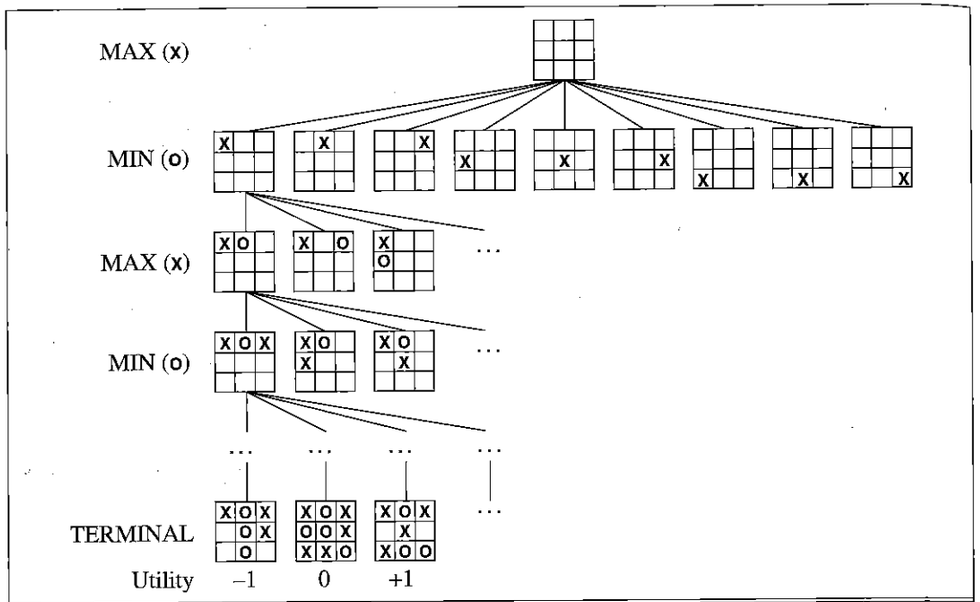

Figure 6.1 A (partial) search tree for the game of tic-tac-toe. The top node is the initial state, and MAX moves first, placing an X in an empty square. We show part of the search tree, giving alternating moves by MIN (O) and MAX, until we eventually reach terminal states, which can be assigned utilities according to the rules of the game.

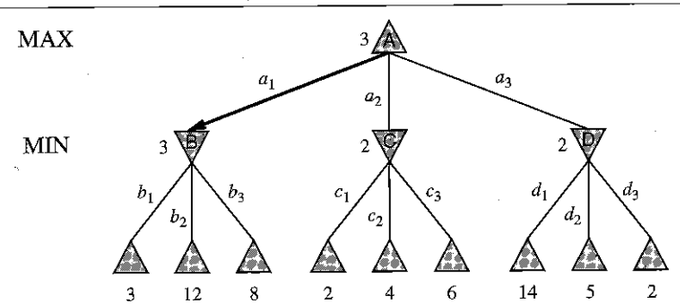

Figure 6.2 A two-ply game tree. The $\triangle$ nodes are "MAX nodes," in which it is MAX's turn to move, and the ∇ nodes are "MIN nodes." The terminal nodes show the utility values for MAX; the other nodes are labeled with their minimax values. MAX's best move at the root is a_1, because it leads to the successor with the highest minimax value, and MIN's best reply is b_1, because it leads to the successor with the lowest minimax value.

Let us apply these definitions to the game tree in Figure 6.2. The terminal nodes on the bottom level are already labeled with their utility values. The first MIN node, labeled B, has three successors with values 3, 12, and 8, so its minimax value is 3. Similarly, the other two MIN nodes have minimax value 2. The root node is a MAX node; its successors have minimax

MINIMAX DECISION

values 3, 2, and 2; so it has a minimax value of 3. We can also identify the **minimax decision** at the root: action a_1 is the optimal choice for MAX because it leads to the successor with the highest minimax value.

This definition of optimal play for MAX assumes that MIN also plays optimally—it maximizes the *worst-case* outcome for MAX. What if MIN does not play optimally? Then it is easy to show (Exercise 6.2) that MAX will do even better. There may be other strategies against suboptimal opponents that do better than the minimax strategy; but these strategies necessarily do worse against optimal opponents.

