Math 281A HW6

December 4, 2017

Problem 1. Show complete sufficiency implies minimal sufficiency.

Proof. Denote T as the complete sufficient statistic and S as the minimal sufficient statistic. Consider statistic $E(T|S)$. First notice that $E(E(T|S) - T) = 0$ always holds for any parameter. and since S is minimal sufficient, there is a measurable function g satisifying $g(T) = S$ therefore $E(T|S) = E(T|g(T))$ which is a function of T. Now $E(E(T|S) - T) = 0$ implies $E(T|S) = T$ almost surely, and this shows that T is a function of S, which means T is minimal sufficient as well. □

Problem 2. Denote the loss function $L(\Delta p) = \mathbb{1}(|\Delta - p| > 1/4)$. Show the loss function for estimator Δ where $\Delta|_{X=0} \sim U(-1/2, 1/2)$ and $\Delta|_{X=1} \sim U(1/2, 3/2)$, where $1/4 < p < 3/4$.

Solution . Consider the case where $X = 1$. The loss will be the portion of interval $(1/2, 3/2)$ outside of the ball centered at p with radius $1/4$, which will be $3/2 - (p + 1/4) = 5/4 - p$. Similarly when $X = 0$, the loss will be $p - 1/4 - (-1/2) = p + 1/4$. Therefore the aggregate loss will be $p(5/4 - p) + (1 - p)(p + 1/4) = -2p^2 - 2p + 1/4$.

Problem 3. Show that any unbiased estimator $\hat{\theta}$ of θ can be improved by multiplying a constant $c < 1$.

Solution . The original MSE is $var(\hat{\theta})$, while the shrinkage estimator has MSE as $c^2 var(\hat{\theta}) + (1 - c)^2\theta^2$. In order to have the latter less than the former, we need

$$var(\hat{\theta}) > c^2[var(\hat{\theta}) + \theta^2] - 2c\theta^2 + \theta^2.$$

The minimal is achieved with $c = \theta^2/[\theta^2 + var(\hat{\theta})]$, which is associated with θ.