Math 281B HW 2.

1. Show that for $X_1 \dots X_n$ iid from continuous density f & cdf F, the pdf $f_{(k)}$ for order statistics $X_{(k)}$ is

$$f_{(k)}(x) = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1} (1-F(x))^{n-k} f(x), \quad x \in \mathbb{R}.$$

We consider a transformation $U_i = F(X_i)$. It's easy to see that U_i follows uniform distribution on $[0,1]$. Denote $U_{(k)}$ as the k^{th} order statistic of $U_1 \dots U_n$. It's easy to see that $U_{(k)} = F(X_{(k)})$.

Consider event $\{t \le U_{(k)} \le t+\Delta t\}$. We need exact $k-1$ $(U_i)_s$ to be less than t and at least one of rest $n-k$ variables to be inside $[t, t+\Delta t]$. Notice that if there are more then one variables lying inside $[t, t+\Delta t]$, the probability is of order $O(\Delta t^2)$.

Therefore

$$P(t \le U_{(k)} \le t+\Delta t) = \binom{n}{k-1}\binom{n-k+1}{n-k} t^{k-1}(1-t)^{n-k}\Delta t + O(\Delta t^2)$$

which means

$$P(t \le X_{(k)} \le t+\Delta t) = P(F(t) \le U_{(k)} \le F(t+\Delta t))$$

$$= \frac{n!}{(k-1)!(n-k)!} F(t)^{k-1}(1-F(t))^{n-k}(F(t+\Delta t)-F(t)) + O[F(t+\Delta t)-F(t)]^2$$

By the definition of density.

$$f_{(k)}(t) = \lim_{\Delta t \to 0} \frac{P(t \le X_{(k)} \le t+\Delta t)}{\Delta t}$$

$$= \frac{n!}{(k-1)!(n-k)!} F(t)^{k-1}(1-F(t))^{n-k} f(t)$$

2. Use the previous fact to show that when F is Cauchy (θ), $n=5$, $X_{(3)}$ has finite variance.

It suffices to show the result with $\theta = 0$ since cauchy is of a location family. In this case $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, $F(x) = \frac{1}{\pi}\arctan x + \frac{1}{2}$.

Also, it suffices to show the existence of $E(X_{(3)}^2)$ since $E(X_{(3)}^2) < \infty$ implies $E(X_{(3)}) < \infty$, which leads to $\mathrm{var}(X_{(3)}) < \infty$.

$$E(X_{(3)}^2) \overset{\text{proportional to}}{\propto} \int_{-\infty}^{+\infty} x^2 \left(\arctan(x) + \tfrac{1}{2}\right)^2 \left(\arctan(x) - \tfrac{1}{2}\right)^2 \frac{1}{1+x^2}\, dx$$

$$\overset{\text{symmetry}}{=} 2\int_0^{+\infty} x^2 \left(\arctan(x) + \tfrac{1}{2}\right)^2 \left(\arctan(x) - \tfrac{1}{2}\right)^2 \frac{1}{1+x^2}\, dx$$

$\frac{1}{2} \le \arctan(x) + \frac{1}{2} \le 1$ when $x \ge 0$

$$\le 2\int_0^{+\infty} x^2 \left(\arctan(x) - \tfrac{1}{2}\right)^2 \frac{1}{1+x^2}\, dx \qquad (*)$$

The tail of $\arctan(x) - \frac{1}{2}$ when x is large is similar to $\frac{1}{x}$:

(proof: $\lim_{x\to+\infty} \frac{-\arctan(X) + \frac{1}{2}}{1/x} = \lim_{x\to\infty} \frac{1/(1+x^2)}{1/x^2} = 1$)

Therefore there is some C such that $\frac{1}{2} - \arctan(x) < \frac{2}{x}$ for all $x > C$. Then

$$(*) \le 2\int_0^C x^2 \left(\tfrac{1}{2} - \arctan(x)\right)^2 \frac{dx}{1+x^2} + 2\int_C^{+\infty} x^2 \cdot \frac{2^2}{x^2} \cdot \frac{dx}{1+x^2} < \infty.$$

3. Check that $C > 0.3$ for Barry - Esseen Theorem.

An easy counter example is Rademacher (0.5) with $n=1$.

The definition is

$$R = \begin{cases} 1 & p = 0.5 \\ -1 & p = 0.5. \end{cases}$$

Then $E(R) = 0$ $\quad var(R) = 1.$

So: $\left| P\left(\frac{R - E(R)}{var(R)} \leq y \right) - \Phi(y) \right|$

$$= \left| P(R \leq y) - \Phi(y) \right|$$

Note that $P(R \leq y) = \begin{cases} 0 & y < -1 \\ 0.5 & -1 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$

So when y is close to 1:

$$\left| P(R \leq y) - \Phi(y) \right| \approx \left| 0.5 - 0.84 \right|$$

$$> \frac{0.3}{\sqrt{n}} \left(\frac{E|R - ER|^3}{[var(R)]^{3/2}} \right) = 0.3$$