Math 281A HW4

November 6, 2017

Problem 1. If X, Y are random variables with joint distribution, Show that

$$var(X) = E(var(X|Y)) + var(E(X|Y)).$$

Proof. We start with

$$var(E(X|Y)) = E(E(X|Y)^2 - E(X))^2 = E(E(X|Y)^2) - (E(X))^2$$

and then

$$E(var(X|Y)) = E(E(X^2 - (E(X|Y))^2|Y)) = E(X^2) - E(E(X|Y)^2).$$

Combine the equations and the result follows. □

Problem 2. If X_1, X_2 are IID standard normal, find the density of X_1/X_2.

Solution . The joint distribution for (X_1, X_2) is $N(0, I)$ with dimension 2. Now consider transformation $Y_1 = X_1/X_2$ and $Y_2 = X_2$. Then the inverse transformation is $X_1 = Y_1Y_2$ and $X_2 = Y_2$, with Jacobian

$$\begin{vmatrix} Y_2 & Y_1 \\ 0 & 1 \end{vmatrix} = Y_2$$

Therefore the joint density of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1y_2, y_2)y_2 = \frac{1}{2\pi}y_2e^{-\frac{y_2^2(y_1^2+1)}{2}}.$$

The marginal density of y_1 is found by integrating against y_2, which gives

$$f_{Y_1}(y_1) = \frac{1}{\pi(y_1^2 + 1)}$$

which is the Cauchy distribution.

Problem 3. Assume X_i are IID Geometric p. For general n, find a good unbiased estimator for $g_1(p) = 1/p$. For $n = 2$, find a good unbiased estimator for $g_2(p) = p$. For g_1, show that CR bound is achieved.

Solution . Note that $E(X_1) = 1/p$ therefore naturally $\bar{X}$ is an unbiased estimator for g_1. For the CR bound, realize that the Fisher information for p with one single observation is $1/(1-p)p^2$, therefore the CR bound is $(1-p)/np^2$. Notice that $var(X_1) = (1-p)/p^2$, and this indicates that the CR bound is achieved.

For g_2, consider using an estimator based on $T = X_1 + X_2$. The sum of two IID geometric distribution forms a random variable following negative binomial distribution with parameter $n = 2$ and p. Therefore, with method one, we can find the needed estimator as $1/(T+1) = 1/(X_1 + X_2 + 1)$.

Problem 4. Denote $E^\lambda(X) = (EX^\lambda)^{1/\lambda}$. Show that

$$\lim_{\lambda \to 0} E^\lambda(X) = exp\{E \log X\}.$$

Proof. Take a natural log at both side and we have

$$\log E^\lambda(X) = \frac{\log EX^\lambda}{\lambda}.$$

Send $\lambda \to 0$, and use L'Hospital rule on the right, we have

$$\lim_{\lambda \to 0} \log E^\lambda(X) = \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} \log EX^\lambda = \lim_{\lambda \to 0} \frac{EX^\lambda \log X}{EX^\lambda} = E \log X.$$

And this finishes the proof. □

Problem 5. If X is distributed as Unif(α, β), show that (i) the order statistics $T = (X_{(1)}, X_{(n)})$ is sufficient; (ii) Calculate $E(\bar{X}|T)$; (iii) Show that this estimator is unbiased for all symmetric distributions, on estimating the mean.

Proof. (i): The density function can be rewritten as $f(x) = \mathbb{1}(X_{(1)} \leq x \leq X_{(n)})1/(\beta - \alpha)$, therefore $T = (X_{(1)}, X_{(n)})$ is sufficient.

(ii): Notice that given T, X_i are following this distribution: with probability $1/n$ each, X_i equal to $X_{(1)}$ or $X_{(n)}$, and with probability $(n-2)/n$ following Unif$(X_{(1)}, X_{(n)})$. Now it is easy to see that $E(X_i|T) = T/2$, which is also equivalent to $E(\bar{X}|T)$.

(iii): By tower rule, $E[E(\bar{X}|T)] = E[\bar{X}]$, which is the mean. □