Transient

The conservation law for the transport of a scalar in an unsteady flow has the general form:

$$\frac{\partial}{\partial t}(\rho\Phi) + div(\rho\, u\, \Phi) = div(\gamma\ grad\ \Phi) + S$$

Where

$\frac{\partial}{\partial t}(\rho\Phi)$➔Unsteady term

$div(\rho\, u\, \Phi)$ ➔Convective term

$div(\gamma\ grad\ \Phi)$ ➔Diffusion term

S ➔Source term

Φ ➔Dependent variable (example mass, pressure, temperature, etc.)

On integrating general differential equation over a Control Volume CV we get,

$$\int_{CV}\left(\frac{\partial}{\partial t}(\rho\phi)dV\right) + \int_{CV}(div(\rho u\phi)dV) = \int_{CV}(\mathrm{div}(\Gamma grad\phi)\, dV) + \int_{CV}(S_\phi dV)$$

Further integrating over a finite time step Δt

$$\int_t^{t+\Delta t}\left[\int_{CV}\left(\frac{\partial}{\partial t}(\rho\phi)\right)dV\right]dt + \int_t^{t+\Delta t}\left(\int_{CV} div(\rho u\phi)dV\right)dt$$

$$= \int_t^{t+\Delta t}\left(\int_{CV} div(\Gamma grad\phi)dV\right)dt + \int_t^{t+\Delta t}\left(\int_{CV}(S_\phi)dV\right)dt$$

1D unsteady heat conduction

Unsteady 1D heat conduction is governed by the equation:

$$\rho C\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + S$$

$$\int_t^{t+\Delta t}\left[\int_{CV}\left(\rho C\frac{\partial T}{\partial t}\right)dV\right]dt = \int_t^{t+\Delta t}\left[\int_{CV}\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)dV\right]dt + \int_t^{t+\Delta t}\left(\int_{CV} SdV\right)dt$$

Which can also be written as:

$$\int_w^e\left[\int_t^{t+\Delta t}\rho C\frac{\partial T}{\partial t}dt\right]dV = \int_t^{t+\Delta t}\left[\left(kA\frac{\partial T}{\partial x}\right)_e - \left(kA\frac{\partial T}{\partial x}\right)_w\right]dt + \int_t^{t+\Delta t}\bar{S}\Delta V dt$$

In above equation ΔV is the volume of the control volume which is equal to AΔx, where Δx is the width of the control volume and $\bar{S}$ is the average source strength. If the temperature at a node is assumed to prevail over the whole control volume, the left hand side can be written as:

$$\int_w^e \left[\int_t^{t+\Delta t} \rho C \frac{\partial T}{\partial t} dt\right] dV = \rho C (T_P - T_P^o)\Delta V$$

Here, parameters with super script 'o' represent value of parameter at time 't' and values without any superscript denotes parameter at time $t + \Delta t$.

So the term $\frac{\partial T}{\partial t}$ can be written as $(T_P - T_P^o/\Delta t)$, and it can be noted that this has been discretised using a first order backward differencing scheme.

If we apply central differencing scheme to the diffusion terms on the right hand side equation, it can be written as:

$$\int_t^{t+\Delta t} \left[\left(kA\frac{\partial T}{\partial x}\right)_e - \left(kA\frac{\partial T}{\partial x}\right)_w\right] dt + \int_t^{t+\Delta t} \bar{S}\Delta V dt = \rho C (T_P - T_P^o)\Delta V$$

To evaluate time integral values of temperatures at time 't' or at time ' $t + \Delta t'$ or combination of both can be used.

The approach can be generalised by means of a weighting parameter θ between 0 and 1 and write the integral I_T of temperature T_P w.r.t. tie as

$$I_T = \int_t^{t+\Delta t} T_P dt = [\theta T_P + (1-\theta)T_P^o]\Delta t$$

θ	I_T
0	$T_P^o \Delta t$
0.5	$0.5(T_P + T_P^o)\Delta t$
1	$T_P \Delta t$

I.e. if $\theta = 0$ the temperature at old time level 't' is used; if $\theta = 1$ the temperature at new time level i.e. $t + \Delta t$ is used; and finally if $\theta = 0.5$ the temperatures at new and old time levels are equally weighted.

$$\rho C \left(\frac{T_P - T_P^o}{\Delta t}\right)\Delta x$$

$$= \theta\left[\frac{K_e(T_E - T_P)}{\delta x_{PE}} - \frac{K_w(T_P - T_W)}{\delta x_{WP}}\right] + (1-\theta)\left[\frac{K_e(T_E^o - T_P^o)}{\delta x_{PE}} - \frac{K_w(T_P^o - T_W^o)}{\delta x_{WP}}\right] + \bar{S}\Delta x$$

On arranging, we get:

$$\left[\rho C\frac{\Delta x}{\Delta t}+\theta\frac{K_e}{\delta x_{PE}}+\frac{K_w}{\delta x_{WP}}\right]T_P = \frac{K_e}{\delta x_{PE}}[\theta T_E+(1-\theta)T_E^o]+\frac{K_w}{\delta x_{WP}}[\theta T_W+(1-\theta)T_W^o] + \left[\rho C\frac{\Delta x}{\Delta t}-(1-\theta)\frac{K_e}{\delta x_{PE}}-(1-\theta)\frac{K_w}{\delta x_W P}\right]T_P^o+\bar{S}\Delta x$$

