Theorem: For any square matrix A with real number elements then

a.
$$(A + A^{T})$$
 is symmetric

b.
$$(A - A^{T})$$
 is skew-symmetric

How to calculate?

$$A = \frac{1}{2}[(A + A^{T}) + (A - A^{T})]$$

$$\Rightarrow A = \frac{1}{2}(A + A^{T}) + \frac{1}{2}(A - A^{T})$$

 \Rightarrow A = Symmertic matrix + Skew Symmertic matrix

Express the matrix $A = \begin{bmatrix} 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix}$ as the sum of a symmetric form that, $A = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix}$ then $A^{T} = \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 5 & -2 & -1 + 1 & 6 - 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 5 & -2 & -1 & 1 \\ 6 & 0 & 3 & -1 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 4 & 3 & 0 & 6 \\ 4 & 3 & 0 & 6 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 4 & 3 & 0 & 6 \\ 4 & 3 & 0 & 6 \end{bmatrix}$ $A^{T} = \begin{bmatrix} 9 & 2 & 3 \\ 4 & 3 & 0$ 6 as the sum of a symmetric and a skew symmetric matrix.

Given that,
$$A = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix}$$
 then

$$A^{T} = \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$$

$$(A + A^{T}) = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \\ 9 + 9 & 2 + 5 & 3 + 4 \end{bmatrix}$$

$$\Rightarrow (A + A^{T}) = \begin{bmatrix} 9+9 & 2+5 & 3+4 \\ 5+2 & -1-1 & 6+0 \\ 4+3 & 0+6 & -8-8 \end{bmatrix}$$

$$\Rightarrow (A + A^{T}) = \begin{bmatrix} 18 & 7 & 7 \\ 7 & -2 & 6 \\ 7 & 6 & -16 \end{bmatrix}$$
So $\frac{1}{2}(A + A^{T}) = \begin{bmatrix} 9 & \frac{7}{2} & \frac{7}{2} \\ \frac{7}{2} & -1 & 3 \end{bmatrix}$

So
$$\frac{1}{2}(A + A^{T}) = \begin{bmatrix} \frac{7}{2} & -1 & 3\\ \frac{7}{2} & 3 & -8 \end{bmatrix}$$

$$(A - A^{T}) = \begin{bmatrix} 9 & 2 & 3 \\ 5 & -1 & 6 \\ 4 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 9 & 5 & 4 \\ 2 & -1 & 0 \\ 3 & 6 & -8 \end{bmatrix}$$

$$\Rightarrow (A - A^{T}) = \begin{bmatrix} 9 - 9 & 2 - 5 & 3 - 4 \\ 5 - 2 & -1 + 1 & 6 - 0 \\ 4 - 3 & 0 - 6 & -9 + 9 \end{bmatrix}$$

$$\Rightarrow (A - A^{T}) = \begin{bmatrix} 0 & -3 & -1 \\ 3 & 0 & 6 \\ 1 & 6 & 0 \end{bmatrix}$$

So
$$\frac{1}{2}(A - A^{T}) = \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{1}{2} \\ \frac{3}{2} & 0 & 3 \\ \frac{1}{2} & -3 & 0 \end{bmatrix}$$

Hence,
$$A = \frac{1}{2}(A + A^{T}) + \frac{1}{2}(A - A^{T})$$

$$\Rightarrow (A + A^{T}) = \begin{bmatrix} 18 & 7 & 7 \\ 7 & -2 & 6 \\ 7 & 6 & -16 \end{bmatrix}$$

$$\text{Hence, } A = \frac{1}{2}(A + A^{T}) + \frac{1}{2}(A - A^{T})$$

$$\Rightarrow A = \begin{bmatrix} 9 & \frac{7}{2} & \frac{7}{2} \\ \frac{7}{2} & -1 & 3 \\ \frac{7}{2} & 3 & -8 \end{bmatrix} + \begin{bmatrix} 0 & -\frac{3}{2} & -\frac{1}{2} \\ \frac{3}{2} & 0 & 3 \\ \frac{1}{2} & -3 & 0 \end{bmatrix}$$

Express the following matrices as the sum of a symmetric and a skew symmetric matrix.

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \qquad A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} \qquad A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$