TriDiagonal Matrix Algorithm (TDMA) or Thomas Algorithm

Set of algebraic linear equations are solved by forward elimination and back substitution. The TDMA or Thomas Algorithm is a simplified form of Gaussian elimination. The *general form* of tridiagonal system of n unknowns may be written as

$$-\beta_i X_{i-1} + D_i X_i - \alpha_i X_{i+1} = C_i \text{ for i = 1 to n,}$$

matrix form:

$$\begin{bmatrix} D_1 & -\alpha_1 & 0 & 0 & . & . & 0 \\ -\beta_2 & D_2 & -\alpha_2 & 0 & . & . & 0 \\ 0 & -\beta_3 & D_3 & -\alpha_3 & . & . & 0 \\ 0 & 0 & -\beta_4 & D_4 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & -\alpha_{n-1} \\ 0 & 0 & 0 & . & . & -\beta_n & D_n \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ . \\ . \\ X_n \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ . \\ . \\ C_n \end{bmatrix}$$

Solution

The above tridiagonal system of n unknowns can be solved by the following steps according to Thomas Algorithm.

Solved by forward substitution,

$$A_i = \frac{\alpha_i}{D_i - \beta_i A_{i-1}} \qquad C_i' = \frac{\beta_i C_{i-1}' + C_i}{D_i - \beta_i A_{i-1}} \qquad \text{for i = 1 to n;}$$

Final solution by backward substitution,

$X_i = A_i X_{i+1} + C_i'$ for i = n to 1;

Note: For last value of X that is $X_n = C'_n$, then $X_{n-1} = A_{n-1} X_n + C'_{n-1}$ and so on.

Example

Solve $\begin{bmatrix} 20 & -5 & 0 & 0 & 0 \\ -5 & 15 & -5 & 0 & 0 \\ 0 & -5 & 15 & -5 & 0 \\ 0 & 0 & -5 & 15 & -5 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} 1100 \\ 100 \\ 100 \\ 100 \\ 100 \end{bmatrix}$

Solution in tabular form

i	β_i	D_i	α_i	C_i	A_i	C'_i	X_i
1	0	20	5	1100	0.2500	55.0000	64.2276
2	5	15	5	100	0.3636	27.2727	36.9106
3	5	15	5	100	0.3793	17.9310	26.5041
4	5	15	5	100	0.3816	14.4737	22.6016
5	5	10	0	100	0	*21.3008*	*21.3008*

References

https://www.cfd-online.com/Wiki/Tridiagonal_matrix_algorithm_-_TDMA_(Thomas_algorithm)

Thomas, L.H. (1949), *Elliptic Problems in Linear Differential Equations over a Network*, Watson Sci. Comput. Lab Report, Columbia University, New York..