COSMOS 2022, Computational Lab 06

This Lab consists entirely of optional exercises. The last one is actually a bit of a 'research project'. By that I mean it's possible no-one has done it before and it's interesting enough to communicate (if you make progress on it!)

[01] MD in 2D

Our 'projectile motion' MD code of Lab 04 was very special: We threw the ball *straight up* so that we only had to compute the height $y(t)$. Let's do the more general projectile motion problem where the thrown object moves horizontally as well as vertically.

I think you will find the procedure charmingly simple: every MD equation you had before morphs into a pair of equations! I'll supply the code for this one, but make sure you understand what all the lines are doing! We'll include friction just for fun!

```
#include <stdio.h>
#include <math.h>
int main()
{
     double x,y,vx,vy,t,dt,Fx,Fy,m,b,g=-9.8;
     int j;
     FILE * fileout;
     fileout=fopen("flitwick","w");
     printf("\nEnter mass m, friction constant b, and time step dt:   ");
     scanf("%lf %lf %lf",&m,&b,&dt);
     printf("\nEnter starting position x,y:   ");
     scanf("%lf %lf",&x,&y);
     printf("\nEnter starting velocity vx,vy:   ");
     scanf("%lf %lf",&vx,&vy);
     t=0.;
     do
     {
          t=t+dt;
          x=x+vx*dt;
          y=y+vy*dt;
          Fx=-b*vx;
          Fy=m*g-b*vy;
          vx=vx+(Fx/m)*dt;
          vy=vy+(Fy/m)*dt;
          fprintf(fileout,"\n   %12.6lf %12.6lf",x,y);
      }while(y>0);
      fclose(fileout);
      return 0;
}
```

Run your code for no air resistance ($b = 0$). What is the range (distance traveled) if initial $x = y = 0$, and initial velocity components $v_{x0} = 10$ m/s $v_{y0} = 17.32$ m/s. This means you threw the ball at an angle of $\theta = 60°$. Use $dt = 0.01$. The value of m will not matter here.

If you have taken a physics course, you might know the range $R = 2\, v_{x0}\, v_{y0}/g$ where v_{x0} and v_{y0} are the initial x and y velocities. How well do your results agree with this formula? Try running with $dt = 0.001$. How well do your results agree now?

Question for you: You may know from a physics course that you should throw a ball at a $\theta = 45°$ angle to optimize the range, the distance traveled in the x direction. I believe this result assumes the initial $y = 0$. Is $\theta = 45°$ still best if there is air resistance? You might want to run with a small dt (maybe 0.001 or 0.0005?) if you try to work this out, because we want especially high accuracy numbers to compare and find the best θ. My understanding is the answer to this question attained importance in World War 1 when guns became powerful enough. It was discovered that a shell would go further if you fired it at $\theta > 45°$ because then it got high enough that the thickness of the air decreases, allowing the shell to go further. The code above does not incorporate that effect. You would need to make b depend on y. Want to try writing such a code? What would a good functional form for $b(y)$ look like?

[02] Kepler Problem

Let's write a code to determine the orbits of the planets around the sun! We need to know Newton's universal law of gravity, which generalizes the rule $F = mg$ with $g = -9.8$ m/s^2, which is true only near the earth's surface.

Newton's universal law of gravity states that *any* two masses M_1 and M_2 which are separated bt a distance r exert forces on each other with magntitdue

$$F = \frac{GM_1M_2}{r^2} \qquad\qquad G = 6.67 \times 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2} \tag{1}$$

The direction of the force is along the line connecting the masses and is attractive.

