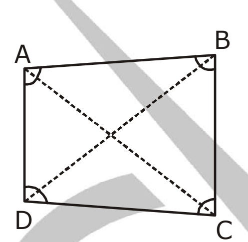
QUADRILATERALS AND PARALLELOGRAM

INTRODUCTION

In the previous chapter, we studied triangles and their properties. We know that a triangle is formed by joining any three non-collinear points in a plane. Similarly, if we join any four points in a plane, taken in order, no three of which are collinear then they form a quadrilateral. If we look around us, we will find a number of objects which are in the shape of a quadrilateral, for example- wall of a room, top of a book, blackboard, top of a table etc.

In the present chapter, we shall study quadrilaterals and their different kinds as parallelogram, rectangle, rhombus and square.

Quadrilateral. A plane figure formed by joining four points in an order, no three of which are collinear is called a quadrilateral. A quadrilateral has four vertices, four sides and four angles. It has two diagonals.



In the adjacent figure, ABCD is a quadrilateral. It has four vertices A, B, C and D. It has four sides AB, BC, CD and DA. It has four angles $\angle A$, $\angle B$, $\angle C$ and $\angle D$. It has two diagonals AC and BD.

A quadrilateral ABCD is also written as 'quad ABCD' or ' $\square$ ABCD' in short.

Terms related to a Quadrilateral.

1. **Adjacent sides.** Any two sides of a quad. which have one common vertex are called adjacent sides or consecutive sides.

   In quad. ABCD following are the pairs of adjacent sides -

   AB, BC; BC, CD; CD, DA; DA, AB.

2. **Opposite sides.** Any two sides of a quad.which have no common vertex are called opposite sides.

   In quad. ABCD following are the pairs of opposite sides-

   AB and CD; BC and AD.

3. **Consecutive angles.** Any two angles of a quad. which have one common arm are called consecutive angles.

   In quad. ABCD, $\angle A$ and $\angle B$; $\angle B$ and $\angle C$; $\angle C$ and $\angle D$; $\angle D$ and $\angle A$ and consecutive angles.

4. **Opposite angles.** Any two angles of a quad. Which have no common arm are called opposite angles.

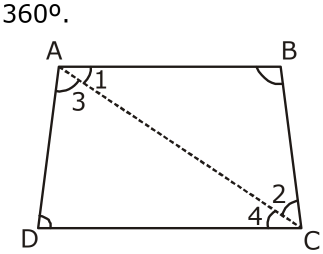
   In quad. ABCD, $\angle A$ and $\angle C$; $\angle B$ and $\angle D$ are pairs of opposite angles.

Angle sum property of a quadrilateral.

Theorem 1. The sum of all angles of a quadrilateral is 360º.

Given. A quad. ABCD.

To prove. $\angle A + \angle B + \angle C + \angle D = 360^\circ$.



Construction. Join AC.

Proof. In ΔABC,

$\angle 1 + \angle B + \angle 2 = 180^\circ$ (angle sum property of a Δ) .....(1)

In ΔADC,

$\angle 3 + \angle D + \angle 4 = 180^\circ$ (Angle sum property of a Δ) .....(2)

Adding equations (1) and (2), we get

$\angle 1 + \angle B + \angle 2 + \angle 3 + \angle D + \angle 4 = 180^\circ + 180^\circ$

$\Rightarrow \quad (\angle 1 + \angle 3) + \angle B + (\angle 2 + \angle 4) + \angle D = 360^\circ$

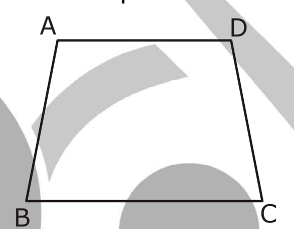
$\Rightarrow \quad \angle A + \angle B + \angle C + \angle D = 360^\circ$

Types of Quadrilaterals.

There are different types of quadrilaterals which are discussed below -

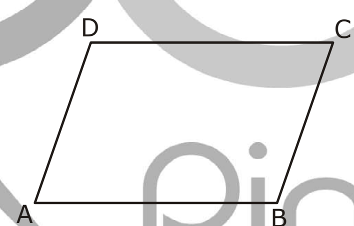
1. **Trapezium.** A quad. in which one pair of opposite sides is parallel is called a trapezium. In the adjacent figure, ABCD is a trapezium with AD||BC. In short we write it as 'trap. ABCD'.

A trap. is said to be isosceles trap. if its non-parallel sides are equal.



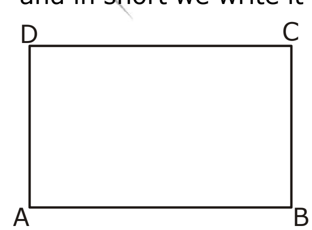
Thus trap. ABCD is isosceles if AB = DC.

2. **Parallelogram.** A quad. in which both the pairs of opposite sides are parallel to each other is called a parallelogram.

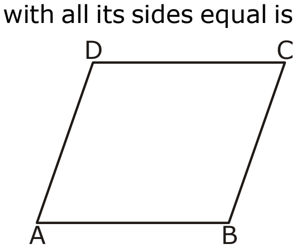


So if AB||CD and AD||BC, then ABCD is a parallelogram and in short we write it as ||gm ABCD.

3. **Rectangle.** A ||gm with one of its angle as 90º is called a rectangle. Here ABCD is a rectangle with AB || DC, AD||BC and $\angle A = 90^\circ$ and in short we write it as rect. ABCD.



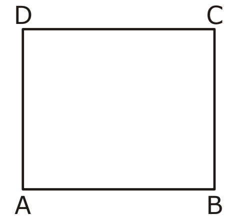
4. **Rhombus.** A quadrilateral with all its sides equal is called a rhombus.



In the adjacent figure AB = BC = CD = DA.

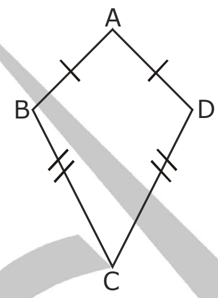
$\therefore$ ABCD is a rhombus and in short we write it as 'rhomb. ABCD'.

5. **Square .** A quadrilateral with all its sides equal and with all of its angles as 90º is called a square. Here in the figure, AB = BC = CD = DA and ∠A =∠B = ∠C = ∠D = ∠90º.



