Econ 612: Assignment 2

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Problem 1

The figures below plot the sample autocorrelation function and the partial autocorrelation function for some time series Y_t with 1000 observations for lags 1 to 20, along with 0.95 confidence intervals for each lag.

a

Autocorrelation is the linear dependence of a variable with itself at two points in time. The h^{th}-order autocorrelation of a stationary time series y_t is:

$$\rho_h = Corr(y_t, y_{t-h}) = \frac{Cov(y_t, y_{t-h})}{Var(y_t)} = \frac{\gamma_h}{\gamma_0}$$

Correlation between two variables can result from a mutual linear dependence on the other variables. Partial autocorrelation is the autocorrelation between y_t and y_{t-h} after removing any linear dependence on $y_1, y_2, \cdot y_{t-h+1}$.

b

Suppose that a researcher models the sample of data with an AR(2) process (mis-specified) as below:

$$Y_t = \phi Y_{t-2} + \epsilon_t$$

where $|\phi| < 1$, ϵ_t is a white noise process with mean zero and variance σ^2.

(i) Derive the first-order and second-order autocorrelations for this model

$$\begin{aligned}
\gamma_0 &= Var(Y_t) \\
&= Var(\phi Y_{t-2} + \epsilon_t) \\
&= \phi^2 Var(Y_{t-2}) + Var(\epsilon_t) + 2\phi Cov(Y_{t-2}, \epsilon_t) \\
&= \phi^2 \gamma_0 + \sigma^2 \\
\gamma_0(1-\phi^2) &= \sigma^2 \\
\gamma_0 &= \frac{\sigma^2}{1-\phi^2}
\end{aligned}$$

$$\begin{aligned}
\gamma_1 &= Cov(Y_t, Y_{t-1}) \\
&= Cov(\phi Y_{t-2} + \epsilon_t, Y_{t-1}) \\
&= \phi Cov(Y_{t-2}, Y_{t-1}) + Cov(\epsilon_t, Y_{t-1}) \\
&= \phi \gamma_1 + 0 \\
\gamma_1(1-\phi) &= 0 \\
\gamma_1 &= 0
\end{aligned}$$

$$\begin{aligned}
\gamma_2 &= Cov(Y_t, Y_{t-2}) \\
&= Cov(\phi Y_{t-2} + \epsilon_t, Y_{t-2}) \\
&= \phi Cov(Y_{t-2}, Y_{t-2}) + Cov(\epsilon_t, Y_{t-2}) \\
&= \phi Var(Y_{t-2}) + 0 \\
\gamma_2 &= \phi \gamma_0 \\
\gamma_2 &= \frac{\phi \sigma^2}{1-\phi^2}
\end{aligned}$$

First-order autocorrelation:

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = 0$$

Second-order autocorrelation:

$$\rho_2 = \frac{\gamma_2}{\gamma_0} = \phi$$

(ii) Find the prediction of $P[y_{t+1}|Y_t], P[y_{t+2}|Y_t], P[y_{t+h}|Y_t]$ and $P[y_\infty|Y_t]$

$$P[y_{t+1}|Y_t] = \phi Y_{t-1}$$

$$P[y_{t+2}|Y_t] = \phi Y_t$$

$$P[y_{t+h}|Y_t] = \begin{cases} \phi^{\frac{h}{2}} Y_t & h \text{ is even} \\ \phi^{\frac{h+1}{2}} Y_{t-1} & h \text{ is odd} \end{cases}$$

$$\begin{aligned} P[y_\infty|Y_t] &= \phi^\infty Y_t \\ &= 0 \end{aligned}$$

b

Given the figures above, what type of ARMA model should be the correct one to model the sample data? How can you tell the researcher empirically that the AR(2) model is mis-specified?

The correct ARMA model for the sample data would be an AR(1) model. One could empirically tell the researcher that the AR(2) is mis-specified by comparing model selection criterion, such as AIC, BIC and HQIC, and MSE.

