INDIVISIBILITY AND UNIFORM COMPUTATIONAL STRENGTH

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Abstract. A countable structure is indivisible if for every coloring with finite range there is a monochromatic isomorphic subcopy of the structure. Each indivisible structure $\mathbb{S}$ naturally corresponds to an indivisibility problem $\mathsf{Ind}\,\mathbb{S}$, which outputs such a subcopy given a presentation and coloring. We investigate the Weihrauch complexity of the indivisibility problems for two structures: the rational numbers $\mathbb{Q}$ as a linear order, and the equivalence relation $\mathscr{E}$ with countably many equivalence classes each having countably many members. We separate the Weihrauch degrees of both $\mathsf{Ind}\,\mathbb{Q}$ and $\mathsf{Ind}\,\mathscr{E}$ from several benchmark problems, showing in particular that $\mathsf{C}_\mathbb{N} \mid_\mathrm{W} \mathsf{Ind}\,\mathbb{Q}$ and hence $\mathsf{Ind}\,\mathbb{Q}$ is strictly weaker than the problem of finding an interval in which some color is dense for a given coloring of $\mathbb{Q}$; and that the Weihrauch degree of $\mathsf{Ind}\,\mathscr{E}_k$ is strictly between those of SRT^2_k and RT^2_k, where $\mathsf{Ind}\,\mathbb{S}_k$ is the restriction of $\mathsf{Ind}\,\mathbb{S}$ to k-colorings.

1. Introduction

Reverse mathematics is a branch of logic which seeks to classify theorems based on their logical strength. This is often done by working over a relatively weak base system for mathematics, such as a fragment of second-order arithmetic, and then looking at the truth of various implications in models of that fragment. It turns out that one can frequently obtain more fine-grained distinctions between theorems by comparing their intrinsic computational or combinatorial power, rather than being limited to considerations of provability only. For theorems expressible as Π^1_2 statements of second-order arithmetic, this can be achieved by identifying a theorem with an instance-solution pair, where instances are all sets X to which the hypothesis of the theorem apply, and solutions to X are all sets Y witnessing the truth of the conclusion of the theorem for X. One then introduces various reducibilities to compare, among different instance-solution pairs, the difficulty of finding a solution from a given instance.

Instance-solution pairs can be formalized as problems:

Definition 1.1. A *problem* is a partial multivalued function on Baire space, $P\colon \subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$. Any $x \in \operatorname{dom} P$ is called an *instance* of P, and any $y \in P(x)$ is a *solution* to x with respect to P.

In the present work, we focus on problems arising from indivisible structures, and use mainly Weihrauch reducibility to gauge their uniform computational content. We assume familiarity with computability theory and the basic terminology of

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computable structure theory, as in the introductory sections of [18]. All our structures are countable and we will take them to have domain $\mathbb{N}$ when convenient.

Definition 1.2. A structure $\mathcal{S}$ is *indivisible* if for any presentation $\mathcal{A}$ of $\mathcal{S}$ and any coloring of $A = \operatorname{dom}\mathcal{A}$ with finitely many colors, there is a monochromatic subset $B \subseteq A$ such that $\mathcal{B} \simeq \mathcal{A}$, where $\mathcal{B}$ is the substructure induced by B. $\mathcal{S}$ is *computably indivisible* if in addition such a set B can always be computed from $\mathcal{A}$ and c.

Indivisibility belongs properly to Ramsey theory, the study of combinatorial structures which cannot be "too disordered" in that any large subset must contain a highly organized substructure. This field goes back to 1930, when Ramsey's theorem first appeared, and principles arising in Ramsey theory have long been objects of interest in reverse mathematics.

Definition 1.3. If $\mathcal{S}$ is an indivisible structure, then $\mathsf{Ind}\,\mathcal{S}$ is the *indivisibility problem* associated to $\mathcal{S}$, for which

- Instances are triples $\langle \mathcal{A}, c, k\rangle$ where $\mathcal{A}$ is a presentation of $\mathcal{S}$ (identified with its atomic diagram) and c is a k-coloring of A, and
- Solutions to $\langle \mathcal{A}, c, k\rangle$ are (characteristic functions of) subsets B of A which are monochromatic for c and such that $\mathcal{B} \simeq \mathcal{A}$.

Here $\mathcal{A}$ is allowed to be an arbitrary presentation of $\mathcal{S}$, not necessarily one having domain $A = \mathbb{N}$. We still treat c as a coloring of all of $\mathbb{N}$ but disregard any color assigned to a point outside A. By convention, $i \in A$ iff the formula $x_i = x_i$ is true in the atomic diagram of $\mathcal{A}$, and if it is false then all atomic formulas involving x_i should also be set to false. We will also use the following variations on $\mathsf{Ind}\,\mathcal{S}$:

- For each fixed k, let $\mathsf{Ind}\,\mathcal{S}_k$ have domain $\{\,\langle \mathcal{A}, c\rangle : \langle \mathcal{A}, c, k\rangle \in \operatorname{dom}(\mathsf{Ind}\,\mathcal{S})\,\}$ and let $\mathsf{Ind}\,\mathcal{S}_k(\langle \mathcal{A}, c\rangle) = \mathsf{Ind}\,\mathcal{S}(\langle \mathcal{A}, c, k\rangle)$. That is, $\mathsf{Ind}\,\mathcal{S}_k$ is the restriction of $\mathsf{Ind}\,\mathcal{S}$ to k-colorings, so that $\mathsf{Ind}\,\mathcal{S}$ is exactly the problem $\bigsqcup_{k\in\mathbb{N}} \mathsf{Ind}\,\mathcal{S}_k$, where $\sqcup$ is defined in the next section.
- Let $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$ be the problem with $\operatorname{dom}(\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}) = \bigcup_{k\in\mathbb{N}} \operatorname{dom}(\mathsf{Ind}\,\mathcal{S}_k)$ and $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}(x) = \mathsf{Ind}\,\mathcal{S}_k(x)$ if $x \in \operatorname{dom}(\mathsf{Ind}\,\mathcal{S}_k)$. Then $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$ is just $\mathsf{Ind}\,\mathcal{S}$ but without any given information on the number of colors used.

One can also define the *strong indivisibility problem* where a solution additionally includes an explicit isomorphism between $\mathcal{B}$ and $\mathcal{A}$. We do not study this notion further; both it and $\mathsf{Ind}\,\mathcal{S}$ are discussed, for example, in [1].

This paper focuses on two indivisible structures in particular. First, the set of rational numbers $\mathbb{Q}$ is viewed as, up to order isomorphism, the unique countable dense linear order with no greatest or least element. $\mathsf{Ind}\,\mathbb{Q}$ is studied in Section 4. The other structure is the so-called countable equivalence relation:

Definition 1.4. The *countable equivalence relation* $\mathscr{E}$ is, up to isomorphism, the structure in the language $\{R\}$ of a single binary relation such that R is an equivalence relation, and such that E is divided into countably many equivalence classes each having countably many members.

To see $\mathscr{E}$ is indivisible, suppose its elements are colored red and blue (it suffices to show for two colors). Then either there are infinitely many equivalence classes each containing infinitely many red points, or there are cofinitely many equivalence classes each containing cofinitely many blue points. Either way one gets a monochromatic subcopy of $\mathscr{E}$. $\mathsf{Ind}\,\mathscr{E}$ is studied in Section 5. Our main results separate $\mathsf{Ind}\,\mathbb{Q}$ and $\mathsf{Ind}\,\mathscr{E}$ from various benchmark problems. In the case of $\mathbb{Q}$, we have

- $\mathsf{Ind}\,\mathbb{Q}$ is Weihrauch reducible to $\mathsf{TC}^*_{\mathbb{N}}$ but incomparable with $\mathsf{C}_{\mathbb{N}}$ (Theorem 4.1 and Corollary 4.9).
- Hence $\mathsf{Ind}\,\mathbb{Q}$ is not uniformly strong enough to find a rational interval where a color appears densely (Corollary 4.2).
- $\mathsf{Ind}\,\mathbb{Q}$ is not limit computable (Proposition 4.6).
- $\mathsf{Ind}\,\mathbb{Q}$ cannot be solved by any problem having a c.e. approximation (Theorem 4.8), in a precise sense given by Definition 4.7.

The hierarchy of Weihrauch reducibilities (and nonreducibilities) relating versions of $\mathsf{TC}^k_{\mathbb{N}}$, $\mathsf{Ind}\,\mathbb{Q}_k$, and RT^1_k is shown in Figure 1 below. Next, $\mathsf{Ind}\,\mathscr{E}$ turns out to be closely related to but distinct from Ramsey's theorem for pairs:

- The Weihrauch degree of $\mathsf{Ind}\,\mathscr{E}_k$ is strictly between those of SRT^2_k and RT^2_k for all $k \geq 2$;
- In particular, $\mathsf{Ind}\,\mathscr{E}_2$ is not Weihrauch reducible to $\mathsf{SRT}^2_{\mathbb{N}}$ and RT^2_2 is not computably reducible to $\mathsf{Ind}\,\mathscr{E}_{\mathbb{N}}$ (Theorem 5.1).

The Weihrauch reducibilities between these principles are displayed in Figure 4.

Before getting to our main results in Sections 4 and 5, we define all the reducibilities under consideration as well as the main problems used as benchmarks in our results. In Section 3 we comment on the issue of presentations and make some general observations. Finally, in Section 6 we briefly mention some directions for future research on $\mathsf{Ind}\,\mathbb{Q}$ and $\mathsf{Ind}\,\mathscr{E}$.

2. Background

Here we collect general definitions and notations used for the rest of the paper. The standard reference on reverse mathematics is Simpson [21]. The books by Hirschfeldt [13] and by Dzhafarov and Mummert [6] are more computability-oriented treatments of the subject, the latter discussing in detail the reducibilities we employ. Our notation and conventions with respect to computability theory are standard and can be found in any modern textbook, e.g., [22]. For any set X and $n \in \mathbb{N}$, $X \upharpoonright n$ is the truncation of X up through its first n bits. If $\sigma \in \mathbb{N}^{<\mathbb{N}}$, then $[\sigma]$ denotes the set of finite or infinite strings extending σ. We will always identify k with the set $\{0, \ldots, k-1\}$. Write W_e for the eth c.e. set with respect to some computable enumeration of Turing machines.

Definition 2.1. Let P and Q be problems. Then

- P is *computably reducible* to Q, written $P \leq_{\mathrm{c}} Q$, if each $p \in \operatorname{dom} P$ computes some $x \in \operatorname{dom} Q$ such that for every $q \in Q(x)$, $p \oplus q$ computes an element of $P(p)$. If we require q by itself to compute an element of $P(p)$, then P is *strongly computably reducible* to Q, written $P \leq_{\mathrm{sc}} Q$.
- P is *Weihrauch reducible* to Q, written $P \leq_{\mathrm{W}} Q$, if there are Turing functionals Δ and Ψ such that if $p \in \operatorname{dom} P$, then $\Delta^p \in \operatorname{dom} Q$; and for any $q \in Q(\Delta^p)$, we have $\Psi^{p \oplus q} \in P(p)$. If we require Ψ to only have oracle access to q, then P is *strongly Weihrauch reducible* to Q, written $P \leq_{\mathrm{sW}} Q$.

Δ and Ψ are sometimes called the forward and return functionals, respectively. We have $P \leq_{\mathrm{sW}} Q \implies P \leq_{\mathrm{W}} Q \implies P \leq_{\mathrm{c}} Q$ and $P \leq_{\mathrm{sW}} Q \implies P \leq_{\mathrm{sc}} Q \implies P \leq_{\mathrm{c}} Q$, but none of these implications reverse and there is no logical relationship between $\leq_W$ and $\leq_{sc}$. All of these reducibilities also imply that Q implies P in ω-models of RCA_0, i.e., models with the standard natural numbers. Write $P|_\square Q$ if P and Q are incomparable under $\leq_\square$, and $P \equiv_\square Q$ if $P \leq_\square Q$ and $Q \leq_\square P$, for $\square \in \{\mathrm{c}, \mathrm{sc}, \mathrm{W}, \mathrm{sW}\}$. See [14, 6] for details on these and other reducibilities, and [3] for a good recent survey of Weihrauch reducibility with a large bibliography and historical remarks. The reducibilities above can be defined between problems on any represented spaces, but we do not need this level of generality. More information can again be found in [3].

