Adjacency labeling schemes for sparse and power law graphs.

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Abstract

We devise adjacency labeling schemes for power-law graphs. These are a sub-family of sparse graphs that received considerable attention in the literature in the last decade. We first find a lower bound for the size of the induced universal graphs required to each of the families. We then provide a constructive upper bound for these induced universal graphs using adjacency labeling schemes of similar asymptotic size.

1 Introduction

Power-law graphs appear in vast number of places. The number of nodes in power-law graphs seen in practice is of magnitude 10^{12}, labels of magnitude $\sqrt{10^{12}}$ is reasonable to handle given the current state of hardware.

1.1 Previous work

A graph G is universal, respectively induced universal to the graph family $\mathcal{F}$ if it contains all graphs in $\mathcal{F}$ as subgraphs, respectively induced subgraphs. The size of the smallest induced universal graph was first studied by Moon [**?**]. Seen from labelling schemes, he showed a lower bound of $n/2$ on the label size for general graphs, and an upper bound of $n/2+\log n$. This gap was recently closed by Alstrup et al. [**?**]. Universal graphs for sparse graphs were investigated first by Babai et al. [**?**] and improved by Alon and Asodi [**?**].

Remark 1. Let N be the smallest number of nodes in an induced universal graph $G = (V, E)$ for a family $\mathcal{F}$ of graphs each with n vertices. If this family contains no isomorphic graphs, then $\binom{N}{n}$ is clearly an upper bound on $|\mathcal{F}|$.

2 Preliminaries

In the following we deal with n-vertex undirected finite and connected graphs. A graph $G = (V, E)$ is *sparse* if $|E| = O(n)$, and more precisely a graph with at most cn edges is called a c-sparse graph. We denote the family of c-sparse graphs with n nodes as $\mathcal{S}_C$. The degree of a vertex v in a graph $G = (V, E)$ is denoted by $\Delta(v)$. For every $1 \leq k \leq n$ the collection of vertices of degree k is denoted V_k. We begin by a definition relating to the fraction of vertices of a certain degree.

Definition 1. *The* degree distribution *of a graph* $G = (V, E)$ *is the mapping* $\mathrm{ddist}_G(k) : \mathbb{N}_0 \longrightarrow \mathbb{Q}$ *defined by* $\mathrm{ddist}_G(k) := \frac{|V_k|}{n}$.

For socalled *power-law graphs* $\mathrm{ddist}_G(k) \sim Ck^{-\alpha}$ for real numbers $C > 0$ and $\alpha > 1$. As $n\mathrm{ddist}_G(k)$ is a positive integer, but $nCk^{-\alpha}$ is, in general, a non-integral real, the "$\sim$" entails some rounding of $nCk^{-\alpha}$.

In the literature, a number of generative models that "grow" graphs whose degree distributions are, with high probability, asymptotically "close" to $\mathrm{ddist}_G(k) \sim Ck^{-\alpha}$ have been proposed for various values of α, most prominently the Barabasi-Albert model [] and ... Common to these are that they work in discrete time-steps with each step involving some randomization (e.g., in the Barabasi-Albert model, each step introduces a fresh node that connects to a fixed number of existing edges with some probability dependent on the degrees of the existing nodes).

We require that $k \mapsto C\frac{n}{k^\alpha}$ be a probability distribution. Hence, in particular $C\sum_{k=1}^{\infty} i^{-\alpha} = 1$, and hence $C = 1/\sum_{k=1}^{\infty} i^{-\alpha} = 1/\zeta(\alpha)$ where ζ is the Riemann zeta function. Thus, C is dictated by the choice of α.

2.1 Proper power-law graphs

We define a class of *proper* power law graphs where the number of vertices of degree k must be $C\frac{n}{k^\alpha}$ rounded either up or down and the number of vertices of degree k is non-increasing with k (note that the function $k \mapsto C\frac{1}{k^\alpha}$ is strictly decreasing, so these demands are more lax than they could have been).

