

# INDIVISIBILITY AND UNIFORM COMPUTATIONAL STRENGTH

KENNETH GILL

**ABSTRACT.** A countable structure is indivisible if for every coloring with finite range there is a monochromatic isomorphic subcopy of the structure. Each indivisible structure  $\mathcal{S}$  naturally corresponds to an indivisibility problem  $\text{Ind } \mathcal{S}$ , which outputs such a subcopy given a presentation and coloring. We investigate the Weihrauch complexity of the indivisibility problems for two structures: the rational numbers  $\mathbb{Q}$  as a linear order, and the equivalence relation  $\mathcal{E}$  with countably many equivalence classes each having countably many members. We separate the Weihrauch degrees of both  $\text{Ind } \mathbb{Q}$  and  $\text{Ind } \mathcal{E}$  from several benchmark problems, showing in particular that  $\text{C}_{\mathbb{N}}|_{\text{W}} \text{Ind } \mathbb{Q}$  and hence  $\text{Ind } \mathbb{Q}$  is strictly weaker than the problem of finding an interval in which some color is dense for a given coloring of  $\mathbb{Q}$ ; and that the Weihrauch degree of  $\text{Ind } \mathcal{E}_k$  is strictly between those of  $\text{SRT}_k^2$  and  $\text{RT}_k^2$ , where  $\text{Ind } \mathcal{S}_k$  is the restriction of  $\text{Ind } \mathcal{S}$  to  $k$ -colorings.

## 1. INTRODUCTION

Reverse mathematics is a branch of logic which seeks to classify theorems based on their logical strength. This is often done by working over a relatively weak base system for mathematics, such as a fragment of second-order arithmetic, and then looking at the truth of various statements in models of that fragment. It turns out that one can frequently obtain more fine-grained distinctions between theorems by comparing their intrinsic computational or combinatorial power, rather than being limited to considerations of provability only. For theorems expressible as  $\Pi_2^1$  statements of second-order arithmetic, this can be achieved by identifying a theorem with an instance-solution pair, where instances are all sets  $X$  to which the hypothesis of the theorem apply, and solutions to  $X$  are all sets  $Y$  witnessing the truth of the conclusion of the theorem for  $X$ . One then introduces various reducibilities to compare, among different instance-solution pairs, the difficulty of finding a solution from a given instance.

Instance-solution pairs can be formalized as problems:

**Definition 1.1.** A *problem* is a partial multivalued function on Baire space,  $P: \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ . Any  $x \in \text{dom } P$  is called an *instance* of  $P$ , and any  $y \in P(x)$  is a *solution* to  $x$  with respect to  $P$ .

In the present work, we focus on problems arising from indivisible structures, and use mainly Weihrauch reducibility to gauge their uniform computational content. We assume familiarity with computability theory and the basic terminology of

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computable structure theory, as in the introductory sections of [18]. All our structures are countable and we will take them to have domain  $\mathbb{N}$  when convenient.

**Definition 1.2.** A structure  $\mathcal{S}$  is *indivisible* if for any presentation  $\mathcal{A}$  of  $\mathcal{S}$  and any coloring of  $A = \text{dom } \mathcal{A}$  with finitely many colors, there is a monochromatic subset  $B \subseteq A$  such that  $B \simeq \mathcal{A}$ , where  $\mathcal{B}$  is the substructure induced by  $B$ .  $\mathcal{S}$  is *computably indivisible* if in addition such a set  $B$  can always be computed from  $\mathcal{A}$  and  $c$ .

Indivisibility belongs properly to Ramsey theory, the study of combinatorial structures which cannot be “too disordered” in that any large subset must contain a highly organized substructure. This field goes back to 1930, when Ramsey’s theorem first appeared, and principles arising in Ramsey theory have been objects of great interest in reverse mathematics.

**Definition 1.3.** If  $\mathcal{S}$  is an indivisible structure, then  $\text{Ind } \mathcal{S}$  is the *indivisibility problem* associated to  $\mathcal{S}$ , for which

- Instances are triples  $\langle \mathcal{A}, c, k \rangle$  where  $\mathcal{A}$  is a presentation of  $\mathcal{S}$  (identified with its atomic diagram) and  $c$  is a  $k$ -coloring of  $A$ , and
- Solutions to  $\langle \mathcal{A}, c, k \rangle$  are (characteristic functions of) subsets  $B$  of  $A$  which are monochromatic for  $c$  and such that  $B \simeq \mathcal{A}$ .

Here  $\mathcal{A}$  is allowed to be an arbitrary presentation of  $\mathcal{S}$ , not necessarily one having domain  $\mathbb{N}$ . We will also use the following variations on  $\text{Ind } \mathcal{S}$ :

- For each fixed  $k$ , let  $\text{Ind } \mathcal{S}_k$  have domain  $\{ \langle \mathcal{A}, c \rangle : \langle \mathcal{A}, c, k \rangle \in \text{dom}(\text{Ind } \mathcal{S}) \}$  and let  $\text{Ind } \mathcal{S}_k(\langle \mathcal{A}, c \rangle) = \text{Ind } \mathcal{S}(\langle \mathcal{A}, c, k \rangle)$ . That is,  $\text{Ind } \mathcal{S}_k$  is the restriction of  $\text{Ind } \mathcal{S}$  to  $k$ -colorings, so that  $\text{Ind } \mathcal{S}$  is exactly the problem  $\bigsqcup_{k \in \mathbb{N}} \text{Ind } \mathcal{S}_k$ , where  $\bigsqcup$  is defined in the next section.
- Let  $\text{Ind } \mathcal{S}_{\mathbb{N}}$  be the problem with  $\text{dom}(\text{Ind } \mathcal{S}_{\mathbb{N}}) = \bigcup_{k \in \mathbb{N}} \text{dom}(\text{Ind } \mathcal{S}_k)$  and  $\text{Ind } \mathcal{S}_{\mathbb{N}}(x) = \text{Ind } \mathcal{S}_k(x)$  if  $x \in \text{dom}(\text{Ind } \mathcal{S}_k)$ . Then  $\text{Ind } \mathcal{S}_{\mathbb{N}}$  is just  $\text{Ind } \mathcal{S}$  but without any given information on the number of colors used.

One can also define the *strong indivisibility problem* where a solution additionally includes an explicit isomorphism between  $B$  and  $\mathcal{A}$ . We do not study this notion further; both it and  $\text{Ind } \mathcal{S}$  are discussed, for example, in [1].

This paper focuses on two indivisible structures in particular. First, the set of rational numbers  $\mathbb{Q}$  is viewed as, up to order isomorphism, the unique countable dense linear order with no greatest or least element.  $\text{Ind } \mathbb{Q}$  is studied in Section 4. The other structure is the so-called countable equivalence relation:

**Definition 1.4.** The *countable equivalence relation*  $\mathcal{E}$  is, up to isomorphism, the structure in the language  $\{R\}$  of a single binary relation such that  $R$  is an equivalence relation, and such that  $E$  is divided into countably many equivalence classes each having countably many members.

$\text{Ind } \mathcal{E}$  is studied in Section 5. Our main results separate  $\text{Ind } \mathbb{Q}$  and  $\text{Ind } \mathcal{E}$  from various benchmark problems in terms of computational strength. In the case of  $\mathbb{Q}$ , we have

- $\text{Ind } \mathbb{Q}$  is Weihrauch reducible to  $\text{TC}_{\mathbb{N}}^*$  but incomparable with  $\text{C}_{\mathbb{N}}$  (Theorem 4.1 and Corollary 4.9).
- Hence  $\text{Ind } \mathbb{Q}$  is not uniformly strong enough to find a rational interval where a color appears densely (Corollary 4.2).

- $\text{Ind } \mathbb{Q}$  is not limit computable (Proposition 4.6).
- $\text{Ind } \mathbb{Q}$  cannot be solved by any problem having a c.e. approximation (Theorem 4.8), in a precise sense given by Definition 4.7.

The hierarchy of Weihrauch reducibilities (and nonreducibilities) relating versions of  $\text{TC}_{\mathbb{N}}^k$ ,  $\text{Ind } \mathbb{Q}_k$ , and  $\text{RT}_k^1$  is shown in Figure 1 below. Next,  $\text{Ind } \mathcal{E}$  turns out to be closely related to but distinct from Ramsey's theorem for pairs:

- The Weihrauch degree of  $\text{Ind } \mathcal{E}_k$  is strictly between those of  $\text{SRT}_k^2$  and  $\text{RT}_k^2$  for all  $k \geq 2$ ;
- In particular,  $\text{Ind } \mathcal{E}_2$  is not Weihrauch reducible to  $\text{SRT}_{\mathbb{N}}^2$  and  $\text{RT}_2^2$  is not computably reducible to  $\text{Ind } \mathcal{E}_{\mathbb{N}}$  (Theorem 5.1).

The Weihrauch reducibilities between these principles are displayed in Figure 4.

Before getting to our main results in Sections 4 and 5, we define all the reducibilities under consideration as well as the main problems used as benchmarks in our results. In Section 3 we comment on the issue of presentations and make some general observations. Finally, in Section 6 we briefly mention some directions for future research on  $\text{Ind } \mathbb{Q}$  and  $\text{Ind } \mathcal{E}$ .

## 2. BACKGROUND

Here we collect the general definitions and notations used for the rest of the paper. The standard reference on reverse mathematics is Simpson [21]. *Slicing the Truth* by Hirschfeldt [13] and the recent book by Dzhafarov and Mummert [7] are more computability-oriented treatments of the subject, the latter discussing in detail the reducibilities we employ. Our notation and conventions with respect to computability theory are standard and can be found in any modern textbook, e.g., [22]. For any set  $X$  and  $n \in \mathbb{N}$ ,  $X \upharpoonright n$  is the truncation of  $X$  up through its first  $n$  bits. If  $\sigma \in \mathbb{N}^{<\mathbb{N}}$ , then  $[\sigma]$  denotes the set of finite or infinite strings extending  $\sigma$ . We will always identify  $k$  with the set  $\{0, \dots, k-1\}$ . Write  $W_e$  for the  $e$ th c.e. set with respect to some computable enumeration of Turing machines.

**Definition 2.1.** Let  $P$  and  $Q$  be problems. Then

- $P$  is *computably reducible* to  $Q$ , written  $P \leq_c Q$ , if each  $p \in \text{dom } P$  computes some  $x \in \text{dom } Q$  such that for every  $q \in Q(x)$ ,  $p \oplus q$  computes an element of  $P(p)$ . If we require  $q$  by itself to compute an element of  $P(p)$ , then  $P$  is *strongly computably reducible* to  $Q$ , written  $P \leq_{sc} Q$ .
- $P$  is *Weihrauch reducible* to  $Q$ , written  $P \leq_W Q$ , if there are Turing functionals  $\Delta$  and  $\Psi$  such that if  $p \in \text{dom } P$ , then  $\Delta^p \in \text{dom } Q$ ; and for any  $q \in Q(\Delta^p)$ , we have  $\Psi^{p \oplus q} \in P(p)$ . If we require  $\Psi$  to only have oracle access to  $q$ , then  $P$  is *strongly Weihrauch reducible* to  $Q$ , written  $P \leq_{sW} Q$ .

We have  $P \leq_{sW} Q \implies P \leq_W Q \implies P \leq_c Q$  and  $P \leq_{sW} Q \implies P \leq_{sc} Q \implies P \leq_c Q$ , but none of these implications reverse and there is no logical relationship between  $\leq_W$  and  $\leq_{sc}$ . All of these reducibilities also imply that  $Q$  implies  $P$  in  $\omega$ -models of  $\text{RCA}_0$ , i.e., models with the standard natural numbers. Write  $P \sqsubseteq Q$  if  $P$  and  $Q$  are incomparable under  $\leq_{\square}$ , and  $P \equiv_{\square} Q$  if  $P \leq_{\square} Q$  and  $Q \leq_{\square} P$ , for  $\square \in \{c, sc, W, sW\}$ . See [14, 7] for details on these and other reductions, and [3] for a good recent survey of Weihrauch reducibility with a large bibliography and historical remarks. The reducibilities above can be defined

between problems on any represented spaces, but we do not need this level of generality. More information can again be found in [3].