The minimax algorithm

MINIMAX ALGORITHM

The **minimax algorithm** (Figure 6.3) computes the minimax decision from the current state. It uses a simple recursive computation of the minimax values of each successor state, directly implementing the defining equations. The recursion proceeds all the way down to the leaves of the tree, and then the minimax values are **backed up** through the tree as the recursion unwinds. For example, in Figure 6.2, the algorithm first recurses down to the three bottom-left nodes, and uses the UTILITY function on them to discover that their values are 3, 12, and 8 respectively. Then it takes the minimum of these values, 3, and returns it as the backed-up value of node B. A similar process gives the backed up values of 2 for C and 2 for D. Finally, we take the maximum of 3, 2, and 2 to get the backed-up value of 3 for the root node.

BACKED UP

The minimax algorithm performs a complete depth-first exploration of the game tree. If the maximum depth of the tree is m, and there are b legal moves at each point, then the time complexity of the minimax algorithm is $O(b^m)$. The space complexity is $O(bm)$ for an algorithm that generates all successors at once, or $O(m)$ for an algorithm that generates successors one at a time (see page 76). For real games, of course, the time cost is totally impractical, but this algorithm serves as the basis for the mathematical analysis of games and for more practical algorithms.

Optimal decisions in multiplayer games

Many popular games allow more than two players. Let us examine how to extend the minimax idea to multiplayer games. This is straightforward from the technical viewpoint, but raises some interesting new conceptual issues.

First, we need to replace the single value for each node with a *vector* of values. For example, in a three-player game with players A, B, and C, a vector $\langle v_A, v_B, v_C \rangle$ is associated with each node. For terminal states, this vector gives the utility of the state from each player's viewpoint. (In two-player, zero-sum games, the two-element vector can be reduced to a single value because the values are always opposite.) The simplest way to implement this is to have the UTILITY function return a vector of utilities.

Now we have to consider nonterminal states. Consider the node marked X in the game tree shown in Figure 6.4. In that state, player C chooses what to do. The two choices lead to terminal states with utility vectors $\langle v_A = 1, v_B = 2, v_C = 6 \rangle$ and $\langle v_A = 4, v_B = 2, v_C = 3 \rangle$. Since 6 is bigger than 3, C should choose the first move. This means that if state X is reached, subsequent play will lead to a terminal state with utilities $\langle v_A = 1, v_B = 2, v_C = 6 \rangle$. Hence,

```
function MINIMAX-DECISION(state) returns an action
  inputs: state, current state in game

  v ← MAX-VALUE(state)
  return the action in SUCCESSORS(state) with value v

function MAX-VALUE(state) returns a utility value
  if TERMINAL-TEST(state) then return UTILITY(state)
  v ← −∞
  for a, s in SUCCESSORS(state) do
    v ← MAX(v, MIN-VALUE(s))
  return v

function MIN-VALUE(state) returns a utility value
  if TERMINAL-TEST(state) then return UTILITY(state)
  v ← ∞
  for a, s in SUCCESSORS(state) do
    v ← MIN(v, MAX-VALUE(s))
  return v
```

Figure 6.3 An algorithm for calculating minimax decisions. It returns the action corresponding to the best possible move, that is, the move that leads to the outcome with the best utility, under the assumption that the opponent plays to minimize utility. The functions MAX-VALUE and MIN-VALUE go through the whole game tree, all the way to the leaves, to determine the backed-up value of a state.

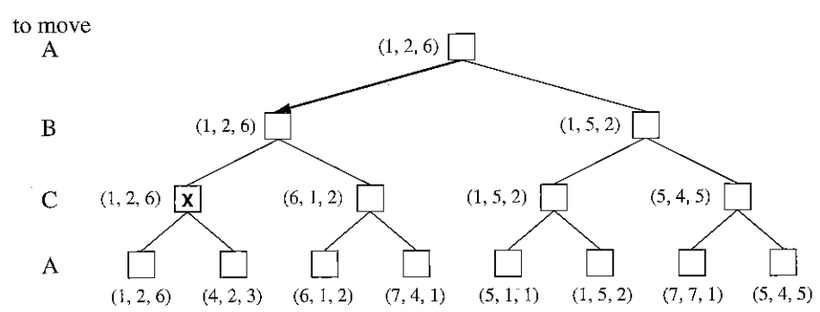

Figure 6.4 The first three ply of a game tree with three players (A, B, C). Each node is labeled with values from the viewpoint of each player. The best move is marked at the root.

the backed-up value of X is this vector. In general, the backed-up value of a node n is the utility vector of whichever successor has the highest value for the player choosing at n.