Looking at Exercise [01] we need Fx and Fy. This simple picture helps:

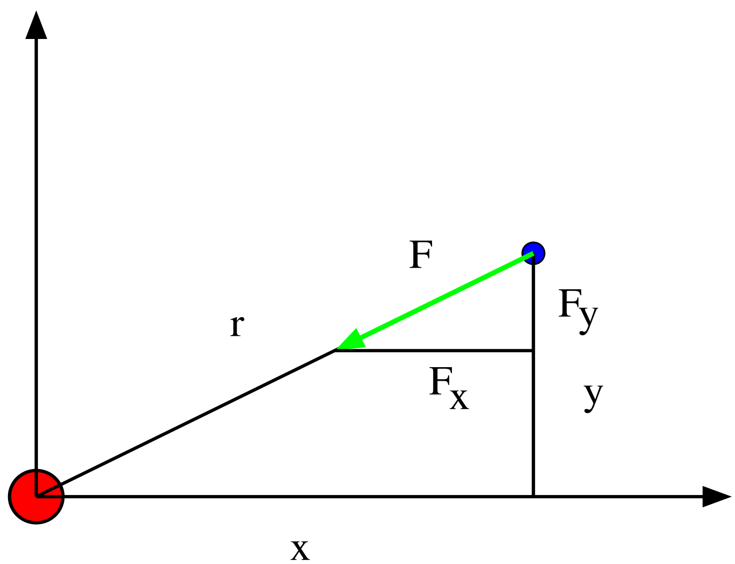

Figure 1: Picture of the geometry of the Kepler problem. x, y, r give the location of the Earth relative to the sun, with $r = \sqrt{x^2 + y^2}$. F_x, F_y, F give the gravitational force and its components.

Because F lies along the line connecting the earth and sun, the two triangles $\{x, y, r\}$ and $\{F_x, F_y, F\}$ are 'similar': their angles are all identical. As a consequence we know the ratios of corresponding sides must be equal: $F_x/F = x/r$ for example. Together with Eq. 1 this gives the equations for F_x and F_y:

$$F_x = -\frac{GM_1M_2}{r^2}\frac{x}{r} \qquad\qquad F_y = -\frac{GM_1M_2}{r^2}\frac{y}{r} \tag{2}$$

Think about where the minus signs come from! Now that we have F_x and F_y we are ready to do MD!

I will let you write this Kepler code. It looks an awful like the code on page 1. Obviously you need to define G instead of g and replace the two equations for F_x and F_y. You will need to provide the masses of the earth and sun (though the mass of the earth cancels out). One more important thing: if the motion is not circular, r can change as the planet moves. *You need to add a line* $r = \mathrm{sqrt}\{x * x + y * y\}$; which recomputes r, right after x and y are updated.

Run your code by starting the earth out on the x-axis: $x_0 = 1.496 \times 10^{11}$ m and $y_0 = 0$ m. The earth's orbital velivity around the sun is $v = 2.978 \times 10^4$ m/s. How should you divide this into v_{x0} and v_{y0} if the earth starts on the x-axis (assume the orbit is circular).

Imprtant note: What time step dt should you use? The rule is that dt needs to be *small compared to the time scale of the motion.* Here the time scale is one year $T = 3.156 \times 10^7$ seconds. So if you make $dt = 10^4$ seconds that is a small number (!) (compared to T). What number of steps N should you use?

You might have some fun by distorting the orbit. Try starting with half the correct velocity: $v = 1.489 \times 10^4$ m/s. What happens to the earth's orbit? Does it fall into the sun? What happens when v is twice the correct value: $v = 3.992 \times 10^4$ m/s.

There is a ton of cool physics in this problem:

Why are we only considering the earth moving in a 2D plane of x and y? Doesn't the earth live in a 3D universe?!?!

What sorts of orbital shapes are possible for Newton's law of gravity?

Why are the Kepler orbits 'closed' (i.e. they write over each other as each orbit is completed). Is that true for any force law?

How did Newton come up with his law of gravity? (This is an especially cool question as it illustrates how experimental observations are entwined with theory.) See pages 5-6.

What velocity is enough to allow the earth to escape the sun? How you you compute it?

How was “Newton’s Univeral Law of Gravity” Discovered?

