AREA

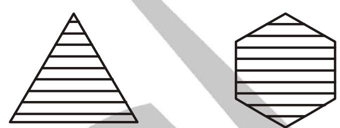
INTRODUCTION

In Previous classes, we have learn to find areas of plane figures, for example, area of triangle, rectangle, square, parallelogram, rhombus etc. In the present chapter, we shall study the relationship between the areas of these geometrical figures particularly when the two figures lie on same base and between same parallel lines.

Let us first understand the meaning of area of planar region and some axioms related to it.

Area of plane region. The part of a plane enclosed by a simple closed figure is called a planar region corresponding to that figure. The measure of this planar region in some unit is called the area of that planar region. Thus area of a figure is a number, associated with the part of the plane enclosed by the figure for example-

1. The part of the plane enclosed by a triangle is called the area of triangular region.



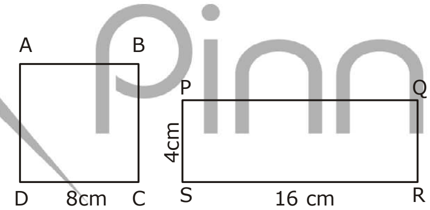
2. The part of the plane enclosed by a polygon is called area of the polygonal region.

Area axioms for plane figures. Following are the axioms related to area of plane figures.

1. Area Axiom of Congruent Figures.

We know that- "two plane figures are congruent' means they have same shape and size. If we place one figure on other, the two figures cover each other exactly. In other words, they have same area. Thus, we can say if R_1 and R_2 are two plane regions such that $R_1 \cong R_2$ then ar (R_1) = ar (R_2) e.g., if $\Delta ABC \cong \Delta PQR$ then ar (ΔABC) = ar (ΔPQR) if quad. ABCD $\cong$ quad. PQRS then ar (ABCD) = ar (PQRS).

But converse of above is not true i.e., if areas of two plane regions are same then they need not be congruent. For example, a square ABCD of side 8 cm has area 64 cm^2 and a rectangle of sides 16 cm and 4 cm also has area 64cm^2 and a rectangle of sides 16 cm and 4 cm also has area 64 cm^2. But , clearly, square ABCD is not congruent to rectangle PQRS.



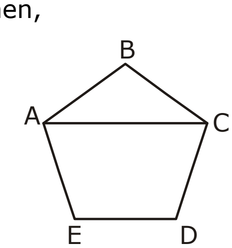
2. Axiom for Area of Union of Two Regions.

If R is a planar region, which is a union of two non- overlapping planar regions R_1 and R_2, then ar (R) = ar (R_1) + ar (R_2)

e.g., if R is a polygonal region ABCDE which is the union of two regions.

R_1 : the triangular region ABC.

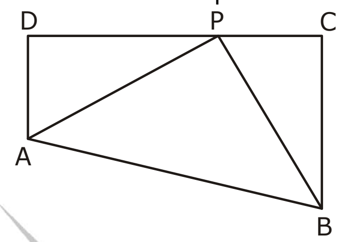
R_2 : the quadrilateral region CEDA. then,



ar (Polygon ABCDE) = ar (ΔABC) + ar (quad. CEDA).

3. Axiom for Area of Included Region.

If R_1 be a plane region included in any other planar region R_2, then ar $(R_1) \leq$ ar (R_2) e.g., in the adjacent figure, triangular region ABP is included inside the quadrilateral region ABCD, therefore



ar (ΔABP) $\leq$ ar (quad. ABCD).

4. Axiom for Area of a Rectangle.

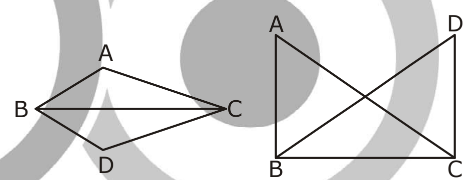
If a rectangle ABCD has length l and breadth b then ar (rect. ABCD) = l × b.

Using above axioms, we can derive the formulae for area of parallelogram, triangle, trapezium and rhombus. It also needs the study of relationship between the areas of these geometric figures under the condition when they lie on the same base and between the same parallels.

