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Matematica discreta

Aritmetica entera y modular
(Sesiones 10-12)

MD, aritmetica

Ejercicio 2

$28x - 36y = 44 \qquad x, y \in \mathbb{Z}$

$ax + by = c$

$a, b, c \in \mathbb{Z}$

$d = mcd(a,b) \Rightarrow d \mid a \quad y \quad d \mid b \quad d \mid c$

$d \mid ax + by$

$d = mcd(a,b) \mid c \qquad c = kd$ algún $k \in \mathbb{Z}$

Bezout

$d = as + bt \qquad c = k.d = \underbrace{ask}_{x} + \underbrace{btk}_{y}$

$x_0 = ks = \frac{c}{d} s$

$y_0 = kt = \frac{c}{d} t$

1. mcd

$d = mcd(a,b) = mcd(36, 28)$

$36 = 28 \times 1 + 8$

$28 = 8 \times 3 + 4$

$8 = 4 \times 2 + 0$

$mcd(36, 28) = 4$

$4 \mid 44 \Rightarrow$ hay soluciones

2_ Bezout

$as + bt = d$

$28s + 36t = 4$

$4 = 28 + 8(-3)$

$4 = 28 + (36 + 28(-1))(-3)$

$4 = 28 + 36(-3) + 28(3) = 28(4) + 36(-3)$

$(s, t) = (4, -3)$

3_ sol particular de la ecuacion

$\left.\begin{array}{l} d = 4 \\ dn = 44 \end{array}\right\} \Rightarrow n = 11$

$(x_0, y_0) = (ns, nt) = (44, -33)$

4_ Determinar α, β

$\begin{cases} a = \alpha d \\ b = \beta d \end{cases} \rightarrow \begin{cases} 28 = \alpha 4 \\ 36 = \beta 4 \end{cases} \rightarrow \begin{cases} \alpha = 7 \\ \beta = 9 \end{cases}$

5_ solución general de la ecuación

$$\begin{cases} x = x_0 + k\beta = 44 + 9k \\ y = y_0 - k\alpha = -33 - 7k \end{cases} \qquad k \in \mathbb{Z}$$

6_ Comprobación $k = 1$

$$\begin{cases} x = 44 + 9 = 53 \\ y = -33 - 7 = -40 \end{cases}$$

$$28(53) + 36(-40) = 44$$

$$1484 - 1440 = 44$$

Práctica 3 =

$$1000 \leq z \leq 1500$$

z = equipos

x = contenedores de 68 equipos

$$\begin{cases} z \equiv 0 \pmod{68} \\ z \equiv 32 \pmod{20} \end{cases} \rightarrow \begin{cases} z = 68x \\ z = 32 + 20y \end{cases} \Rightarrow 68x - 20y = 32$$

$$ax + by = c$$

1_ mcd

$d = mcd(a,b) = mcd(68,20)$

$68 = 20 \times 3 + 8$

$20 = 8 \times 2 + 4$

$8 = 4 \times 2 + 0$

$d = 4$

$4 \mid 32 \Rightarrow$ hay soluciones

2_ Bezout

$as + bt = d$

$68s + 20t = 4$

$4 = 20 + 8(-2)$

$4 = 20 + (68 + 20(-3))(-2)$

$4 = 20 + 68(-2) + 20(6) = 68(-2) + 20(7)$

$(s,t) = (-2,7)$

3_ sol Particular

$\left.\begin{array}{l} d = 4 \\ d_n = 32 \end{array}\right\} \Rightarrow n = 8$

$(x_0, y_0) = (ns, nt) = (-16, 56)$

4 – Determinar α, β

$$\begin{cases} a = \alpha d \\ b = \beta d \end{cases} \rightarrow \begin{cases} 68 = \alpha 4 \\ 20 = \beta 4 \end{cases} \rightarrow \begin{cases} \alpha = 17 \\ \beta = 5 \end{cases}$$

5 – Solución general

$$\begin{cases} x = x_0 + K\beta = -16 + 5\cancel{\beta}K \\ y = y_0 - K\alpha = 56 - 17K \end{cases} \qquad K \in \mathbb{Z}$$

