Fall 2018
EE 236A
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Linear Programming

Homework 3 (17 points)
Due: 12 pm, Thursday Oct. 18th

"*Linear Programming Exercises*" refers to http://www.seas.ucla.edu/∼vandenbe/ee236a/homework/problems.pdf

Problem 1 (`4 points`, Exer. 12 in *Linear Programming Exercises*): We are given p matrices $A_i \in \mathbb{R}^{n\times n}$, and we would like to find a single matrix $X \in \mathbb{R}^{n\times n}$ that we can use as an approximate right inverse for each matrix A_i, *i.e.* we would like to have

$$A_i X \approx I, \quad i = 1, \dots, p \tag{1}$$

We can do this by solving the following optimization problem with X as variable:

$$\text{minimize} \max_{i=1,\dots,p} ||I - A_i X||_\infty \tag{2}$$

Here $||H||_\infty$ is the infinity-norm or the max-row-sum norm of a matrix H, defined as

$$||H||_\infty = \max_{i=1,\dots,m} \sum_{j=1}^{n} |H_{ij}| \tag{3}$$

if $H \in \mathbb{R}^{m\times n}$. Express (2) as an LP. You don't have to reduce the LP to a canonical form, as long as you are clear about what the variables are, what the meaning is of any auxiliary variables that you introduce, and why the LP is equivalent to the problem (2).

Problem 2 (`2 points`, Exer. 33 (b)(e) in *Linear Programming Exercises*): Which of the following sets S are polyhedra? If possible, express S in inequality form, i.e., give matrices A and b such that $S = \{x | Ax \leq b\}$.

(a) $S = \{x \in \mathbf{R}^n |\ x \geq 0,\ \mathbf{1}^T x = 1,\ \sum_{i=1}^n x_i a_i = b_1,\ \sum_{i=1}^n x_i a_i^2 = b_2\}$, where $a_i \in \mathbf{R}$, $i = 1, \cdots, n$, $b_1 \in \mathbf{R}$, and $b_2 \in \mathbf{R}$ are given.

(b) $S = \{x \in \mathbf{R}^n |\ ||x - x_0|| \leq ||x - x_1||\}$, where $x_0, x_1 \in \mathbf{R}^n$ are given. S is the set of points that are closer to x_0 than to x_1.

Problem 3 (`3 points`, Exer. 35 (a) in *Linear Programming Exercises*):

Is $\tilde{x} = (1,1,1,1)$ an extreme point of the polyhedron $\mathcal{P}$ defined by the linear inequalities

$$\begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{bmatrix} ?$$

If it is, find a vector c such that $\tilde{x}$ is the unique minimizer of $c^T x$ over $\mathcal{P}$.

Hint: If the objective function is parallel to one of the hyperplanes defining the feasibility region, Do you get an unique minimizer ? Try to think of the solution of this problem graphically.

Problem 4 (`4 points`, Exer. 47 in *Linear Programming Exercises*)

Consider the polyhedron

$$\mathcal{P} = \{x \in \mathbf{R}^4 | Ax \leq b, Cx = d\}$$

where

$$A = \begin{bmatrix} -1 & -1 & -3 & -4 \\ -4 & -2 & -2 & -9 \\ -8 & -2 & 0 & -5 \\ 0 & -6 & -7 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} -8 \\ -17 \\ -15 \\ -17 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 13 & 11 & 12 & 12 \end{bmatrix}, \quad d = 48$$

Prove that $\hat{x} = (1,1,1,1)$ is an extreme point of $\mathcal{P}$.

Problem 5 (`4 points`, Exer. 24 in *Linear Programming Exercises*): Describe how you would use linear programming to solve the following problem. You are given an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array} \tag{4}$$

in which the coefficients of $A \in \mathbb{R}^{m \times n}$ are uncertain. Each coefficient A_{ij} can take arbitrary values in the interval

$$[\bar{A}_{ij} - \Delta A_{ij}, \bar{A}_{ij} + \Delta A_{ij}],$$

where $\bar{A}_{ij}$ and ΔA_{ij} are given with $\Delta A_{ij} \geq 0$. The optimization variable x in (4) must be feasible for all values of A. In other words, we want to solve

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \text{ for all } A \in \mathcal{A}, \end{array} \tag{5}$$

where $\mathcal{A} \subseteq \mathbf{R}^{m \times n}$ is the set

$$\mathcal{A} = \{A \in \mathbf{R}^{m \times n} \mid \bar{A}_{ij} - \Delta A_{ij} \leq A_{ij} \leq \bar{A}_{ij} + \Delta A_{ij}, \quad i = 1, \cdots, m, \ j = 1, \cdots, n\}. \tag{6}$$

If you know more than one solution method, you should give the most efficient one.