Which can be written as:

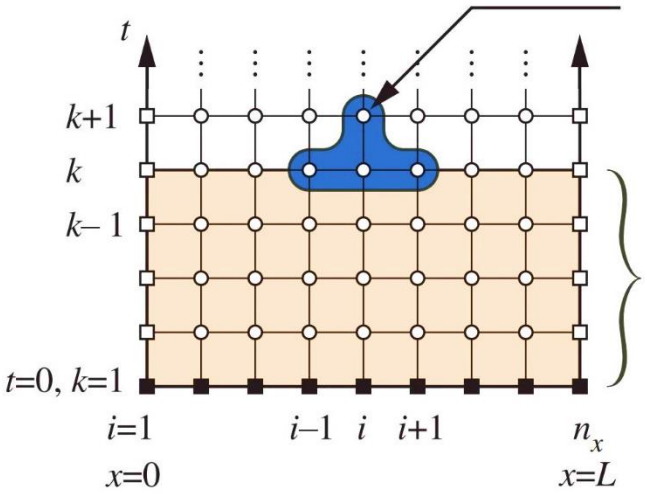
$$a_P T_P = a_W[\theta T_W+(1-\theta)T_W^o]+a_E[\theta T_E+(1-\theta)T_E^o]+\left[a_p^o-(1-\theta)a_W-(1-\theta)a_E\right]T_P^o+b$$

where,

a_P	a_P^o	a_W	a_E	b
$\theta(a_W+a_E)+a_P^o$	$\rho C\frac{\Delta x}{\Delta t}$	$\frac{K_w}{\delta x_{WP}}$	$\frac{K_c}{\delta x_{PE}}$	$\bar{S}\Delta x$

The exact form of the final discretised equation depends on the value of θ. When θ is zero, we use temperatures at the old time level t on the right hand side of equation to evaluate T_P at the new time and the resulting scheme is called as **Explicit**. When $0<\theta<1$ temperatures at the new time level are used on both sides of the equation and the resulting schemes are called as Implicit. The extreme case of $\theta=1$ is termed as **Fully implicit** and the case corresponding to $\theta=1/2$ is called **Crank Nicolson** scheme.

Explicit scheme (FTCS – Forward Time Centered Space)



FTCS scheme enables explicit calculation of u at this node

Solution is known for these nodes

Figure Credit: Gerald Recktenwald (gerry@pdx.edu)

In explicit scheme we have $\theta = 0$, and taking source term as: $\bar{S}\Delta x = S_c + S_P T_P^o$

$$a_P T_P = a_W T_W^o + a_E T_E^o + \left[a_p^o - (a_W + a_E - S_P)\right] T_P^o + S_c$$

where,

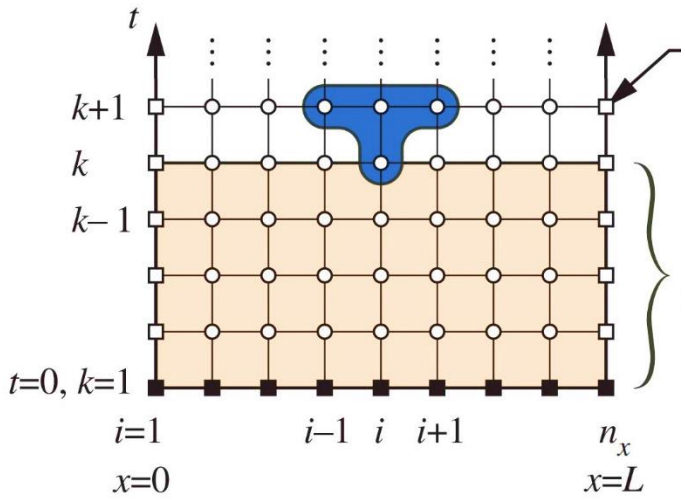
a_P	a_P^o	a_W	a_E
a_P^o	$\rho C \frac{\Delta x}{\Delta t}$	$\frac{K_w}{\delta x_{WP}}$	$\frac{K_e}{\delta x_{PE}}$

RHS of equation only contains values at the old time step so the left hand side can be calculated by forward marching in time. The scheme is based on backward differencing and its Taylor series truncation error accuracy is first-order wrt time. As all coefficients need to be positive in the discretised equation .The coefficient of T_P^o may be viewed as the neighbour coefficient connecting the values at the old time level to those at the new time level. Fof this coefficient to be positive we must have $a_P^o - a_W - a_E > 0$. For constant K and uniform grid spacing, $\delta x_{PE} = \delta x_{WP} = \Delta x$, this condition may be written as:

$$\rho C \frac{\Delta x}{\Delta t} > \frac{2K}{\Delta x} \text{ or } \Delta t < \rho C \frac{(\Delta x)^2}{2K}$$

This inequality sets a stringent maximum limit to the time step size and represents a serious limitation for the explicit scheme. It becomes very expensive to improve spatial accuracy because the maximum possible time step needs to be reduced as the square of Δx. Consequently, this method is not recommended for general transient problems. Explicit schemes with greater formal accuracy than the above one have been designed. Nevertheless, provided that the time step size is chosen with care, the explicit scheme described above is efficient for simple conduction calculations.

