Tracking the Traveling Salesman Problem

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Abstract

This paper introduces a method for calculating the average and standard deviation of Hamiltonian cycle weights in complete graphs without enumerating all possible tours. By classifying edges as chosen, neighboring, independent, or connecting, and using a thought-experiment framework, we derive polynomial-time equations for the mean and variance of all Hamiltonian cycles that include specific subpaths. These results offer a new way to approach the traveling salesman problem (TSP) statistically, allowing us to make meaningful observations about the overall structure of solutions without needing to solve the problem exactly. The approach is combinatorial in nature but motivated by ideas from statistical mechanics, opening the door to further connections between graph theory and thermodynamic models.

1 Introduction

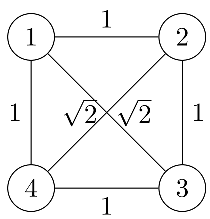
The traveling salesman problem (TSP) is a classic example of a hard combinatorial problem, where finding the shortest possible tour that visits every vertex exactly once becomes quickly intractable as the graph grows [3]. Most approaches focus on finding or approximating this optimal tour. In this paper, however, we take a different approach: instead of focusing on a single tour, we study the statistical behavior of all Hamiltonian cycles in a complete graph.

By examining how often each edge appears across all tours and using combinatorial arguments rooted in symmetry and edge classification, we derive exact formulas for the average weight of all Hamiltonian cycles, as well as for the subset of cycles that include specific edges. We also extend the method to calculate the standard deviation of these cycle weights in polynomial time.

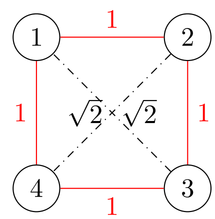
What makes this interesting is that we're able to describe the overall energy landscape of the TSP without solving it. This perspective is not only computationally efficient, but also closely related to how systems behave in statistical physics — where individual configurations matter less than the overall distribution. The goal of this paper is to explore that connection and to show how a shift in perspective can lead to new insights in a well-studied problem.

2 Average weight of all Hamiltonian cycles in a complete graph

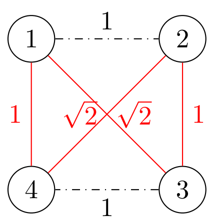
To begin, consider the simple graph with four vertices in Figure 1.



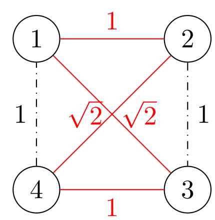
(a) Complete graph



(b) Hamiltonian tour of length 4



(c) Hamiltonian tour of length $2+2\sqrt{2}$



(d) Hamiltonian tour of length $2+2\sqrt{2}$

Figure 1: tsp with 4 vertex

One thing that might not be so obvious is that each edge in a TSP complete graph appears an equal number of times across all possible Hamiltonian tours, and that is true for all complete graph no matter the number of vertices. For example, in Figure 1 E(1,2) is repeated twice across all possible Hamiltonian tours. It appears in tours visualized by graph b and graph d in Figure 1. Also, in Figure 1 you will find E(2,3), E(2,4), E(3,4), E(1,3), and E(1,4) all repeated twice across all the possible Hamiltonian tours.

Therefore, if we let the number of times an edge repeats across all possible Hamiltonian tours as r and we let $\sum W(e)$ be the sum of all the edge's weights, then for a graph of n vertices, the average Hamiltonian cycle length, H_{avg} can be calculated using this formula $(2/(n-1)!)(r\sum W(e))$. The equation is divided by $(n-1)!/2$ because that is the total number of unique Hamiltonian cycles that a complete graph of n vertices contains [4]. We can simplify our equation by finding the equation for r depending on the value of n or the number of vertices the graph contain.

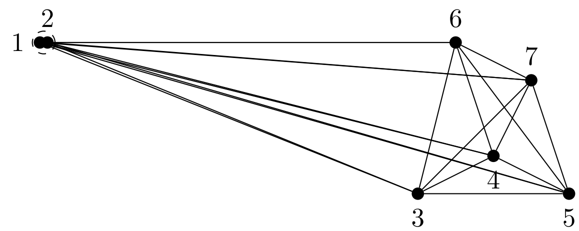


Figure 2: special graph

To do so, let us preform a thought experiment where we have a special graph with two points that are close together while very far away from all the other points in the graph, an example is illustrated in Figure 2. From Figure 2, one can ask how many times does E(1,2) repeat across all possible Hamiltonian tours in the special graph or more essentially, what is the value of r for this graph? V(1) and V(2) are so close to each other and far away from all the other vertices that we can consider E(1,2) as a single vertex, V(”E(1,2)”) and we get a new graph. This new graph has $(n-2)!/2$ unique Hamiltonian tours, minus 2 because we lost a vertex when we consider E(1,2) as V(”E(1,2)”).

Since each tour in the new graph visits the new vertex V(”E(1,2)”), the total the number of tours in the new graph is all the tours that contains E(1,2) in the original graph except that its connected to E(1,2) in the original graph in two different way. For example, in the new graph we can traverse V(”E(1,2)”) from V(3) to V(”E(1,2)”) than to V(4) but, in the original graph we would have to traverse E(1,2) via V(3) to V(1), to V(2), and to V(4) or from V(3) to V(2), to V(1), and to V(4). So, we have $r = 2((n-2)!/2)$=$(n-2)!$ and we can have our first important equation by introducing the formula for r into the equation for the average Hamiltonian tour cycle: $(2/(n-1)!)((n-2)! \sum W(e))$ [1].

$$H_{\text{avg}} = \frac{2}{n-1} \sum W(e) \tag{1}$$

where:

- H_{avg} is the average Hamiltonian cycle weight for all the unique Hamiltonian cycles contained in a complete graph.
- n is the number of vertices.
- $\sum W(e)$ is the sum of all the edge's weights.

[1] we have encounter equation 1 before in our research but we are unable to reference a specific author.

2.1 Example of Equation 1

Back in Figure 1 we can calculate H_{avg} using brute force, $((2+2\sqrt{2})+(2+2\sqrt{2})+(4))/3=(1/3)(8+4\sqrt{2})$, or using equation 1, $(2/3)(1+1+1+1+\sqrt{2}+\sqrt{2})=(1/3)(8+4\sqrt{2})$. although both method seem similar, but as n increase, equation 1 is able to calculate H_{avg} in polynomial time whereas the brute force method becomes untractable [3].

3 Relation of Eulerian cycles with Hamiltonian Cycles

Also very interesting about Equation 1, is that if we consider a complete graph as a really a complete directed graph where we consider that the weight of each edge is same for any direction we decide to traverse it and a complete directed graph we divide by 2 to cancel out all the Hamiltonian cycles that is the same as a cycle going the opposite direction. Then, in a complete graph, every vertex has an even degree because every edge has a parallel edge going the opposite direction. So, it follows from Euler's theorem that every complete graph has an Eulerian cycle and its weight is $\sum W(\vec{e})$.

For the rest of the paper, when we refer to a complete graph, we mean a complete undirected graph, and we only mention this definition of a complete graph because these graphs have $(n-1)!$ unique Hamiltonian cycles and all of their edges will repeat $(n-2)!$ times. The average Hamiltonian cycle weight these complete graphs will have is $\sum W(\vec{e})/(n-1)$, which is equal to Equation 1 when we say that the direction doesn't matter. But, from that definition of a complete graph we can say that in a complete graph the weight of the Eulerian cycle divided by n minus one is the same as all the Hamiltonian cycles divided by n minus one factorial. We are not going to need this information for the rest of this paper, but we thought readers of this paper might found this relation fascinating.

4 Average weight of all Hamiltonian cycles that pass through one edge

Equation 1 gives the average tour weight for all Hamiltonian cycles in a complete graph, but suppose that we wanted to know the average tour weight for all Hamiltonian cycles that pass through a specific edge or $H_{\text{avgE}\mathit{i,j}}$. We can use our thought experiment and Figure 2 again to find $H_{\text{avgE}\mathit{1,2}}$, which is the average tour weight for all unique Hamiltonian cycles that pass through E(1,2).

4.1 Classification of edges

If we were to write out all the unique Hamiltonian cycles that contain E(1,2), then we would realize that some edges repeat more than others. In fact, we can categorize them into three categories, chosen edges (CEs), neighboring edges (NEs) and independent edges (IEs). **Chosen edges CEs**, are the edges that the Hamiltonian cycles must pass through, in this case simply E(1,2), but in general we could have asked what the average Hamiltonian cycle weight for all Hamiltonian cycles in our special graph that pass through E(1,2), E(4,5), ... in the same cycle or $H_{\text{avgE}\mathit{1,2}\text{E}\mathit{4,5....}}$, then the CEs would be edges E(1,2), E(4,5),... We could have also chosen to calculate $H_{\text{avgE}\mathit{1,2}\text{E}\mathit{1,3}}$ and have the CEs as E(1,2) and E(1,3), but we should not say their are two edges for the CEs rather one, E(2,1,3) and that we are calculating $H_{\text{avgE}\mathit{2,1,3}}$, as the number of the CEs will be important in the future. We can also define $\sum W(ce)$ as the sum of the weight all the chosen edges.