There are a few algebraic operations on problems which we will have occasion to use. Our formulation mostly follows [3]. Let P and Q be problems. Then

- $P \times Q$ is the *parallel product* of P and Q, defined by $\operatorname{dom}(P \times Q) = \operatorname{dom} P \times \operatorname{dom} Q$, and $(P \times Q)(p, q) = P(p) \times Q(q)$.
- P^n is the n-fold parallel product of P with itself. The instances can be regarded as including the data n.
- P^* is the *finite parallelization* of P, with $\operatorname{dom}(P^*) = \{\, \{n\} \times (\operatorname{dom} P)^n : n \in \mathbb{N}\,\}$ and $P^*(\langle n, p_1, \ldots, p_n\rangle) = \{n\} \times P(p_1) \times \ldots \times P(p_n)$. So, P^* is the problem that can solve any specified finite number of instances of P at once.
- $\hat{P}$ is the *parallelization* of P. Instances are sequences $\langle p_0, p_1, p_2, \ldots\rangle \in \operatorname{dom}(P)^{\mathbb{N}}$, and the set of solutions to this instance is the Cartesian product $\prod_{i\in\mathbb{N}} P(p_i)$.
- $P \sqcup Q$ is the *coproduct* of P and Q. Here $\operatorname{dom}(P \sqcup Q) = \operatorname{dom} P \sqcup \operatorname{dom} Q$, and $(P \sqcup Q)(0, p) = \{0\} \times P(p)$ and $(P \sqcup Q)(1, q) = \{1\} \times Q(q)$. So $P \sqcup Q$ is the problem which is capable of solving any instance of P or of Q, but only one at a time.
- P' is the *jump* of P. An instance of P' is a sequence $\langle p_0, p_1, p_2, \ldots\rangle$ of reals converging to some $p \in \operatorname{dom} P$ in the Baire space topology (equivalently, converging entrywise). The solutions to this instance are just the elements of $P(p)$. So P' answers the same question about $p \in \operatorname{dom} P$ that P does, but only has access to a limit representation of p. We also set $P^{(n+1)} = (P^{(n)})'$ with $P^{(0)} = P$.
- $P * Q$ is the *compositional product* of P and Q. This can be characterized intuitively as the strongest problem under $\leq_W$ obtainable as the composition of f and g, ranging over all $f \leq_W P$ and $g \leq_W Q$. (A more precise definition of a representative of the degree of $P * Q$ is given in Section 3 below.)

The following "benchmark" problems will be used as a basis for comparison with the problems we study:

- LPO, the *limited principle of omniscience*, has $\operatorname{dom} \mathsf{LPO} = \mathbb{N}^{\mathbb{N}}$, and $\mathsf{LPO}(0^{\mathbb{N}}) = 0$ and $\mathsf{LPO}(p) = 1$ otherwise. $\mathsf{LPO}^{(n)}$ can be thought of as answering a single Σ^0_{n+1} question.
- $\lim$ maps a convergent sequence of reals $\langle p_0, p_1, p_2, \ldots\rangle$ to its limit.
- $\mathsf{C}_{\mathbb{N}}$, closed choice on $\mathbb{N}$, outputs an element of a nonempty set $A \subseteq \mathbb{N}$ given an enumeration of its complement.
- $\mathsf{TC}_{\mathbb{N}}$, the totalization of $\mathsf{C}_{\mathbb{N}}$, extends $\mathsf{C}_{\mathbb{N}}$ by allowing $A = \emptyset$ and outputting any number in this case.
- RT^n_k, Ramsey's theorem for n-tuples and k colors, has instances $c\colon [\mathbb{N}]^n \to k$, and solutions to c are (characteristic functions of) infinite c-homogeneous

sets. Here $[X]^n$ is the set of n-element subsets of X, and a set $X \subseteq \mathbb{N}$ is *homogeneous* for c if X is infinite and c is monochromatic on $[X]^n$.

- SRT^2_k, stable Ramsey's theorem for pairs, is the restriction of RT^2_k to *stable colorings*, i.e., colorings c such that $\lim_m c\{n, m\}$ exists for all n.
- $\mathsf{RT}^n_{\mathbb{N}}$ has instances $c \in \bigcup_{k\in\mathbb{N}} \operatorname{dom} \mathsf{RT}^n_k$. Solutions to c are again infinite c-homogeneous sets. Notice that in this formulation, the number of colors used is not included as part of an instance. $\mathsf{SRT}^2_{\mathbb{N}}$ is defined similarly.
- For $k \in \mathbb{N} \cup \{\mathbb{N}\}$, cRT^n_k is the "color version" of RT^n_k, which only outputs the colors of RT^n_k-solutions. We have $\mathsf{RT}^1_k \equiv_{\mathrm{W}} \mathsf{cRT}^1_k$ for all k, since the color can be used to compute the set of points of that color and vice versa.
- RT^n_+ is $\bigsqcup_{k\in\mathbb{N}} \mathsf{RT}^n_k$, and cRT^n_+ and SRT^2_+ are defined similarly.

Clearly $\mathsf{RT}^n_2 \leq_{\mathrm{sW}} \mathsf{RT}^n_3 \leq_{\mathrm{sW}} \dots \leq_{\mathrm{sW}} \mathsf{RT}^n_+ \leq_{\mathrm{sW}} \mathsf{RT}^n_{\mathbb{N}}$ for all n, and likewise for SRT^2_k. Also $\mathsf{Ind}\,\mathcal{S}_2 \leq_{\mathrm{sW}} \mathsf{Ind}\,\mathcal{S}_3 \leq_{\mathrm{sW}} \dots \leq_{\mathrm{sW}} \mathsf{Ind}\,\mathcal{S} \leq_{\mathrm{sW}} \mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$ for any indivisible $\mathcal{S}$.

3. Uniform computable categoricity

Recall that a computable structure $\mathcal{A}$ is uniformly computably categorical if there is a Turing functional which computes an isomorphism from $\mathcal{B}$ to $\mathcal{A}$ given the atomic diagram of any presentation $\mathcal{B}$ of $\mathcal{A}$. If $\mathcal{S}$ is indivisible and uniformly computably categorical, then from the point of view of $\leq_{\mathrm{W}}$ or $\leq_{\mathrm{c}}$ we can as a convention regard the instances of $\mathsf{Ind}\,\mathcal{S}$ as only including the data k and $c\colon \mathbb{N} \to k$, since in a reduction, the functionals Δ and Ψ can just build in the translations between any given presentation $\mathcal{A}$ and some fixed computable presentation of $\mathcal{S}$. Instances of $\mathsf{Ind}\,\mathcal{S}_k$ and $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$ are viewed simply as colorings c of $\mathbb{N}$. One can state this more formally as

Proposition 3.1. *If $\mathcal{S}$ is uniformly computably categorical and indivisible, then $\mathsf{Ind}\,\mathcal{S}_k \equiv_{\mathrm{W}} P_k$ where P_k is the restriction of $\mathsf{Ind}\,\mathcal{S}_k$ to instances of the form $\langle \mathcal{S}, c\rangle$. Analogous statements hold for $\mathsf{Ind}\,\mathcal{S}$ and $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$.*

This convention is not a priori justified when considering $\leq_{\mathrm{sW}}$ or $\leq_{\mathrm{sc}}$, because the return functional Ψ could need oracle access to the presentation $\mathcal{A}$ in order to translate back to a solution of $\langle \mathcal{A}, c, k\rangle$ if Δ modified $\mathcal{A}$.

Both of the structures we focus on in this paper are uniformly computably categorical and so we will follow the above convention without further comment. That $\mathbb{Q}$ is uniformly computably categorical follows from the classical back-and-forth argument, which is effective. To see that $\mathscr{E}$ is uniformly computably categorical, for any presentation $\mathcal{A}$ of $\mathscr{E}$, decompose any $n \in A$ as a pair $\langle x, y\rangle_{\mathcal{A}}$ so that n is the yth element of the xth distinct equivalence class, in order of discovery within the atomic diagram of $\mathcal{A}$. Then if also $m = \langle z, w\rangle_{\mathcal{A}}$, we have that n and m are equivalent iff $x = z$, and $\langle \cdot, \cdot\rangle_{\mathcal{A}}$ is uniformly $\mathcal{A}$-computable. (This definition does not uniquely specify $\langle \cdot, \cdot\rangle_{\mathcal{A}}$, but one can make some canonical choice.) If we take $\mathscr{E}$ to be computable and $\mathcal{A}$ is a given copy of $\mathscr{E}$, without loss of generality with $E = A = \mathbb{N}$, then the map $\langle x, y\rangle_{\mathcal{A}} \mapsto \langle x, y\rangle_{\mathscr{E}}$ is a uniformly $\mathcal{A}$-computable isomorphism between $\mathcal{A}$ and $\mathscr{E}$.

As an aside, the problems corresponding to uniformly computably categorical indivisible structures seem to have some nice properties. For example, it is not hard to see that the indivisibility and strong indivisibility problems of such a structure must be Weihrauch equivalent. There is also the following easy observation:

Proposition 3.2. *If $\mathcal{S}$ is indivisible and uniformly computably categorical, then* $\mathsf{Ind}\,\mathcal{S} \equiv_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}^*$.

Proof. It is known that for any problem f with a computable point in its domain, $f \equiv_{\mathrm{W}} f \times f$ iff $f \equiv_{\mathrm{W}} f^*$ (see [3]). If (by assumption) $\mathcal{S}$ has a computable presentation then $\mathrm{dom}(\mathsf{Ind}\,\mathcal{S})$ has a computable point, so it suffices to show in our case that $\mathsf{Ind}\,\mathcal{S} \times \mathsf{Ind}\,\mathcal{S} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}$. Since $\mathcal{S}$ is uniformly computably categorical, as noted above, we can assume each instance consists only of a coloring together with a number of colors. Then suppose $\langle a, j\rangle$ and $\langle b, k\rangle$ are two instances of $\mathsf{Ind}\,\mathcal{S}$ with $a\colon \mathbb{N} \to j$ and $b\colon \mathbb{N} \to k$. Define $c\colon \mathbb{N} \to 2^{j+1}3^{k+1}$ by $c(n) = 2^{a(n)+1}3^{b(n)+1}$. If H is a solution to c with color i, then there are $i_1 \in j$, $i_2 \in k$ with $i = 2^{i_1+1}3^{i_2+1}$, and hence $a(H) = i_1$ and $b(H) = i_2$, so that H is simultaneously a solution to both $\langle a, j\rangle$ and $\langle b, k\rangle$. $\square$

Furthermore, we have the following result which extends the conclusion of Proposition 62 of [20] from $\mathbb{N}$ to any indivisible, uniformly computably categorical $\mathcal{S}$. Our argument is really only a slight adaptation of the original, but we give it in detail for completeness. It will be useful for the proof to use the representative $f \star g$ of the Weihrauch degree of $f * g$, defined by

$$(f \star g)(x, y) = \langle \mathrm{id} \times f\rangle \circ \Phi_x \circ g(y),$$

where id is the identity map on $\mathbb{N}^{\mathbb{N}}$ and Φ is a universal functional [3].

Proposition 3.3. *For any indivisible $\mathcal{S}$, we have* $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S} * \mathsf{C}_{\mathbb{N}}$. *If $\mathcal{S}$ is also uniformly computably categorical, then* $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S} * \mathsf{C}_{\mathbb{N}}$.

Proof, after [20]. For the first statement, let $\mathcal{S}$ be an arbitrary indivisible structure and let $\langle \mathcal{A}, c\rangle$ be a given instance of $\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$. Build a $\mathsf{C}_{\mathbb{N}}$-instance by, whenever a new color is seen in c, enumerating all numbers less than that color. From a $\mathsf{C}_{\mathbb{N}}$-solution k we get an instance $\langle \mathcal{A}, c, k\rangle$ of $\mathsf{Ind}\,\mathcal{S}$, and a solution to this instance is also a solution to $\langle \mathcal{A}, c\rangle$.