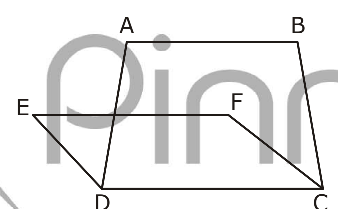
GEOMETRIC FIGURES ON THE SAME BASE AND BETWEEN THE SAME PARALLELS

Let us first understand the meaning of 'same base'. Two geometric figures are said to have same base if they have one side common. For example, in the following cases figures are on same base.

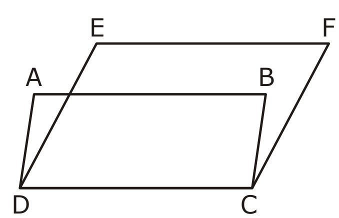
(i) ΔABC and ΔDBC are on the same base BC in each of the two figures given here.



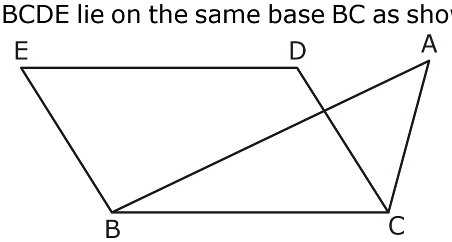
(ii) In the adjacent figure trapezium ABCD and parallelogram CDEF are on the same base CD.



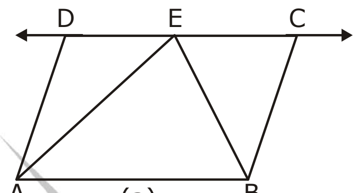
(iii) Two parallelograms ABCD and CDEF are on the same base CD in the figure given below-



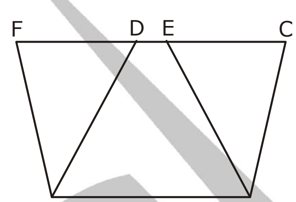
(iv) ΔABC and parallelogram BCDE lie on the same base BC as shown in the given figure.



Now two plane geometric figures are said to be on the same base and between the same parallels if each of these have one side common and their opposite sides or vertex lie along or on a line parallel to the base and on the same side of base. For example, in fig. (a) parallelogram ABCD and ΔABE lie on the same base AB and between the same parallels AB and DC.

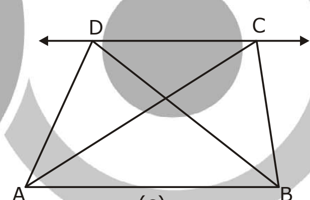


(a)



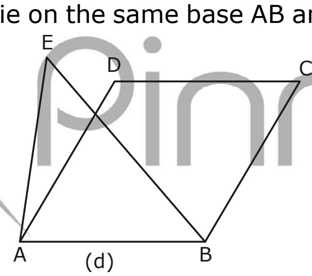
(b)

In fig. (b) parallelograms ABCD and ABEF lie on the same base AB and between the same parallels AB and FC.



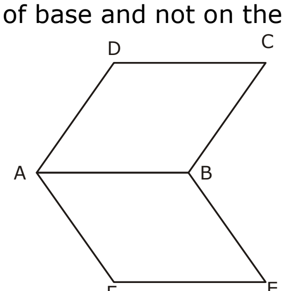
(c)

In fig. (c) triangles ABC adn ABD lie on the same base AB and between the same parallels AB and DC.



(d)

It should be noted that the figures (d) and (e) lie on the same but not between the same base parallels. In fig. (d) vertex E of ΔABE and opposite side of parallelogram ABCD do not lie on the same line parallel to base AB while in Fig. (e) parallelogram ABCD and parallelogram ABEF have their sides opposite to base on different sides of base and not on the same side of base.

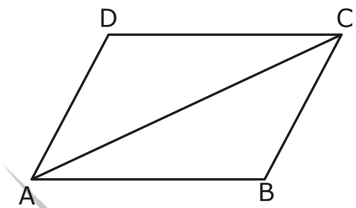


(e)

So figures (d) and (e) cannot be considered as the figures on same base and between the same parallels.

SOLVED PROBLEMS

Ex.1 Diagonal of a parallelogram divides it into two triangles of equal area.



Sol. **Given.** ABCD is a ||gm and AC is diagonal.

To prove : ar (ABC) = ar (ADC).

