A NOTE ON THE INDIVISIBILITY OF THE HENSON GRAPHS

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Abstract. We show that in contrast to the Rado graph, the Henson graphs are not computably indivisible.

1. Introduction

The Rado graph is, up to graph isomorphism, the unique countable undirected graph that satisfies the following property: if A and B are any finite disjoint sets of vertices, there is a vertex not in A or B which is connected to every member of A and to no member of B. It is homogeneous and universal for the class of finite graphs.

Our interest here lies with the closely related family of *Henson graphs*, introduced by C. Ward Henson in 1971 [Hen71]. For each $n \geq 3$, the Henson graph H_n is up to isomorphism the unique countable graph which satisfies the following property analogous to that characterizing the Rado graph: for any finite disjoint sets of vertices A and B, if A does not contain a copy of K_{n-1}, then there is a vertex $x \notin A \cup B$ connected to every member of A and to no member of B. (Here we write K_m for the complete graph on m vertices.) The graph H_n is homogeneous and universal for the class of K_n-free finite graphs.

We presume familiarity with the basic terminology of computable structure theory, as for example in the first chapter of [Mon21]. A structure $\mathcal{S}$ is said to be *indivisible* if for any presentation $\mathcal{A}$ of $\mathcal{S}$ and any coloring c of $\operatorname{dom} \mathcal{A}$ with finite range, there is a monochromatic subset of $\operatorname{dom} \mathcal{A}$ which induces a substructure isomorphic to $\mathcal{S}$. We call the monochromatic subset in question a *homogeneous set* for c. $\mathcal{S}$ is *computably indivisible* if there is a homogeneous set computable from $\mathcal{A}$ and c, for any presentation $\mathcal{A}$ and coloring c of $\operatorname{dom} \mathcal{A}$.

For the rest of the paper, we fix a computable presentation of H_n with domain $\mathbb{N}$ and thus focus only on the coloring. Viewed as a structure in the language of a single binary relation, the Rado graph is known to be indivisible, and computably so (folklore). Each of the Henson graphs is also indivisible. Henson himself proved that a weak form of indivisibility holds for each H_n. Full indivisibility was first shown for $n = 3$ by Komjáth and Rödl [KR86], and then for all n by El-Zahar and Sauer [ES89]. (A clarified and corrected version of the proof of Komjáth and Rödl can be found in [Gil23].) Work on the Ramsey theory of the Henson graphs has progressed beyond vertex colorings; recently, Natasha Dobrinen has undertaken a deep study of the structure of H_n and shown that for each n, H_n has finite big Ramsey degrees, developing many novel techniques in the process [Dob20; Dob22].

Our far more modest result concerns only vertex colorings and states that unlike the Rado graph, none of the Henson graphs is computably indivisible:

Theorem 1. *For every $n \geq 3$, there is a computable 2-coloring of H_n with no c.e. homogeneous set.*

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This theorem naturally raises the question of how complicated a homogeneous set for a coloring of H_n can or must be. An analysis of the proof of the indivisibility of H_3 by Komjáth and Rödl in [KR86] demonstrates that a homogeneous set can always be computed in the first jump of the coloring. For H_n in general, the proof of El-Zahar and Sauer in [ES89] shows that the $(2n-3)$rd jump of a coloring suffices to compute a homogeneous set. The latter is a strictly worse upper bound for $n = 3$, and it is currently unknown whether a similar discrepancy exists for any $n \geq 4$. Where vertex colorings of H_n fall on the spectrum of coding vs. cone avoidance is another intriguing question.

