Math 281A HW3

October 31, 2017

Problem 1. Let X_i be IID $N(0, \sigma^2)$. Find $i(\sigma^2)$, and show that the usual unbiased estimator achieves it.

Solution . Denote $\theta = \sigma^2$. The density for X_i is

$$f_\theta(x) = \frac{1}{\sqrt{2\pi\theta}} \exp\{-\frac{x^2}{2\theta}\}$$

and the second order partial derivative of log density against θ is

$$\frac{\partial^2 \log f}{\partial \theta^2} = \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}.$$

Now it is easy to see that

$$i(\theta) = -E(\frac{\partial^2 \log f}{\partial \theta^2}) = -\frac{1}{2\theta^2} + \frac{\theta}{\theta^3} = \frac{1}{2\theta^2} = \frac{1}{2\sigma^4}.$$

The usual estimator here is $\frac{1}{n}\sigma_i X_i^2$ (Note that it is not sample variance since we already know that the mean is zero. Sample variance applies to the situation when the mean is unknown.). The variance is

$$var(\frac{1}{n}\sigma_i X_i^2) = \frac{1}{n} var(X_i^2) = \frac{\sigma^4}{n} var(Z^2) = \frac{2\sigma^4}{n} = \frac{1}{ni(\sigma^2)},$$

where Z is standard normal, and the last equation shows that the estimator reaches the CR bound. Note that a fact that the square of the standard normal is Chi-square(1) is used here, and Chi-square(1) has variance 2.

Problem 2. If X is normal with mean zero and standard deviation σ, determine $i(\sigma)$.

Solution . Directly use the previous problem's solution, we have

$$i(\sigma) = i(\sqrt{\theta}) = i(\theta)(1/2\sqrt{\theta})^{-2} = \frac{1}{2\theta^2} * 4\theta = \frac{2}{\theta} = \frac{2}{\sigma^2}.$$

The formula used here is from the fact that CR bound of $g(\theta)$ is $[g'(\theta)]^2/I(\theta)$, then naturally the corresponding Fisher information for $g(\theta)$ as a whole, is $I(\theta)[g'(\theta)]^{-2}$.

Problem 3. Show that if X is from a scale family then $\log X$ is from a location family.

Proof. Rewrite $X = \theta Z$, then $logX = \log\theta + \log Z$. Denote F as the CDF of $\log Z$, we have

$$F_\theta(x) = P(\log X \le x) = P(\log Z \le x - \log\theta) = F(x - \log\theta)$$

Reparametrize $\log\theta = \phi$ and the result is clear. □

Problem 4. If X is not identically zero and Y is independent of X, and Y has positive density at 0. Show that $E(X/Y) = \infty$.

Proof. Clearly, the problem is solved once we show that $E(1/|Y|) = \infty$. This is done by realizing that there exists some $\varepsilon > 0$ such that $f(y) > f(0)/2$ in $(0, \varepsilon)$ by continuity. Now

$$E(1/|Y|) \ge \int_0^\infty \frac{1}{y} f(y)dy \ge \int_0^\varepsilon \frac{1}{y} f(y)dy \ge 0.5 \int_0^\varepsilon \frac{1}{y} f(0)dy = \infty.$$

□

Problem 5. If X is distributed as $Pois(\lambda)$, show that the Fisher information about $\sqrt{\lambda}$ is independent of λ.

Proof. Use the fact the Fisher information about λ is $1/\lambda$, and the formula of Problem 2, we have

$$i(\sqrt{\lambda}) = i(\lambda)(1/2\sqrt{\lambda})^{-2} = 4.$$

□

Problem 6. Y is uniformly distributed as $U[\theta - 0.5, \theta + 0.5]$, and $X = Y + \varepsilon$ where ε is $N(0, \sigma^2)$. Find out the Fisher information of θ contained in X when $\sigma \to 0$.

Solution . The density function of X can be found by convolution as follows

$$\begin{aligned} f(x) &= \int f_Y(t) f_\varepsilon(x-t)dt \\ &= \int_{\theta-0.5}^{\theta+0.5} \frac{1}{\sigma}\phi(\frac{x-t}{\sigma})dt \\ &= \Phi(\frac{x-\theta+0.5}{\sigma}) - \Phi(\frac{x-\theta-0.5}{\sigma}). \end{aligned}$$

By the formula of Fisher information, namely as $E[(f'(X))^2/f(X)]$, we have the Fisher information as

$$\frac{1}{\sigma^2}\int \frac{(\phi(\frac{x-\theta+0.5}{\sigma}) - \phi(\frac{x-\theta-0.5}{\sigma}))^2}{\Phi(\frac{x-\theta+0.5}{\sigma}) - \Phi(\frac{x-\theta-0.5}{\sigma})}dx = \frac{1}{\sigma^2}\int \frac{(\phi(\frac{x+0.5}{\sigma}) - \phi(\frac{x-0.5}{\sigma}))^2}{\Phi(\frac{x+0.5}{\sigma}) - \Phi(\frac{x-0.5}{\sigma})}dx$$

Change variable $s = x/\sigma$, and we have the above equal to

$$\frac{1}{\sigma}\int \frac{(\phi(s+\frac{1}{2\sigma}) - \phi(s-\frac{1}{2\sigma}))^2}{\Phi(s+\frac{1}{2\sigma}) - \Phi(s-\frac{1}{2\sigma})}ds$$

Take an interval $(1/2\sigma, \infty)$, we see that on this interval the denominator is bounded from above by $0.5-\delta$, with δ small, and the nominator is monotone increasing with σ approaching zero. Therefore the above is bounded from below by

$$\frac{1}{\sigma}\int_{1/2}^{\infty}\frac{(\phi(s+\frac{1}{2})-\phi(s-\frac{1}{2}))^2}{1/2-\delta}ds \to \infty$$