COSMOS 2022, Computational Lab 08

[01] The inverse A^{-1} of the (square) matrix A is the matrix which obeys $A^{-1}A = I$. Here I is the 'identity matrix' which has the number '1' down the diagonal and zero everywhere else. Here are the 2×2 and 3×3 identity matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad\qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In Lab 07 you saw the multiplying a general matrix A by I leaves A unchanged. Thus I plays the role of the number 1 for the space of numbers.

[02] Given the vector (3×1 matrix)

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Show $I\,v = v$. (This is just another example of what you saw in Lab 07 where you proved $I\,A = A$.)

[03] The general rule for the inverse of the 2x2 matrix A:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad\qquad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Check that $A^{-1}A = I$.

Note 1: What we mean by multiplying a matrix by a number is that each entry in the matrix is multiplied by the number.

Note 2: Clearly A^{-1} exists only if the determinant, $\det A = ad - bc$ is nonzero.

Check that $A\,A^{-1} = I$. This is *not obvious* because we know that $A\,B$ is usually not the same as $B\,A$ so just because $A^{-1}A = I$ there is no reason to expect $A\,A^{-1} = I$ also. But we get lucky!

[04] Solve the linear equations

$$\begin{aligned} 8x + 3y &= 1 \\ 2x + y &= 1 \end{aligned}$$

without using matrices. That is, what are x and y that solve these two equations?

[05] Show, using the rules for matrix multiplication, that you can rewrite the problem in [04] as:

$$\begin{pmatrix} 8 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

[06] Compute the inverse of

$$A = \begin{pmatrix} 8 & 3 \\ 2 & 1 \end{pmatrix}$$

Compute

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Compare with what you found in [04].

[07] What is the determinant of the identity matrix I?

What is the inverse of the identity matrix I?

[08] Using the matrices from [06], what are $\det A$ and $\det A^{-1}$?

You should have found $\det A^{-1} = 1/\det A$. This is always true!

[09] (extra) Can you prove the relation in [08] using $A^{-1} A = I$ and what you found here in [07] about $\det I$, and what you found in [03] of Lab 07: $\det\big(A\,B\big) = \det A \, \det B$?

[10] Consider the two matrices:

$$A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} -1 & 1 \\ -2 & 4 \end{pmatrix}$$

Compute the product $A\,B$.

Compute the inverses A^{-1} and B^{-1}.

Show that the inverse of $A\,B$ is $B^{-1}A^{-1}$.

This is sort of weird! When you invert the product you get the product of the inverses, as you would expect, but you have to reverse the order!

[11] (extra) Prove the result in [10]. That is, prove that $\big(A\,B\big)^{-1} = B^{-1}A^{-1}$.

Hint 1: The definition of $\big(A\,B\big)^{-1}$ is that it must satisfy $\big(A\,B\big)^{-1}\big(AB\big) = I$. (This is the meaning of 'inverse'!)

Hint 2: Recall from Lab [07] that matrix multiplication is associative.