236A - HW #2

Noyan Evirgen
205220656

Scanned with CamScanner

Nogan Evirgen

HW 2

Problem 1

a) $\min \|Ax-b\|_1$ s.t. $\|x\|_\infty \le 1$ | $\min \|z\|_1$ s.t. $z=Ax-b$, $\max_i \{|x_i|\} \le 1$

$$\|z\|_1 = |z_1|+|z_2|+\dots+|z_m| = \underbrace{\max\{-z_1,z_1\}}_{t_1}+\dots+\underbrace{\max\{-z_m,z_m\}}_{t_m}$$

Also,

$$\max_i\{|x_i|\} = \max_i\{-x_i,x_i\} \le 1$$

$\Rightarrow$

$$\begin{array}{l} \min \|Ax-b\|_1 \\ \text{s.t. } \|x\|_\infty \le 1 \end{array} \quad\Bigg|\quad \begin{array}{l} \min\ t_1+t_2+\dots+t_m \\ \text{s.t. } z=Ax-b \\ -t_1 \le z_1 \le t_1 \\ \vdots \\ -t_m \le z_m \le t_m \\ -1 \le x_1 \le 1 \\ \vdots \\ -1 \le x_n \le 1 \end{array} \quad\Bigg|\quad \begin{array}{l} \min\ t_1+t_2+\dots+t_m \\ \text{s.t. } z=Ax-b \\ -t_i \le z_i \le t_i,\ i=1,\dots,m \\ -1 \le x_j \le 1,\ j=1,\dots,n \end{array}$$

b)

$$\begin{array}{l} \min \|x\|_1 \\ \text{s.t. } \|Ax-b\|_\infty \le 1 \end{array} \quad\Bigg|\quad \begin{array}{l} \min\{\max\{-x_1,x_1\}+\dots+\max\{-x_n,x_n\}\} \\ \text{s.t. } z=Ax-b \\ \|z\|_\infty \le 1 \end{array}$$

$$t_i \to \max\{-x_i,x_i\} \Rightarrow \begin{array}{l} \min\ t_1+t_2+\dots+t_n \\ \text{s.t. } z=Ax-b \\ \max_i\{|z_i|\} \le 1 \\ -t_1 \le x_1 \le t_1 \\ \vdots \\ -t_n \le x_n \le t_n \end{array} \quad\Bigg|\quad \begin{array}{l} \min\ t_1+t_2+\dots+t_n \\ \text{s.t. } z=Ax-b \\ -t_j \le x_j \le t_j,\ j=1,\dots,n \\ -1 \le z_i \le 1,\ i=1,\dots,m \end{array}$$

Scanned with CamScanner

c) $\min\{\|Ax-b\|_1+\|x\|_\infty\}$ | $\min\{\|z\|_1+\max_i\{|x_i|\}\}$ s.t. $z=Ax-b$

Neyan Evirgen

$$\min\left[\underbrace{\max\{-z_1,z_1\}}_{t_1}+\dots+\underbrace{\max\{-z_m,z_m\}}_{t_m}+\underbrace{\max_i\{x_i,-x_i\}}_{k}\right]$$

s.t $z=Ax-b$

$$\begin{aligned}\min\quad & t_1+t_2+\dots+t_m+k\\ \text{s.t.}\quad & z=Ax-b\\ & -t_1\le z_1\le t_1\\ & \vdots\\ & -t_m\le z_k\le t_m\\ & -k\le x_i\le k\\ & i=1,\dots,n\end{aligned}$$

$$\begin{aligned}\min\quad & t_1+t_2+\dots+t_m+k\\ \text{s.t.}\quad & z=Ax-b\\ & -t_j\le z_j\le t_j \quad j=1,\dots,m\\ & -k\le x_i\le k \quad ,i=1,\dots,n\end{aligned}$$

Scanned with CamScanner

Noyan Evirgen

Problem 2

Let r_i, $i \in \{1,2,3,4,5,f1,f2\}$ be the cost associated by outlets and factories.

Let X_{ij} be the weight of transported cheese from i to j.