We shouldn't say $H_{\text{avgE}\mathit{1,2}\text{E}\mathit{1,5}\text{E}\mathit{4,5}}$, rather $H_{\text{avgE}\mathit{2,1,5,4}}$ because you can only visit a vertex once in a Hamiltonian cycle, that is the same reason $H_{\text{avgE}\mathit{1,5,2}\text{E}\mathit{3,5}}$ is zero. It doesn't make sense to ask what is the average Hamiltonian cycle weight for all Hamiltonian cycles in our graph that contains E(1,5,2) and E(3,5) because V(5) is being visited twice. Also, when calculating $H_{\text{avgE}\mathit{2,1,5,4}}$, V(1) and V(5) can't be part of our neighboring and independent edges because all the cycles we considering must pass through the CEs and V(1) and V(5) was already visited in the CEs. So, if we have two CEs that share an adjacent vertex we must consider them as really one edge and the adjacent vertex or certain cases vertices must be excluded from the neighboring and independent edges.

Neighboring edges NEs, are edges that are not chosen nor do they included any of the excluded vertices, but are adjacent to any of the chosen edges. Since in our example the chosen edge is simply E(1,2) the NEs from Figure 2 are E(1,3), E(1,4), E(1,5), E(1,6), E(1,7), E(2,3), E(2,4), E(2,5), E(2,6), and E(2,7). But for example, we could have had the chosen edges as E(1,2) and E(3,7,4) then the new NEs would have been E(1,5), E(1,6), E(2,5), E(2,6), E(3,5), E(3,6), E(4,5), and E(4,6). Notice we exclude V(7) in the NEs because it was already visited in the CE, E(3,7,4). In addition, we can define $\sum W(ne)$ as the sum of the weight of all the neighboring edges.

Finally, **independent edges IEs**, are the edges that are not chosen, does not include any of the exclude vertices, and are not adjacent to any of the chosen edges. With E(1,2) as chosen, the IEs are E(3,4), E(3,5), E(3,6), E(3,7), E(4,5), E(4,6), E(4,7), E(5,6), E(5,7), and E(6,7), with no excluded vertices in this case. Also, we can define $\sum W(ie)$ as the sum of the weight of all the independent edges.

4.2 Repetition of edges

Now we simply have to figure out how many times the CEs, the NEs, and the IEs repeat in all the Hamiltonian cycles that pass through E(1,2) and we will be able to calculate the average tour weight for all the Hamiltonian cycles that pass through E(1,2) or $H_{\text{avgE}\mathit{1,2}}$. If we let t, q, and p be the number of times the CEs, NEs, and IEs repeats respectively then the math look like this: $(1/(n-2)!)(t\sum W(ce)+q\sum W(ne)+p\sum W(ie))$. the equation is divided $(n-2)!$ because that is the total number of unique Hamiltonian cycles that pass through E(1,2), as we established earlier each edge repeat $(n-2)!$ times across all possible unique Hamiltonian cycles for a complete graph of n vertices.

The amount of times the CEs repeats, t, is trivial as it is the same amount as the total of all the Hamiltonian cycles we are averaging since all the Hamiltonian cycles must pass through the CEs. In the case where we are looking for the average tour weight that must pass through a single edge like E(1,2), the CE will repeat $(n-2)!$ times. In the case with more than one CE or with excluded vertices, it is a bit more complicated as we will see later.

To calculate how many times the NEs repeats or q, we must use the thought experiment again. In our special graph at Figure 2, we have E(1,2) as the CE so E(2,6) could be one of the NEs. since each of the NEs repeats the same amount of time, all we got to do is calculated how many unique Hamiltonian cycles possible with E(1,2) and E(2,6) or E(1,2,6). Using the thought experiment again, we can imagine V(1), V(2), and V(6) as very close to each other and we can imagine all the other vertices in our special graph as far away as possible from V(1), V(2), and V(6). Then we can consider E(1,2,6) as V("E(1,2,6)") to make a new graph with $(1/2)(n-3)!$ unique Hamiltonian cycles. We use minus 3 because we lost two vertices when we considered E(1,2,3) as V("E(1,2,6)"). But we know that all the Hamiltonian cycles in the new graph are the same as all the Hamiltonian cycles in the original graph in Figure 2, that contain E(1,2,6) except that the original graph has twice the amount Hamiltonian cycles because each of the cycles from the new graph that pass through V("E(1,2,6)") can approach E(1,2,6) from the original graph in two different ways, Via E(1,2,6) or E(6,2,1). Therefore we have $(2)(1/2)(n-3)!$ which is equal to $(n-3)!$ as the total amount of Hamiltonian cycles that pass through E(1,2,6) or as the amount of times the NEs will repeat when we have one CE and no excluded vertices.

Similarly, if we wanted to calculate how many times the IEs will repeat when we are calculating $H_{\text{avgE}\mathit{1,2}}$, then all we have to do is calculate how many unique Hamiltonian cycles we can make from the graph that pass through E(1,2) and any other IE in the same cycle, as all the IEs repeat the same amount of time. So, we can pick E(4,7) as an IE and begin calculating for the total amount of unique possible Hamiltonian cycles that pass through E(1,2) and E(4,7) in the same cycle. We can use the thought experiment again but instead of imagining a group of vertices close to each other, we must imagine now two different group of vertices that have each vertices of their group close to each other while far away from each other and all other vertices in the graph. So that E(1,2) and E(4,7) becomes V("E(1,2)") and V("E(4,7)") to give a new graph with $(1/2)(n-3)!$ unique Hamiltonian cycles. Again minus three because we lost two vertices when we consider the two edges as two vertices.

Unlike before, the new graph has four times less the amount of Hamiltonian cycles from all the Hamiltonian cycles in the original graph that pass through E(1,2) and E(4,7) because each of the two imagine vertices could be crossed in two different ways when being visited as an edge in the original graph. Thus, we have the value for p or the amount of times the IE will repeat as $(2)(2)(1/2)(n-3)!$ which is $2(n-3)!$

Now that we have the values for t, q, and p as $(n-2)!$, $(n-3)!$, and $2(n-3)!$ respectively we can plug them into our equation for $H_{\text{avgE}\mathit{1,2}}$ to get $((1/(n-2)!)((n-2)!\sum W(ce)+(n-3)!\sum W(ne)+2(n-3!)\sum W(ie))$ which is $\sum W(ce)+(1/n-2)\sum W(ne)+(2/n-2)\sum W(ie)$. Notice that this equation will work every-time we are calculating the average Hamiltonian cycle weight with one CE and no excluded vertices. So that if we let W(i,j) be the weight of E(i,j) we get our second important equation:

$$H_{\text{avgE}\mathit{i,j}} = W(i,j) + \frac{1}{n-2}\sum W(ne) + \frac{2}{n-2}\sum W(ie) \tag{2}$$

where:

- $H_{\text{avgE}\mathit{i,j}}$ is the average Hamiltonian cycle weight for all Hamiltonian cycles passing through E(i,j).
- n is the number of vertices.
- W(i,j) is the weight of E(i,j).
- $\sum W(ne)$ is the sum of all the weights of the neighboring edges.
- $\sum W(ie)$ s the sum of all the weights of the independent edges.

4.3 Example of equation 2

If we wanted to explicitly write the equation for $H_{\text{avgE}\mathit{1,2}}$ in Figure 2, it would look something like this: $H_{\text{avgE}\mathit{1,2}} = W(1,2) + (1/n-2)(W(1,3) + W(1,4)+W(1,5)+W(1,6)+W(1,7)+W(2,3)+W(2,4)+W(2,5)+W(2,6)+W(2,7))+(2/n-2)(W(3,4)+W(3,5)+W(3,6)+W(3,7)+W(4,5)+W(4,6)+W(4,7)+W(5,6)+W(5,7)+W(6,7))$. We could have plug in 7 for n since the special graph in Figure 2 has 7 vertices but you get the idea. If we wanted to calculate $H_{\text{avgE}\mathit{1,2}}$ using brute force instead, we would first have to determine and calculate all the 120 Hamiltonian cycles that include E(1,2), then add up their weights to divide by 120. It should be obvious that Equation 2 is far more efficient than the brute force method as it calculate the average in polynomial time.

5 Average weight of all Hamiltonian cycles that pass through more than one edge in a cycle

Now we can ask, what is the general formula for the average Hamiltonian cycle weight that must pass through any amount of CE with any amount of excluded vertices. In other words, what is H_{avgCEs}? To answer this question, we must now introduce a new type of edge that is edges that are adjacent to two of the CEs, connecting the two CEs. We can call them **connecting chosen edges CCEs** and the sum of all their weights as $\sum W(cce)$. Also, we must now keep track of how many CEs we have and how many vertices are there that formed the CEs and denote them by $|CE|$ and $|VCE|$ respectively. So that, if the CEs are E(1,2,5) and E(3,7) then the CCEs would be E(1,3), E(1,7), E(5,3) and E(5,7) because these are the edges connecting the two CEs, $|CE|$ would be 2 because there are only two CEs and $|VCE|$ would be 5 because the five vertices, V(1), V(2), V(3), V(5), and V(7) are all used to form the CEs. With the CCEs and the variables $|CE|$ and $|VCE|$ defined, we can now calculate the total number of unique Hamiltonian cycles that pass through any amount of CEs with any amount of excluded edges in one cycle or, in other words, we can now calculate the total amount of unique Hamiltonian cycles which we plan to average.

