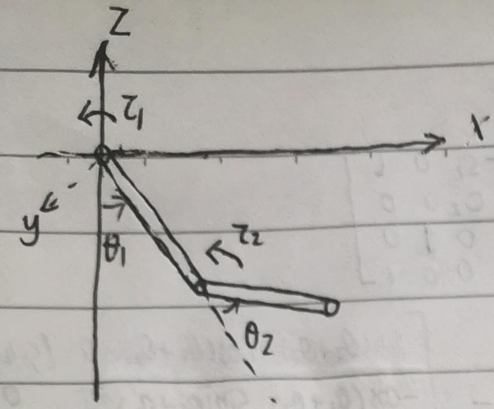
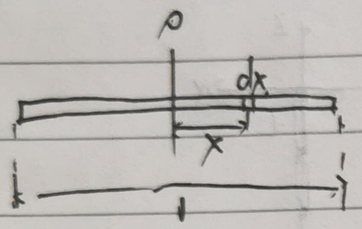


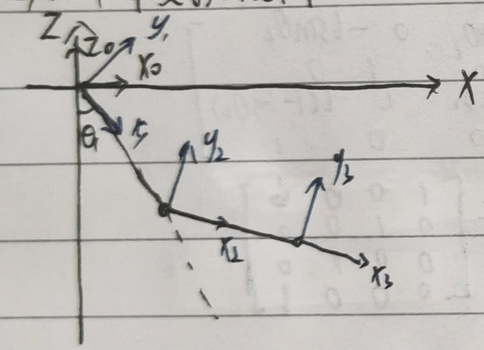
$l = l_1 = l_2 = 1m \quad m_1 = m_2 = 5kg.$

1>. 连杆绕质心的转动惯量



$$J = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^2 \cdot \frac{5}{1}\, dx = \frac{5\times 1}{12} = \frac{5}{12}\ kg\cdot m$$

2>. 计算变换矩阵.



MD-H 连杆参数表.

i	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	$-\frac{\pi}{2}$	0	$(\theta_1 - \frac{\pi}{2})$
2	l	0	0	θ_2
3	l	0	0	

① D-H法. i) $${}^0_1T = Rot(X,d)\cdot Rot(Z,\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ -\cos\theta_1 & \sin\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\cos\theta_1 & \sin\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ii) $${}^1_2T = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

iii) 手爪 $${}^2_3T = \begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

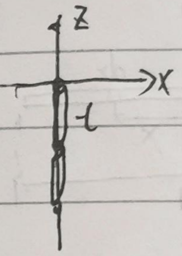
No.

Date.

~~运动学方程(末端连杆)~~ ${}^0_2T = {}^0_1T\,{}^1_2T = \begin{bmatrix} s_1 & c_1 & 0 & 0\\ 0 & 0 & -1 & 0\\ -c_1 & s_1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix} c_2 & -s_2 & 0 & l\\ s_2 & c_2 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$

$$= \begin{bmatrix} \sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2, & -\sin\theta_1\sin\theta_2+\cos\theta_1\cos\theta_2, & 0 & l\sin\theta_1\\ 0 & 0 & -1 & 0\\ -\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2, & \sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2, & 0, & -l\cos\theta_1\\ 0 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix} \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & l\sin\theta_1\\ 0 & 0 & -1 & 0\\ -\cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2), & 0, & -l\cos\theta_1\\ 0 & 0 & 0 & 1\end{bmatrix}$$

②指数积方法: $\omega_1 = \begin{bmatrix}0\\1\\0\end{bmatrix} = \omega_2$ $r_1 = \begin{bmatrix}0\\0\\0\end{bmatrix}$ $r_2 = \begin{bmatrix}0\\0\\-l\end{bmatrix}$ $r_3 = \begin{bmatrix}0\\0\\-2l\end{bmatrix}$



$[\omega] = \begin{bmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{bmatrix}$ $e^{[V_1]\theta_1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \sin\theta_1 & \cos\theta_1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$ $\cancel{e^{[V_2]\theta_2}}$

$$e^{[V_2]\theta_2} = \begin{bmatrix} R(\theta_2) & p(\theta_2)\\ 0 & 1\end{bmatrix} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & -l\sin\theta_2\\ 0 & 0 & 1 & 0\\ \sin\theta_2 & \cos\theta_2 & 0 & -l(1-\cos\theta_2)\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

$${}^0_2T(0,0) = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ -1 & 0 & 0 & -l\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

$${}^0_3T(\theta_1,\theta_2) = e^{[V_1]\theta_1}\, e^{[V_2]\theta_2}\; {}^0_2T(0,0)\; {}^2_3T(0,0)$$

$$= \begin{bmatrix} s_1c_2+c_1s_2 & -s_1s_2+c_1c_2 & 0 & ls_1\\ 0 & 0 & -1 & 0\\ s_1s_2-c_1c_2 & s_1c_2+c_1s_2 & 0 & -lc_1\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix} 1 & & & l\\ & 1 & & \\ & & 1 & \\ & & & 1\end{bmatrix}.$$

$$= \begin{bmatrix} s(1+2) & c(1+2) & 0 & l\,s(1+2)+ls_1\\ 0 & 0 & -1 & 0\\ -c(1+2) & s(1+2) & 0 & -l\,c(1+2)-lc_1\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

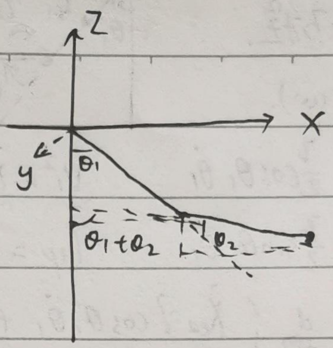
运动学方程:

手爪 ${}^0_3T = {}^0_2T\,{}^2_3T = \begin{bmatrix} \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & l[\sin(\theta_1+\theta_2)+\sin\theta_1].\\ 0 & 0 & -1 & 0\\ -\cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0 & -l[\cos(\theta_1+\theta_2)+\cos\theta_1]\\ 0 & 0 & 0 & 1\end{bmatrix} = {}^0_3T(\theta_1,\theta_2).$

No.