There are a few algebraic operations on problems which we will have occasion to use. Our formulation mostly follows [3]. Let  $P$  and  $Q$  be problems. Then

- $P \times Q$  is the *parallel product* of  $P$  and  $Q$ , defined by  $\text{dom}(P \times Q) = \text{dom } P \times \text{dom } Q$ , and  $(P \times Q)(p, q) = P(p) \times Q(q)$ .
- $P^n$  is the  $n$ -fold parallel product of  $P$  with itself. The instances can be regarded as including the data  $n$ .
- $P^*$  is the *finite parallelization* of  $P$ , with  $\text{dom}(P^*) = \{ \{n\} \times (\text{dom } P)^n : n \in \mathbb{N} \}$ . The set of solutions to  $\langle n, p_1, \dots, p_n \rangle$  is  $\{n\} \times P(p_1) \times \dots \times P(p_n)$ . So,  $P^*$  is the problem that can solve any specified finite number of instances of  $P$  at once.
- $\hat{P}$  is the *parallelization* of  $P$ . Instances are sequences  $\langle p_0, p_1, p_2, \dots \rangle \in \text{dom}(P)^\mathbb{N}$ , and the set of solutions to this instance is the Cartesian product  $\prod_{i \in \mathbb{N}} P(p_i)$ .
- $P \sqcup Q$  is the *coproduct* of  $P$  and  $Q$ . Here  $\text{dom}(P \sqcup Q) = \text{dom } P \sqcup \text{dom } Q$ , and  $(P \sqcup Q)(0, p) = \{0\} \times P(p)$  and  $(P \sqcup Q)(1, q) = \{1\} \times Q(q)$ . So  $P \sqcup Q$  is the problem which is capable of solving any instance of  $P$  or of  $Q$ , but only one at a time.
- $P'$  is the *jump* of  $P$ . An instance of  $P'$  is a sequence  $\langle p_0, p_1, p_2, \dots \rangle$  of reals converging to some  $p \in \text{dom } P$  in the Baire space topology (equivalently, converging entrywise). The solutions to this instance are just the elements of  $P(p)$ . So  $P'$  answers the same question about  $p \in \text{dom } P$  that  $P$  does, but only has access to a limit representation of  $p$ .
- $P^{(n)}$  is the  $n$ -fold jump of  $P$ , obtained simply by iterating the jump operation  $n$  times.
- $P * Q$  is the *compositional product* of  $P$  and  $Q$ . This can be characterized intuitively as the strongest problem under  $\leq_W$  obtainable as the composition of  $f$  and  $g$ , ranging over all  $f \leq_W P$  and  $g \leq_W Q$ . (A more precise definition of a representative of the Weihrauch degree of  $P * Q$  is given in Section 3 below; see [3].)

The following “benchmark” problems will be used as a basis for comparison with the problems we study:

- LPO, the *limited principle of omniscience*, has instances  $p \in \mathbb{N}^\mathbb{N}$ , and outputs 0 if  $p = 0^\mathbb{N}$  and 1 otherwise. LPO can be thought of as answering a single  $\Sigma_1^0$  question.
- LPO $^{(n)}$ , the  $n$ th jump of LPO, can be thought of as answering a single  $\Sigma_{n+1}^0$  question.
- $\text{lim}$  maps a convergent sequence of reals  $\langle p_0, p_1, p_2, \dots \rangle$  to its limit.  $\text{lim}$  is strongly Weihrauch equivalent to the Turing jump problem, which maps  $p \in \mathbb{N}^\mathbb{N}$  to  $p'$ . It is also strongly Weihrauch equivalent to  $\text{LPO}$ .
- $\mathbb{C}_\mathbb{N}$ , closed choice on the natural numbers, outputs an element of a nonempty set  $A \subseteq \mathbb{N}$  given an enumeration of its complement.
- $\text{TC}_\mathbb{N}$ , the totalization of  $\mathbb{C}_\mathbb{N}$ , is a slight extension of the latter. An instance is an enumeration of the complement of any  $A \subseteq \mathbb{N}$ , and it either outputs an element of  $A$  if  $A \neq \emptyset$ , or any number if  $A = \emptyset$ .

- $\text{RT}_k^n$ , Ramsey's theorem for  $n$ -tuples and  $k$  colors, has instances  $c: [\mathbb{N}]^n \rightarrow k$ , and solutions to  $c$  are (characteristic functions of) infinite  $c$ -homogeneous sets. Here  $[X]^n$  is the set of  $n$ -element subsets of  $X$ , and a set  $X \subseteq \mathbb{N}$  is *homogeneous* for  $c$  if  $X$  is infinite and  $c$  is monochromatic on  $[X]^n$ .
- $\text{SRT}_k^2$ , stable Ramsey's theorem for pairs, is the restriction of  $\text{RT}_k^2$  to *stable colorings*, i.e., colorings  $c$  such that  $\lim_m c\{n, m\}$  exists for all  $n$ .
- $\text{RT}_{\mathbb{N}}^n$  has instances  $c \in \bigcup_{k \in \mathbb{N}} \text{dom } \text{RT}_k^n$ . Solutions to  $c$  are again infinite  $c$ -homogeneous sets. Notice that in this formulation, the number of colors used is not included as part of an instance.  $\text{SRT}_{\mathbb{N}}^2$  is defined similarly.
- For  $k \in \mathbb{N} \cup \{\mathbb{N}\}$ ,  $\text{cRT}_k^n$  is the "color version" of  $\text{RT}_k^n$ , which only outputs the colors of  $\text{RT}_k^n$ -solutions. We have  $\text{RT}_k^1 \equiv_W \text{cRT}_k^1$  for all  $k$ , since the color can be used to compute the set of points of that color and vice versa.
- $\text{RT}_+^n$  is  $\bigsqcup_{k \in \mathbb{N}} \text{RT}_k^n$ , and  $\text{cRT}_+^n$  and  $\text{SRT}_+^2$  are defined similarly.

### 3. UNIFORM COMPUTABLE CATEGORICITY

Recall that a computable structure  $\mathcal{A}$  is uniformly computably categorical if there is a Turing functional which computes an isomorphism from  $\mathcal{B}$  to  $\mathcal{A}$  given the atomic diagram of any presentation  $\mathcal{B}$  of  $\mathcal{A}$ . If  $\mathcal{S}$  is indivisible and uniformly computably categorical, then from the point of view of  $\leq_W$  or  $\leq_c$  we can regard the instances of  $\text{Ind } \mathcal{S}$  as only including the data  $k$  and  $c: \mathbb{N} \rightarrow k$ , since in a reduction, the functionals  $\Delta$  and  $\Psi$  can just build in the translations between any given presentation  $\mathcal{A}$  and some fixed computable presentation of  $\mathcal{S}$ . Instances of  $\text{Ind } \mathcal{S}_k$  and  $\text{Ind } \mathcal{S}_{\mathbb{N}}$  are viewed simply as colorings  $c$  of  $\mathbb{N}$ . This can more formally be stated as

**Proposition 3.1.** *If  $\mathcal{S}$  is uniformly computably categorical and indivisible, then  $\text{Ind } \mathcal{S}_k \equiv_W P_k$  where  $P_k$  is the restriction of  $\text{Ind } \mathcal{S}_k$  to instances of the form  $\langle \mathcal{S}, c \rangle$ . Analogous statements hold for  $\text{Ind } \mathcal{S}$  and  $\text{Ind } \mathcal{S}_{\mathbb{N}}$ .*

This convention is not a priori justified when considering  $\leq_{sW}$  or  $\leq_{sc}$ , because  $\Psi$  could need oracle access to the presentation  $\mathcal{A}$  in order to correctly translate to a solution of  $\langle \mathcal{A}, c, k \rangle$ .

Both of the structures we focus on in this paper are uniformly computably categorical and so we will follow the above convention without further comment. That  $\mathbb{Q}$  is uniformly computably categorical follows from the classical back-and-forth argument, which is effective. To see that  $\mathcal{E}$  is uniformly computably categorical, for any presentation  $\mathcal{A}$  of  $\mathcal{E}$ , decompose any  $n \in A$  as a pair  $\langle x, y \rangle_{\mathcal{A}}$  so that  $n$  is the  $y$ th element of the  $x$ th distinct equivalence class, in order of discovery within the atomic diagram of  $\mathcal{A}$ . Then if also  $m = \langle z, w \rangle_{\mathcal{A}}$ , we have that  $n$  and  $m$  are equivalent iff  $x = z$ , and  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$  is uniformly  $\mathcal{A}$ -computable. (This definition does not uniquely specify  $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ , but one can make some canonical choice.) If we take  $\mathcal{E}$  to be computable and  $\mathcal{A}$  is a given copy of  $\mathcal{E}$ , without loss of generality with  $E = A = \mathbb{N}$ , then the map  $\langle x, y \rangle_{\mathcal{A}} \mapsto \langle x, y \rangle_{\mathcal{E}}$  is a uniformly  $\mathcal{A}$ -computable isomorphism between  $\mathcal{A}$  and  $\mathcal{E}$ .

As an aside, the problems corresponding to uniformly computably categorical indivisible structures seem to have some nice properties. For example, it is not hard to see that the indivisibility and strong indivisibility problems of such a structure must be Weihrauch equivalent. There is also the following easy observation:

**Proposition 3.2.** *If  $\mathcal{S}$  is indivisible and uniformly computably categorical, then  $\text{Ind } \mathcal{S} \equiv_W \text{Ind } \mathcal{S}^*$ .*

*Proof.* It is known that for any problem  $f$  with a computable point in its domain,  $f \equiv_W f \times f$  iff  $f \equiv_W f^*$  (see [3]). If (by assumption)  $\mathcal{S}$  has a computable presentation then  $\text{dom}(\text{Ind } \mathcal{S})$  has a computable point, so it suffices to show in our case that  $\text{Ind } \mathcal{S} \times \text{Ind } \mathcal{S} \leq_W \text{Ind } \mathcal{S}$ . Since  $\mathcal{S}$  is uniformly computably categorical, as noted above, we can assume each instance consists only of a coloring together with a number of colors. Then suppose  $\langle a, j \rangle$  and  $\langle b, k \rangle$  are two instances of  $\text{Ind } \mathcal{S}$  with  $a: \mathbb{N} \rightarrow j$  and  $b: \mathbb{N} \rightarrow k$ . Define  $c: \mathbb{N} \rightarrow 2^{j+1}3^{k+1}$  by  $c(n) = 2^{a(n)+1}3^{b(n)+1}$ . If  $H$  is a solution to  $c$  with color  $i$ , then there are  $i_1 \in j$ ,  $i_2 \in k$  with  $i = 2^{i_1+1}3^{i_2+1}$ , and hence  $a(H) = i_1$  and  $b(H) = i_2$ , so that  $H$  is simultaneously a solution to both  $\langle a, j \rangle$  and  $\langle b, k \rangle$ .  $\square$

Furthermore, we have the following result which generalizes Proposition 62 of [20] from  $\mathbb{N}$  to any indivisible, uniformly computably categorical  $\mathcal{S}$ . Our argument is only a slight adaptation of the original, but we give it in detail for completeness. It will be useful for the proof to use the representative  $f \star g$  of the Weihrauch degree of  $f * g$ , defined by

$$(f \star g)(x, y) = \langle \text{id} \times f \rangle \circ \Phi_x \circ g(y),$$

where  $\text{id}$  is the identity map on  $\mathbb{N}^{\mathbb{N}}$  and  $\Phi$  is a universal functional [3].

**Proposition 3.3.** *For any indivisible  $\mathcal{S}$ , we have  $\text{Ind } \mathcal{S}_{\mathbb{N}} \leq_W \text{Ind } \mathcal{S} * \mathbb{C}_{\mathbb{N}}$ . If  $\mathcal{S}$  is also uniformly computably categorical, then  $\text{Ind } \mathcal{S}_{\mathbb{N}} \equiv_W \text{Ind } \mathcal{S} * \mathbb{C}_{\mathbb{N}}$ .*

*Proof, after [20].* To prove the first statement, let  $\mathcal{S}$  be an arbitrary indivisible structure and let  $\langle \mathcal{A}, c \rangle$  be a given instance of  $\text{Ind } \mathcal{S}_{\mathbb{N}}$ . Build a  $\mathbb{C}_{\mathbb{N}}$ -instance by, whenever a new color is seen in  $c$ , enumerating all numbers less than that color. From a  $\mathbb{C}_{\mathbb{N}}$ -solution  $k$  we get an instance  $\langle \mathcal{A}, c, k \rangle$  of  $\text{Ind } \mathcal{S}$ , and a solution to this instance is also a solution to  $\langle \mathcal{A}, c \rangle$ .