6 – Comprobación $k=1$

$$\begin{cases} x = -16 + 5 = -11 \\ y = 56 - 17 = 39 \end{cases}$$

$$68(-11) + 20(39) = 32$$
$$-748 + 780 = 32 \quad \checkmark$$

$$z = 68x \rightarrow 68(-16+5K) = -1088 + 340K$$

$$z = 32 - 20y \Rightarrow 32 - 20(56 - 17K) = 32 - 1120 + 340K$$
$$= -1088 + 340K$$

$$z = -1088 \pmod{340} = \underbrace{1292 \text{ equipos}} \quad \checkmark$$

$\downarrow$
suponer $z = 1000$

$$340K = 1000 + 1088$$

$$K = \frac{2088}{340} = 6{,}\ldots$$

$$340 \cdot 6 = 2040 \rightarrow 2040 - 1088 = 952 < 1000$$

$\downarrow$ $+340 \rightarrow K = 7 \,/\, z = 1292$

Ejercicio 3

$[33]^{-1} \quad \mathbb{Z}_{50}$

1- $mcd(33,50) = d = mcd(a,b)$

$50 = 33 \times 1 + 17$

$33 = 17 \times 1 + 16$

$17 = 16 \times 1 + 1$

$16 = 1 \times 16 + 0$

$d = mcd = 1 \rightarrow \exists$ inverso

2- Bezout

$as + bt = d$

$50s + 33t = 1$

$1 = 17 + 16(-1)$

$1 = 17 + (33 + 17(-1))(-1) = 33(-1) + 17(2)$

$1 = 33(-1) + (50 + 33(-1))(2)$

$1 = 50(2) + 33(-3)$

$(s,t) = (2,-3)$

$[1] = \cancel{[50]}[2] + [33][-3] \rightarrow [1] = [33][-3]$

0

$[33]^{-1} = [-3] = [50-3] = [47]$

El Inverso de $[33]$ en $\mathbb{Z}_{50}$ es $[47]$

ejercicio 4: $\mathbb{Z}_7$

$$\begin{cases} x + [5]y = [2] \\ [2]x - y = [3] \end{cases}$$

$$\left(\begin{array}{cc|c} 1 & 5 & 2 \\ 2 & -1 & 3 \end{array}\right) \xrightarrow{\cancel{F_2 = F_1 \cdot 2 - F_2}}$$

$$\cancel{\left(\begin{array}{cc|c} 1 & 5 & 2 \\ 0 & -11 & -1 \end{array}\right) \xrightarrow{F_2 = F_2 \cdot -\frac{1}{11}} \left(\begin{array}{cc|c} 1 & 5 & 2 \\ 0 & 1 & \frac{1}{11} \end{array}\right)}$$

$$= \left(\begin{array}{cc|c} 1 & 5 & 2 \\ 2 & 6 & 3 \end{array}\right) \xrightarrow[F_2 = F_2 - 2F_1]{\cancel{F_2 = \ldots}} \left(\begin{array}{cc|c} 1 & 5 & 2 \\ 0 & -4 & -1 \end{array}\right) =$$

$$\left(\begin{array}{cc|c} 1 & 5 & 2 \\ 0 & 3 & 6 \end{array}\right) \xrightarrow{F_2 = \frac{F_2}{3}} \left(\begin{array}{cc|c} 1 & 5 & 2 \\ 0 & 1 & 2 \end{array}\right)$$

$$\boxed{y = [2]}$$

$$x + [5]y = [2]$$
$$x + [10] = [2]$$
$$x = [-8]$$
$$\boxed{x = [6]}$$

~~Ejercicio 5~~

$\cancel{x^2 + [3]x + [4] = 0 \quad x \in \mathbb{Z}_{11}}$

otra forma:

$\mathbb{Z}_7$

$$x + [5]y = [2]$$
$$[2]x + [6]y = [3]$$

~~Eq 1 x 2~~

$$\cancel{[2]x + [10]y = [2]}$$
$$\cancel{- \quad [2]x + [6]y = [3]}$$
$$\cancel{[-4]y = [1]}$$
$$\cancel{[3]y = [1]}$$

Eq 1 x 5

$$[5]x + [25]y = [10]$$
$$+ \quad [2]x + [6]y = [3]$$
$$[7]x + [31]y = [13]$$
$$\downarrow \qquad \downarrow$$
$$0$$
$$[3]y = [6]$$

