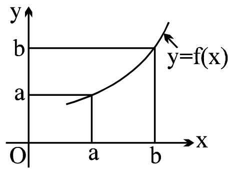
have real and distinct root

Ex.2 Show that the function $f(x) = (x-a)^2 (x-b)^2 + x$ takes the value $\frac{a+b}{2}$ for some value of $x \in [a, b]$

Sol. $f(a) = a$; $f(b) = b$; Also find f is continuous in [a, b] and $\frac{a+b}{2} \in [a, b]$

Hence using intermediate value theroem (I.V.T.)

$\exists$ some $c \in [a, b]$ such that $f(c) = \frac{a+b}{2}$



Ex.3 Suppose that f(x) is continuous in [0, 1] and f(0) = 0, f(1) = 0.
Prove that $f(c)=1-2c^2$ for some $c \in (0, 1)$

Sol. Let $g(x) = f(x) + 2x^2 - 1$ in [0, 1]
$g(0) = -1$ and $g(1) = 1$
$\therefore$ by I.V.T. $\exists$ some $c \in (0, 1)$; $g(c) = 0$ $\Rightarrow$ $f(c) = 1 - 2c^2$

Ex.4 Let f be a continuous function defined on [0, 1] with range [0, 1].
Show that there exists some 'c' in [0, 1] such that f(c) = 1 – c.

Sol. Consider $g(x) = f(x) - 1 + x$
$g(0) = f(0) - 1 \le 0$ (as $f(0) \le 1$)
$g(1) = f(1) \ge 0$ (as $f(1) \ge 0$)
Hence g(0) and g(1) and of opposite signs
hence $\exists$ at least one $c \in [0, 1]$ such that $g(c) = 0$
$\therefore$ $g(c) = f(c) - 1 + c = 0$; $f(c) = 1 - c$

Here it should be noted that, the sign of equality in $f(0)-1\le 0$ and in $f(1)\ge 0$ gives $c = 0$ and 1 respectively.

Ex.5 Let $f : [0, 2] \to R$ be continuous and f(0) = f(2).
Prove that there exists x_1 and x_2 in (0, 2) such that $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$

Sol. Consider continuous function g(x) as
$g(x) = f(x+1) - f(x)$ $(x_2 = x_1 + 1)$
now, $g(0) = f(1) - f(0) = f(1) - f(2)$ ....(1)
$g(1) = f(2) - f(1) = f(2) - f(1)$ ....(2)
hence g(0) and g(1) are of opposite signs, hence $\exists$ some $c \in (0, 1)$ where $g(c) = 0$
i.e. $f(c+1) = f(c)$ $[c + 1 \in (1, 2)$ as $c \in (0, 1)]$
put $c = x_1$; $c + 1 = x_2$
$\therefore$ $f(x_2) = f(x_1)$ where $x_2 - x_1 = 1$, obviously $x_1, x_2 \in (0, 2)$

Ex.6 Prove that the function $f(x) = a\sqrt{x-1} + b\sqrt{2x-1} - \sqrt{2x^2-3x+1}$ where a + 2b = 2 and a, b $\in$ R always has a root in (1, 5) $\forall$ b $\in$ R.

Sol. Let b > 0, then f(1) = b > 0
and $f(5) = 2a + 3b - 6 = 2(a + 2b) - b - 6 = 4 - b - 6 = -(2 + b) < 0$
Hence by IVT, $\exists$ some $c \in (1, 5)$ for which $f(c) = 0$
If b = 0 then a = 2

$f(x) = 2\sqrt{x-1} - \sqrt{2x^2-3x+1} = 0$

$\Rightarrow$ $4(x-1) = 2x^2 - 3x + 1 = (2x-1)(x-1) \Rightarrow (2x-5)(x-1) = 0$ $\Rightarrow x = \frac{5}{2}$

Hence $f(x) = 0$ if $x = \frac{5}{2}$ which lies in (1, 5)

