Math 281 HW 6.

Let $X_1 \cdots X_n$ be iid from $U[0,\theta]$.

Let $L_n^1 = (0.025^{-\frac{1}{n}} - 0.975^{\frac{1}{n}}) X_{(n)}$ $\quad L_n^2 = \left(\frac{1}{1-0.975^{\frac{1}{n}}} - \frac{1}{1-0.025^{\frac{1}{n}}}\right) X_{(1)}$

which are two lengths of two confidence intervals.

1). Find a sequence of k_n^i s.t. $\{k_n^i L_n^i\}$ is tight, but not convergence in probability.

2). Show that L_n^1 is $o_p(1)$ but L_n^2 is not $o_p(1)$.

Step 1: Find the asymptotic distribution of $X_{(1)}$ & $X_{(n)}$

Consider the event $\{X_{(1)} > \frac{t}{n}\} = \{nX_{(1)} > t\}$.

we have $P(X_{(1)} > \frac{t}{n}) = P(nX_{(1)} > t)$

$$= \left[1 - P(X_1 < \tfrac{t}{n})\right]^n = \left[1 - \frac{t/\theta}{n}\right]^n \to e^{-\frac{t}{\theta}}$$

This indicates that $nX_{(1)} \xrightarrow{d} Exp(\frac{1}{\theta})$, which is exponential distribution with parameter $\frac{1}{\theta}$.

By symmetry. $n(\theta - X_{(n)}) \xrightarrow{d} Exp(\frac{1}{n})$

This also indicates that $X_{(n)} \xrightarrow{P} \theta$.

Step 2: Determine the order of the coefficients.

Consider $a_n = \left(0.025^{-\frac{1}{n}} - 0.975^{-\frac{1}{n}}\right) = \left(e^{-\frac{1}{n}\log 0.025} - e^{-\frac{1}{n}\log 0.975}\right)$

$$= e^{-\frac{1}{n}\log 0.975}\left(e^{\frac{1}{n}(\log 0.975 - \log 0.025)} - 1\right)$$

$e^x - 1 \sim x$ when $x \to 0$

$$\sim \frac{1}{n}(\log 0.975 - \log 0.025) \sim \frac{1}{n}$$

$$b_n = \left(\frac{1}{1-0.975^{\frac{1}{n}}} - \frac{1}{1-0.025^{\frac{1}{n}}}\right) = \left(\frac{1}{1-e^{\frac{1}{n}\log 0.975}} - \frac{1}{1-e^{\frac{1}{n}\log 0.025}}\right)$$

$$= \frac{e^{\frac{1}{n}\log 0.975} - e^{\frac{1}{n}\log 0.025}}{\left(e^{\frac{1}{n}\log 0.975}-1\right)\left(e^{\frac{1}{n}\log 0.025}-1\right)} = \frac{e^{\frac{1}{n}\log 0.025}\left(e^{\frac{1}{n}(\log 0.975 - \log 0.025)}-1\right)}{\left(e^{\frac{1}{n}\log 0.975}-1\right)\left(e^{\frac{1}{n}\log 0.025}-1\right)}$$

$e^x - 1 \sim x$ when $x \to 0$.

$$\sim \frac{\cancel{\frac{1}{n}(0.97}\ \frac{1}{n}(\log 0.975 - \log 0.025)}{\frac{1}{n^2}(\log 0.975)(\log 0.025)} \sim n.$$

Step 3: combine the results. we get

$$L_n^{1} \sim \frac{1}{n} X_{(n)} \qquad L_n^2 \sim n X_{(1)}.$$

Therefore it suffices to take $k_n^1 = n\cos n$, and $k_n^2 = 1$.

Also, since $X_{(n)} \xrightarrow{P} \theta$, then $L_n^1 \sim \frac{1}{n} X_{(n)} \xrightarrow{P} 0$, which is just $L_n^1 = o_p(1)$. Meanwhile $L_n^2 \sim n X_{(1)} \xrightarrow{d} \mathrm{Exp}(\frac{1}{\theta})$, which is clearly not $o_p(1)$.