Ejercicio 6

$$3^{25} \cdot 7^{68} \equiv x \pmod{23}$$

$$3^{22} \cdot 7^{22} \equiv 1 \pmod{23}$$

$$3^{25} = 3^{22} \cdot 3^{3}$$

$$7^{68} = 7^{66} \cdot 7^{2}$$

$$3^{25} \cdot 7^{68} \equiv \underbrace{3^{3} \cdot 7^{2}}_{\downarrow} \pmod{23}$$

$$4x + 1323/23 = 57$$

$$1323 = 23 \cdot 57 + 12 \rightarrow \text{resto}$$

~~Practica 5:~~

~~$z = 700$ cn~~

~~$z = x \pmod{56}$~~

~~$z = y \pmod{21}$~~

Practica Si

$$56x + 21y = 700$$

$$ax + by = c$$

1_ mcd

$d = mcd(a,b) = mcd(56,21)$

$56 = 21 \times 2 + 14$

$21 = 14 \times 1 + 7$

$14 = 7 \times 2 + 0$

$d = 7$

$7 / 700 \rightarrow$ hay solución

2 – Bezout

$$as + bt = d$$

$$56s + 21t = 7$$

$$7 = 21 + 14(-1)$$

$$7 = 21 + (56 + 21(-2))(-1)$$

$$7 = 56(-1) + 21(3)$$

$$(s,t) = (-1,3)$$

3 – Sol particular

$d = 7$

$dn = 700 = c \rightarrow n = 100$

$$(x_0, y_0) = (ns, nt) = (-100, 300)$$

4 – Determinar α, β

$$\begin{cases} a = \alpha d \\ b = \beta d \end{cases} \rightarrow \begin{cases} 56 = \alpha 7 \\ 21 = \beta 7 \end{cases} \rightarrow \begin{cases} \alpha = 8 \\ \beta = 3 \end{cases}$$

5 – Sol general

$$\begin{cases} x = x_0 + \beta k \\ y = y_0 - \alpha k \end{cases} \rightarrow \begin{matrix} = -100 + \overset{3}{\cancel{8}}k \\ = 300 - 8k \end{matrix} \quad k \in \mathbb{Z}$$

6_ Comprobación

$K = 1$

$\cancel{x = -100 + 8 \cdot 1 = -92}$

$\cancel{y = 300 - 3 \cdot 1 = 297}$

$\cancel{56x + 21y = 700}$

$\cancel{56(-92) + 21(297) = 1058 \neq 700}$

$x = -100 + 3 \cdot 1 = -97$

$y = 300 - 8 \cdot 1 = 292$

$56(-97) + 21(292) = 700$

Se puede llevar el trabajo de la siguiente forma

$56x + 21y = 700$

siendo $x = -100 + 3k$

$y = 300 - 8k$ $\quad k \in \mathbb{Z}$

$K = 34$ ↑

$x = -100 + 3 \cdot 34 = 2$

$y = 300 + -272 = 28$

$56(2) + 21(28) = 700$

Para obtener un resultado coherente utilizamos $k \geq 34$

~~Practica 6:~~

$$\cancel{z \equiv 7 \pmod{11} \rightarrow z = 11x + 7}$$

$$\cancel{z \equiv 4 \pmod{17} \Rightarrow z = 17x + 4}$$

$$\cancel{11x + 7 = 17x + 4}$$

$$\cancel{17x - 11x = 7 - 4}$$

$$\cancel{6x = 3}$$

$$\cancel{x = \frac{1}{2}}$$

Practica 6:

$$z \equiv 7 \pmod{11}$$

$$z \equiv 4 \pmod{17}$$

$[a] = [b]$ en $\mathbb{Z}_n$

$\longleftrightarrow a \equiv b \pmod{n}$

· $a - b = kn$ en $\mathbb{Z}n$

$$\begin{cases} z = 11x + 7 \\ z = 17y + 4 \end{cases} \rightarrow 11x + 7 = 17y + 4$$

$$\underbrace{11x - 17y = -3}_{ax + by = c}$$

1_ mcd

$a = 11$
$b = -17$

$d = mcd(a, b) = mcd(11, -17)$

$17 = 11 \times 1 + 6$

$11 = 6 \times 1 + 5$

$6 = 5 \times 1 + 1$

$5 = 1 \times 5 + 0$

$d = 1$

$1 \mid -3 \rightarrow$ hay soluciones

2_ Bezout

$as + bt = d$

$11s \; -17t = 1$

$1 = 6 + 5(-1)$

$1 = 6 + (11 + 6(-1))(-1) = 6(2) + 11(-1)$

$1 = 11(-1) + (17 + 11(-1))(2)$

$1 = 17(2) + 11(-3)$

$(s, t) = (-3, -2)$

3_ sol particular

$d = 1$

$nd = -3 \rightarrow n = -3$

$(x_0, y_0) = (ns, nt) = (9, 6)$

4 - Determinar α, β

$\begin{cases} a = \alpha d \\ b = \beta d \end{cases} \Rightarrow \boxed{\begin{matrix} 11 = \alpha 1 \\ -17 = \beta 1 \end{matrix}}$