We want to

minimize $r_1 + r_2 + r_3 + r_4 + r_5 + r_{f1} + r_{f2}$

s.t.

$$r_1 = C_{13}(X_{13}) + C_{12}(X_{12}) + C_{f11}(X_{1f1})$$
$$r_2 = C_{12}(X_{21}) + C_{23}(X_{23}) + C_{25}(X_{25})$$
$$r_3 = C_{13}(X_{31}) + C_{23}(X_{32}) + C_{34}(X_{34})$$
$$r_4 = C_{34}(X_{43}) + C_{f24}(X_{4f2})$$
$$r_5 = C_{25}(X_{52}) + C_{f15}(X_{5f1}) + C_{f25}(X_{5f2})$$
$$r_{f1} = C_{f11}(X_{f11}) + C_{f15}(X_{f15}) + P_1(X_{f11} + X_{f15} - X_{1f1} - X_{5f1})$$
$$r_{f2} = C_{f24}(X_{f24}) + C_{f25}(X_{f25}) + P_2(X_{f24} + X_{f25} - X_{4f2} - X_{5f2})$$

$$X_{f11} + X_{31} + X_{21} - (X_{1f1} + X_{13} + X_{12}) \geq 1250$$
$$X_{12} + X_{32} + X_{52} - (X_{21} + X_{23} + X_{25}) \geq -700$$
$$X_{13} + X_{23} + X_{43} - (X_{31} + X_{32} + X_{34}) \geq 400$$
$$X_{34} + X_{f24} - (X_{43} + X_{4f2}) \geq 1800$$
$$X_{25} + X_{f15} + X_{f25} - (X_{52} + X_{5f1} + X_{5f2}) \geq 1000$$

$$X_{f11} + X_{f15} - X_{1f1} - X_{5f1} \geq 0$$
$$X_{f24} + X_{f25} - X_{4f2} - X_{5f2} \geq 0$$
$$X_{ij} \geq 0 \text{ for all } i,j$$

Scanned with CamScanner

Problem 3

Noyan Evirgen

$$B = A_0 + x_1 A_1 + \dots + x_p A_p$$

$$\Longrightarrow \max_{\|y\|_1=1} \|(A_0 + x_1 A_1 + \dots + x_p A_p) y\|_1 = \max_{\|y\|_1=1} \|By\|_1 \quad \text{s.t. } B = A_0 + x_1 A_1 + \dots + x_p A_p$$

$$= \max_{j=1,\dots,n} \sum_{i=1,\dots,m} |B_{ij}| \quad \text{s.t. } B = A_0 + x_1 A_1 + \dots + x_p A_p$$

$$\boxed{\begin{aligned}\max_{j=1,\dots,n} \sum_{i=1,\dots,m} |B_{ij}| &= \max_{j=1,\dots,n} \left(|B_{1j}| + |B_{2j}| + \dots + |B_{mj}|\right) \\ &= \max_{j=1,\dots,n} \Big[\underbrace{\max\{-B_{1j}, B_{1j}\}}_{t_1} + \dots + \underbrace{\max\{-B_{mj}, B_{mj}\}}_{t_m}\Big]\end{aligned}}$$

$$\Longrightarrow \max_{j=1,\dots,n} \sum_{i=1,\dots,m} |B_{ij}| \quad \text{s.t. } B = A_0 + x_1 A_1 + \dots + x_p A_p$$

$$\begin{aligned} \max \quad & t_1 + t_2 + \dots + t_m \\ \text{s.t.} \quad & B = A_0 + x_1 A_1 + \dots + x_p A_p \\ & -t_1 \le B_{1j} \le t_1 \\ & -t_2 \le B_{2j} \le t_2 \\ & \quad \vdots \qquad\qquad\qquad , \; j = 1, \dots, n \\ & -t_m \le B_{mj} \le t_m \end{aligned}$$

Problem 4

a) $\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$ and $\|y\|_\infty = \max_i \{|y_i|\} \le 1$

$x^T y = \sum_{i=1}^n x_i y_i \le \sum_{i=1}^n |x_i||y_i|$ If $\max_i \{|y_i|\} < 1$, then $|y_i| \le 1$ for all i

$$\Longrightarrow x^T y \le \sum_{i=1}^n |x_i||y_i| \le \sum_{i=1}^n |x_i| = \|x\|_1$$

If we choose $y = sgn(x)$ where $sgn(x) = \begin{cases} -1 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$

$$\Longrightarrow x^T y = \sum_{i=1}^n x_i y_i = \sum_{i=1}^n |x_i| = \|x\|_1$$ also satisfying $\|y\|_\infty \le 1$

Scanned with CamScanner

Noyan Evirgen

b) Since $v \in \{-\rho, \rho\}^n$ has every combination, one of the $v \in \{-\rho,\rho\}^n$ needs to satisfy,

$$v^* = \rho \cdot \operatorname{sgn}(x) \quad \text{where} \quad \operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$$

Therefore,

$\bar{a}_i^T x + v^{*T} x \le b_i$, $i=1,\dots,m$ needs to be satisfied.