∴ ABCD is a square and in short we write it 'sq. ABCD.

6. **Kite.** A quad. is said to be in the shape of a kite if its two pairs of adjacent sides are equal.



Here ABCD is a kite with AB = AD and BC = CD.

Thus we have seen that rectangle, square, rhombus are all some special types of parallelograms, but trapezium and kite are not parallelograms.

A square is a rectangle but every rectangle is not a square.

A parallelogram is a trapezium but every trapezium is not a parallelogram.

Properties of a parallelogram.

In a parallelogram, each of the following properties hold :

(i) Diagonals of a ||gm divide it into two congruent triangles.

(ii) Opposite sides of a parallelogram are equal.

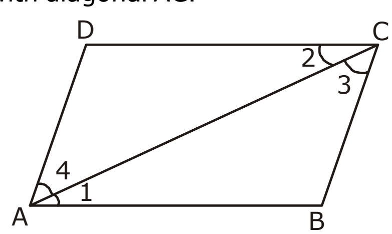
(iii) Opposite angles of a parallelogram are equal.

(iv) Diagonals of a parallelogram bisect each other.

All above necessary properties of a ||gm can be proved in the form of theorems as discussed below-

Theorem 2. A diagonal of a parallelogram divides it into two congruent triangles (CBSE 2010) (NCERT Example)

Given. A parallelogram ABCD with diagonal AC.



To prove. $\Delta ABC \cong \Delta CDA$.

Proof. Since ABCD is a parallelogram,

∴ AB || DC and AD || BC.

Now in ΔABC and ΔCDA,

AB || DC and AC is transversal,

∴ $\angle 1 = \angle 2$ (alt. int. ∠'s)

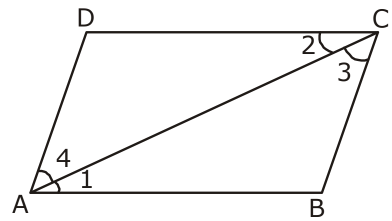
Also AD || BC and AC is transversal,

$\therefore$ $\angle 3 = \angle 4$ (alt. int. $\angle$'s)

and AC = CA (common)

$\therefore$ Δ ABC $\cong$ Δ CDA (ASA congruence condition.)

Theorem 3. In a parallelogram opposite sides are equal.



Given. A parallelogram ABCD.

To Prove. AB = DC and AD = BC.

Construction. Join AC.

Proof. As ABCD is a ||gm, AB||DC and AD||BC.

Now AB||DC and AC is transversal,

$\therefore$ $\angle 1 = \angle 2$ (alt. int. $\angle$'s)

Also AD || BC and AC is transversal,

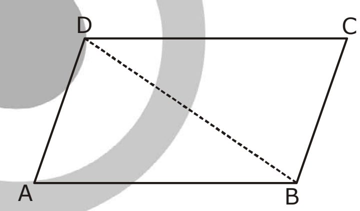
$\therefore$ $\angle 3 = \angle 4$ (alt. int. $\angle$'s)

and AC = CA (common)

$\therefore$ Δ ABC $\cong$ Δ CDA (ASA congruence condition.)

$\therefore$ AB = CD and BC = DA (cpct).

Theorem 4. In a parallelogram opposite angles are equal.



Given. A ||gm ABCD.

To prove. $\angle A = \angle C$ and $\angle B = \angle D$.

Construction. Join BD.

Proof. In ΔABD and $\angle$BCD,

AB = DC

and AD = BC

($\because$ opposite sides of a ||gm are equal)

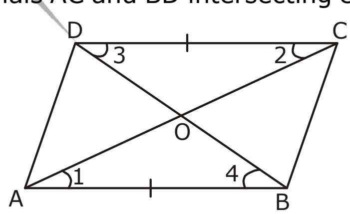
BD = BD (Common)

$\therefore$ $\angle BDA = \angle BCD$ (cpct)

Similarly, we can prove that $\angle ABC = \angle ADC$,

Theorem 5. In a parallelogram diagonals bisect each other .

Given. A ||gm ABCD with diagonals AC and BD intersecting each other at O.



To prove. OA = OC and OB = OD.

Proof. In ΔAOB and ΔCOD,

AB = CD (opposite sides of a ||gm)

$\because$ AB||CD and AC transversal

$\therefore$ $\angle 1 = \angle 2$ (alt. int. $\angle$'s)

$\because$ AB||CD and BD transversal

$\therefore$ $\angle 3 = \angle 4$ (alt. int. $\angle$'s)

$\therefore$ Δ AOB $\cong$ Δ COD (ASA congruence condition)

$\therefore$ AB = CO and BO = DO (cpct).

SOLVED PROBLEMS

Ex.1 Three angles of a quadrilateral measure 56°, 100° and 88°. Find the measure of the fourth angle.

Sol. Let the measure of the fourth angle be x.

$\therefore$ $56° + 100° + 88° + x° = 360°$ $\therefore$ [Sum of all the angles of quadrileteral is 360°]

$\Rightarrow$ $244°+x=360°$

$\Rightarrow$ $x = 360° - 244° = 116°$

Hence, the measure of the fourth angle is 116°.

Ex.2 The angles of a quadrilateral are in the ratio 3 : 5 : 9 : 13. Find all the angles of the quadrilateral.

Sol. Let the four angles of the quadrilateral be 3x, 5x, 9x and 13x . **[NCERT]**

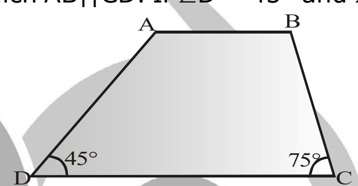
$\therefore$ $3x + 5x + 9x + 13x = 360°$ $\therefore$ [Sum of all the angles of quadrileteral is 360°]

$\Rightarrow$ $30x = 360°$

$\Rightarrow$ $x =12°$

Hence, the angles of the quadrilateral are $3 \times 12° = 36°$, $5 \times 12° = 60°$, $9 \times 12° = 108°$ and $13 \times 12° = 156°$.

Ex.3 In figure, ABCD is a trapezium in which AB||CD. If $\angle D = 45°$ and $\angle C = 75°$, find $\angle A$ and $\angle B$.