5.1 The total number of Hamiltonian cycles that pass through any amount of chosen edges in one cycle

Beginning with the original complete graph, there would be $(1/2)(n-1)!$ unique Hamiltonian cycles, but as we apply the techniques from the thought experiment to calculate just the unique Hamiltonian cycles that pass through the CEs in one cycle, this equation would have to be modified. First of all, when we consider each of the CEs as a vertex and imagine a new graph with fewer vertices, then the new imaginary graph has $(1/2)(n-1+|CE|-|VCE|)!$ unique Hamiltonian Cycles. We simply took out all the vertices that formed the CEs (-$|VCE|$) and replace them by a single vertex for each of their CEs (+$|CE|$). But, this imaginary graph is the same as all the Unique Hamiltonian cycles from the original graph that pass through the CEs in one cycle except that each of the CEs could be cross two different ways when being visited as an edge in the original graph. So we must multiply the equation by $2^{|CE|}$ to get $2^{(|CE|-1)}(n-1+|CE|-|VCE|)!$ as the equation for the total amount of Hamiltonian cycles that pass through the CEs, which we will call $\Omega(n,|CE|,|VCE|)$ [2]

5.2 Repetition of the CEs, NEs, IEs, and CCEs

Now to calculate the average tour weight for all the tours that pass through any CEs with any excluded vertices in one cycle or H_{avgCEs}, we simply have

[2]it should be obvious that $\Omega(n,|CE|,|VCE|)$ is not computable when n is too large, but it can be approximated using Stirling's formula.

to calculate how many times each type of edge repeat, so that if a, b, c, and d are how many times the CEs, CCEs, NEs, and IEs repeat respectively, then the equation would look like this: $((1/\Omega(n, |CE|, |VCE|))(a \sum(ce) + b \sum(cce) + c \sum W(ne) + d \sum W(ie))$. The value for a is trivial because each Hamiltonian cycle we are considering must pass through the CEs, Therefore the CEs will repeat the same amount of time as the total amount of unique Hamiltonian Cycles that pass through the CEs, which is $\Omega(n, |CE|, |VCE|)$ or $2^{(|CE|-1)}(n-1+|CE|-|VCE|)!$ times.

The value for b is the same as the amount of unique Hamiltonian cycles that pass through the CEs and any CCE in one cycle, since all the CCEs will repeat the same amount of times. So, when we do the thought experiment one CE will be lost because of the CCE connecting two CEs to form one CE and our equation is the same as $\Omega(n, |CE|, |VCE|)$ except that we have to subtract one from $|CE|$ to get $b = \Omega(n, |CE|-1, |VCE|) = 2^{(|CE|-2)}(n-2+|CE|-|VCE|)!$.

The amount of time each NE will repeat is the same as the amount of unique Hamiltonian cycles there is which pass through the CEs and any of the NEs in one cycle, as all the NEs will repeat the same amount of times. The equation is the same as the equation for $\Omega(n, |CE|, |VCE|)$ except that we are adding a vertex to one of the CEs to form a NE. So, the value for $|VCE|$ is increase by one while the value for $|CE|$ stays the same which mean $c = \Omega(n, |CE|, |VCE|+1) = 2^{(|CE|-1)}(n-2+|CE|-|VCE|)!$.

The value for d is the total amount of Hamiltonian cycles that pass through the CEs and any of the IEs in one cycle because all the IEs will repeat the same amount of times. But when the thought experiment is done, The equation is the same as the equation for the total amount of unique Hamiltonian cycles that pass through the CEs or $\Omega(n, |CE|, |VCE|)$ except that one CE was added to account for the IE and $|VCE|$ was increase by two from the two vertices in the IE which is use to form the new CE. Therefore $d = \Omega(n, |CE|+1, |VCE|+2) = 2^{|CE|}(n-2+|CE|-|VCE|)!$.

Finally, to solve for H_{avgCEs} we simply have to plug in the values for a, b, c, and d to get $H_{\text{avgCEs}} = (1/\Omega(n, |CE|, |VCE|))(\Omega(n, |CE|, |VCE|) \sum(ce) + \Omega(n, |CE|-1, |VCE|) \sum(cce) + \Omega(n, |CE|, |VCE|+1) \sum W(ne) + \Omega(n, |CE|+1, |VCE|+2) \sum W(ie))$. We can simplify to get our third equation:

$$H_{\text{avgCEs}} = \sum W(ce) + \frac{1}{2(n-1+|CE|-|VCE|)} \sum W(cce) + \frac{1}{(n-1+|CE|-|VCE|)} \sum W(ne) + \frac{2}{(n-1+|CE|-|VCE|)} \sum W(ie) \tag{3}$$

where:

- H_{avgCEs} is the average Hamiltonian cycle weight for all the the unique Hamiltonian cycles that include any CEs.
- Chosen Edges CES are the edges that the Hamiltonian cycles must pass through each time.
- $|CE|$ is the amount of CEs.
- $|VCE|$ is the amount of vertices that formed the CEs.
- n is the number of vertices for the complete graph.
- $\sum W(ce)$ is the sum of all the weight of the chosen edges, CEs.
- $\sum W(cce)$ is the sum of all the weight of the connecting chosen edges, CCEs.
- $\sum W(ne)$ is the sum of all the weight of the neighboring edges, NEs.
- $\sum W(ie)$ is the sum of all the weight of the independent edges, IEs.

5.3 Example of equation 3

For example lets say there is a complete graph with 10 vertices and we wanted to know what is the average Hamiltonian cycle weight of all the cycles that passes through E(3,4), E(5,8), and E(4,9) in one cycle. Then, all we got to do is solve for $H_{\text{avgE}\mathit{3,4,9}\text{E}\mathit{5,8}}$ with n=10, $|CE|$=2, and $|VCE|$=5. Using Equation 3 our math should look like this: $H_{\text{avgE}\mathit{3,4,9}\text{E}\mathit{5,8}} = W(3,4,9) + W(5,8) + (1/12)(W(3,5) + W(3,8) + W(5,9) + W(8,9)) + (1/6)(W(3,1) + W(3,2) + W(3,6) + W(3,7) + W(3,10)+W(9,1)+W(9,2)+W(9,6)+W(9,7)+W(9,10)+W(5,1)+W(5,2)+ W(5,6)+W(5,7)+W(5,10)+W(8,1)+W(8,2)+W(8,6)+W(8,7)+W(8,10))+ (1/3)(W(1,2) + W(1,6) + W(1,7) + W(1,10) + W(2,6) + W(2,7) + W(2,10) + W(6,7) + W(6,10) + W(7,10))$.

If we wanted to compute the same thing using brute force, first we would have to identify all the Hamiltonian cycles that passes through E(3,4,9) and E(5,8) in one cycle from all the other Hamiltonian cycles that doesn't. Since we have an equation, $\Omega(n, |CE|, |VCE|)$, that can compute how many of those cycles their would be, we can calculate it. With n=10, $|CE|$=2, and $|VCE|$=5, there would have been 1,440 of such cycles whose weight we would have to add up and then divide by 1,440 to get $H_{\text{avgE}\mathit{3,4,9}\text{E}\mathit{5,8}}$. It should be obvious that Equation 3 is far more superior than using the brute force method.

6 Standard deviation of all Hamiltonian cycles weight in a complete graph

Another important quantity that we can calculate in polynomial time is the standard deviation σ, of all the Hamiltonian cycles weight in a complete graph,

H_σ and the standard deviation of all the Hamiltonian cycles weight that only pass through the CEs, $H_{\sigma\text{CEs}}$. The computation is similar to the computation for the averages but a little more complicated.

6.1 Classification and repetition of every product for the variance of all Hamiltonian cycles weight in a complete graph

To compute H_σ, we have to use Equation 1 for the average Hamiltonian cycle weight of all the Hamiltonian cycles in a complete graph. So,we get $(H_\sigma)^2 = \sum_{j=1}^{(1/2)(n-1)!}[(2/(n-1)!)((2/n-1)\sum W(e) - W(hc)_j)^2]$, where $W(hc)_j$ is the jth Hamiltonian cycle weight out of the $(1/2)(n-1)!$ total cycles. If we were to foil out this equation, we would find that we would end up with four different types of products: an edge times itself $W(e)^2$ and an edge times another edge $W(e)_1 * W(e)_2$. But in all the Hamiltonian cycles of a complete graph two adjacent edge and two independent edge in a cycle will repeat at different amount of times, that we should get two different products from $(-W(hc)_j) * (-W(hc)_j)$ when foiling $(H_\sigma)^2$. So that, the third and fourth products are an edge times another edge that its neighbor $W(e)_1 * W(e)_{2_n}$, meaning they are adjacent and an edge times another edge that is independent of it $W(e)_1 * W(e)_{2_i}$. We can also take the sum of every pair product possible for each type of product and denote them by $\sum W(e)^2$, $\sum W(e)_1 * W(e)_2$, $\sum W(e)_1 * W(e)_{2n}$, and $\sum W(e)_1 * W(e)_{2i}$, where $\sum W(e)_1 * W(e)_{2n} + \sum W(e)_1 * W(e)_{2i} = \sum W(e)_1 * W(e)_2$.

Now as we foil the equation for $(H_\sigma)^2$, $\sum_{j=1}^{(1/2)(n-1)!}[(2/(n-1)!)((2/n-1)\sum W(e) - W(hc)_j)^2]$, we see that there will be a $(4/(n-1)^2)\sum W(e)^2$ product from $(2/n-1)\sum W(e) * (2/n-1)\sum W(e)$ and that product will occur $(1/2)(n-1)!$ times since each difference we take between the average and an jth Hamiltonian cycle will have the $(2/n-1)\sum W(e)$ factor. Also we will get a $-(4/(n-1))\sum W(e)^2$ product from $(2/n-1)\sum W(e) * (-W(hc)_j)$ and that product will occur $(n-2)!$ times as it is the amount of time an edge will appear in all of the $-W(hc)_j$ factors. Finally, there will be a product of $\sum W(e)^2$ for every product of $(-W(hc)_j) * (-W(hc)_j)$ and that product will occur $(n-2)!$ times as it is the amount of time an edge will appear in the factors of $(-W(hc)_j$.