To prove the second statement, let Bound be the problem that outputs an upper bound on an enumeration of a finite set. This is Weihrauch equivalent to $\mathsf{C}_{\mathbb{N}}$ and it will be convenient to show that $\mathsf{Ind}\,\mathcal{S} \star \mathsf{Bound} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$. Let (x, Y) be an instance of $\mathsf{Ind}\,\mathcal{S} \star \mathsf{Bound}$, so Y is a finite set represented by an (infinite) enumeration. Since $\mathcal{S}$ is uniformly computably categorical, we can treat Φ_x as computing only a coloring with an upper bound on its range. For each $i \in \mathbb{N}$, let p_i be the ith prime number. Build $d \in \mathrm{dom}(\mathsf{Ind}\,\mathcal{S}_{\mathbb{N}})$ as follows: find a number i_0 such that $\Phi_x(i_0)$ outputs, in addition to a partial coloring c_0 of $\mathbb{N} = \mathrm{dom}\,\mathcal{S}$, an upper bound k_0 on the range of c_0. This i_0 is an initial guess for an element of $\mathsf{Bound}(Y)$. Set $d(n) = p_{i_0}^{c_0(n)}$ for all n for which we see $c_0(n)$ defined by $\Phi_x(i_0)$. In general, if i_s has been found, let $d(n) = p_{i_s}^{c_s(n)}$ whenever $c_s(n)$ is defined and n had not been colored at a previous stage. If numbers are enumerated into Y so that $\max Y \geq i_s$, find an $i_{s+1} > \max Y$ such that $\Phi_x(i_{s+1})$ outputs a number k_{s+1}; such an i_{s+1} must exist, so we can hold off on extending d until k_{s+1} appears. When it does, continue with $d(n) = p_{i_{s+1}}^{c_{s+1}(n)}$ for all n which had not been previously colored and for which $c_{s+1}(n)$ is defined.

Eventually this process stabilizes at some i_ℓ, since Y is finite, and once it does $\Phi_x(i_\ell)$ must produce a total coloring. Then any solution H of d has color $p_{i_\ell}^a$ for some $a < k_\ell$. By rerunning the procedure in the last paragraph, the return functional can recover d, hence find the color of H, and from that learn i_ℓ and output $\Phi_x(i_\ell)$ to satisfy id. Finally, H is in fact a solution to $\Phi_x(i_\ell)$, because it can only include points colored after i_ℓ stabilizes and so the fact that d and $\Phi_x(i_\ell)$ differ on finitely many other points is of no consequence. □

The condition of uniform computable categoricity in the above statements is not necessary, since they all hold for $\mathsf{Ind}\,\mathbb{N} = \mathsf{RT}^1_+$ while $(\mathbb{N}, <)$ is not computably categorical. We do not know a characterization of the structures $\mathcal{S}$ for which these propositions hold. A good source of (counter)examples could be the class of nonscattered (countable) linear orders, all of which are indivisible as a consequence of the fact that every countable linear order embeds into $\mathbb{Q}$. The only infinite uniformly computably categorical linear order is $\mathbb{Q}$ itself, yet we have

Example 3.4. If $\mathcal{L} = (n + \mathbb{Q} + m, <)$ for some $n, m \in \mathbb{N}$, then $\mathsf{Ind}\,\mathcal{L}_k \equiv_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_k$. Similarly for $\mathsf{Ind}\,\mathcal{L}$ and $\mathsf{Ind}\,\mathcal{L}_{\mathbb{N}}$.

Proof. $[\leq_{\mathrm{W}}]$ Given an instance $\langle \mathcal{A}, c\rangle$ of $\mathsf{Ind}\,\mathcal{L}_k$, read far enough in the atomic diagram of $\mathcal{A}$ to find at least $n+m+2$ distinct elements and put them in increasing order. The $(n+1)$st and $(n+2)$nd smallest elements found, call them x and y, are both in $\mathbb{Q}$. Build an instance of $\mathsf{Ind}\,\mathbb{Q}_k$ by restricting the domain of $\mathcal{A}$ to the interval (x, y) and using c as-is. If H is a solution to this instance, a solution to $\langle \mathcal{A}, c\rangle$ can be found by taking $(z, w) \cap H$ for some $z < w \in H$ and adding to it n more points below z and m more points above w, all chosen from H.

$[\geq_{\mathrm{W}}]$ If $\langle \mathcal{A}, c\rangle \in \mathrm{dom}(\mathsf{Ind}\,\mathbb{Q}_k)$, build $\langle \mathcal{B}, d\rangle \in \mathrm{dom}(\mathsf{Ind}\,\mathcal{L}_k)$ by shifting all indices for elements of A up by $n + m$, then making elements $0, \ldots, n-1$ of B less than all other elements and $n, \ldots, n+m-1$ greater than all other elements. That is, let the relation $x_{i+n+m} < x_{j+n+m}$ hold in $\mathcal{B}$ iff $x_i < x_j$ holds in $\mathcal{A}$, and also make $x_0 < x_1 < \cdots x_{n-1} < x_i < x_n < x_{n+1} < \cdots < x_{n+m-1}$ hold in $\mathcal{B}$ for all $i \geq n + m$ (omitting any i such that $i - n - m \notin A$). Then let $d(x + n + m) = c(x)$ and color $0, \ldots, n+m-1 \in B$ arbitrarily. If H is an $\mathsf{Ind}\,\mathcal{L}_k$-solution to this instance, then one can delete $0, \ldots, n+m-1$ from H if necessary and obtain the solution $\tilde{H} = \{\, x - n - m : x \in H \,\}$ to $\langle \mathcal{A}, c\rangle$. □

Consequently, Propositions 3.2 and 3.3 hold for any such $\mathcal{L}$. If one adds constant symbols to the language for every member of a pair of adjacent elements of an order $\mathcal{L}$, then the above argument can be extended (for instance) to any infinite $\mathcal{L}$ having only finitely many such pairs.

4. The rational numbers

4.1. **Prior related work.** $\mathsf{Ind}\,\mathbb{Q}$ is exactly the problem form of the reverse-mathematical principle ER^1 studied by Frittaion and Patey in [9]. There the authors show, among other things, that the implication $\mathsf{ER}^1 \to \mathsf{RT}^1_+$ over RCA_0 is strict. This has a kind of uniform counterpart in our Corollary 4.9(i), since trivially $\mathsf{RT}^1_+ \leq_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}$ (and $\mathsf{RT}^1_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathcal{S}_{\mathbb{N}}$) for any indivisible $\mathcal{S}$, and this is strict if $\mathcal{S} = \mathbb{Q}$. But separations over RCA_0 have no direct bearing in the present setting: both $\mathbb{Q}$ and $\mathbb{N}$ are computably indivisible and hence both $\mathsf{Ind}\,\mathbb{Q}$ and RT^1_+ hold in ω-models of RCA_0, indeed are computably equivalent to the identity map on Baire space. To obtain meaningful distinctions between them one must pass to a stronger reducibility, and so we focus on the uniform content of $\mathsf{Ind}\,\mathbb{Q}$ via its Weihrauch degree.

The Weihrauch degrees of a family of problems related to $\mathsf{Ind}\,\mathbb{Q}$ have recently been an object of study in the work of Pauly, Pradic, and Soldà [20]. In their terminology, if c is any coloring of $\mathbb{Q}$, then an open interval $I \subseteq \mathbb{Q}$ is a *c-shuffle* if for every color occurring in I, the set of points of that color is dense in I. They then investigate several corresponding families of problems:

- $\mathsf{Shuffle}$ has instances (k, c), where $k \in \mathbb{N}$ and $c\colon \mathbb{Q} \to k$. If I is a (code for a) rational interval and $C \subseteq k$, then $(I, C) \in \mathsf{Shuffle}(k, c)$ iff I is a c-shuffle with exactly the colors of C.
- $\mathsf{iShuffle}$ is the weakening of $\mathsf{Shuffle}$ which, for an instance (k, c), returns an interval I such that there exists $C \subseteq k$ with $(I, C) \in \mathsf{Shuffle}(k, c)$.
- $\mathsf{cShuffle}$ is the weakening of $\mathsf{Shuffle}$ which, for an instance (k, c), returns a $C \subseteq k$ such that there is an I with $(I, C) \in \mathsf{Shuffle}(k, c)$.
- $(\eta)^1_{<\infty}$, which had previously been studied in [9] as a reverse-mathematical principle, is a different weakening of $\mathsf{Shuffle}$ which, for an instance (k, c), returns an interval I and a single color $n < k$ such that the points of c-color n are dense in I.

$\mathsf{Shuffle}_k$, $\mathsf{iShuffle}_k$, and $\mathsf{cShuffle}_k$ are defined to be the restrictions of the first three problems to instances of the form (k, c). One can also define $\mathsf{Shuffle}_\mathbb{N}$, $\mathsf{iShuffle}_\mathbb{N}$, and $\mathsf{cShuffle}_\mathbb{N}$ in a natural way. The authors show that $\mathsf{cShuffle} \equiv_\mathrm{W} (\mathsf{LPO}')^*$, $\mathsf{iShuffle} \equiv_\mathrm{W} \mathsf{TC}^*_\mathbb{N} \equiv_\mathrm{W} (\eta)^1_{<\infty} \equiv_\mathrm{W} \mathsf{i}(\eta)^1_{<\infty}$ where the latter problem is the version of $(\eta)^1_{<\infty}$ which only returns an interval I, and $\mathsf{Shuffle} \equiv_\mathrm{W} (\mathsf{LPO}')^* \times \mathsf{TC}^*_\mathbb{N}$. More specifically, they show $\mathsf{iShuffle}_k \equiv_\mathrm{W} \mathsf{TC}^{k-1}_\mathbb{N}$, the $(k-1)$-fold parallel product of $\mathsf{TC}_\mathbb{N}$. (They also show that $\mathsf{cShuffle}_k \leq_\mathrm{W} (\mathsf{LPO}')^{2k-1}$. The reverse direction of the equivalence between $\mathsf{cShuffle}$ and $(\mathsf{LPO}')^*$ is established by showing that $\mathsf{LPO}' \leq_\mathrm{W} \mathsf{cShuffle}$ and $\mathsf{cShuffle} \equiv_\mathrm{W} \mathsf{cShuffle}^*$. The precise relationship between number of colors and number of parallel instances of LPO' is left open.)

After a draft of this article was circulated, the author was made aware of contemporaneous work by Dzhafarov, Solomon, and Valenti [7], which concerns the *tree pigeonhole principle* TT^1_+: given any coloring of $2^{<\mathbb{N}}$ with bounded range, there is an infinite monochromatic subset H isomorphic to $2^{<\mathbb{N}}$ as a partial order. Problems TT^1_k and $\mathsf{TT}^1_\mathbb{N}$ are defined analogously. For any such H, we have $(H, <_{KB}) \simeq (\mathbb{Q} + 1, <)$ where $\leq_{KB}$ is the Kleene-Brouwer order, so by Example 3.4 one gets $\mathsf{TT}^1_k \equiv_\mathrm{W} \mathsf{Ind}\,\mathbb{Q}_k$, $\mathsf{TT}^1_+ \equiv_\mathrm{W} \mathsf{Ind}\,\mathbb{Q}$, and $\mathsf{TT}^1_\mathbb{N} \equiv_\mathrm{W} \mathsf{Ind}\,\mathbb{Q}_\mathbb{N}$. The results of [7] include (in effect) significant strengthenings of some of the corollaries of Theorems 4.1 and 4.8 below, but otherwise do not overlap with ours, and are obtained by entirely different methods.

4.2. **The Weihrauch degree of $\mathsf{Ind}\,\mathbb{Q}$.** It is clear that $\mathsf{Ind}\,\mathbb{Q}_k \leq_\mathrm{W} \mathsf{iShuffle}_k$ and $\mathsf{Ind}\,\mathbb{Q} \leq_\mathrm{W} \mathsf{iShuffle}$, since for any color i found in a shuffle, the set of points of that color in the shuffle is isomorphic to $\mathbb{Q}$. The main result of this section is that these reductions are strict: being able to uniformly find a monochromatic copy of $\mathbb{Q}$ is not sufficient to uniformly find an interval in which every color appearing is dense, or indeed an interval where even one color is dense. This follows from Theorem 4.1 below, as $\mathsf{C}_\mathbb{N}$ is strictly Weihrauch reducible to $\mathsf{TC}_\mathbb{N} \equiv_\mathrm{W} \mathsf{iShuffle}_2$.

The relationships between $\mathsf{Ind}\,\mathbb{Q}_k$, $\mathsf{TC}^k_\mathbb{N}$, and RT^1_k are summarized in Figure 1. Proposition 63 of [20] establishes that $\mathsf{iShuffle}_\mathbb{N} \equiv_\mathrm{W} \mathsf{iShuffle} * \mathsf{C}_\mathbb{N}$, and we have an analogous characterization of $\mathsf{Ind}\,\mathbb{Q}_\mathbb{N}$ from Proposition 3.3.

We will identify without comment natural numbers with the rationals they encode via a fixed computable presentation of $\mathbb{Q}$.