Date.

37. 计算雅可比矩阵.



$$\begin{cases} x = l\sin(\theta_1+\theta_2) + l\sin\theta_1 \\ y = 0 \\ z = -l\cos(\theta_1+\theta_2) - l\cos\theta_1 \end{cases}$$

$$\dot{x} = \begin{bmatrix} l\cos(\theta_1+\theta_2)+l\cos\theta_1 & l\cos(\theta_1+\theta_2) \\ 0 & 0 \\ l\sin(\theta_1+\theta_2)+l\sin\theta_1 & l\sin(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix} = J(q)\cdot\dot{q} \qquad \dot{q} = \begin{bmatrix} \dot\theta \\ \dot\theta \\ \dot{i} \end{bmatrix} =$$

对关节1. $J_1 = \begin{bmatrix} Z_1 \times ({}^0_1R\,{}^1P_1) \\ Z_1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} l\cos(\theta_1+\theta_2)+l\cos\theta_1 \\ 0 \\ l\sin(\theta_1+\theta_2)+l\sin\theta_1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} l\cos(\theta_1+\theta_2)+l\cos\theta_1 \\ 0 \\ l\sin(\theta_1+\theta_2)+l\sin\theta_1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$

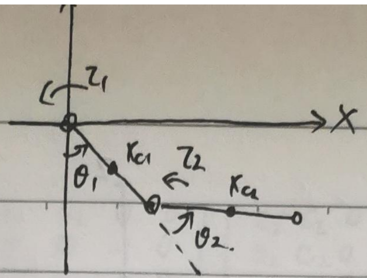
对关节2. 有 $J_2 = \begin{bmatrix} l\cos(\theta_1+\theta_2) \\ 0 \\ l\sin(\theta_1+\theta_2) \\ 0 \\ 1 \\ 0 \end{bmatrix}.$ $\Rightarrow$ 速度雅可比矩阵 $J = [J_1, J_2] = \begin{bmatrix} l\cos(\theta_1+\theta_2)+l\cos\theta_1, & l\cos(\theta_1+\theta_2) \\ 0 & 0 \\ l\sin(\theta_1+\theta_2)+l\sin\theta_1, & l\sin(\theta_1+\theta_2) \\ 0, & 0 \\ 1, & 1 \\ 0, & 0 \end{bmatrix}$

(X-Z).

若2R机器人若只考虑二维情况 $J = \begin{bmatrix} l\cos(\theta_1+\theta_2)+l\cos\theta_1 & l\cos(\theta_1+\theta_2) \\ l\sin(\theta_1+\theta_2)+l\sin\theta_1 & l\sin(\theta_1+\theta_2) \end{bmatrix}.$

No.

Date.　　$m=5 \quad l=1.$



47. Lagrange法求机器人封闭动力学方程.

质心: x_{c1}, z_{c1} 等效到质点(心).

平动动能

杆1: $\begin{cases} x_{c1} = \frac{l}{2}\sin\theta_1 \\ z_{c1} = -\frac{l}{2}\cos\theta_1 \end{cases} \overset{d}{\Longrightarrow} \begin{cases} \dot{x}_{c1} = \frac{l}{2}\cos\theta_1\,\dot\theta_1 \\ \dot{z}_{c1} = \frac{l}{2}\sin\theta_1\,\dot\theta_1 \end{cases}$　　$V_1^2 = \dot{x}_{c1}^2 + \dot{z}_{c1}^2 = \frac{l^2}{4}\dot\theta_1^2.$

$T_{1p} = \frac{1}{2}mV_1^2 = \frac{ml^2}{8}\dot\theta_1^2.$

杆2: $\begin{cases} x_{c2} = l\sin\theta_1 + \frac{l}{2}\sin(\theta_1+\theta_2) \\ z_{c2} = -l\cos\theta_1 - \frac{l}{2}\cos(\theta_1+\theta_2). \end{cases} \overset{d}{\Longrightarrow} \begin{cases} \dot{x}_{c2} = l\cos\theta_1\,\dot\theta_1 + \frac{l}{2}\cos(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2) \\ \dot{z}_{c2} = l\sin\theta_1\,\dot\theta_1 + \frac{l}{2}\sin(\theta_1+\theta_2)(\dot\theta_1+\dot\theta_2). \end{cases}$

$V_2^2 = \dot{x}_{c2}^2 + \dot{z}_{c2}^2 = \frac{5}{4}l^2\dot\theta_1^2 + \frac{l^2}{4}\dot\theta_2^2 + \frac{l^2}{2}\dot\theta_1\dot\theta_2 + l^2\cos\theta_2\,\dot\theta_1(\dot\theta_1+\dot\theta_2).$　　$T_{2p} = \frac{1}{2}mV_2^2.$

转动动能　杆1: $T_{1r} = \frac{1}{2}J\omega_1^2 = \frac{1}{2}\cdot\frac{ml^2}{12}\dot\theta_1^2 = \frac{ml^2}{24}\dot\theta_1^2.$　　$T_{2r} = \frac{1}{2}J\omega_2^2 = \frac{1}{2}\cdot\frac{ml^2}{12}(\dot\theta_1+\dot\theta_2)^2.$