Definition 2. *Let $\alpha > 1$ be a real number. We say that a graph $G = (V, E)$ is an* α-proper power-law graph *if (i) for every $1 \leq i \leq n$: $|V_i| = n \cdot \mathrm{ddist}_G(k) \in \{\lfloor \frac{1}{\zeta(\alpha)}\frac{n}{i^\alpha} \rfloor, \lceil \frac{1}{\zeta(\alpha)}\frac{n}{i^\alpha} \rceil\}$, and (ii) for every $1 \leq i \leq n-1$: $|V_i| \geq |V_{i+1}|$.*

Proposition 1. *The maximum degree of a node in an α-proper power-law graph is at most $\left(\frac{1}{\zeta(\alpha)(\alpha-1)} + 2\right)\sqrt[\alpha]{n} + 2$.*

Proof. Let $n > 0$ be an arbitrary integer and let $k' \triangleq \lceil \sqrt[\alpha]{n} \rceil$. Furthermore, let $S_{k'} = \sum_{i=1}^{k'} |V_i|$, that is $S_{k'}$ is the number of nodes of degree at most k'.

Let $S_{k'}^{-} = \sum_{i=1}^{k'} \lfloor \frac{n}{\zeta(\alpha)} i^{-\alpha} \rfloor$. Then, $S_{k'} \geq S_{k'}^{-}$. We now bound $S_{k'}^{-}$ from below.

For every i with $1 \leq i \leq k'$ we have $\lfloor \frac{n}{\zeta(\alpha)} i^{-\alpha} \rfloor + 1 \geq \frac{1}{\zeta(\alpha)} x^{-\alpha}$, and hence

$$\begin{aligned}
S^{-}_{k'} + k' = &\sum_{i=1}^{k'} \left(\left\lfloor \frac{n}{\zeta(\alpha)} i^{-\alpha} \right\rfloor + 1 \right) \geq \sum_{i=1}^{k'} \frac{n}{\zeta(\alpha)} i^{-\alpha} = \frac{n}{\zeta(\alpha)} \sum_{i=1}^{k'} i^{-\alpha} \geq \\
&n \left(1 - \frac{1}{\zeta(\alpha)} \sum_{i=k'+1}^{\infty} i^{-\alpha} \right) \geq n \left(1 - \frac{1}{\zeta(\alpha)} \int_{k'}^{\infty} x^{-\alpha} dx \right) = \\
&n \left(1 - \frac{1}{\zeta(\alpha)} \left[\frac{1}{\alpha - 1} x^{-\alpha+1} \right]_{\infty}^{k'} \right) = n \left(1 - \frac{1}{\zeta(\alpha)(\alpha-1)} \left(\lceil n^{\frac{1}{\alpha}} \rceil \right)^{-\alpha+1} \right) \geq \\
&n \left(1 - \frac{1}{\zeta(\alpha)(\alpha-1)} \left(n^{\frac{1}{\alpha}} \right)^{-\alpha+1} \right) = n - \frac{n}{\zeta(\alpha)(\alpha-1)} n^{-1+\frac{1}{\alpha}} = \\
&n - \frac{1}{\zeta(\alpha)(\alpha-1)} \sqrt[\alpha]{n}
\end{aligned}$$

Thus, $S_{k'} \geq S^{-}_{k'} \geq n - \frac{1}{\zeta(\alpha)(\alpha-1)} \sqrt[\alpha]{n} - \lceil \sqrt[\alpha]{n} \rceil$

As for every $1 \leq i \leq n-1$: $|V_i| \geq |V_{i+1}|$, there are thus at most $1/(\zeta(\alpha)(\alpha - 1)) \sqrt[\alpha]{n} + \lceil \sqrt[\alpha]{n} \rceil$ nodes of degree strictly more than $k' = \lceil \sqrt[\alpha]{n} \rceil$. Hence, the maximum degree of any α-proper power-law graph is at most $\left(\frac{1}{\zeta(\alpha)(\alpha-1)} + 2 \right) \sqrt[\alpha]{n} + 2$. □