Now we show  $\text{Ind } \mathcal{S} * \mathbb{C}_{\mathbb{N}} \leq_W \text{Ind } \mathcal{S}_{\mathbb{N}}$  holds when  $\mathcal{S}$  is uniformly computably categorical. Let **Bound** be the problem that outputs an upper bound on an enumeration of a finite set. This is Weihrauch equivalent to  $\mathbb{C}_{\mathbb{N}}$  and it will be convenient to show instead that  $\text{Ind } \mathcal{S} \star \text{Bound} \leq_W \text{Ind } \mathcal{S}_{\mathbb{N}}$ . Let  $(x, Y)$  be an instance of  $\text{Ind } \mathcal{S} \star \text{Bound}$ , so  $Y$  is a finite set represented by an (infinite) enumeration. For each  $i \in \mathbb{N}$ , let  $p_i$  be the  $i$ th prime number. Build  $d \in \text{dom}(\text{Ind } \mathcal{S}_{\mathbb{N}})$  as follows: find a number  $i_0$  such that  $\Phi_x(i_0)$  outputs, in addition to a partial coloring  $c_0$  of  $\mathbb{N} = \text{dom } \mathcal{S}$ , an upper bound  $k_0$  on the range of  $c_0$ . This  $i_0$  is an initial guess for an element of  $\text{Bound}(Y)$ . Set  $d(n) = p_{i_0}^{c_0(n)}$  for all  $n$  for which we see  $c_0(n)$  defined by  $\Phi_x(i_0)$ . In general, if  $i_s$  has been found, let  $d(n) = p_{i_s}^{c_s(n)}$  whenever  $c_s(n)$  is defined and  $n$  had not been colored at a previous stage. If numbers are enumerated into  $Y$  so that  $\max Y \geq i_s$ , find an  $i_{s+1} > \max Y$  such that  $\Phi_x(i_{s+1})$  outputs a number  $k_{s+1}$ ; such an  $i_{s+1}$  must exist, so we can hold off on extending  $d$  until  $k_{s+1}$  appears. Once it does, continue with  $d(n) = p_{i_{s+1}}^{c_{s+1}(n)}$  for all  $n$  which had not been previously colored and for which  $c_{s+1}(n)$  is defined.

Eventually this process stabilizes at some  $i_\ell$ , since  $Y$  is finite, and once it does  $\Phi_x(i_\ell)$  must produce a total coloring. Then any solution  $H$  of  $d$  has color  $p_{i_\ell}^a$  for some  $a < k_\ell$ . By rerunning the procedure in the last paragraph, the return functional can recover  $d$ , hence find the color of  $H$ , and from that learn  $i_\ell$  and

output  $\Phi_x(i_\ell)$  to satisfy id. Finally,  $H$  is in fact a solution to this instance, because it can only include points colored after  $i_\ell$  stabilizes and so the fact that  $d$  and  $\Phi_x(i_\ell)$  differ on finitely many other points has no bearing on it.  $\square$

We do not know if any of these implications reverse, i.e., if the conclusions of any of the above propositions imply  $\mathcal{S}$  is uniformly computably categorical.

#### 4. THE RATIONAL NUMBERS

We will identify without comment natural numbers with the rationals they encode via a fixed computable presentation of  $\mathbb{Q}$ . Since  $\mathbb{Q}$  is computably indivisible,  $\text{Ind } \mathbb{Q}$  is computably equivalent to the identity map on Baire space, and so we focus our attention entirely on its uniform content via its Weihrauch degree.

**4.1. The shuffle problems.** The Weihrauch degrees of a family of problems related to  $\text{Ind } \mathbb{Q}$  have recently been an object of study in the work of Pauly, Pradic, and Soldà [20]. In their terminology, if  $c$  is any coloring of  $\mathbb{Q}$ , then an open interval  $I \subseteq \mathbb{Q}$  is a *c-shuffle* if for every color occurring in  $I$ , the set of points of that color is dense in  $I$ . They then investigate several corresponding families of problems:

- **Shuffle** has instances  $(k, c)$ , where  $k \in \mathbb{N}$  and  $c: \mathbb{Q} \rightarrow k$ . If  $I$  is a (code for a) rational interval and  $C \subseteq k$ , then  $(I, C) \in \text{Shuffle}(k, c)$  iff  $I$  is a *c-shuffle* with exactly the colors of  $C$ .
- **iShuffle** is the weakening of **Shuffle** which, for an instance  $(k, c)$ , returns an interval  $I$  such that there exists  $C \subseteq k$  with  $(I, C) \in \text{Shuffle}(k, c)$ .
- **cShuffle** is the weakening of **Shuffle** which, for an instance  $(k, c)$ , returns a  $C \subseteq k$  such that there is an  $I$  with  $(I, C) \in \text{Shuffle}(k, c)$ .
- $(\eta)_{<\infty}^1$ , which had previously been studied in [9] as a reverse-mathematical principle, is a different weakening of **Shuffle** which, for an instance  $(k, c)$ , returns an interval  $I$  and a single color  $n < k$  such that the points of *c*-color  $n$  are dense in  $I$ .

$\text{Shuffle}_k$ ,  $\text{iShuffle}_k$ , and  $\text{cShuffle}_k$  are defined to be the restrictions of the first three problems to instances of the form  $(k, c)$ . One can also define  $\text{Shuffle}_{\mathbb{N}}$ ,  $\text{iShuffle}_{\mathbb{N}}$ , and  $\text{cShuffle}_{\mathbb{N}}$  in a natural way. The authors show that  $\text{cShuffle} \equiv_W (\text{LPO}')^*$ ,  $\text{iShuffle} \equiv_W \text{TC}_{\mathbb{N}}^* \equiv_W (\eta)_{<\infty}^1 \equiv_W \text{i}(\eta)_{<\infty}^1$  where the latter problem is the version of  $(\eta)_{<\infty}^1$  which only returns an interval  $I$ , and  $\text{Shuffle} \equiv_W (\text{LPO}')^* \times \text{TC}_{\mathbb{N}}^*$ . More specifically, they show  $\text{iShuffle}_k \equiv_W \text{TC}_{\mathbb{N}}^{k-1}$ , the  $(k-1)$ -fold parallel product of  $\text{TC}_{\mathbb{N}}$ . (They also show that  $\text{cShuffle}_k \leq_W (\text{LPO}')^{2k-1}$ . The reverse direction of the equivalence between  $\text{cShuffle}$  and  $(\text{LPO}')^*$  is established by showing that  $\text{LPO}' \leq_W \text{cShuffle}$  and  $\text{cShuffle} \equiv_W \text{cShuffle}^*$ . The precise relationship between number of colors and number of parallel instances of  $\text{LPO}'$  is left open.)

**4.2. The Weihrauch degree of  $\text{Ind } \mathbb{Q}$ .** It is clear that  $\text{Ind } \mathbb{Q}_k \leq_W \text{iShuffle}_k$  and  $\text{Ind } \mathbb{Q} \leq_W \text{iShuffle}$ , since for any color  $i$  found in a shuffle, the set of points of that color in the shuffle is isomorphic to  $\mathbb{Q}$ . The main result of this section is that these reducibilities are strict: being able to uniformly find a monochromatic copy of  $\mathbb{Q}$  is not sufficient to uniformly find an interval in which every color appearing is dense, or indeed an interval where even one color is dense. This follows from Theorem 4.1 below, as  $\mathbb{C}_{\mathbb{N}}$  is strictly Weihrauch reducible to  $\text{TC}_{\mathbb{N}} \equiv_W \text{iShuffle}_2$ .

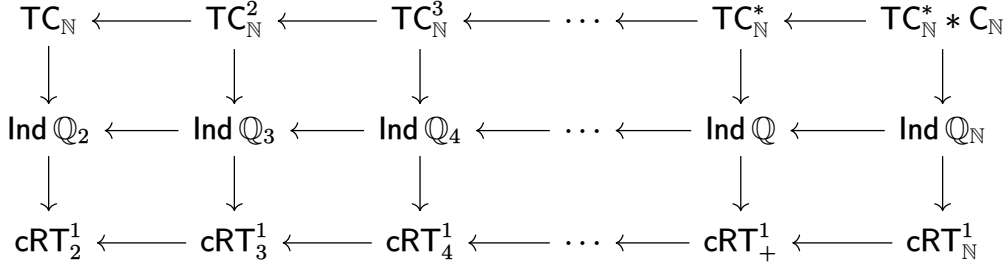


FIGURE 1. Reducibilities between versions of  $\text{cRT}^1$ ,  $\text{Ind } \mathbb{Q}$ , and  $\text{iShuffle} \equiv_W \text{TC}_N^*$ . An arrow from  $Q$  to  $P$  signifies that  $P <_W Q$ . No other reductions hold other than those implied by transitivity.

The relationship between  $\text{Ind } \mathbb{Q}_k$ ,  $\text{TC}_N^k$ , and  $\text{RT}_k^1$  is summarized in Figure 1. Proposition 63 of [20] establishes that  $\text{iShuffle}_N \equiv_W \text{iShuffle} * C_N$ , and we have an analogous characterization of  $\text{Ind } \mathbb{Q}_N$  from Proposition 3.3.

**Theorem 4.1.**  $C_N \not\leq_W \text{Ind } \mathbb{Q}$ .

*Proof.*  $C_N$  is a fractal, so if  $C_N \leq_W \bigsqcup_{i \in \mathbb{N}} f_i$  then  $C_N \leq_W f_i$  for some  $i$  (see [3]). Since  $\text{Ind } \mathbb{Q} = \bigsqcup_k \text{Ind } \mathbb{Q}_k$ , it suffices to show  $C_N \not\leq_W \text{Ind } \mathbb{Q}_k$  for any given  $k$ .

So, fix  $k$  and suppose  $C_N \leq_W \text{Ind } \mathbb{Q}_k$  via  $\Delta$  and  $\Psi$ . Recall that a valid instance of  $C_N$  cannot enumerate every natural number, so that  $f \in \text{dom } C_N$  necessitates  $\text{ran } f \neq \mathbb{N}$ . We regard  $\Psi$  as computing a  $C_N$ -solution  $n$  if  $\Psi^{f \oplus H}(0) \downarrow = n$ , for an instance  $f$  of  $C_N$  and  $\text{Ind } \mathbb{Q}_k$ -solution  $H$  of  $\Delta^f$ . A solution  $n$  is valid only if  $n \notin \text{ran } f$ .

We first give an informal outline of the proof. Let  $\sigma$  and  $R$  be a string and a finite set of red points witnessing that  $\Psi$  converges and outputs a number  $m$ . We want to extend  $\sigma$  to  $\sigma'$  by enumerating  $m$ , so that the latter is no longer a solution to any  $C_N$ -instance  $g \in [\sigma']$ . If there is such a  $g$  such that  $R$  extends to an  $\text{Ind } \mathbb{Q}_k$ -solution of  $\Delta^g$ , the reduction fails at this step. Now, the elements of  $R$  partition  $\mathbb{Q}$  into a finite set of intervals, call them  $R$ -intervals. If there is no  $g$  as above, then for every such  $g$ ,  $\Delta^g$  makes the set of red points scattered in at least one  $R$ -interval. (Recall that a linear order is scattered if it contains no densely ordered subset.)

The particular  $R$ -interval where this happens depends on  $g$ , so we need to account for all of them. Extend  $\sigma'$  to a string  $\tau$  witnessing convergence of  $\Psi$  on a blue set  $B_i$  in as many different  $R$ -intervals as possible. Diagonalize a second time by enumerating all the outputs of  $\Psi$  on all of the  $B_i$ . If  $\tau'$  is the resulting string, then again it might happen that no  $B_i$  extends to a  $\Delta^h$ -solution for any  $h \in [\tau']$ . That means, as before with  $R$ , that for each  $i$ , for each  $h \in [\tau']$ , there is a  $B_i$ -interval in which the set of blue points is scattered for  $\Delta^h$ . In particular, for each  $h$ , there is some  $i$  and some  $B_i$ -interval where both red *and* blue points are scattered for  $\Delta^h$ .