动能. $T_1 = T_{1p} + T_{1r} = \frac{ml^2}{6}\dot\theta_1^2 = \frac{5}{6}\dot\theta_1^2$

$T_2 = T_{2p} + T_{2r} = \frac{2ml^2}{3}\dot\theta_1^2 + \frac{ml^2}{6}\dot\theta_2^2 + \frac{ml^2}{3}\dot\theta_1\dot\theta_2 + \frac{ml^2}{2}\cos\theta_2\,\dot\theta_1(\dot\theta_1+\dot\theta_2).$

$T = T_1 + T_2 = \frac{25}{6}\dot\theta_1^2 + \frac{5}{6}\dot\theta_2^2 + \frac{5}{3}\dot\theta_1\dot\theta_2 + \frac{5}{2}\cos\theta_2\,\dot\theta_1(\dot\theta_1+\dot\theta_2).$

势能　以$z=0$为零势能面.

$U = U_1 + U_2 = -\frac{mgl}{2}\cos\theta_1 - mgl\cos\theta_1 - \frac{mgl}{2}\cos(\theta_1+\theta_2) = -\frac{15g}{2}\cos\theta_1 - \frac{5}{2}g\cos(\theta_1+\theta_2)$

有Lagrange函数　$L = T - U$　$\tau_i = \frac{d}{dt}\frac{\partial T}{\partial \dot q_i} - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i}.$

$\tau_1 = (\frac{25}{3} + 5\cos\theta_2)\ddot\theta_1 + (\frac{5}{3} + \frac{5}{2}\cos\theta_2)\ddot\theta_2 - \frac{5}{2}\sin\theta_2\,\dot\theta_2^2 - 5\sin\theta_2\,\dot\theta_1\dot\theta_2 + \frac{15}{2}g\sin\theta_1 + \frac{5}{2}g\sin(\theta_1+\theta_2).$

$\tau_2 = (\frac{5}{3} + \frac{5}{2}\cos\theta_2)\ddot\theta_1 + \frac{5}{3}\ddot\theta_2 + \frac{5}{2}\sin\theta_2\,\dot\theta_1^2 + \frac{5}{2}g\sin(\theta_1+\theta_2).$

$$\tau = \overbrace{\begin{bmatrix} \frac{25}{3} + 5\cos\theta_2 & \frac{5}{3} + \frac{5}{2}\cos\theta_2 \\ \frac{5}{3} + \frac{5}{2}\cos\theta_2 & \frac{5}{3} \end{bmatrix}}^{D}\begin{bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 & -\frac{5}{2}\sin\theta_2 \\ \frac{5}{2}\sin\theta_2 & 0 \end{bmatrix}}^{h}\begin{bmatrix} \dot\theta_1^2 \\ \dot\theta_2^2 \end{bmatrix} + \overbrace{\begin{bmatrix} -5\sin\theta_2\,\dot\theta_1\dot\theta_2 \\ 0 \end{bmatrix}}^{G_1}$$

$$+ \overbrace{\begin{bmatrix} \frac{15}{2}g\sin\theta_1 + \frac{5}{2}g\sin(\theta_1+\theta_2). \\ \frac{5}{2}g\sin(\theta_1+\theta_2) \end{bmatrix}}^{G_2}.$$

∴封闭动力学方程.

验证惯性矩阵与动能之间的关系.

$$① .\ T=\frac{1}{2}\dot{q}^T D(q)\dot{q}=\frac{1}{2}\sum_{i,j}^{n} d_{ij}(q)\dot{q}_i\dot{q}_j \quad (q=\theta).$$

此时 $n=2$.

右: 连杆线速度 $v_i=J_{vi}(q)\dot{q}$ 角速度 $\omega_i=J_{\omega i}(q)\dot{q}$

动能 $T=\frac{1}{2}mv^Tv+\frac{1}{2}\omega^T I\omega$ I: 惯性张量矩阵. R: 惯性坐标系变换矩阵.

$$=\frac{1}{2}\dot{q}^T\underbrace{\sum_{i=1}^{n}\left[m_i(J_{vi}(q))^T J_{vi}(q)+(J_{\omega i}(q))^T R_i(q)I_i(R_i(q))^T J_{\omega i}(q)\right]}_{=D(q).}\dot{q}$$

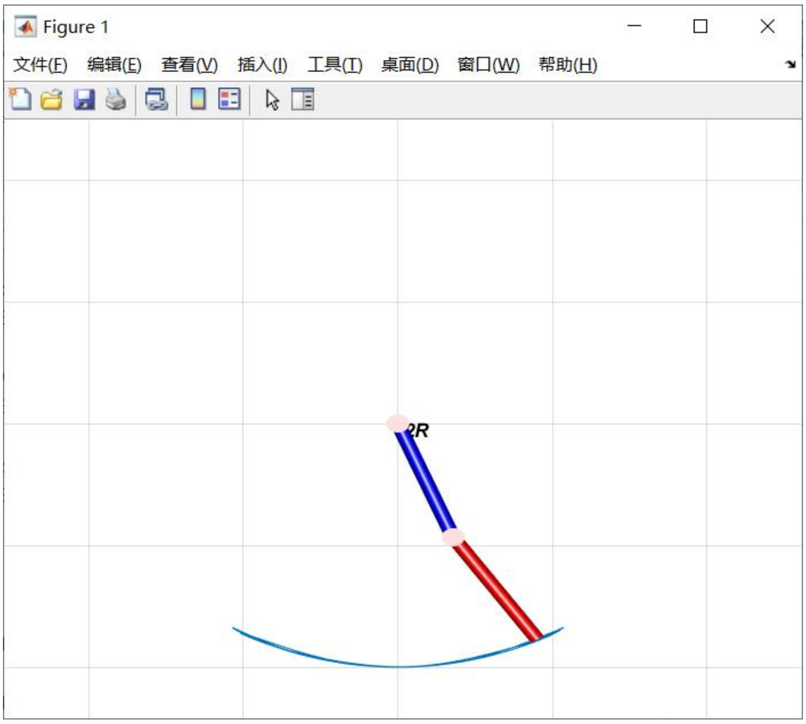
$$\text{等式①右}=\frac{1}{2}\left[D_{11}(\theta)\dot{\theta}_1\dot{\theta}_2+D_{12}(\theta)\dot{\theta}_1\dot{\theta}_2+D_{21}(\theta)\dot{\theta}_1\dot{\theta}_2+D_{22}(\theta)\dot{\theta}_1\dot{\theta}_2\right].$$