5 - Sol general

$\begin{cases} x = x_0 + \beta k = \boxed{9 - 17k} \\ y = y_0 - \alpha k = \boxed{+6 - 11k} \end{cases}$ $k \in \mathbb{Z}$

6 - Comprobación

$k = 1$

$x = 9 - 17 = -8$

$y = +6 - 11 = -5$

$11(-8) - 17(-5) = -3$

~~Los enteros que verifican las condiciones se obtienen de la siguiente forma~~

~~$11x - 17y = -3$~~

$z = 11x + 7 = 11(9 - 17k) + 7 = 99 - 187k + 7 = 106 - 187k$

$z = 17y + 4 = 17(6 - 11k) + 4 = 102 - 187k + 4 = 106 - 187k$

$z \equiv 106 \pmod{187}$

$z = [106]$ en $\mathbb{Z}_{187}$

$S = \{[106]\}$ en $\mathbb{Z}_{187}$

$\mathbb{Z}_n$ $\mathbb{Z}_n^*$ conjunto de invers

Si $mcd(a,n)=1 \rightarrow$ hay inverso

Si $[y] \in \mathbb{Z}_n^* \rightarrow [y]^{\varphi(n)} = [1]$

practica 7:

1_ $\varphi(35) = \varphi(5 \times 7) = \varphi(5) \cdot \varphi(7) = 4 \cdot 6 = [24]$ en $\mathbb{Z}_{35}$

$[27]^{\varphi(35)}$

2_ $mcd(27,35)$

$35 = 27 \times 1 + 8$

$27 = 8 \times 3 + 3$

$8 = 3 \times 2 + 2$

$3 = 2 \times 1 + 1$

$2 = 1 \times 2 + 0$

Teorema de Euler

$mcd = 1 \rightarrow [27] \in \mathbb{Z}_{35}^* \rightarrow [27]^{\varphi(35)} = [1]$ //

3_ $2^{99} / 35$

$2^{99} = x \pmod{35}$

$2^{99} = 2^{35} \cdot 2^{35} \cdot 2^{29}$

$2^{99} = 2^{34} \cdot 2^{34} \cdot 2^{31}$

$2^{34} \ldots$

3 - $2^{99} \equiv x \pmod{35}$

$2 \in \mathbb{Z}^*$
$35 \in \mathbb{Z}^*$

φ Euler

$2^{\varphi(35)} \equiv 1 \pmod{35}$

$\varphi(35) = \varphi(5)\varphi(7)$
$= 4 \cdot 6 = 24$

$mcd(2,35) = 1$

$$2^{99} = (2^{24})^4 \cdot 2^3 \equiv 8 \equiv \boxed{8 \pmod{35}}$$

4 - $27^3 = 19683$ en $\mathbb{Z}_{35}$
$= 35 \cdot 562 + 13$
$\equiv 13 \pmod{35} \rightarrow [27]^3 = [13]$ en $\mathbb{Z}_{35}$

5- $27^{99} \equiv x \pmod{35}$

$27 \in \mathbb{Z}^+$

$35 \in \mathbb{Z}^+$

$\text{mcd}(27, 35) = 1$

$35 = 27 \times 1 + 8$

$27 = 8 \times 3 + 3$

$8 = 3 \times 2 + 2$

$3 = 2 \times 1 + 1$

$2 = 1 \times 2 + 0$

$\text{mcd} = 1$

$\xrightarrow{\text{Euler}}$ $27^{\varphi(35)} \equiv 1 \pmod{35}$

$\varphi(35) = \varphi(5)\,\varphi(7)$

$= 4 \cdot 6 = 24$

$27^{99} \equiv 27^{24 \times 4 + 3}$

$\equiv (27^{24})^4 \cdot 27^3 \equiv 1^4 \cdot 27^3 \equiv 19683$

$\equiv 35 \times 562 + 13$

$\equiv 13 \pmod{35}$

R: $35 - 13 =$ faltan 22 años