From here the argument recurses into further and further subintervals, resulting in a tree structure. This is illustrated in Figure 2. The endgame of the proof is that if the string  $\alpha$  is reached after having to iterate the procedure  $k - 1$  times, so that each diagonalization so far prevented the set of rationals witnessing convergence of  $\Psi$  from being extendible to a solution, then for every  $f \in [\alpha]$  there is some interval (out of finitely many choices) where the first  $k - 1$  colors are all scattered. Then



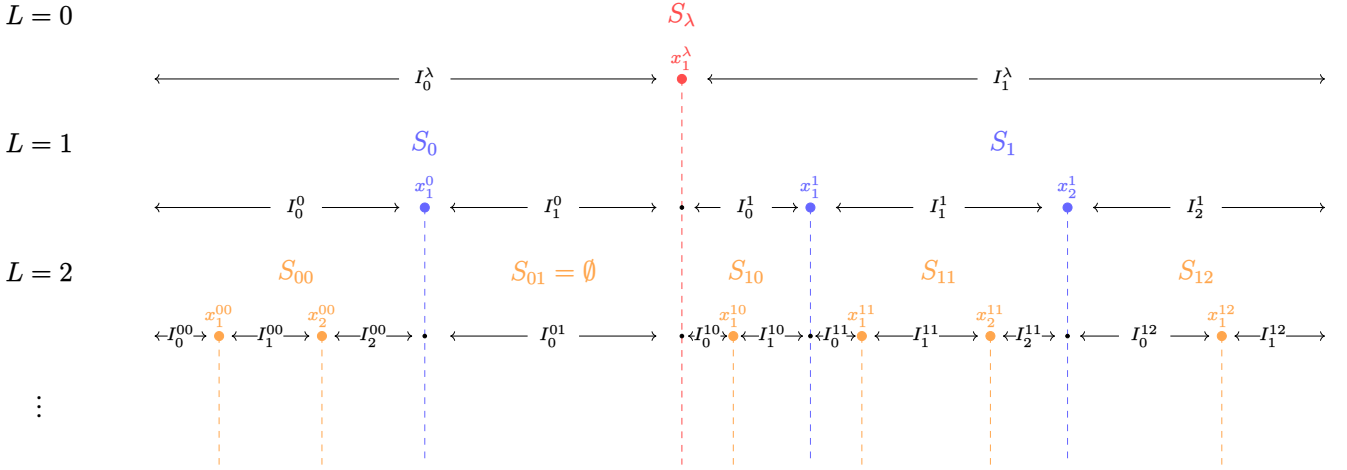


FIGURE 2. An example of how the diagonalization procedure might look, with points vertically displaced to show which are added at each stage  $L$ . The point  $x_1^\lambda$  is carried through to become the largest element of  $X_0$  and  $X_{01}$  and the smallest element of  $X_1$  and  $X_{10}$ .  $x_1^0$ ,  $x_1^1$ , and  $x_2^1$  are carried through in a similar fashion, as are  $x_0^\lambda = -\infty$  and  $x_2^\lambda = +\infty$ . Here no orange subset of  $I_1^0$  could be found witnessing convergence of  $\Psi$  on any extension of  $\sigma_1$ , so we set  $X_{01} = \{x_1^0, x_1^\lambda\} = \{x_1^0, x_2^0\}$ . At the end of the procedure, at least one of  $I_0^\lambda, I_1^\lambda$  will have red points scattered; at least one of  $I_0^0, I_1^0$  and one of  $I_0^1, I_1^1, I_2^1$  will have blue points scattered;  $I_0^{01}$  will have orange points scattered, as will at least one of the  $I_i^{00}$ s, one of the  $I_i^{10}$ s, one of the  $I_i^{11}$ s, and one of the  $I_i^{12}$ s; and so on.

the last remaining color must be dense there, so diagonalizing one final time is guaranteed to succeed in at least one interval.

Now we proceed to the formal proof. Pick any  $f \in \text{dom } \mathbb{C}_\mathbb{N}$  and any solution  $H$  to  $\Delta^f$ . Without loss of generality suppose  $\Delta^f(H) = 0$ . Then  $\Psi^{f \oplus H}(0) \downarrow = m_\lambda$  for some  $m_\lambda$ , where  $\lambda$  denotes the empty string. If  $m_\lambda \in \text{ran } f$ , then the reduction fails already, so assume otherwise. Let  $u$  be the use of this computation, let  $S_\lambda = H \upharpoonright u$  (identifying  $H$  with its characteristic function), let  $\sigma_0 = f \upharpoonright u'$  where  $u' = \max\{u, \text{the use of } \Delta^f(S_\lambda)\}$ , and let  $\sigma'_0 = \sigma_0 \hat{\ } m_\lambda$ . Write  $S_\lambda = \{x_1^\lambda < \dots < x_{\ell(\lambda)}^\lambda\}$ , let  $x_0^\lambda = -\infty$  and  $x_{\ell(\lambda)+1}^\lambda = +\infty$ , and let  $X_\lambda = S_\lambda \cup \{x_0^\lambda, x_{\ell(\lambda)+1}^\lambda\}$ . Write  $I_i^\lambda$  for the interval  $(x_i^\lambda, x_{i+1}^\lambda)$ .

Suppose first there is a  $g \in [\sigma'_0] \cap \text{dom } \mathbb{C}_\mathbb{N}$  such that  $S_\lambda$  is extendible to a solution  $K$  of  $\Delta^g$ . If so, then  $\Psi^{g \oplus K}(0) \downarrow = m_\lambda$  but  $m_\lambda \in \text{ran } g$ , defeating the reduction in this case. Otherwise, if  $S_\lambda$  is not extendible to a solution of  $\Delta^g$  for any such  $g$ , then for each  $g$  there is at least one  $i \geq 0$  such that the set of color-0 points in  $I_i^\lambda$  with respect to  $\Delta^g$  is scattered. A scattered set of rationals is in particular nowhere dense with respect to the Euclidean topology, so if the color-0 points in  $I_i^\lambda$  are scattered for  $\Delta^g$  then the set of points of colors  $1, 2, \dots, k-1$  is dense in that interval for  $\Delta^g$ .

Let  $L > 0$  and suppose  $\sigma'_{L-1}$  has already been found, and the sets  $X_\beta$  and intervals  $I_i^\beta$  have been defined for all  $|\beta| = L-1$  and  $0 \leq i \leq \ell(\beta)$ . We claim that there is a  $\sigma_L \in [\sigma'_{L-1}]$  with the following property: for all  $\rho \in [\sigma_L]$ , if any  $i \geq 0$  and  $\beta$  with  $|\beta| = L-1$  are such that there exists a finite set  $S \subset I_i^\beta$  which has  $\Delta^\rho$ -color  $L$  and such that  $\Psi^{\rho \oplus S}(0) \downarrow$ , then the same is true of  $\sigma_L$  in place of  $\rho$ , for the same  $i$  (and possibly a different  $S$ ). To see why this is so, consider the

function  $a: [\sigma'_{L-1}] \rightarrow \mathbb{N}$  which sends a string  $\tau$  to the number of distinct intervals  $I_i^\beta$ , for  $|\beta| = L - 1$  and  $0 \leq i \leq \ell(\beta)$ , such that there exists  $S_{\beta \smallfrown i} \subset I_i^\beta$  witnessing  $\Delta^\tau(S_{\beta \smallfrown i}) \downarrow = L$  and  $\Psi^{\tau \oplus S_{\beta \smallfrown i}}(0) \downarrow$ . Since there are only finitely many intervals at hand,  $a$  is bounded, and any  $\sigma_L \in [\sigma'_{L-1}]$  witnessing its maximum value has the stated property.

Continue by picking any such  $\sigma_L$ , and fix a choice of  $S_\alpha$  as above for any  $\alpha = \beta \smallfrown i$  such that a set  $S_\alpha$  could be found. Extend  $\sigma_L$  to  $\sigma'_L$  by enumerating the output  $m_\alpha$  of  $\Psi^{\sigma_L \oplus S_\alpha}(0)$  for each such  $\alpha$ . Write  $S_\alpha = \{x_1^\alpha < x_2^\alpha < \dots < x_{\ell(\alpha)}^\alpha\}$ , let  $x_0^\alpha = x_i^\beta$  and  $x_{\ell(\alpha)+1}^\alpha = x_{i+1}^\beta$ , and let  $X_\alpha = S_\alpha \cup \{x_0^\alpha, x_{\ell(\alpha)+1}^\alpha\}$ . If for some  $\alpha = \beta \smallfrown i$  of length  $L$  no  $S_\alpha$  could be found, simply let  $\ell(\alpha) = 0$  and  $X_\alpha = \{x_0^\alpha, x_1^\alpha\} = \{x_i^\beta, x_{i+1}^\beta\}$ . Then let  $I_i^\beta = (x_i^\beta, x_{i+1}^\beta)$  for all  $\beta$  of length  $L - 1$ .

If there is a  $g \in [\sigma'_L] \cap \text{dom } \mathbb{C}_\mathbb{N}$  and an  $\alpha$  of length  $L$  such that  $S_\alpha$  is extendible to a color- $L$  solution of  $\Delta^g$ , then this  $g$  witnesses the failure of the reduction since for some solution  $U \supset S_\alpha$  of  $\Delta^g$ , we have  $\Psi^{g \oplus U}(0) \downarrow = m_\alpha$  while  $m_\alpha \in \text{ran } g$ . Suppose, then, that no such  $g$  exists. This means that for all  $g \in [\sigma'_L] \cap \text{dom } \mathbb{C}_\mathbb{N}$ , for all  $\beta$  with  $|\beta| = L - 1$ , there is an  $i \geq 0$  such that the set of color- $L$  points in  $I_i^\beta$  is scattered with respect to  $\Delta^g$ . If  $L < k - 1$ , then continue inductively as above by finding sets  $S_\xi$  for  $|\xi| = L + 1$  and a string  $\sigma_{L+1} \supset \sigma'_L$ . But if  $L = k - 1$ , then we claim that in fact for any  $g \in [\sigma'_L] \cap \text{dom } \mathbb{C}_\mathbb{N}$ , there must be an  $S_\alpha$  with  $|\alpha| = L$  that is extendible to a color- $(k - 1)$  solution of  $\Delta^g$ . To see this, first recall that by assumption at this stage, for any such  $g$ , in particular there is an  $i_0$  such that color-0 points are scattered in  $I_{i_0}^\lambda$ . In turn, there is an  $i_1$  such that color-1 points are also scattered in  $I_{i_1}^{i_0}$ ; an  $i_2$  such that color-2 points are also scattered in  $I_{i_2}^{i_0 i_1}$ ; and so on, so that there is ultimately an  $i_{k-2}$  such that the points of colors  $0, 1, \dots, k - 2$  are all scattered in the interval  $I_{i_{k-2}}^\beta$ , where  $\beta = i_0 i_1 \dots i_{k-3}$  (or  $\beta = \lambda$  if  $k = 2$ ). It follows that the set of points of color  $k - 1$  is dense in  $I_{i_{k-2}}^\beta$  with respect to  $\Delta^g$ . In particular, the set of color- $(k - 1)$  points in  $I_{i_{k-2}}^\beta$  is a solution of  $\Delta^g$ , so if  $\alpha = i_0 i_1 \dots i_{k-2}$ , then we must have been able to find a set  $S_\alpha$  of color  $k - 1$  as above such that  $\Psi^{\sigma_L \oplus S_\alpha}(0)$  converges. Hence for this  $g$  and  $\alpha$ , there is a solution  $U \supset S_\alpha$  of  $\Delta^g$  with  $\Psi^{g \oplus U}(0) \downarrow = m_\alpha$  and  $m_\alpha \in \text{ran } g$ , defeating the reduction and completing the proof.  $\square$

**Corollary 4.2.** *For all  $k \geq 2$ , the Weihrauch reducibility of  $\text{Ind } \mathbb{Q}_k$  to  $\text{iShuffle}_k$  is strict.*

Since  $\text{iShuffle} \equiv_W \text{i}(\eta)_{<\infty}^1$ , as mentioned earlier, this means that  $\text{Ind } \mathbb{Q}$  is not strong enough to find an interval where even one color is dense.

**Corollary 4.3.**  $\text{cRT}_{k+1}^1 \not\leq_W \text{Ind } \mathbb{Q}_k$  for all  $k \geq 2$ . Since  $\text{RT}_k^1 \leq_W \text{Ind } \mathbb{Q}_k$ , it follows that  $\text{Ind } \mathbb{Q}_{k+1} \not\leq_W \text{Ind } \mathbb{Q}_k$ .