$$=\frac{1}{2}\left[\frac{25}{3}+5\cos\theta_2+\frac{5}{3}+\frac{5}{2}\cos\theta_2+\frac{5}{3}+\frac{5}{2}\cos\theta_2+\frac{5}{3}\right]\dot{\theta}_1(\dot{\theta}_2+\dot{\theta}_1).$$

$$=\frac{25}{6}\dot{\theta}_1^2+\frac{5}{6}\dot{\theta}_2^2+\frac{5}{3}\dot{\theta}_1\dot{\theta}_2+\frac{5}{2}\cos\theta_2\dot{\theta}_1(\dot{\theta}_1+\dot{\theta}_2).$$

$$=T$$

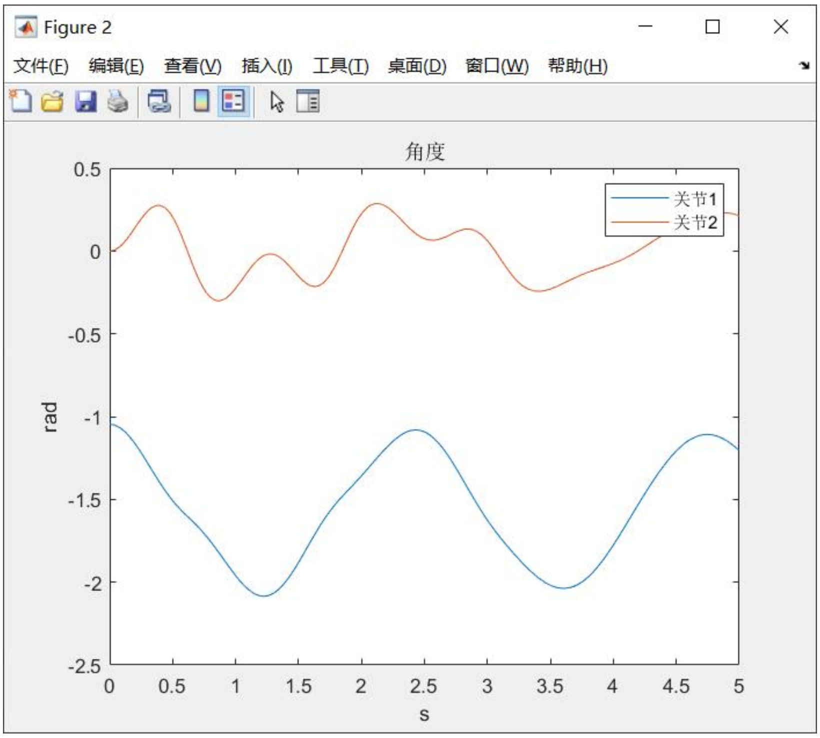
仿真 1



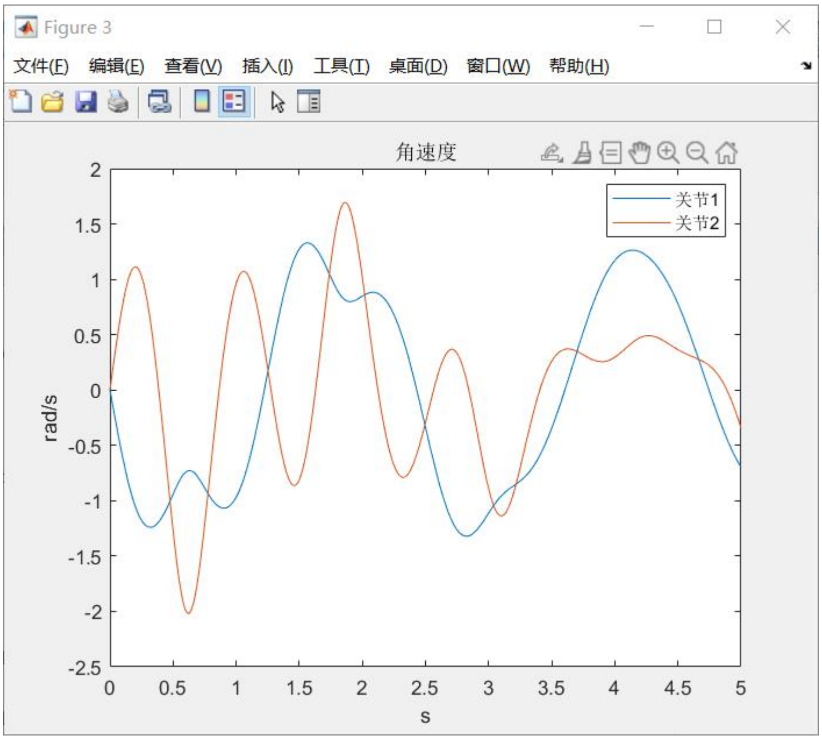
5s 运动轨迹图

① **关节角度、角速度、角加速度**

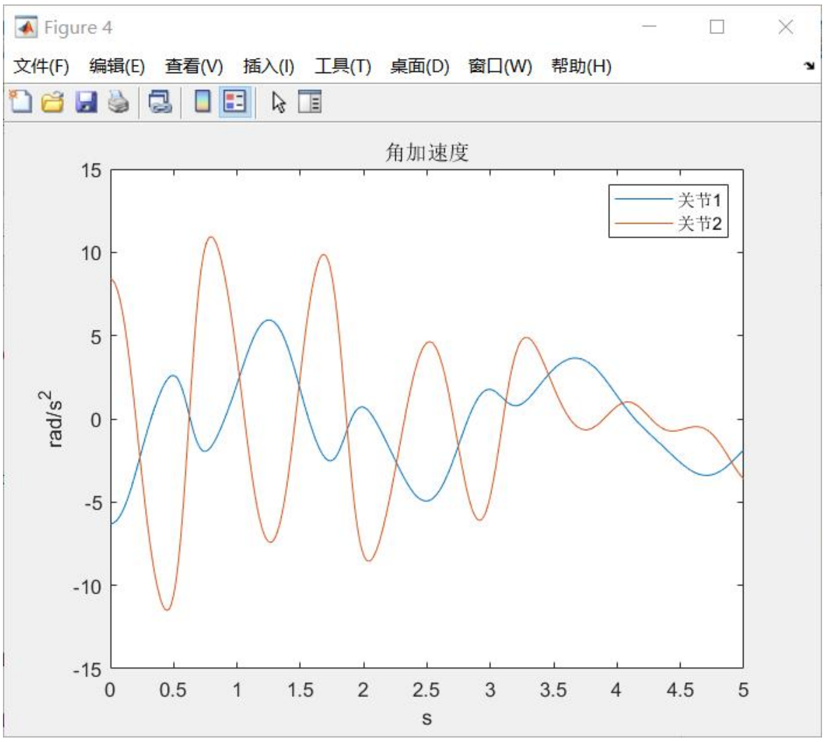
关节角度：



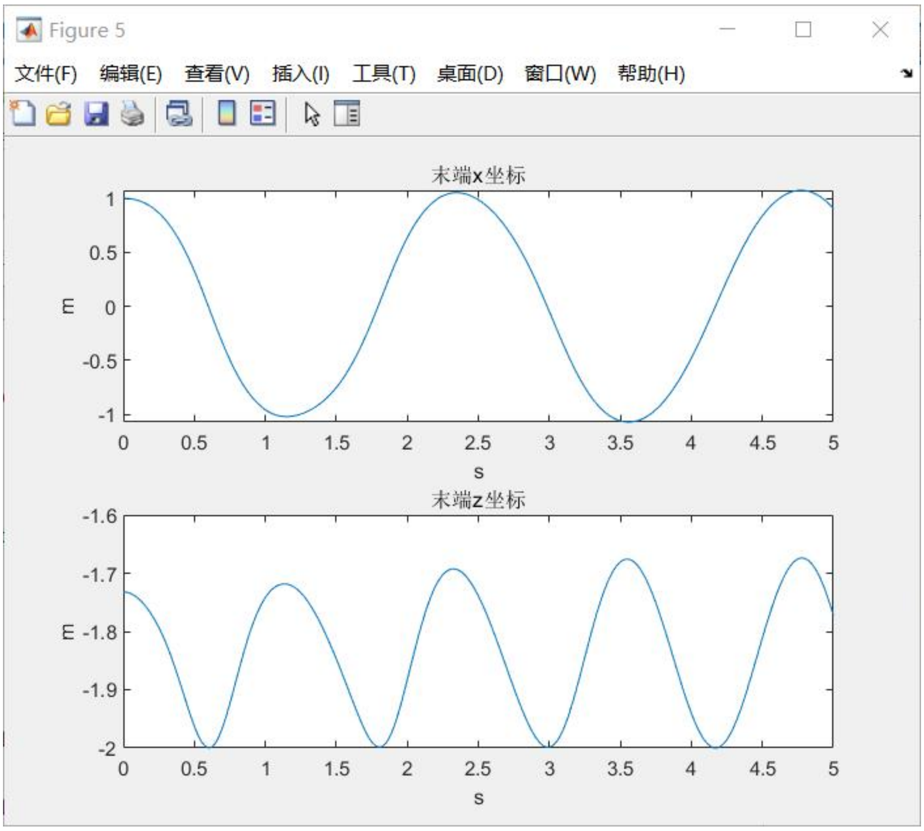
关节角速度：