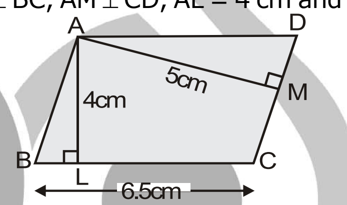
Proof. We know that diagonal of a parallelogram divides it into two congruent triangles.

$\because$ $\Delta ABC \cong \Delta CDA$

$\therefore$ ar (ΔABC) = ar (ΔCDA)

(by area axiom of congruent figures)

Ex.2 In fig, ABCD is a parallelogram, AL $\perp$ BC, AM $\perp$ CD, AL = 4 cm and AM = 5 cm. If BC = 6.5 cm, then find CD.

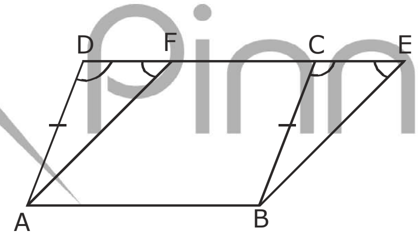


Sol. We have, BC × AL = CD × AM (Each equal to area of the parallelogram ABCD)

$\Rightarrow$ 6.5 × 4 = CD × 5

$\Rightarrow$ CD = $\frac{6.5 \times 4}{5}$ cm $\Rightarrow$ CD = 5.2 cm.

Ex.3 Parallelogram on the same base and between the same parallels are equal in area. **[NCERT]** **(CBSE 2010)**



Sol. **Given.** Two parallelograms ABCD and ABEF on the same base AB and between the same parallels AB and DE.

To Prove : ar (||gm ABCD) = ar (||gm ABEF).

Proof. In ΔADF and ΔBCE.

$\because$ AD||BC being opposite sides of a parallelogram and DC a transversal,

$\therefore$ $\angle$ADF = $\angle$BCE (corresponding angles)

Also since AF ||BE being opposite sides of a parallelogram and DE a transversal,

$\therefore$ $\angle$AFD = $\angle$BEC (corresponding angles)

$\therefore$ $\angle$DAF = $\angle$CBE ($\because$ if two angles of two triangles are equal, third will also be equal)

and, AD = BC (opp. sides of a||gm)

$\therefore$ $\Delta ADF \cong \Delta BCE$ (ASA congruence condition)

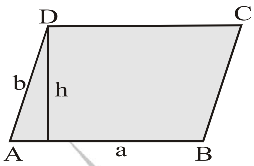
$\therefore$ ar (ΔADF) = ar (ΔBCE)

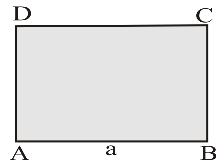
$\Rightarrow$ ar (ΔADF) + ar (||gm ABCF) = ar (ΔBCE) + ar (||gm ABCF)

$\Rightarrow$ ar (||gm ABCD) = ar (||gm ABEF)

Ex.4 Prove that of all parallelograms of which the sides are given, the parallelogram which is rectangle has the greatest area.

Sol. Let ABCD be a parallelogram in which AB = a and AD = b. Let h be the altitude corresponding to the base AB. Then,





ar (||gm ABCD) = AB × h = ah

Since the sides a and b are given. Therefore, with the same sides a and b we can construct infinitely many parallelograms with different heights.

Now, ar (||gm ABCD) = ah

$\Rightarrow$ ar (||gm ABCD) is maximum or greatest when h is maximum. [$\because$ a is given i.e., a is constant]

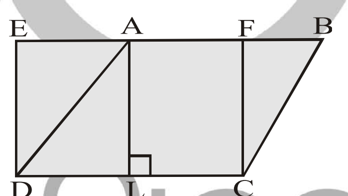
But, the maximum value which h can attain is AD = b and this is possible when AD is perpendicular to AB i.e. the ||gm ABCD becomes a rectangle.

Thus, (ar ||gm ABCD) is greatest when AD $\perp$ AB i.e. when (||gm ABCD) is a rectangle.

Ex.5 In fig, ABCD is a parallelogram and EFCD is a rectangle. Also AL $\perp$ DC. Prove that

(i) ar (ABCD) = (EFCD) (ii) ar (ABCD) = DC × AL

Sol. (i) We know that a rectangle is also a parallelogram.