Fully Implicit scheme (Backward Time Centered Space)



BTCS scheme requires simultaneous calculation of u at all nodes on the k+1 mesh line i.e., solve TDMA for each time step

Solution is known for these nodes

Figure Credit: Gerald Recktenwald (gerry@pdx.edu)

When the value of θ is set to 1 we obtain the fully implicit scheme. Taking the source term as: $\bar{S}\Delta x = S_c + S_P T_P$

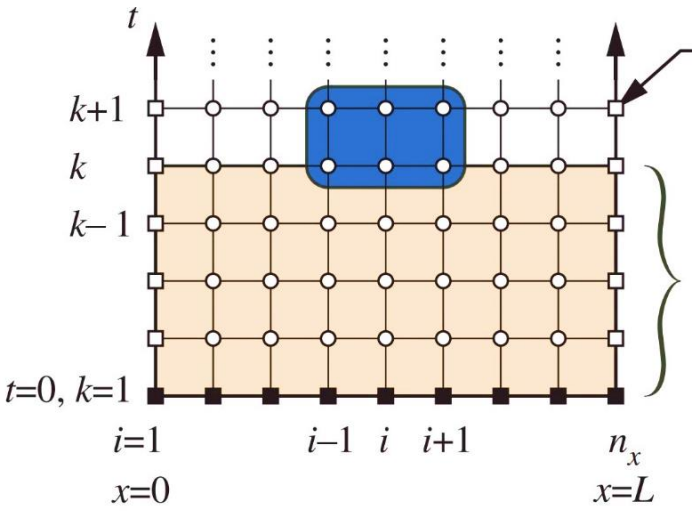
$$a_P T_P = a_W T_W + a_E T_E + a_p^o T_P^o + S_c$$

where,

a_P	a_P^o	a_W	a_E
$a_P^o + a_W + a_E - S_P$	$\rho C \frac{\Delta x}{\Delta t}$	$\frac{K_w}{\delta x_{WP}}$	$\frac{K_c}{\delta x_{PE}}$

Both sides of the equation contain temperatures at the new time step, and a system of algebraic equations must be solved at each time level. The time marching procedure starts with a given initial field of temperatures T°. The system of equations is solved after selecting time step Δt. Next the solution T is assigned to T° and the procedure is repeated to progress the solution by a further time step. It can be seen that all coefficients are positive, which makes the implicit scheme unconditionally stable for any size of time step. Since the accuracy of the scheme is only first-order in time, small time steps are needed to ensure the accuracy of results. The implicit method is recommended for general purpose transient calculations because of its robustness and unconditional stability.

Crank-Nicolson scheme



Crank-Nicolson scheme requires simultaneous calculation of u at all nodes on the k+1 mesh line i.e., solve TDMA for each time step

Solution is known for these nodes

Figure Credit: Gerald Recktenwald (gerry@pdx.edu)

The Crank Nicolson method results from setting $\theta = 1/2$. The source term is linearised as

$$\bar{S}\Delta x = S_c + (1/2)S_P T_P + (1/2)S_P T_P^0$$

$$a_P T_P = a_E\left(\frac{T_E + T_E^\circ}{2}\right) + a_W\left(\frac{T_W + T_W^o}{2}\right) + \left(a_P^0 - \frac{a_E}{2} - \frac{a_W}{2}\right)T_P^0 + S_c + \frac{1}{2}S_P T_P^0$$

where,

a_P	a_P^o	a_W	a_E
$a_P^o + \frac{1}{2}(a_W + a_E) - \frac{1}{2}S_P$	$\rho C\frac{\Delta x}{\Delta t}$	$\frac{K_w}{\delta x_{WP}}$	$\frac{K_e}{\delta x_{PE}}$

Since more than one unknown value of T at the new time level is present, the method is implicit, and simultaneous equations for all node points need to be solved at each time step. Although schemes with $1/2 \leq \theta \leq 1$, including the Crank-Nicolson scheme, are unconditionally stable for all values of the time step (Fletcher, 1991), it is more important to ensure that all coefficients are positive for physically realistic and bounded results. This is the case if the coefficient of T_P^0 satisfies the following condition:

$a_P^0 => \left[\frac{a_e + a_W}{2}\right]$ which leads to, $\Delta t < \rho C\frac{\Delta x^2}{k}$

This time step limitation is only slightly less restrictive than associated with the explicit method. The Crank-Nicolson method is based on central differencing and hence it is second-order accurate in time. With sufficiently small time steps it is possible to achieve considerably greater accuracy than with the explicit method. The overall accuracy of a computation depends also on the spatial differencing practice so the Crank-Nicolson scheme is normally used in conjunction with spatial central differencing.