*Proof.* Theorem 10 of [20] states that  $\text{cRT}_{k+1}^1 \leq_W \text{TC}_\mathbb{N}^m$  if and only if  $k \leq m$ . Because  $\text{Ind } \mathbb{Q}_k \leq_W \text{TC}_\mathbb{N}^{k-1}$ , this means in particular that  $\text{Ind } \mathbb{Q}_k \not\leq_W \text{cRT}_{k+1}^1$ .  $\square$

**Corollary 4.4.**  $\text{Ind } \mathbb{Q}_\mathbb{N} \not\leq_W \text{Ind } \mathbb{Q}$ .

*Proof.*  $\mathbb{C}_\mathbb{N} \leq_{sW} \text{RT}_\mathbb{N}^1 \leq_{sW} \text{Ind } \mathbb{Q}_\mathbb{N}$ , but  $\mathbb{C}_\mathbb{N} \not\leq_W \text{Ind } \mathbb{Q}$ .  $\square$

It is clear that  $\text{Ind } \mathbb{Q}_\mathbb{N} \not\leq_W \text{TC}_\mathbb{N}^*$  and  $\text{Ind } \mathbb{Q}_\mathbb{N} \leq_W \text{iShuffle}_\mathbb{N} \equiv_W \text{TC}_\mathbb{N}^* * \mathbb{C}_\mathbb{N}$ . Only one more fact is needed to complete Figure 1:

**Corollary 4.5.**  $\text{TC}_{\mathbb{N}} \not\leq_W \text{Ind } \mathbb{Q}_{\mathbb{N}}$ , and hence  $\text{iShuffle}_{\mathbb{N}} \not\leq_W \text{Ind } \mathbb{Q}_{\mathbb{N}}$ .

*Proof.* It is known that  $\text{TC}_{\mathbb{N}}$  is a closed fractal, see [3] for the definition. Le Roux and Pauly showed in [16] that if  $f$  is a closed fractal and  $f \leq_W g * \mathbb{C}_{\mathbb{N}}$ , then  $f \leq_W g$ . Recalling from Proposition 3.3 that  $\text{Ind } \mathbb{Q}_{\mathbb{N}} \equiv_W \text{Ind } \mathbb{Q} * \mathbb{C}_{\mathbb{N}}$ , it follows that if  $\text{TC}_{\mathbb{N}} \leq_W \text{Ind } \mathbb{Q}_{\mathbb{N}}$  then  $\text{TC}_{\mathbb{N}} \leq_W \text{Ind } \mathbb{Q}$ , contradicting Theorem 4.1.  $\square$

Theorem 4.1 and its corollaries show that  $\text{Ind } \mathbb{Q}$  is uniformly rather weak, but it is not too easy to solve either. In particular, first of all, it is not limit computable. Limit computability of a problem can be characterized as Weihrauch reducibility to  $\text{lim}$ , and we have

**Proposition 4.6.**  $\text{Ind } \mathbb{Q}_2 \not\leq_W \text{lim}$ .

*Proof.* We show that  $\text{Ind } \mathbb{Q}_2 \not\leq_W \widehat{\text{LPO}}$ , which is strongly Weihrauch equivalent to  $\text{lim}$  (see [3]). Suppose  $\text{Ind } \mathbb{Q}_2 \leq_W \widehat{\text{LPO}}$  via the functionals  $\Delta$  and  $\Psi$ . We view the unique solution  $a^p$  of an  $\widehat{\text{LPO}}$ -instance  $p = \langle p_0, p_1, \dots \rangle$  as an infinite binary string, with  $a^p(i) = 1$  iff there is a 1 anywhere in the string  $p_i$ . We will refer to  $p_i$  as “the  $i$ th  $\widehat{\text{LPO}}$ -instance” of  $p$ .

We use lowercase Greek letters for finite partial colorings of  $\mathbb{Q}$ , viewed as finite strings. For a given  $\sigma \in 2^{<\mathbb{N}}$ , let

$$F(\sigma) = \{i : \exists \tau \supset \sigma \text{ such that } \Delta^\tau \text{ outputs a 1} \\ \text{somewhere in its } i\text{th } \widehat{\text{LPO}}\text{-instance}\}.$$

Thus if  $i \in F(\sigma)$  and  $\tau \supset \sigma$  is as above, then for any  $c \in [\tau]$ ,  $a^c(i) = 1$ . If  $i \notin F(\sigma)$ , then for any  $c \in [\sigma]$ ,  $a^c(i) = 0$ . This means that for any finite partial coloring  $\sigma$  and any  $s$ , we can extend  $\sigma$  to some  $\tau$  such that  $a^c \upharpoonright s$  is the same string for all  $c \in [\tau]$ .

So, to defeat the Weihrauch reduction given by  $\Delta$  and  $\Psi$ , first build an instance  $c = \bigcup_s \sigma_s$  of  $\text{Ind } \mathbb{Q}_2$  together with its solution  $a = \bigcup_s a_s$  of  $\Delta^c$  by finite extension, starting with the empty string  $\sigma_0 = \lambda$  and  $a_0 = \lambda$ . At stage  $s + 1$ , let  $i$  be the least element of  $F(\sigma_s)$  greater than or equal to  $s$ , if such an element exists. If it does, let  $\sigma_{s+1}$  be an extension of  $\sigma_s$  corresponding to  $i$  as in the definition of  $F(\sigma_s)$ , i.e., an extension  $\tau$  such that  $\Delta^\tau$  puts a 1 somewhere in its  $i$ th  $\widehat{\text{LPO}}$ -instance. Then let  $a_{s+1} = a_s \hat{\ } 00 \dots 1$ , where the 1 is in bit  $i$ . Otherwise, if no such  $i$  exists, let  $\sigma_{s+1} = \sigma_s \hat{\ } 0$  and  $a_{s+1} = a_s \hat{\ } 0$ . In the latter case, every  $c \in [\sigma_s]$  gives  $\Delta^c$  only finitely many  $\widehat{\text{LPO}}$ -instances having a 1, and accordingly this results in  $a$  having cofinitely many 0s.

Because this procedure builds a valid instance  $c$  of  $\text{Ind } \mathbb{Q}_2$  together with a valid solution  $a$  to  $\Delta^c$ , there must be a stage  $s$  and at least two distinct  $p, q \in \mathbb{Q}$  with  $\Psi^{c \oplus a}(p)[s] \downarrow = \Psi^{c \oplus a}(q)[s] \downarrow = 1$ . That is,  $\Psi$  must eventually put at least two points into a purported solution to  $c$ . If  $c(p) \neq c(q)$ , the reduction already fails at this point, so assume  $p$  and  $q$  were given the same color (we may also assume both had been colored by stage  $s$ ). Then at all future stages, instead of continuing to extend  $\sigma_s$  as in the last paragraph, simply color every remaining rational with color  $1 - c(p)$ . This produces a new coloring  $d$  with  $d \upharpoonright s = c \upharpoonright s$  and such that  $\Delta^d$  has solution  $b$  with  $b \upharpoonright s = a \upharpoonright s$ . Therefore  $\Psi^{d \oplus b}$  still includes  $p$  and  $q$  in its solution to  $d$ . Since only finitely many rationals share the  $d$ -color of  $p$  and  $q$ , the reduction fails to produce a densely ordered monochromatic set.  $\square$

Next, we introduce a class of problems that is incomparable with the class of limit computable problems, but all of which are still weak in a precise sense. A problem is *first-order* if it has codomain  $\mathbb{N}$ .

**Definition 4.7.** We say a problem  $P$  is *pointwise c.e. traceable*, or p.c.e.t. for short, if  $P$  is Weihrauch equivalent to a first-order problem  $Q$  with the property that there is a Turing functional  $\Gamma$  such that for all  $q \in \text{dom } Q$ ,  $\Gamma^q(0) \downarrow = e$  for some index  $e$  (depending on  $q$ ) with  $W_e^q$  finite and  $Q(q) \cap W_e^q \neq \emptyset$ .

Roughly, this means that we can uniformly enumerate a finite list of solutions to any instance  $p$  of  $P$ , possibly along with finitely many non-solutions.

**Theorem 4.8.** *If  $P$  is any pointwise c.e. traceable problem, then  $\text{Ind } \mathbb{Q}_2 \not\leq_W P$ .*

Although this result was found independently in the course of the present work, the core idea of its proof was sketched for a special case in [9, §5] and may have appeared elsewhere. To provide motivation for our argument, we will outline it for the same special case treated in [9], which in effect is  $P = \text{cRT}_2^1$ . Suppose  $\text{Ind } \mathbb{Q}_2 \leq_W \text{cRT}_2^1$  with the reduction being given by functionals  $\Delta$  and  $\Psi$ . For any coloring  $c$  of  $\mathbb{Q}$  we build, there are only two possible solutions to  $\Delta^c$ , namely 0 and 1. As we build  $c$ , at least one of  $\Psi^{c \oplus 0}$  and  $\Psi^{c \oplus 1}$  must begin to output elements of a solution to  $c$ . Eventually, if enough points appear, one can find disjoint intervals  $I$  and  $J$  whose endpoints were output by  $\Psi^{c \oplus 0}$  and  $\Psi^{c \oplus 1}$ , respectively. From this stage on, we color the rest of the points in  $I$  and  $J$  the opposite color as their respective endpoints. When  $\Psi$  outputs any two points  $x$  and  $y$ , it also commits to outputting infinitely many points of the same color in  $(x, y)$ , and the coloring  $c$  makes this impossible both in  $I$  and in  $J$ . Hence both  $\Psi^{c \oplus 0}$  and  $\Psi^{c \oplus 1}$  fail to compute a  $c$ -solution.

It is intuitively at least believable that this idea ought to translate with a little work to any problem  $P$  with a fixed finite number of solutions. To further extend the approach to a general p.c.e.t. problem, one can watch for numbers to appear in  $W_e^p$ , where  $p = \Delta^c \in \text{dom } P$  and  $e$  is the index as in the definition of a p.c.e.t. problem. Only finitely many numbers will ever appear for any coloring  $c$  of  $\mathbb{Q}$ , so we can diagonalize against all its elements at once as above, adjusting the set of disjoint intervals as needed whenever a new element of  $W_e^p$  is enumerated. In the end, at least one number in  $W_e^p$  will be a correct solution to  $p$ , so we can be sure of having legitimately diagonalized against at least one solution.

*Proof of Theorem 4.8.* Suppose  $\text{Ind } \mathbb{Q}_2 \leq_W P$  via  $\Delta$  and  $\Psi$ . Without loss of generality, we can assume  $P$  is first-order. We build a coloring  $c = \lim_s c_s$  of  $\mathbb{Q}$  by finite extension for which this purported reduction fails to produce a solution to  $c$ . Begin with  $c(0) = 0$ .

If  $\Gamma$  is a functional as in the definition of a p.c.e.t. problem, let  $N(s) = W_x^{\Delta(c)}[s]$  if  $\Gamma^{\Delta(c)}(0)[s] \downarrow = x$ , or  $\emptyset$  if this computation diverges. Since  $\Psi$  only has oracle access to  $c$  together with a single number from  $N(s)$ , we can view  $\Psi$  as a finite set of functionals  $\Psi_i$ , one per  $i \in N(s)$ , each only having oracle access to  $c$ . Then we can diagonalize against all such  $\Psi_i$  without regard to the eventual correctness of any particular solution  $i$ . This is enough to defeat the reduction, because we only need to show  $\Psi_i^c$  fails to compute a  $c$ -solution for one  $\Delta^c$ -solution  $i$ .

