Fall 2018
EE 236A
Christina Fragouli
TA: Debraj Basu

Linear Programming

Solutions of Homework 2

<u>Problem 1</u> (`3 points`, Exer. 9 in *Linear Programming Exercises*): Formulate the following problems as LPs:

(a) minimize $\|Ax - b\|_1$ subject to $\|x\|_\infty \leq 1$.

(b) minimize $\|x\|_1$ subject to $\|Ax - b\|_\infty \leq 1$.

(c) minimize $\|Ax - b\|_1 + \|x\|_\infty$.

In each problem, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbf{R}^m$ are given, and $x \in \mathbf{R}^n$ is the optimization variable.

Solution:

(a) The problem is equivalent to the LP

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} y_i \\ \text{subject to} & -y \preceq Ax - b \preceq y \\ & -\mathbf{1} \preceq x \preceq \mathbf{1}. \end{array}$$

The variables are $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$. In matrix notation:

$$\begin{array}{ll} \text{minimize} & \bar{c}^T \bar{x} \\ \text{subject to} & \bar{A}\bar{x} \preceq \bar{b}, \end{array}$$

where

$$\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \bar{c} = \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & -I \\ -A & -I \\ I & 0 \\ -I & 0 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} b \\ -b \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}.$$

(b) If we introduce a new variable $y \in \mathbf{R}^n$, we can express the problem as

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & -y \preceq x \preceq y \\ & -\mathbf{1} \preceq Ax - b \preceq \mathbf{1}, \end{array}$$

which is an LP in x and y. In matrix notation, the problem is

$$\begin{array}{ll} \text{minimize} & \bar{c}^T \bar{x} \\ \text{subject to} & \bar{A}\bar{x} \preceq \bar{b}, \end{array}$$

where

$$\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \bar{c} = \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} I & -I \\ -I & -I \\ A & 0 \\ -A & 0 \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} 0 \\ 0 \\ b + \mathbf{1} \\ b - \mathbf{1} \end{bmatrix}.$$

Another good solution is to write x as the difference of two nonnegative vectors $x = x^+ - x^-$, and to express the problem as

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T x^+ + \mathbf{1}^T x^- \\ \text{subject to} & -\mathbf{1} \preceq Ax^+ - Ax^- - b \preceq \mathbf{1} \\ & x^+ \succeq 0, \quad x^- \succeq 0, \end{array}$$

which is an LP in $x^+ \in \mathbf{R}^n$ and $x^- \in \mathbf{R}^n$. In matrix notation,

$$\begin{array}{ll} \text{minimize} & \bar{c}^T \bar{x} \\ \text{subject to} & \bar{A}\bar{x} \preceq \bar{b}, \end{array}$$

where

$$\bar{x} = \begin{bmatrix} x^+ \\ x^- \end{bmatrix}, \quad \bar{c} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & -A \\ -A & A \\ -I & 0 \\ 0 & -I \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} \mathbf{1} + b \\ \mathbf{1} - b \\ 0 \\ 0 \end{bmatrix}.$$

(c) We can introduce new variables $y \in \mathbf{R}^m$ and $t \in \mathbf{R}$ and write the problem as

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y + t \\ \text{subject to} & -y \preceq Ax - b \preceq y \\ & -t\mathbf{1} \preceq x \preceq t\mathbf{1}, \end{array}$$

which is an LP in x, y, and t. In matrix notation:

$$\begin{array}{ll} \text{minimize} & \bar{c}^T \bar{x} \\ \text{subject to} & \bar{A}\bar{x} \preceq \bar{b}, \end{array}$$

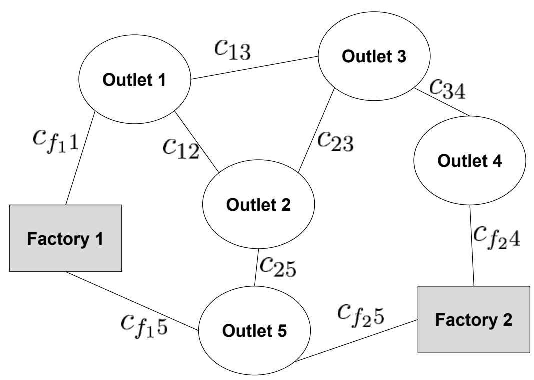
where

$$\bar{x} = \begin{bmatrix} x \\ y \\ t \end{bmatrix}, \quad \bar{c} = \begin{bmatrix} 0 \\ \mathbf{1} \\ 1 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & -I & 0 \\ -A & -I & 0 \\ I & 0 & -\mathbf{1} \\ -I & 0 & -\mathbf{1} \end{bmatrix}, \quad \bar{b} = \begin{bmatrix} b \\ -b \\ 0 \\ 0 \end{bmatrix}.$$