Theorem 4.1. $\mathsf{C}_\mathbb{N} \not\leq_\mathrm{W} \mathsf{Ind}\,\mathbb{Q}$.

$$\begin{tikzcd}
\mathsf{TC}_{\mathbb{N}} & \mathsf{TC}_{\mathbb{N}}^{2} \arrow[l] \arrow[d] & \mathsf{TC}_{\mathbb{N}}^{3} \arrow[l] \arrow[d] & \cdots \arrow[l] & \mathsf{TC}_{\mathbb{N}}^{*} \arrow[l] \arrow[d] & \mathsf{iShuffle}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^{*} * \mathsf{C}_{\mathbb{N}} \arrow[l] \arrow[d] \\
\mathsf{Ind}\,\mathbb{Q}_2 & \mathsf{Ind}\,\mathbb{Q}_3 \arrow[l] \arrow[d] & \mathsf{Ind}\,\mathbb{Q}_4 \arrow[l] \arrow[d] & \cdots \arrow[l] & \mathsf{Ind}\,\mathbb{Q} \arrow[l] \arrow[d] & \mathsf{Ind}\,\mathbb{Q}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q} * \mathsf{C}_{\mathbb{N}} \arrow[l] \arrow[d] \\
\mathsf{cRT}_2^1 & \mathsf{cRT}_3^1 \arrow[l] & \mathsf{cRT}_4^1 \arrow[l] & \cdots \arrow[l] & \mathsf{cRT}_+^1 \arrow[l] & \mathsf{cRT}_{\mathbb{N}}^1 \equiv_{\mathrm{W}} \mathsf{cRT}_+^1 * \mathsf{C}_{\mathbb{N}} \arrow[l]
\arrow[from=1-1, to=2-1] \arrow[from=2-1, to=3-1]
\end{tikzcd}$$

FIGURE 1. Reducibilities between versions of cRT^1, $\mathsf{Ind}\,\mathbb{Q}$, and $\mathsf{iShuffle} \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}^{*}$. An arrow from Q to P signifies that $P <_{\mathrm{W}} Q$. No other reductions hold other than those implied by transitivity.

Proof. $\mathsf{C}_{\mathbb{N}}$ is a fractal, so if $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \bigsqcup_{i \in \mathbb{N}} f_i$ then $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} f_i$ for some i (see [3] for the terminology and a reference for the result). Since $\mathsf{Ind}\,\mathbb{Q} = \bigsqcup_k \mathsf{Ind}\,\mathbb{Q}_k$, it suffices to show $\mathsf{C}_{\mathbb{N}} \not\leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_k$ for any given k.

So, fix k and suppose $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_k$ via Δ and Ψ. Recall that a valid instance of $\mathsf{C}_{\mathbb{N}}$ cannot enumerate every natural number, so that $f \in \operatorname{dom} \mathsf{C}_{\mathbb{N}}$ necessitates $\operatorname{ran} f \neq \mathbb{N}$. We regard Ψ as computing a $\mathsf{C}_{\mathbb{N}}$-solution n if $\Psi^{f \oplus H}(0)\downarrow = n$, for an instance f of $\mathsf{C}_{\mathbb{N}}$ and $\mathsf{Ind}\,\mathbb{Q}_k$-solution H of Δ^f. A solution n is valid only if $n \notin \operatorname{ran} f$.

We first give an informal outline of the proof. Let σ and R be a string and a finite set of red points witnessing that Ψ converges and outputs a number m. We want to extend σ to σ' by enumerating m, so that the latter is no longer a solution to any $\mathsf{C}_{\mathbb{N}}$-instance $g \in [\sigma']$. If there is such a g such that R extends to an $\mathsf{Ind}\,\mathbb{Q}_k$-solution of Δ^g, the reduction fails at this step. Now, the elements of R partition $\mathbb{Q}$ into a finite set of intervals, call them R-intervals. If there is no g as above, then for every such g, Δ^g makes the set of red points scattered in at least one R-interval. (Recall that a linear order is scattered if it contains no densely ordered subset.)

The particular R-interval where this happens depends on g, so we need to account for all of them. Extend σ' to a string τ witnessing convergence of Ψ on a blue set B_i in as many different R-intervals as possible. Diagonalize a second time by enumerating all the outputs of Ψ on all of the B_i. If τ' is the resulting string, then again it might happen that no B_i extends to a Δ^h-solution for any $h \in [\tau']$. That means, as before with R, that for each i, for each $h \in [\tau']$, there is a B_i-interval in which the set of blue points is scattered for Δ^h. In particular, for each h, there is some i and some B_i-interval where both red *and* blue points are scattered for Δ^h.

From here the argument recurses into further and further subintervals, resulting in a tree structure. This is illustrated in Figure 2. The endgame of the proof is that if the string α is reached after having to iterate the procedure $k-1$ times, so that each diagonalization so far prevented the set of rationals witnessing convergence of Ψ from being extendible to a solution, then for every $f \in [\alpha]$ there is some interval (out of finitely many choices) where the first $k-1$ colors are all scattered. Then the last remaining color must be dense there, so diagonalizing one final time is guaranteed to succeed in at least one interval.

Now we proceed to the formal proof. Pick any $f \in \operatorname{dom} \mathsf{C}_{\mathbb{N}}$ and any solution H to Δ^f. Without loss of generality suppose $\Delta^f(H) = 0$. Then $\Psi^{f \oplus H}(0)\downarrow = m_\lambda$ for some m_λ, where λ denotes the empty string. If $m_\lambda \in \operatorname{ran} f$, then the reduction

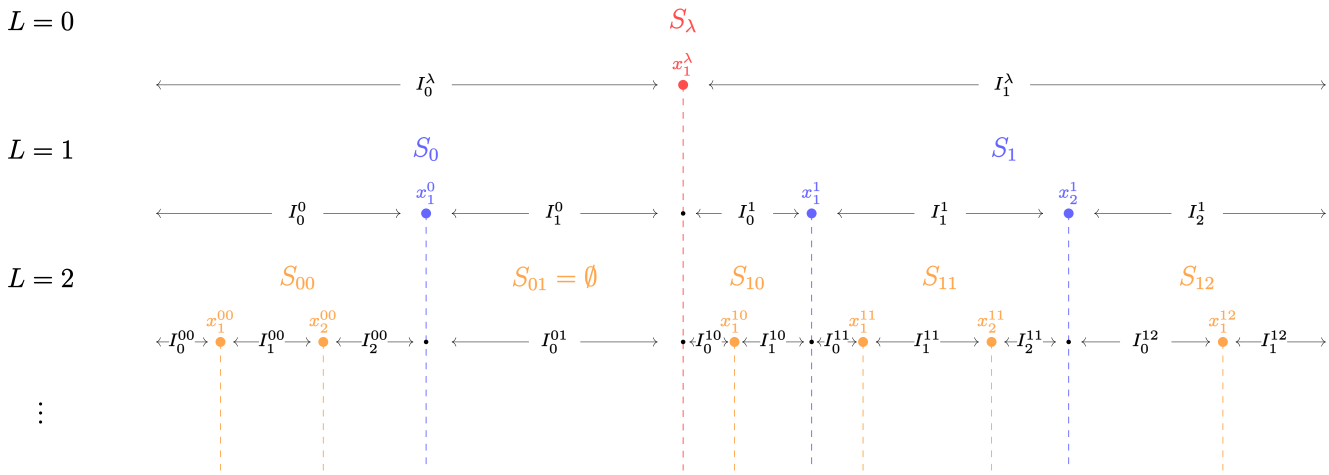

FIGURE 2. An illustration of an example diagonalization, with points vertically displaced to show which are added at each stage L. The point x_1^λ is carried through to become the largest element of X_0 and X_{01} and the smallest element of X_1 and X_{10}. x_1^0, x_1^1, and x_2^1 are carried through in a similar fashion, as are $x_0^\lambda = -\infty$ and $x_2^\lambda = +\infty$. Here no orange subset of I_1^0 could be found witnessing convergence of Ψ on any extension of σ_1, so we set $X_{01} = \{x_1^0, x_1^\lambda\} = \{x_1^0, x_2^0\}$. At the end of the procedure, at least one of I_0^λ, I_1^λ will have red points scattered; at least one of I_0^0, I_1^0 and one of I_0^1, I_1^1, I_2^1 will have blue points scattered; I_0^{01} will have orange points scattered, as will at least one of the I_i^{00}s, one of the I_i^{10}s, one of the I_i^{11}s, and one of the I_i^{12}s; and so on.

fails already, so assume otherwise. Let u be the use of this computation, let $S_\lambda = H \restriction u$ (identifying H with its characteristic function), let $\sigma_0 = f \restriction u'$ where $u' = \max\{u, \text{the use of } \Delta^f(S_\lambda)\}$, and let $\sigma_0' = \sigma_0 \frown m_\lambda$. Write $S_\lambda = \{x_1^\lambda < \ldots < x_{\ell(\lambda)}^\lambda\}$, let $x_0^\lambda = -\infty$ and $x_{\ell(\lambda)+1}^\lambda = +\infty$, and let $X_\lambda = S_\lambda \cup \{x_0^\lambda, x_{\ell(\lambda)+1}^\lambda\}$. Write I_i^λ for the interval $(x_i^\lambda, x_{i+1}^\lambda)$.

Suppose first there is a $g \in [\sigma_0'] \cap \operatorname{dom} \mathsf{C}_{\mathbb{N}}$ such that S_λ is extendible to a solution K of Δ^g. If so, then $\Psi^{g \oplus K}(0)\downarrow = m_\lambda$ but $m_\lambda \in \operatorname{ran} g$, defeating the reduction in this case. Otherwise, if S_λ is not extendible to a solution of Δ^g for any such g, then for each g there is at least one $i \geq 0$ such that the set of color-0 points in I_i^λ with respect to Δ^g is scattered. A scattered set of rationals is in particular nowhere dense with respect to the Euclidean topology, so if the color-0 points in I_i^λ are scattered for Δ^g then the set of points of colors $1, 2, \ldots, k-1$ is dense in that interval for Δ^g.

Let $L > 0$ and suppose σ_{L-1}' has already been found, and the sets X_β and intervals I_i^β have been defined for all $|\beta| = L-1$ and $0 \leq i \leq \ell(\beta)$. We claim that there is a $\sigma_L \in [\sigma_{L-1}']$ with the following property: for all $\rho \in [\sigma_L]$, if any $i \geq 0$ and β with $|\beta| = L-1$ are such that there exists a finite set $S \subset I_i^\beta$ which has Δ^ρ-color L and such that $\Psi^{\rho \oplus S}(0)\downarrow$, then the same is true of σ_L in place of ρ, for the same i (and possibly a different S). To see why this is so, consider the function $a\colon [\sigma_{L-1}'] \to \mathbb{N}$ which sends a string τ to the number of distinct intervals I_i^β, for $|\beta| = L-1$ and $0 \leq i \leq \ell(\beta)$, such that there exists $S_{\beta \frown i} \subset I_i^\beta$ witnessing $\Delta^\tau(S_{\beta \frown i})\downarrow = L$ and $\Psi^{\tau \oplus S_{\beta \frown i}}(0)\downarrow$. Since there are only finitely many intervals at hand, a is bounded, and any $\sigma_L \in [\sigma_{L-1}']$ witnessing its maximum value has the stated property.

Continue by picking any such σ_L, and fix a choice of S_α as above for any $\alpha = \beta^\frown i$ such that a set S_α could be found. Extend σ_L to σ'_L by enumerating the output m_α of $\Psi^{\sigma_L \oplus S_\alpha}(0)$ for each such α. Write $S_\alpha = \{x^\alpha_1 < x^\alpha_2 < \cdots < x^\alpha_{\ell(\alpha)}\}$, let $x^\alpha_0 = x^\beta_i$ and $x^\alpha_{\ell(\alpha)+1} = x^\beta_{i+1}$, and let $X_\alpha = S_\alpha \cup \{x^\alpha_0, x^\alpha_{\ell(\alpha)+1}\}$. If for some $\alpha = \beta^\frown i$ of length L no S_α could be found, simply let $\ell(\alpha) = 0$ and $X_\alpha = \{x^\alpha_0, x^\alpha_1\} = \{x^\beta_i, x^\beta_{i+1}\}$. Then let $I^\beta_i = (x^\beta_i, x^\beta_{i+1})$ for all β of length $L - 1$.