As we continue to foil $\sum_{j=1}^{(1/2)(n-1)!}[(2/(n-1)!)((2/n-1)\sum W(e) - W(hc)_j)^2]$, there will be a product of $(8/(n-1)^2)\sum W(e)_1 * W(e)_2$ from $(2/n-1)\sum W(e) * (2/n-1)\sum W(e)$ and it will occur $(1/2)(n-1)!$ since that is how many Hamiltonian cycles we have to subtract from the average. There will also be a product of $2(-4/n-1)\sum W(e)_1 * W(e)_2$ from $(2/n-1)\sum W(e) * (-W(hc)_j)$. The product is multiply by two because you can get the same product in two different ways, from multiplying E_1 from one of the jth Hamiltonian cycles to E_2 from the average formula or from multiplying E_2 from one of the jth Hamiltonian cycles to E_1 from the average formula.

There will be a product of $2\sum W(e)_1 * W(e)_{2_n}$ from $(-W(hc)_j) * (-W(hc)_j)$ and that product will occur $(n-3)!$ times or the total amount of Hamiltonian cycles there are that pass through two adjacent edges. We could also have calculated that using $\Omega(n, |CE|, |VCE|)$, with $|ce|$=1, $|vce|$=3, and n remain unspecified. the final product will be $2\sum W(e)_1 * W(e)_{2_i}$ from $(-W(hc)_j) * (-W(hc)_j)$ and that product will occur $2(n-3)!$ times. The amount of time it occur can be confirm using the formula $\Omega(n, |CE|, |VCE|)$ with $|CE|$=2, $|VCE|$=4, and n remain unspecified.

We can mathematically summarize the work we just did in the last four paragraphs by having $\sum_{j=1}^{(1/2)(n-1)!}[(2/(n-1)!)((2/n-1)\sum W(e) - W(hc)_j)^2] = (2/(n-1)!)(\sum W(e)^2((4/(n-1)^2)((n-1)!/2) - (4/n-1)(n-2)! + (n-2)!) + \sum W(e)_1 * W(e)_2((8/(n-1)^2)((n-1)!/2) - (8/n-1)(n-2)!) + \sum W(e)_1 * W(e)_{2_n}(2(n-3)!) + \sum W(e)_1 * W(e)_{2_i}(4(n-3)!)) = (H_\sigma)^2$. We can simplify to get $(H_\sigma)^2 = \sum W(e)^2((-4/(n-1)^2) + (2/n-1)) + \sum W(e)_1 * W(e)_2((-8/(n-1)^2)) + \sum W(e)_1 * W(e)_{2_n}(4/(n-1)(n-2)) + \sum W(e)_1 * W(e)_{2_i}(8/(n-1)(n-2))$. Our simplify equation for $(H_\sigma)^2$ is computable in polynomial time but we can rewrite it in an even simpler form.

Using the fact that $(\sum W(e))^2 = \sum W(e)^2 + 2(\sum W(e)_1 * W(e)_2)$ we get $\sum W(e)_1 * W(e)_2 = (1/2)(\sum W(e))^2 - (1/2)\sum W(e)^2$. Now we can substitute $(1/2)(\sum W(e))^2 - (1/2)\sum W(e)^2$ for $\sum W(e)_1 * W(e)_2$ to get $(H_\sigma)^2 = \sum W(e)^2(2/n-1) + (4/(n-1)(n-2))(\sum W(e)_1 * W(e)_{2_n} + 2\sum W(e)_1 * W(e)_{2_i}) - (\sum W(e))^2(4/(n-1)^2)$, but $(\sum W(e))^2(4/(n-1)^2) = (H_{\text{avg}})^2$. Therefore we can write our next important equation in two different version:

$$\begin{aligned}(H_\sigma)^2 &= (\frac{-4}{(n-1)^2} + \frac{2}{n-1})\sum W(e)^2 + \frac{-8}{(n-1)^2}\sum W(e)_1 * W(e)_2 \\ &+ \frac{4}{(n-1)(n-2)}\sum W(e)_1 * W(e)_{2_n} + \frac{8}{(n-1)(n-2)}\sum W(e)_1 * W(e)_{2_i} \\ &= (\frac{2}{n-1})\sum W(e)^2 + (\frac{4}{(n-1)(n-2)})(\sum W(e)_1 * W(e)_{2_n} + 2\sum W(e)_1 * W(e)_{2_i}) - (H_{\text{avg}})^2\end{aligned} \tag{4}$$

where:

- H_σ is the standard deviation of the weight of all unique Hamiltonian cycles in a complete graph.
- n is the number of vertices in the compete graph.
- $\sum W(e)^2$ is the sum of all the products of every edge and themselves.
- $\sum W(e)_1 * W(e)_2$ is the sum of all the products of every edge and another edge.
- $\sum W(e)_1 * W(e)_{2_n}$ is the sum of all the products of every edge and their neighboring edges.

- $\sum W(e)_1 * W(e)_{2_i}$ is the sum of all the products of every edge and of every edge independent of them.

- H_{avg} is the average Hamiltonian cycle weight of all the Hamiltonian cycles in a complete graph.

6.2 Example of equation 4

For example, in Figure 1 we have already established that $H_{\text{avg}} = (1/3)(8 + 4\sqrt{2})$, n=4, and we know $\sum W(e)_1 * W(e)_{2_n} + \sum W(e)_1 * W(e)_{2_i} = \sum W(e)_1 * W(e)_2$ then we can use either version of Equation 4 under the assumption that the weight of $E(i,j)$ is denoted by $W(i,j)$ which is the same as $W(j,i)$, the math to calculate H_σ, $\sum W(e)^2$, $\sum W(e)_1 * W(e)_2$, $\sum W(e)_1 * W(e)_{2_n}$, and $\sum W(e)_1 * W(e)_{2_i}$ looks like this: $\sum W(e)^2 = W(1,2)^2 + W(1,3)^2 + W(1,4)^2 + W(2,3)^2 + W(2,4)^2 + W(3,4)^2 = 1+2+1+1+2+1 = 8$, $\sum W(e)_1 * W(e)_{2_n} = (W(1,2))(W(1,3)) + (W(1,2))(W(1,4)) + (W(1,3))(W(1,4)) + (W(2,1))(W(2,3)) + (W(2,1))(W(2,4)) + (W(2,3)(W(2,4)) + (W(3,1))(W(3,2)) + (W(3,1))(W(3,4)) + (W(3,2))(W(3,4)) + (W(4,1))(W(4,2)) + (W(4,1))(W(4,3)) + (W(4,2))(W(4,3)) = \sqrt{2} + 1 + \sqrt{2} + 1 + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + 1 + \sqrt{2} + 1 + \sqrt{2} = 4 + 8\sqrt{2}$, $\sum W(e)_1 * W(e)_{2_i} = (W(1,2))(W(3,4)) + (W(1,3))(W(2,4)) + (W(1,4))(W(2,3)) = 1 + 2 + 1 = 4$, and $\sum W(e)_1 * W(e)_2 = 4 + 8\sqrt{2} + 4 = 8 + 8\sqrt{2}$. Now we simply have to plug the values for n, H_{avg}, $\sum W(e)^2$, $\sum W(e)_1 * W(e)_2$, $\sum W(e)_1 * W(e)_{2_n}$, and $\sum W(e)_1 * W(e)_{2_i}$ into either version of Equation 4 to get $(H_\sigma)^2 = (2/9)(8) - (8/9)(8 + 8\sqrt{2}) + (2/3)(4 + 8\sqrt{2}) + (4/3)(4) = (2/3)(8) + (2/3)(12 + 8\sqrt{2}) - ((1/3)(8 + 4\sqrt{2}))^2 = (8/3) - (16/9)\sqrt{2}$ or $H_\sigma = \sqrt{(8/3) - (16/9)\sqrt{2}}$.

7 Standard deviation of all the weight of Hamiltonian cycles that pass through the CEs in one cycle

As we mentioned before, the standard deviation of the weight of all the unique Hamiltonian cycles that pass through the CEs in one cycle, $H_{\sigma\text{CEs}}$, can be calculated in polynomial time, but we have to use Equation 3 for the average of all the unique Hamiltonian cycles that pass through the CEs in one cycle. So, $(H_{\sigma\text{CEs}})^2 = \sum_{j=1}^{\Omega(n,|CE|,|VCE|)} [(1/\Omega(n,|CE|,|VCE|))(\sum W(ce) + (2(n-1+|CE|-|VCE|))^{-1} \sum W(cce) + (n-1+|CE|-|VCE|)^{-1} \sum W(ne) + 2(n-1+|CE|-|VCE|)^{-1} \sum W(ie)) - W(hce)_j)^2]$, where $W(hce)_j$ is the jth Hamiltonian cycle weight out of the $\Omega(n,|CE|,|VCE|)$ total unique Hamiltonian cycles that pass through the CEs in one cycle. To foil the equation for $H_{\sigma\text{CEs}}$ as we did for the equation H_σ will take several pages, so we summarize the work in Table 1 below:

Table 1: The foiling of $\sum_{j=1}^{\Omega(n,|CE|,|VCE|)}[(1/\Omega(n,|CE|,|VCE|))((\sum W(ce) + (2(n-1+|CE|-|VCE|))^{-1}\sum W(cce) + (n-1+|CE|-|VCE|)^{-1}\sum W(ne) + ((n-1+|CE|-|VCE|)/2)^{-1}\sum W(ie)) - W(hce)_j)^2]$