Problem 2 (`3 points`): Linear programming can be used to optimize the cost of goods transportation between different selling points. The following is a simplified version of such an approach. Solodrex manufacutres a brand of cheese in 2 factories and sells its production through 5 sales outlets in California. The demands of the market have changed in different areas this month and therefore, this weekend Solodrex intends to produce and redistibute cheese stocks to its 5 sales outlets. Current stocks and the needed stocks at each outlet are given in the table below.

	Current Stock (lb)	Needed Stock (lb)
Outlet 1	1,250	2,500
Outlet 2	1,700	1,000
Outlet 3	1,400	1,800
Outlet 4	1,200	3,000
Outlet 5	1,000	2,000

The two factories of Solodrex (Factory 1 and Factory 2) can manufacture cheese at a cost of p_1 and p_2 \$ per lb, respectively. The manufactured stock as well as the stock available at each outlet can be moved through the roads connecting them which are shown in the figure. The cost per lb of transportation through these roads (in either direction) is also shown.



Write an LP that will enable Solodrex to minimize the cost needed to meet the new market requirement.

Solution: Let $q_1,\ q_2$ be the amount of cheese produced from Factory 1 and Factory 2, respectively. Let t_{ij} be the amount of cheese moved from node i to node j. We denote by $N(i)$ the set of neighboring nodes (outlets and/or factories) to the i-th node. For example $N(1) = \{f_1, 2, 3\}$. For Factory 1, we have $N(f_1) = \{1, 5\}$. We denote the current avaialable stock vector by s and the demanded stocks vector by n. These vectors are given in Table 1.

1. we have the following simple constraints on the variables:

$$\begin{aligned} t_{ij} &\geq 0 \quad \forall i, j \in \{1, 2, 3, 4, 5, f_1, f_2\},\ i \neq j \\ q_i &\geq 0, \quad i = 1, 2 \end{aligned}$$

2. The factories have to at least produce the amount of cheese that is not available in the market, therefore we have:

$$q_1 + q_2 \geq \sum_{i=1}^{5} n_i - \sum_{i=1}^{5} s_i$$

3. The goods transported from the factory are at most equal to the amount produced in that factory plus any good transported into the factory:

$$\sum_{j \in N(f_i)} t_{f_i j} \leq q_i + \sum_{j \in N(f_i)} t_{j f_i}, \quad i = 1, 2$$

4. For each Outlet, the difference between the incoming and outgoing amount of cheese should be greater than or equal to the difference between its needed and current stocks.

$$\sum_{j \in N(i)} t_{ji} - \sum_{j \in N(i)} t_{ij} \geq n_i - s_i, \quad i = 1, 2, 3, 4, 5$$

Finally, the objective function should minimize the sum of transportation costs and production costs:

$$\text{minimize} \ \ (p_1 q_1 + p_2 q_2) \ + \sum_{\substack{i,j \in \{f_1, f_2, 1, 2, \cdots, 5\}, \\ i \neq j}} c_{ij}(t_{ij} + t_{ji})$$

Problem 3 (`3 points`, Exer. 10 in *Linear Programming Exercises*): Formulate the following problem as an LP. Given $p+1$ matrices $A_0, A_1, \cdots, A_p \in \mathbf{R}^{m \times n}$, find the vector $x \in \mathbf{R}^p$ that minimizes

$$\max_{\|y\|_1 = 1} \|(A_0 + x_1 A_1 + \cdots + x_p A_p) y\|_1. \tag{1}$$

Hint: *you can use the identity* $\max_{\|y\|_1=1} \|Ay\|_1 = \max_{j=1,\cdots,n} \sum_{i=1,\cdots,m} |A_{ij}|$.

Solution: We first note that for $A \in \mathbf{R}^{m \times n}$,

$$\max_{\|y\|_1=1} \|Ay\|_1 = \max_{j=1,\cdots,n} \sum_{i=1,\cdots,m} |A_{ij}|$$

(i.e., we add the absolute values in each column of A, and then take the maximum of those column sums).

