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Matematica discreta

Aritmetica entera y modular
(Sesiones 10-12)

MD, aritmética

Ejercicio 2

$28x - 36y = 44 \quad x, y \in \mathbb{Z}$

$ax + by = c$

$a, b, c \in \mathbb{Z}$

$d = mcd(a,b) \Rightarrow d \mid a \text{ y } d \mid b \quad d \mid c$

$d \mid ax + by$

$d = mcd(a,b) \mid c \qquad c = kd$ algún $k \in \mathbb{Z}$

Bezout

$d = as + bt \qquad c = k \cdot d = \underbrace{aks}_{x} + \underbrace{btk}_{y}$

$x_0 = ks = \frac{c}{d} s$

$y_0 = kt = \frac{c}{d} t$

1. mcd

$d = mcd(a,b) = mcd(36, 28)$

$36 = 28 \cdot 1 + 8$

$28 = 8 \cdot 3 + 4$

$8 = 4 \cdot 2 + 0$

$\therefore \quad mcd(36, 28) = 4$

$4 \mid 44 \Rightarrow$ hay soluciones

2- Bezout

$as + bt = d$

$28s + 36t = 4$

$4 = 28 + 8(-3)$

$4 = 28 + (36 + 28(-1))(-3)$

$4 = 28 + 36(-3) + 28(3) = 28(4) + 36(-3)$

$(s, t) = (4, -3)$

3- Sol particular de la ecuación

$d = 4$
$dn = 44$ $\Big\} \Rightarrow n = 11$

$(x_0, y_0) = (ns, nt) = (44, -33)$

4- Determinar α, β

$$\begin{cases} a = \alpha d \\ b = \beta d \end{cases} \rightarrow \begin{cases} 28 = \alpha 4 \\ 36 = \beta 4 \end{cases} \rightarrow \begin{cases} \alpha = 7 \\ \beta = 9 \end{cases}$$

5_ solución general de la ecuación

$$\begin{cases} x = x_0 + k\beta = 44 + 9k \\ y = y_0 - k\alpha = -33 - 7k \end{cases} \quad k \in \mathbb{Z}$$

6_ Comprobación $k = 1$

$$\begin{cases} x = 44 + 9 = 53 \\ y = -33 - 7 = -40 \end{cases}$$

$$28(53) + 36(-40) = 44$$

$$1484 - 1440 = 44$$

Práctica 3 =

$$1000 \leq z \leq 1500$$

z = equipos

x = contenedores de 68 equipos

$$\begin{cases} z \equiv 0 \pmod{68} \\ z \equiv 32 \pmod{20} \end{cases} \rightarrow \begin{cases} z = 68x \\ z = 32 + 20y \end{cases} \Rightarrow 68x - 20y = 32$$

$$ax + by = c$$

1- mcd

$d = mcd(a,b) = mcd(68,20)$

$68 = 20 \times 3 + 8$

$20 = 8 \times 2 + 4$

$8 = 4 \times 2 + 0$

$d = 4$

$4 \mid 32 \Rightarrow$ hay soluciones

2- Bezout

$as + bt = d$

$68s + 20t = 4$

$4 = 20 + 8(-2)$

$4 = 20 + [68 + 20(-3)](-2)$

$4 = 20 + 68(-2) + 20(6) = 68(-2) + 20(7)$

$(s,t) = (-2,7)$

3- Sol Particular

$d = 4 \mid b \Rightarrow n = 8$

$d_0 = 32$

$(x_0, y_0) = (ns, nt) = (-16, 56)$

4- Determinar α, β

$$\begin{cases} a = \alpha d \\ b = \beta d \end{cases} \rightarrow \begin{cases} 68 = \alpha 4 \\ 20 = \beta 4 \end{cases} \rightarrow \begin{cases} \alpha = 17 \\ \beta = 5 \end{cases}$$

5 - Solución general

$$\begin{cases} x_0 = x_0 + k\beta = -16 + 5k \\ y_0 = y_0 - k\alpha = 56 - 17k \end{cases} \qquad k \in \mathbb{Z}$$