Products	**Factors**	**Occurrence Ω**
$(2(n-1+\lvert CE\rvert-\lvert VCE\rvert))^{-2}\sum W(cce)^2$	$(2(n-1+\lvert CE\rvert-\lvert VCE\rvert))^{-1}\sum(cce) * (2(n-1+\lvert CE\rvert-\lvert VCE\rvert))^{-1}\sum(cce)$	$2^{(\lvert CE\rvert-1)}(n-1+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert,\lvert VCE\rvert)$
$-(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-1}\sum W(cce)^2$	$(2(n-1+\lvert CE\rvert-\lvert VCE\rvert))^{-1}\sum(cce) * -W(hce)_j$	$2^{(\lvert CE\rvert-2)}(n-2+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert+1,\lvert VCE\rvert)$
$\sum W(cce)^2$	$-W(hce)_j * -W(hce)_j$	$2^{(\lvert CE\rvert-2)}(n-2+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert+1,\lvert VCE\rvert)$
$(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-2}\sum W(ne)^2$	$(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-1}\sum W(ne) * (n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-1}\sum W(ne)$	$2^{(\lvert CE\rvert-1)}(n-1+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert,\lvert VCE\rvert)$
$-2(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-1}\sum W(ne)^2$	$(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-1}\sum W(ne) * -W(hce)_j$	$2^{(\lvert CE\rvert-1)}(n-2+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert,\lvert VCE\rvert+1)$
$\sum W(ne)^2$	$-W(hce)_j * -W(hce)_j$	$2^{(\lvert CE\rvert-1)}(n-2+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert,\lvert VCE\rvert+1)$
$4(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-2}\sum W(ie)^2$	$2(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-1}\sum W(ie) * 2(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-1}\sum W(ie)$	$2^{(\lvert CE\rvert-1)}(n-1+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert,\lvert VCE\rvert)$
$-4(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-1}\sum W(ie)^2$	$2(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-1}\sum W(ie) * -W(hce)_j$	$2^{(\lvert CE\rvert)}(n-2+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert+1,\lvert VCE\rvert+2)$
$\sum W(ie)^2$	$-W(hce)_j * -W(hce)_j$	$2^{(\lvert CE\rvert)}(n-2+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert+1,\lvert VCE\rvert+2)$
$(1/2)(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-2}\sum W(cce)_1 * W(cce)_2$	$(2(n-1+\lvert CE\rvert-\lvert VCE\rvert))^{-1}\sum(cce) * (2(n-1+\lvert CE\rvert-\lvert VCE\rvert))^{-1}\sum(cce)$	$2^{(\lvert CE\rvert-1)}(n-1+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert,\lvert VCE\rvert)$
$-2(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-1}\sum W(cce)_1 * W(cce)_2$	$(2(n-1+\lvert CE\rvert-\lvert VCE\rvert))^{-1}\sum(cce) * -W(hce)_j$	$2^{(\lvert CE\rvert-2)}(n-2+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert-1,\lvert VCE\rvert)$
$2\sum W(cce)_1 * W(cce)_{2_n}$	$-W(hce)_j * -W(hce)_j$	0
$2\sum W(cce)_1 * W(cce)_{2_i}$	$-W(hce)_j * -W(hce)_j$	$2^{(\lvert CE\rvert-3)}(n-3+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert-2,\lvert VCE\rvert)$
$2(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-2}\sum W(ne)_1 * W(ne)_2$	$(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-1}\sum W(ne) * (n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-1}\sum W(ne)$	$2^{(\lvert CE\rvert-1)}(n-1+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert,\lvert VCE\rvert)$
$-4(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-1}\sum W(ne)_1 * W(ne)_2$	$(n-1+\lvert CE\rvert-\lvert VCE\rvert)^{-1}\sum W(ne) * -W(hce)_j$	$2^{(\lvert CE\rvert-1)}(n-2+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert,\lvert VCE\rvert+1)$
$2\sum W(ne)_1 * W(ne)_{2_n}$	$-W(hce)_j * -W(hce)_j$	$2^{(\lvert CE\rvert-2)}(n-3+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert-1,\lvert VCE\rvert+1)$
$2\sum W(ne)_1 * W(ne)_{2_i}$	$-W(hce)_j * -W(hce)_j$	$2^{(\lvert CE\rvert-1)}(n-3+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert,\lvert VCE\rvert+2)$

Products	**Factors**	**Occurrence Ω**
$8(n-1+\|CE\|-\|VCE\|)^{-2}\sum W(ie)_1 * W(ie)_2$	$2(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ie) * 2(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ie)$	$2^{(\|CE\|-1)}(n-1+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|,\|VCE\|)$
$-8(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ie)_1 * W(ie)_2$	$2(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ie) * -W(hce)_j$	$2^{(\|CE\|)}(n-2+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|+1,\|VCE\|+2)$
$2\sum W(ie)_1 * W(ie)_{2_n}$	$-W(hce)_j * -W(hce)_j$	$2^{(\|CE\|)}(n-3+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|+1,\|VCE\|+3)$
$2\sum W(ie)_1 * W(ie)_{2_i}$	$-W(hce)_j * -W(hce)_j$	$2^{(\|CE\|+1)}(n-3+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|+2,\|VCE\|+4)$
$(n-1+\|CE\|-\|VCE\|)^{-2}\sum W(cce) * W(ne)$	$(2(n-1+\|CE\|-\|VCE\|))^{-1}\sum(cce) * (n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ne)$	$2^{(\|CE\|-1)}(n-1+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|,\|VCE\|)$
$-(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(cce) * W(ne)$	$(2(n-1+\|CE\|-\|VCE\|))^{-1}\sum(cce) * -W(hce)_j$	$2^{(\|CE\|-1)}(n-2+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|,\|VCE\|+1)$
$-2(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(cce) * W(ne)$	$(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ne) * -W(hce)_j$	$2^{(\|CE\|-2)}(n-2+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|-1,\|VCE\|)$
$2\sum W(cce) * W(ne)_n$	$-W(hce)_j * -W(hce)_j$	0
$2\sum W(cce) * W(ne)_i$	$-W(hce)_j * -W(hce)_j$	$2^{(\|CE\|-2)}(n-3+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|-1,\|VCE\|+1)$
$2(n-1+\|CE\|-\|VCE\|)^{-2}\sum W(cce) * W(ie)_n$	$(2(n-1+\|CE\|-\|VCE\|))^{-1}\sum(cce) * 2(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ie)$	0
$-(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(cce) * W(ie)_n$	$(2(n-1+\|CE\|-\|VCE\|))^{-1}\sum(cce) * -W(hce)_j$	0
$-4(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(cce) * W(ie)_n$	$2(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ie) * -W(hce)_j$	0
$2\sum W(cce) * W(ie)_n$	$-W(hce)_j * -W(hce)_j$	0
$2(n-1+\|CE\|-\|VCE\|)^{-2}\sum W(cce) * W(ie)_i$	$(2(n-1+\|CE\|-\|VCE\|))^{-1}\sum(cce) * 2(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ie)$	$2^{(\|CE\|-1)}(n-1+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|,\|VCE\|)$
$-(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(cce) * W(ie)_i$	$(2(n-1+\|CE\|-\|VCE\|))^{-1}\sum(cce) * -W(hce)_j$	$2^{(\|CE\|)}(n-2+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|+1,\|VCE\|+2)$
$-4(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(cce) * W(ie)_i$	$2(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ie) * -W(hce)_j$	$2^{(\|CE\|-2)}(n-2+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|-1,\|VCE\|)$
$2\sum W(cce) * W(ie)_i$	$-W(hce)_j * -W(hce)_j$	$2^{(\|CE\|-1)}(n-3+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|,\|VCE\|+2)$
$4(n-1+\|CE\|-\|VCE\|)^{-2}\sum W(ne) * W(ie)$	$(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ne) * 2(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ie)$	$2^{(\|CE\|-1)}(n-1+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|,\|VCE\|)$
$-2(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ne) * W(ie)$	$(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ne) * -W(hce)_j$	$2^{(\|CE\|)}(n-2+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|+1,\|VCE\|+2)$
$-4(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ne) * W(ie)$	$2(n-1+\|CE\|-\|VCE\|)^{-1}\sum W(ie) * -W(hce)_j$	$2^{(\|CE\|-1)}(n-2+\|CE\|-\|VCE\|)!$ or $\Omega(n,\|CE\|,\|VCE\|+1)$

Products	Factors	Occurrence Ω
$2\sum W(ne) * W(ie)_n$	$-W(hce)_j * -W(hce)_j$	$2^{(\lvert CE\rvert-1)}(n-3+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert,\lvert VCE\rvert+2)$
$2\sum W(ne) * W(ie)_i$	$-W(hce)_j * -W(hce)_j$	$2^{(\lvert CE\rvert)}(n-3+\lvert CE\rvert-\lvert VCE\rvert)!$ or $\Omega(n,\lvert CE\rvert+1,\lvert VCE\rvert+3)$

Their is no products or factors involving the $\sum W(ce)$ in Table 1 because we are subtracting $W(hce)_j$ from $H_{\sigma\text{CEs}}$, but all unique Hamiltonian cycles that pass through the CEs in one cycle will have the $\sum W(ce)$ factor, so it will always cancel out from the $\sum W(ce)$ factor from $H_{\sigma\text{CEs}}$. Regardless, we could have added it in Table 1 and let it cancel out later naturally, but to save space in Table 1 we just omitted it. Also, the products in Table 1 with 0 as their occurrence and $-W(hce)_j * -W(hce)_j$ as their factors simply mean that their are no Hamiltonian cycles that pass through both factors that would make that product. The only product with 0 has Occurrence with factors other than $-W(hce)_j * -W(hce)_j$ is $\sum W(cce) * W(ie)_n$ because none of the CCEs and the IEs will be neighbors, in other words, none of the CCEs and the IEs will be adjacent by the definition we give them earlier in this paper.