We prove this as follows. Since $\|z\|_1 = \max_{\|u\|_\infty \le 1} u^T z$, we can write

$$\begin{aligned}
\max_{\|y\|_1=1} \|Ay\|_1 &= \max_{\|u\|_\infty \le 1} \max_{\|y\|_1=1} u^T Ay \\
&= \max_{\|u\|_\infty \le 1} \max_{\|y\|_1=1} \sum_{j=1}^{n} y_j \left(\sum_{i=1}^{m} A_{ij} u_j \right) \\
&= \max_{\|u\|_\infty \le 1} \max_{j=1,\cdots,n} \left| \sum_{i=1}^{m} A_{ij} u_j \right| \\
&= \max_{j=1,\cdots,n} \max_{\|u\|_\infty \le 1} \left| \sum_{i=1}^{m} A_{ij} u_j \right| \\
&= \max_{j=1,\cdots,n} \sum_{i=1}^{m} |A_{ij}| \qquad (2)
\end{aligned}$$

Using this expression we can formulate the problem as

$$\text{minimize} \quad \max_{j=1,\cdots,n} \sum_{i=1}^{m} |(A_0 + A_1 x_1 + \cdots + A_p x_p)_{ij}|$$

which can be formulated as an LP

$$\begin{array}{ll}
\text{minimize} & t \\
\text{subject to} & -s_{ij} \le (A_0 + A_1 x_1 + \cdots + A_p x_p)_{ij} \le s_{ij} \\
& \sum_{i=1}^{m} s_{ij} \le t, \; j = 1, \cdots, n.
\end{array}$$

The variables are $x, t \in \mathbb{R}$ and $S \in \mathbb{R}^{m \times n}$.

Problem 4 (4 `points`, Exer. 22 in *Linear Programming Exercises*):

(a) Let $x \in \mathbb{R}^n$ be a given vector. Prove that $x^T y \leq \|x\|_1$ for all y with $\|y\|_\infty \leq 1$. Is the inequality tight, i.e., does there exists a vector y that satisfies $\|y\|_\infty \leq 1$ and $x^T y = \|x\|_1$?

(b) Consider the set of inequalities

$$a_i^T x \leq b_i, \quad i = 1, \cdots, m, \tag{3}$$

Suppose you don't know the coefficients a_i exactly. Instead you are given nominal values $\bar{a}_i$, and you know that the actual coefficient vectors satisfy

$$\|a_i - \bar{a}_i\|_\infty \leq \rho, \tag{4}$$

for a given $\rho > 0$. In other words the actual coefficients a_{ij} can be anywhere in the intervals $[\bar{a}_{ij} - \rho, \bar{a}_{ij} + \rho]$, or equivalently, each vector a_i can lie anywhere in a rectangle with corners $\bar{a}_{ij} + v$ where $v \in \{-\rho, \rho\}^n$ (i.e., v has components ρ or $-\rho$).

The set of inequalities (3) must be satisfied for all possible values of a_i, i.e., we replace (3) with the constraints

$$a_i^T x \leq b_i, \qquad \text{for all } a_i \in \{\bar{a}_i + v \mid \|v\|_\infty \leq \rho\} \text{ and for } i = 1, \cdots, m. \tag{5}$$

A straight forward but very inefficient way to express this constraint is to enumerate the 2^n corners of the rectangle of possible values a_i and to require that

$$\bar{a}_i^T x + v^T x \leq b_i, \qquad \text{for all } v \in \{-\rho, \rho\}^n \text{ and for } i = 1, \cdots, m. \tag{6}$$

This is a system of $m2^n$ inequalities.

Use the result in (a) to show that (5) is in fact equivalent to the much more compact set of nonlinear inequalities

$$\bar{a}_i^T x + \rho\|x\|_1 \leq b_i, \quad i = 1, \cdots, m. \tag{7}$$

(c) Consider the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \cdots, m \end{array} \tag{8}$$

Again we are interested in situations where the coefficient vectors a_i are uncertain, but satisfy bounds $\|a_i - \bar{a}_i\|_\infty \leq \rho$ for given ρ and $\bar{a}_i$. We want to minimize $c^T x$ subject to the constraint that the inequalities $a_i^T x \leq b_i$ area satisfied for *all* possible values of a_i. We call this a *robust* LP.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \rho\|x\|_1 \leq b_i, \quad i = 1, \cdots, m \end{array} \tag{9}$$

Express (9) as an LP.