If there is a $g \in [\sigma'_L] \cap \operatorname{dom} \mathsf{C}_\mathbb{N}$ and an α of length L such that S_α is extendible to a color-L solution of Δ^g, then this g witnesses the failure of the reduction since for some solution $U \supset S_\alpha$ of Δ^g, we have $\Psi^{g\oplus U}(0){\downarrow}= m_\alpha$ while $m_\alpha \in \operatorname{ran} g$. Suppose, then, that no such g exists. This means that for all $g \in [\sigma'_L] \cap \operatorname{dom} \mathsf{C}_\mathbb{N}$, for all β with $|\beta| = L - 1$, there is an $i \geq 0$ such that the set of color-L points in I^β_i is scattered with respect to Δ^g. If $L < k - 1$, then continue inductively as above by finding sets S_ξ for $|\xi| = L + 1$ and a string $\sigma_{L+1} \supset \sigma'_L$. But if $L = k - 1$, then we claim that in fact for any $g \in [\sigma'_L] \cap \operatorname{dom} \mathsf{C}_\mathbb{N}$, there must be an S_α with $|\alpha| = L$ that is extendible to a color-$(k - 1)$ solution of Δ^g. To see this, first recall that by assumption at this stage, for any such g, in particular there is an i_0 such that color-0 points are scattered in $I^\lambda_{i_0}$. In turn, there is an i_1 such that color-1 points are also scattered in $I^{i_0}_{i_1}$; an i_2 such that color-2 points are also scattered in $I^{i_0 i_1}_{i_2}$; and so on, so that there is ultimately an i_{k-2} such that the points of colors $0, 1, \ldots, k - 2$ are all scattered in the interval $I^\beta_{i_{k-2}}$, where $\beta = i_0 i_1 \cdots i_{k-3}$ (or $\beta = \lambda$ if $k = 2$). It follows that the set of points of color $k - 1$ is dense in $I^\beta_{i_{k-2}}$ with respect to Δ^g. In particular, the set of color-$(k - 1)$ points in $I^\beta_{i_{k-2}}$ is a solution of Δ^g, so if $\alpha = i_0 i_1 \ldots i_{k-2}$, then we must have been able to find a set S_α of color $k - 1$ as above such that $\Psi^{\sigma_L \oplus S_\alpha}(0)$ converges. Hence for this g and α, there is a solution $U \supset S_\alpha$ of Δ^g with $\Psi^{g\oplus U}(0){\downarrow}= m_\alpha$ and $m_\alpha \in \operatorname{ran} g$, defeating the reduction and completing the proof. $\square$

Corollary 4.2. *For all $k \geq 2$, the Weihrauch reducibility of* $\mathsf{Ind}\,\mathbb{Q}_k$ *to* $\mathsf{iShuffle}_k$ *is strict.*

Since $\mathsf{iShuffle} \equiv_{\mathrm{W}} \mathsf{i}(\eta)^1_{<\infty}$, as mentioned earlier, this means that $\mathsf{Ind}\,\mathbb{Q}$ is not strong enough to find an interval where even one color is dense.

Corollary 4.3. $\mathsf{cRT}^1_{k+1} \not\leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_k$ *for all $k \geq 2$. Hence* $\mathsf{Ind}\,\mathbb{Q}_{k+1} \not\leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_k$.

Proof. Theorem 10 of [20] states that $\mathsf{cRT}^1_{k+1} \leq_{\mathrm{W}} \mathsf{TC}^m_\mathbb{N}$ if and only if $k \leq m$. Because $\mathsf{Ind}\,\mathbb{Q}_k \leq_{\mathrm{W}} \mathsf{TC}^{k-1}_\mathbb{N}$, this means in particular that $\mathsf{Ind}\,\mathbb{Q}_k \not\geq_{\mathrm{W}} \mathsf{cRT}^1_{k+1}$. The second statement follows since $\mathsf{RT}^1_k \leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_k$. $\square$

Corollary 4.4. $\mathsf{Ind}\,\mathbb{Q}_\mathbb{N} \not\leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}$.

Proof. $\mathsf{C}_\mathbb{N} \leq_{\mathrm{W}} \mathsf{RT}^1_\mathbb{N} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_\mathbb{N}$, but $\mathsf{C}_\mathbb{N} \not\leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}$. $\square$

It is clear that $\mathsf{Ind}\,\mathbb{Q}_\mathbb{N} \not\leq_{\mathrm{W}} \mathsf{TC}^*_\mathbb{N}$, since the latter cannot solve $\mathsf{RT}^1_\mathbb{N}$, and $\mathsf{Ind}\,\mathbb{Q}_\mathbb{N} \leq_{\mathrm{W}} \mathsf{iShuffle}_\mathbb{N} \equiv_{\mathrm{W}} \mathsf{TC}^*_\mathbb{N} * \mathsf{C}_\mathbb{N}$. Only one more fact is needed to complete Figure 1:

Corollary 4.5. $\mathsf{TC}_\mathbb{N} \not\leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_\mathbb{N}$, *and hence* $\mathsf{iShuffle}_\mathbb{N} \not\leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_\mathbb{N}$.

Proof. It is known that $\mathsf{TC}_\mathbb{N}$ is a closed fractal, see [3] for the definition. Le Roux and Pauly showed in [16] that if f is a closed fractal and $f \leq_{\mathrm{W}} g * \mathsf{C}_\mathbb{N}$, then $f \leq_{\mathrm{W}} g$. Recalling from Proposition 3.3 that $\mathsf{Ind}\,\mathbb{Q}_\mathbb{N} \equiv_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q} * \mathsf{C}_\mathbb{N}$, it follows that if $\mathsf{TC}_\mathbb{N} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_\mathbb{N}$ then $\mathsf{TC}_\mathbb{N} \leq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}$, contradicting Theorem 4.1. $\square$

Theorem 4.1 and its corollaries show that $\mathsf{Ind}\,\mathbb{Q}$ is uniformly rather weak, but it is not too easy to solve either. In particular, first of all, it is not limit computable. Limit computability of a problem can be characterized as Weihrauch reducibility to lim, and we have

Proposition 4.6. $\mathsf{Ind}\,\mathbb{Q}_2 \not\leq_{\mathrm{W}} \lim$.

Proof. We show that $\mathsf{Ind}\,\mathbb{Q}_2 \not\leq_{\mathrm{W}} \widehat{\mathsf{LPO}}$, which is strongly Weihrauch equivalent to lim (see [3]). Suppose $\mathsf{Ind}\,\mathbb{Q}_2 \leq_{\mathrm{W}} \widehat{\mathsf{LPO}}$ via the functionals Δ and Ψ. We view the unique solution a^p of an $\widehat{\mathsf{LPO}}$-instance $p = \langle p_0, p_1, \ldots\rangle$ as an infinite binary string, with $a^p(i) = 1$ iff there is a 1 anywhere in the string p_i. We will refer to p_i as "the ith LPO-instance" of p.

We use lowercase Greek letters for finite partial colorings of $\mathbb{Q}$, viewed as finite strings. For a given $\sigma \in 2^{<\mathbb{N}}$, let

$$F(\sigma) = \{i \ : \exists \tau \supset \sigma \text{ such that } \Delta^\tau \text{ outputs a 1 somewhere in its } i\text{th } \mathsf{LPO}\text{-instance}\}.$$

Thus if $i \in F(\sigma)$ and $\tau \supset \sigma$ is as above, then for any $c \in [\tau]$, $a^c(i) = 1$. If $i \notin F(\sigma)$, then for any $c \in [\sigma]$, $a^c(i) = 0$. This means that for any finite partial coloring σ and any s, we can extend σ to some τ such that $a^c \restriction s$ is the same string for all $c \in [\tau]$.

So, to defeat the Weihrauch reduction given by Δ and Ψ, first build an instance $c = \bigcup_s \sigma_s$ of $\mathsf{Ind}\,\mathbb{Q}_2$ together with its solution $a = \bigcup_s a_s$ of Δ^c by finite extension, starting with the empty string $\sigma_0 = \lambda$ and $a_0 = \lambda$. At stage $s+1$, let i be the least element of $F(\sigma_s)$ greater than or equal to s, if such an element exists. If it does, let σ_{s+1} be an extension of σ_s corresponding to i as in the definition of $F(\sigma_s)$, i.e., an extension τ such that Δ^τ puts a 1 somewhere in its ith LPO-instance. Then let $a_{s+1} = a_s^\frown 00\cdots 1$, where the 1 is in bit i. Otherwise, if no such i exists, let $\sigma_{s+1} = \sigma_s^\frown 0$ and $a_{s+1} = a_s^\frown 0$. In the latter case, every $c \in [\sigma_s]$ gives Δ^c only finitely many LPO-instances having a 1, and accordingly this results in a having cofinitely many 0s.

Because this procedure builds a valid instance c of $\mathsf{Ind}\,\mathbb{Q}_2$ together with a valid solution a to Δ^c, there must be a stage s and at least two distinct $p, q \in \mathbb{Q}$ with $\Psi^{c\oplus a}(p)[s]{\downarrow}= \Psi^{c\oplus a}(q)[s]{\downarrow}= 1$. That is, Ψ must eventually put at least two points into a purported solution to c. If $c(p) \neq c(q)$, the reduction already fails at this point, so assume p and q were given the same color (we may also assume both had been colored by stage s). Then at all future stages, instead of continuing to extend σ_s as in the last paragraph, simply color every remaining rational with color $1 - c(p)$. This produces a new coloring d with $d \restriction s = c \restriction s$ and such that Δ^d has solution b with $b \restriction s = a \restriction s$. Therefore $\Psi^{d\oplus b}$ still includes p and q in its solution to d. Since only finitely many rationals share the d-color of p and q, the reduction fails to produce a densely ordered monochromatic set. ☐

Next, we introduce a class of problems that is incomparable with the class of limit computable problems, but all of which are still weak in a precise sense. A problem is *first-order* if it has codomain $\mathbb{N}$.

Definition 4.7. We say a problem P is *pointwise c.e. traceable*, or p.c.e.t. for short, if P is Weihrauch equivalent to a first-order problem Q with the property that there is a Turing functional Γ such that for all $q \in \operatorname{dom} Q$, $\Gamma^q(0)\downarrow = e$ for some index e (depending on q) with W_e^q finite and $Q(q) \cap W_e^q \neq \emptyset$.

Roughly, this means that we can uniformly enumerate a finite list of solutions to any instance p of P, possibly along with finitely many non-solutions. Examples of pointwise c.e. traceable problems include $\mathsf{RT}^1_{\mathbb{N}}$ and any problem with finite computable codomain.

Theorem 4.8. *If P is any pointwise c.e. traceable problem, then $\mathsf{Ind}\,\mathbb{Q}_2 \not\leq_{\mathrm{W}} P$.*

Although this result was found independently in the course of the present work, the core idea of its proof was outlined in [9, §5], where the authors call it a "disjoint extension commitment" of ER^1 (or $\mathsf{Ind}\,\mathbb{Q}$), and may have appeared elsewhere. To motivate our argument, we will sketch it for the same special case treated in [9], namely $P = \mathsf{cRT}^1_2$. Suppose $\mathsf{Ind}\,\mathbb{Q}_2 \leq_{\mathrm{W}} \mathsf{cRT}^1_2$ with the reduction being given by functionals Δ and Ψ. If we build a coloring c of $\mathbb{Q}$, then Δ^c can build an instance with a solution of color 0, of color 1, or without loss of generality both. Eventually Ψ has to commit to initial segments of solutions H_0, H_1 to c corresponding to $0, 1 \in \mathsf{cRT}^1_2(\Delta^c)$. When enough rational points are output by Ψ, one can find disjoint intervals I_0 and I_1 with endpoints in H_0 and H_1, respectively, and make the remaining points in both intervals the opposite color as their respective endpoints. This defeats the reduction since when Ψ outputs two points, it commits to outputting infinitely many of the same color between them (this is exactly the "commitment" alluded to above), which we have made impossible in both I_0 and I_1.

It is intuitively clear that this idea should translate to any problem P with finite codomain. To further extend the approach to a general p.c.e.t. problem, one can watch for numbers to appear in W_e^p, where $p = \Delta^c \in \operatorname{dom} P$ and e is the index as in the definition of a p.c.e.t. problem. Only finitely many numbers will ever appear for any c, so we can diagonalize against all its elements at once as above, adjusting the set of disjoint intervals as needed whenever a new element of W_e^p is enumerated.

Proof of Theorem 4.8. Suppose $\mathsf{Ind}\,\mathbb{Q}_2 \leq_{\mathrm{W}} P$ via Δ and Ψ. Without loss of generality, we can assume P is first-order. We build a coloring $c = \lim_s c_s$ of $\mathbb{Q}$ by finite extension for which this purported reduction fails to produce a solution to c. Begin with $c(0) = 0$.