Define a recurrence relation  $e(n)$  by letting  $e(1) = 3$  and  $e(n+1) = 2ne(n) + 2$  for  $n \geq 1$ . We claim that if any functionals  $\Phi_0, \dots, \Phi_{k-1}$  each enumerate at least  $e(k)$

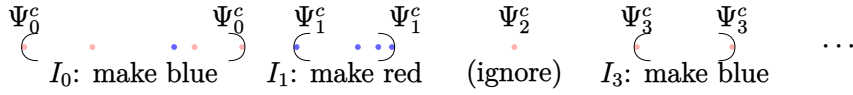


FIGURE 3. An example of how the construction might work in diagonalizing against some functionals  $\Psi_0, \Psi_1, \dots$ . The points shown have already been colored red or blue by  $c$  at this stage, and enumerated by  $\Psi_i^c$  for some  $i$ .  $\Psi_0^c$ ,  $\Psi_1^c$ , and  $\Psi_3^c$  have enumerated enough points so that we can find three disjoint intervals  $I_0, I_1, I_3$  whose endpoints are respectively among the domains of  $\Psi_0^c, \Psi_1^c, \Psi_3^c$  up to this point. Since the endpoints of  $I_0$  are red, we plan to make all future points in  $I_0$  blue, and similarly for  $I_1$  and  $I_3$ . At this stage,  $\Psi_2^c$  has not enumerated enough points to find an  $I_2$  disjoint from  $I_0 \cup I_1 \cup I_3$ , so we ignore it for now.

points by stage  $s$ , then one can choose at stage  $s$  a collection of  $k$  disjoint rational intervals  $I_0, \dots, I_{k-1}$  such that for each  $i$ , the endpoints of  $I_i$  were enumerated by  $\Phi_i$ . The proof is by induction on  $k$ . The case  $k = 1$  is immediate, so suppose the claim holds for all  $i \leq k$  and that we have  $k + 1 > 1$  functionals  $\Phi_0, \dots, \Phi_{k-1}, \Phi_k$  each having enumerated at least  $e(k + 1)$  points. Now, for purposes of the inductive step, we are only considering the first  $k$  functionals  $\Phi_0, \dots, \Phi_{k-1}$  to have enumerated  $2e(k)$  points each; any extra points can be discarded. Then the  $2ke(k)$  total points enumerated by  $\Phi_0, \dots, \Phi_{k-1}$  divide  $\mathbb{Q}$  into  $2ke(k) + 1$  open intervals. If  $\Phi_k$  enumerates at least  $2ke(k) + 2$  points, by the pigeonhole principle, at least one of those  $2ke(k) + 1$  open intervals will contain two such points  $a_k, b_k \in \mathbb{Q}$ . We then choose  $I_k = (a_k, b_k)$ . This splits the set of points enumerated by the  $\Phi_i$ ,  $i < k$ , into two groups, those below  $a_k$  and those above  $b_k$ . Using the pigeonhole principle again, for each  $i < k$ , one of those groups contains at least  $2e(k)/2 = e(k)$  points enumerated by  $\Phi_i$ . The inductive hypothesis can now be applied to the set of (at most  $k$ ) functionals with at least  $e(k)$  points enumerated below  $a_k$ , and separately to the set of functionals with at least  $e(k)$  points enumerated above  $b_k$ , to produce a set of disjoint rational intervals as required.

Now we proceed to the main argument, which is essentially by finite injury (though without a priority order). Let  $A(s)$  be the set of  $i \in N(s)$  such that  $\Psi_i^c[s]$  has output at least one point. Let  $V(s)$  be the set of  $i \in A(s)$  such that  $\Psi_i^c[s]$  has output at least  $e(|A(s)|)$  points. At stage  $s + 1$ , for each  $i \in V(s)$ , if  $\Psi_i$  does not have a follower, assign it a follower which is a (code for a) rational interval  $I_i$  as furnished by the claim in the previous paragraph, applied to the functionals  $\{\Psi_i : i \in V(s)\}$ . Thus the endpoints of  $I_i$  were enumerated by  $\Psi_i^c$  by stage  $s$ , and  $I_i \cap I_j = \emptyset$  if  $i \neq j \in V(s)$ . If there is a  $j \in A(s + 1) \setminus A(s)$ , cancel the followers of all  $\Psi_i$  which have one at stage  $s + 1$ . Now color  $c_{s+1}(s + 1) = 1 - c_s(a_i)$  if  $a_i$  is an endpoint of  $I_i$  for some  $i \in V(s)$ , and color  $c_{s+1}(s + 1) = 0$  otherwise. (See Figure 3 for an example of how this might look at a particular stage.)

Because  $A(s)$  eventually stabilizes, each  $\Psi_i$  will have a follower canceled at most finitely many times, and so the set of intervals  $I_i$  also stabilizes. Any  $\Psi_i$  which outputs infinitely many points will have  $i \in V(s)$  for large enough  $s$ , so for each such  $i$ , there are two points output by  $\Psi_i^c$  between which only finitely many are of the same color. Therefore  $\Psi_i^c$  does not compute a solution to  $c$ .  $\square$

Note that the construction above can be done computably.

**Corollary 4.9.** (i)  $\text{Ind } \mathbb{Q}_2 \not\leq_W \text{cRT}_{\mathbb{N}}^n$  and hence  $\not\leq_W \mathbb{C}_{\mathbb{N}}$ .  
(ii) For all  $n \geq 0$ ,  $\text{Ind } \mathbb{Q}_2 \not\leq_W (\text{LPO}^{(n)})^*$ .

(iii)  $\text{Ind } \mathbb{Q}_2 \not\leq_W \text{cShuffle}$ .

*Proof.* (i)  $\text{cRT}_{\mathbb{N}}^n$  is p.c.e.t. since the set of solutions to  $c$  is contained in  $\text{ran } c$ , and  $\mathbb{C}_{\mathbb{N}} \leq_W \text{cRT}_{\mathbb{N}}^1$ . (ii) If there are  $k$  instances of  $\text{LPO}^{(n)}$  given in parallel, at most  $2^k$  distinct solutions are possible, and they can each be encoded as a single natural number. So in particular  $(\text{LPO}^{(n)})^*$  is p.c.e.t. (iii) This follows from (ii) as  $\text{cShuffle} \equiv_W (\text{LPO}')^*$  [20].  $\square$

Note that the finite parallelization here cannot be replaced with the infinite parallelization, because  $\text{Ind } \mathbb{Q}_2 \leq_W \text{TC}_{\mathbb{N}} \leq_W \lim' \equiv_W \widehat{\text{LPO}'} \equiv_W \widehat{\text{LPO}'}.$

Before continuing to the next section, we pause to make some observations about p.c.e.t. problems. Pointwise c.e. traceability can be viewed as a generalization of c.e. traceability, which is the special case where  $P$  is a single-valued function from  $\mathbb{N}$  to  $\mathbb{N}$  (see for instance [22, §11.4] for the definition). It is also related to the notion of a pointwise finite problem, defined in [11], which is a problem such that every instance has finitely many solutions. However, a p.c.e.t. problem is not necessarily pointwise finite, one counterexample being  $\mathbb{C}_{\mathbb{N}}$ . And the existence of functions  $\mathbb{N} \rightarrow \mathbb{N}$  which are not c.e. traceable shows that pointwise finite problems  $\mathbb{N}$  may not be p.c.e.t., even when restricted to those that are first-order.

The class of p.c.e.t. problems has some attractive algebraic properties:

- If  $P$  and  $Q$  are p.c.e.t., then so are  $P \times Q$  and  $P^*$ .
- If  $P$  is p.c.e.t. and  $Q \leq_W P$ , then  $Q$  is p.c.e.t.

(These properties hold for problems  $P$  and  $Q$  on any represented spaces, as is straightforward to show.) On the other hand,

- $P * Q$  may not be p.c.e.t. even if both  $P$  and  $Q$  are. For example,  $\mathbb{C}_{\mathbb{N}}$  and  $\text{LPO}'$  are both p.c.e.t. but  $\mathbb{C}_{\mathbb{N}} * \text{LPO}' \geq_W \text{TC}_{\mathbb{N}} \geq_W \text{Ind } \mathbb{Q}_2$  [2, Corollary 8.10].
- Neither  $P^{u*}$  nor  $P^\diamond$  need be p.c.e.t. if  $P$  is. It follows from [23, Theorem 7.2] that  $(\text{LPO}')^{u*} \equiv_W (\text{LPO}')^\diamond \equiv_W \mathbb{C}'_{\mathbb{N}}$ , and it is known that  $\mathbb{C}'_{\mathbb{N}} >_W \text{TC}_{\mathbb{N}}$  [2]. (See [23] for definitions of the undefined notation here.)

Just as the fact that  $\text{LPO}'$  and  $\lim$  are Weihrauch incomparable (which is known in the literature, see e.g. [2]) shows that pointwise c.e. traceability is logically incomparable with limit computability, the fact that  $\text{LPO}'$  is p.c.e.t. shows it to be strictly more general than computability with finitely many mind changes: the latter can be characterized by Weihrauch reducibility to  $\mathbb{C}_{\mathbb{N}}$ , which is p.c.e.t. and known to be Weihrauch incomparable with  $\text{LPO}'$ .

## 5. THE COUNTABLE EQUIVALENCE RELATION

**5.1. The Weihrauch degree of  $\text{Ind } \mathcal{E}$ .** The countable equivalence relation  $\mathcal{E}$  was defined in the introduction. We identify  $E = \text{dom } \mathcal{E}$  with  $\mathbb{N} \times \mathbb{N}$ , viewing  $(x, y)$  as the  $y$ th element of the  $x$ th equivalence class. We also refer to the  $x$ th equivalence class as the “ $x$ th column” of  $\mathcal{E}$ .

The proof of this theorem, in particular Lemma 5.7, was obtained jointly with Linda Westrick.

**Theorem 5.1.** *For all  $k \geq 2$ ,  $\text{SRT}_k^2 \leq_W \text{Ind } \mathcal{E}_k \leq_W \text{RT}_k^2$ , but  $\text{RT}_2^2 \not\leq_c \text{Ind } \mathcal{E}_{\mathbb{N}}$  and  $\text{Ind } \mathcal{E}_2 \not\leq_W \text{SRT}_{\mathbb{N}}^2$ .*

Since  $\text{SRT}_j^2 \not\leq_c \text{RT}_k^2$  whenever  $j > k \geq 2$  [14, 19], we obtain Figure 4, a very similar diagram to that shown for  $\text{Ind } \mathbb{Q}$  in Figure 1.

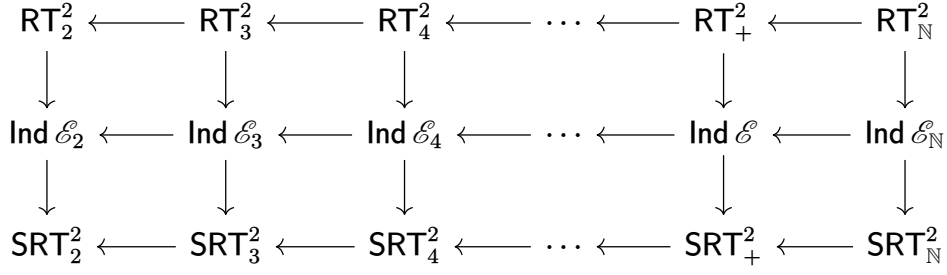


FIGURE 4. Reducibilities between  $\text{SRT}_k^2$ ,  $\text{Ind } \mathcal{E}_k$ , and  $\text{RT}_k^2$ . An arrow from  $Q$  to  $P$  signifies that  $P <_W Q$ . No other Weihrauch reductions hold other than those implied by transitivity. The diagram remains true if  $<_W$  is replaced by  $<_c$ , except that the reductions  $\text{SRT}_k^2 \leq_c \text{Ind } \mathcal{E}_k$  are not known to be strict.

$\text{RT}_2^2$  and  $\text{SRT}_2^2$  have been of great interest in reverse mathematics over a period of decades, and their separation in  $\omega$ -models of  $\text{RCA}_0$  was a major open problem until recently [17]. In 2001, Cholak, Jockusch, and Slaman [5] proved that  $\text{RT}_2^2$  can be logically decomposed over  $\text{RCA}_0$  into the conjunction of  $\text{SRT}_2^2$  with  $\text{COH}$ , where  $\text{COH}$  is the cohesive principle:

**Definition 5.2.** The *cohesive principle*  $\text{COH}$  states that for every sequence of sets of natural numbers  $(R_n)$ , there is an infinite set  $C$  such that for each  $n$ ,  $C$  is up to finite error a subset of either  $R_n$  or its complement.

It is known that there are no implications between  $\text{SRT}_2^2$  and  $\text{COH}$  over  $\text{RCA}_0$  or even in  $\omega$ -models of  $\text{RCA}_0$  [12, 17]. In terms of Weihrauch degrees, it is true that  $\text{COH} <_W \text{RT}_2^2$ ,  $\text{SRT}_2^2 <_W \text{RT}_2^2$ , and  $\text{COH} \upharpoonright_W \text{SRT}_2^2$ . However, an exact characterization has not yet been found of the Weihrauch degree of  $\text{RT}_2^2$  in terms of those of  $\text{SRT}_2^2$  and  $\text{COH}$ . For example, it is known that  $\text{RT}_2^2 \upharpoonright_W \text{SRT}_2^2 \times \text{COH}$  and  $\text{RT}_2^2 <_W \text{SRT}_2^2 * \text{COH}$  [6]. Against this backdrop, then, it is natural to wonder what relationship  $\text{Ind } \mathcal{E}_2$  has to  $\text{COH}$ , if any. We will briefly elaborate on this question in Section 6.

**5.2. Proof of Theorem 5.1.** We use the notation  $\{x, y\}$  for unordered pairs, and omit the extra parentheses from  $c(\{x, y\})$  and  $c((x, y))$  to reduce visual clutter.