Thus, parallelogram ABCD and rectangle EFCD are on the same base CD and between the same parallels CD and BE.

$\therefore$ ar (||gm ABCD) = ar (EFCD)

(ii) From (i), we have ar (ABCD) = ar (EFCD)

$\Rightarrow$ ar (ABCD) = CD × FC

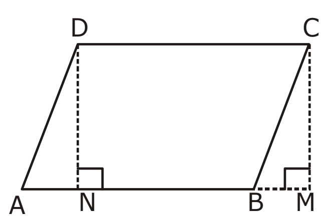
[$\because$ Area of a rectangle = Base × Height]

$\Rightarrow$ ar (ABCD) = CD × AL

[$\because$ AL = FC as ALCF is a rectangle]

$\Rightarrow$ ar (ABCD) = DC × AL

Ex.6 Area of a parallelogram is equal to the product of its base and corresponding altitude. **[NCERT]**



Sol. **Given.** A parallelogram ABCD in which DN is altitude corresponding to base AB.

To Prove : ar (||gm ABCD) = AB × DN.

Construction. Draw CM perpendicular to AB which meets AB produced at M.

Proof. DN and CM are both perpendicular to same line AB,

$\therefore$ DN ||CM

Also DN = CM

($\because$ each is distance between two parallel lines AB and DC)

$\therefore$ DCMN is a parallelogram. ($\because$ A quad. is a parallelogram if one pair of opp. side is equal and parallel)

Also $\angle$DNM = 90º, So DCMN is a rectangle.

$\therefore$ ar (||gm DCMN) = DC × DN

($\because$ Area of rect. = length × breadth)

But, AB = DC

($\because$ opp. sides of a parallelogram are equal)

$\therefore$ ar (||gm DCMN) = AB × DN ...(1)

Also as both the parallelograms (DCMN and ABCD) lie on the same base DC and between the same parallels DC and AM.

$\therefore$ ar (|| gm DCMN) = ar (||gm ABCD)...(2)

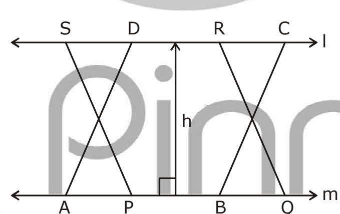
($\because$ parallelograms on same base and between same parallels are equal in area)

From equations (1) and (2), we get

ar (ABCD) = AB × DN = base

× corresponding altitude.

corollary. Parallelograms on equal base and between the same parallels are equal in area.



Proof. If ABCD adn PQRS be two parallelograms on equal base AB and PQ ann between same parallel lines l and m, and h be the perpendicular distance between l and m, then

ar (||gm ABCD) = AB × h ...(1)

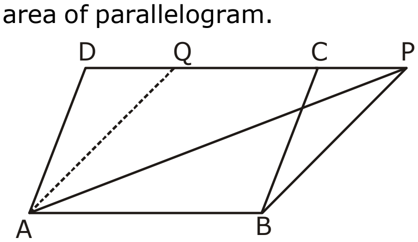
and ar (||gm PQRS) = PQ× h ...(2)

But AB = PQ (given) ...(3)

From equations (1), (2) and (3), we get

ar (ABCD) = ar (||gm PQRS)

Ex.7 If a triangle and a parallelogram lie on the same base and between the same parallel then area of triangle is equal to half of the area of parallelogram. **[NCERT]**



Sol. **Given.** ΔABP and a ||gm ABCD on same base AB and between the same parallels AB and DP.

To prove : ar (ΔABP) = $\frac{1}{2}$ar (||gm ABCD).

Construction. Through A , draw a line AQ parallel to BP intersecting DP at Q.

Proof. AB||DC ($\because$ opp. sides of a parallelogram are parallel).

$\therefore$ AB || QP

Also, AQ || BP (by construction)

$\therefore$ ABPQ is a parallelogram.

Thus ABCD and ABPQ are two parallelograms on the same base AB and between the same parallels AB and DP.

$\therefore$ ar (||gm ABCD) = are (||gm ABPQ) ...(1)

Also in parallelogram ABPQ, AP is the diagonal

$\therefore$ ar (Δ ABP) = ar (Δ AQP)

($\because$ diagonal of a parallelogram divides it into two triangles of equal area).

