# Solution to AMM Problem 11248

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Feb 27, 2007

We consider the euclidian vector space $\mathcal{C}([0,1])$ of the continuous real-valued functions on $[0,1]$ with the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$. What we want is a polynomial $P(x) = \sum_{k=0}^{n-1} a_k x^k$ for which

$$\sum_{k=0}^{n-1} a_k = n \tag{1}$$

and

$$\int_0^1 (P(x))^2 dx = 1. \tag{2}$$

If we have such a $P(x)$, we get with the Cauchy-Schwarz inequality that

$$n^2 = |\sum_{k=0}^{n-1} a_k \int_0^1 x^k f(x)dx|^2 = |\langle P, f\rangle|^2 \le \underbrace{\|P\|^2}_{1} \cdot \|f\|^2 = \int_0^1 (f(x))^2 dx$$

and are done.

We propose that conditions (1) and (2) are satisfied with the coefficents

$$a_k = (-1)^{n+k+1} \prod_{i=1}^{k} (\frac{n^2}{i^2} - 1) \tag{3}$$

It is relatively easy to see that (1) holds:

$$\sum_{k=0}^{n-1} a_k = (-1)^{n+1} \sum_{k=0}^{n-1} (-1)^k \frac{(n-1)(n+1)\cdots(n-k)(n+k)}{(k!)^2} =$$

$$= (-1)^{n+1} \sum_{k=0}^{n-1} \binom{n-1}{k}\binom{-n-1}{k} = (-1)^{n+1} \sum_{k=0}^{n-1} \binom{n-1}{n-1-k}\binom{-n-1}{k} =$$

$$= (-1)^{n+1} \binom{-2}{n-1} = (-1)^{n+1} \frac{(-2)(-3)\cdots(-n)}{(n-1)!} = (-1)^{2n} \frac{n!}{(n-1)!} = n$$

We will now verify equation (2).

$$\int_0^1 (P(x))^2 dx = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} a_i a_j \int_0^1 x^{i+j} dx = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \frac{a_i a_j}{i+j+1}$$

Thus, the double-sum we have to compute is

$$\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} F(n,i,j) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} (-1)^{i+j} \frac{(n+i)!(n+j)!}{(i+j+1)(i!)^2(j!)^2 n^2 (n-i-1)!(n-j-1)!}$$

with $F(n,i,j) = 0$ for $i > n-1$ or $j > n-1$.

We first show the following recurrence:

$$\frac{n^2(2n+3)}{(2n+1)(n+2)^2} F(n,i,j) - \frac{4(n+1)^3}{(2n+1)(n+2)^2} F(n+1,i,j) + F(n+2,i,j) =$$
$$= G_1(n,i+1,j) - G_1(n,i,j) + G_2(n,i,j+1) - G_2(n,i,j)$$

with

$$G_1(n,i,j) = -2(-1)^{i+j} \frac{i^2 j(2n+3)(n+i)!(n+j)!}{(n+1)(n+2)^2(n-i+1)!(n-j+1)!(i!)^2(j!)^2}$$
$$G_2(n,i,j) = -2(-1)^{i+j} \frac{i j^2(2n+3)(n+i)!(n+j)!}{(n+1)(n+2)^2(n-i+1)!(n-j+1)!(i!)^2(j!)^2}$$

and $G_k(n,i,j) = 0$ for $i > n+1$ or $j > n+1$.
It is for $0 \le i < n+1$, $0 \le j \le n+1$

$$G_1(n,i+1,j) - G_1(n,i,j) =$$
$$= \frac{2(-1)^{i+j} j(2n+3)(n+i)!(n+j)!}{(n+1)(n+2)^2(n-i)!(n-j+1)!(i!)^2(j!)^2}((n+i+1) + \frac{i^2}{n-i+1}) =$$
$$= \frac{2(-1)^{i+j} j(2n+3)(n+i)!(n+j)!}{(n+1)(n+2)^2(n-i)!(n-j+1)!(i!)^2(j!)^2} \cdot \frac{(n+1)^2}{n-i+1} =$$
$$= \frac{2(-1)^{i+j} j(2n+3)(n+1)(n+i)!(n+j)!}{(n+2)^2(n-i+1)!(n-j+1)!(i!)^2(j!)^2}$$

and for $i = n+1$

$$G_1(n,n+2,j) - G_1(n,n+1,j) = -G_1(n,n+1,j) =$$
$$= \frac{2(-1)^{i+j}(n+1)^2 j(2n+3)(n+i)!(n+j)!}{(n+1)(n+2)^2(n-i+1)!(n-j+1)!(i!)^2(j!)^2} =$$
$$= \frac{2(-1)^{i+j} j(2n+3)(n+1)(n+i)!(n+j)!}{(n+2)^2(n-i+1)!(n-j+1)!(i!)^2(j!)^2}$$