If Γ is a functional as in the definition of a p.c.e.t. problem, let $N(s) = W_x^{\Delta(c)}[s]$ if $\Gamma^{\Delta(c)}(0)[s]\downarrow = x$, or $\emptyset$ if this computation diverges. Since Ψ only has oracle access to c together with a single number from $N(s)$, we can view Ψ as a finite set of functionals Ψ_i, one per $i \in N(s)$, each only having oracle access to c. Then we can diagonalize against all such Ψ_i without regard to the eventual correctness of any particular solution i. This is enough to defeat the reduction, because we only need to show Ψ_i^c fails to compute a c-solution for one Δ^c-solution i.

Define a recurrence relation $e(n)$ by letting $e(1) = 3$ and $e(n+1) = 2ne(n)+2$ for $n \geq 1$. We claim that if any functionals $\Phi_0, \ldots, \Phi_{k-1}$ each enumerate at least $e(k)$ points by stage s, then one can choose at stage s a collection of k disjoint rational intervals $I_0, \ldots, I_{k-1}$ such that for each i, the endpoints of I_i were enumerated by Φ_i. The proof is by induction on k. The case $k = 1$ is immediate, so suppose the claim holds for all $i \leq k$ and that we have $k+1 > 1$ functionals $\Phi_0, \ldots, \Phi_{k-1}, \Phi_k$ each

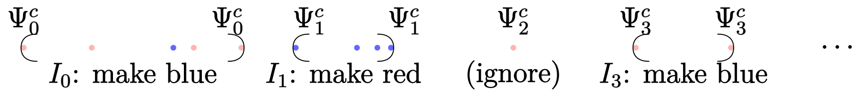

FIGURE 3. An example of how the construction might work in diagonalizing against some functionals $\Psi_0, \Psi_1, \dots$. The points shown have already been colored red or blue by c at this stage, and enumerated by Ψ_i^c for some i. Ψ_0^c, Ψ_1^c, and Ψ_3^c have enumerated enough points so that we can find three disjoint intervals I_0, I_1, I_3 whose endpoints are respectively among the domains of $\Psi_0^c, \Psi_1^c, \Psi_3^c$ up to this point. Since the endpoints of I_0 are red, we plan to make all future points in I_0 blue, and similarly for I_1 and I_3. At this stage, Ψ_2^c has not enumerated enough points to find an I_2 disjoint from $I_0 \cup I_1 \cup I_3$, so we ignore it for now.

having enumerated at least $e(k+1)$ points. Now, for purposes of the inductive step, we are only considering the first k functionals $\Phi_0, \dots, \Phi_{k-1}$ to have enumerated $2e(k)$ points each; any extra points can be discarded. Then the $2ke(k)$ total points enumerated by $\Phi_0, \dots, \Phi_{k-1}$ divide $\mathbb{Q}$ into $2ke(k)+1$ open intervals. If Φ_k enumerates at least $2ke(k)+2$ points, by the pigeonhole principle, at least one of those $2ke(k)+1$ open intervals will contain two such points $a_k, b_k \in \mathbb{Q}$. We then choose $I_k = (a_k, b_k)$. This splits the set of points enumerated by the Φ_i, $i < k$, into two groups, those below a_k and those above b_k. Using the pigeonhole principle again, for each $i < k$, one of those groups contains at least $2e(k)/2 = e(k)$ points enumerated by Φ_i. The inductive hypothesis can now be applied to the set of (at most k) functionals with at least $e(k)$ points enumerated below a_k, and separately to the set of functionals with at least $e(k)$ points enumerated above b_k, to produce a set of disjoint rational intervals as required.

Now we proceed to the main argument, which is essentially by finite injury (though without a priority order). Let $A(s)$ be the set of $i \in N(s)$ such that $\Psi_i^c[s]$ has output at least one point. Let $V(s)$ be the set of $i \in A(s)$ such that $\Psi_i^c[s]$ has output at least $e(|A(s)|)$ points. At stage $s+1$, for each $i \in V(s)$, if Ψ_i does not have a follower, assign it a follower which is a (code for a) rational interval I_i as furnished by the claim in the previous paragraph, applied to the functionals $\{\, \Psi_i : i \in V(s) \,\}$. Thus the endpoints of I_i were enumerated by Ψ_i^c by stage s, and $I_i \cap I_j = \emptyset$ if $i \neq j \in V(s)$. If there is a $j \in A(s+1) \setminus A(s)$, cancel the followers of all Ψ_i which have one at stage $s+1$. Now color $c_{s+1}(s+1) = 1 - c_s(a_i)$ if a_i is an endpoint of I_i for some $i \in V(s)$, and color $c_{s+1}(s+1) = 0$ otherwise. (See Figure 3 for an example of how this might look at a particular stage.)

Because $A(s)$ eventually stabilizes, each Ψ_i will have a follower canceled at most finitely many times, and so the set of intervals I_i also stabilizes. Any Ψ_i which outputs infinitely many points will have $i \in V(s)$ for large enough s, so for each such i, there are two points output by Ψ_i^c between which only finitely many are of the same color. Therefore Ψ_i^c does not compute a solution to c. □

Note that the construction above can be done computably.

Corollary 4.9. *(i)* $\mathsf{Ind}\,\mathbb{Q}_2 \not\leq_{\mathrm{W}} \mathsf{cRT}^n_{\mathbb{N}}$ *and hence* $\not\leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.
(ii) $\mathsf{Ind}\,\mathbb{Q}_2 \not\leq_{\mathrm{W}} (\mathsf{LPO}^{(n)})^*$.
(iii) $\mathsf{Ind}\,\mathbb{Q}_2 \not\leq_{\mathrm{W}} \mathsf{cShuffle}$.

Proof. (i) $\mathsf{cRT}^n_{\mathbb{N}}$ is p.c.e.t. since the set of solutions to c is contained in $\operatorname{ran} c$, and $\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{cRT}^1_{\mathbb{N}}$. (ii) If there are k instances of $\mathsf{LPO}^{(n)}$ given in parallel, at most 2^k distinct solutions are possible, and they can each be encoded as a single natural number. So in particular $(\mathsf{LPO}^{(n)})^*$ is p.c.e.t. (iii) This follows from (ii) as $\mathsf{cShuffle} \equiv_{\mathrm{W}} (\mathsf{LPO}')^*$ [20]. □

Before continuing to the next section, we pause to make some observations about p.c.e.t. problems. Pointwise c.e. traceability can be viewed as a generalization of c.e. traceability, which is the special case where P is a single-valued function from $\mathbb{N}$ to $\mathbb{N}$ (see for instance [22, §11.4] for the definition). It is also related to the notion of a pointwise finite problem, defined in [11], which is a problem such that every instance has finitely many solutions. However, a p.c.e.t. problem is not necessarily pointwise finite, one counterexample being $\mathsf{C}_{\mathbb{N}}$. And the existence of functions $\mathbb{N} \to \mathbb{N}$ which are not c.e. traceable shows that even first-order pointwise finite problems may not be p.c.e.t.

The class of p.c.e.t. problems has some attractive algebraic properties:

- If P and Q are p.c.e.t., then so are $P \times Q$ and P^*.
- If P is p.c.e.t. and $Q \leq_{\mathrm{W}} P$, then Q is p.c.e.t.

(These properties hold for problems P and Q on any represented spaces, as is straightforward to show.) On the other hand,

- $P * Q$ may not be p.c.e.t. even if both P and Q are. For example, $\mathsf{C}_{\mathbb{N}}$ and LPO' are both p.c.e.t. but $\mathsf{C}_{\mathbb{N}} * \mathsf{LPO}' \geq_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}} \geq_{\mathrm{W}} \mathsf{Ind}\,\mathbb{Q}_2$ [2, Corollary 8.10].
- Neither P^{u*} nor $P^{\diamond}$ need be p.c.e.t. if P is. It follows from [23, Theorem 7.2] that $(\mathsf{LPO}')^{u*} \equiv_{\mathrm{W}} (\mathsf{LPO}')^{\diamond} \equiv_{\mathrm{W}} \mathsf{C}'_{\mathbb{N}}$, and it is known that $\mathsf{C}'_{\mathbb{N}} >_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}}$ [2]. (See [23] for definitions of the undefined notation used here.)

Just as the fact that LPO' and lim are Weihrauch incomparable (which is known in the literature, see e.g. [2]) shows that pointwise c.e. traceability is logically incomparable with limit computability, the fact that LPO' is p.c.e.t. shows it to be strictly more general than computability with finitely many mind changes: the latter can be characterized by Weihrauch reducibility to $\mathsf{C}_{\mathbb{N}}$, which is p.c.e.t. and known to be Weihrauch incomparable with LPO'.

5. The countable equivalence relation

The countable equivalence relation $\mathscr{E}$ was defined in the introduction. We identify $E = \operatorname{dom} \mathscr{E}$ with $\mathbb{N} \times \mathbb{N}$, viewing (x, y) as the yth element of the xth equivalence class. We also refer to the xth equivalence class as the "xth column" of $\mathscr{E}$.

The proof of this theorem, in particular Lemma 5.6, was obtained jointly with Linda Westrick.

Theorem 5.1. *For all $k \geq 2$, $\mathsf{SRT}^2_k \leq_{\mathrm{W}} \mathsf{Ind}\,\mathscr{E}_k \leq_{\mathrm{W}} \mathsf{RT}^2_k$, but $\mathsf{RT}^2_2 \not\leq_{\mathrm{c}} \mathsf{Ind}\,\mathscr{E}_{\mathbb{N}}$ and $\mathsf{Ind}\,\mathscr{E}_2 \not\leq_{\mathrm{W}} \mathsf{SRT}^2_{\mathbb{N}}$.*

Since $\mathsf{SRT}^2_j \not\leq_{\mathrm{c}} \mathsf{RT}^2_k$ whenever $j > k \geq 2$ [14, 19], we obtain Figure 4, a very similar diagram to that shown for $\mathsf{Ind}\,\mathbb{Q}$ in Figure 1.

The rest of the section is occupied with the proof of Theorem 5.1, which is broken into four lemmas. The first two establish that $\mathsf{SRT}^2_k \leq_{\mathrm{W}} \mathsf{Ind}\,\mathscr{E}_k \leq_{\mathrm{W}} \mathsf{RT}^2_k$, the third shows $\mathsf{RT}^2_2 \not\leq_{\mathrm{c}} \mathsf{Ind}\,\mathscr{E}_{\mathbb{N}}$ and thus gives a separation between $\mathsf{Ind}\,\mathscr{E}_k$ and RT^2_k, and the fourth shows $\mathsf{Ind}\,\mathscr{E}_2 \not\leq_{\mathrm{W}} \mathsf{SRT}^2_{\mathbb{N}}$ and so separates $\mathsf{Ind}\,\mathscr{E}_k$ from SRT^2_k. We use the notation $\{x, y\}$ for unordered pairs, and omit the extra parentheses from $c(\{x, y\})$ and $c((x, y))$ to reduce visual clutter.

$$
\begin{tikzcd}
\mathsf{RT}^2_2 \arrow[d] & \mathsf{RT}^2_3 \arrow[l] \arrow[d] & \mathsf{RT}^2_4 \arrow[l] \arrow[d] & \cdots \arrow[l] & \mathsf{RT}^2_+ \arrow[l] \arrow[d] & \mathsf{RT}^2_{\mathbb{N}} \arrow[l] \arrow[d] \\
\mathsf{Ind}\,\mathscr{E}_2 \arrow[d] & \mathsf{Ind}\,\mathscr{E}_3 \arrow[l] \arrow[d] & \mathsf{Ind}\,\mathscr{E}_4 \arrow[l] \arrow[d] & \cdots \arrow[l] & \mathsf{Ind}\,\mathscr{E} \arrow[l] \arrow[d] & \mathsf{Ind}\,\mathscr{E}_{\mathbb{N}} \arrow[l] \arrow[d] \\
\mathsf{SRT}^2_2 & \mathsf{SRT}^2_3 \arrow[l] & \mathsf{SRT}^2_4 \arrow[l] & \cdots \arrow[l] & \mathsf{SRT}^2_+ \arrow[l] & \mathsf{SRT}^2_{\mathbb{N}} \arrow[l]
\end{tikzcd}
$$

FIGURE 4. Reducibilities between SRT^2_k, $\mathsf{Ind}\,\mathscr{E}_k$, and RT^2_k. An arrow from Q to P signifies that $P <_{\mathrm{W}} Q$. No other Weihrauch reductions hold other than those implied by transitivity. The diagram remains true if $<_{\mathrm{W}}$ is replaced by $<_{\mathrm{c}}$, except that the reductions $\mathsf{SRT}^2_k \leq_{\mathrm{c}} \mathsf{Ind}\,\mathscr{E}_k$ are not known to be strict.

