Math 281A HW2

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Problem 1. X and Y have finite variance, then $cov(X,Y)^2 \leq var(X)var(Y)$. The equality only holds if and only if $X = aY + b$ happens with probability 1.

Proof. Cauchy-Schwartz inequality tells us

$$
\begin{aligned}
cov(X,Y)^2 &= \int (X - E(X))(Y - E(Y))dP \\
&\leq \int [(X - E(X))^2 dP][(Y - E(Y))^2 dP] = var(X)var(Y)
\end{aligned}
$$

If $X = aY + b$ holds with probability one, without loss of generality we set $var(Y) = 1$. It is easy to see that $var(X) = a^2$ and $cov(X,Y) = a$, which satisfies the previous equality.

Now assume the previous equality is true. Also we assume $var(Y) = 1$ and $var(X) = a^2$. From the equality we see that $cov(X,Y) = a$. Now consider random variable $X - aY$. We see that

$$var(X - aY) = var(X) + a^2 var(Y) - 2a * cov(X,Y) = 0.$$

This indicates that there exists a constant b such that $P(X - aY = b) = 1$, which finishes the proof.

□

Problem 2. Another proof for Cauchy-Schwartz inequality is that

$$\int (f + \lambda g)dP = 0$$

has at most one root regarding to λ.

Proof. Expand the left hand side we have

$$\int f^2 dP + 2\lambda \int fg dP + \lambda^2 \int g^2 dP.$$

This is a quadratic regarding to λ. In order to have only one root, we have

$$(2 \int fg dP)^2 - 4 \int g^2 dP \int f^2 dP \leq 0$$

as desired.

□

Problem 3. If δ and δ' have finite variance, so does $\delta - \delta'$.

Proof. It is trivial by the inequality $var(\delta - \delta') \le 4\max(var(\delta), var(\delta'))$ $\square$

Problem 4. Show that UMVU estimator is unique (with probability one).

Proof. Assume δ and γ are two UMVUEs. Consider $(\delta+\gamma)/2$. It is trivial to show that this is also an unbiased estimator. Therefore $var((\delta+\gamma)/2) \ge var(\delta) = var(\gamma)$, which implies $cov(\delta, \gamma) \ge var(\delta) = var(\gamma)$. By the result in (1), we know that $\delta = a\gamma + b$ with probability one. Since they have same expectation (unbiased) and variance (both UMVU), we know that $a = 1, b = 0$. Therefore the uniqueness is proved. $\square$

Problem 5. Assume X follows $bin(n, p)$. Use method I to find the UMVUE for p^3.

Solution . Let the estimator be $\delta(X)$. We have

$$E_p(\delta) = \sum_{x=0}^{n} \delta(x) \binom{n}{x} p^x (1-p)^{n-x} = p^3.$$

Denote $\rho = p/(1-p)$, we have

$$\sum_{x=0}^{n} \delta(x) \binom{n}{x} \rho^x = \rho^3 (1+\rho)^{n-3}$$

Note that for any t,

$$(1+\rho)^t = \sum_{x=0}^{t} \binom{t}{x} \rho^x,$$

then we have

$$\sum_{x=0}^{n} \delta(x) \binom{n}{x} \rho^x = \sum_{x=0}^{n-3} \binom{n-3}{x} \rho^{x+3} = \sum_{x=3}^{n} \binom{n-3}{x-3} \rho^x.$$

Compare the coefficients, we have $\delta(x) = 0$ when $x < 3$, and $\delta(x) = \binom{n-3}{x-3} / \binom{n}{x}$ for $3 \le x \le n$.

Problem 5. $\{X_i\}$ are IID $N(\xi, \sigma^2)$, with σ known. Find the UMVUEs for ξ^2, ξ^3, ξ^4.

Solution . Since $\bar{X}$ is the sufficient and complete statistic for ξ, unbiased estimators based on $\bar{X}$ are UMVUEs.

(i) From $E(\bar{X})^2 = \xi^2 + \sigma^2/n$ we know $\bar{X}^2 - \sigma^2/n$ is the UMVUE for ξ^2.

(ii) Calculation shows that $E(\bar{X})^3 = \xi^3 + 3\sigma^2\xi/n$, therefore the UMVUE for ξ^3 is $\bar{X}^3 - 3\bar{X}\sigma^2/n$.

(iii) Similarly consider $E(\bar{X})^4 = 6\sigma^2\xi^2/n + 3\sigma^4/n^2 + \xi^4$, and we have the UMVUE as $\bar{X}^4 - 3\sigma^4/n^2 - 6\sigma^2(\bar{X}^2 - \sigma^2/n)/n$.

Problem 6. Show that $S_{XY}/(n-1)$ is an unbiased estimator for $cov(X, Y)$, where

$$S_{XY} = \sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y}).$$

Proof.

$$
\begin{aligned}
S_{XY} &= \sum_i X_i Y_i - n\bar{X}\bar{Y} = \sum_i X_i Y_i - \frac{1}{n}(\sum_i X_i Y_i + \sum_{i \neq j} X_i Y_j) \\
&= \frac{n-1}{n} \sum_i X_i Y_i + \frac{1}{n}(\sum_{i \neq j} X_i Y_j)
\end{aligned}
$$

Taking expectations, noting that the latter part has $n(n-1)$ copies, we have

$$
E(S_{XY}) = (n-1)E(X_i Y_i) - (n-1)E(X_i)E(Y_i) = (n-1)cov(X, Y).
$$

□

Problem 7. If $\{X_i\}$ are IID $Pois(\lambda)$, what is the rationale behind estimator

$$
\log(n/(\sum_i \mathbb{1}(X_i = 0) + 1))
$$

Solution . Note that $P(X = 0) = e^{-\lambda}$, so $\lambda = \log(1/P(X = 0))$. Using the fact that $P(X = 0) = E(\mathbb{1}(X = 0))$, and use the sample mean to approximate the expectation, naturally we have an estimator as

$$
\log(n/(\sum_i \mathbb{1}(X_i = 0)))
$$

Note that the denominator can be zero with positive probability, which makes the MSE to be infinity. Therefore we add 1 to the denominator, making the estimator as

$$
\log(n/(\sum_i \mathbb{1}(X_i = 0) + 1)).
$$