Math281B HW5.

1. $X \sim Bin(n, p)$, and $\hat{p} = \frac{\sqrt{n}}{1+\sqrt{n}}\left(\frac{X}{n}\right) + \frac{1}{1+\sqrt{n}} \cdot \frac{1}{2}$, the minimax for p.

prove that $\sqrt{n}(\hat{p} - p) \xrightarrow{d} N\left(\frac{1}{2} - p, p(1-p)\right)$

$$\sqrt{n}(\hat{p} - p) = \sqrt{n}\left(\frac{\sqrt{n}}{1+\sqrt{n}}\left(\frac{X}{n}\right) + \frac{1}{1+\sqrt{n}} \cdot \frac{1}{2} - p\right)$$

$$= \frac{\sqrt{n}}{1+\sqrt{n}} \cdot \underbrace{\sqrt{n}\left(\frac{X}{n} - p\right)}_{(\mathrm{I})} + \frac{\sqrt{n}}{1+\sqrt{n}} \underbrace{\left(\frac{1}{2} - p\right)}_{(\mathrm{II})}$$

Since $\sqrt{n}\left(\frac{X}{n} - p\right) \xrightarrow{d} N(0, p(1-p))$ by CLT,

$\frac{\sqrt{n}}{1+\sqrt{n}} \to 1$. By Slutsky's theorem,

$(\mathrm{I}) \xrightarrow{d} N(0, p(1-p))$, $(\mathrm{II}) \to \left(\frac{1}{2} - p\right)$.

Therefore

$$\sqrt{n}(\hat{p} - p) \xrightarrow{d} N\left(\frac{1}{2} - p, p(1-p)\right).$$

2. $R_{\frac{x}{n}}(p) = \frac{1}{n} p(1-p)$ $R_{\hat{p}}(p) = \frac{1}{4(1+\sqrt{n})^2}$

Determine $I_n = \{p : R_{\hat{p}}(p) \le R_{\frac{x}{n}}(p)\}$ and its behavior when $n \to \infty$.

If $p \ne \frac{1}{2}$, $p(1-p) < \frac{1}{4}$, then

$$\frac{R_{\frac{x}{n}}(p)}{R_{\hat{p}}(p)} = \frac{p(1-p)}{1/4} \cdot \frac{(1+\sqrt{n})^2}{n} \xrightarrow{n\to\infty} 4p(1-p) < 1.$$

Therefore, eventually $R_{\frac{x}{n}}(p) < R_{\hat{p}}(p)$.

If $p = \frac{1}{2}$, $R_{\frac{x}{n}}(p) = R_{\frac{x}{n}}(\frac{1}{2}) = \frac{1}{4n} > R_{\hat{p}}(\frac{1}{2}) = \frac{1}{4(1+\sqrt{n})^2}$

To sum up: $I_n \to \{\frac{1}{2}\}$.

3. $X \sim Bin(n,p)$, $L(p,d) = \frac{(d-p)^2}{p(1-p)}$. show that :

1. $\frac{X}{n}$ has constant risk

2. $\frac{X}{n}$ is the Bayes solution with respect to uniform prior.

1. $R_{\frac{X}{n}}(p) = E\left(L(p, \frac{X}{n})\right) = E\left(\frac{(\frac{X}{n}-p)^2}{p(1-p)}\right) = \frac{1}{n}.$

2. we have $\pi(p) = 1$ and $f(x|p) = \binom{n}{x} p^x (1-p)^{n-x}$

Therefore $\pi(p|x) \propto p^x (1-p)^{n-x}$

which indicates that the posterior is Beta $(x+1, n-x+1)$

To minimize the risk, which is equivalent to

$$\min_d \int_0^1 B(x+1, n-x+1) \frac{(d-p)^2}{p(1-p)} p^x (1-p)^{n-x} \, dp$$

$$= \min_d \int_0^1 B(x+1, n-x+1) (d-p)^2 p^{x-1} (1-p)^{n-x-1} \, dp. \quad (*)$$

(i) $X > 0$, $X < n$, : $(*)$ is equivalent to minimize squared loss under Beta $(X, n-X)$, which is the mean of such distribution : $\frac{X}{n}$.

(ii) $X=0$: It's easy to see that any d such that $d \neq 0$ on a positive ~~measure~~ measure set will result in $(*) = +\infty$.
Therefore $d = 0 = \frac{X}{n}$.

(iii) $X=n$: Similar to (ii), $d = 1 = \frac{X}{n}$.

To sum up: $\frac{X}{n}$ is the ~~posterior~~ Bayes estimator.

4. Suppose δ is unbiased for $g(\theta)$. Then there is $c \in (0,1)$ such that $c\delta$ dominates δ.

By the bias-variance relationship:

$$R_{c\delta}(\theta) = var(c\delta) + Bias^2(c\delta)$$

$$= c^2\, var(\delta) + (1-c)^2 g^2(\theta)$$

$$= c^2(R_\delta(\theta) + g^2(\theta)) - 2g^2(\theta)c + g^2(\theta)$$

This is minimized by $c = \dfrac{g^2(\theta)}{R_\delta(\theta) + g^2(\theta)}$

Since $c=1$ is a root for equation $R_{c\delta}(\theta) = R_\delta(\theta)$ for all θ, then for given θ, any number in $\left(\dfrac{g^2(\theta)}{R_\delta(\theta) + g^2(\theta)}, 1\right)$

will make $R_{c\delta}(\theta) < R_\delta(\theta)$.

To make such a choice independent of θ, just take

$$c = \bigcap_{\theta \in \Theta} \left(\frac{g^2(\theta)}{R_\delta(\theta) + g^2(\theta)}, 1\right)$$

$$= \left(\sup_\theta \frac{g^2(\theta)}{R_\delta(\theta) + g^2(\theta)}, 1\right).$$

5. $X \sim poi(\lambda)$. for which (a, b) such that $aX+b$ admissible?

First: $R_\delta(X) = var(\delta) + bias^2(\delta) = a^2\lambda + ((a-1)\lambda + b)^2$

We consider a prior $Gamma(\alpha, \beta)$ on λ (shape-rate parameterization)

it's easy to see that:

$$\pi(\lambda|X) \propto \pi(\lambda) \cdot f(x|\lambda) \propto \lambda^{\alpha+X-1} e^{-(\beta+1)\lambda}$$

So the posterior is $Gamma(\alpha+X, \beta+1)$, with Bayes estimator

$$\frac{\alpha+X}{\beta+1} = \frac{1}{\beta+1}X + \frac{\alpha}{\beta+1}.$$ This is admissible, and by taking different (α, β), we see that $a<1$, $b>0$ makes $aX+b$ admissible.

Now we rule out other options. First, $(a, b) > 0$, otherwize it's dominated by $\delta^+ = \delta \mathbb{1}(\delta > 0)$ since $\lambda > 0$.

when $a \geq 1$. $R_\delta(\lambda) \geq a^2\lambda \geq \lambda = R_X(\lambda)$, so it's dominated by X.

The case $\delta = X$ is not admissible by question (4), since X is unbiased to λ.