In your high school courses, you probably encountered “The Scientific Method”. The discovery of the way masses attract each other is one of the most beautiful examples of this process. It begins (like many great discoveries) with new experimental abilities- Galileo’s invention of the telescope. Next, came Kepler’s careful acquisition of data: how far “R” are the planets from the sun, and how long “T” do they take to complete an orbit. Then, Kepler noticed an amazing pattern in his data. Even though Pluto is a hundred times farther from the sun than Mercury, and even though it takes a thousand times longer to go around the sun, there is a certain combination of R and T which is almost the *same* for all the planets!

We ask you to discover this pattern. Specifically, on the next page we provide a table of R and T, and a calculator, and ask you to put together different combinations to see if one of them is more constant than the others.

One important question to ask yourself (which is sometimes not emphasized in discussions of the scientific method) is how perfect a pattern needs to be in order to be labeled a pattern. What do you think is a reasonable amount of variation? In this specific case of planets going around the sun, what might cause imperfections in the data?

The story of gravity does not stop with Kepler’s pattern. Newton came along and wrote down a rule for the force between planets and the sun which explained it. The key ingredient is that the force between two masses falls off with the square of R, ie $F \sim 1/R^2$: a planet three times as far away from the sun feels a force only $1/3^2 = 1/9$ as big. It turns out that this rule, in combination with a rule for the acceleration of an object in a circular orbit, explains Kepler’s numbers.

The truly miraculous ending to this story is this: Kepler’s pattern and Newton’s law do not only apply to the planets going around the sun! They apply to the moon going around the earth, to the sun orbiting the black hole at the center of the Milky Way. They apply to the bending of light from the most distant remnants of the Big Bang. Newton’s law even applies to the force you feel (your weight) as you sit in a classroom, or the gravitational force between you and your neighbor (although this force is very small).

Emulating this sort of process is the goal of every scientist. One question for you to ask, as you prepare for undergraduate research, is where you see yourself in this “Scientific Method”. Are you a Galileo who wants to build new instruments? Are you a Kepler who wants to take data and notice patterns? Are you a Newton who wants to write down new theories?

Planet	Distance from Sun R (miles)	Length of Year T (seconds)	R/T	T/R^2	R^3/T^2
Mercury	36,000,000	8,000,000	4.50×10^0	6.17×10^{-9}	8.08×10^8
Venus	67,000,000	19,400,000			
Earth	93,000,000	31,500,000			
Mars	140,000,000	59,000,000			
Jupiter	480,000,000	370,000,000			
Saturn	890,000,000	930,000,000			
Uranus	1,800,000,000	2,700,000,000			
Neptune	2,800,000,000	5,200,000,000			
Pluto	3,650,000,000	7,800,000,000			

[03] Kirkwood Gap Problem

As you probably know, there is an 'asteroid belt' between Mars and Jupiter, the small fragments of a planet torn asunder by a tug-of-war between Jupiter and the sun. You might imagine that the distribution of the number of asteroids a given distance from the sun is featureless. Amazingly, it is not. There are specific distances where there are no asteroids. These are known as 'Kirkwood gaps'.

You can read about Kirkwood gaps here:

`https://en.wikipedia.org/wiki/Kirkwood_gap`

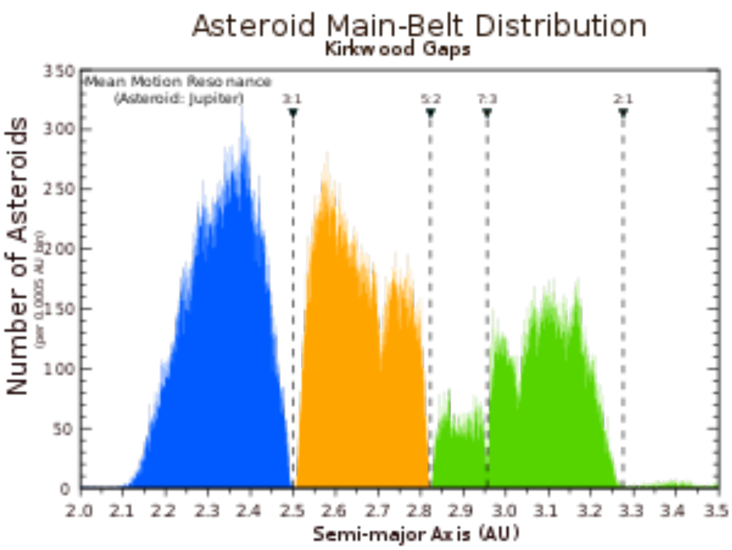

Figure 2: Picture of Kirkwood gaps from the wikipedia article.