If b < 0 , f(1) = b < 0 and $f(2) = a + b\sqrt{3} - \sqrt{3}$

$= (a + 2b) + (\sqrt{3} - 2)b - \sqrt{3} = (2 - \sqrt{3}) - (2 - \sqrt{3})b = (2 - \sqrt{3})(1 - b) > 0$ (as b < 0)

Hence f(1) and f(2) have opposite signs
$\therefore$ $\exists$ some $c \in (1, 2) \subset (1, 5)$ for which f(c) = 0

IN-CHAPTER EXERCISE - 3

1 Test the continuity of the function of f(x) at the origin : $f(x)=\begin{cases}\dfrac{x}{|x|}, & x\neq 0\\ 1, & x=0\end{cases}$

2 A function f(x) is defined as, $f(x)=\begin{cases}\dfrac{x^2-x-6}{x-3}; & \text{if } x\neq 3\\ 5; & \text{if } x=3\end{cases}$. Show that f(x) continuous at x = 3

3 If $f(x)=\begin{cases}\dfrac{x^2-1}{x-1}; & \text{for } x\neq 1\\ 2; & \text{for } x=1\end{cases}$, find whether f(x) is continuous at x = 1.

4 If $f(x)=\begin{cases}e^{1/x}, & \text{if } x\neq 0\\ 1, & \text{if } x=0\end{cases}$, find whether f is continuous at x = 0.

5 Let $f(x)=\begin{cases}\dfrac{1-\cos x}{x^2}, & \text{when } x\neq 0\\ 1, & \text{when } x=0\end{cases}$. Show that f(x) is discontinuous at x = 0.

6 Discuss the continuity of the following functions at the indicated point(s) :

(i) $f(x)=\begin{cases}|x|\cos\left(\dfrac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases}$ at x = 0

(ii) $f(x)=\begin{cases}x^2\sin\left(\dfrac{1}{x}\right), & x\neq 0\\ 0, & x=0\end{cases}$ at x = 0

(iii) $f(x)=\begin{cases}(x-a)\sin\left(\dfrac{1}{x-a}\right), & x\neq 0\\ 0, & x=0\end{cases}$ at x = a

(iv) $f(x)=\begin{cases}\dfrac{e^x-1}{\log(1+2x)}, & x\neq 0\\ 7, & x=0\end{cases}$ at x = 0

(v) $f(x)=\begin{cases}\dfrac{1-x^n}{1-x}, & x\neq 1\\ n-1, & x=0\end{cases}$ $n\in N$, at x = 1

(vi) $f(x)=\begin{cases}\dfrac{|x^2-1|}{x-1}, & \text{for } x\neq 1\\ 2, & \text{for } x=1\end{cases}$ at x = 1

7 Show that $f(x)=\begin{cases}\dfrac{\sin 3x}{\tan 2x}, & \text{if } x<0\\ \dfrac{3}{2}, & \text{if } x=0\\ \dfrac{\log(1+3x)}{e^{2x}-1}, & \text{if } x>0\end{cases}$ is continuous at x = 0.

8 Discuss the continuity of the function f(x) at the point x = 1/2, where $f(x)=\begin{cases}x, & 0\le x<1/2\\ 1/2, & x=1/2\\ 1-x, & 1/2<x\le 1\end{cases}$

9 Discuss the continuity of $f(x) = \begin{cases} 2x-1, & x<0 \\ 2x+1, & x \ge 0 \end{cases}$ at $x = 0$

10 For what value of k is the function $f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \ne 1 \\ k, & x = 1 \end{cases}$, continuous at $x = 1$?

11 Determine the value of the constant k so that the function $f(x) = \begin{cases} \dfrac{x^2-3x+2}{x-1}, & \text{if } x \ne 1 \\ k, & \text{if } x = 1 \end{cases}$ is continuous at $x = 1$

12 For what value of k is the function $f(x) = \begin{cases} \dfrac{\sin 5x}{3x}, & \text{if } x \ne 0 \\ k, & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$?