Since $v^{*T}x = x^T v^*$,

$$\left.\begin{aligned} &\bar{a}_i^T x + x^T v^* \le b_i \\ \Rightarrow\ &\bar{a}_i^T x + \rho(x^T \operatorname{sgn}(x)) \le b_i \\ \Rightarrow\ &\bar{a}_i^T x + \rho\|x\|_1 \le b_i \quad (1) \end{aligned}\right\} i=1,\dots,m$$

For $\underset{v \neq v^*}{v \in \{-\rho,\rho\}^n}$,

$$\bar{a}_i^T x + x^T v \le \bar{a}_i^T x + \rho\|x\|_1 \quad (2) \quad \underline{\text{since } x^T v \le \rho\|x\|_1}$$
$i=1,\dots,m$

Combining (1) and (2)

$$\bar{a}_i^T x + v^T x \le \underline{\bar{a}_i^T x + \rho\|x\|_1 \le b_i} \quad i=1,\dots,m$$

c) $\min\ c^T x$
s.t. $\bar{a}_i^T x + \rho\|x\|_1 \le b_i$, $i=1,\dots,m$

$\min\ c^T x$
s.t. $\bar{a}_i^T x + \rho(|x_1|+|x_2|+\dots+|x_n|) \le b_i$
$i=1,\dots,m$

$$\rho(\underbrace{\max\{-x_1,x_1\}}_{t_1} + \underbrace{\max\{-x_2,x_2\}}_{t_2} + \dots + \underbrace{\max\{-x_n,x_n\}}_{t_n})$$

$\Rightarrow \min\ c^T x$
s.t. $\bar{a}_i^T x + \rho(t_1+t_2+\dots+t_n) \le b_i$, $i=1,\dots,m$
$-t_1 \le x_1 \le t_1$
$-t_2 \le x_2 \le t_2$
$\vdots$
$-t_n \le x_n \le t_n$ (5)

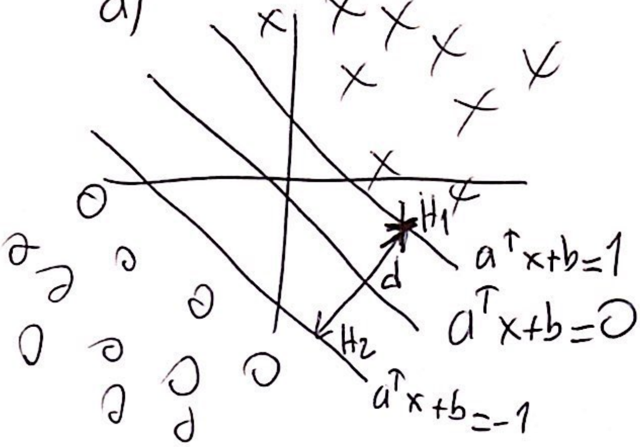
Scanned with CamScanner

Noyan Evirgen

Problem 5

Since $\min_{i=1,\dots,N} s_i(a^T v_i + b) = 1$ $\min_{i=1,\dots,N} (a^T v_i + b) = \pm 1$

a)



Distance $d = \frac{|(1-b)-(-1-b)|}{\|a\|_\infty} = \frac{2}{\|a\|_\infty}$

Since distance between $a^T x = b_1$ and $a^T x = b_2$ is $\frac{|b_1 - b_2|}{\|a\|_\infty}$

b) As shown at a), the margin is $\frac{2}{\|a\|_\infty}$

Therefore we, $\max \frac{1}{\|a\|_\infty}$

s.t.

$s_i(a^T v_i + b) = 1$, $i = 1, \dots, m$

Scanned with CamScanner