2. Proof of the theorem

Write $x \in G$, for a graph G, to mean x is a vertex of G. By abuse of notation, if $V \subset G$ is any set of vertices, we will identify V with the induced subgraph of G on V. Furthermore, we always identify natural numbers with the elements of H_n they encode via our fixed computable presentation of H_n, and sets of naturals with the corresponding induced subgraphs of H_n. If $A = \{a_1 < \cdots < a_n\}$ and $B = \{b_1 < \cdots < b_n\}$ are two sets of vertices in a graph G, write $A \simeq^* B$ if the map $a_i \mapsto b_i$ is an isomorphism of induced subgraphs. If the vertices of G are given some linear ordering, denote by $G \restriction m$ the induced subgraph of G on its first m vertices. If $x \in G$, let $G(x)$ denote the induced subgraph of G consisting of the neighbors of x. A set of the form $G(x)$ is referred to as a "neighbor set". Let $\mathscr{T}_n$ be the set of finite K_n-free simple connected graphs.

We will need two lemmas. The first is a consequence of the following theorem of Jon Folkman, which appears as Theorem 2 in [Fol70]. For a graph G, let $\delta(G)$ be the largest n such that G contains a subgraph isomorphic to K_n.

Theorem 2 (Folkman). *For each $k > 0$ and finite graph F, there is a finite graph G such that*

(a) $\delta(G) = \delta(F)$, and

(b) for any partition of the vertices of G as $G_1 \sqcup \cdots \sqcup G_k$, there is an i such that G_i contains a subgraph isomorphic to F.

Part (a) implies that G is K_n-free if F is.

Lemma 3. *For each n and k, there is a $G \in \mathscr{T}_n$ which is not an induced subgraph of $\bigcup_{i=1}^{k} H_n(x_i)$ for any vertices $x_1, \ldots, x_k \in H_n$. In particular, no finite union of neighbor sets in H_n can contain an isomorphic copy of H_n.*

Proof. By applying Theorem 2 with $F = K_{n-1}$, there is a K_n-free G such that for every partition of G into k sets, at least one set contains a K_{n-1}. Since a neighbor set in H_n cannot contain a K_{n-1}, this means that G is not contained in any union of k neighbor sets. □

Note that the graph G can be found computably from n and k by a brute-force search. The next fact is a restatement of Lemma 1 of [ES89]:

Lemma 4 (El-Zahar & Sauer). *Let Δ be a finite induced subgraph of H_n with d vertices. Let Γ be any member of $\mathscr{T}_n$ with $d+1$ vertices put in increasing order such that $\Delta \simeq^* \Gamma \restriction d$. Then there are infinitely many choices of $x \in H_n$ such that $\Delta \cup \{x\} \simeq^* \Gamma$.*

Proof of Theorem 1. The proof is by a finite injury priority argument. We build a computable $c\colon H_n \to 2$, viewing 2 as the set $\{R, B\}$ (red and blue), to meet requirements

$$R_e\colon\ (|W_e| = \infty \wedge |c(W_e)| = 1) \implies \text{Lemma 4 fails if } H_n \text{ is replaced with } W_e \subset H_n.$$

These are given the priority order $R_0 > R_1 > R_2 > \cdots$. We also define a computable function p in stages, where $p(x, s)$ is the planned color of vertex x at stage s, beginning with $p(x, 0) = R$

(red) for all x. This function will be used to keep track of vertices which requirements "reserve" to be a certain color. Only one vertex will actually be colored at each stage, starting with $c(0) = R$.

A requirement R_e is said to be active at stage s if $e \leq s$ and $W_{e,s}$ contains at least one element that was enumerated after the most recent stage in which R_e was injured (to be explained below). If R_e was never injured, we say it is active simply if $W_{e,s} \neq \emptyset$. Each requirement R_e will amass a finite list of vertices $\{x_e^1, x_e^2, \dots, x_e^\ell\}$ in W_e as its followers, together with a target graph Γ_e (also explained below). When a follower x_e^m is added, R_e will set the function $p(x,s)$ for some vertices $x \in H_n(x_e^m)$; we say R_e *reserves* x when it sets $p(x,s)$. Weaker requirements cannot reserve vertices which are currently reserved by stronger requirements. The followers, target graph, and reservations of R_e are canceled when R_e is injured by a stronger requirement. (Canceling a reserved vertex just means the vertex is no longer considered to be reserved by R_e, and does not change the values of p or c.) We may as well assume each W_e is monochromatic, and will refer to R_e as either a red or blue requirement accordingly.

