# Math 281 A Final

December 10, 2017

**Exercise 1.** (a) Broken stick theorem tells us (or by brutal calculation), the distance of $X_{(1)}$ and the endpoint $\theta - 1$, is $1/n + 1$. Therefore $\hat{\theta} = X_{(1)} + n/(n+1)$ is an unbiased estimator.

(b) A better estimator could be $\tilde{\theta} = (X_{(1)} + X_{(n)})/2 + 1/2$. The unbiasedness comes from symmetry, and the symmetry also tells us $var(X_{(1)}) = var(X_{(n)})$. Therefore

$$var(\tilde{\theta}) = \frac{1}{4}[var(X_{(1)}) + var(X_{(n)})] + \frac{1}{2}cov(X_{(1)}, X_{(n)}) \leq var(X_{(1)}) = var(\hat{\theta}).$$

Notice that the equivalence only holds when $X_{(1)}$ and $X_{(n)}$ are almost surely linearly dependent, which obviously is not the case. So it is strictly better, thus $\hat{\theta}$ is not UMVU.

(c) (i) The result comes from Lehmann-Scheffe Theorem. (ii) It is not sufficient. Given $X_{(1)}$, the distribution of $X$ is still depending on $\theta$. The exact form of distribution of $X|X_{(1)}$ is with probability $1/n$ on $X_{(1)}$, and with probability $(n-1)/n$, uniformly on $[X_{(1)}, \theta]$.

If one find it difficult showing this, it can be shown by given some certain numbers. For example if given $X_{(1)} = 1$, the density of conditional distribution of $X$ at 1.5 is positive when $\theta = 1.7$, and 0 when $\theta = 1.4$. So it is still depend on $\theta$. This violates the definition of sufficiency.

(d) Yes. No matter how one permutes the data, the value of the smallest one among the data stay fixed.

**Exercise 2.** (a) Using Method I, suppose $f(X)$ is unbiased for $e^{-3\lambda}$, and we have

$$\sum_t f(t)\frac{\lambda^t}{t!}e^{-\lambda} = e^{-3\lambda}$$

which gives

$$\sum_t f(t)\frac{\lambda^t}{t!} = e^{-2\lambda} = \sum_t \frac{(-2\lambda)^t}{t!}.$$

Comparing the coefficients gives $f(t) = (-2)^t$.

(b)It is weird since it takes negative values, while the parameter to be estimated is a positive value. The plug in ensures that the estimator is positive.

(c)The variance is finite if and only if the second moment is finite, which is $E(f(X)^2) = E(4^X) = E(e^{X\log 4}) = M(\log 4)$. Since the moment generating function for Poisson distribution is finite on the positive real line, the variance of this estimator is finite.

**Exercise 3.** One non-random estimator which converges to $\mu$ in probability and have infinite variance for all $n$, is $\bar{X} + 1/X_{(n)}$.

It is easy to see that the estimator has infinite variance, since $X_{(n)}$ has positive density at 0. By the same argument from a homework, $E(1/X_{(n)}^2) = \infty$.

Now we show that this estimator is consistent. Notice that $\bar{X} \to \mu$ in probability by Law of Large Numbers. And for any positive number $M$, we will have $P(X_{(n)} \leq M) \to 0$ as $n \to \infty$. This is because

$P(X_{(n)} \leq M) = F(M)^n$ where $F$ is the c.d.f. of the Laplace distribution. Notice that $F(M) < 1$ for all $M$, so the right hand side converges to 0 when $n \to \infty$.

Now for any $\varepsilon > 0$,

$$P(|\bar{X} + 1/X_{(n)} - \mu| \geq 2\varepsilon) \leq P(|\bar{X} - \mu| \geq \varepsilon) + P(|1/X_{(n)}| \geq \varepsilon) = P(|\bar{X} - \mu| \geq \varepsilon) + P(|X_{(n)}| \leq \varepsilon^{-1})$$

The right hand side converges to 0 when $n \to \infty$ by the previous argument, so it is consistent.