13 Determine the value of the constant k so that the function $f(x) = \begin{cases} kx^2, & \text{if } x \le 2 \\ 3, & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$

14 Find the value of a so that the function $f(x) = \begin{cases} ax+5, & \text{if } x \le 2 \\ x-1, & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$.

15 Prove that the function $f(x) = \begin{cases} \dfrac{x}{|x|+2x^2}, & x \ne 0 \\ k, & x = 0 \end{cases}$

remains discontinuous at $x = 0$, regardless the choice of k.

16 Find the value of k if f(x) is continuous at $x = \pi/2$, where $f(x) = \begin{cases} \dfrac{k\cos x}{\pi-2x}, & x \ne \pi/2 \\ 3, & x = \pi/2 \end{cases}$

17 Let $f(x) = \dfrac{\log\left(1+\dfrac{x}{a}\right)-\log\left(1-\dfrac{x}{b}\right)}{x}$, $x \ne 0$ Find the value of f at $x = 0$ so that f becomes continuous at $x = 0$.

18 If $f(x) = \begin{cases} \dfrac{2^{x+2}-16}{4^x-16}, & \text{if } x \ne 2 \\ k, & \text{if } x = 2 \end{cases}$, is continuous at $x = 2$, find k.

19 If $f(x) = \begin{cases} \dfrac{\cos^2 x-\sin^2 x-1}{\sqrt{x^2+1}-1}, & x \ne 0 \\ k, & x = 0 \end{cases}$, is continuous at $x = 0$, find k.

20 Extend the definition of the function $f(x) = \dfrac{1-\cos 7(x-\pi)}{5(x-\pi)^2}$ by defining continuity at the point $x = \pi$

21 In each of the following, find the value of the constant k so that the given function is continuous at the indicated point ;

(i) $f(x) = \begin{cases} \dfrac{1-\cos 2kx}{x^2}, & \text{if } x \neq 0 \\ 8, & \text{if } x = 0 \end{cases}$ at $x = 0$ (ii) $f(x) = \begin{cases} (x-1)\tan\dfrac{\pi x}{2}, & \text{if } x \neq 1 \\ k, & \text{if } x = 1 \end{cases}$ at $x = 1$

22 If $f(x) = \begin{bmatrix} 2x^2 + k, & \text{if } x \geq 0 \\ -2x^2 + k, & \text{if } x < 0 \end{bmatrix}$, then what should be the value of k so that f(x) is continuous at $x = 0$.

23 Prove that the function $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x < 0 \\ x+1, & x \geq 0 \end{cases}$ is everywhere continuous.

24 Find the points of discontinuity, if any, of the following functions :

(i) $f(x) = \begin{cases} x^3 - x^2 + 2x - 2, & \text{if } x \neq 1 \\ 4, & \text{if } x = 1 \end{cases}$ (ii) $f(x) = \begin{cases} \dfrac{x^4 - 16}{x-2}, & \text{if } x \neq 2 \\ 16, & \text{if } x = 2 \end{cases}$

(iii) $f(x) = \begin{cases} \dfrac{\sin x}{x}, & \text{if } x < 0 \\ 2x+3, & \text{if } x \geq 2 \end{cases}$ (iv) $f(x) = \begin{cases} \dfrac{\sin 3x}{x}, & \text{if } x \neq 0 \\ 4, & \text{if } x = 0 \end{cases}$

(v) $f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & \text{if } x \neq 0 \\ 5, & \text{if } x = 0 \end{cases}$ (vi) $f(x) = \begin{cases} \dfrac{x^4 + x^3 + 2x^2}{\tan^{-1} x}, & \text{if } x \neq 0 \\ 10, & \text{if } x = 0 \end{cases}$

(vii) $f(x) = \begin{cases} \dfrac{e^x - 1}{\log_e(1+2x)}, & \text{if } x \neq 0 \\ 7, & \text{if } x = 0 \end{cases}$ (viii) $f(x) = \begin{cases} |x-3|, & \text{if } x \geq 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & \text{if } x < 1 \end{cases}$

