University of Pennsylvania
ESE 5000: Linear Systems Theory
HOMEWORK V

Due: on December 9 at 11:59pm EST on Gradescope

INSTRUCTIONS
Read the following instructions carefully before beginning to work on the homework.

- You must submit your solutions on Gradescope. It must be submitted on Gradescope as a **single PDF file**. LaTeX is preferred but handwritten submissions are allowed as long as they are readable by our staff to be graded.
- Please start a new problem on a fresh page and mark on Gradescope all the pages corresponding to each problem. Failure to do so may result in your work not graded completely.
- Clearly indicate the name and Penn email of **all your collaborators** on your submitted solutions. You may discuss the problems but you cannot share solutions and whatever you submit must be your own work. **Failure to do so will result in penalties according to Penn's Code of Conduct policies**.
- **Late days policy reminder**: 3 days with no penalty, after which -35% per day.
- **Regrade requests are handled via Gradescope within 3 days from publication of grades. After that no regrade request is admissible.** When requesting a regrade, the entire submission may be regraded by the TAs.
- Please submit only one PDF. If you wrote any code, submit it as well by attaching it to your PDF submission.

Problem 1. (**Pole placement**, 15 points) Consider the following system.

$$\dot{x} = \begin{bmatrix} 2 & -4 & 3 \\ 1 & -3 & -1 \\ 1 & -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

a) (3 points) Decompose the system into controllable and uncontrollable subsystems.

b) (2 points) Justify whether or not the system is stabilizable.

c) (4 points) Design a feedback gain $\hat{K}$ for the transformed system, you obtained in a, to place the poles to $-1, -2, -3$.

d) (3 points) Compute feedback gain K corresponding to $\hat{K}$ to be applied to the original system in order to have the same pole placement as in c.

e) (3 points) Justify whether or not it is possible to place the poles to $-2, -2, -3$.

Problem 2. (**Algebraic Riccati equation (ARE) in discrete time**, 15 points) Consider the discrete-time system

$$x_{k+1} = Ax_k + Bu_k,$$

where

$$A = \begin{bmatrix} 0 & -2 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and the infinite-horizon cost function

$$J = \sum_{k=0}^{\infty} \left(2x_{1,k}^2 \; + \; 3x_{2,k}^2 \; + \; 3u_k^2 \right).$$

This corresponds to the quadratic weights

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \qquad R = 3.$$

Design an LQR feedback gain K_{lqr} for the system using the following steps.

a) (9 points) Find the symmetric positive definite matrix $P \in \mathbb{R}^{2\times 2}$ that satisfies the discrete-time Algebraic Riccati Equation (ARE):

$$P = A^T PA + Q - A^T PB\,(R + B^T PB)^{-1} B^T PA.$$

b) (3 points) Compute the LQR feedback gain

$$K_{\text{lqr}} = -(R + B^T PB)^{-1} B^T PA.$$

c) (3 points) Compute the closed-loop system matrix

$$A_{\text{lqr}} = A + BK_{\text{lqr}}$$

so that the closed-loop dynamics are

$$x_{k+1} = A_{\text{lqr}} x_k$$

under the control law $u_k = K_{\text{lqr}} x_k$.

Problem 3. (**Detectability**, 12 points) Consider the following system.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{bmatrix} x$$
$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x$$

a) (6 points) Decompose the system into observable and unobservable subsystems.

b) (6 points) Justify whether or not the system is detectable.

Problem 4. (**Separation principle**, 15 points) Consider the linear, time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

Assume that (A, B) is controllable and (A, C) is observable.

(a) (2 point) Let matrix L be given such that $A + LC$ has all negative eigenvalues. Determine the observer equations for the system.

(b) (4 points) Suppose a linear feedback $u(t) = K\hat{x}(t)$ controller is applied with K such that $A + BK$ has all negative eigenvalues. Express the plant and observer as a single LTI system with state $\begin{bmatrix} x \\ \hat{x} \end{bmatrix}$.

(c) (12 points) Prove that the combined plant–observer system obtained in part (b) is stable. Hint: Consider applying a similarity transformation that uses the estimation error $x - \hat{x}$ as part of the new coordinates.

Problem 5 (**Robust Observers**, 15 points)**.** Consider the linear, time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + w(t) \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $w(t)$ denotes a disturbance signal (unknown and uncontrollable input). Suppose that (A, C) is observable and let L be a stabilizing observer gain matrix.

Determine the observer error dynamics $\dot{e} = f(e, w)$ and prove that there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that

- $c_1\|e\|_2^2 \leq V(e) \leq c_2\|e\|_2^2$ and
- $\|e\|_2 \geq c_3\|w\|_2 \implies \dot{V}(e, w) \leq -c_4\|e\|_2^2$

for all $e, w \in \mathbb{R}^n$, for some constants $c_1, c_2, c_3, c_4 > 0$, where $\dot{V}(e, w) = \nabla V(e)^\top f(e, w)$.

Hint: Determine a suitable Lyapunov equation and use it to construct V and to derive the constants c_1, c_2, c_3, c_4.

Remark:

- This confirms that the observer error system is (exponentially) *input-to-state* (ISS) stable when $w(t)$ is treated as its input, with V a certifying *ISS Lyapunov function.* Conceptually, ISS stability of a system tells us that the system is asymptotically stable when the input is zero and that its trajectories converge to a ball whose radius is an increasing function of the size ($\|\cdot\|_\infty$-wise) of the input signal.