Lemma 5.2. $\mathsf{Ind}\,\mathscr{E}_k \leq_{\mathrm{W}} \mathsf{RT}^2_k$.

Proof. Let $c\colon \mathbb{N} \to k$ be a coloring of E. Define an RT^2_k-instance $d\colon [\mathbb{N}]^2 \to k$ by

$$d\{x, y\} = c(x, y) \quad \text{if } x < y.$$

Let $\tilde{H}$ be an infinite homogeneous set for d, and let H be the $\tilde{H}$-computable set

$$H = \{\, (x, y) \,:\, x \leq y \in \tilde{H} \,\} \subseteq \mathbb{N}^2.$$

It is clear that H is monochromatic for c. For each $x \in \tilde{H}$, $(x, y) \in H$ for all of the infinitely many $y > x$ in $\tilde{H}$, so H induces a substructure isomorphic to $\mathscr{E}$. $\square$

Lemma 5.3. $\mathsf{SRT}^2_k \leq_{\mathrm{W}} \mathsf{Ind}\,\mathscr{E}_k$.

Proof. Let $c\colon [\mathbb{N}]^2 \to k$ be an instance of SRT^2_k. Define $d\colon E \to k$ by

$$d(x, y) = \begin{cases} c\{x, y\}, & \text{if } x < y \\ 0, & \text{otherwise.} \end{cases}$$

Let H be an $\mathsf{Ind}\,\mathscr{E}_k$-solution for d. We build an SRT^2_k-solution for c as follows: write the least element of H as (x_1, y). Then find another column $x_2 > x_1$ represented in H such that $c\{x_1, x_2\} = c\{x_1, y\}$. There is such an x_2 by stability of the coloring c: the d-color of H must be the same as the stable c-color of column x_1, so for all large enough x, $c\{x_1, x\} = c\{x_1, y\} = d(x_1, y)$. Since there are infinitely many equivalence classes of E represented in H, and the equivalence class of x_1 has infinitely many points of this color, there is eventually a column n represented in H with $d(x_1, x) = d(x_1, y)$, and we can take x_2 to be this x. (Note (x_1, x_2) need not be in H.) Next, find a column $x_3 > x_2$ represented in H such that

$$c\{x_1, x_2\} = c\{x_1, x_3\} = c\{x_2, x_3\}.$$

Again, this must be possible by stability and the existence of infinitely many different columns represented in H. Proceed in the same way, for each i finding a column x_i of H with $x_i > x_{i-1}$ such that $c\{x_j, x_k\} = c\{x_1, x_2\}$ for all $1 \leq j < k \leq i$. The set $\{\, x_i \,:\, i \in \mathbb{N} \,\}$ is then an infinite homogeneous set for c. $\square$

Corollary 5.4. *For all $k \geq 2$, $\mathsf{Ind}\,\mathscr{E}_{k+1} \not\leq_{\mathrm{c}} \mathsf{Ind}\,\mathscr{E}_k$ and $\mathsf{Ind}\,\mathscr{E} \not\leq_{\mathrm{c}} \mathsf{Ind}\,\mathscr{E}_k$. Also $\mathsf{Ind}\,\mathscr{E}_{\mathbb{N}} \not\leq_{\mathrm{c}} \mathsf{Ind}\,\mathscr{E}$.*

Proof. All three statements follow from the above lemmas together with the fact that $\mathsf{SRT}^2_{k+1} \not\leq_{\mathrm{c}} \mathsf{RT}^2_k$ [19, Corollary 3.6]. □

Lemma 5.5. $\mathsf{RT}^2_2 \not\leq_{\mathrm{c}} \mathsf{Ind}\,\mathscr{E}_{\mathbb{N}}$.

Proof. We show that $\mathsf{Ind}\,\mathscr{E}_{\mathbb{N}}$ admits Δ^0_2 solutions, that is, every computable instance has a Δ^0_2 solution. Since RT^2_2 does not share this property [15, Corollary 3.2], the separation follows.

Let c be a computable coloring of E with $\operatorname{ran} c \subseteq k$ for some k. Suppose c stabilizes in infinitely many columns. Nonuniformly pick a color i which is the stable color of infinitely many columns, and use $\emptyset'$ to enumerate the set S of such columns by asking, for each x, y, if there is $z > y$ with $c(x, z) \neq i$, and putting $x \in S$ if the answer is 'no'. As a relativization of the fact that every infinite c.e. set has an infinite computable subset, there is an infinite Δ^0_2 subset of S, and the set of points of color i in all such columns is a Δ^0_2 solution to c.

If instead c stabilizes in only finitely many columns, nonuniformly delete those columns; the remaining columns each have infinitely many points of at least two different colors. Suppose there are colors $i_1 \neq i_2$ such that infinitely many columns include only finitely many points of colors other than i_1 or i_2. The set S of such columns can again be enumerated by $\emptyset'$ by asking, for each x, y, if there is $z > y$ with $c(x, z) \notin \{i_1, i_2\}$, and putting $x \in S$ if the answer is 'no'. S has an infinite Δ^0_2 subset, and the set of points of color i_1 (or of i_2) in the columns in that subset is a Δ^0_2 solution to c because, by assumption, all columns of S have infinitely many points both of color i_1 and of color i_2.

In general, let $n \geq 1$ be the least possible number of distinct colors $i_1, \ldots, i_n$ such that there is an infinite set S of columns each of which includes only finitely many points of colors other than $i_1, \ldots, i_n$. That is, n is such that only finitely many columns have infinitely many points of $n-1$ or fewer different colors. Nonuniformly delete these columns, so that every column has infinitely many points of each of at least n colors, and fix a particular choice of $i_1, \ldots, i_n$ as above. Then for this set of colors $\{i_1, \ldots, i_n\}$, $\emptyset'$ can again enumerate S by letting $x \in S$ if we find a y such that there is no $z > y$ having $c(x, z) \notin \{i_1, \ldots, i_n\}$. There is an infinite Δ^0_2 subset of S, and the set of points of color i_j in the columns of this subset, for any $1 \leq j \leq n$, is a Δ^0_2 solution of c. □

Lemma 5.6. $\mathsf{Ind}\,\mathscr{E}_2 \not\leq_{\mathrm{W}} \mathsf{SRT}^2_{\mathbb{N}}$.

Proof. Suppose $\mathsf{Ind}\,\mathscr{E}_2 \leq_{\mathrm{W}} \mathsf{SRT}^2_{\mathbb{N}}$ via the functionals Δ and Ψ. We use lowercase Latin letters for instances of $\mathsf{Ind}\,\mathscr{E}_2$ and lowercase Greek letters for their initial segments.

We will find (noncomputably) a finite sequence of strings $\sigma_0 \subset \sigma_1 \subset \cdots$ such that for some s, there is an $\mathsf{Ind}\,\mathscr{E}_2$-instance $c \in [\sigma_s]$ which defeats the Weihrauch reduction. To accomplish this, we build certain monochromatic sets $H^i = \lim_s h^i_s$ of $\mathsf{SRT}^2_{\mathbb{N}}$-columns by finite extension, for each color i used by Δ^c, to use as oracles for Ψ. Then we diagonalize against Δ and Ψ by adding, along the way, finitely many locks on the columns of E (to be explained below), represented by a lock function $L = \bigcup_s L_s \colon \subseteq \mathbb{N} \to 2$ that will be updated finitely many times. These locks are in fact the same as those used in Cohen forcing with locks, but we do not formally use a notion of forcing here and the argument does not necessarily produce a generic c. The overall strategy, roughly speaking, is to pick the sets h^i_s

in such a way that for each s, either (a) for some $c \in [\sigma_s]$ respecting the current lock function, h^i_s is guaranteed to extend to a Δ^c-solution of color i; or (b) there are no Δ^c-solutions of color i for any such c. We will see that (a) must eventually occur for some i, at which stage the procedure ends.

For any string σ and lock function L, let $[\sigma, L]$ be the set of (finite or infinite) $\tau \in [\sigma]$ such that whenever $L(x) \in \{0, 1\}$ is defined, if (x, y) is newly colored by τ, then $\tau(x, y) = L(x)$. "Newly colored" simply means that τ is defined on (x, y) while σ is not. So $[\sigma, L]$ is the set of extensions of σ which respect the lock function L by only adding points of color $L(x)$ in column x. Observe that $[\sigma, L'] \subset [\sigma, L]$ if $L' \supset L$.

The central claim that makes this proof work is as follows: for any σ, any $\mathsf{SRT}^2_{\mathbb{N}}$-column n, and any lock function L,

$$\exists \tau \in [\sigma, L]\ \exists i \in \mathbb{N}\ \forall \rho \in [\tau, L]\ (\text{if } \Delta^\rho \text{ newly colors } \{n, m\} \text{ where } m \geq n, \\ \text{then } \Delta^\rho\{n, m\} = i).$$

We say τ L-forces column n to be color i when this happens. If the claim were false for some σ, n, and L, then for every $\tau \in [\sigma, L]$ and $i \in k$, there would be a further $\rho(\tau) \in [\tau, L]$ such that $\Delta^{\rho(\tau)}$ adds a point not of color i into column n. Letting $\tau_0 = \sigma$ and $\tau_{s+1} = \rho(\tau_s)$ for all s, the $\mathsf{Ind}\,\mathscr{E}_2$-instance $d = \lim_s \tau_s$ is such that Δ^d has infinitely many points of at least two different colors in column n, so is not an $\mathsf{SRT}^2_{\mathbb{N}}$-instance, a contradiction. A key point is that this argument works for any L independently of σ, so that even if we change L partway through the proof we can always extend any σ_s to a σ_{s+1} which L-forces a fresh column to be some color. A column already L-forced remains L'-forced to be the same color for any $L' \supset L$.

For each σ, lock function L, and $i \in \mathbb{N}$, let

$$C_i(\sigma, L) = \{\, n : \sigma\ L\text{-forces } \mathsf{SRT}^2_{\mathbb{N}}\text{-column } n \text{ to be color } i \,\}.$$

Now we describe the procedure to find (σ_s), (h^i_s), and (L_s) in detail. First of all, we claim there is a σ_0 such that $\Delta^c([\mathbb{N}]^2) = \Delta^{\sigma_0}([\mathbb{N}]^2)$ for all $c \in [\sigma_0]$. In other words, σ_0 already makes Δ use all the colors it will ever use. To see this, start with τ_0 such that $\Delta^{\tau_0}\{x_0, y_0\}\downarrow = i_0$ for some x_0, y_0 and for some color $i_0 \in \mathbb{N}$. Extend τ_s to τ_{s+1} if there are x_{s+1}, y_{s+1}, and $i_{s+1} \notin \{i_0, \ldots, i_s\}$ with $\Delta^{\tau_{s+1}}\{x_{s+1}, y_{s+1}\}\downarrow = i_{s+1}$. Since any instance of $\mathsf{SRT}^2_{\mathbb{N}}$ only uses finitely many colors, the sequence (τ_s) stabilizes, and $\sigma_0 = \lim_s \tau_s$ has the stated property.

Fix σ_0 as above and, without loss of generality, let k be such that the range of Δ^{σ_0} is $k = \{0, \ldots, k-1\}$. Let $D \subseteq k$ be the set of "already-diagonalized" colors, starting with $D = \emptyset$; we will use D to keep track of which colors i have already witnessed convergence of Ψ on H^i. As long as $D \subsetneq k$, the colors in D at stage s will be exactly those of which no Δ^c-solution can be found for any $c \in [\sigma_s, L_s]$. For any σ and L, let $b(\sigma, L)$ be the largest possible number of distinct colors found among the columns freshly L-forced by any $\tau \in [\sigma, L]$. More specifically, let

$$b(\sigma, L) = \max_{\tau \in [\sigma, L]} \#\{\, i \in k \setminus D : \max C_i(\tau, L) > \max C_i(\sigma, L) \,\}.$$

Let $M(\sigma, L)$ be the set of $\tau \in [\sigma, L]$ witnessing this maximum. As long as $D \neq k$, we have $b(\sigma, L) > 0$ and thus $M(\sigma, L) \neq \emptyset$ for all σ and L.

Start with σ_0 as above, $L_0 = D = \emptyset$, and $h^i_0 = \emptyset$ for each $i \in k$. At stage $s > 0$, we use L_{s-1} to obtain σ_s, use σ_s to obtain h^i_s, and use h^i_s to obtain L_s. Specifically, first of all, let σ_s be any element of $M(\sigma_{s-1}, L_{s-1})$. Thus σ_s L_{s-1}-forces a fresh column to stabilize to a color i for as many different i as possible, excluding colors in D.

