Math 281B HW 4.

Ch4. 1.2. The pdf of Beta(a, b) is $f(x) = \dfrac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$

for $x \in [0,1]$. Therefore

$$f'(x) = \frac{x^{a-2}(1-x)^{b-2}}{B(a,b)} \left((a-1)(1-x) - (b-1)x\right).$$

ⓐ increasing: Take $a = 2$, $b = 1$

ⓑ decreasing: Take $a = 1$ $b = 2$

ⓒ increasing in $(0, p_0)$, decreasing in $(p_0, 1)$.

Take $a = 1 + p_0$, $b = 2 - p_0$.

In this case, $f'(x) = \dfrac{x^{a-2}(1-x)^{b-2}}{B(a,b)}(p_0 - x)$.

ⓓ decreasing in $(0, p_0)$. increasing in $(p_0, 1)$

Take $a = 1 - p_0$, $b = p_0$

In this case $f'(x) = \dfrac{x^{a-2}(1-x)^{b-2}}{B(a,b)}(x - p_0)$.

1.6.

p has a posterior of Beta$(a+x, b+n-x)$. from the conjugate prior of Beta(a, b).

Therefore $E(p|x) = \dfrac{a+x}{a+b+n}$

$$var(p|x) = \frac{(a+x)(b+n-x)}{(a+b+n)^2(n+a+b+1)}$$

for the target $p(1-p)$, the Bayes estimator

is $E(p(1-p)|x) = E(p|x) - E(p^2|x)$

$$= E(p|x) - var(p|x) - [E(p|x)]^2$$

Solve algebra here $= \dfrac{(x+a)(n+b-x)}{(a+b+n)(a+b+n+1)}$

1.9. Take a look at the kernel of the posterior:

$$f(\lambda|x) \propto \pi(\lambda)\cdot f(x/\lambda) \propto \lambda^{g-1} e^{-\lambda/\alpha} \cdot \lambda^{\Sigma X_i} e^{-n\lambda}$$

$$= \lambda^{g+\Sigma X_i - 1} e^{\cancel{-\lambda(\alpha+n)} -\lambda(\frac{1}{\alpha}+n)}$$

This is the kernel of $Gamma(g+\Sigma X_i, (\frac{1}{\alpha}+n)^{-1})$.

Therefor λ has posterior of $Gamma(g+\Sigma X_i, (\frac{1}{\alpha}+n)^{-1})$

The Bayes estimator, which is the posterior mean, is

$$\delta = \frac{g+\Sigma X_i}{\frac{1}{\alpha}+n} = \frac{\alpha g + \alpha n \bar{X}}{\alpha n + 1} = \frac{1}{\alpha n+1}\alpha g + \frac{\alpha n}{\alpha n+1}\bar{X}$$

prior mean   sample mean

when $n \to \infty$, the first part shrinks, and $\delta \to \bar{X}$

when $\alpha \to \infty$, $g \to 0$, $\delta \to \bar{X}$. (prior contains less information)

If both happens. $\delta \to X$

Ch5 1.9

$$\delta^* = \begin{cases} \frac{X}{n} & pr = 1-\varepsilon \\ \frac{1}{2} & pr = \varepsilon. \end{cases}$$

$$R(p, \delta^*) = E((\delta^* - p)^2) = (1-\varepsilon)\frac{p-p^2}{n} + \varepsilon\left(\frac{1}{2} - p\right)^2$$

Set $\varepsilon = \frac{1}{n+1}$, one can show that

$$R(p, \delta^*) = \frac{1}{4(n+1)} \quad \text{which is a constant.}$$

$$\sup_p R\left(p, \frac{X}{n}\right) = \sup_p \frac{p(1-p)}{n} = \frac{1}{4n} \quad \left(p = \frac{1}{2}\right)$$

Therefore $R(p, \delta^*) < \sup_p R\left(p, \frac{X}{n}\right)$.

1.10 $\delta = \dfrac{X + \frac{1}{2}\sqrt{n}}{n + \sqrt{n}}$. $E(\delta) = \dfrac{np + \frac{1}{2}\sqrt{n}}{n + \sqrt{n}}$

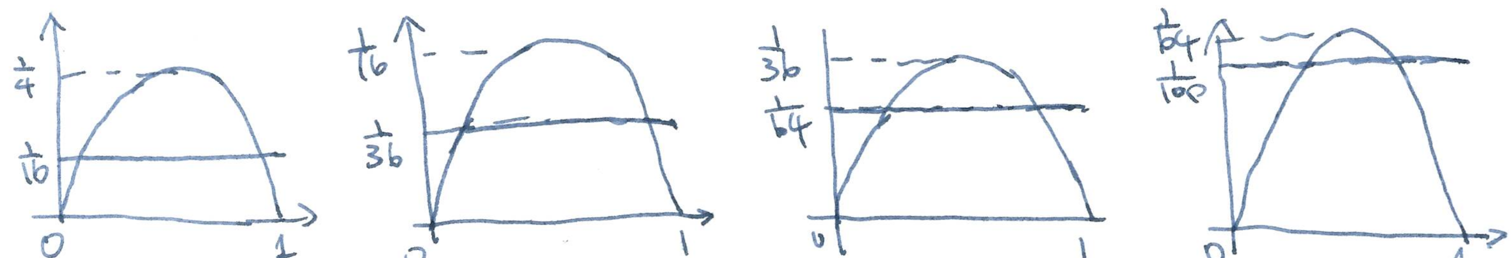
$$\text{Bias}(\delta) = E(\delta) - p = \frac{(\frac{1}{2} - p)\sqrt{n}}{n + \sqrt{n}}$$

Therefore $p < \frac{1}{2}$ it's ~~un under~~ overestimating

$p > \frac{1}{2}$ it's underestimating.

1.12. $R(p, \delta) = \frac{1}{4}\left(\dfrac{1}{1+\sqrt{n}}\right)^2$

$R(p, \frac{X}{n}) = \dfrac{p(1-p)}{n}$



n=1 n=4 n=9 n=16

~~a~~ as n increases the difference of two peaks of the estimators gets ~~g~~ closer.