Sol. We have, AB || CD and AD is a transversal.

so, $\angle A + \angle D = 180°$ [Interior angles on the same side of the transversal]

$\Rightarrow$ $\angle A + 45° = 180°$ [$\because \angle D = 45°$]

$\Rightarrow$ $\angle A = 180° - 45° = 135°$

Similarly, AB||CD and BC is a transversal.

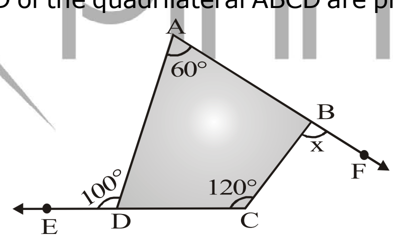
so, $\angle B + \angle C = 180°$

$\Rightarrow$ $\angle B + 75° = 180°$ [$\because \angle C = 75°$]

$\Rightarrow$ $\angle B = 105°$

Hence $\angle A = 135°$ and $\angle B = 105°$

Ex.4 In the given figure, sides AB and CD of the quadrilateral ABCD are produced. Find the value of x.



Sol. Since, $\angle ADE + \angle ADC = 180°$ [Linear pair]

$\Rightarrow$ $100° + \angle ADC = 180°$ [$\because \angle ADE = 100°$]

$\Rightarrow$ $\angle ADC = 180° - 100° = 80°$

In quadrilateral ABCD

$\angle ADC + \angle A + \angle ABC + \angle C = 360°$ $\therefore$ [Sum of all the angles of quadrileteral is 360°]

$\Rightarrow$ $80° + 60° + \angle ABC + 120° = 360°$

$\Rightarrow$ $\angle ABC + 260° = 360°$

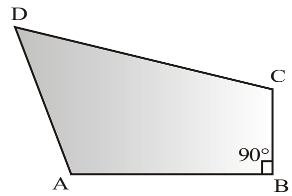
$\Rightarrow$ $\angle ABC = 100°$

But, $\angle ABC + x = 180°$ [Linear pair]

$\therefore$ $100° + x = 180°$

Hence, x = 80°.

Ex.5 In quadrilateral ABCD $\angle B = 90°$, $\angle C - \angle D = 60°$ and $\angle A - \angle C - \angle D = 10°$. Find $\angle A$, $\angle C$ and $\angle D$.



Sol. $\angle A + \angle B + \angle C + \angle D = 360°$ (Sum of the four angles of a quadrilateral is 360°)

$\Rightarrow \quad \angle A + \angle C + \angle D = 360° - \angle B$

$\angle A + \angle C + \angle D = 360° - 90°$

$\angle A + \angle C + \angle D = 270° \quad ...(1)$

It is given that

$\angle A - \angle C - \angle D = 10° \quad ...(2)$

$\angle C - \angle D = 60° \quad ...(3)$

Adding (1) and (2), we get

$(\angle A + \angle C + \angle D) + (\angle A - \angle C - \angle D) = 270° + 10°$

$\angle A + \angle C + \angle D + \angle A - \angle C - \angle D = 280°$

$2\angle A = 280°$

$\angle A = \frac{280°}{2} \quad \Rightarrow \quad \angle A = 140°$

From (1), $140° + \angle C + \angle D = 270°$

$\Rightarrow \quad \angle C + \angle D = 270° - 140°$

$\Rightarrow \quad \angle C + \angle D = 130° \quad ...(4)$

Adding (3) and (4), we get

$(\angle C - \angle D) + \angle C + \angle D = 60° + 130°$

$\angle C - \angle D + \angle C + \angle D = 190°$

$2 \times \angle C = 190°$

$\angle C = \frac{190°}{2} \quad \Rightarrow \quad \angle C = 95°$

Subtracting (3) from (4), we get

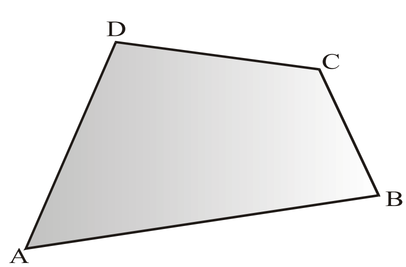
$(\angle C + \angle D) - (\angle C - \angle D) = 130° - 60°$

$\angle C + \angle D - \angle C + \angle D = 70°$

$2 \times \angle D = 70°$

$\angle D = \frac{70°}{2} \quad \Rightarrow \quad \angle D = 35°$

Ex.6 In quadrilateral ABCD $\angle A + \angle C = 140°$, $\angle A : \angle C = 1 : 3$ and $\angle B : \angle D = 5 : 6$. Find the $\angle A$, $\angle B$, $\angle C$ and $\angle D$.



Sol. $\angle A + \angle C = 140°$ (Given)

$\angle A : \angle C = 1 : 3$ (Given)

sum of ratio = 1 + 3 = 4

$\Rightarrow \quad \angle A = \frac{1}{4} \times 140° = 35°$

and $\quad \angle C = \frac{3}{4} \times 140° = 35° \times 3 = 105°$

Sum of all the angles of quadrilateral is 360°

We have $\angle A + \angle B + \angle C + \angle D = 360°$

$\Rightarrow \quad 35° + \angle B + 105° + \angle D = 360°$

$\Rightarrow \quad \angle B + \angle D + 140° = 360°$

$\Rightarrow \quad \angle B + \angle D = 360° - 140°$

$\Rightarrow \quad \angle B + \angle D = 220°$

It is given that,

$\angle B : \angle D = 5 : 6$

sum of ratios = 5 + 6 = 11

$\Rightarrow \quad \angle B = \frac{5}{11} \times 220° = 20° \times 5 = 100°$

and $\quad \angle D = \frac{6}{11} \times 220° = 20° \times 6 = 120°$

Hence, $\angle A = 35°$, $\angle B = 100°$, $\angle C = 105°$ and $\angle D = 120°$

Ex.7 Prove that if the diagonals of a quadrilateral bisect each other at right angles, then it is a rhombus.

Sol. **Given:** ABCD is a quadrilateral where diagonals AC and BD meet at 0, such that AO = OC, OB = OD and AC $\perp$ BD **[NCERT]**

To Prove: Quadrilateral ABCD is a rhombus, i.e., AB = BC = CD = DA

Proof : In ΔAOB and ΔAOD, OB=OD [Common]

AO=AO [Given]

$\angle AOB = \angle AOD$ [Each = 90°]

$\therefore$ $\Delta AOB \cong \Delta AOD$ [SAS Rule]

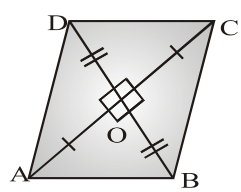
$\therefore$ AB = AD [C.P.C.T.]