Proposition 2. *Let $C > 0$ and $\alpha > 2$. Then, any α-proper power-law graph is sparse.*

Proof. By Proposition 1, the maximum degree of a node in an α-proper power-law graph is at most $k' \triangleq \left(\frac{1}{\zeta(\alpha)(\alpha-1)} + 2 \right) \sqrt[\alpha]{n} + 2$, whence the total number of edges is at most $\frac{1}{2} \sum_{i=1}^{k'} kV_k$. By definition, $V_k \leq \lceil \frac{1}{\zeta(\alpha)} \frac{n}{k^n} \rceil \leq \frac{1}{\zeta(\alpha)} \frac{n}{k^\alpha} + 1$, and thus

$$\begin{aligned}
\frac{1}{2} \sum_{i=1}^{k'} kV_k \leq \frac{1}{2} \sum_{i=1}^{k'} k \left(\frac{1}{\zeta(\alpha)} \frac{n}{k^\alpha} + 1 \right) &\leq \frac{k'}{2} + \frac{n}{\zeta(\alpha)} \sum_{k=1}^{\infty} k^{-\alpha+1} \\
&\leq \left(\frac{1}{2\zeta(\alpha)(\alpha-1)} + 1 \right) \sqrt[\alpha]{n} + 1 + \frac{n\zeta(\alpha-1)}{\zeta(\alpha)}
\end{aligned}$$

□

FIXME! JGS: I don't know what happens at $1 \leq \alpha \leq 2$.

2.2 Approximate power-law graphs

Definition 3. *Let $\alpha > 1$ and $\epsilon > 0$ be real numbers. An undirected graph (V, E) is said to be an* ϵ-approximate α-power-law graph *if, for all $1 \leq k \leq n = |V|$, we have* $\left| \frac{V_k}{n} - \frac{1}{\zeta(\alpha)} k^{-\alpha} \right| < \epsilon \frac{1}{\zeta(\alpha)} k^{-\alpha}$.

That is, for an ϵ-approximate α-power-law graph, the probability that node has degree k differs from a power law with exponent α by strictly less than ϵ weighted by the "pure" power-law probability that the node has degree k.

FIXME: Perhaps a drawing of a histogram is appropriate.

As shown by Bollobàs et al. [], the Barabasi-Albert model will, for each ϵ, produce graphs that with probability 1 for all sufficiently large n will be ϵ-approximate 3-power-law graphs (but only for the range $1 \leq k < \sqrt[15]{n}$; see the definition of "ϵ-approximate α-power-law graph with cutoff" below.).

We require the graphs to have a degree distribution that is a good approximation to a power-law for *all* $1 \leq k \leq n-1$. We stress that the adjacency labelling schemes we devise in Section ... work with the with upper bounds even if nodes above a certain degree threshold (e.g. $k \geq \sqrt[15]{n}$) are ignored (cf. the result by Bollobàs et al.).

Proposition 3. *Let $\alpha > 1$ and $\epsilon > 0$ be real numbers. For all $n \geq \zeta(\alpha)k^{\alpha}/\epsilon$ there is an ϵ-approximate α-powerlaw graph with n nodes.*

Proof. TBD □

FIXME! JGS: the lower bound on n above doesn't matter. The important thing is that there is such a bound.