But ar (ABP) + ar (AQP) = ar (ABPQ)

$\Rightarrow$ ar (ABP) + ar (ABP) = ar (ABPQ)

$\Rightarrow$ 2ar (ABP) = ar (ABPQ) ...(2)

From equations (1) and (2), we get

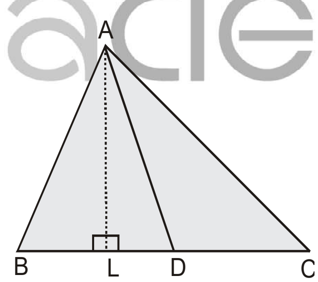
2ar (Δ ABP) = ar (||gm ABCD)

$\Rightarrow$ ar (Δ ABP) = $\frac{1}{2}$ ar (||gm ABCD)

Hence area of triangle is half the area of parallelogram.

Ex.8 Show that a median of a triangle divides it into two triangles of equal area. **[NCERT]**

Sol. **Given :** ΔABC in which AD is a median.



To prove : ar(ΔABD) = ar (ΔADC).

Construction : Draw AL $\perp$ BC.

Proof : Since AD is the median ΔABC.

Therefore, D is the mid-point of BC.

$\Rightarrow$ BD = DC

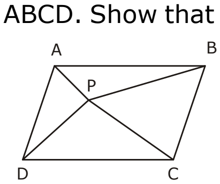
$\Rightarrow$ BD × AL = DC × AL [Multiplying both sides by AL]

$\Rightarrow$ $\frac{1}{2}$(BD × AL) = $\frac{1}{2}$(DC × AL) [Multiplying both sides by $\frac{1}{2}$]

$\Rightarrow$ ar (ΔABD) = ar (ΔADC)

Hence, proved.

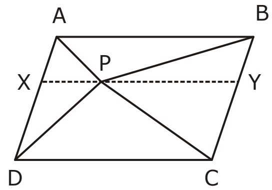
Ex.9 In the given figure, P is a point in the interior of a parallelogram ABCD. Show that **[NCERT]**



(i) ar (ΔAPB) + ar (Δ PCD) = $\frac{1}{2}$ ar (||gm ABCD)

(ii) ar (ΔAPD) + ar (ΔPBC) = ar (ΔAPB) ar (ΔPCD).

Sol. Let us draw a line through P parallel to AB which meet AD at X and BC at Y.



$\because$ AD || BC (opp. sides of a parallelogram)

$\therefore$ AX || BY

Also AB || XY (by construction)

$\therefore$ ABYX is a parallelogram.

Similarly CDXY is a parallelogram.

Now parallelogram ABYX and ΔAPB lie on the same base AB and between the same parallels AB and XY,

$\therefore$ ar (Δ APB) = $\frac{1}{2}$ ar (||gm ABYX) ...(1)

And parallelogram CDXY and ΔPDC lie on the same base DC and between the same parallels DC and XY,

$\therefore$ ar (Δ PCD) = $\frac{1}{2}$ ar (||gm CDYX) ...(2)

Adding equations (1) and (2), we get

ar (ΔAPB) + ar (Δ PCD) = $\frac{1}{2}$ ar (||gm ABYX) + $\frac{1}{2}$ ar (||gm CDXY)

ar (ΔAPB) + ar (Δ PCD) = $\frac{1}{2}$ (ar (||gm ABYX) + ar (||gm CDXY))

ar (ΔAPB) + ar (Δ PCD) = $\frac{1}{2}$ ar (||gm ABCD)

$\therefore$ ar (Δ APB) + ar (Δ PCD) = $\frac{1}{2}$ ar (||gm ABCD) ...(3)

Also, ar (ΔAPB) + ar (ΔPBC) + ar (ΔAPD) + ar (Δ PCD) = ar (||gm ABCD).