Similarly, we can prove that

AB = BC ...(i)

BC = CD ...(ii)

CD = AD ...(iii)



From (i), (ii), (iii) and (iv), we obtain

AB = BC = CD = DA

$\therefore$ Quadrilateral ABCD is a rhombus.

Ex.8 Prove that the diagonals of a square are equal and bisect each other at right angles. **[NCERT]**

Sol. **Given:** ABCD is a square.

To Prove: (i) AC = BD (ii) AC and BD bisect each other at right angles.

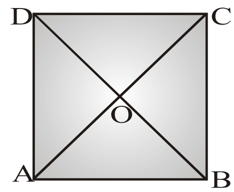
Proof: In ΔABC and ΔBAD,

AB = BA [Common]

BC = AD [Opp. sides of square ABCD]

$\angle ABC = \angle BAD$ [Each = 90° ($\because$ ABCD is a square] $\therefore$ $\Delta ABC \cong \Delta BAD$ [SAS Rule]

$\therefore$ AC = BD [C.P.C.T.] ... (i)



In ΔAOD and ΔBOC

AD = CB [Opp. sides of square ABCD]

$\angle OAD = \angle OCB$ [Alternate angles as AD||BC and transversal AC intersects them]

$\angle ODA = \angle OBC$ [Alternate angles as AD||BC and transversal BD intersects them]

$\Delta AOD \cong \Delta BOC$ [ASA Rule]

$\therefore$ OA = OC and OB = OD [C.P.C.T.] ...(ii)

So, O is the mid point of AC and BD.

Now, In ΔAOB and ΔCOB

AB = BC [Given]

OA = OC [from (ii)]

OB = OB [Common]

$\therefore$ $\Delta AOB \cong \Delta COB$ [By SSS Rule]

$\therefore$ $\angle AOB = \Delta BOC$ [C.P.C.T]

But $\angle AOB + \angle BOC = 180°$ [Linear pair]

$\angle AOB + \angle AOB = 180°$

[AOB = BOC proved earlier]

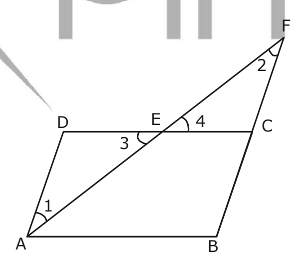
$\Rightarrow$ $2\angle AOB = 180°$

$\Rightarrow$ $\angle AOB = \frac{180°}{2} = 90°$

$\therefore$ $\angle AOB = \angle BOC = 90°$

$\therefore$ AC and BD bisect each other at right angles.

Ex.9 In the adjacent figure ABCD is a parallelogram. E is the mid point of CD. AE is joined and produced to meet BC at F. Show that BF = 2AD.



Sol. In ΔADE and ΔFCE,

$\because$ AD||BF and AF transversal

$\therefore$ $\angle 1 = \angle 2$ (alt. int. $\angle$'s)

$\angle 3 = \angle 4$ (vert. opp. $\angle$'s

DE = EC ($\because$ E is mid point of CD)

$\therefore$ $\Delta ADE \cong \Delta FCE$ (AAS congruence condition)

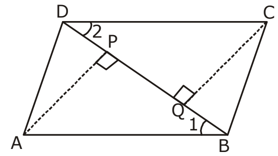
$\therefore$ AD = CF (cpct)

But AD = BC (opp. sides of a ||gm)

$\Rightarrow$ AD + AD = BC +CF

$\Rightarrow$ 2AD = BF.

Ex.10 ABCD is a parallelogram and AP and CQ are perpendiculars from vertices A and C on diagonal BD. Show that : (i) $\Delta APB \cong \Delta CQD$, (ii) AP = CQ. **[NCERT]**



Sol. In Δ APB and Δ CQD ,

$\because$ AB ||DC and BD transversal

$\therefore$ $\angle 1 = \angle 2$ (alt. int. $\angle$'s)

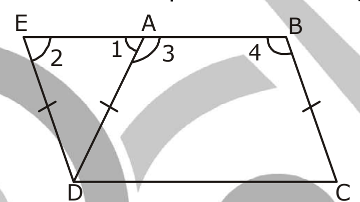
AB = DC (opp. sides of a ||gm)

$\angle APB = \angle CQD$ (each 90º)

$\therefore$ $\Delta APB \cong \Delta CQS$ (AAS congruence condition)

$\therefore$ AP = CQ (cpct).

Ex.11 In the given figure ABCD is an isosceles trapezium with AB||DC and AD = BC. Prove that $\angle A = \angle B$.



[NCERT]

Sol. Let us draw DE||BC, which meets BA produced at E.

Now since AB||DC $\therefore$ EB||DC and ED ||BC (by construction)

$\therefore$ BCDE is a parallelogram.

$\therefore$ BC = De (opp. sides of a||gm)

But BC = AD (given)

$\therefore$ AD = DE

$\Rightarrow$ $\angle 1 = \angle 2$

(angles opp. to equal sides are equal)

Now $\angle 1 + \angle 3 = 180º$ (Linear pair)

and $\angle 2 + \angle 4 = 180º$ (co. int. $\angle$'s)

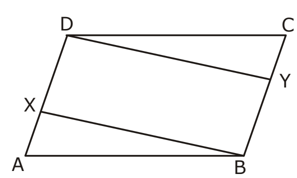
$\therefore$ $\angle 1 + \angle 3 = \angle 2 + \angle 4$

$\Rightarrow$ $\angle 1 = \angle 4$ (using $\angle 1 = \angle 2$)

or $\angle A = \angle B$.

Ex.12 ABCD is a parallelogram. X and Y are points on sides AD and BC such that AX = $\frac{1}{3}$BC and CY = $\frac{1}{3}$ BC.

Prove that BXDY is a parallelogram.



Sol. ABCD is a parallelogram.