Lemma 1. *Let $\alpha > 1$ and $\epsilon > 0$ be real numbers. The maximum degree of a node in an ϵ-approximate α-power-law graph is at most $\left\lfloor \sqrt[\alpha]{\frac{(1+\epsilon)n}{\zeta(\alpha)}} \right\rfloor$.*

FIXME! JGS: There is a silly special case above where $\sqrt[\alpha]{n}$ is an integer where the formulation needs to be changed. Will just engender more notational confusion :-/

Proof. The assumption that $|\frac{V_k}{n} - \frac{1}{\zeta(\alpha)}k^{-\alpha}| < \frac{1}{\zeta(\alpha)}k^{-\alpha}\epsilon$ implies that $V_k/n - \epsilon\frac{1}{\zeta(\alpha)k^{-\alpha}} \leq \frac{1}{\zeta(\alpha)k^{-\alpha}}$, hence that $V_k \leq \frac{n}{\zeta(\alpha)}k^{-\alpha}(1+\epsilon)$. But $\frac{n}{\zeta(\alpha)}k^{-\alpha}(1+\epsilon) < 1 iff k > \sqrt[\alpha]{\frac{(1+\epsilon)n}{\zeta(\alpha)}}$, whence $k > \sqrt[\alpha]{\frac{(1+\epsilon)n}{\zeta(\alpha)}}$ implies $V_k = 0$ (as V_k is a non-negative integer). □

Proposition 4. *Let $\alpha > 2$ and $\epsilon > 0$ be real numbers. Any ϵ-approximate α-power-law graph is $\frac{(1+\epsilon)\zeta(\alpha-1)}{2\zeta(\alpha)}$-sparse.*

Proof. By Lemma 1, the maximum degree of a node in an ϵ-approximate α-power-law graph is at most $k' \triangleq \left\lfloor \sqrt[\alpha]{\frac{(1+\epsilon)n}{\zeta(\alpha)}} \right\rfloor$. Hence, the total number of edges is at most $\frac{1}{2}\sum_{k=1}^{k'} kV_k$.

As in the proof of Lemma 1, observe that $V_k \leq \frac{n}{\zeta(\alpha)}k^{-\alpha}(1+\epsilon)$, we hence have

$$\frac{1}{2}\sum_{k=1}^{k'} kV_k \leq \frac{n(1+\epsilon)}{2\zeta(\alpha)}\sum_{k=1}^{k'} k^{-\alpha+1} \leq \frac{n(1+\epsilon)}{2\zeta(\alpha)}\sum_{k=1}^{\infty} k^{-\alpha+1} \leq \frac{n(1+\epsilon)\zeta(\alpha-1)}{2\zeta(\alpha)}$$

As $\zeta(\alpha-1)$ is well-defined for $\alpha > 2$, the result follows. □

FIXME: JGS has no idea what happens for $1 < \alpha \leq 2$. Conceivably, the graphs could be non-sparse.

2.3 Approximate power-law graphs with cutoff

Generative models may fail to generate graphs whose degree distributions follow power-laws exactly, cf. the result by Bollobàs et al. [] mentioned above. This is the motivation for the following definition.

Definition 4. *Let $\alpha > 1$ and $\epsilon > 0$ be real numbers and let $h : \mathbb{N} \longrightarrow \mathbb{N}$ be a non-decreasing, unbounded function with $h(n) \leq n$. An undirected graph (V, E) is said to be an* ϵ-approximate α-power-law graph with cutoff h *if, for all $1 \leq k \leq h(n) = |V|$, we have $\left|\frac{V_k}{n} - \frac{1}{\zeta(\alpha)}k^{-\alpha}\right| < \epsilon\frac{1}{\zeta(\alpha)}k^{-\alpha}$.*

The difference between Definitions 3 and 4 is that a graph with cutoff with n nodes need only be approximate power-law below the "cutoff" of $h(n)$.

For specific ϵ, α and h, for example, the $h(n) = \sqrt[15]{n}$ of Bollobàs et al. [], one may consider the family of all ϵ-approximate α-power-law graphs with cutoff h. The below result shall later be key to the observation that such families in general do not have short labelling schemes for adjacency.

Lemma 2. *Let $\alpha > 1$ and $\epsilon > 0$ be real numbers and let $h : \mathbb{N} \longrightarrow \mathbb{N}$ be a non-decreasing, unbounded function with $h(n) \leq n$. Then, for every natural number m and every undirected graph G with $n - h(m) -?$ nodes there is an ϵ-approximate α-power-law graph with cutoff h having blah nodes that contains the complement G as a subgraph.*

Proof. TBD ☐

3 The Labeling Schemes

We first handle sparse graphs.

