Math 281B HW1 Solutions

1. Show that if $X \sim \chi^2_{n,\lambda}$, then $EX = n+\lambda$

$\text{var}\, X = 2n + 4\lambda$.

Denote $Y_i \sim N(\delta_i, 1)$, by definition, $X \overset{d}{=} \sum_{i=1}^n Y_i^2$

Therefore $EX = E(\sum_{i=1}^n Y_i^2) = \sum_{i=1}^n EY_i^2 = \sum_{i=1}^n (\text{var}(Y_i) + (EY_i)^2)$

$= \sum_{i=1}^n (1+\delta_i^2) = n + \lambda$

Similarly

$\text{var}(X) = \text{var}(\sum_{i=1}^n Y_i^2) = \sum_{i=1}^n \text{Var}(Y_i^2)$ (independence)

$= \sum_{i=1}^n (E(Y_i^4) - (E(Y_i^2))^2)$

$= \sum_{i=1}^n (\delta_i^4 - 6\delta_i^2 + 3 - (1+\delta_i^2)^2) = \sum_{i=1}^n (2 + 4\delta_i^2)$

$= 2n + 4\lambda$

2. Prove the result in 1) with the fact that if $K \sim Poi(\frac{\lambda}{2})$ and $X|K \sim \chi^2_{n+2k}$ then $X \sim \chi^2_{n,\lambda}$.

Using the conditional properties:

$$EX = E(E(X|K)) = E(n+2k) = n + 2\cdot(\tfrac{\lambda}{2}) = n+\lambda$$

$$var(X) = E(var(X|K)) + var(E(X|K))$$

$$= E(2n+4k) + var(n+2k)$$

$$= 2n+2\lambda + var(2k) = 2n+4\lambda \ .$$

3. $X \sim Bin(n, p)$. $g(p) = p(1-p)$. Compare the limiting behaviours of:

MLE: $(\bar{X})(1-\bar{X})$ UMVUE: $\frac{n}{n-1}\bar{X}(1-\bar{X})$.

when $p = \frac{1}{2}$ and $p \neq \frac{1}{2}$.

i) $p \neq \frac{1}{2}$: Based on CLT: $\sqrt{n}(\bar{X} - p) \xrightarrow{d} N(0, p(1-p))$

and $g'(p) = 1 - 2p$, by delta method,

we have for MLE:

$$\sqrt{n}(\bar{X}(1-\bar{X}) - g(p)) \xrightarrow{d} N(0, p(1-p)(1-2p)^2)$$

Similarly, for UMVUE:

$$\sqrt{n}\left(\frac{n}{n-1}\bar{X}(1-\bar{X}) - g(p)\right) = \frac{\sqrt{n} \cdot n}{n-1}(\bar{X}(1-\bar{X}) - g(p))$$

$$+ \frac{\sqrt{n}}{n-1} g(p)$$

The first part converges weakly to $N(0, p(1-p)(1-2p)^2)$ by Slutsky's theorem, and the second part converges to 0.

So $\sqrt{n}\left(\frac{n}{n-1}\bar{X}(1-\bar{X}) - g(p)\right) \xrightarrow{d} N(0, p(1-p)(1-2p)^2)$

ii) $p = \frac{1}{2}$, By higher order delta method:

$$n(\bar{X}(1-\bar{X}) - g(p)) \xrightarrow{d} -\frac{1}{4}\chi_1^2.$$

By the same decomposition before,

$$n\left(\frac{n}{n-1}\bar{X}(1-\bar{X}) - g(p)\right) = \frac{n^2}{n-1}((\bar{X})(1-\bar{X}) - g(p)) + \frac{n}{n-1} g(p)$$

the first part converges to $-\frac{1}{4}\chi_1^2$ while the second part converges to $\frac{1}{4}$. Therefore $n(\bar{X}(1-\bar{X}) - g(p)) \xrightarrow{d} \frac{1}{4} - \frac{1}{4}\chi_1^2$

4. $X_1 \cdots X_n \overset{iid}{\sim} N(\theta, 1)$. Compare the limiting behaviours of

MLE: $\Phi(a-\bar{X})$ UMVUE: $\Phi((a-X)\sqrt{\frac{n}{n-1}})$

of $g(\theta) = \Phi(a-\theta)$

First realize that $\sqrt{n}(\bar{X}-\theta) \sim N(0,1)$, and $g(\theta)$ is differentiable with $\theta$, and equipped with non-zero derivative.