25 In the following, determine the value(s) of constant(s) involved in the definition so that the given function is continuous :

(i) $f(x) = \begin{cases} \dfrac{\sin 2x}{5x}, & \text{if } x \neq 0 \\ 3k, & \text{if } x = 0 \end{cases}$ (ii) $f(x) = \begin{cases} k(x^2 + 3x), & \text{if } x < 0 \\ \cos 2x, & \text{if } x \geq 0 \end{cases}$

(iii) $f(x) = \begin{cases} 2, & \text{if } x \leq 3 \\ ax + b, & \text{if } 3 < x < 5 \\ 9, & \text{if } x \geq 5 \end{cases}$ (iv) $f(x) = \begin{cases} 4, & \text{if } x \leq -1 \\ ax^2 + b, & \text{if } -1 < x < 0 \\ \cos x, & \text{if } x \geq 0 \end{cases}$

(v) $f(x) = \begin{cases} \dfrac{\sqrt{1+px} - \sqrt{1-px}}{x}, & \text{if } -1 \leq x < 0 \\ \dfrac{2x+1}{x-2}, & \text{if } 0 \leq x \leq 1 \end{cases}$

26 The function $f(x) = \begin{cases} x^2/a & , \text{ if } 0 \le x < 1 \\ a & , \text{ if } 1 \le x < \sqrt{2} \\ \dfrac{2b^2 - 4b}{x^2} & , \text{ if } \sqrt{2} \le x < \infty \end{cases}$

is continuous on $[0, \infty)$, then find the most suitable values of a and b.

27 The function f(x) is defined as $f(x) = \begin{cases} x^2 + ax + b & , 0 \le x < 2 \\ 3x + 2 & , 2 \le x \le 4 \\ 2ax + 5b & , 4 < x \le 8 \end{cases}$

If f is continuous on [0, 8] , find the values of a and b.

28 If $f(x) = \dfrac{\tan\left(\dfrac{\pi}{4} - x\right)}{\cot 2x}$ for $x \ne \pi/4$, find the value which can be assigned to f(x) at $x = \pi/4$ so that the function f(x) becomes continuous every where in $(0, \pi/2)$.

29 Discuss the continuity of the function $f(x) = \begin{cases} 2x - 1 & , \text{ if } x < 2 \\ \dfrac{3x}{2} & , \text{ if } x > 2 \end{cases}$

30 Discuss the continuity of $f(x) = \sin |x|$ at $x = 0$.

ANSWER KEY

1. Discontinuous **3.** Continuous **4.** Discontinuous

6 (i) Continuous (ii) Continuous (iii) Continuous (iv) Discontinuous (v) Discontinuous (vi) Discontinuous

8. Continuous **9.** Discontinuous **10.** 2 **11.** – 1

12. 5/3 **13.** 3/4 **14.** – 2 **16.** 6

17. $\dfrac{a+b}{ab}$ **18.** 1/2 **19.** – 4 **20.** $f(\pi) = \dfrac{49}{10}$

21. (i) $k = \pm 2$ (ii) $k = \dfrac{-2}{\pi}$ **22.** k can be any real number

24. (i) x = 1 (ii) x = 2 (iii) x = 0 (iv) x = 0 (v) x = 0 (vi) x = 0 (vii) x = 0 (viii) Nowhere discontinuous

25. (i) $k = \dfrac{2}{15}$ (ii) No value of k can make f(x) continuous at x = 0 (iii) a = 7/2, b = – 17/2 (iv) a = 3, b = 1 (v) p = – 1/2

26. a = – 1 , b = 1 or a = 1 , b = 1 ± $\sqrt{2}$

27. a = 3, b = – 2 **28.** $f\left(\dfrac{\pi}{4}\right) = \dfrac{1}{2}$ **29.** Everywhere continuous

30. Let g(x) = sin x and h (x) = |x| As, g(x) and h(x) are everywhere continuous, therefore, f(x) is also everywhere continuous.