CENG 384 - Signals and Systems for Computer Engineers
Spring 2018-2019
Written Assignment 3

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January 26, 2020

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1. (a)

$$N = 4, \quad w_0 = \frac{2\pi}{4} \tag{1}$$

$$X(n) = \sum_{k=\langle N \rangle} a_k e^{jkw_0 n}, \quad a_k = \frac{1}{N} \sum_{k=\langle N \rangle} x(n) e^{-jkw_0 n} \tag{2}$$

$$x(0) = 0, \quad x(1) = 1, \quad x(2) = 2, \quad x(3) = 1 \tag{3}$$

$$\begin{aligned}
k = 0, \quad a_0 &= \frac{1}{4}[x(0) + x(1)e^{-j0w} + x(2)e^{-2j0w} + x(3)e^{-3j0w}] \\
&= \frac{1}{4}[0 + 1 + 2 + 1] = 1 \\
k = 1, \quad a_1 &= \frac{1}{4}[x(0) + x(1)e^{-j\frac{2\pi}{4}} + x(2)e^{-j2\frac{2\pi}{4}} + x(3)e^{-j3\frac{2\pi}{4}}] \\
&= \frac{1}{4}[0 + e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + e^{-j\frac{3\pi}{2}}] \\
&= \frac{1}{4}[-j - 2 + j] = \frac{-1}{2} \\
k = 2, \quad a_2 &= \frac{1}{4}[x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi+} + x(3)e^{-j3\pi}] \\
&= \frac{1}{4}[0 + e^{-j\pi} + 2e^{-j2\pi} + e^{-j3\pi}] \\
&= \frac{1}{4}[-1 + 2 - 1] = 0 \\
k = 3, \quad a_3 &= \frac{1}{4}[x(0) + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}}] \\
&= \frac{1}{4}[0 + e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi} + e^{-j\frac{9\pi}{2}}] \\
&= \frac{1}{4}[j - 2 - j] = \frac{-1}{2}
\end{aligned} \tag{4}$$

The figure below is the graph of the spectral coefficients of Fourier Series for signal x[n]. The graph repeats itself to infinity with period 4.

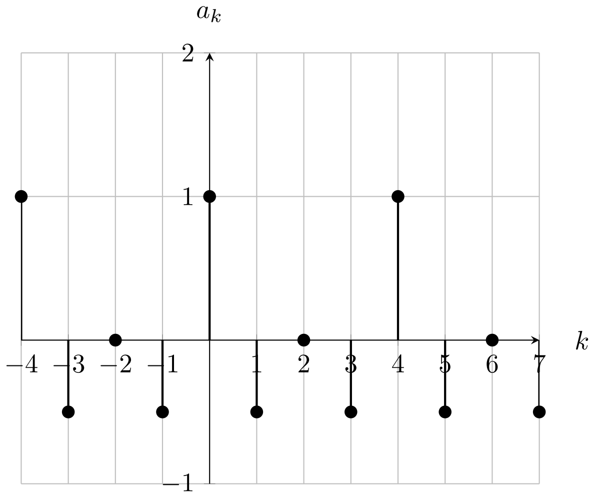

Figure 1: k vs. a_k

(b) i. In terms of x[n], y[n] equals to following,

$y[n] = x[n] - \sum_{n_0=\infty}^{-\infty} \delta(n - 4n_0 + 1)$

ii.

$$N = 4, \quad w_0 = \frac{2\pi}{4} \tag{5}$$

$$b_k = \frac{1}{N} \sum_{k=\langle N \rangle} y(n) e^{-jkw_0 n} \tag{6}$$

$$y(0) = 0, \quad y(1) = 1, \quad y(2) = 2, \quad y(3) = 0 \tag{7}$$

$$\begin{aligned}
b_k &= \frac{1}{4}[0e^{-jkw_0(0)} + 1e^{-jkw_0(1)} + 2e^{-jkw_0(2)} + 0] \\
b_k &= \frac{1}{4}[e^{-jk\frac{\pi}{2}} + 2e^{-jk\pi}] \\
k = 0, \quad b_0 &= \frac{1}{4}[e^{-j0} + 2e^{-j0}] = \frac{3}{4} \\
k = 1, \quad b_1 &= \frac{1}{4}[e^{-j\frac{\pi}{2}} + 2e^{-j\pi}] \\
&= \frac{1}{4}[-j - 2] = -0.25j - 0.5 \\
k = 2, \quad b_2 &= \frac{1}{4}[e^{-j\pi} + 2e^{-j2\pi}] = 0.25 \\
k = 3, \quad b_3 &= \frac{1}{4}[e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi}] \\
&= \frac{1}{4}[j - 2] = 0.25j - 0.5
\end{aligned} \tag{8}$$