Therefore with $g'(\theta) = -\phi(a-\theta)$:

$$\sqrt{n}(MLE - g(\theta)) \xrightarrow{d} N(0, \phi^2(a-\theta))$$

Also. extended delta method gives

$$\sqrt{n}(UMVUE - g(\theta) \xrightarrow{d} N(0, \phi^2(a-\theta)).$$

5. $X_1 \cdots X_n \overset{iid}{\sim} Poi(\lambda)$. $g(\lambda) = \lambda e^{-\lambda}$ find the limiting behaviours of

MLE: $g(\bar{x})$ UMVUE: $\bar{x}(1-\frac{1}{n})^{n\bar{x}-1}$.

CLT tells $\sqrt{n}(\bar{X}-\lambda) \xrightarrow{d} N(0, \lambda)$

when $\lambda \neq 1$ $(g'(\lambda) = e^{-\lambda}(1-\lambda) \neq 0)$:

$$\sqrt{n}(g(\bar{x}) - g(\lambda)) \xrightarrow{d} N(0, \lambda(1-\lambda)^2 e^{-2\lambda}).$$

when $\lambda = 1$, second order delta method gives

$$n(g(\bar{x}) - g(\lambda)) \xrightarrow{d} -\frac{1}{2e}\chi_1^2$$

For UMVUE, by $(1-\frac{1}{n})^n = e^{-1}(1-\frac{1}{2n} + o(\frac{1}{n^2}))$.

we have $\bar{X}(1-\frac{1}{n})^{n\bar{X}-1} = [\frac{1}{1-\frac{1}{n}}]\bar{X}e^{-\bar{X}}(1-\frac{1}{2n} + o(\frac{1}{n^2}))^{\bar{X}}$

$$= \bar{X}e^{-\bar{X}}(1-\frac{\bar{X}}{2n} + O_p(\frac{1}{n^2}))(1+\frac{1}{n} + o(\frac{1}{n^2}))$$

$$= g(\bar{X})(1+\frac{1}{n}(1-\frac{\bar{X}}{2}) + O_p(\frac{1}{n^2}))$$

Therefore $\sqrt{n}(UMVUE - g(\bar{x})) = \frac{1}{\sqrt{n}}g(\bar{X})(1-\frac{\bar{X}}{2}) = o_p(1)$.

Slutsky theorem gives $\sqrt{n}(UMVUE - g(\lambda)) \xrightarrow{d} N(0, \lambda(1-\lambda)^2 e^{-2\lambda})$

when $\lambda \neq 1$.

Note that when $\lambda = 1$, $n(UMVUE - g(\bar{X})) = g(\bar{x})(1-\frac{\bar{X}}{2})$

$\xrightarrow{P} g(1)(1-\frac{1}{2}) = \frac{1}{2e}$.

Therefore $n(UMVUE - g(\lambda)) \xrightarrow{d} \frac{1}{2e} - \frac{1}{2e}\chi_1^2$.

6. $X_1 \cdots X_n \overset{iid}{\sim} N(\theta, 1)$. $\delta_{1n} = \Phi(a - \bar{x})$

$$\delta_{2n} = \frac{\# X_i \le a}{n}$$

$$ARE(\delta_2, \delta_1) = \frac{\phi^2(-\theta)}{\Phi(-\theta)(1-\Phi(-\theta))}$$

Does $e_{21}$ gets better or worse when $\theta \to \pm\infty$?

Note that $e_{21}$ is symmetric against zero, we only consider $-\theta \to +\infty$. or $\theta \to -\infty$.

By Mill's ratio: $\left( \frac{1-\Phi(x)}{\phi(x)} \sim \frac{1}{x} \quad \text{as } x \to +\infty \right)$

when $\theta \to -\infty$, $\Phi(-\infty) \to 1$

$$e_{21} \sim \phi(-\theta)(-\theta)$$

$$= -\theta \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{\theta^2}{2}}$$

$$(\lim \theta \to -\infty) = 0.$$

Therefore it's getting worse.