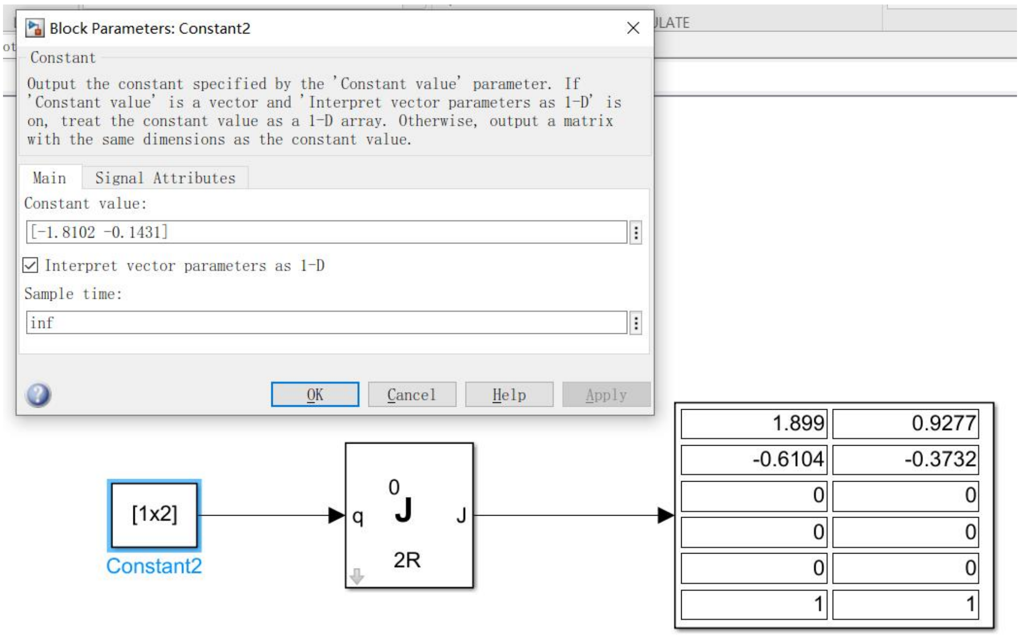
关节角加速度



② 末端执行器的笛卡尔坐标轨迹

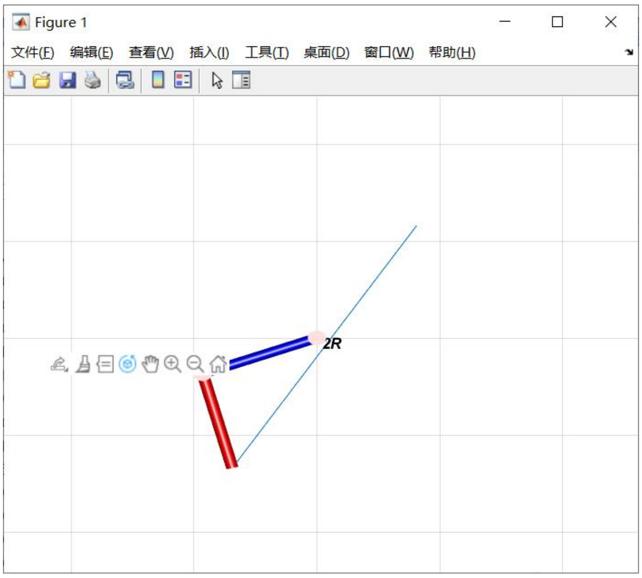


③ t=3.2s 时的雅可比矩阵



t=3.2s 时索引 i=320，查询 qd_t 队列表得角度θ=[-1.8102 -0.1431]，带入第 3 问计算得到的雅可比矩阵[$\cos(\theta_1+\theta_2)+\cos(\theta_1), \cos(\theta_1+\theta_2); \sin(\theta_1+\theta_2)+\sin(\theta_1), \sin(\theta_1+\theta_2)$]后，发现第二列结果相差一个正负号，可能是方向选取时有所不同。

仿真 2



10s 运动轨迹图

① 计算力矩控制规则

57-②-1).

由47.得 $\tau = D(\theta)\ddot{\theta} + h(\theta,\dot{\theta})\dot{\theta} + G(\theta,\dot{\theta})$.

$$\begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix} = \begin{bmatrix} l\cos\theta_1\dot{\theta}_1 + l\cos(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2) \\ l\sin\theta_1\dot{\theta}_1 + l\sin(\theta_1+\theta_2)(\dot{\theta}_1+\dot{\theta}_2)\end{bmatrix} = \begin{bmatrix}-0.15\\ -0.25\end{bmatrix}$$ → 可解微分方程得到 $\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$, $\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$.

已知 $\theta_d(0) = \begin{bmatrix}\frac{5}{9}pi\\ \frac{1}{2}pi\end{bmatrix}$ 及 $\dot{\theta}_d(0)$.

选择力矩 $\tau = D(\theta_d)\ddot{\theta}_d + h(\theta_d,\dot{\theta}_d)\dot{\theta}_d + N(\theta_d,\dot{\theta}_d)$