The figures below are the magnitude and phase graphs of the spectral coefficients of Fourier Series for signal y[n]. The graphs repeat themselves to infinity with period 4.
$|b_0| = 0.75$, $|b_2| = 0.25$ $|b_3| = |b_4| = \frac{\sqrt{5}}{4} = 0.55$
The phases are $\angle b_0 = \angle b_2 = 0$ and $\angle b_1 = \pi + arctan(\frac{1}{2}) = 206°$, $\angle b_3 = \pi - arctan(\frac{-1}{2}) = 154°$

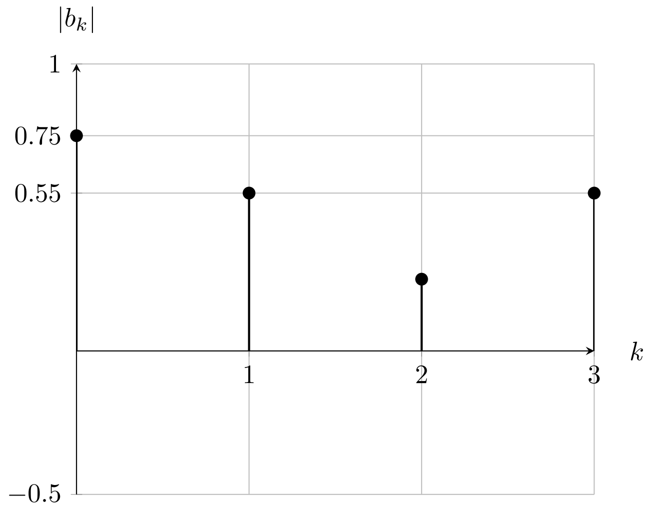

Figure 2: k vs. $|b_k|$

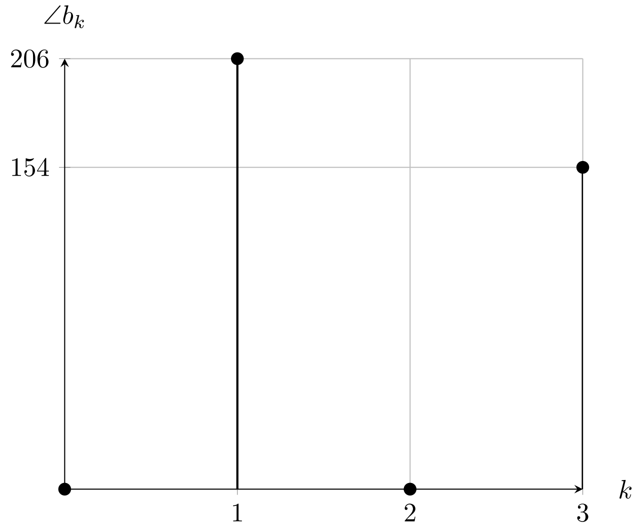

Figure 3: k vs. $\angle b_k$

2. The period of x[n] is 4 and from fact b we can conclude $\sum_{k=1}^{4} x[k] = 4$ and we get $a_0 = \frac{1}{4}\sum_{k=1}^{4} x[k] = 1$. According to fact c $|a_1 - a_{11}| = 1$ and because the period is 4 $a_{11} = a_3$ we get $|a_1 - a_3| = 1$. Also w is $w = \frac{2\pi}{4} = \frac{\pi}{2}$. From fact e we get

   $\sum_{k=0}^{3} x[k](e^{-jwk} + e^{-3jwk}) = 4$

   $\frac{1}{4}\sum_{k=0}^{3} x[k]e^{-jwk} + \frac{1}{4}\sum_{k=0}^{3} x[k]e^{-3jwk} = 1$

   $a_1 + a_3 = 1$ we also have the equation $|a_1 - a_3| = 1$ from above. Fact a says that some of the coefficients are complex numbers so, $a_1 = \frac{1}{2} - \frac{j}{2}$ and $a_3 = \frac{1}{2} + \frac{j}{2}$
   Because of fact d says one of the coefficients is zero, $a_2 = 0$.