$\therefore$ AD||BC (opp. sides of a||gm)

$\Rightarrow$ XD||BY ...(1)

Also, AD = BC (opp. sides of a||gm are equal)

$\Rightarrow \quad \frac{1}{3}AD = \frac{1}{3}BC$

$\Rightarrow \quad AX = CY$

$\left(\because \text{given that } AX = \frac{1}{3}AD \text{ and } CY = \frac{1}{3}BC\right)$

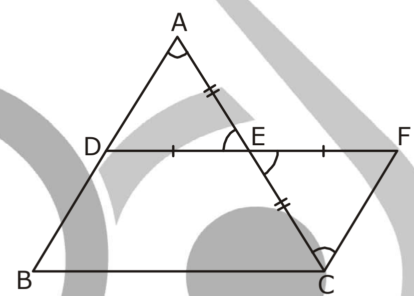
$\Rightarrow \quad AD - AX = BC - CY$

$\Rightarrow \quad XD = BY \qquad ...(2)$

From equations (1) and (2), we get XBYD is a ||gm.

(if in a quad. a pair of opp. sides is equal and parallel then it is a||gm)

Ex.13 In ΔABC, D and E are mid points of sides AB and AC respectively. DE is joined and produced till F such that DE = EF. CF is joined. Show that BCFD is a parallelogram.



Sol. In ΔADE and ΔCFE

AE = EC ($\because$ E is the mid point of AC)

DE = EF (by construction)

$\angle$AED = $\angle$CEF (vert. int. $\angle$'s)

$\therefore$ Δ ADE $\cong$ $\angle$CFE (SAS congruence condition)

$\therefore$ $\angle$FCE = $\angle$EAD (cpct)

But it is a pair of alt. int. $\angle$'s, therefore CF||AD

or CF||BD ($\because$ BD and AD are coincident lines)

Also CF = AD (cpct)

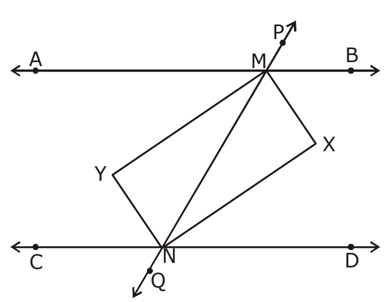
But Ad = BD ($\because$ D is mid point of AB)

$\therefore$ CF = BD

Hence in quad. BCFD, CF||BD and CF = BD.

$\therefore$ BCFD is a parallelogram.

Ex.14 Two parallel lines l and m are intersected by a transversal p. Show that the quadrilateral formed by the bisectors of interior angles is a rectangle. **[NCERT]**



Sol. Let two parallel lines I and m be intersected by a transversal p at M and N respectively. The bisectors of $\angle$BMN and $\angle$DNM intersect each other at X and bisectors of $\angle$AMN and $\angle$CNM intersect each other at Y. We are required to show that MXNY is a rectangle.Now since AB||CD and PQ transversal, therefore

$\angle BMN = \angle MNC$ (alt. int. $\angle$'s)

$\Rightarrow \quad \frac{1}{2}\angle BMN = \frac{1}{2}\angle MNC$

$\Rightarrow \quad \angle XMN = \angle MNY$

($\because$ MX bisects $\angle$BMN and NY bisects $\angle$MNC)

But these form a pair of alternate angles for lines MX and NY with MN transversal.

$\therefore \quad$ MX || NY

Similarly we can prove that MY||NX.

$\because$ Both the pairs of opposite sides are parallel,

$\therefore \quad$ MXNY is a parallelogram.

Also, $\angle BMN + \angle MND = 180º$ (Co-interior $\angle$'s)

$\Rightarrow \quad \frac{1}{2}\angle BMN + \frac{1}{2}\angle MND = 90º$

$\Rightarrow \quad \angle XMN + \angle XNM + \angle MXN = 180º$

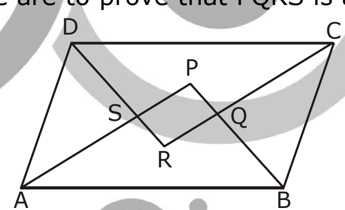
$\Rightarrow \quad 90º + \angle MXN = 180º$ (using eqn. (1))

$\Rightarrow \quad \angle MXN = 90º$

Hence MXYN is a||gm with one of its angle as 90º, So it is a rectangle.

Ex.15 Show that the bisectors of angles of a parallelogram form a rectangle.

Sol. Let ABCD be a ||gm. Let bisectors of $\angle$A and $\angle$B intersect each other at P, bisectors of $\angle$B and $\angle$C intersect each other at Q, bisectors of $\angle$C and $\angle$D intersect each other at R and Bisectors of $\angle$D and $\angle$A intersect each other at S. We are to prove that PQRS is a rectangle.



Now since AP bisects $\angle$A,

$\therefore \quad \angle PAB = \frac{1}{2}\angle A$

and since BP bisects $\angle$B,

$\therefore \quad \angle PAB = \frac{1}{2}\angle B$

But $\angle A + \angle B = 180º$ (co-interior angles)

$\therefore \quad \frac{1}{2}\angle A + \frac{1}{2}\angle B = 90º$

$\Rightarrow \quad \angle PAB + \angle PBA = 90º \quad .....(1)$

$\angle PAB + \angle PBA + \angle P = 180º$

(angle sum property of triangle)

$\Rightarrow \quad 90º + \angle P = 180º \qquad$ (using eqn.1)

$\Rightarrow \quad \angle P = 90º$

Similarly we can prove that $\angle$Q, $\angle$R, $\angle$S each is 90º.

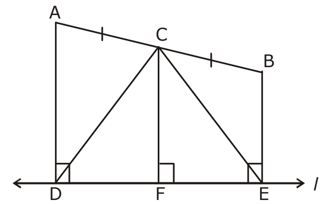
Now we have a quad. PQRS in which opposite angles are equal, so PQRS is a parallelogram.

Also $\angle P = 90º$

$\therefore \quad$ PQRS is a rectangle.

($\because$ a parallelogram with any one of its angle as right angle is a rectangle)

Ex.16 In the given figure, on line I perpendicular lines AD and BE are drawn from points A and B. IF C is the mid point of AB, prove that CD = CE.



Sol. Let us draw CF perpendicular to I. Now AD, CF and BE all are perpendicular to I

AD||CF||BE

and these lines make equal intercepts on transversal

AB ($\because$ AC = BC, C being mid point of AB).