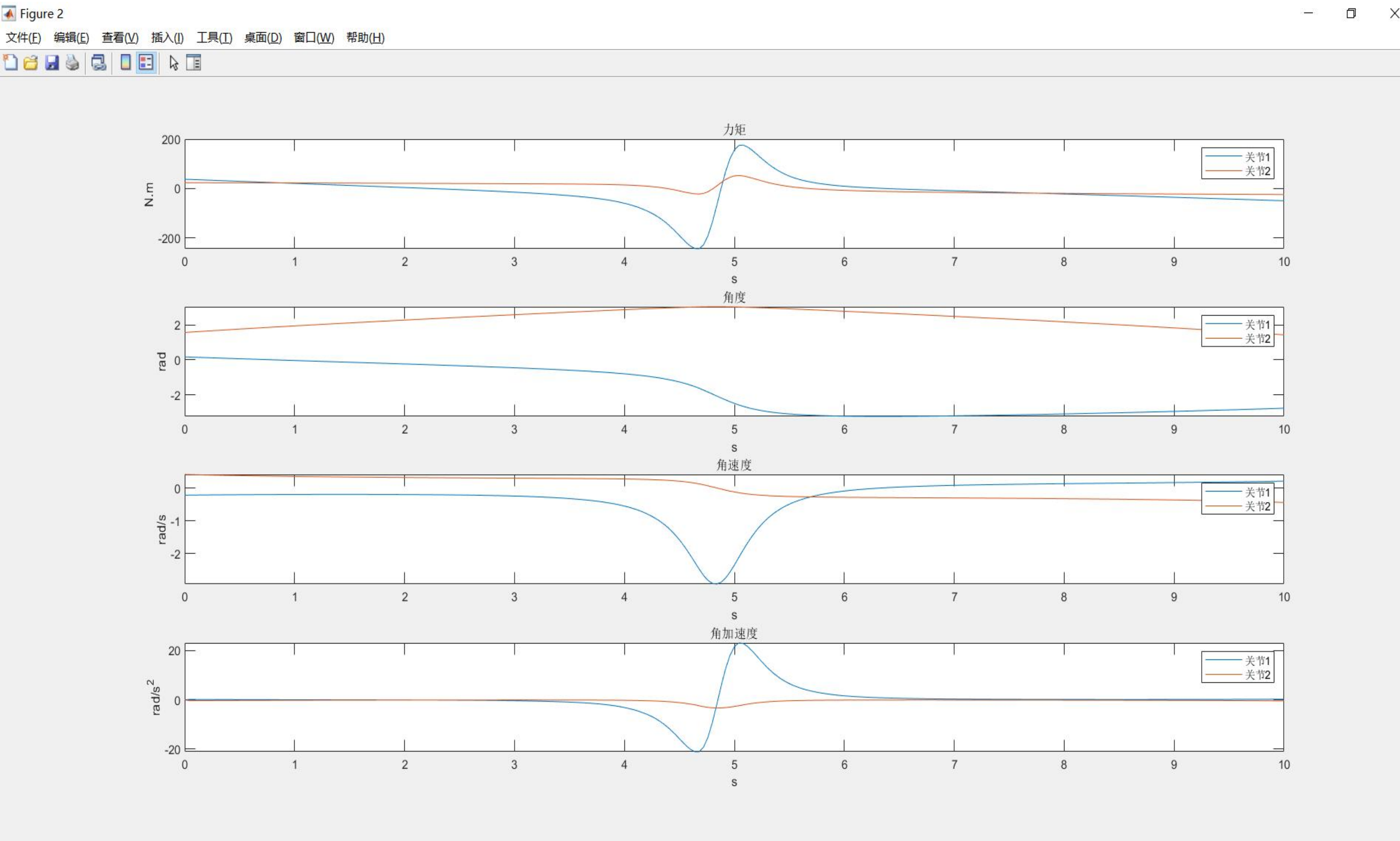
设误差 $e = \theta - \theta_d$ $\quad \tau = D(\theta)(\ddot{\theta}_d - K_v\dot{e} - K_p e) + h(\theta,\dot{\theta})\dot{\theta} + N(\theta,\dot{\theta})$.

$$D(\theta)(\ddot{e} + K_v\dot{e} + K_p e) = 0.$$

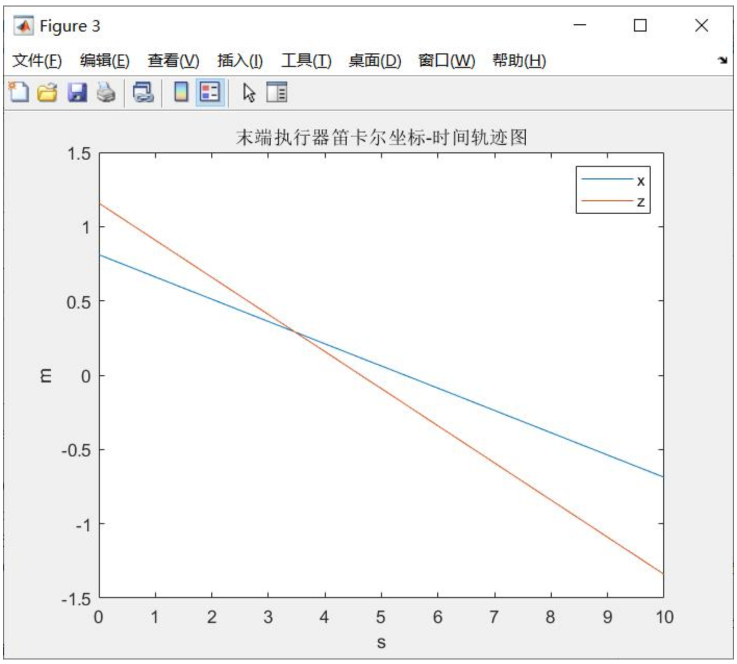
$D(\theta)$正定，则 $K_v\dot{e} + K_p e + \ddot{e} = 0$.

K_v、K_p 为正定对称阵.