Problem 2: Autocovariance and MLE

a

ARMA model:

$$\phi(L)X_t = \theta(L)Z_t$$

with $\phi(L) = 1 - L + \frac{1}{4}L^2$, $\theta(L) = 1 + L + \frac{1}{2}L^2$
So

$$\begin{aligned} (1 - L + \frac{1}{4}L^2)X_t &= (1 + L + \frac{1}{2}L^2)Z_t \\ 1 + L + \frac{1}{2}L^2 &= (1 - L + \frac{1}{4}L^2)\psi(L) \end{aligned}$$

$$\begin{aligned} \iff 1 + L + \frac{1}{2}L^2 &= (1 - L + \frac{1}{4}L^2)(\psi_0 + \psi_1 L + \psi_2 L^2 + \cdots) \\ &= \psi_0 - \psi_1 L + \psi_2 L^2 + ... - \psi_0 L - \psi_1 L^2 - \psi_2 L^3 + \cdots + \psi_0 \frac{1}{4}L^2 + \frac{1}{4}\psi_1 L^3 + \frac{1}{4}\psi_2 L^4 + \cdots \end{aligned}$$

$$\begin{aligned} \psi_0 &= 1 \\ \psi_1 - \psi_0 &= 1 \implies \psi_1 = 2 \\ \psi_2 - \psi_1 + \frac{1}{4}\psi_0 &= \frac{1}{2} \implies \psi_2 = \frac{9}{4} \\ \psi_3 - \psi_2 + \frac{1}{4}\psi_1 &= 0 \implies \psi_3 = \frac{1}{4} \\ \psi_4 - \psi_3 + \frac{1}{4}\psi_2 &= 0 \implies \psi_4 = -\frac{5}{16} \\ \psi_5 - \psi_4 + \frac{1}{4}\psi_3 &= 0 \implies \psi_5 = \frac{4}{16} \end{aligned}$$

$$\begin{aligned} \gamma_0 &= \sigma^2(\psi_0\psi_0 + \psi_1\psi_1 + \psi_2\psi_2) \\ &= \sigma^2(1^2 + 2^2 + \left(\frac{9}{4}\right)^2) \\ &= \frac{161}{16}\sigma^2 \end{aligned}$$

$$\begin{aligned} \gamma_1 &= \sigma^2(\psi_0\psi_1 + \psi_1\psi_2) \\ &= \sigma^2(2 + \frac{9}{2}) \\ &= \frac{13}{2}\sigma^2 \end{aligned}$$

Consider the ARMA(2,2) process: $(1 - L + \frac{1}{4}L^2)X_t = (1 + L + \frac{1}{2}L^2)Z_t$ where Z_t is $WN(0, \sigma^2)$. Write down the autocovariance function of X for lags $h = 0, 1, 2, 3, 4, 5$.

$$
\begin{aligned}
(1 - L + \frac{1}{4}L^2)X_t &= (1 + L + \frac{1}{2}L^2)Z_t \\
X_t - X_{t-1} + \frac{1}{4}X_{t-2} &= Z_t + Z_{t-1} + \frac{1}{2}Z_{t-2} \\
X_t &= X_{t-1} - \frac{1}{4}X_{t-2} + Z_t + Z_{t-1} + \frac{1}{2}Z_{t-2}
\end{aligned}
$$

$$
\begin{aligned}
\gamma_0 &= Var(X_t) \\
&= Var(X_{t-1} - \frac{1}{4}X_{t-2} + Z_t + Z_{t-1} + \frac{1}{2}Z_{t-2}) \\
&= Var(X_{t-1}) + \frac{1}{16}Var(X_{t-2}) + Var(Z_t) + Var(Z_{t-1}) + \frac{1}{4}Var(Z_{t-2}) - 2 \cdot \frac{1}{4}Cov(X_{t-1}, X_{t-2}) + 2Cov(X_{t-1}, \\
&+ 2Cov(X_{t-1}, Z_{t-1}) + 2 \cdot \frac{1}{2}Cov(X_{t-1}, Z_{t-2}) - 2 \cdot \frac{1}{4}Cov(X_{t-2}, Z_t) - 2 \cdot \frac{1}{4}Cov(X_{t-2}, Z_{t-1}) - 2 \cdot \frac{1}{4}\frac{1}{2}Cov(X_{t-2} \\
&2Cov(Z_t, Z_{t-1}) + 2 \cdot \frac{1}{2}Cov(Z_t, Z_{t-2}) + 2 \cdot \frac{1}{2}Cov(Z_{t-1}, Z_{t-2}) \\
&= Var(X_{t-1}) + \frac{1}{16}Var(X_{t-2}) + \sigma^2 + \sigma^2 + \frac{1}{4}\sigma^2 - \frac{1}{2}Cov(X_{t-1}, X_{t-2}) \\
-\frac{1}{16}\gamma_0 &= \frac{9}{4}\sigma^2 - \frac{1}{2}\gamma_1 \\
-\frac{1}{16}\gamma_0 &= \frac{9}{4}\sigma^2 - \frac{1}{2} \cdot \frac{4}{5}\gamma_0 \text{ from } \gamma_1 \\
\gamma_0(\frac{1}{2} \cdot \frac{4}{5} - \frac{1}{16}) &= \frac{9}{4}\sigma^2 \\
\frac{27}{80}\gamma_0 &= \frac{9}{4}\sigma^2 \\
\gamma_0 &= \frac{20}{3}\sigma^2
\end{aligned}
$$