$\therefore$ Intercepts made on transversal I should also be equal i.e.,

DF = FE (by intercept theorem)

Now in ΔCDF and ΔCEF,

DF = EF (proved above)

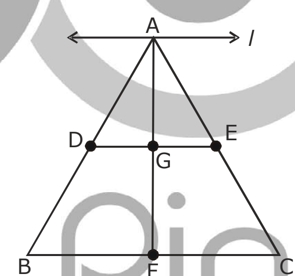
CF = CF (common)

$\angle CFD = \angle CFE$ (each 90º)

$\therefore$ $\Delta CDF \cong \Delta CEF$ (SAS congruence condition)

CD = CE (cpct).

Ex.17 In ΔABC, D and E are the mid points of sides AB and AC. F is any point of BC. Line joining A and F meets DE at G. Show that G is the mid point of AF.



Sol. Let us draw a line *l* through A, parallel to BC. Also DE||BC (by mid point theorem).

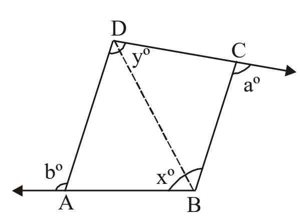
$\therefore \ell$ ||DE||BC and AB transversal on which intercepts made by these parallel lines are equal as AD = BD ($\because$ given that D is the mid point of AB).

AF is another transversal, therefore by intercept theorem, intercepts made by the parallel lines on AF are also equal i.e., AG = GF.

or G is the mid point of AF.

Ex.18 The sides BA and DC of a quadrilateral ABCD are produced as shown in fig.

Prove that a + b = x + y.



Sol. Join BD. In ΔABD, we have

$\angle ABD + \angle ADB = b^o$ ....(i)

In ΔCBD, we have

$\angle CBD + \angle CDB = a^o$ ....(ii)

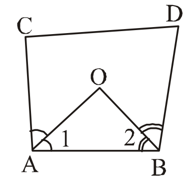
Adding (i) and (ii), we get

$(\angle ABD + \angle CBD) + (\angle ADB + \angle CDB) = a^o + b^o$

$\Rightarrow$ $x^o + y^o = a^o + b^o$

Hence, x + y = a + b

Ex.19 In a quadrilateral ABCD, AO and BO are the bisectors of $\angle A$ and $\angle B$ respectively. Prove that $\angle AOB = \frac{1}{2}(\angle C + \angle D)$.



Sol. In ΔAOB, we have

$\angle AOB + \angle 1 + \angle 2 = 180^\circ$

$\Rightarrow \angle AOB = 180^\circ - (\angle 1 + \angle 2)$

$\Rightarrow \angle AOB = 180^\circ - \left(\frac{1}{2}\angle A + \frac{1}{2}\angle B\right)$ $\left[\because \angle 1 = \frac{1}{2}\angle A \text{ and } \angle 2 = \frac{1}{2}\angle B\right]$

$\Rightarrow \angle AOB = 180^\circ - \frac{1}{2}(\angle A + \angle B)$

$\Rightarrow \angle AOB = 180^\circ - \frac{1}{2}[360^\circ - (\angle C + \angle D)]$

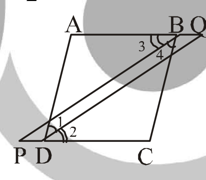
$[\because \angle A + \angle B + \angle C + \angle D = 360^\circ$

$\therefore \angle A + \angle B = 360^\circ - (\angle C + \angle D)]$

$\Rightarrow \angle AOB = 180^\circ - 180^\circ + \frac{1}{2}(\angle C + \angle D)$

$\Rightarrow \angle AOB = \frac{1}{2}(\angle C + \angle D)$

Ex.20 In figure bisectors of $\angle B$ and $\angle D$ of quadrilateral ABCD meet CD and AB produced at P and Q respectively. Prove that $\angle P + \angle Q = \frac{1}{2}(\angle ABC + \angle ADC)$



Sol. In ΔPBC, we have

$\therefore \angle P + \angle 4 + \angle C = 180^\circ$

$\Rightarrow \angle P + \frac{1}{2}\angle B + \angle C = 180^\circ$ ...(i)

In ΔQAD, we have

$\angle Q + \angle A + \angle 1 = 180^\circ$

$\Rightarrow \angle Q + \angle A + \frac{1}{2}\angle D = 180^\circ$ ...(ii)

Adding (i) and (ii), we get

$\angle P + \angle Q + \angle A + \angle C + \frac{1}{2}\angle B + \frac{1}{2}\angle D = 180^\circ + 180^\circ$

$\Rightarrow \angle P + \angle Q + \angle A + \angle C + \frac{1}{2}\angle B + \frac{1}{2}\angle D = 360^\circ$

$\Rightarrow \angle P + \angle Q + \angle A + \angle C + \frac{1}{2}(\angle B + \angle D) = \angle A + \angle B + \angle C + \angle D$

$[\therefore$ In a quadrilateral ABCD

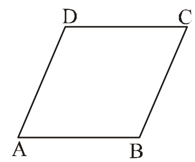
$\angle A + \angle B + \angle C + \angle D = 360^\circ]$

$\Rightarrow \angle P + \angle Q = \frac{1}{2}(\angle B + \angle D)$

$\Rightarrow \angle P + \angle Q = \frac{1}{2}(\angle ABC + \angle ADC)$

Ex.21 In a parallelogram ABCD, prove that sum of any two consecutive angles is 180º.

Sol. Since ABCD is a parallelogram. Therefore, AD || BC.



Now, AD || BC and transversal AB intersects them at A and B respectively.

$\therefore \quad \angle A + \angle B = 180^\circ$

[$\because$ Sum of the interior angles on the same side of the transversal is 180º]

Similarly, we can prove that

$\angle B + \angle C = 180^\circ$, $\angle C + \angle D = 180^\circ$ and $\angle D + \angle A = 180^\circ$.

Ex.22 In a parallelogram ABCD, $\angle D = 115^\circ$, determine the measure of $\angle A$ and $\angle B$.