We now detail the construction, and afterwards show that all requirements are injured at most finitely often and are met. First, if no R_e is active at stage $s+1$ for $e \leq s$, set $p(x, s+1) = p(x,s)$ for all x, set $c(s+1) = p(s+1, s+1)$, and end the stage. If a requirement R_e is already active at stage $s+1$ and has no follower, give it a follower x_e^1 which is any element of $W_{e,s}$ that was enumerated after the stage in which R_e was last injured, or otherwise any element of $W_{e,s}$ if R_e was never injured. Then for every $y \in H_n(x_e^1)$ which is not currently reserved by a stronger requirement and has not yet been colored, reserve y by setting $p(y, s+1)$ to be the opposite color as $c(x_e^1)$.

If R_e is active and has a follower at stage $s+1$ but no target graph, let its target graph be some $\Gamma_e \in \mathscr{T}_n$ which cannot be contained in $k+1$ neighbor sets, where k is the total number of all followers of stronger currently active requirements. Such a Γ_e may be furnished by Lemma 3. Order Γ_e in such a way that each vertex (except the first) is connected to at least one previous vertex.

Next, suppose that at least one requirement is active and has a follower and target graph at stage $s+1$. Go through the following procedure for each such R_e in order from strongest to weakest. Let m be the number of followers of R_e at stage s; we will at this point have $\{x_e^1, \dots, x_e^m\} \simeq^* \Gamma_e \restriction m$. Suppose there is some $x \in W_{e,s+1}$ with x greater than the stage at which x_e^m was enumerated into W_e, and such that $\{x_e^1, \dots, x_e^m, x\} \simeq^* \Gamma_e \restriction (m+1)$. If so, then give R_e the new follower $x_e^{m+1} = x$, and for all $y \in H_n(x_e^{m+1})$ with $y > s+1$ such that y is not currently reserved by any stronger requirement, have R_e reserve $p(y, s+1) = R$ if R_e is blue, or $p(y, s+1) = B$ if R_e is red. Injure all weaker requirements by canceling their followers, target graphs, and reservations. After this is done for all active R_e, end the stage by making $p(z, s+1) = p(z,s)$ for any z for which $p(z, \cdot)$ was not modified earlier in the stage, and then letting $c(s+1) = p(s+1, s+1)$. If instead no x as above was found for any active R_e, then set $p(x, s+1) = p(x,s)$ for all x, set $c(s+1) = p(s+1, s+1)$, and end the stage. This completes the construction.

Each requirement only need accumulate a finite list of followers, so in particular R_0 will only injure other requirements finitely many times. After the last time a requirement is injured, it only injures weaker requirements finitely often, so inductively we have that every requirement is only injured finitely many times before acquiring its final list of followers and target graph. And each requirement is satisfied: suppose (without loss of generality) R_e is blue. For each $i \geq 2$, the vertex x_e^i is an element of $H_n(x_e^j)$ for some $j < i$, by assumption on how we have ordered Γ_e. If x_e^j was enumerated into W_e at stage s, then when this x_e^j was chosen as a follower, R_e reserved every element of $H_n(x_e^j)$ greater than s by making its planned color red—except for those vertices which were already reserved (to be blue) by stronger (red) requirements. Therefore, if x_e^i is blue, then since in particular the construction requires $x_e^i > s$, we must have x_e^i a neighbor of some follower of a stronger (red) requirement. (We asked for x_e^i to be greater than the stage t at which x_e^{i-1} was enumerated. Such an x_e^i can be found for any t by Lemma 4.) So this copy we are building of Γ_e inside W_e is contained entirely in a union of neighbor sets of followers of stronger active requirements, except possibly for x_e^1 which may lie outside of any such neighbor set. If R_e is never injured again, then the number k of such followers never changes again; it is the same as it was when the target graph Γ_e was chosen not to fit inside $k+1$ neighbor sets. The latter number is large enough to also cover x_e^1, so that this copy of Γ_e can never be completed inside W_e, implying Lemma 4 fails in W_e. □

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