Linear Programming
(ECE 236A) – HW 3

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Problem 1

$$\min \|H\|_\infty = \min \max_{i=1,\dots,m} \sum_{j=1}^{n} |H_{ij}| = \min \max \{|H_{i1}|+|H_{i2}|+\dots+|H_{in}|\}$$

$$= \min \max \{ \underbrace{\max\{-H_{i1}, H_{i1}\}}_{t_{i1}} + \dots + \underbrace{\max\{-H_{in}, H_{in}\}}_{t_{in}} \}$$

$$= \min \max_{i=1,\dots,m} t_{i1}+t_{i2}+\dots+t_{in}$$

$$\text{s.t. } -t_{i1} \le H_{i1} \le t_{i1}$$

$$\vdots$$

$$-t_{in} \le H_{in} \le t_{in}$$

$$\min \max_{i=1,\dots,m} \sum_{j=1}^{m} t_{ij}$$

$$\text{s.t. } -t_{ij} \le H_{ij} \le t_{ij}$$

$$\max_{i=1,\dots,m} \sum_{j=1}^{m} t_{ij} \longrightarrow t$$ (Change of variable)

$$\min \ t$$

$$\text{s.t. } -t_{ij} \le H_{ij} \le t_{ij}$$

$$\sum_{j=1}^{m} t_{ij} \le t$$

equivalent to $\min \|H\|_\infty$ where $\|H\|_\infty = \max_{i=1,\dots,m} \sum_{j=1}^{n} |H_{ij}|$

We try to minimize $\max_{i=1,\dots,p} \|I - A_i X\|_\infty$

$$\Rightarrow \quad \min \ t$$

$$\text{s.t. } -(B_k)_{ij} \le (I - A_k X)_{ij} \le (B_k)_{ij}, \quad k=1,\dots,p$$

$$\sum_{j=1}^{n} (B_k)_{ij} \le t \qquad i=1,\dots,n$$

$$j=1,\dots,n$$

Since we are optimizing for the $\max_{i=1,\dots,p}$, all of them (All the "k"s) should satisfy the constraint.

Problem 2

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(a) S is a polyhedron.

$-x_i \le 0$, $i=1,\dots,n$

$1^T x \le 1$

$-1^T x \le -1$

$\sum_i a_i x_i \le b_1$

$-\sum_i a_i x_i \le -b_1$

$\sum_i a_i^2 x_i \le b_2$

$-\sum_i a_i^2 x_i \le -b_2$

$$\underbrace{\begin{bmatrix} \left.\begin{matrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{matrix}\right\} n \\ \begin{matrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & -1 & -1 & \cdots & -1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ -a_1^2 & -a_2^2 & -a_3^2 & \cdots & -a_n^2 \end{matrix} \end{bmatrix}}_{\in \mathbb{R}^{(n+6)\times n}} x \le \underbrace{\begin{bmatrix} n\left\{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right. \\ 1 \\ -1 \\ b_1 \\ -b_1 \\ b_2 \\ -b_2 \end{bmatrix}}_{\in \mathbb{R}^{(n+6)}}$$

(b) S is polyhedron.

Square the ineq. ↗

$$\|x-x_0\|^2 \le \|x-x_1\|^2 \iff -2x_0^T x + \|x_0\|^2 \le -2x_1^T x + \|x_1\|^2$$

$$\underbrace{-2(x_0-x_1)^T}_{A} x \le \underbrace{\|x_1\|^2 - \|x_0\|^2}_{b}$$

Problem 3

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Active indizes are 1, 2, 3, 4.

Therefore $A_J = \begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \end{bmatrix}$

rank$(A_J) = 4$ therefore $(1,1,1,1)$ is an extreme point.

$$1^T A_J \tilde{x} = \sum_{i \in J} b_i = -13$$

Also,

$$1^T A_J x \le \sum_{i \in J} b_i = -13 \implies \underbrace{-1^T A_J}_{c^T} x \ge -\sum_{i \in J} b_i = 13$$

Then, $c^T x \ge -1^T A_J \tilde{x} = -\sum_{i \in J} b_i = 13$

$\boxed{c^T x \ge 13}$ for $c = -A_J^T 1$

$c^T = -1^T A_J$

$$c^T = \begin{bmatrix} -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 4 \\ -15 \end{bmatrix}$$

For $c^T = [8 \ 16 \ 4 \ -15]^T$

$\tilde{x} = (1,1,1,1)$ minimizes over P

$c^T \tilde{x} = 13$

Problem 4

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Active indices are 2, 3, 4.

Because $A\hat{x} = \begin{bmatrix} -9 \\ -17 \\ -15 \\ -17 \end{bmatrix}$ and $b = \begin{bmatrix} -8 \\ -17 \\ -15 \\ -17 \end{bmatrix}$

Therefore we want → $\operatorname{rank}\left(\begin{bmatrix} -4 & -2 & -2 & -9 \\ -3 & -2 & 0 & -5 \\ 0 & -6 & -7 & -4 \\ 13 & 11 & 12 & 22 \end{bmatrix}\right) = 4$

Thus, $\hat{x}$ is an extreme point.

Problem 5

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If we satisfy for the A^*, where A^* is the "worst" matrix one can find for the constraint, then $Ax \leq b$ for all $A \in \mathcal{A}$ is satisfied.

By looking it can be seen that

$$A^* = \bar{A}_{ij} + \Delta A_{ij} \cdot \operatorname{sgn}(x_j) \quad \text{where} \quad \operatorname{sgn}(x) = \begin{cases} -1, & \text{for } x<0 \\ 0, & \text{for } x=0 \\ 1, & \text{for } x>0 \end{cases}$$

Since $A^* x \geq Ax$ for all $A \in \mathcal{A}$.

$$(A^* x)_i = \sum_{j=1}^{n} (\bar{A}_{ij} + \Delta A_{ij} \operatorname{sgn}(x_j)) x_j = \sum_{j=1}^{n} \bar{A}_{ij} x_j + \sum_{j=1}^{n} \Delta A_{ij} |x_j|$$

$$= (\bar{A}x)_i + (\Delta A |x|)_i$$

$$\rightarrow \begin{array}{ll} \min & c^T x \\ \text{st.} & \bar{A}x + \Delta A |x| \leq b \end{array} \Bigg| \begin{array}{ll} \min & c^T x \\ \text{st.} & \bar{A}x + \Delta A t \leq b \\ & -t \leq x \leq t \end{array}$$