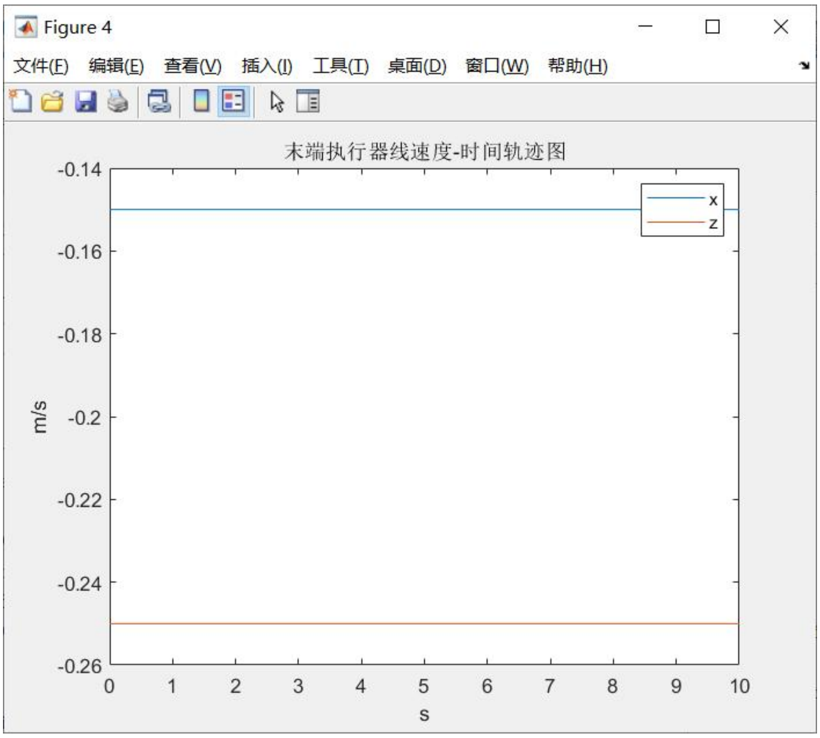
② 关节力矩、角度、角速度、角加速度轨迹



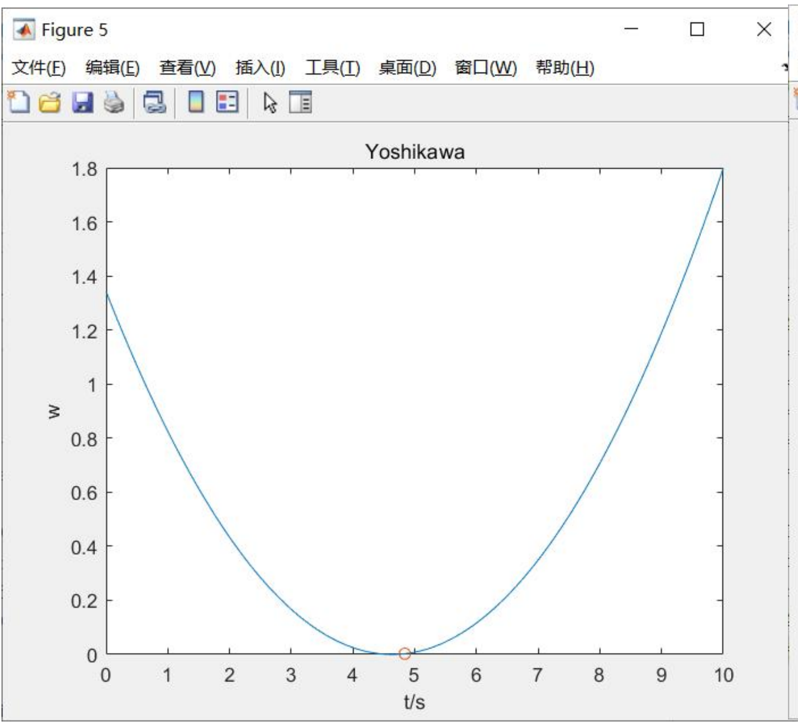
③ 末端执行器的笛卡尔坐标轨迹



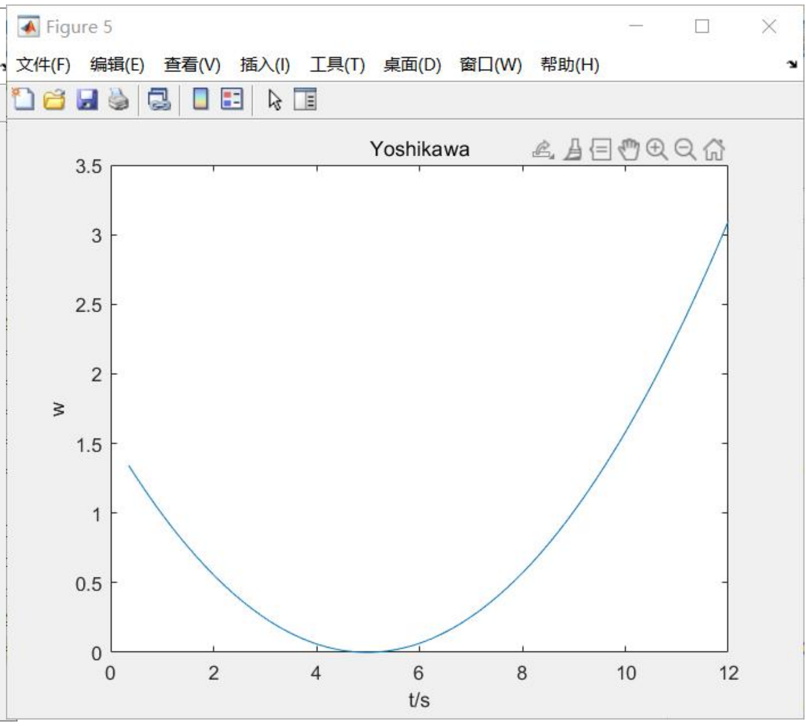
④ 末端执行器笛卡尔坐标速度轨迹



⑤ **Yoshikawa 可操作性度量值 w 与机器人位型**



t=10s



t=12s

圈中标记出的点为奇异位型，即速度雅可比矩阵行列式值为 0 的情况，w 从0^+开始变化的时刻，机器人处于奇异位型。