To finish foiling $\sum_{j=1}^{\Omega(n,|CE|,|VCE|)}[(1/\Omega(n,|CE|,|VCE|)$ $((\sum W(ce)+(2(n-1+|CE|-|VCE|))^{-1}\sum W(cce)+(n-1+|CE|-|VCE|)^{-1}\sum W(ne)+$ $((n-1+|CE|-|VCE|)/2)^{-1}\sum W(ie))-W(hce)_j)^2]$ and to make the calculation easier to read, we can let $n-1+|CE|-|VCE| = \omega$ which means $\Omega(n,|CE|,|VCE|) = 2^{|CE|-1}\omega! = 2^{\omega-n+|VCE|}\omega!$. Then, we simply have to multiply each product by their occurrence from Table 1, divide them by $\Omega(n,|CE|,|VCE|)$, and simplify to obtain $(H_{\sigma\text{CEs}})^2 = ((-1/4\omega^2)+(1/2\omega))\sum W(cce)^2+((-1/\omega^2)+(1/\omega))\sum W(ne)^2+((-4/\omega^2)+(2/\omega))\sum W(ie)^2-(1/2\omega^2)\sum W(cce)_1 * W(cee)_2+(1/2\omega(\omega-1))\sum W(cce)_1 * W(cce)_{2_i} - (2/\omega^2)\sum W(ne)_1 * W(ne)_2 + (1/\omega(\omega-1))\sum W(ne)_1 * W(ne)_{2_n}+(2/\omega(\omega-1))\sum W(ne)_1 * W(ne)_{2_i}-(8/\omega^2)\sum W(ie)_1 * W(ie)_2+(4/\omega(\omega-1))\sum W(ie)_1 * W(ie)_{2_n}+(8/\omega(\omega-1))\sum W(ie)_1 * W(ie)_{2_i}-(1/\omega^2)\sum W(cce) * W(ne)+(1/\omega(\omega-1))\sum W(cce) * W(ne)_i+((-2/\omega^2)+(2/\omega(\omega-1)))\sum W(cce) * W(ie)_i-(4/\omega^2)\sum W(ne) * W(ie)+(2/\omega(\omega-1))\sum W(ne) * W(ie)_n+(4/\omega(\omega-1))\sum W(ne) * W(ie)_i$.

At this point the simplify equation for $(H_{\sigma\text{CEs}})^2$ is computable in polynomial time, but if we substitute $\sum W(cce)_1 * W(cce)_2$, $\sum W(ne)_1 * W(ne)_2$, and $\sum W(ie)_1 * W(ie)_2$ with $((1/2)(\sum W(cce))^2-(1/2)\sum W(cce)^2)$, $((1/2)(\sum W(ne))^2-(1/2)\sum W(ne)^2)$, and $((1/2)(\sum W(ie))^2-(1/2)\sum W(ie)^2)$ respectively, then we will be able to substitute $(1/4)\omega^{-2}(\sum W(cce))^2+\omega^{-2}(\sum W(ne))^2+4\omega^{-2}(\sum W(ie))^2+\omega^{-2}\sum W(cce) * W(ne)+2\omega^{-2}\sum W(cce) * W(ie)+4\omega^{-2}\sum W(ne) * W(ie)$ into the equation with $(H_{\text{avgCEs}}-\sum W(ce))^2$ and we get to write the 5th important equation in two different ways.

$$(H_{\sigma\text{CEs}})^2 = (-\frac{1}{4\omega^2}+\frac{1}{2\omega})\sum W(cce)^2+(-\frac{1}{\omega^2}+\frac{1}{\omega})\sum W(ne)^2+(-\frac{4}{\omega^2}+\frac{2}{\omega})\sum W(ie)^2$$
$$-\frac{1}{2\omega^2}\sum W(cce)_1 * W(cee)_2+\frac{1}{2\omega(\omega-1)}\sum W(cce)_1 * W(cce)_{2_i}-\frac{2}{\omega^2}\sum W(ne)_1 * W(ne)_2$$
$$+\frac{1}{\omega(\omega-1)}\sum W(ne)_1 * W(ne)_{2_n}+\frac{2}{\omega(\omega-1)}\sum W(ne)_1 * W(ne)_{2_i}-\frac{8}{\omega^2}\sum W(ie)_1 * W(ie)_2$$
$$+\frac{4}{\omega(\omega-1)}\sum W(ie)_1 * W(ie)_{2_n}+\frac{8}{\omega(\omega-1)}\sum W(ie)_1 * W(ie)_{2_i}-\frac{1}{\omega^2}\sum W(cce) * W(ne)$$
$$+\frac{1}{\omega(\omega-1)}\sum W(cce) * W(ne)_i + (-\frac{2}{\omega^2}+\frac{2}{\omega(\omega-1)})\sum W(cce) * W(ie)_i$$
$$-\frac{4}{\omega^2}\sum W(ne) * W(ie)+\frac{2}{\omega(\omega-1)}\sum W(ne) * W(ie)_n+\frac{4}{\omega(\omega-1)}\sum W(ne) * W(ie)_i$$
$$= \frac{1}{2\omega}\sum W(cce)^2 + \frac{1}{\omega}\sum W(ne)^2$$
$$+\frac{2}{\omega}\sum W(ie)^2$$
$$+\frac{1}{\omega(\omega-1)}(\frac{1}{2}\sum W(cce)_1 * W(cce)_{2_i} + \sum W(ne)_1 * W(ne)_{2_n}$$
$$+2\sum W(ne)_1 * W(ne)_{2_I}+4\sum W(ie)_1 * W(ie)_{2_n}+8\sum W(ie)_1 * W(ie)_{2_i}+\sum W(cce) * W(ne)_i$$
$$+2\sum W(cce) * W(ie)_i + 2\sum W(ne) * W(ie)_n + 4\sum W(ne) * W(ie)_i)$$
$$-(H_{\text{avgCEs}} - \sum W(ce))^2 \quad (5)$$

where:

- $H_{\sigma\text{CEs}}$ is the standard deviation of the weight of all unique Hamiltonian cycles in a complete graph that pass through the CEs in one cycle.
- Chosen Edges CES are the edges that the Hamiltonian cycles must pass through each time in one cycle and the sum of their weights are $\sum W(ce)$.
- $\omega = n - 1 + |CE| - |VCE|$
- $|CE|$ is the amount of CEs.
- $|VCE|$ is the amount of vertices that formed the CEs.
- n is the number of vertices for the complete graph.
- $\sum W(cce)^2$ is the sum of all the products of every connecting chosen edges CCEs and their selves.
- $\sum W(ne)^2$ is the sum of all the products of every neighboring edges NEs with their selves.

- $\sum W(ie)^2$ is the sum of all the products of every independent edges IEs with their selves.
- $\sum W(cce)_1 * W(cce)_2$ is the sum of all the products of each of the CCEs with another CCE.
- $\sum W(cce)_1 * W(cce)_{2_i}$ is the sum of all the products of each of the CCEs with another independent CCE.
- $\sum W(ne)_1 * W(ne)_2$is the sum of all the products of each of the NEs with another NE.
- $\sum W(ne)_1 * W(ne)_{2_n}$ is the sum of all the products of each of the NEs with another adjacent NE.
- $\sum W(ne)_1 * W(ne)_{2_i}$ is the sum of all the products of each of the NEs with another independent NE.
- $\sum W(ie)_1 * W(ie)_2$is the sum of all the products of each of the IEs with another IE.
- $\sum W(ie)_1 * W(ie)_{2_n}$ is the sum of all the products of each of the IEs with another adjacent IE.
- $\sum W(ie)_1 * W(ie)_{2_i}$ is the sum of all the products of each of the IEs with another independent IE.
- $\sum W(cce) * W(ne)$ is the sum of all the products of each of the CCEs with an NE.
- $\sum W(cce) * W(ne)_n$ is the sum of all the products of each of the CCEs with an adjacent NE.
- $\sum W(cce) * W(ne)_i$ is the sum of all the products of each of the CCEs with an independent NE.
- $\sum W(cce) * W(ie)_i$ is the sum of all the products of each of the CCEs with an independent IE.
- $\sum W(ne) * W(ie)$ is the sum of all the products of each of the NEs with an IE.
- $\sum W(ne) * W(ie)_n$ is the sum of all the products of each of the NEs with an adjacent IE.
- $\sum W(ne) * W(ie)_i$ is the sum of all the products of each of the NEs with an independent IE.
- H_{avgCEs} is the average Hamiltonian cycle weight of all the Hamiltonian cycles in a complete graph that pass through the CEs in one cycle.

7.1 Example of equation 5

Here is a real life example, a traveling salesman has to travel to 7 cities and end at the same city he starts in, but between city a where he lives, and city b lives his girlfriend who likes flowers sold only at a store between city e and city f and adventurous stories about his travel. So, he wants to travel a different Hamiltonian cycle each time he travels to get the most adventure possible, but every time his in city e he has to go to city f next to get the flowers which are on the way, and if his in city f first, then he has to go to city e next to get the flowers. However, before returning home, he will always make sure to stop in city b first so that he can see his girlfriend on the way home.

While planning his trips, the salesman creates a table for the cost between each city in Table 2 below, and he wants to know how much it will cost him to try all the Hamiltonian cycles between the seven cities that always have cycles that pass between city e and f and between city b and a in one trip. Furthermore, the salesman is curious about the standard deviation of the cost of all these trips. The salesman is smart enough to know that the cycles will not necessarily follow a normal distribution, but he wants an idea of how the cost of those trips will fluctuate.