   So, $x[n] = 1 + (\frac{1}{2} - \frac{j}{2})e^{j\frac{\pi}{2}n} + (\frac{1}{2} + \frac{j}{2})e^{j\frac{3\pi}{2}n}$

   $e^{j\frac{\pi}{2}n} = cos(\frac{\pi}{2}n) + jsin(\frac{\pi}{2}n) = j$ and $e^{j\frac{3\pi}{2}n} = cos(\frac{3\pi}{2}n) + jsin(\frac{3\pi}{2}n) = -j$

   $x[0] = 2 \quad x[1] = 2 \quad x[2] = 0 \quad x[3] = 0$

   The graph of the x[n] is below and it repeats itself to infinity,

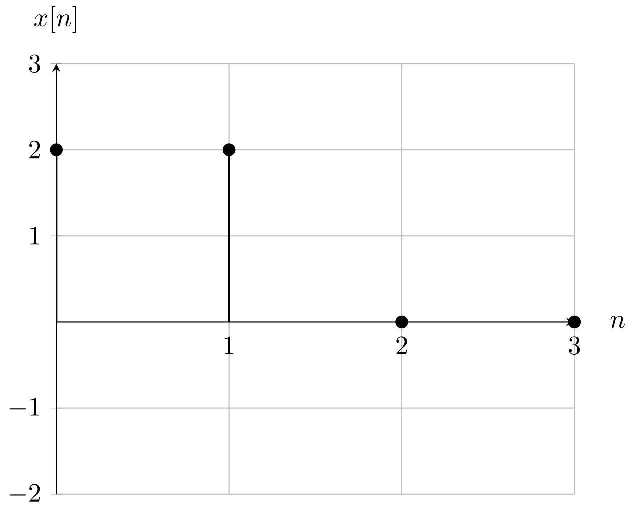

Figure 4: n vs. $x[n]$

3. $(x(t) + r(t)) * h(t) = x(t) * h(t) + r(t) * h(t) = x(t)$

   So, h(t) must be a low pass filter. Because it passes x(t) but not r(t). We also know that r(t) is composed of only very high frequency components. Therefore, h(t) must be low pass filter to contaminate x(t) by the noise r(t). h(t) is a low pass filter and H(jw) is 1 between $-Kw_0$ and Kw_0 where K is the period of the x(t). So the graph of H(jw) must be something like below,

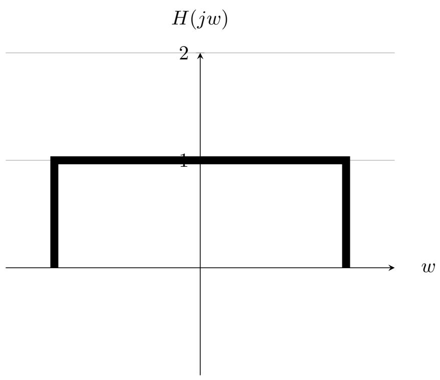

Figure 5: w vs. $H(jw)$.

   We can calculate the h(t) as below,

$$\begin{aligned} h(t) &= \frac{1}{2\pi}\int_{-Kw_0}^{Kw_0} H(jw)e^{jwt}dw \\ &= \frac{1}{2\pi}\left(\frac{e^{jKw_0t} - e^{-jKw_0t}}{jt}\right) \\ &= \frac{sin(Kw_0t)}{\pi t} \end{aligned} \tag{9}$$

4. (a) From the diagram we can find the following equation $y^{''}(t) + 5y^{'}(t) + 6y(t) = 4x^{'}(t) + x(t)$. The Fourier Transform of this function is

   $(jw)^2Y(jw) + 5(jw)Y(jw) + 6Y(jw) = 4(jw)X(jw) + X(jw)$

   The frequency response is

   $H(jw) = \frac{Y(jw)}{X(jw)} = \frac{4(jw)+1}{(jw)^2+5(jw)+6}$

   $H(jw) = \frac{11}{jw+3} - \frac{7}{jw+2}$

   (b) From the equation $e^{-at} = \frac{1}{a+jw}$, the impulse response is

$h(t) = 11e^{-3t} - 7e^{-2t}$

(c) $Y(jw) = H(jw)X(jw)$

$H(jw) = \frac{4(jw)+1}{(jw)^2+5(jw)+6}$ and $X(jw) = \frac{1}{1+4jw}$

$Y(jw) = \frac{1}{(jw)^2+5jw+6} = \frac{1}{jw+2} - \frac{1}{jw+3}$

$y(t) = e^{-2t} - e^{-3t}$