COSMOS 2022, Computational Lab 09

[01] We saw that the matrix

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

will rotate a vector by an angle θ in the clockwise direction. What matrix will rotate by θ in the *counter-clockwise* direction?

[02] What is the determinant of the matrix R? What does its value tell you about whether you can invert R? Now go ahead and compute R^{-1}.

[03] Compare what you found in [2] for R^{-1} with what you found for the matrix which rotates *clockwise* by θ in [01]. What do you notice? Does it make sense?

[04] We saw that the matrix

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

will "project" a vector $\vec{v}$ onto the x-axis. What matrix will project oto the y-axis?

[05] What is the determinant of P? What is the inverse of P? Can you make an argument for what you concluded about P^{-1}?

[06] If you add the matrix that projects on the x-axis to to the matrix that projects on the y-axis what do you get?

[07] We saw that the matrix

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

will "project" a vector $\vec{v}$ onto the x-axis. Thinking 'physically' (no mathematics allowed!) what do you think should happen if you project $\vec{v}$ onto the x-axis to get $P\,\vec{v}$, and then repeat the process by projecting *again* to form $P(P\,\vec{v})$? Now do some math: compute P^2, the square of the matrix P. What do you notice? Does it make sense in terms of what you decided should happen when you project twice in a row?

[08] (quite hard!) We know the matrix which projects a vector $\vec{v}$ onto the x-axis, given by the equation $y = 0$. What matrix will project a vector $\vec{v}$ onto the line $y = x$?

[09] (quite hard!) Generalize [08]: What matrix will project a vector $\vec{v}$ onto a line making an angle θ with the x-axis?

[10] (quite hard?) Suppose you have vectors $\vec{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ living in three dimansional space. What three-by-three matrix would rotate $\vec{v}$ around the z-axis by an angle θ? How about the x-axis? How about the y-axis?

[11] Compute the eigenvalues of

$$M = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Hint: You need to find the values of λ that make the determinant of

$$M = \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$$

equal to zero.

[12] Compute the eigenvectors of

$$M = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

Hint: Form the matrix

$$M - \lambda I = \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$$

for the λ values you derived in [10]. Then figure out which vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ obeys

$$\left(M - \lambda I \right) \vec{v} = \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

[13] Compute the eigenvalues of

$$P = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

[14] Compute the eigenvectors of

$$M = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$