Math 281 A Midterm

November 14, 2017

Exercise I. (a) For a single sample, the density function is $p_\lambda(x) = \frac{\lambda^x}{x!}e^{-\lambda}$. Take a double derivative to its log and take negative expectations gives that $i(\lambda) = 1/\lambda$. Therefore the information of the sample is $ni(\lambda) = n/\lambda$.

(b) The CR bound is just the reverse of the Fisher information, which is λ/n.

(c) Since $var(X_1) = \lambda$ for Poisson variables, $var(\bar{X}) = \lambda/n$, which hits the CR bound, and this shows that it is UMVU.

(d) Use the canonical form of the exponential family, we get $T(x) = \sum_i X_i$ and $\eta = \log(\lambda)$. Sufficiency is done by the factorization theorem, and completeness is proven by the fact that $\log(\lambda)$ contains an open interval in $\mathbb{R}$ when $\lambda > 0$.

(e) Y follows the zero-deflated Poisson distribution. To get its probability function, just divide the probability that the X_i is non-zero, to make the summation of the probabilities to 1.

$$P(Y = y) = \frac{\lambda^y}{y!(1 - e^{-\lambda})}e^{-\lambda}$$

Take a log, and take twice partial derivative to λ, we have

$$\frac{\partial^2 P}{\partial \lambda^2} = -\frac{y}{\lambda^2} + \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2}$$

The expectation of Y can be calculated using the fact that $E(Y)P(X > 0) = E(X)$, which yields $E(Y) = \lambda/(1 - e^{-\lambda})$. Plug in and we have

$$i_Y(\lambda) = \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{\lambda(1 - e^{-\lambda})^2}$$

(f) Rao-Blackwell theorem shows that $E(\delta|T)$ has a variance no larger than δ if T is sufficient, and δ is unbiased. The Lehmann-Scheffe says that any unbiased estimator based on the complete and sufficient statistic is UMVU. Rao-Blackwell does not address the UMVU property as Lehmann-Scheffe does, but when T is also complete in Rao-Blackwell, the conditional expectation is indeed UMVUE.

(g) $(X_1 - X_2)^2$ is an unbiased estimator for 2λ, and $\sum_i X_i$ is complete and sufficient by (d). Therefore $\delta(X)$ is UMVU for 2λ. Notice that $2\bar{X}$ is also an UMVUE for 2λ by (c), therefore by uniqueness of UMVUE, $\delta(X) = 2\bar{X}$.

(h) $\delta(X)$ is a function of $\bar{X}$ by construction, thus is free of λ and it is a statistic.

(i) As is shown in (c), $var(\delta(X)) = var(2\bar{X}) = 4var(\bar{X}) = 4\lambda/n$.

(j) No, the conditional tuples are not complete since $E(\bar{X} - \sum_i (X_i - \bar{X})^2/(n-1)) = 0$ which ruins the completeness property.

(k) Yes it is a statistic due to the fact that it is a function of the tuple $(\sum_i X_i, \sum_i (X_i - \bar{X})^2)$, and these two are statistics.

(l) Due to the tower property:

$$E(\varepsilon(X)) = E(E((X_1 - X_2)^2| \sum_i X_i, \sum_i (X_i - \bar{X})^2)) = E((X_1 - X_2)^2) = 4\lambda = E(\delta(X)).$$

(m) First part, by the decomposition of the variance:

$$
\begin{aligned}
var((X_1 - X_2)^2) &= var(E((X_1 - X_2)^2 | \sum_i X_i, \sum_i (X_i - \bar{X})^2)) + E(var((X_1 - X_2)^2 | \sum_i X_i, \sum_i (X_i - \bar{X})^2)) \\
&\geq var(E((X_1 - X_2)^2 | \sum_i X_i, \sum_i (X_i - \bar{X})^2)) = var(\varepsilon(X))
\end{aligned}
$$

Second part is automatically true since $\delta(X)$ is UMVU. And both of them are unbiased.

Exercise 2. (i) It is easy to see that $f_\theta(x) = f_0(x - \theta)$ which is the definition of location family.

(ii) By the expansion of the canonical form, $(\sum_i X_i, \sum_i X_i^2, \sum_i X_i^3)$ is a sufficient statistic.

(iii) For location family the variance is fixed. Therefore denote C as the variance of X_1 when $\theta = 0$, and construct

$$
g(\sum_i X_i, \sum_i X_i^2, \sum_i X_i^3) = S^2 - C
$$

where S^2 is the sample variance. Clearly $E(g) = 0$ for all θ and $g \neq 0$.