Sol. Since the sum of any two consecutive angles of a parallelogram is 180º. Therefore,

$\angle A + \angle D = 180^\circ$ and $\angle A + \angle B = 180^\circ$

Now, $\angle A + \angle D = 180^\circ$

$\Rightarrow \quad \angle A + 115^\circ = 180^\circ$

$[\because \angle D = 115^\circ \text{ (given)}]$

$\Rightarrow \quad \angle A = 65^\circ$

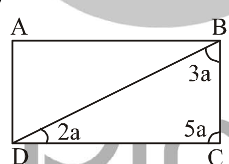
and $\quad \angle A + \angle B = 180^\circ$

$\Rightarrow \quad 65^\circ + \angle B = 180^\circ$

$\Rightarrow \quad \angle B = 115^\circ$

Thus, $\angle A = 65^\circ$ and $\angle B = 115^\circ$

Ex.23 In figure find the four angles A, B, C and D in the parallelogram ABCD.



Sol. In ΔBCD we have

$\angle BDC + \angle DCB + \angle CBD = 180^\circ$

$\Rightarrow \quad 2a + 5a + 3a = 180^\circ$

$\Rightarrow \quad 10a = 180^\circ$

$\Rightarrow \quad a = 18^\circ$

$\therefore \quad \angle C = 5a = 5 \times 18^\circ = 90^\circ$

Since opposite angles are equal in a parallelogram. Therefore,

$\angle A = \angle C \quad \Rightarrow \quad \angle A = 90^\circ$

Since the sum of the angles of a quadrilateral is 360º. Therefore,

$\angle A + \angle B + \angle C + \angle D = 360^\circ$

$\Rightarrow \quad 2(\angle A + \angle B) = 360^\circ$

$[\because \angle A = \angle C \text{ and } \angle B = \angle D]$

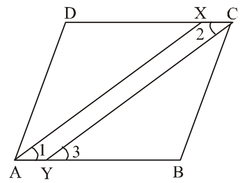
$\Rightarrow \quad \angle A + \angle B = 180^\circ$

$\Rightarrow \quad 90^\circ + \angle B = 180^\circ \; [\because \angle A = 90^\circ]$

$\Rightarrow \quad \angle B = 90^\circ$

Hence, $\angle A = 90^\circ$, $\angle B = 90^\circ$, $\angle C = 90^\circ$ and $\angle D = 90^\circ$

Ex.24 ABCD is a parallelogram and line segments AX & CY are bisector of $\angle A$ and $\angle C$. Show that AX || CY.



Sol. Since opposite angles are equal in a parallelogram. Therefore, in parallelogram ABCD, we have

$\angle A = \angle C$

$\Rightarrow \quad \frac{1}{2}\angle A = \frac{1}{2}\angle C$

$\Rightarrow \quad \angle 1 = \angle 2$ ....(i)

[$\because$ AX and CY are bisectors of $\angle A$ and $\angle C$ respectively]

Now, AB || DC and the transversal CY intersects them.

$\therefore \quad \angle 2 = \angle 3$ ....(ii)

[$\because$ Alternate interior angles are equal]

From (i) and (ii), we get

$\angle 1 = \angle 3$

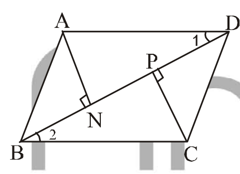
Thus, transversal AB intersects AX and YC at A and Y such that $\angle 1 = \angle 3$

i.e. corresponding angles are equal.

$\therefore \quad$ AX || CY

Ex.25 In figure AN and CP are perpendiculars to the diagonal BD of a parallelogram ABCD. Prove that:

(i) $\Delta ADN \cong \Delta CBP$ (ii) AN = CP **[NCERT]**



Sol. Since ABCD is a parallelogram.

$\therefore \quad$ AD || BC

Now, AD || BC and transversal BD intersects them at B and D.

$\therefore \quad \angle 1 = \angle 2$

[$\because$ Alternate interior angles are equal]

Now, in Δs ADN and CBP, we have

$\angle 1 = \angle 2$

$\angle AND = \angle CPD$ and, AD = BC

[$\because$ Opposite sides of a $||^{gm}$ are equal]

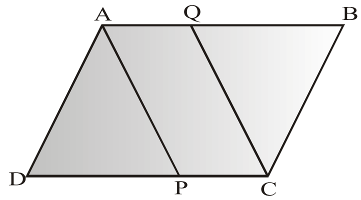
So, by AAS criterion of congruence

$\Delta ADN \cong \Delta CBP$

AN = CP

[$\because$ Corresponding parts of congruent triangles are equal]

Ex.26 In figure ABCD is a parallelogram and AP and CQ are bisectors of $\angle A$ and $\angle C$. Prove that AP||CQ.



Sol. We have $\angle A = \angle C$ [Opposite angles of a $||^{gm}$]

$\frac{1}{2}\angle A = \frac{1}{2}\angle C$

$\Rightarrow \quad \angle PAQ = \angle PCQ$ ...(1)

Now, AB || CD and CQ is a transversal. Therefore ,

$\angle PCQ = \angle BQC$ [Alternate angles] ...(2)

$\Rightarrow \quad \angle PAQ = \angle BQC$ [From (1)]

But, these are corresponding angles formed when AP and CQ are intersected by transversal AB.

$\therefore$ AP||CQ

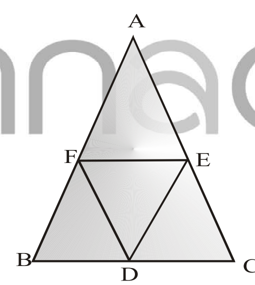
Hence proved.

Ex.27 In the following figure, D, E and F are respectively the mid-points of sides BC, CA and AB of an equilateral triangle ABC. Prove that ΔDEF is also an equilateral triangle. **[NCERT]**

Sol. **Given :** D, E and F are respectively the mid-points of sides BC, CA and AB of an equilateral triangle ABC.

To prove : ΔDEF is also an equilateral triangle.

Proof : since the segment joining the mid points of two sides of a triangle is half of the third side. Therefore D and E are the mid point of BC and AC respectively.



$\therefore \quad DE = \frac{1}{2}AB$ ....(i)

E and F are the mid point of AC and AB respectively

$\therefore \quad EF = \frac{1}{2}BC$ ...(ii)

F and D are the mid point of AB and BC respectively

$\Rightarrow \quad FD = \frac{1}{2}AC$ ...(iii)

$\because \Delta ABC$ is an equilateral triangle

$\Rightarrow \quad AB = BC = CA$

$\Rightarrow \quad \frac{1}{2}AB = \frac{1}{2}BC = \frac{1}{2}CA$

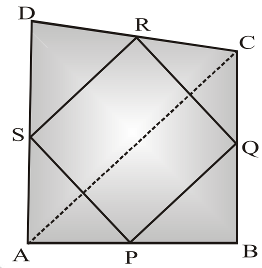
$\Rightarrow \quad DE = EF = FD$

using (i), (ii) & (iii)

Hence, ΔDEF is an equilateral triangle.

