Fall 2018
EE 236A
Christina Fragouli
TA: Debraj Basu

Linear Programming

Homework 2 (17 points)
Due: 12 pm, Thursday Oct. 11th

"*Linear Programming Exercises*" refers to http://www.seas.ucla.edu/∼vandenbe/ee236a/homework/problems.pdf

Problem 1 (`3 points`, Exer. 9 in *Linear Programming Exercises*): Formulate the following problems as LPs:

(a) minimize $\|Ax - b\|_1$ subject to $\|x\|_\infty \leq 1$.

(b) minimize $\|x\|_1$ subject to $\|Ax - b\|_\infty \leq 1$.

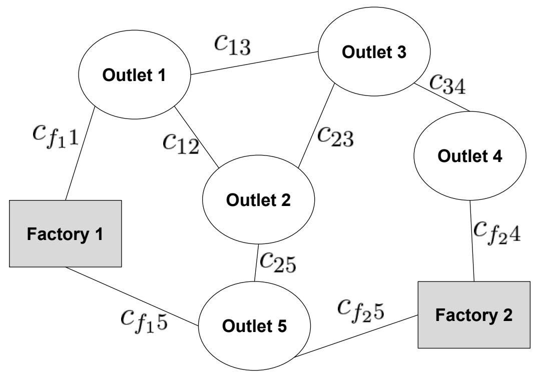
(c) minimize $\|Ax - b\|_1 + \|x\|_\infty$.

In each problem, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbf{R}^m$ are given, and $x \in \mathbf{R}^n$ is the optimization variable.

Problem 2 (`3 points`): Linear programming can be used to optimize the cost of goods transportation between different selling points. The following is a simplified version of such an approach. Solodrex manufacutres a brand of cheese in 2 factories and sells its production through 5 sales outlets in California. The demands of the market have changed in different areas this month and therefore, this weekend Solodrex intends to produce and redistibute cheese stocks to its 5 sales outlets. Current stocks and the needed stocks at each outlet are given in the table below.

	Current Stock (lb)	Needed Stock (lb)
Outlet 1	1,250	2,500
Outlet 2	1,700	1,000
Outlet 3	1,400	1,800
Outlet 4	1,200	3,000
Outlet 5	1,000	2,000

The two factories of Solodrex (Factory 1 and Factory 2) can manufacture cheese at a cost of p_1 and p_2 $ per lb, respectively. The manufactured stock as well as the stock available at each outlet can be moved through the roads connecting them which are shown in the figure. The cost per lb of transportation through these roads (in either direction) is also shown.
Write an LP that will enable Solodrex to minimize the cost needed to meet the new market requirement.



Problem 3 (3 points, Exer. 10 in *Linear Programming Exercises*): Formulate the following problem as an LP. Given $p+1$ matrices $A_0, A_1, \cdots, A_p \in \mathbf{R}^{m\times n}$, find the vector $x \in \mathbf{R}^p$ that minimizes

$$\max_{\|y\|_1=1} \|(A_0 + x_1 A_1 + \cdots + x_p A_p)y\|_1. \tag{1}$$

Hint: *you can use the identity* $\max_{\|y\|_1=1} \|Ay\|_1 = \max_{j=1,\cdots,n} \sum_{i=1,\cdots,m} |A_{ij}|$.

Problem 4 (4 points, Exer. 22 in *Linear Programming Exercises*):

(a) Let $x \in \mathbb{R}^n$ be a given vector. Prove that $x^T y \le \|x\|_1$ for all y with $\|y\|_\infty \le 1$. Is the inequality tight, i.e., does there exists a vector y that satisfies $\|y\|_\infty \le 1$ and $x^T y = \|x\|_1$?

(b) Consider the set of inequalities

$$a_i^T x \le b_i, \quad i = 1, \cdots, m, \tag{2}$$

Suppose you don't know the coefficients a_i exactly. Instead you are given nominal values $\bar{a}_i$, and you know that the actual coefficient vectors satisfy

$$\|a_i - \bar{a}_i\|_\infty \le \rho, \tag{3}$$

for a given $\rho > 0$. In other words the actual coefficients a_{ij} can be anywhere in the intervals $[\bar{a}_{ij} - \rho, \bar{a}_{ij} + \rho]$, or equivalently, each vector a_i can lie anywhere in a rectangle with corners $\bar{a}_{ij} + v$ where $v \in \{-\rho, \rho\}^n$ (i.e., v has components ρ or $-\rho$).

The set of inequalities (2) must be satisfied for all possible values of a_i, i.e., we replace (2) with the constraints

$$a_i^T x \le b_i, \quad \text{for all } a_i \in \{\bar{a}_i + v \mid \|v\|_\infty \le \rho\} \text{ and for } i = 1, \cdots, m. \tag{4}$$

A straight forward but very inefficient way to express this constraint is to enumerate the 2^n corners of the rectangle of possible values a_i and to require that

$$\bar{a}_i^T x + v^T x \le b_i, \quad \text{for all } v \in \{-\rho, \rho\}^n \text{ and for } i = 1, \cdots, m. \tag{5}$$

This is a system of $m2^n$ inequalities.

Use the result in (a) to show that (4) is in fact equivalent to the much more compact set of nonlinear inequalities

$$\bar{a}_i^T x + \rho\|x\|_1 \le b_i, \quad i = 1, \cdots, m. \tag{6}$$

(c) Consider the LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \le b_i, \quad i = 1, \cdots, m \end{array} \tag{7}$$

Again we are interested in situations where the coefficient vectors a_i are uncertain, but satisfy bounds $\|a_i - \bar{a}_i\|_\infty \le \rho$ for given ρ and $\bar{a}_i$. We want to minimize $c^T x$ subject to the constraint that the inequalities $a_i^T x \le b_i$ area satisfied for *all* possible values of a_i. We call this a *robust* LP.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \bar{a}_i^T x + \rho \|x\|_1 \le b_i, \quad i = 1, \cdots, m \end{array} \tag{8}$$

Express (8) as an LP.

Problem 5 (`4 points`, Exer. 25 in *Linear Programming Exercises*): In the lecture we discussed the problem of finding a strictly separating hyperplane for a set of points with binary labels:

$$s_i(a^T v_i + b) > 0, \quad i = 1, \cdots, N. \tag{9}$$

The variables are $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. The n-vectors v_i and the labels $s_i \in \{-1, 1\}$ are given. We can define the *margin of separation* of a strictly separating hyperplane as the maximum value of t such that

$$s_i(a^T(v_i + w) + b) \ge 0 \quad \text{for all } w \text{ with } \|w\|_\infty \le t, \quad i = 1, \cdots, N.$$

The idea is that if we replace each point v_i with a hypercube (a ball in $\|\cdot\|_\infty$-norm) centered at v_i and with radius t, then the hyperplane separates the N hypercubes.

(a) Suppose a and b define a strictly separating hyperplane (i.e., satisfy (9)), and that the coefficients are normalized so that

$$\min_{i=1,\cdots,N} s_i(a^T v_i + b) = 1.$$

What is the margin of separation of the hyperplane?

(b) Formulate the problem of finding a strictly separating hyperplane with maximum margin of separation as a linear program.