Next, write $m_s^i = \max C_i(\sigma_s, L_s)$. (Notice that we may have $m_\ell^i = m_{\ell+1}^i$ for some values of ℓ, if $\sigma_{\ell+1}$ did not L_ℓ-force any fresh column to be color i.) Then let $h_s^i = h_{s-1}^i \cup \{m_n^i\}$, where n is least such that

- $\max h_{s-1}^i < m_n^i \leq m_s^i$, and
- for all $m \in h_{s-1}^i$, we have $\Delta^{\sigma_s}\{m, m_n^i\}{\downarrow}= i$ with m_n^i greater than the largest element of column m which Δ^{σ_s} makes a color other than i.

If no such n exists, take $h_s^i = h_{s-1}^i$. By convention we take $\max\emptyset = -1$. The set h_s^i is a valid initial segment of some solution to Δ^c for *any* $c \in [\sigma_s, L_{s-1}]$ for which infinitely many $\mathsf{SRT}^2_{\mathbb{N}}$-columns stabilize to i. This is because $[h_s^i]^2$ is Δ^c-monochromatic by design, and starting from any Δ^c-solution G, one can truncate G to $\hat{G}$ by selecting only the columns whose indices are higher than the point at which any of the columns in h_s^i stabilize, producing an infinite $\hat{G}$ with $[h_s^i \cup \hat{G}]^2$ monochromatic. Hence either h_s^i extends to a solution of color i (for some c as above), or there is no solution of color i (for any c as above).

To end stage s, for each $i \notin D$, search for x, y with $x \notin \operatorname{dom} L_{s-1}$ (i.e., with column x not already locked) such that

$$\Psi^{\sigma_s \oplus h_s^i}(x, y){\downarrow}= 1, \tag{1}$$

and such that (x, y) was already colored by c. Such numbers must eventually be found for some $i \in k \setminus D$ and s if Ψ is to compute an $\mathsf{Ind}\,\mathscr{E}_2$-solution, and we can take $x \notin \operatorname{dom} L_{s-1}$ because Ψ will output elements of infinitely many columns of E, while L_{s-1} is finite. That we can also take $i \in k \setminus D$ is justified below. If x and y are found at stage s, set $L_s(x) = 1 - \sigma_s(x, y)$, put $i \in D$, and end the stage. That is, we lock column x to be the opposite color that Ψ already committed to there, and with $i \in D$ we no longer search for further extensions of h_s^i at future stages. If instead no such x and y are found for any i, let $L_s = L_{s-1}$ and end the stage.

Suppose the numbers x, y as above are found at stage s and with oracle h_s^i. If there is a $d \in [\sigma_s, L_s]$ for which h_s^i extends to a solution H of Δ^d, then this d witnesses the failure of the Weihrauch reduction, because $\Psi^{d \oplus H}$ outputs a point in E whose color is shared by only finitely many other points in the same column. If such a d exists then the procedure ends at this stage. In particular, this must happen if $D = k$, because then an oracle h_s^i witnessing convergence of Ψ was already found for every $i \in k$, and at least one of these must extend to a valid solution of Δ^d for any $d \in [\sigma_s, L_s]$.

Otherwise, for this i and indeed for all $i \in D$, there is no $d \in [\sigma_s, L_s]$ with h_s^i extending to a solution of Δ^d. As remarked above, this implies that for all $d \in [\sigma_s, L_s]$, Δ^d has only finitely many columns stabilize to j, for each $j \in D$. Then it must still be possible to L_s-force infinitely many columns to be color i for at least one $i \in k \setminus D$. Convergence of Ψ on h_s^i must eventually occur for any such i, because no $d \in [\sigma_s]$ will use any colors outside of k, which explains why we could take $i \in k \setminus D$ in (1) above. Hence the recursive procedure we have described can be continued for the remaining colors in k. This completes the proof since the procedure is guaranteed to end in success if $D = k$. $\square$

It is clear that the proofs of Lemmas 5.2 and 5.3 do not depend on k and thus extend to show $\mathsf{SRT}^2_+ \leq_W \mathsf{Ind}\,\mathscr{E} \leq_W \mathsf{RT}^2_+$ and $\mathsf{SRT}^2_{\mathbb{N}} \leq_W \mathsf{Ind}\,\mathscr{E}_{\mathbb{N}} \leq_W \mathsf{RT}^2_{\mathbb{N}}$. Hence the above suffices to establish all reductions and nonreductions shown in Figure 4.

6. Further directions

Solutions of $\mathsf{Ind}\,\mathbb{Q}$ are rather nebulous: for example, if x and y are two elements of a solution H, one can delete the whole interval $[x, y]$ from H and still obtain a solution. The seeming weakness of $\mathsf{Ind}\,\mathbb{Q}$ may be a consequence of this nebulosity, and it is unclear how much power it derives from the fact that it outputs a second-order object, i.e., a set of rationals. An investigation of these properties can be put on precise footing with the following notions introduced in [8] and [11], respectively:

- The *first-order part* of a problem P, ${}^1P$, is the strongest first-order problem Weihrauch reducible to P.
- The *deterministic part* of P, $\operatorname{Det} P$, is the strongest problem Weihrauch reducible to P for which every instance has a unique solution.

Question 6.1. How strong exactly are the first-order and deterministic parts of $\mathsf{Ind}\,\mathbb{Q}_k$ and $\mathsf{Ind}\,\mathbb{Q}$? Is $\mathsf{Ind}\,\mathbb{Q}$ Weihrauch equivalent to a first-order problem?

The second question was answered in the negative in recent work by Dzhafarov, Solomon, and Valenti on the tree pigeonhole principle TT^1_+ [7], which as mentioned earlier is equivalent to $\mathsf{Ind}\,\mathbb{Q}$. They showed that none of the problems TT^1_k, TT^1_+, or $\mathsf{TT}^1_{\mathbb{N}}$ are Weihrauch equivalent to any first-order problem, and indeed ${}^1\mathsf{TT}^1_{\mathbb{N}} \equiv_W \mathsf{RT}^1_{\mathbb{N}}$. Moreover, RT^1_k turns out to be strictly weaker than ${}^1\mathsf{TT}^1_k$ for all $k \geq 2$; whether ${}^1\mathsf{TT}^1_k$ and ${}^1\mathsf{TT}^1_+$ have a precise characterization in terms of known Weihrauch degrees is open. Regarding $\operatorname{Det}(\mathsf{Ind}\,\mathbb{Q})$, Manlio Valenti pointed out to the author that since $\operatorname{Det}\mathsf{RT}^1_k \equiv_W \lim_k$ and $\operatorname{Det}(\mathsf{TC}^*_{\mathbb{N}}) \equiv_W \mathsf{C}_{\mathbb{N}} \equiv_W \lim_{\mathbb{N}}$, where $\lim_k$ maps an eventually constant element of $k^{\mathbb{N}}$ to its limit and similarly for $\lim_{\mathbb{N}}$, we must have $\lim_k <_W \operatorname{Det}(\mathsf{Ind}\,\mathbb{Q}) <_W \lim_{\mathbb{N}}$ for all k. (The second reduction is strict by Theorem 4.1.) Precise characterizations of the degrees of $\operatorname{Det}(\mathsf{Ind}\,\mathbb{Q}_k)$ and of $\operatorname{Det}(\mathsf{Ind}\,\mathbb{Q})$ have not been established. On the other hand, $\operatorname{Det}(\mathsf{Ind}\,\mathbb{Q}_{\mathbb{N}}) \equiv_W \mathsf{C}_{\mathbb{N}}$ since $\mathsf{C}_{\mathbb{N}} \leq_W \mathsf{Ind}\,\mathbb{Q}_{\mathbb{N}}$ and, using [11, Theorem 3.9], $\operatorname{Det}(\mathsf{TC}^*_{\mathbb{N}} * \mathsf{C}_{\mathbb{N}}) \leq_W \operatorname{Det}(\mathsf{TC}^*_{\mathbb{N}}) * \mathsf{C}_{\mathbb{N}} \equiv_W \mathsf{C}_{\mathbb{N}} * \mathsf{C}_{\mathbb{N}} \equiv_W \mathsf{C}_{\mathbb{N}}$.

Turning to $\mathscr{E}$, Theorem 5.1 is intriguing given the great interest in RT^2_2 in reverse mathematics over several decades. The celebrated logical decomposition of RT^2_2 into SRT^2_2 and COH by Cholak, Jockusch, and Slaman [5], where COH is the so-called cohesive principle, naturally prompted the question of the logical separation of SRT^2_2 from COH and (equivalently) from RT^2_2. This separation was fully achieved in ω-models of RCA_0 a few years ago by Monin and Patey [17], who showed SRT^2_2 does not imply COH in ω-models. The converse nonimplication had been established earlier by Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [12].

Against this backdrop, one is led to wonder what relationship $\mathsf{Ind}\,\mathscr{E}_2$ has to COH as well. Clearly COH cannot imply $\mathsf{Ind}\,\mathscr{E}_2$ in ω-models. But whether COH is reducible in any sense to $\mathsf{Ind}\,\mathscr{E}_2$ is open, as is the question of the separation between $\mathsf{Ind}\,\mathscr{E}_k$ and SRT^2_k under $\leq_c$. We conjecture that

Conjecture 6.2. $\mathsf{COH} \not\leq_c \mathsf{Ind}\,\mathscr{E}_{\mathbb{N}}$ and $\mathsf{Ind}\,\mathscr{E}_2 \not\leq_c \mathsf{SRT}^2_{\mathbb{N}}$, or at least $\mathsf{Ind}\,\mathscr{E}_k \not\leq_c \mathsf{SRT}^2_k$.

Of course, if either statement is false, then the other would immediately follow. We briefly mention a way to view $\mathsf{Ind}\,\mathscr{E}_k$ in terms of RT^1_k which might suggest a route to the resolution of both conjectures. Several authors have observed that

$\mathsf{COH} \equiv_{\mathrm{W}} (\widehat{\mathsf{RT}^1_2})^{\mathrm{FE}}$, where P^{FE} is the "finite error" version of P with $\operatorname{dom} P^{\mathrm{FE}} = \operatorname{dom} P$ and $x \in P^{\mathrm{FE}}(p)$ iff there is a $y \in P(p)$ with x and y differing only on a finite set. The following definition was inspired by the principle RCOH, a weakening of COH introduced by Cholak, Dzhafarov, Hirschfeldt, and Patey in [4].

Definition 6.3. The *weak parallelization* of a problem P is the problem $\widetilde{P}$ such that $\operatorname{dom} \widetilde{P} = \operatorname{dom} \widehat{P}$, and where the solutions to the instance $\langle p_0, p_1, \ldots \rangle$ are all sets of the form

$$\bigcup_{n \in A} \{ \langle n, x \rangle \ : \ x \in P(p_n) \},$$

where A is an infinite subset of $\mathbb{N}$.

In other words, $\widetilde{P}$ picks infinitely many parallel instances of P to solve out of a given instance of $\widehat{P}$. Thus $\mathsf{Ind}\,\mathscr{E}_k$ is an a priori stronger variant of $\widetilde{\mathsf{RT}^1_k}$ in which the solutions of the parallel RT^1_k-instances represented in an $\widetilde{\mathsf{RT}^1_k}$-solution are all of the same color. However, by nonuniformly picking a color shared by infinitely many columns of a solution of $\widetilde{\mathsf{RT}^1_k}$, it is not hard to see that $\mathsf{Ind}\,\mathscr{E}_k \equiv_{\mathrm{c}} \widetilde{\mathsf{RT}^1_k}$. Moreover, $\mathsf{SRT}^2_k \equiv_{\mathrm{c}} \widetilde{\mathsf{SRT}^1_k}$ by additionally computing a homogeneous set using a standard argument as in the proof of Lemma 5.3. Rephrasing in these terms, we have

Conjecture 6.2'. $(\widehat{\mathsf{RT}^1_2})^{\mathrm{FE}} \not\leq_{\mathrm{c}} \widetilde{\mathsf{RT}^1_2}$ *and* $\widetilde{\mathsf{RT}^1_k} \not\leq_{\mathrm{c}} \widetilde{\mathsf{SRT}^1_k}$.

Finally, there are of course many indivisible structures other than $\mathbb{Q}$ and $\mathscr{E}$ which could be investigated along similar lines as in the present work, for example the Rado graph. As observed earlier, every nonscattered countable linear order is indivisible, and it could be interesting to derive properties of their indivisibility problems from their properties as orders and vice versa.

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