Solution:

(a) The results follows from

$$\begin{aligned} x^T y &\leq \sum_{i=1}^{n} x_i y_i \\ &\leq \max_{i=1,\cdots,n} |y_i| \sum_{i=1}^{n} |x_i| \\ &= \|x\|_1 \|y\|_\infty \\ &\leq \|x\|_1. \end{aligned}$$

We have equality if we choose $y_i = 1$ if $x_i \geq 0$, and $y_i = -1$ if $x_i < 0$.

(b) x satisfies the constraint (5) if and only if for all i,

$$\|v\|_\infty \leq \rho \quad \Rightarrow \quad \bar{a}_i^T x + v^T x \leq b_i,$$

i.e., if and only if

$$\bar{a}_i^T x + \max_{\|v\|_\infty \leq \rho} v^T x \leq b_i, \quad i = 1, \cdots, m.$$

It follows from the result in (a) that

$$\max_{\|v\|_\infty \leq \rho} v^T x = \rho \|x\|_1,$$

and the maximum value is attained when $v_i = \rho$ if $x_i \geq 0$ and $v_i < -\rho$ if $x_i < 0$. Therefore, (5) is equivalent to

$$\bar{a}_i^T x + \rho \|x\|_1 \leq b_i, \quad i = 1, \cdots, m.$$

(c) We can express the optimization problem (9) as the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \rho \mathbf{1}^T y \leq b_i, \quad i = 1, \cdots, m \\ & -y \leq x \leq y. \end{array} \tag{10}$$

The variables are $x, y \in \mathbf{R}^n$. An alternative formulation is to split x as $x = x^+ + x^-$, with x^+ and x^- nonnegative:

$$\begin{array}{ll} \text{minimize} & c^T x^+ - c^T x^- \\ \text{subject to} & \bar{a}_i^T x^+ - \bar{a}_i^T x^- + \rho \mathbf{1}^T x^+ + \rho \mathbf{1}^T x^- \leq b_i, \quad i = 1, \cdots, m \\ & x^+ \leq 0, \; x^- \leq 0. \end{array} \tag{11}$$

The variables are $x^+, x^- \in \mathbf{R}^n$.

<u>Problem 5</u> (`4 points`, Exer. 25 in *Linear Programming Exercises*): In the lecture we discussed the problem of finding a strictly separating hyperplane for a set of points with binary labels:

$$s_i(a^T v_i + b) > 0, \quad i = 1, \cdots, N. \tag{12}$$

The variables are $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The n-vectors v_i and the labels $s_i \in \{-1, 1\}$ are given. We can define the *margin of separation* of a strictly separating hyperplane as the maximum value of t such that

$$s_i(a^T(v_i + w) + b) \geq 0 \quad \text{for all } w \text{ with } \|w\|_\infty \leq t, \quad i = 1, \cdots, N.$$

The idea is that if we replace each point v_i with a hypercube (a ball in $\|\cdot\|_\infty$-norm) centered at v_i and with radius t, then the hyperplane separates the N hypercubes.

(a) Suppose a and b define a strictly separating hyperplane (i.e., satisfy (12)), and that the coefficients are normalized so that

$$\min_{i=1,\cdots,N} s_i(a^T v_i + b) = 1.$$

What is the margin of separation of the hyperplane?

(b) Formulate the problem of finding a strictly separating hyperplane with maximum margin of separation as a linear program.

Solution:

(a) The inequality $s_i(a^T v_i + b) + s_i a^T w \geq 0$ holds for all w with $\|w\|_\infty \leq t$ if and only if

$$s_i(a^T v_i + b) - t\|a\|_1 \geq 0.$$

The normalized hyperplane therefore has a margin of separation t if

$$\min_i s_i(a^T v_i + b) - t\|a\|_1 = 1 - t\|a\|_1 = 0.$$

In other words, the margin is $1/\|a\|_1$.

(b) Maximizing the margin $1/\|a\|_1$ is the same as minimizing $\|a\|_1$. The problem is therefore equivalent to

$$\begin{array}{ll} \text{minimize} & \|a\|_1 \\ \text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \cdots, N. \end{array}$$

It is easily seen that at the optimum at least one of the inequalities will be tight. (Otherwise we can further decrease $\|a\|_1$ by scaling a and b with a factor less than 1.) The problem can be further formulated as a linear program

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T y \\ \text{subject to} & s_i(a^T v_i + b) \geq 1, \quad i = 1, \cdots, N \\ & -y \preceq a \preceq y. \end{array}$$