The proof is broken into four lemmas. The first two establish that  $\text{SRT}_k^2 \leq_W \text{Ind } \mathcal{E}_k \leq_W \text{RT}_k^2$ , the third shows  $\text{RT}_2^2 \not\leq_c \text{Ind } \mathcal{E}_\mathbb{N}$  and thus gives a separation between  $\text{Ind } \mathcal{E}_k$  and  $\text{RT}_k^2$ , and the fourth shows  $\text{Ind } \mathcal{E}_2 \not\leq_W \text{SRT}_\mathbb{N}^2$  and so separates  $\text{Ind } \mathcal{E}_k$  from  $\text{SRT}_k^2$ .

**Lemma 5.3.**  $\text{Ind } \mathcal{E}_k \leq_W \text{RT}_k^2$ .

*Proof.* Fix  $k$  and let  $c: \mathbb{N} \rightarrow k$  be a coloring of  $E$ . Define an  $\text{RT}_k^2$ -instance  $d: [\mathbb{N}]^2 \rightarrow k$  by

$$d\{x, y\} = c(x, y) \quad \text{if } x < y.$$

Let  $\tilde{H}$  be an infinite homogeneous set for  $d$ , and let  $H$  be the  $\tilde{H}$ -computable set

$$H = \{ (x, y) : x \leq y \in \tilde{H} \} \subseteq \mathbb{N}^2.$$

It is clear that  $H$  is monochromatic for  $c$ . For each  $x \in \tilde{H}$ ,  $(x, y) \in H$  for all of the infinitely many  $y > x$  in  $\tilde{H}$ , so  $H \simeq E$ .  $\square$

**Lemma 5.4.**  $\text{SRT}_k^2 \leq_w \text{Ind } \mathcal{E}_k$ .

*Proof.* Let  $c: [\mathbb{N}]^2 \rightarrow k$  be an instance of  $\text{SRT}_k^2$ . Define  $d: E \rightarrow k$  by

$$d(x, y) = \begin{cases} c\{x, y\}, & \text{if } x < y \\ 0, & \text{otherwise.} \end{cases}$$

Let  $H$  be an  $\text{Ind } \mathcal{E}_k$ -solution for  $d$ . We build an  $\text{SRT}_k^2$ -solution for  $c$  as follows: write the least element of  $H$  as  $(x_1, y)$ . Then find another column  $x_2 > x_1$  represented in  $H$  such that  $c\{x_1, x_2\} = c\{x_1, y\}$ . There is such an  $x_2$  by stability of the coloring  $c$ : the  $d$ -color of  $H$  must be the same as the stable  $c$ -color of column  $x_1$ , so for all large enough  $x$ ,  $c\{x_1, x\} = c\{x_1, y\} = d(x_1, y)$ . Since there are infinitely many equivalence classes of  $E$  represented in  $H$ , and the equivalence class of  $x_1$  has infinitely many points of this color, there is eventually a column  $n$  represented in  $H$  with  $d(x_1, x) = d(x_1, y)$ , and we can take  $x_2$  to be this  $x$ . (Note  $(x_1, x_2)$  need not be in  $H$ .) Next, find a column  $x_3 > x_2$  represented in  $H$  such that

$$c\{x_1, x_2\} = c\{x_1, x_3\} = c\{x_2, x_3\}.$$

Again, this must be possible by stability and the existence of infinitely many different columns represented in  $H$ . Proceed in the same way, for each  $i$  finding a column  $x_i$  of  $H$  with  $x_i > x_{i-1}$  such that  $c\{x_j, x_k\} = c\{x_1, x_2\}$  for all  $1 \leq j < k \leq i$ . The set  $\{x_i : i \in \mathbb{N}\}$  is then an infinite homogeneous set for  $c$ .  $\square$

**Corollary 5.5.** For all  $k \geq 2$ ,  $\text{Ind } \mathcal{E}_{k+1} \not\leq_c \text{Ind } \mathcal{E}_k$  and  $\text{Ind } \mathcal{E} \not\leq_c \text{Ind } \mathcal{E}_k$ . Also,  $\text{Ind } \mathcal{E}_{\mathbb{N}} \not\leq_c \text{Ind } \mathcal{E}$ .

*Proof.* All three statements follow from the above lemmas together with the fact that  $\text{SRT}_{k+1}^2 \not\leq_c \text{RT}_k^2$  [19, Corollary 3.6].  $\square$

**Lemma 5.6.**  $\text{RT}_2^2 \not\leq_c \text{Ind } \mathcal{E}_{\mathbb{N}}$ .

*Proof.* We show that  $\text{Ind } \mathcal{E}_{\mathbb{N}}$  admits  $\Delta_2^0$  solutions, that is, every computable instance has a  $\Delta_2^0$  solution. Since  $\text{RT}_2^2$  does not share this property [15, Corollary 3.2], the separation follows.

Let  $c$  be a computable coloring of  $E$  with  $\text{ran } c \subseteq k$  for some  $k$ . Suppose  $c$  stabilizes in infinitely many columns. Nonuniformly pick a color  $i$  which is the stable color of infinitely many columns, and use  $\emptyset'$  to enumerate the set  $S$  of such columns by asking, for each  $x, y$ , if there is  $z > y$  with  $c(x, z) \neq i$ , and putting  $x \in S$  if the answer is ‘no’. As a relativization of the fact that every infinite c.e. set has an infinite computable subset, there is an infinite  $\Delta_2^0$  subset of  $S$ , and the set of points of color  $i$  in all such columns is a  $\Delta_2^0$  solution to  $c$ .

If instead  $c$  stabilizes in only finitely many columns, nonuniformly delete those columns; the remaining columns each have infinitely many points of at least two different colors. Suppose there are colors  $i_1 \neq i_2$  such that infinitely many columns include only finitely many points of colors other than  $i_1$  or  $i_2$ . The set  $S$  of such columns can again be enumerated by  $\emptyset'$  by asking, for each  $x, y$ , if there is  $z > y$  with  $c(x, z) \notin \{i_1, i_2\}$ , and putting  $x \in S$  if the answer is ‘no’.  $S$  has an infinite  $\Delta_2^0$  subset, and the set of points of color  $i_1$  (or of  $i_2$ ) in the columns in that subset is a  $\Delta_2^0$  solution to  $c$  because, by assumption, all columns of  $S$  have infinitely many points both of color  $i_1$  and of color  $i_2$ .

In general, let  $n \geq 1$  be the least possible number of distinct colors  $i_1, \dots, i_n$  such that there is an infinite set  $S$  of columns each of which includes only finitely many



points of colors other than  $i_1, \dots, i_n$ . That is,  $n$  is such that only finitely many columns have infinitely many points of  $n-1$  or fewer different colors. Nonuniformly delete these columns, so that every column has infinitely many points of each of at least  $n$  colors, and fix a particular choice of  $i_1, \dots, i_n$  as above. Then for this set of colors  $\{i_1, \dots, i_n\}$ ,  $\emptyset'$  can again enumerate  $S$  by letting  $x \in S$  if we find a  $y$  such that there is no  $z > y$  having  $c(x, z) \notin \{i_1, \dots, i_n\}$ . There is an infinite  $\Delta_2^0$  subset of  $S$ , and the set of points of color  $i_j$  in the columns of this subset, for any  $1 \leq j \leq n$ , is a  $\Delta_2^0$  solution of  $c$ .  $\square$

**Lemma 5.7.**  $\text{Ind } \mathcal{E}_2 \not\leq_W \text{SRT}_{\mathbb{N}}^2$ .

*Proof.* Suppose  $\text{Ind } \mathcal{E}_2 \leq_W \text{SRT}_{\mathbb{N}}^2$  via the functionals  $\Delta$  and  $\Psi$ . We use lowercase Latin letters for instances of  $\text{Ind } \mathcal{E}_2$  and lowercase Greek letters for their initial segments.

We will find (noncomputably) a finite sequence of strings  $\sigma_0 \subset \sigma_1 \subset \dots$  such that for some  $s$ , there is an  $\text{Ind } \mathcal{E}_2$ -instance  $c \in [\sigma_s]$  which defeats the Weihrauch reduction. To accomplish this, we build certain monochromatic sets  $H^i = \lim_s h_s^i$  of  $\text{SRT}_{\mathbb{N}}^2$ -columns by finite extension, for each color  $i$  used by  $\Delta^c$ , to use as oracles for  $\Psi$ . Then we diagonalize against  $\Delta$  and  $\Psi$  by adding, along the way, finitely many locks on the columns of  $E$  (to be explained below), represented by a lock function  $L = \bigcup_s L_s: \subseteq \mathbb{N} \rightarrow 2$  that will be updated finitely many times. These locks are in fact the same as those used in Cohen forcing with locks, but we do not formally use a notion of forcing here and the argument does not necessarily produce a generic  $c$ . The overall strategy, roughly speaking, is to pick the sets  $h_s^i$  in such a way that for each  $s$ , either (a) for some  $c \in [\sigma_s]$  respecting the current lock function,  $h_s^i$  is guaranteed to extend to a  $\Delta^c$ -solution of color  $i$ ; or (b) there are no  $\Delta^c$ -solutions of color  $i$  for any such  $c$ . We will see that (a) must eventually occur for some  $i$ , at which stage the procedure ends.

For any string  $\sigma$  and lock function  $L$ , let  $[\sigma, L]$  be the set of (finite or infinite)  $\tau \in [\sigma]$  such that whenever  $L(x) \in \{0, 1\}$  is defined, if  $(x, y)$  is newly colored by  $\tau$ , then  $\tau(x, y) = L(x)$ . “Newly colored” simply means that  $\tau$  is defined on  $(x, y)$  while  $\sigma$  is not. So  $[\sigma, L]$  is the set of extensions of  $\sigma$  which respect the lock function  $L$  by only adding points of color  $L(x)$  in column  $x$ . Observe that  $[\sigma, L'] \subset [\sigma, L]$  if  $L' \supset L$ .

The central claim that makes this proof work is as follows: for any  $\sigma$ , any  $\text{SRT}_{\mathbb{N}}^2$ -column  $n$ , and any lock function  $L$ ,

$$\begin{aligned} \exists \tau \in [\sigma, L] \exists i \in \mathbb{N} \forall \rho \in [\tau, L] \text{ (if } \Delta^\rho \text{ newly colors } \{n, m\} \text{ where } m \geq n, \\ \text{then } \Delta^\rho \{n, m\} = i). \end{aligned}$$

We say  $\tau$   $L$ -forces column  $n$  to be color  $i$  when this happens. If the claim were false for some  $\sigma$ ,  $n$ , and  $L$ , then for every  $\tau \in [\sigma, L]$  and  $i \in k$ , there would be a further  $\rho(\tau) \in [\tau, L]$  such that  $\Delta^{\rho(\tau)}$  adds a point not of color  $i$  into column  $n$ . Letting  $\tau_0 = \sigma$  and  $\tau_{s+1} = \rho(\tau_s)$  for all  $s$ , the  $\text{Ind } \mathcal{E}_2$ -instance  $d = \lim_s \tau_s$  is such that  $\Delta^d$  has infinitely many points of at least two different colors in column  $n$ , so is not an  $\text{SRT}_{\mathbb{N}}^2$ -instance, a contradiction. A key point is that this argument works for any  $L$  independently of  $\sigma$ , so that even if we change  $L$  partway through the proof we can always extend any  $\sigma_s$  to a  $\sigma_{s+1}$  which  $L$ -forces a fresh column to be some color. Also, a column already  $L$ -forced remains  $L'$ -forced to be the same color for any  $L' \supset L$ .

For each  $\sigma$ , lock function  $L$ , and  $i \in \mathbb{N}$ , let

$$C_i(\sigma, L) = \{n : \sigma \text{ } L\text{-forces } \text{SRT}_{\mathbb{N}}^2\text{-column } n \text{ to be color } i\}.$$

Now we describe the procedure to find  $(\sigma_s)$ ,  $(h_s^i)$ , and  $(L_s)$  in detail. First of all, we claim there is a  $\sigma_0$  such that  $\Delta^c([\mathbb{N}]^2) = \Delta^{\sigma_0}([\mathbb{N}]^2)$  for all  $c \in [\sigma_0]$ . In other words,  $\sigma_0$  already makes  $\Delta$  use all the colors it will ever use. To see this, start with  $\tau_0$  such that  $\Delta^{\tau_0}\{x_0, y_0\} \downarrow = i_0$  for some  $x_0, y_0$  and for some color  $i_0 \in \mathbb{N}$ . Extend  $\tau_s$  to  $\tau_{s+1}$  if there are  $x_{s+1}, y_{s+1}$ , and  $i_{s+1} \notin \{i_0, \dots, i_s\}$  with  $\Delta^{\tau_{s+1}}\{x_{s+1}, y_{s+1}\} \downarrow = i_{s+1}$ . Since any instance of  $\text{SRT}_{\mathbb{N}}^2$  only uses finitely many colors, the sequence  $(\tau_s)$  stabilizes, and  $\sigma_0 = \lim_s \tau_s$  has the stated property.