Table 2: Cost between each cities

ab 1	ag 3	bg 2	de 3
ac 3	bc 4	cd 1	df 1
ad 5	bd 9	ce 2	dg 4
ae 7	be 3	cf 4	ef 1
af 4	bf 1	cg 5	eg 3
fg 3			

Table 3: Useful information

CEs	E(a,b) and E(e,f)
CCEs	E(a,e), E(a,f), E(b,e), and E(b,f)
NEs	E(a,c), E(a,d), E(a,g), E(b,c), E(b,d), E(b,g), E(e,c), E(e,d), E(e,g), E(f,c), E(f,d), and E(f,g)
IEs	E(c,d), E(c,g), and E(d,g)

To calculate the cost of his trips, the salesman simply has to find the average cost of all the trips and multiply it by the total number of trips he has to take. The total number of trips he has to take can be calculated using $\Omega(n, |CE|, |VCE|)$, with $n = 7$, $|CE| = 2$, and $|VCE| = 4$ to obtain 48. The average cost of his trips is $H_{\text{avgEa,bEe,f}} = W(a,b) + W(e,f) + (1/8)(W(a,e) + W(a,f) + W(b,e) + W(b,f)) + (1/4)(W(a,c) + W(a,d) + W(a,g) + W(b,c) + W(b,d) + W(b,g) + W(e,c) + W(e,d) + W(e,g) + W(f,c) + W(f,d) + W(f,g)) + (1/2)(W(c,d) + W(c,g) + W(d,g)) = 155/8$ from Equation 3. That puts his cost at $48(155/8) = 930$ of whatever units he is measuring his cost.

Calculating the standard deviation of all Hamiltonian cycle trips that pass between city a and b and between city e and f in one cycle is the same as calculating $H_{\sigma\text{Ea,bEe,f}}$. We can use the second version of Equation 5 with $\sum W(cce)^2 = (W(a,e))^2 + (W(a,f))^2 + (W(b,e))^2 + (W(b,f))^2 = 75$, $\sum W(ne)^2 = (W(a,c))^2 +$

$(W(a,d))^2 + (W(a,g))^2 + (W(b,c))^2 + (W(b,d))^2 + (W(b,g))^2 + (W(e,c))^2 + (W(e,d))^2 + (W(e,g))^2 + (W(f,c))^2 + (W(f,d))^2 + (W(f,g))^2 = 192$, and $\sum W(ie)^2 = (W(c,d))^2 + (W(c,g))^2 + (W(d,g))^2 = 42$.

$\sum W(cce)_1 * W(cce)_{2_i} = 0$ because there is no sum product from $-W(hce)_i * -W(hce)_i$ since there is no cycle with both E(a,e) and E(b,f) because it would make a non-Hamiltonian cycle with the CEs. In fact, all pair factors that made up the sum products in the second version of Equation 5 must be included in a Hamiltonian cycle that pass through the CEs in one cycle. If they are not in a Hamiltonian cycle that pass through the CEs in one cycle, then their sum is zero. For a similar reason $\sum W(cce)_1 * W(cce)_{2_n} + \sum W(cce)_1 * W(cce)_{2_i}$ does not necessarily equal $\sum W(cce)_1 * W(cce)_2$ and that is why the second version of equation 5 is more efficient to use since we already calculated $H_{\text{avgEa,bEe,f}}$, rather than using the first version where we would have to do extra work to calculate $\sum W(cce)_1 * W(cce)_2$. Regardless both version should calculate $H_{\sigma \text{E}a,b\text{E}e,f}$ in polynomial time.

To continue $\sum W(ne)_1 * W(ne)_{2_n} = (W(a,c)*(W(e,c)+W(a,c)*(W(f,c)+W(a,d)*(W(e,d)+W(a,d)*(W(f,d)+W(a,g)*(W(e,g)+W(a,g)*(W(f,g)+W(b,c)*(W(e,c)+W(b,c)*(W(f,c)+W(b,d)*(W(e,d)+W(b,d)*(W(f,d)+W(b,g)*(W(e,g)+W(b,g)*(W(f,g) = 128$, $\sum W(ne)_1 * W(ne)_{2_i} = W(a,c) * W(b,d) + W(a,c) * W(e,d) + W(a,c) * W(f,d) + W(a,c) * W(b,g) + W(a,c) * W(e,g) + W(a,c) * Wf,g) + W(a,d) * W(b,c) + W(a,d) * W(e,c) + W(a,d) * W(f,c) + W(a,d) * W(b,g) + W(a,d) * W(e,g) + W(a,d) * W(f,g) + W(a,g) * W(b,d) + W(a,g) * W(e,d) + W(a,g) * W(f,d) + W(a,g) * W(b,c) + W(a,g) * W(e,c) + W(a,g) * W(f,c) + W(b,c) * W(e,d) + W(b,c) * W(f,d) + W(b,c) * W(e,g) + W(b,c) * W(f,g) + W(b,d) * W(e,c) + W(b,d) * W(f,c) + W(b,d) * W(e,g) + W(b,d) * W(f,g) + W(b,g) * W(e,c) + W(b,g) * W(f,c) + W(b,g) * W(e,d) + W(b,g) * W(f,d) + W(e,c) * W(f,d) + W(e,c) * W(f,g) + W(e,d) * W(f,c) + W(e,d) * W(f,g) + W(e,g) * W(f,c) + W(e,g) * W(f,d) = 434$, $\sum W(ie)_1 * W(ie)_{2_n} = W(c,d)*W(c,g)+W(c,d)*W(d,g)+W(d,g)*W(c,g) = 29$, and $\sum W(ie)_1 * W(ie)_{2_i} = o$.

$\sum W(cce) * W(ne)_i = W(a,e)*W(b,c)+W(a,e)*W(b,d)+W(a,e)*W(b,g)+W(a,e) * W(f,c) + W(a,e) * W(f,d) + W(a,e) * W(f,g) + W(a,f) * W(b,c) + W(a,f) * W(b,d) + W(a,f) * W(b,g) + W(a,f) * W(e,c) + W(a,f) * W(e,d) + W(a,f) * W(e,g) + W(b,e) * W(a,c) + W(b,e) * W(a,d) + W(b,e) * W(a,g) + W(b,e) * W(f,c) + W(b,e) * W(f,d) + W(b,e) * W(f,g) + W(b,f) * W(a,c) + W(b,f) * W(a,d) + W(b,f) * W(a,g) + W(b,f) * W(e,c) + W(b,f) * W(e,d) + W(b,f) * W(e,g) = 329$ and because there are no CCEs and IEs that are adjacent and we will multiply all the CCEs with all the IEs, we can write $\sum W(cce) * W(ie)_i = (\sum W(cce))(\sum W(ie)) = (W(a,e) + W(a,f) + W(b,e) + W(b,f))(W(c,d) + W(c,g) + W(d,g)) = 150$.

$\sum W(ne) * W(ie)_n = W(a,c)*W(c,d)+W(a,c)*W(c,g)+W(b,c)*W(c,d)+W(b,c)*W(c,g)+W(e,c)*W(c,d)+W(e,c)*W(c,g)+W(f,c)*W(c,d)+W(f,c)*W(c,g)+W(a,d)*W(d,c)+W(a,d*W(d,g)+W(b,d)*W(d,c)+W(b,d)*W(d,g)+W(e,d*W(d,c)+W(e,d)*W(d,g)+W(f,d)*W(d,c)+W(f,d)*W(d,g)+W(a,g)*W(g,c)+W(a,g)*W(g,d)+W(b,g)*W(g,c)+W(b,g)*W(g,d)+W(e,g)*W(g,c)+W(e,g)*W(g,d)+W(f,g)*W(g,c)+W(f,g)*W(g,d) = 267$, and $\sum W(ne) * W(ie)_i = W(a,c)*W(d,g)+W(a,d)*W(c,g)+W(a,g)*W(c,d)+W(b,c)*W(d,g)+W(b,d)*W(c,g)+W(b,g)*W(c,d)+W(e,c)*W(d,g)+W(e,d)*W(c,g)+W(e,g)*W(c,d)+W(f,c)*W(d,g)+W(f,d)*W(c,g)+W(f,g)*W(c,d) = 153$.

Substituting the average and the sums values into the second version of Equation 5 with $n = 7$, $|CE| = 2$, and $|VCE| = 4$ or $\omega = 4$ will give $H_{\sigma\text{CEs}} = \sqrt{3277/192}$. So, in conclusion the salesman will have 48 unique Hamiltonian cycle trips he could take that is guaranteed to pass between city a and b while also passing between city e and city f. each trip on average will cost him 155/8 units for a total cost of 930 units for all 48 cycles. Finally, the cost of the trips will have a standard deviation of $\sqrt{3277/192}$.

8 Trivial approximation algorithm for TSP

An easy and obvious way to try to solve TSP graphs is by taking the average of all the unique Hamiltonian cycles that pass through every pair of edges that pass through a vertex and take the pair of edges with the smallest average as CEs. Then, you can look for the next pair of edges with the smallest average of Hamiltonian cycles that pass through those pair of edges and the CEs. One can continue doing that until they have a cycle which is guaranteed to be smaller than or equal to the H_{avgCEs}. But, we can do better with the help of statistical physics.

9 Statistical algorithm for TSP

It turns out physicists has a powerful tool for optimization problems, that is the Boltzmann distribution. We quite certain this has been done by many researchers over the years, but we can describe the tsp problem as a classical system in a Boltzmann distribution. At this point we have to confess that the authors of this paper are not experts nor scholars in the traditional sense, but rather individuals with a strong aptitude for problem-solving. We respectfully ask that experts who read this paper forgive their oversights and consider the ideas presented with an open mind. Also readers who may not be familiar with statistical physics, we highly recommend the undergraduate textbook Statistical Physics (Second Edition) by F. Mandl, as much of the foundational information for the system we are about to describe is from this source.