Hence Proved

Ex.28 ABCD is a quadrilateral in which P, Q, R and S are mid-points of the sides AB, BC, CD and DA. AC is a diagonal. **[NCERT]**



Show that :

(i) SR $\parallel$ AC and SR = $\frac{1}{2}$ AC (ii) PQ = SR

(iii) PQRS is a parallelogram.

Sol. **GIVEN :** ABCD is a quadrilateral in which P, Q, R and S are mid-points of AB, BC, CD and DA. AC is a diagonal.

TO PROVE :

(i) SR $\parallel$ AC and SR = $\frac{1}{2}$ AC

(ii) PQ = SR

(iii) PQRS is a parallelogram.

PROOF : (i) In ΔDAC,

$\because$ S is the mid-point of DA and R is the mid-point of DC

$\therefore$ SR $\parallel$ AC and SR = $\frac{1}{2}$ AC [By Mid-point theorem]

(ii) In ΔBAC,

$\because$ P is the mid-point of AB and Q is the mid-point of BC

$\therefore$ PQ $\parallel$ AC and PQ = $\frac{1}{2}$ AC [By Mid-point theorem]

But from (i) SR = $\frac{1}{2}$ AC & (ii) PQ = $\frac{1}{2}$ AC

$\Rightarrow$ PQ = SR

(iii) PQ $\parallel$ AC [From (ii)]

SR $\parallel$ AC [From (i)]

$\therefore$ PQ $\parallel$ SR

[Two lines parallel to the same line are parallel to each other]

Also, PQ = SR [From (ii)]

$\therefore$ PQRS is a parallelogram.

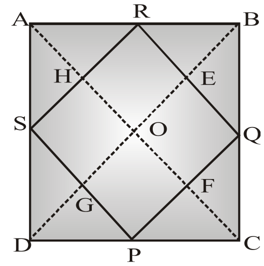
[A quadrilateral is a parallelogram if a pair of opposite sides is parallel and is of equal length]

Ex.29 Show that the quadrilateral formed by joining the mid-point of the consecutive sides of a square is also a square. **[NCERT]**

Sol. **Given :** ABCD is a square. R, Q, P and S are the mid-points of the consecutive isdes AB, BC, CD and DA respectively. PQ, QR, RS and SP are joined.

To prove : Quadrilateral PQRS is a square.

Construction : Join AC and BD



Proof : $RQ \parallel AC$ and $RQ = \frac{1}{2} AC$ ... (1)

$SP \parallel AC$ and $SP = \frac{1}{2} AC$ ...(2)

From (1) & (2)

$\therefore$ RQ = SP and $RQ \parallel SP$

Similarly, SR = PQ and $SR \parallel PQ$

$\therefore$ PQRS is a parallelogram

$\because$ $RQ \parallel AC$ $\therefore$ $RE \parallel HO$

$\because$ $SR \parallel PQ$ $\therefore$ $HR \parallel OE$

$\therefore$ OERH is a parallelogram.

$\therefore$ $\angle R = \angle HOE$ [Opposite $\angle$s of a $\parallel$ gm]

But $\angle HOE = 90°$ [Diagonal of square bisect at 90°]

$\therefore$ $\angle R = 90°$ $\therefore$ Quadrilateral PQRS is a rectangle.

But AC = BD [Diagonal of a square are equal]

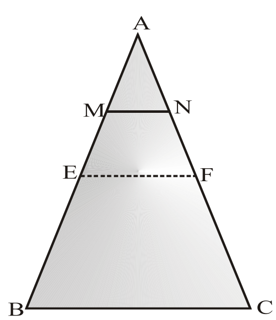
$\therefore$ HF = GE or PQ = QR, so all sides are equal

$\therefore$ PQ = QR = RS = SP $\therefore$ Quadrilateral PQRS is a square.

Hence Proved.

Ex.30 In triangle ABC, points M and N on sides AB and AC respectively are taken so that $AM = \frac{1}{4} AB$ and $AN = \frac{1}{4} AC$. Prove that $MN = \frac{1}{4} BC$.

Sol. **Given :** In triangle ABC, points M and N on the sides AB and AC respectively are taken so that $AM = \frac{1}{4} AB$ and $AN = \frac{1}{4} AC$.



To prove : $MN = \frac{1}{4} BC$.

Construction : Join EF where E and F are the mid points of AB and AC respectively.

Proof : $\because$ E is the mid-point of AB and F is the mid-point of AC.

$\therefore$ EF $\parallel$ BCand EF = $\frac{1}{2}$BC ...(1)

Now, AE = $\frac{1}{2}$AB and AM = $\frac{1}{4}$AB

$\therefore$ AM = $\frac{1}{2}$AE

Similarly, AN = $\frac{1}{2}$AF

$\Rightarrow$ M and N are the mid-points of AE and AF resepctively.

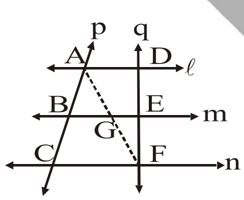
$\therefore$ MN $\parallel$ EF and MN = $\frac{1}{2}$EF = $\frac{1}{2}\left(\frac{1}{2}BC\right)$ [From (1)]

MN = $\frac{1}{4}$BC.

Hence Proved.

Ex.31 In figure ℓ,m and n are three parallel lines intersected by transversals p and q such that ℓ, m and n cut-off equal intercepts AB and BC on p. Show that ℓ, m and n cut off equal intercepts DE and EF on q also.

Sol.



Given : AB = BC

To prove : DE = EF

Construction : Join AF, it intersect line m at G

In ΔACF, B is the mid-point of AC ($\because$ AB = BC and BG $\parallel$ CF. Therefore, G is the mid-point of AF.

In ΔAFD, G is the mid-point of AF and GD $\parallel$ AD.

$\therefore$ E is the mid-point of DE

$\Rightarrow$ DE = EF

Hence, ℓ, m and n cut of equal intercepts DE and EF on q.