Proposition 5. *There exist a $\sqrt{2cn}\log n + \log n$ adjacency labeling scheme for $\mathcal{S}_c$.*

Proof. Let $G = (V, E)$ be a c-sparse graph. We first assign each vertex $v \in V$ a unique identifier $ID(v)$, using $\log n$ bits. A vertex of degree at least $\sqrt{2cn}$ is called *fat* and *thin* otherwise. From hereon, we use the terminology degree threshold to describe the value separating these two groups. The first bit of $\mathcal{L}(v), v \in V$ is set to zero if v is fat and to one if it is thin. Since there are at most $2cn$ edges, the number of fat vertices is at most $\sqrt{2cn}$. Let (u, v) be an edge in G such that $ID(u) < ID(v)$. If u and v are both either thin or fat $ID(v)$ will appear in $\mathcal{L}(u)$ and vice versa. If u is fat and v is thin, $ID(u)$ will appear in $ID(v)$. Since there are at most $\sqrt{2cn}$ fat vertices, the size of the largest label is bounded by $\sqrt{2cn}\log n + \log n$. Similarly, thin vertices enjoy the same label size as they have at degree at most $\sqrt{2cn}$. Decoding the label is now obvious, and will take $O(\sqrt{n})$ operations. ☐

Remark 2. It is easy to see that $f(n)$-sparse graphs[1] enjoy a $\sqrt{2f(n)n}\log n$ labeling scheme by setting the degree threshold to $\sqrt{2f(n)n}$.

Recall that $\mathcal{P}_{C,\alpha} \in \mathcal{S}_{2C}$ when $\alpha \geq 2$. This yields a $\sqrt{4Cn}\log n$ labeling scheme for $\mathcal{P}_{C,\alpha}$. We now show that this label can be significantly improved. To do so we first need to account for the number of vertices of degree at least k for any $1 \leq k \leq n$.

Proposition 6. *Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be a mapping such that $f(n) = o(n)$. Let $C > 0$ and $\alpha > 1$ be real numbers. Then there is $N \in \mathbb{N}$ such that if G is a power-law graph with $\mathrm{ddist}_G(k) = Ck^{-\alpha}$ and at least N vertices, then the fraction of vertices in G with degree at least $f(n)+1$ is bounded above by $O(f(n)^{-(\alpha-1)})$.*

Proof. For $1 \leq j \leq n-1$, the fraction of vertices of degree at least j is $C\sum_{i=j}^{n-1} 1/i^{\alpha}$. Note that $d/dx(Cx^{-\alpha}) = -\alpha C x^{-(\alpha+1)} < 0$ and thus for all $j \geq 1$ we have $\int_j^{j+1} Cx^{-\alpha}dx > C(j+1)^{-\alpha}$. Hence, the fraction of vertices of degree at least $j+1$ is at most

$$\int_j^{n-1} Cx^{-\alpha}dx = \left[\frac{C}{-(\alpha-1)}x^{-(\alpha-1)}\right]_j^{n-1} = \frac{C}{\alpha-1}\left(j^{-(\alpha-1)} - (n-1)^{-(\alpha-1)}\right)$$

In particular, the fraction of vertices of degree at least $f(n)+1$ is at most $\frac{C}{\alpha-1}\left(f(n)^{-(\alpha-1)} - (n-1)^{-(\alpha-1)}\right) \leq \frac{C}{\alpha-1}f(n)^{-(\alpha-1)} = O(f(n)^{-(\alpha-1)})$. ☐