Anyone who plays multiplayer games, such as Diplomacy$^{\text{TM}}$, quickly becomes aware that there is a lot more going on than in two-player games. Multiplayer games usually involve

ALLIANCES

alliances, whether formal or informal, among the players. Alliances are made and broken

as the game proceeds. How are we to understand such behavior? Are alliances a natural consequence of optimal strategies for each player in a multiplayer game? It turns out that they can be. For example suppose A and B are in weak positions and C is in a stronger position. Then it is often optimal for both A and B to attack C rather than each other, lest C destroy each of them individually. In this way, collaboration emerges from purely selfish behavior. Of course, as soon as C weakens under the joint onslaught, the alliance loses its value, and either A or B could violate the agreement. In some cases, explicit alliances merely make concrete what would have happened anyway. In other cases there is a social stigma to breaking an alliance, so players must balance the immediate advantage of breaking an alliance against the long-term disadvantage of being perceived as untrustworthy. See Section 17.6 for more on these complications.

If the game is not zero-sum, then collaboration can also occur with just two players. Suppose, for example, that there is a terminal state with utilities $\langle v_A = 1000, v_B = 1000 \rangle$, and that 1000 is the highest possible utility for each player. Then the optimal strategy is for both players to do everything possible to reach this state—that is, the players will automatically cooperate to achieve a mutually desirable goal.

6.3 ALPHA–BETA PRUNING

ALPHA–BETA PRUNING

The problem with minimax search is that the number of game states it has to examine is exponential in the number of moves. Unfortunately we can't eliminate the exponent, but we can effectively cut it in half. The trick is that it is possible to compute the correct minimax decision without looking at every node in the game tree. That is, we can borrow the idea of **pruning** from Chapter 4 in order to eliminate large parts of the tree from consideration. The particular technique we will examine is called **alpha–beta pruning**. When applied to a standard minimax tree, it returns the same move as minimax would, but prunes away branches that cannot possibly influence the final decision.

Consider again the two-ply game tree from Figure 6.2. Let's go through the calculation of the optimal decision once more, this time paying careful attention to what we know at each point in the process. The steps are explained in Figure 6.5. The outcome is that we can identify the minimax decision without ever evaluating two of the leaf nodes.

Another way to look at this is as a simplification of the formula for MINIMAX-VALUE. Let the two unevaluated successors of node C in Figure 6.5 have values x and y and let z be the minimum of x and y. The value of the root node is given by

$$\begin{aligned}
\text{MINIMAX-VALUE}(root) &= \max(\min(3,12,8), \min(2,x,y), \min(14,5,2)) \\
&= \max(3, \min(2,x,y), 2) \\
&= \max(3,z,2) \qquad \text{where } z \leq 2 \\
&= 3.
\end{aligned}$$

In other words, the value of the root and hence the minimax decision are *independent* of the values of the pruned leaves x and y.