Likewise,

$$G_2(n,i,j+1)-G_2(n,i,j)=\frac{2(-1)^{i+j}i(2n+3)(n+1)(n+i)!(n+j)!}{(n+2)^2(n-i+1)!(n-j+1)!(i!)^2(j!)^2}$$

Putting this together, we have for $0 \le i,j \le n+1$

$$G_1(n,i+1,j)-G_1(n,i,j)+G_2(n,i,j+1)-G_2(n,i,j)=$$
$$=\frac{2(-1)^{i+j}(i+j)(2n+3)(n+1)(n+i)!(n+j)!}{(n+2)^2(n-i+1)!(n-j+1)!(i!)^2(j!)^2}$$

Conversely, for $0 \le i,j \le n+1$:

$$\frac{n^2(2n+3)}{(2n+1)(n+2)^2}F(n,i,j)-\frac{4(n+1)^3}{(2n+1)(n+2)^2}F(n+1,i,j)+F(n+2,i,j)=$$
$$=\frac{n^2(2n+3)}{(2n+1)(n+2)^2}\cdot\frac{(-1)^{i+j}(n+i)!(n+j)!(n-i+1)(n-i)(n-j+1)(n-j)}{(i+j+1)(i!)^2(j!)^2n^2(n-i+1)!(n-j+1)!}-$$
$$-\frac{4(n+1)^3}{(2n+1)(n+2)^2}\cdot\frac{(-1)^{i+j}(n+i+1)!(n+j+1)!(n-i+1)(n-j+1)}{(i+j+1)(i!)^2(j!)^2(n+1)^2(n-i+1)!(n-j+1)!}+$$
$$+\frac{(-1)^{i+j}(n+i+2)!(n+j+2)!}{(i+j+1)(i!)^2(j!)^2(n+2)^2(n-i+1)!(n-j+1)!}=$$
$$=\frac{(-1)^{i+j}(n+i)!(n+j)!}{(n+2)^2(i+j+1)(i!)^2(j!)^2(n-i+1)!(n-j+1)!}\cdot$$
$$\cdot\Big(\frac{2n+3}{2n+1}(n-i+1)(n-i)(n-j+1)(n-j)-$$
$$-\frac{4(n+1)(n+i+1)(n+j+1)(n-i+1)(n-j+1)}{2n+1}+$$
$$+(n+i+1)(n+i+2)(n+j+1)(n+j+2)\Big)=$$
$$=\frac{(-1)^{i+j}(n+i)!(n+j)!}{(n+2)^2(i+j+1)(i!)^2(j!)^2(n-i+1)!(n-j+1)!}\cdot\frac{1}{2n+1}\cdot$$
$$\cdot(6i+6j+22in+24in^2+22jn+24jn^2+44ijn+48ijn^2+16ijn^3+6i^2+22i^2n+$$
$$+8j^2n^3+24j^2n^2+8i^2n^3+22j^2n+12ij+6j^2+8in^3+8jn^3+24i^2n^2)=$$
$$=\frac{(-1)^{i+j}(n+i)!(n+j)!}{(n+2)^2(i+j+1)(i!)^2(j!)^2(n-i+1)!(n-j+1)!}\cdot$$
$$\cdot\frac{2(2n+1)(2n+3)(i+j+1)(i+j)(n+1)}{2n+1}=$$
$$=\frac{2(-1)^{i+j}(i+j)(2n+3)(n+1)(n+i)!(n+j)!}{(n+2)^2(n-i+1)!(n-j+1)!(i!)^2(j!)^2}=$$
$$=G_1(n,i+1,j)-G_1(n,i,j)+G_2(n,i,j+1)-G_2(n,i,j)$$

We note that

$$\sum_{i=0}^{n+1}\sum_{j=0}^{n+1}(G_1(n,i+1,j)-G_1(n,i,j)+G_2(n,i,j+1)-G_2(n,i,j))=$$

$$=\sum_{j=0}^{n+1}(G_1(n,n+2,j)-G_1(n,0,j))+\sum_{i=0}^{n+1}(G_2(n,i,n+2)-G_2(n,i,0))=$$

$$=\sum_{j=0}^{n+1}0+\sum_{i=0}^{n+1}0=0$$

With the notation $S(n-1)=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}F(n,i,j)$, summation over $i$ and $j$ $(0\le i,j\le n+1)$ yields

$$\frac{n^2(2n+3)}{(2n+1)(n+2)^2}S(n-1)-\frac{4(n+1)^3}{(2n+1)(n+2)^2}S(n)+S(n+1)=0 \qquad (4)$$

We will prove $S(n)=1$ for all $n$ by induction. We calculate that $S(1)=S(2)=1$. Let now $S(n-1)=S(n)=1$. Then equation (4) is

$$S(n+1)=\frac{4(n+1)^3}{(2n+1)(n+2)^2}-\frac{n^2(2n+3)}{(2n+1)(n+2)^2}=\frac{2n^3+9n^2+12n+4}{(2n+1)(n+2)^2}=1$$

and we have shown equation (2).