Math 281B HW3.

(1). $X_1 \dots X_n$ iid with median $\theta$ and density $f$ s.t.

$$\lim_{x \to \theta^-} f(x) = \alpha_1 \neq \lim_{x \to \theta^+} f(x) = \alpha_2 .$$

Find the asymptotic distribution of sample median.

Without loss of generality, we set $\theta = 0$. We make up two density $g, h$, which are symmetric of 0. and

$$\begin{cases} g(x) = f(x), \quad x < 0, & g(0) = \lim_{x \to 0^-} f(x) = \alpha_1 \\ h(x) = f(x), \quad x > 0, & h(0) = \lim_{x \to 0^+} f(x) = \alpha_2 \end{cases}$$

Now imagine $Y_1 \dots Y_n \overset{iid}{\sim} g$, $Z_1 \dots Z_n \overset{iid}{\sim} h$, it's easy ~~for~~ to see that

$$\dot{Y} \mid \dot{Y} \leq 0 \overset{d}{=} \dot{X} \mid \dot{X} \leq 0$$

and

$$\dot{Z} \mid \dot{Z} \geq 0 \overset{d}{=} \dot{X} \mid \dot{X} \geq 0 .$$

$$\sqrt{n}\dot{Y} \xrightarrow{d} N(0, \tfrac{1}{4\alpha_1^2}) \qquad \sqrt{n}\dot{Z} \xrightarrow{d} N(0, \tfrac{1}{4\alpha_2})$$

Therefore

$$\sqrt{n}\dot{X} \xrightarrow{d} X_1^- + X_2^+ \quad \text{where} \quad \begin{matrix} X_1^- \sim N(0, \frac{1}{4\alpha_1^2})^- \\ X_2^+ \sim N(0, \frac{1}{4\alpha_2^2})^+ \end{matrix}$$

or. the negative side of ~~X~~ $\sqrt{n}\dot{X}$ asymtotically follows $N(0, \frac{1}{4\alpha_1^2})$ and the positive side is $N(0, \frac{1}{4\alpha_2^2})$

2. Assume the density is $f(x) = |x| \mathbb{1}(|x| \le 1)$, show that $\sqrt{n}\dot{x}$ is not bounded in probability

Denote $Y_i = X_i^2 \operatorname{sgn}(X_i)$. It's easy to show that

$Y_i \overset{iid}{\sim} \text{Unif}(-1, 1)$.

$$( P(Y_i \le y) = P(X_i^2 \le y) = P(X_i \le \sqrt{y})$$

$$= \frac{1}{2} + \frac{1}{2}(\sqrt{y})^2 = \frac{1}{2} + y \text{ when } y \ge 0. \text{ Similar to } y < 0)$$

Then $\sqrt{n}\dot{Y} \xrightarrow{d} N(0, \frac{1}{4f_y^2(0)}) = N(0, 1)$

Note that the order of data is preserved from $X_i$ to $Y_i$,

So $\sqrt{n}(\dot{X}^2 \operatorname{sgn}(\dot{X})) \xrightarrow{d} N(0, 1)$.

which implies $\sqrt{n}\dot{X}^2$ is $\Omega(1)$. or $n^{\frac{1}{4}}\dot{X}$ is $\Omega(1)$.

Then $\sqrt{n}\dot{x} = n^{\frac{1}{4}}(n^{\frac{1}{4}}\dot{x})$ is clearly not ~~bun~~ bounded in prob.

3. Find the covariance & correlation of the asymptotic distribution of the first & third quantiles.

We have

$$\sqrt{n}\left[\begin{pmatrix}\hat{\theta}_1\\ \hat{\theta}_3\end{pmatrix}-\begin{pmatrix}\theta_1\\ \theta_3\end{pmatrix}\right] \xrightarrow{d} N\left[\begin{pmatrix}0\\0\end{pmatrix},\ \begin{bmatrix}\dfrac{F(\theta_1)(1-F(\theta_1))}{f(\theta_1)^2} & \dfrac{F(\theta_1)(1-F(\theta_3))}{f(\theta_1)f(\theta_3)}\\[2ex] \dfrac{F(\theta_1)(1-F(\theta_3))}{f(\theta_1)f(\theta_3)} & \dfrac{F(\theta_3)(1-F(\theta_3))}{f(\theta_3)^2}\end{bmatrix}\right]$$

with the fact that $F(\theta_1) = 1-F(\theta_3) = \frac{1}{4}$

we have the covariance as $\dfrac{1}{16 f(\theta_1) f(\theta_3)}$

and correlation as $\frac{1}{3}$.

4. $X_1 \cdots X_n$ iid Cauchy$(\theta)$, Show $ARE(\dot{X}, \hat{\theta}_{MLE}) = \frac{8}{\pi^2}$.

It's easy that (WLOG, $\theta = 0$).

$$\sqrt{n}(\dot{X}) \xrightarrow{d} N(0, \frac{1}{4f^2(0)}) = N(0, \frac{\pi^2}{4})$$

The ~~R~~ Fisher information of a single observation is

$$I(\theta) = -E_\theta\left[\frac{\partial^2 \log f_\theta(x)}{\partial \theta^2}\right] = E_\theta\left[\frac{2(1-(\theta-x)^2)}{(1+(\theta-x)^2)^2}\right]$$

$$= E_0\left(\frac{2(1-x^2)}{(1+x^2)^2}\right) = \int \frac{1}{\pi}\frac{2(1-x^2)}{(1+x^2)^2} \cdot \frac{1}{1+x^2}\,dx = \frac{1}{2}$$

Since $\sqrt{n}(\hat{\theta}_{MLE}) \xrightarrow{d} N(0, \frac{1}{I(\theta)}) = N(0, 2)$

So $ARE(\dot{x}, \theta_{MLE}) = 2/\frac{\pi^2}{4} = 8/\pi^2$.