Lemma 3. *Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be a computable mapping such that $f(n) = o(n)$ and let $C > 0, \alpha > 1$ be real numbers. Then the family of power-law graphs with $\mathrm{ddist}_G(k) = Ck^{-\alpha}$ has a adjacency labeling scheme such that for all sufficiently large n, the maximum size of a label is bounded above by:*

$$\log n + \max(f(n)\log n, O(n/f(n)^{\alpha-1})).$$

Proof. We set the degree threshold at $f(n)$. By that we mean that a vertex $v \in V$ is *small* if $\Delta(v) \leq f(n)$ and *large* otherwise. By Proposition 6 there are at most $c'n/f(n)^{\alpha-1}$ large vertices for some c'. We assign each vertex a unique identifier from $1\ldots n$, such that the large vertices are assigned the last $c'n/f(n)^{\alpha-1}$ identifiers according to their degree in non-decreasing order.

The label of a small vertex consists of two parts: its unique identifier and a list of the identifiers of all its neighbors. The size of such label is thus at most $(f(n)+1)\log n$. The label of a large vertex consists also of two parts: a unique identifier and a bit string of length $c' \cdot n \cdot f(n)^{-(\alpha-1)}$ such that position i is 1 in this bit string if the vertex is adjacent to the i'th largest large vertex. The size of a such label is at most $c' \cdot n/f(n)^{\alpha+1} + \log n$.

Let $\mathcal{L}(v), \mathcal{L}(w)$ be two labels assigned by our suggested decoder to vertices $v, w \in V$. If v and w are both small or both large, then there is an edge from v to w if and only if w is listed in the label of v and vice versa. Assume w.l.o.g that v is small and w is large, then there is an edge from v to w if and only if w is listed in the label of w ☐

[1]Graphs with n vertices and $f(n)$ edges

From Lemma 3 it follows that the smallest label size is attained when $f(n)\log n = c'n/f(n)^{\alpha-1}$ for some constant c'. We rearrange the equation and get $f(n)^\alpha = c'n$ and thus the optimum label size occurs when $f(n) = \sqrt[\alpha]{c'n}$, and the resulting label size is $O(\sqrt[\alpha]{n}\log n)$.

This also means that there are $O(\sqrt[\alpha]{n})$ large vertices, which correlates nicely with the following fact. The largest degree in an (α, C)-power law graph is bounded by $O(\sqrt[\alpha]{n})$. To see this observe that the number of vertices of degree k must to be at least 1. Thus $nC\frac{1}{k^\alpha} \geq 1$, which implies that $k \leq \sqrt[\alpha]{nC}$ for some constant C.

Conjecture 1. *Any family of graphs such that* ddist(k) *has "high" positive skewness will have labeling schemes for adjacency with sublinar maximum labeling size. A reasonable way forward would be to consider the third moment of some standard distributions and see what happens.*

4 Lower Bounds

We begin this section by showing that the upper bounds achieved for sparse graphs are fairly close to the best possible. By Moon [?] it follows that any adjacency labeling scheme for general graphs requires at least $\lfloor n/2 \rfloor$ bits. For brevity, we assume now that n is an even number. We present the following extension, due to Spinard [?].

Proposition 7. *Any adjacency labeling scheme for c-sparse graphs requires labels of size strictly larger than* $\frac{\sqrt{n}}{2\sqrt{c}}$ *bits.*

Proof. Assume for contradiction that there exist a labeling scheme for adjacency assigning labels of size strictly less than $\frac{\sqrt{n}}{2\sqrt{c}}$. Let G be an n-vertices graph. Let G' be the graph resulting by adding $\frac{n(n-1)}{c}$ isolated vertices to G, and note that now G' is $c-sparse$. The graph G is an induced subgraph of G'. It now follows that the nodes of G have adjacency labels of size less than $\frac{\sqrt{n^2/c}}{2\sqrt{c}} = n/2$ bits. As G was an arbitrary graph, we obtain a contradiction. □

4.1 Lower bounds for $\mathcal{P}_{C,\alpha}$.

We now show that a similar lower bound can be attained for power-law graphs where $\alpha > 2$. To do so, we first must argue that constructing power law graphs in this fashion is possible.