$\Rightarrow$ ar (ΔAPD) + ar (ΔPBC) + $\frac{1}{2}$ ar (||gm ABCD) = ar (||gm ABCD) (using (3))

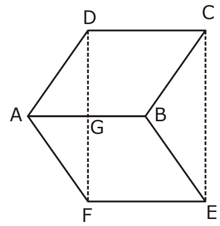
$\Rightarrow$ ar (ΔAPD) + ar (Δ PBC) = ar (||gmABCD) – $\frac{1}{2}$ ar(||gm ABCD)

$\Rightarrow$ ar (ΔAPD) + ar (ΔPBC) = $\frac{1}{2}$ ar (||gm ABCD) ...(4)

From equations (3) and (4), we get

ar (ΔAPD) + ar (ΔPBC) = ar (ΔAPB) + ar (ΔPCD).

Ex.10 ABCD and ABEF are two parallelograms on the opposite sides of AB as shown in the figure. CE and DF are joined. Prove that : (i) ar (Δ ADF) = ar (Δ BCE) (ii) ar (CDFE) = ar (||gm ABCD) + ar (||gm ABEF).



Sol. $\because$ ABCD is a parallelogram,

$\therefore$ AB||DC and AB = DC ...(1)

Also as ABEF is a parallelogram,

$\therefore$ AB||EF and AB = EF ...(2)

From equations (1) and (2), we get

DC||EF and DC = EF.

$\therefore$ DCEF is a parallelogram.

($\because$ one pair of opp. sides of a quad. are equal and parallel)

$\therefore$ DF = CE.

Now in Δ ADF and Δ BCE

AD = BC (opp. sides of a ||gm ABCD)

AF = BE (opp. sides of a ||gm ABEF)

DF = CE (proved earlier)

$\therefore$ ΔADF $\cong$ ΔBCE (SSS congruence condition)

$\therefore$ ar (ΔADF) = ar (ΔBCE) ...(3)

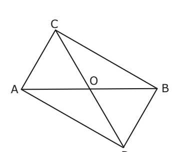
Again ar (||gm ABCD) + ar (||gn ABEF) = ar (ΔADF) + ar (||gm BCDG) + ar (||gm BGFE)

ar (||gm ABCD) + ar (||gn ABEF) = ar (ΔBCE) + ar (||gm BCDG) + ar (||gm BGFE)

ar (||gm ABCD) + ar (||gn ABEF) = = ar (||gm CDFE) (using eqn. (3)

Hence ar (||gm CDFE) = ar (||gm ABCD) + ar (||gm ABEF).

Ex.11 In the given figure ABC and ABD are two triangles on the same base AB. If line segment CD is bisected by AB at O, show that : ar (ΔABC) = ar (ΔABD).



Sol. Given that AB bisects CD, i.e., O is the mid point of CD. Now in ΔADC, AO is the median.

$\therefore$ ar (ΔACO) = ar (ΔADO) .....(1)

($\because$ median of a triangle divides it into two triangles of equal area)

Also in ΔBCD, BO is the median,

$\therefore$ ar (ΔBCO) = ar (ΔBDO) .....(2)

($\because$ median of a triangle divides it into two triangles of equal area)

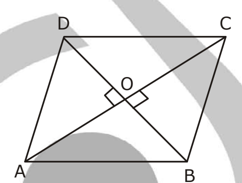
Adding equations (1) and (2) we get

ar (ΔACO) + ar (ΔBCO) = ar (ΔADO) + ar (ΔBDO)

$\Rightarrow$ ar (ΔABC) = ar (ΔABD).

Ex.12 Prove that area of a rhombus is equal to the half of the product of its diagonals.

Sol. Let ABCD be a rhombus. Let its diagonals AC and BD intersect each other at O.



We know that diagonals of a rhombus bisect each other at 90º.

$\therefore$ DO is altitude of ΔADC and BO is the altitude of ΔABC.

Now ar (ΔABC) = $\frac{1}{2}$ × AC × OB ...(1)

and ar (ΔADC) = $\frac{1}{2}$ × AC × OD ...(2)

($\because$ area of triangle = $\frac{1}{2}$ × base×altitude)

Adding equations (1) and (2) we get

ar (ΔABC) + ar (ΔADC) = $\frac{1}{2}$ × AC ×OB + $\frac{1}{2}$ × AC ×OD

$\Rightarrow$ ar (||gm ABCD) = $\frac{1}{2}$ × AC (OB × OD)

$\Rightarrow$ ar (||gm ABCD) = $\frac{1}{2}$ × AC × BD

or area of rhombus = $\frac{1}{2}$ × (product of diagonals).