COSMOS 2022, Computational Lab 07

[01] Consider the matrices A and B:

$$A = \begin{pmatrix} -3 & 2 \\ 1 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 2 \\ 7 & -5 \end{pmatrix}$$

What is $A + B$? What is $B + A$? Is matrix addition commutative?

What is $A \times B$? What is $B \times A$? Is matrix multiplication commutative?

[02] Consider the matrices A, B and C:

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}$$

What is $\big(A \times B\big) \times C$? Note: The position of the parentheses emphasizes you must multiply $A \times B$ *first* and only then multiply by C.

What is $A \times \big(B \times C\big)$? Note: This time the position of the parentheses emphasizes you must multiply $B \times C$ *first* and only then multiply by A.

What law of arithmetic does matrix multiplication appear to satisfy from this example?

[03] The determinant of a 2×2 matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is given by $\det M \equiv ad - bc$. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$$

What is $\det A$? What is $\det B$?

What is $A \times B$? What is $\det \big(A \times B\big)$?

What theorem might hold relating $\det A$, $\det B$, and $\det \big(A \times B\big)$?

If your conjectured theorem is true, what must be true of $\det \big(A \times B\big)$ and $\det \big(B \times A\big)$?

To test, go ahead and compute $B \times A$. What is $\det \big(B \times A\big)$? Is $\det \big(A \times B\big) = \det \big(B \times A\big)$?

Pretty cool! Even though $A \times B$ and $B \times A$ are completely different matrices, their determinants are the same!

[04] The commutator of two matrices A and B is written as $\big[A, B\big]$ and is defined to be

$$\big[A, B\big] \equiv A \times B - B \times A$$

Note: The rule for subtracting two matrices, is to subtract their corresponding entries, very similar to the rule for adding two matrices. Compute the commutator of

$$A = \begin{pmatrix} -1 & 2 & 0 \\ 1 & 3 & 1 \\ -2 & 0 & 4 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{pmatrix}$$

What is $\big[A, B\big]$?

[05] Consider the matrices A and B:

$$A = \begin{pmatrix} -3 & 2 \\ 1 & -4 \\ 0 & 1 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 3 & 2 \\ 7 & -5 \\ 4 & 4 \end{pmatrix}$$

What is $A + B$? What is $A \times B$?

[06] Consider the matrices A and B:

$$A = \begin{pmatrix} -3 & 2 & 0 \\ 1 & -4 & 1 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 3 & 2 \\ 7 & -5 \\ 4 & 4 \end{pmatrix}$$

What is $A + B$? What is $A \times B$? If A has n rows and m columns and B has p rows and q columns, what has to be true for $A + B$ to be defined? What has to be true for $A \times B$ to be defined? What are the dimensions (numbers of rows and columns) of $A \times B$?

[07] The 'identity' matrix is a square matrix with the number 1 down the diagonal and 0 everywhere else. For example, the 2×2 identity matrix is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

What is $I \times A$, where A is the matrix in [01]?

What is $I \times B$, where B is the matrix in [01]?

What is $A \times I$, where A is the matrix in [01]?

[08] The formula for the determinant of a 3×3 matrix

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is $\det M \equiv a(ei - hf) - b(di - gf) + c(dh - eg)$.

Compute $\det A$ and $\det B$ for the matrices of problem [04].

Then compute $\det\big(A \times B\big)$. Does the rule you noticed in problem [03] work for 3×3 matrices?