A *degree sequence* is a sequence of integers $d_1 \dots d_n$ such that $0 \leq d_i \leq n-1$. We denote $v_1 \dots v_n$ the vertices the vertices of a graph $G = (V, E)$ in non-increasing order according to their degree. We denote $v_i \in V$ the i'th vertex in this ordering. The *degree sequence* of G is $d_1 \geq d_2 \dots \geq d_n$ where d_i is the degree of v_i. While every graph has a degree sequence, not all degree sequences have graphs. We say that a degree sequence is *realizable* if it has a corresponding graph. The *Erdös-Gallai theorem* [?] states the following: A degree sequence $d_1 \dots d_N$ is realizable if and only if:

1. For every $1 \leq k \leq N-1$: $\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{N} \min\{k, d_i\}$.

2. $\sum_{i=1}^{n} d_i$ is even.

We now show the following:

Lemma 4. *Any degree sequence $d_1 \geq d_2 \cdots \geq d_n$ that abides to an (α, C) power-law distribution is realizable.*

Proof. First, since graphs in $\mathcal{P}_{C,\alpha}$ are sparse, for every $k > O(\sqrt[\alpha]{n})$ the condition holds trivially by observing that $k(k+1) > O(n)$. The maximum degree of a vertex in a power law graph is $O(\sqrt[\alpha]{n})$. When $k \leq O(\sqrt[\alpha]{n})$ we have that:

$$\sum_{i=1}^{k} d_i < O(\sqrt[\alpha]{n}\sqrt[\alpha]{n}) = O(n^{2/\alpha}) = o(n)$$

since $\alpha > 2$. It is easy to see that in this case $\sum_{i=k+1}^{n} \min\{k, d_i\} = O(n)$, and thus the inequality holds.

Finally, if $\sum_{i=1}^{n} d_i$ is not even we add a single edge to the graph. □

Lemma 5. *Any n vertices graph $G = (V, E)$ can be extended to an $N = cn^{\alpha+1}$ vertices graph $G' = (V', E')$ where $G' \in \mathcal{P}_{C,\alpha}$, such that G is an induced subgraph of G'.*

Proof. The number of vertices with degree at least n in G' is $O(\frac{N}{n^\alpha}) = O(n)$. By Lemma 4, we can create G' by first inserting the degree sequence corresponding the vertices of G, and then complete the power lower distribution naively. □

We can now conclude with our lower bound:

Proposition 8. *Any adjacency labeling scheme for $\mathcal{P}_{C,\alpha}$ requires least $O(\sqrt[\alpha+1]{n})$ bits.*

Proof. Assume for contradiction that there exist a labeling scheme for adjacency assigning labels of size at most $o(\sqrt[\alpha+1]{n}/c)$ bits. Let G be a graph of n-vertices. We now construct the graph G' as described in Lemma 5. The graph G can be reconstructed by an adjacency labeling scheme for adjacency for G' using only the labels of vertices belonging to G inside G'. By the assumption, there is thus a labeling of G' using $o(\sqrt[\alpha+1]{n^{\alpha+1}}) = o(n)$ bits. As G is an arbitrary graph, we obtain a contradiction.

4.2 Constructive models and the implication of the lower bound.

The two most used power-law constructors are Waxman [] and N-level Hierarchical []. The Barabasi-Albert model generates power-law graphs, such that given a parameter m, vertices are inserted to an initially empty graph and attached to at most m existing vertices according to a power-law distribution. We now prove that the constructions by Waxman and Barabasi-Albert can not possibly construct all power-law graphs.

It is easy to see that graphs constructed by the Barabasi-Albert model has an $m \log n$ adjacency labeling scheme. Upon vertex insertion, simply store the identifiers of all vertices attached. This along with proposition 8 suggests that there are a lot more power-law graphs than ones that can be created by preferential attachment, as in the Barabasi-Albert model.

$\square$