$$
\begin{aligned}
\gamma_1 &= Cov(X_t, X_{t-1}) \\
&= Cov(X_{t-1} - \frac{1}{4}X_{t-2} + Z_t + Z_{t-1} + \frac{1}{2}Z_{t-2}, X_{t-1}) \\
&= Cov(X_{t-1}, X_{t-1}) - \frac{1}{4}Cov(X_{t-2}, X_{t-1}) + Cov(Z_t, X_{t-1}) + Cov(Z_{t-1}, X_{t-1}) + \frac{1}{2}Cov(Z_{t-2}, X_{t-1}) \\
&= \gamma_0 - \frac{1}{4}\gamma_1 \\
\frac{5}{4}\gamma_1 &= \gamma_0 \\
\gamma_1 &= \frac{4}{5}\gamma_0 = \frac{4}{5} \cdot \frac{20}{3}\sigma^2 = \frac{16}{3}\sigma^2
\end{aligned}
$$

$$
\begin{aligned}
\gamma_2 &= Cov(X_t, X_{t-2}) \\
&= Cov(X_{t-1} - \frac{1}{4}X_{t-2} + Z_t + Z_{t-1} + \frac{1}{2}Z_{t-2}, X_{t-2}) \\
&= Cov(X_{t-1}, X_{t-2}) - \frac{1}{4}Cov(X_{t-2}, X_{t-2}) + Cov(Z_t, X_{t-2}) + Cov(Z_{t-1}, X_{t-2}) + \frac{1}{2}Cov(Z_{t-2}, X_{t-2}) \\
&= \gamma_1 - \frac{1}{4}\gamma_0 \\
\gamma_2 &= \frac{16}{3}\sigma^2 - \frac{1}{4} \cdot \frac{20}{3}\sigma^2 \\
\gamma_2 &= \frac{11}{3}\sigma^2
\end{aligned}
$$

$$
\begin{aligned}
\gamma_3 &= Cov(X_t, X_{t-3}) \\
&= Cov(X_{t-1} - \frac{1}{4}X_{t-2} + Z_t + Z_{t-1} + \frac{1}{2}Z_{t-2}, X_{t-3}) \\
&= Cov(X_{t-1}, X_{t-3}) - \frac{1}{4}Cov(X_{t-2}, X_{t-3}) + Cov(Z_t, X_{t-3}) + Cov(Z_{t-1}, X_{t-3}) + \frac{1}{2}Cov(Z_{t-2}, X_{t-3}) \\
&= \gamma_2 - \frac{1}{4}\gamma_1 \\
\gamma_3 &= \frac{11}{3}\sigma^2 - \frac{1}{4}\frac{16}{3}\sigma^2 \\
\gamma_3 &= \frac{7}{3}\sigma^2
\end{aligned}
$$

$$\begin{aligned}\gamma_4 &= Cov(X_t, X_{t-4})\\ &= \gamma_3 - \frac{1}{4}\gamma_2\\ &= \frac{7}{3}\sigma^2 - \frac{1}{4}\cdot\frac{11}{3}\sigma^2\\ \gamma_4 &= \frac{17}{12}\sigma^2\end{aligned}$$

$$\begin{aligned}\gamma_5 &= Cov(X_t, X_{t-5})\\ &= \gamma_4 - \frac{1}{4}\gamma_3\\ &= \frac{17}{12}\sigma^2 - \frac{1}{4}\cdot\frac{7}{3}\sigma^2\\ \gamma_5 &= \frac{5}{6}\sigma^2\end{aligned}$$