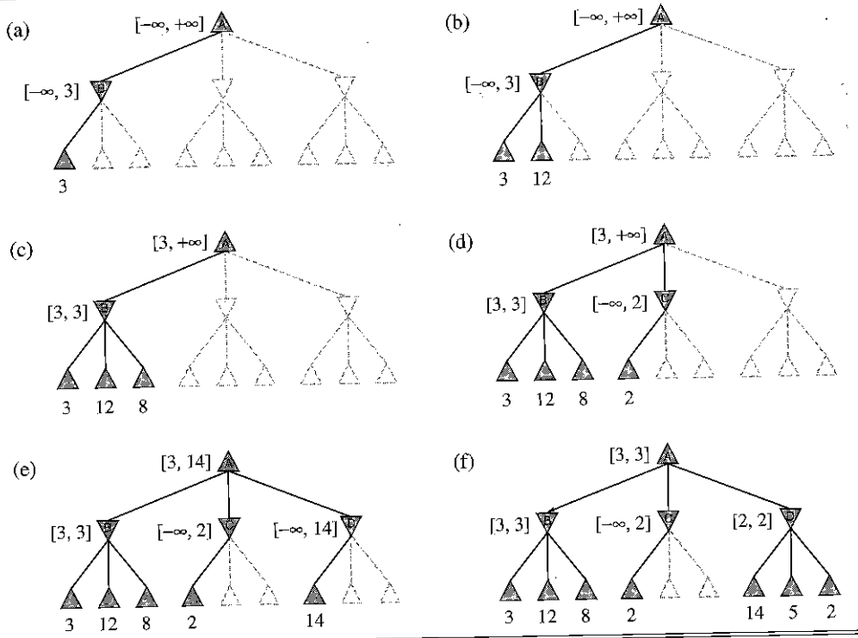

Figure 6.5 Stages in the calculation of the optimal decision for the game tree in Figure 6.2. At each point, we show the range of possible values for each node. (a) The first leaf below B has the value 3. Hence, B, which is a MIN node, has a value of *at most* 3. (b) The second leaf below B has a value of 12; MIN would avoid this move, so the value of B is still at most 3. (c) The third leaf below B has a value of 8; we have seen all B's successors, so the value of B is exactly 3. Now, we can infer that the value of the root is *at least* 3, because MAX has a choice worth 3 at the root. (d) The first leaf below C has the value 2. Hence, C, which is a MIN node, has a value of *at most* 2. But we know that B is worth 3, so MAX would never choose C. Therefore, there is no point in looking at the other successors of C. This is an example of alpha–beta pruning. (e) The first leaf below D has the value 14, so D is worth *at most* 14. This is still higher than MAX's best alternative (i.e., 3), so we need to keep exploring D's successors. Notice also that we now have bounds on all of the successors of the root, so the root's value is also at most 14. (f) The second successor of D is worth 5, so again we need to keep exploring. The third successor is worth 2, so now D is worth exactly 2. MAX's decision at the root is to move to B, giving a value of 3.

Alpha–beta pruning can be applied to trees of any depth, and it is often possible to prune entire subtrees rather than just leaves. The general principle is this: consider a node n somewhere in the tree (see Figure 6.6), such that Player has a choice of moving to that node. If Player has a better choice m either at the parent node of n or at any choice point further up, then n *will never be reached in actual play.* So once we have found out enough about n (by examining some of its descendants) to reach this conclusion, we can prune it.

Remember that minimax search is depth-first, so at any one time we just have to consider the nodes along a single path in the tree. Alpha–beta pruning gets its name from the

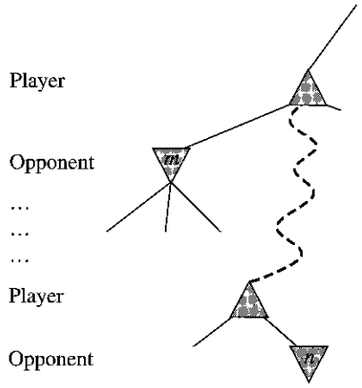

Figure 6.6 Alpha–beta pruning: the general case. If m is better than n for Player, we will never get to n in play.

following two parameters that describe bounds on the backed-up values that appear anywhere along the path:

$\alpha =$ the value of the best (i.e., highest-value) choice we have found so far at any choice point along the path for MAX.

$\beta =$ the value of the best (i.e., lowest-value) choice we have found so far at any choice point along the path for MIN.

Alpha–beta search updates the values of α and β as it goes along and prunes the remaining branches at a node (i.e., terminates the recursive call) as soon as the value of the current node is known to be worse than the current α or β value for MAX or MIN, respectively. The complete algorithm is given in Figure 6.7. We encourage the reader to trace its behavior when applied to the tree in Figure 6.5.

The effectiveness of alpha–beta pruning is highly dependent on the order in which the successors are examined. For example, in Figure 6.5(e) and (f), we could not prune any successors of D at all because the worst successors (from the point of view of MIN) were generated first. If the third successor had been generated first, we would have been able to prune the other two. This suggests that it might be worthwhile to try to examine first the successors that are likely to be best.

If we assume that this can be done,[2] then it turns out that alpha–beta needs to examine only $O(b^{m/2})$ nodes to pick the best move, instead of $O(b^m)$ for minimax. This means that the effective branching factor becomes $\sqrt{b}$ instead of b—for chess, 6 instead of 35. Put another way, alpha–beta can look ahead roughly twice as far as minimax in the same amount of time. If successors are examined in random order rather than best-first, the total number of nodes examined will be roughly $O(b^{3m/4})$ for moderate b. For chess, a fairly simple ordering function (such as trying captures first, then threats, then forward moves, and then backward moves) gets you to within about a factor of 2 of the best-case $O(b^{m/2})$ result. Adding dynamic