9.1 Describing the TSP as thermal-dynamic system

If we consider each configuration of a Hamiltonian cycle for a complete graph as a state and each weight of the configuration as the energy for that state, then the number of state and energy for the classical system is given by $\Omega(n, |CE|, |VCE|)$ or more specifically $\Omega(n, 0, 0)$ which is equal to $(n-1)!/2$ [6]. Historically physicists has the states go from 0 to the maximum state, but for this system the minimum state is state 1 and we will use temperature (T) units so that $K_b = 1$ and $\beta = 1/K_bT = 1/T$. Therefore, the partition function (Z) for the macrostate is given by $Z = \sum_{j=1}^{\Omega(n,0,0)} e^{-\beta * W(hc)_{\mathrm{j}}}$, where $W(hc)_j$ is the weight of the jth Hamiltonian cycle and for a microstate, that is states that are representing Hamiltonian cycles that pass through some specific edges, $Z_{CEs} = \sum_{j=1}^{\Omega(n,|CE|,|VCE|)} e^{-\beta * W(hce)_j}$, where $W(hce)_j$ is the weight of the jth Hamiltonian cycle that pass through the specific edges [6].

To keep from writing two equations one for the macrostate and one for the microstates, we will describe everything in terms of the CEs or the microstates and the reader can set the values for $|CE|$ and $|VCE|$ to zero to get the equations for the macrostates. The next property we can calculate is the entropy (S) for the system at equilibrium where $\beta = 0$ by Boltzmann's definition $S_{CEs} = \ln \Omega(n, |CE|, |VCE|)$ [6]or at any temperature using Shannon's definition $S_{CEs} = \sum_{1}^{\Omega(n,|CE|,|VCE|)} P(hce_{\mathrm{j}}) \ln P(hce_{\mathrm{j}})$ [8], where $P(hce_{\mathrm{j}})$ is the probability that the jth Hamiltonian cycle is among the smallest cycles, since the cycles with the lowest weights will have the highest probabilities for $\beta > 0$ and its given by $P(hce_{\mathrm{j}}) = e^{-\beta * W(hce)_{\mathrm{j}}}/Z$ [6].

From the third law of thermal dynamics [6] and playing around with the numbers we have made those observations which we will now state without proof. Using the Shannon's definition of entropy you will find a number of state associate with each entropy calculated at a specific temperature (Ω_β) and its given by $\Omega_\beta = e^{S_{CEs}}$ and its value will fall between 1 and $\Omega(n, |CE|, |VCE|)$ [8]. In fact as β goes to infinity or as we approach zero temperature, Ω_β will approach 1 or if the TSP complete Graph has k Hamiltonian cycles with the same smallest weight then Ω_β will approach K [7]. That is to say the probabilities are dominated by the states with the smallest cycles, in fact if you were to take $P(hce_{\mathrm{j}})$ for any jth Hamiltonian cycle that is not the smallest cycle at a temperature close to zero then $P(hce_{\mathrm{j}})$ will be very close to zero, but for the smallest cycles $P(hce_{\mathrm{j}})$ will be close to 1 or $1/k$ if there are k Hamiltonian cycles with the same smallest cycles [7].

If we let the β approach zero which is the same as saying temperature goes to infinity or the system is at equilibrium then every state has the same probabilities and Ω_β will approach $\Omega(n, |CE|, |VCE|)$ [7]. Another varying property with temperature for the system is the Helmholtz free energy given by $-(1/\beta) \ln Z$, which at low temperature approach the weight of the smallest Hamiltonian cycle or if there are k Hamiltonian cycles with the same smallest weight than the free energy approach the weight of those cycles minus $(1/\beta)\ln k$. At high temperature the Helmholtz free energy approach approach $-S/\beta$, where the entropy is the Boltzmann definition of entropy or Shannon's definition at equilibrium [7]. The Helmholtz free energy seem to saying the same thing has Ω_β, that is at low temperature the cycles with the lowest cycles dominates and at high temperature the system becomes entropy like and the probabilities even out [6].

Using calculus, the expected value and the variance can be calculated by $-\partial \ln Z/\partial\beta$ and $\partial^2 \ln Z/\partial^2\beta$ respectively [6]. The expected value and variance are not the same as the average value and variance we calculated in polynomial time, rather they are weighted by the probabilities. But, as β approaches zero or when the system is close to equilibrium, both the expected value and the weighted variance approach the same average and variance we calculated in polynomial time [6].

9.2 Approximating the Partition function with a second degree Maclaurin approximation around $\beta = 0$

We can approximate the natural log of the partition function, $\ln Z$, close to equilibrium using a second-order Maclaurin approximation around $\beta = 0$ [9]. The approximation is $\ln Z(\beta=0)+\beta*\partial[\ln Z(\beta=0)]/\partial\beta+(\beta^2/2)*\partial^2[\ln Z(\beta=0)]/\partial^2\beta$ which is equal to $S_{CEs} - \beta * H_{\mathrm{avgCEs}} + (\beta^2/2) * H_{\sigma\mathrm{CEs}}$, where the entropy is the Boltzmann definition of entropy. This approximation is good for temperatures close to equilibrium and is not very useful for prediction at temperatures close to zero where the smallest Hamiltonian weights dominate. Nevertheless, we can use the approximation in a fashion similar to that of simulated annealing [5].

Like in simulated annealing, we have heated up the system, in this case to equilibrium, where all probabilities are equal. Then its possible to lower the temperature a little by plugging a small value for β close to zero into the Maclaurin approximation, where the symmetry of the probabilities is broken and its possible to make predictions from the probabilities [9] [5]. A probability that can be calculated is the probability that an edge(s) is among the smallest Hamiltonian cycles, P(CEs). Its calculated by adding all the probabilities for each Hamiltonian cycles that pass through that edge(s) or $Z_{CEs}/Z \approx e^{S_{CEs}-\beta*H_{\mathrm{avgCEs}}+(\beta^2/2)*H_{\sigma\mathrm{CEs}}}/e^{S-\beta*H_{\mathrm{avg}}+(\beta^2/2)*H_\sigma}$ $\approx e^{S_{CEs}-S-\beta(H_{\mathrm{avgCEs}}-H_{\mathrm{avg}})+(\beta^2/2)(H_{\sigma\mathrm{CEs}}-H_\sigma)}$ [6]. We should note that for graphs with many vertices the entropy is almost impossible to calculate, but it can be approximate using the Stirling formula [2] since S is given by $\ln\Omega(n,0,0)$ which is $\ln[(n-1)!/2]$ and S_{CEs} is given by $\ln\Omega(n,|CE|,|VCE|)$ which is $\ln[2^{|CE|}(n-1+|CE|-|VCE|)!]$.

With P(CEs) its possible to create a probabilistic algorithm to solve tsp graph, and it work similarly to the algorithm we briefly describe using the average. Fist find the $P(CEs)$ for every pair of edges that pass through a vertex, then do that for every vertex and choose the pairs of edges with the highest probability as part of the cycle. Once a pair of edges is chosen, find the next pair of edges that pass through a vertex and the chosen edges with the highest probability and choose those pair edges as part of the cycle as well. It is possible to continue doing this until a Hamiltonian cycle is complete. We should note that this does not guarantee the optimal cycle, but it does guarantee a small weight cycle close to the optimal.

Also, we should mention that the Maclaurin approximation should work for other systems in a Boltzmann distribution that is not about the TSP as long as the system is kept close to equilibrium [9]. As we mentioned earlier, the authors of this paper are not experts in physics and are unable to show a real-life example of classical systems or even some quantum systems that satisfy this approximation, but we will make the following general statement for systems in a Boltzmann distribution; the natural log of the partition function for a system in a Boltzmann distribution close to equilibrium can be approximated by the entropy of the system minus β times the average of all the energies needed for each state plus one half β squared times the variance of all the energies needed for each state [9] [6].

Of course we can continue to higher degrees of the Maclaurin approximation, but we should note that higher degree approximation will give a more accurate approximation of the natural log of partition function at high temperature but it will still fail to make accurate predictions for the system at absolute zero temperature because we are essentially taking a Taylor approximation at infinite temperature which will take an infinite degree of approximation to have a radius of convergence at zero temperature [1]. Maybe in a special case where all higher derivatives to a certain degree vanish and its possible to approximate the natural log of the partition function up to that degree, then theoretically the approximation is exact and its possible to make predictions at absolute zero assuming the natural log of the partition function is analytic at every temperature [1].

Also, we did not reference any papers on the Maclaurin approximation of the partition function around $\beta = 0$ because we did not find it in the literature we study for this paper. But if its a well-used approximation in physics or if its has been discovered by someone else, then we sincerely apologize for not referencing it properly and the original author who discovered it obviously deserves all the credit.

10 Conclusion

At its heart, this paper was not just about solving a mathematical problem, it was about seeing structure where others expect chaos. The traveling salesman problem is known for its complexity, but instead of trying to find a single optimal tour, we focus on understanding the entire landscape of all possible tours. What emerged was a set of clean, polynomial-time formulas that describe not just averages, but the deeper statistical patterns behind the scenes.

By re-imagining how edges behave across all Hamiltonian cycles using thought experiments, symmetry, and edge classification, we found that we could predict collective behavior without brute force. This feels more like physics than pure math. It suggests that what we call "hard problems" are not just hard because of computation, but because we have been asking them the wrong way. This work is not the end, but is a new perspective.

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