6 - Comprobación $k=1$

$$\begin{cases} x = -16 + 5 = -11 \\ y = 56 - 17 = 39 \end{cases}$$

$$68(-11) + 20(39) = 32$$
$$-748 + 780 = 32$$ //

$$z = 68x \rightarrow 68(-16+5k) = -1088 + 340k$$

$$z = 32 - 20y \Rightarrow 32 - 20(56 - 17k) = 32 - 1120 + 340k$$
$$= -1088 + 340k$$

$$z = -1088 \pmod{340} = 1292 \text{ equipos}$$ //

Bien

↓ subset $z = 1000$

$$340k = 1000 + 1088$$

$$k = \frac{2088}{340} = 6,\widehat{1}$$

$$340 \cdot 6 = 2040 \rightarrow 2040 - 1088 = 952 < 1000$$

↓ $+340 \rightarrow k = 7, z = 1292$

Ejercicio 3

$[33]^{-1}$ en $\mathbb{Z}_{50}$

1_ $mcd(33,50) = d = mod(a,b)$

$50 = 33 \cdot 1 + 17$

$33 = 17 \cdot 1 + 16$

$17 = 16 \cdot 1 + 1$

$16 = 1 \cdot 16 + 0$

$d = mcd = 1 \rightarrow \exists$ inverso

2_ Bezout

$as + bt = d$

$50s + 33t = 1$

$1 = 17 + 16(-1)$

$1 = 17 + (33 + 17(-1))(-1) = 33 + 17(2)$

$1 = 33(-1) + (50 + 33(-1))(2)$

$1 = 50(2) + 33(-3)$

$(s,t) = (2,-3)$

$[1] = [50][2] + [33][-3] \rightarrow [1] = [33][-3]$

$[33] = [-3] = [50-3] = [47]$

El inverso de $[33]$ en $\mathbb{Z}_{50}$ es $[47]$

ejercicio 4: $\mathbb{Z}_7$

$$\begin{cases} x + [5]y = [2] \\ [2]x - y = [3] \end{cases}$$

$$\left(\begin{array}{cc|c} 1 & 5 & 2 \\ 2 & -1 & 3 \end{array}\right) \quad F_2 \leftarrow F_1 - F_2$$

$$\left(\begin{array}{cc|c} 1 & 5 & 2 \\ 0 & -11 & -1 \end{array}\right) \xrightarrow{F_2 = F_2 \cdot -\frac{1}{11}} \left(\begin{array}{cc|c} 1 & 5 & 2 \\ 0 & 1 & \frac{1}{11} \end{array}\right)$$

$$= \left(\begin{array}{cc|c} 1 & 5 & 2 \\ 2 & 6 & 3 \end{array}\right) \xrightarrow[F_2 = F_2 - 2F_1]{} \left(\begin{array}{cc|c} 1 & 5 & 2 \\ 0 & -4 & -1 \end{array}\right) =$$

$$\left(\begin{array}{cc|c} 1 & 5 & 2 \\ 0 & 3 & 6 \end{array}\right) \xrightarrow{F_2 = \frac{F_2}{3}} \left(\begin{array}{cc|c} 1 & 5 & 2 \\ 0 & 1 & 2 \end{array}\right)$$

$$\boxed{y = [2]}$$

$$x + [5]y = [2]$$

$$x + [10] = [2]$$

$$x = [-8]$$

$$\boxed{x = [6]}$$

~~Esercizio 5~~

~~$x^2 + [3]x + [4] = 0 \quad x \in \mathbb{Z}_{11}$~~

altra formai

$\mathbb{Z}_7$

$$x + [5]y = [2]$$
$$[2]x + [6]y = [3]$$

~~eq 1 x 2~~

$$[2]x + [10]y = [4]$$
$$[2]x + [6]y = [3]$$

$$[4]y = [1]$$

$$[3]y = [1]$$

eq 1 x 5

$$[5]x + [25]y = [10]$$
$$[2]x + [6]y = [3]$$

$$[7]x + [31]y = [13]$$

$$0$$

$$[3]y = [6]$$

Ejercicio 6

$3^{25} \cdot 7^{68} \equiv X \pmod{23}$

$3^{22} \cdot 7^{22} \equiv 1 \pmod{23}$

$3^{25} = 3^{22} \cdot 3^{3}$

$7^{68} = 7^{66} \cdot 7^{2}$

$3^{25} \cdot 7^{68} = \underbrace{3^{3} \cdot 7^{2}} \pmod{23}$

↓

$7323 / 23 = 57 \rightarrow$

$7323 = 23 \cdot 57 + 12$ → resto

Practica 5:

$z = 700$ cm

$z = x \pmod{56}$

$z = y \pmod{21}$

Practica 5i

$$56x + 21y = 700$$

$$ax + 6y = 0$$

1- mcd

$d = mcd(a,6) = mcd(56,21)$

$56 = 21 \times 2 + 14$

$21 = 14 \times 1 + 7$

$14 = 7 \times 2 + 0$

$d = 7$

$7 / 700 \rightarrow$ hay solución

2- Bezout

$$as + bt = d$$

$$56s + 21t = 7$$

$$7 = 21 + 14(-1)$$

$$7 = 21 + (56 + 21(-2))(-1)$$

$$7 = 56(-1) + 21(3)$$

$$(s,t) = (-1,3)$$

3- Sol particular

$$d = 7 \quad \rightarrow n = 100$$
$$d_n = 700 = c$$

$$(x_0, y_0) = (ns, nt) = (-100, 300)$$

4- Determinar α, β

$$\begin{cases} a = \alpha d \\ b = \beta d \end{cases} \rightarrow \begin{cases} 56 = \alpha 7 \\ 21 = \beta 7 \end{cases} \rightarrow \begin{cases} \alpha = 8 \\ \beta = 3 \end{cases}$$

5- Sol general

$$\begin{cases} x = x_0 + \beta k \\ y = y_0 - \alpha k \end{cases} \rightarrow \begin{array}{l} = -100 + 3k \\ = 300 - 8k \end{array} \quad k \in \mathbb{Z}$$

6_ Comprobación

$K=1$

$x = -100 + 3.1 = -92$

$y = 300 - 8.1 = 292$

$56x + 21y = 700$

$56(-92) + 21(292) = 1058 \neq 700$

$x = -100 + 3.1 = -97$

$y = 300 - 8.1 = 292$

$56(-97) + 21(292) = 700$ //

Se puede llevar el Trabajo de la siguiente forma

$$56x + 21y = 700$$

siendo $x = -100 + 3k$

$y = 300 - 8k$ $\quad k \in \mathbb{Z}$

$K = 34$ ↑

$x = -100 + 3 \cdot 34 = 2$

$y = 300 + -272 = 28$

$56(2) + 21(28) = 700$

Para obtener un resultado coherente utilizamos $K \geq 34$

Practica 6:

$z \equiv 7 \pmod{11} \rightarrow z = 11x + 7$

$z \equiv 4 \pmod{17} \rightarrow z = 17x + 4$

$11x + 7 = 17x + 4$

$17x - 11x = 7 - 4$

$6x = 3$

$x = \frac{1}{2}$

Practica 6:

$z \equiv 7 \pmod{11}$

$z \equiv 4 \pmod{17}$

$[a] = [b]$ en $\mathbb{Z}_n$

$\leftrightarrow a \equiv b \pmod{n}$

$\cdot\ a - b = K_n$ en $\mathbb{Z}_n$

$$\begin{cases} z = 11x + 7 \\ z = 17y + 4 \end{cases} \rightarrow 11x + 7 = 17y + 4$$

$11x - 17y = -3$

$ax + by = c$

$a = 11$
$b = -17$

1_ mcd

$d = mcd(a,b) = mcd(11,-17)$

$17 = 11 \times 1 + 6$
$11 = 6 \times 1 + 5$
$6 = 5 \times 1 + 1$
$5 = 1 \times 5 + 0$

$d = 1$

$1 | -3 \rightarrow$ hay soluciones

2_ Bezout

$as + bt = d$
$11s - 17t = 1$

$1 = 6 + 5(-1)$
$1 = 6 + (11 + 6(-1))(-1) = 6(2) + 11(-1)$
$1 = 11(-1) + (17 + 11(-1))(2)$
$1 = 17(2) + 11(-3)$

$(s,t) = (-3,-2)$

3_ sol particular

$d = 1$
$nd = -3 \rightarrow n = -3$

$(x_0, y_0) = (ns, nt) = (9, -6)$

4 - Determinar α, β

$\begin{cases} a = \alpha d \\ b = \beta d \end{cases} \rightarrow \begin{array}{l} 11 = \alpha 1 \\ -17 = \alpha \beta 1 \end{array}$

5 - Sol general

$\begin{cases} x = x_0 + \beta k = 9 - 17k \\ y = y_0 - \alpha k = 6 - 11k \end{cases} \quad k \in \mathbb{Z}$