To summarize,

$$\begin{aligned}\gamma_0 &= \frac{20}{3}\sigma^2\\ \gamma_1 &= \frac{16}{3}\sigma^2\\ \gamma_2 &= \frac{11}{3}\sigma^2\\ \gamma_3 &= \frac{7}{3}\sigma^2\\ \gamma_4 &= \frac{17}{12}\sigma^2\\ \gamma_5 &= \frac{5}{6}\sigma^2\end{aligned}$$

b

Assume that ϵ_t are independent and that ϵ_t has the following density function (This is called the Laplace distribution):

$$f_{\epsilon_t} = \frac{1}{2}\exp(-|e|)$$

Consider the MA(0) process:

$$Y_t = \mu + \epsilon_t$$

Try to find the MLE for μ

Y_t $Laplace(e+c)$

$$f_Y = \frac{1}{2}\exp(-|e+\mu|)$$

$$\begin{aligned}\log\mathcal{L}(\theta|y_1, y_2, ..., y_t) &= \sum_{t=1}^{T}\log f(y_t;\theta)\\ &= \sum_{t=1}^{T}\log\left(\frac{1}{2}\exp(-|e+\mu|)\right)\\ &= \sum_{t=1}^{T}(-|e+\mu|) - T\log(2) & =\end{aligned}$$

$$\begin{aligned}\frac{\partial\log\mathcal{L}(\theta|y_1, y_2, \cdots, y_T)}{\partial\mu} &= \sum_{t=1}^{T} - \frac{e+\mu}{|e+\mu|}\\ 0 &= \sum_{t=1}^{T} - \frac{e+\mu}{|e+\mu|}\\ 0 &= -\sum_{t=1}^{T}\frac{e}{|e+\mu|} - \sum_{t=1}^{T}\frac{\mu}{|e+\mu|}\\ 0 &= -\sum_{t=1}^{T}\frac{e}{|e+\mu|} - T\mu\sum_{t=1}^{T}\frac{1}{|e+\mu|}\end{aligned}$$

The MLE for μ is the sample median

Problem 3: Spectrum Analysis

a

Consider an AR(1) process $X_t = \phi X_{t-1} + Z_t$ with $|\phi| < 1, Z_t$ is $WN(0,1)$

(i) Argue that the spectral density of X_t exists

If a time series $\{X_t\}$ has autocovariance γ satisfying $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$, then we define its spectral density as

$$f(v) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i v h}$$

for $-\infty < v < \infty$

The autocovariance function for this AR(1) process is:

$$\gamma(h) = \frac{\phi^h}{1-\phi^2}$$

$$\sum_{h=-\infty}^{\infty} \left| \frac{\phi^h}{1-\phi^2} \right| = \frac{\infty}{|\phi^2 - 1|}$$

Any covariance-stationary stochastic process can also be represented in what is called the frequency domain.

Derive the spectral density function f For $X_t = \phi X_{t-1} + Z_t$, $\gamma(h) = \frac{\phi^h}{1-\phi^2}$

$$\begin{aligned}
f(v) &= \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i v h} \\
&= \frac{1}{1-\phi^2} \sum_{h=-\infty}^{\infty} \phi^h e^{-2\pi i v h} \\
&= \frac{1}{1-\phi^2} \Big(1 + \sum_{h=1}^{\infty} \phi^h (e^{-2\pi i v h} + e^{2\pi i v h})\Big) \\
&= \frac{1}{1-\phi^2} \Big(1 + \frac{\phi e^{-2\pi i v h} + e^{2\pi i v h}}{}\Big)
\end{aligned}$$

$$\begin{aligned}
Var(Y_t) &= Var(0.4Y_{t-1} + 0.8Y_{t-2} + \epsilon_t + \delta\epsilon_{t-1}) \\
&= 0.4^2 Var(Y_{t-1}) + 0.8^2 Var(Y_{t-2}) + Var(\epsilon_t) + \delta^2 Var(\epsilon_{t-1}) + 2 \cdot 0.4 \cdot 0.8 Cov(Y_{t-1}, Y_{t-2}) \\
&+ 2 \cdot 0.4 Cov(Y_{t-1}, \epsilon_t) + 2 \cdot 0.4\delta Cov(Y_{t-1}, \epsilon_{t-1}) + 2 \cdot 0.8 Cov(Y_{t-2}, \epsilon_t) + 2 \cdot 0.8\delta Cov(Y_{t-2}, \epsilon_{t-1}) + 2 \cdot \delta Cov(\epsilon_t, \epsilon_{t-1})\gamma_0
\end{aligned}$$