- In particular, if the disturbance $w(t)$ is bounded, say by $\varepsilon > 0$, then the observer error system is *practically* stable in that $e(t)$ behaves as an asymptotically stable system for a finite interval of time, but ultimately enters and remains within a ball whose radius is an increasing function of ε but will typically not actually converge towards any particular point, let alone the origin.

- ISS stability is also closely related to the notion of *robust* stability, that hinges upon asymptotic stability of the closed-loop system where the norm of the feedback controller is bounded by a constant factor of the norm of the state. Conceptually, in this case, this tells us that the if the disturbance $w(t)$ becomes smaller as the observer error becomes smaller, at a linear rate, then the error system will indeed be asymptotically stable.

Problem 6 (25 points)**. Make sure you include the Matlab/Python code.**

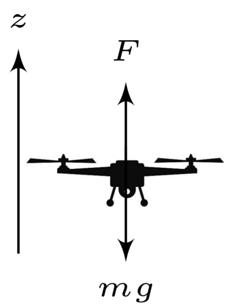

Figure 1: A UAV moves in the z-axis direction under the gravitational force and the actuator force F. The control objective is to hover at the height $z = z_d$.

Consider a one dimensional UAV of Figure 1, which is moving only in the z-axis direction. Its state comprises of the height z and the velocity $\dot{z}$. Hence the equations of motion are:

$$m\ddot{z} = F - mg$$

where F is the actuator input, m is the mass, and g is the gravitational acceleration. The initial height is $z(0) = 0$ and the initial velocity $\dot{z}(0) = 0$. In this problem, we will study state space methods for control design. In particular, the control objective for the UAV is to hover at the desired height z_d, with zero velocity.

1. (6 points) **Discretization** Prior to controlling the system, we need to discretize the continuous-time dynamics given above, using sampling period T_s. Assume that F is piecewise constant in the intervals $[kT_s, (k+1)T_s]$. Prove that the discretized system satisfies the difference equation:

$$x_{k+1} = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} T_s^2/2 \\ T_s \end{bmatrix} \left(\frac{F(kT_s)}{m} - g \right), \quad x_0 = [0,0]^T \tag{2}$$

   where $x_k = [z(kT_s), \dot{z}(kT_s)]^T$. Now, assume that $T_s = 1,\ m = 2,\ g = 9.8$ and that we apply control $F(t) = mg + mu_k$ for $t \in [kT_s, (k+1)T_s)$ to obtain

$$x_{k+1} = Ax_k + Bu_k, \quad x_0 = [0,0]^T \tag{3}$$

   Find the numerical values of A, B. Using that, we will design controllers that enable the UAV to hover around the desired height.

2. (12 points) **Open-loop vs Closed-loop control** For the following questions keep the assumption that $T_s = 1,\ m = 1,\ g = 10$, i.e. use system (3).

   (a) (3 points) Verify that (3) is controllable. Find the minimum norm control sequence :

$$\min_{u_0,\ldots,u_{49}} \sum_{k=0}^{49} u_k^2 \quad s.t.\ x_{50} = [2,0]^T$$

   This is an open-loop controller.

   (b) (3 points) Prove that a feedback law

$$u_k = K(x_k - x_d) \tag{4}$$

   where $x_d = [3,0]^T$, forces the system (3) to x_d, provided that the poles of $A + BK$ are strictly inside the unit circle. Then, design the feedback controller (i.e. find K) so that the closed-loop poles are $0.5, 0.5$.

(c) (3 points) For $k = 0, .., 50$, simulate system (3) under the open-loop controller 2a. Repeat using the closed loop controller 2b. Plot the states evolution under each control method on the same plot (create one plot for height and one for velocity). Compare the two system responses.

(d) (3 points) Assume that due to wind disturbances, there is some noise that is added to the velocity dynamics at each time k:

$$x_{k+1} = Ax_k + Bu_k + \begin{bmatrix} 0 \\ 0.02 \end{bmatrix} w_k \quad x_0 = [0, 0]^T \tag{5}$$

where $w_k \sim \mathcal{N}(0, 1)$. For $k = 0, ..., 50$, simulate system 5 and plot on the same graph the UAV's height (only the first component of the state vector) when under open-loop and closed-loop control. Which one is more robust? **Make sure you use the same noise sequences for the open-loop and the closed-loop.** [Note: use a specific random seed for noise generation (Python) or `rng` (Matlab) to ensure the same noise.]

3. (7 points) **Optimal Control** The LQR control offers an effective way to tune the feedback control law. By carefully selecting matrices Q, R, we can control the tradeoff between how small the state is and how large the control effort is without damaging stability. Here, we will study the infinite-horizon LQR control with cost:

$$J = \sum_{k=0}^{\infty} (x - x_d)^T Q (x - x_d) + u_k^T R u_k \tag{6}$$

where $x_d = [2, 0]^T$ (remember we want the system to hover $\dot{z}_d = 0$ at the desire height $z_d = 2$).

Simulate system (3) under the infinite-horizon LQR control (6), with $Q = \mathbb{I}, R = 1$ for $k = 0, ..., 50$ (useful function **dlqr** for Matlab and useful package **python-control** for Python (use control.dlqr), be careful of the sign). Repeat for $Q = 100\mathbb{I}$ and $R = 1$; $Q = \mathbb{I}$ and $R = 0.1$ Plot the states x_k and the input u_k for all the three cases. Compare the plots and explain any differences.