Fix  $\sigma_0$  as above and, without loss of generality, let  $k$  be such that the range of  $\Delta^{\sigma_0}$  is  $k = \{0, \dots, k-1\}$ . Let  $D \subseteq k$  be the set of “already-diagonalized” colors, starting with  $D = \emptyset$ ; we will use  $D$  to keep track of which colors  $i$  have already witnessed convergence of  $\Psi$  on  $H^i$ . As long as  $D \subsetneq k$ , the colors in  $D$  at stage  $s$  will be exactly those of which no  $\Delta^c$ -solution can be found for any  $c \in [\sigma_s, L_s]$ . For any  $\sigma$  and  $L$ , let  $b(\sigma, L)$  be the largest possible number of distinct colors found among the columns freshly  $L$ -forced by any  $\tau \in [\sigma, L]$ . More specifically, let

$$b(\sigma, L) = \max_{\tau \in [\sigma, L]} \#\{i \in k \setminus D : \max C_i(\tau, L) > \max C_i(\sigma, L)\}.$$

Let  $M(\sigma, L)$  be the set of  $\tau \in [\sigma, L]$  witnessing this maximum. As long as  $D \neq k$ , we have  $b(\sigma, L) > 0$  and thus  $M(\sigma, L) \neq \emptyset$  for all  $\sigma$  and  $L$ .

Start with  $\sigma_0$  as above,  $L_0 = D = \emptyset$ , and  $h_0^i = \emptyset$  for each  $i \in k$ . At stage  $s > 0$ , we use  $L_{s-1}$  to obtain  $\sigma_s$ , use  $\sigma_s$  to obtain  $h_s^i$ , and use  $h_s^i$  to obtain  $L_s$ . Specifically, first of all, let  $\sigma_s$  be any element of  $M(\sigma_{s-1}, L_{s-1})$ . Thus  $\sigma_s$   $L_{s-1}$ -forces a fresh column to stabilize to a color  $i$  for as many different  $i$  as possible, excluding colors in  $D$ .

Next, write  $m_s^i = \max C_i(\sigma_s, L_s)$ . (Notice that we may have  $m_\ell^i = m_{\ell+1}^i$  for some values of  $\ell$ , if  $\sigma_{\ell+1}$  did not  $L_\ell$ -force any fresh column to be color  $i$ .) Then let  $h_s^i = h_{s-1}^i \cup \{m_n^i\}$ , where  $n$  is least such that

- $\max h_{s-1}^i < m_n^i \leq m_s^i$ , and
- for all  $m \in h_{s-1}^i$ , we have  $\Delta^{\sigma_s}\{m, m_n^i\} \downarrow = i$  with  $m_n^i$  greater than the largest element of column  $m$  which  $\Delta^{\sigma_s}$  makes a color other than  $i$ .

By convention we take  $\max \emptyset = -1$ . If no such  $n$  exists, take  $h_s^i = h_{s-1}^i$ . The set  $h_s^i$  is a valid initial segment of some solution to  $\Delta^c$  for *any*  $c \in [\sigma_s, L_{s-1}]$  for which infinitely many  $\text{SRT}_{\mathbb{N}}^2$ -columns stabilize to  $i$ . This is because  $[h_s^i]^2$  is  $\Delta^c$ -monochromatic by design, and starting from any  $\Delta^c$ -solution  $G$ , one can truncate  $G$  to  $\hat{G}$  by selecting only the columns whose indices are higher than the point at which any of the columns in  $h_s^i$  stabilize, producing an infinite  $\hat{G}$  with  $[h_s^i \cup \hat{G}]^2$  monochromatic. Hence either  $h_s^i$  extends to a solution of color  $i$  (for some  $c$  as above), or there is no solution of color  $i$  (for any  $c$  as above).

To end stage  $s$ , for each  $i \notin D$ , search for  $x, y$  with  $x \notin \text{dom } L_{s-1}$  (i.e., with column  $x$  not already locked) such that

$$\Psi^{\sigma_s \oplus h_s^i}(x, y) \downarrow = 1, \tag{1}$$

and such that  $(x, y)$  was already colored by  $c$ . Such numbers must eventually be found for some  $i \in k \setminus D$  and  $s$  if  $\Psi$  is to compute an  $\text{Ind } \mathcal{C}_2$ -solution, and we can take  $x \notin \text{dom } L_{s-1}$  because  $\Psi$  will output elements of infinitely many columns of  $E$ , while  $L_{s-1}$  is finite. That we can also take  $i \in k \setminus D$  is justified below. If  $x$  and  $y$

are found at stage  $s$ , set  $L_s(x) = 1 - \sigma_s(x, y)$ , put  $i \in D$ , and end the stage. That is, we lock column  $x$  to be the opposite color that  $\Psi$  already committed to there, and with  $i \in D$  we no longer search for further extensions of  $h_s^i$  at future stages. If instead no such  $x$  and  $y$  are found for any  $i$ , let  $L_s = L_{s-1}$  and end the stage.

Suppose the numbers  $x, y$  as above are found at stage  $s$  and with oracle  $h_s^i$ . If there is a  $d \in [\sigma_s, L_s]$  for which  $h_s^i$  extends to a solution  $H$  of  $\Delta^d$ , then this  $d$  witnesses the failure of the Weihrauch reduction, because  $\Psi^{d \oplus H}$  outputs a point in  $E$  whose color is shared by only finitely many other points in the same column. If such a  $d$  exists then the procedure ends at this stage. In particular, this must happen if  $D = k$ , because then an oracle  $h_s^i$  witnessing convergence of  $\Psi$  was already found for every  $i \in k$ , and at least one of these must extend to a valid solution of  $\Delta^d$  for any  $d \in [\sigma_s, L_s]$ .

Otherwise, for this  $i$  and indeed for all  $i \in D$ , there is no  $d \in [\sigma_s, L_s]$  with  $h_s^i$  extending to a solution of  $\Delta^d$ . As remarked above, this implies that for all  $d \in [\sigma_s, L_s]$ ,  $\Delta^d$  has only finitely many columns stabilize to  $j$ , for each  $j \in D$ . Then it must still be possible to  $L_s$ -force infinitely many columns to be color  $i$  for at least one  $i \in k \setminus D$ . Convergence of  $\Psi$  on  $h_s^i$  must eventually occur for any such  $i$ , because no  $d \in [\sigma_s]$  will use any colors outside of  $k$ , which explains why we could presume that  $i \in k \setminus D$  in (1) above. Hence the recursive procedure we have described can be continued for the remaining colors in  $k$ . This completes the proof since the procedure is guaranteed to end in success if  $D = k$ .  $\square$

It is clear that the proofs of Lemmas 5.3 and 5.4 do not depend on  $k$  and thus extend to show  $\text{SRT}_+^2 \leq_w \text{Ind } \mathcal{E} \leq_w \text{RT}_+^2$  and  $\text{SRT}_\mathbb{N}^2 \leq_w \text{Ind } \mathcal{E}_\mathbb{N} \leq_w \text{RT}_\mathbb{N}^2$ . Hence this completes the proofs of all reductions and nonreductions shown in Figure 4.

## 6. FURTHER DIRECTIONS

Solutions of  $\text{Ind } \mathbb{Q}$  are rather nebulous: for example, if  $x$  and  $y$  are two elements of a solution  $H$ , one can delete the whole interval  $[x, y]$  from  $H$  and still obtain a solution. The seeming weakness of  $\text{Ind } \mathbb{Q}$  may be a consequence of this nebulosity, and it is unclear how much power (if any)  $\text{Ind } \mathbb{Q}$  derives from the fact that it outputs a second-order object, i.e., a set of rationals. An investigation of these properties can be put on precise footing with the following notions introduced in [8] and [11], respectively:

- The *first-order part* of a problem  $P$  is the strongest first-order problem Weihrauch reducible to  $P$ .
- The *deterministic part* of  $P$  is the strongest problem Weihrauch reducible to  $P$  for which every instance has a unique solution.

**Question 6.1.** How strong exactly are the first-order and deterministic parts of  $\text{Ind } \mathbb{Q}$ ? Is  $\text{Ind } \mathbb{Q}$  Weihrauch equivalent to a first-order problem?

It is naturally of interest to establish the relationship between  $\text{Ind } \mathcal{E}_2$  and  $\text{COH}$  in view of the logical decomposition of  $\text{RT}_2^2$  into  $\text{SRT}_2^2$  and  $\text{COH}$ . It follows from Lemma 5.4 and the fact that  $\text{COH}$  does not imply  $\text{SRT}_2^2$  in  $\omega$ -models [12] that  $\text{COH}$  does not imply  $\text{Ind } \mathcal{E}_2$  in  $\omega$ -models either. Whether  $\text{COH}$  is reducible in any sense to  $\text{Ind } \mathcal{E}_2$  is open, as is the question of the separation between  $\text{Ind } \mathcal{E}_k$  and  $\text{SRT}_k^2$  under  $\leq_c$ . We conjecture that

**Conjecture 6.2.**  $\text{COH} \not\leq_c \text{Ind } \mathcal{E}_\mathbb{N}$  and  $\text{Ind } \mathcal{E}_2 \not\leq_c \text{SRT}_\mathbb{N}^2$ , or at least  $\text{Ind } \mathcal{E}_k \not\leq_c \text{SRT}_k^2$ .

Of course, if either statement is false, then the other would immediately follow. We briefly mention a way to relate  $\text{Ind } \mathcal{E}_2$  to COH which might suggest a route to the resolution of both conjectures. It has been observed by multiple authors that COH and  $(\widetilde{\text{RT}}_2^1)^{\text{FE}}$  are Weihrauch equivalent, where for any problem  $P$ ,  $P^{\text{FE}}$  is the “finite error” version of  $P$  with  $\text{dom } P^{\text{FE}} = \text{dom } P$  and  $x \in P^{\text{FE}}(p)$  iff there is a  $y \in P(p)$  with  $x$  and  $y$  equal on cofinitely many points. The following definition was inspired by the principle RCOH, a weakening of COH introduced by Cholak, Dzhafarov, Hirschfeldt, and Patey in [4].

**Definition 6.3.** The *weak parallelization* of a problem  $P$  is the problem  $\tilde{P}$  such that  $\text{dom } \tilde{P} = \text{dom } P$ , and where the solutions to the instance  $\langle p_0, p_1, \dots \rangle$  are all sets of the form

$$\bigcup_{n \in A} \{ \langle n, x \rangle : x \in P(p_n) \},$$

where  $A$  is an infinite subset of  $\mathbb{N}$ .

In other words,  $\tilde{P}$  picks infinitely many parallel instances of  $P$  to solve out of a given instance of  $P$ . Thus  $\text{Ind } \mathcal{E}_k$  is an a priori stronger variant of  $\widetilde{\text{RT}}_k^1$  in which the solutions of the parallel  $\text{RT}_k^1$ -instances represented in an  $\widetilde{\text{RT}}_k^1$ -solution are all of the same color. However, by nonuniformly picking a color shared by infinitely many columns of a solution of  $\widetilde{\text{RT}}_k^1$ , it is not hard to see that  $\text{Ind } \mathcal{E}_k \equiv_c \widetilde{\text{RT}}_k^1$ . Moreover,  $\text{SRT}_k^2 \equiv_c \widetilde{\text{SRT}}_k^1$ , by additionally computing a homogeneous set using a standard argument as in the proof of Lemma 5.4. Rephrasing Conjecture 6.2 in these terms, we can then ask

**Question 6.4.** What is the relationship between  $(\widetilde{\text{RT}}_k^1)^{\text{FE}}$  and  $\widetilde{\text{RT}}_k^1$ , and between  $\widetilde{\text{RT}}_k^1$  and  $\widetilde{\text{SRT}}_k^1$ , under  $\leq_c$  and under implication in  $\omega$ -models of  $\text{RCA}_0$ ?

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*Email address:* `gillmathpsu@posteo.net`