Math 281A HW1

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Problem 1. Let X_i be IID (μ, σ^2). Let τ be the skewness and κ be the kurtosis. Show that

$$\tau(\bar{X}) = \tau/\sqrt{n}, \quad \kappa(\bar{X}) = \kappa/n$$

Proof. Without loss of generality, set $\mu = 0$. First calculate $E(\bar{X})^3$:

$$E(\bar{X})^3 = \frac{1}{n^3}E(\sum_i X_i)^3 = \frac{1}{n^2}\mu_3$$

The last equation is due to the fact that all the components, expect those third moments, have forms of $X_i^2X_j$ or $X_iX_jX_k$, where all of them have expectation 0. Note that $\bar{X}$ has variance σ/n, and therefore

$$\tau(\bar{X}) = \frac{\mu_3/n^2}{(\sigma^2/n)^{3/2}} = \frac{\mu_3}{\sigma^3\sqrt{n}} = \tau/\sqrt{n}.$$

Similarly, by the same argument,

$$E(\bar{X})^4 = \frac{1}{n^4}E(\sum_i X_i)^4 = \frac{1}{n^3}\mu_4 + \frac{6n(n-1)}{n^4}\sigma^4 = (\frac{\kappa+3}{n^3} + \frac{6n(n-1)}{n^4})\sigma^4.$$

Therefore

$$\kappa(\bar{X}) = E(\bar{X})^4/(\sigma^4/n^2) - 3 = \frac{\sigma^4(\kappa+3n)}{n^3}\frac{n^2}{\sigma^4} - 3 = \kappa/n.$$

□

Problem 2. Show that

$$var(\frac{1}{n-1}\sum_i(X_i - \bar{X})^2) = \frac{\mu_4}{n} - \frac{n-3}{n(n-1)}\sigma^4.$$

Proof. We can assume that all X_i have mean zero here. Use the formula of the variance of quadratic form:

$$var(X'AX) = (\mu_4 - 3\mu_2^2)a'a + 2\mu_2^2 tr(A^2) + 4\mu_2\theta'A^2\theta + 2\mu_3\theta'Aa$$

where $a = tr(A)$. Plug in $A = I - II'/n$ and $X'AX = \sum_i(X_i - \bar{X})^2$. Calculation will finish the proof.

□

Problem 3. Let X_i be uncorrelated random variables with common expectation θ and variances. Then among all estimator $\sum_i \alpha_i X_i$ of θ that $\sum_i \alpha_i = 1$, the mean $\bar{X}$ has the least variance.

Proof. It is equivalent to $\min \sum_i \alpha_i^2$ with respect to $\sum_i \alpha_i = 1$. A Lagrangian multiplier will do the job. Also this can be shown by assuming there exists α_i and α_j that not equal, averaging these two coefficients will produce an estimator that haves less variance. □

Problem 4. Minimize the variance of $\sum_i \alpha_i X_i$ under the following situations: (i) the variance of X_i is σ^2/a_i (a_i known); (ii) X_i have common variance σ^2 but correlated with common correlation coefficient ρ.

Solution . (i) The target function becomes $\min \alpha^2/a$ with respect to $\sum_i \alpha_i = 1$.Use the Lagrangian multiplier we have

$$2\alpha_i/a_i = \lambda$$

and this gives

$$2\alpha_i = \lambda a_i.$$

Using $\sum_i \alpha_i = 1$, we get $\lambda = 2/\sum_i a_i$. Therefore $\alpha = a_i/\sum_i a_i$.

(ii) Similar to (3), one my show that $\alpha = 1/n$ by using symmetry or Lagrangian multiplier.

Problem 5. Let X, Y have common mean θ, variance σ^2 and τ^2, and correlation ρ. Determine the condition that (i) $var(X) < var[(X+Y)/2]$; (ii) The value that minimizes $var[\alpha X + (1 - \alpha Y)]$ is negative.

Solution . (i) Direct calculation shows that $var[(X+Y)/2] = (\sigma^2 + \tau^2 + 2\tau\sigma\rho)/4 > \sigma^2$, which is equivalent to $\tau^2 + 2\tau\rho\sigma > 3\sigma^2$. This is interpreted as the contribution of variance from Y should be larger than the piece from X, with correlations counted.

(ii) By calculating $var[\alpha X + (1 - \alpha Y)] = \alpha^2\sigma^2 + (1-\alpha)^2\tau^2 + 2\alpha(1-\alpha)\rho\tau\sigma$, we may see that when

$$\alpha = \frac{\tau^2 - \rho\tau\sigma}{\sigma^2 + \tau^2 - 2\rho\tau\sigma},$$

the variance is minimized. It is clear that the condition is $\tau - \rho\sigma < 0$. This means including one piece of Y will decrease the total variance compared to the contribution from the correlation of X.