[2] Obviously, it cannot be done perfectly; otherwise the ordering function could be used to play a perfect game!

```
function ALPHA-BETA-SEARCH(state) returns an action
  inputs: state, current state in game

  v ← MAX-VALUE(state, −∞, +∞)
  return the action in SUCCESSORS(state) with value v

function MAX-VALUE(state, α, β) returns a utility value
  inputs: state, current state in game
          α, the value of the best alternative for MAX along the path to state
          β, the value of the best alternative for MIN along the path to state

  if TERMINAL-TEST(state) then return UTILITY(state)
  v ← −∞
  for a, s in SUCCESSORS(state) do
    v ← MAX(v, MIN-VALUE(s, α, β))
    if v ≥ β then return v
    α ← MAX(α, v)
  return v

function MIN-VALUE(state, α, β) returns a utility value
  inputs: state, current state in game
          α, the value of the best alternative for MAX along the path to state
          β, the value of the best alternative for MIN along the path to state

  if TERMINAL-TEST(state) then return UTILITY(state)
  v ← +∞
  for a, s in SUCCESSORS(state) do
    v ← MIN(v, MAX-VALUE(s, α, β))
    if v ≤ α then return v
    β ← MIN(β, v)
  return v
```

Figure 6.7 The alpha–beta search algorithm. Notice that these routines are the same as the MINIMAX routines in Figure 6.3, except for the two lines in each of MIN-VALUE and MAX-VALUE that maintain α and β (and the bookkeeping to pass these parameters along).

move-ordering schemes, such as trying first the moves that were found to be best last time, brings us quite close to the theoretical limit.

In Chapter 3, we noted that repeated states in the search tree can cause an exponential increase in search cost. In games, repeated states occur frequently because of **transpositions**—different permutations of the move sequence that end up in the same position. For example, if White has one move a_1 that can be answered by Black with b_1 and an unrelated move a_2 on the other side of the board that can be answered by b_2, then the sequences $[a_1, b_1, a_2, b_2]$ and $[a_1, b_2, a_2, b_1]$ both end up in the same position (as do the permutations beginning with a_2). It is worthwhile to store the evaluation of this position in a hash table the first time it is encountered, so that we don't have to recompute it on subsequent occurrences.

TRANSPOSITIONS

TRANSPOSITION TABLE

The hash table of previously seen positions is traditionally called a **transposition table**; it is essentially identical to the *closed* list in GRAPH-SEARCH (page 83). Using a transposition table can have a dramatic effect, sometimes as much as doubling the reachable search depth in chess. On the other hand, if we are evaluating a million nodes per second, it is not practical to keep *all* of them in the transposition table. Various strategies have been used to choose the most valuable ones.

6.4 IMPERFECT, REAL-TIME DECISIONS

The minimax algorithm generates the entire game search space, whereas the alpha–beta algorithm allows us to prune large parts of it. However, alpha–beta still has to search all the way to terminal states for at least a portion of the search space. This depth is usually not practical, because moves must be made in a reasonable amount of time—typically a few minutes at most. Shannon's 1950 paper, *Programming a computer for playing chess*, proposed instead that programs should cut off the search earlier and apply a heuristic **evaluation function** to states in the search, effectively turning nonterminal nodes into terminal leaves. In other words, the suggestion is to alter minimax or alpha–beta in two ways: the utility function is replaced by a heuristic evaluation function EVAL, which gives an estimate of the position's utility, and the terminal test is replaced by a **cutoff test** that decides when to apply EVAL.

CUTOFF TEST

Evaluation functions

An evaluation function returns an *estimate* of the expected utility of the game from a given position, just as the heuristic functions of Chapter 4 return an estimate of the distance to the goal. The idea of an estimator was not new when Shannon proposed it. For centuries, chess players (and aficionados of other games) have developed ways of judging the value of a position, because humans are even more limited in the amount of search they can do than are computer programs. It should be clear that the performance of a game-playing program is dependent on the quality of its evaluation function. An inaccurate evaluation function will guide an agent toward positions that turn out to be lost. How exactly do we design good evaluation functions?

First, the evaluation function should order the *terminal* states in the same way as the true utility function; otherwise, an agent using it might select suboptimal moves even if it can see ahead all the way to the end of the game. Second, the computation must not take too long! (The evaluation function could call MINIMAX-DECISION as a subroutine and calculate the exact value of the position, but that would defeat the whole purpose: to save time.) Third, for nonterminal states, the evaluation function should be strongly correlated with the actual chances of winning.

One might well wonder about the phrase "chances of winning." After all, chess is not a game of chance: we know the current state with certainty, and there are no dice involved. But if the search must be cut off at nonterminal states, then the algorithm will necessarily be *uncertain* about the final outcomes of those states. This type of uncertainty is induced by