You can imagine how you might write an MD code to simulate this: You need two moving objects: the asteroid and Jupiter. (Should you allow the sun to move too? Jupiter is pretty heavy!) Then you would track four positions: $x_{\rm asteroid}, y_{\rm asteroid}, x_{\rm Jupiter}, y_{\rm Jupiter}$. You would need to figure out the two gravitational forces on the asteroid, etc. But it's really not fundamentally different from the code you just wrote in [02].

A number of years ago I looked briefly in the literature to see if someone had done this problem already. I did not find anything (but really I should look more thoroughly). If no one has done it, well, one could imagine doing it and writing a nice paper.

This problem could probably take months of your time. I am not even sure it is doable. But I give it to you anyway as an example of a research problem, i.e. a problem that fits two criteria: (i) It is interesting/important; and (ii) no-one has done it already (to the best of my knowledge).

I actually spent a few days on it. I got some tantalizing results, but not a very complete or compelling picture. It was very easy to find a broad peak in the orbital deviation from circularity whose maximum was fairly near the 2:1 resonance. (See the wikipedia article for what this means.) “Easy” means that it is seen even for short simulations of around 13 Jupiter years. Longer simulations of 130 years make the peak higher, but then an increase to 1300 years produces no further change.

Meanwhile a second peak in the neighborhood of 5:3 developed with longer simulations, but its maximum is not in the right place.

Also developing in longer simulations (1300 and 13000 Jupiter years) produced the most encouraging result: a very sharp peak exactly where it should be at 3:1!

Here are my results: Something sort of right is happening, but there is a lot that is not fitting in too well.

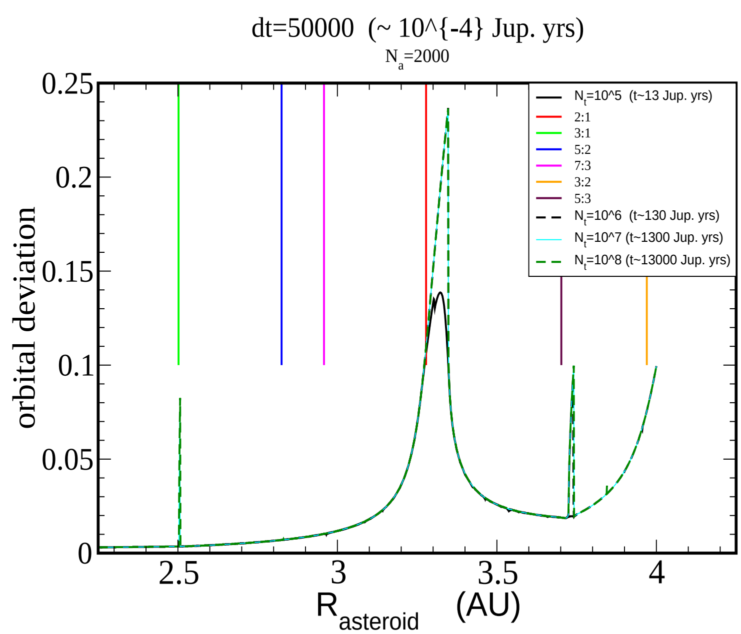

Figure 3: My results. I measured the maximal amount an asteroid’s orbit deviated from its starting cirular orbit under the pull of Jupiter and the sun. What I wanted to see was large values (peaks) of this orbital deviation lining up with resonance positions. The 3:1 did. The 2:1, maybe kinda. I saw something roughly at the 5:3, but not really in the right place. 5:2, 7:3 and 3:2 were all missing.