6 - Comprobación

$k = 1$

$x = 9 - 17 = -8$

$y = 6 - 11 = -5$

$11(-8) - 17(-5) = -3$

Los enteros que verifican las condiciones se obtienen de la siguiente forma

$11x - 17y = 3$

$z = 11x + 7 = 11(9 - 17k) + 7 = 99 - 187k + 7 = 106 - 187k$

$z = 17y + 4 = 17(6 - 11k) + 4 = 102 - 187k + 4 = 106 - 187k$

$z \equiv 106 \pmod{187}$

$z = [106]$ en $\mathbb{Z}_{187}$

$S = \{[106]\}$ en $\mathbb{Z}_{187}$

$\mathbb{Z}_n$ $\mathbb{Z}_n^*$ conjunto de inversos

Si $mcd(a,n)=1$ → hay inverso

$\forall [y] \in \mathbb{Z}_n^* \rightarrow [y]^{\phi(n)} = [1]$

Practica 7:

1- $\phi(35) = \phi(5 \times 7) = \phi(5) \cdot \phi(7) = 4 \cdot 6 = [24]$ en $\mathbb{Z}_{35}$

2- $mcd(27, 35)$ $\quad [27]^{\phi(35)}$

$35 = 27 \times 1 + 8$

$27 = 8 \times 3 + 3$

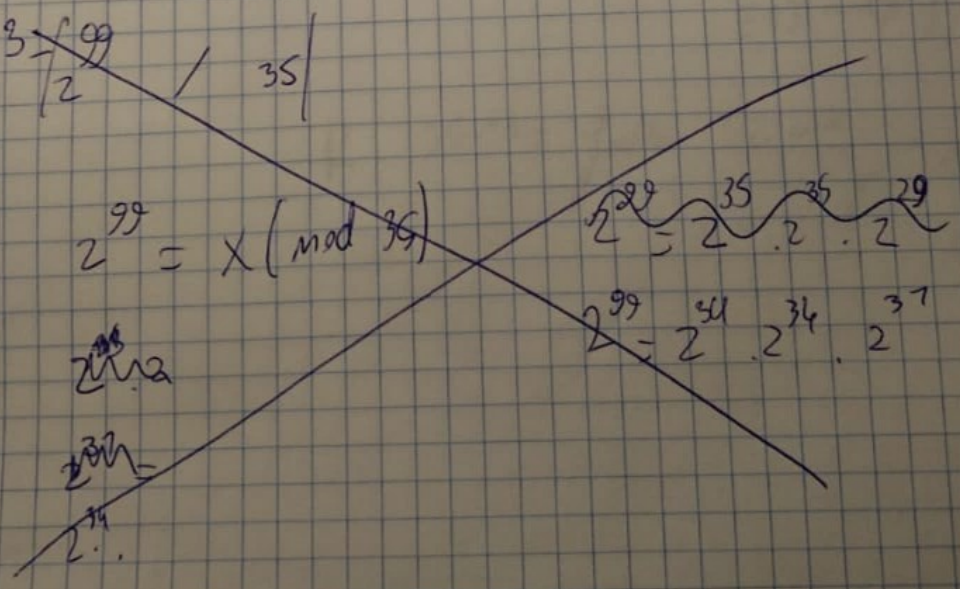
$8 = 3 \times 2 + 2$

$3 = 2 \times 1 + 1$

$2 = 1 \times 2 + 0$

Teorema de Euler

$mcd = 1 \rightarrow [27] \in \mathbb{Z}_{35}^* \rightarrow [27]^{\phi(35)} = [1]$ //



3- $2^{99} = x \pmod{35}$

$2 \in \mathbb{Z}^*$
$35 \in \mathbb{Z}^*$ → Euler

$2^{\ell(35)} = 1 \pmod{35}$
$\ell(35) = \ell(5)\ell(7)$
$= 4 \cdot 6 = 24$

$mcd(2, 35) = 1$

$2^{99} = \left|2^{24}\right|^4 \cdot 2^3 = 8 \equiv 8 \pmod{35}$ //

4- $27^3 = 19683$ en $\mathbb{Z}_{35}$
$= 35 \cdot 562 + 13$
$= 13 \pmod{35}$ → $[27]^3 = [13]$ en $\mathbb{Z}_{35}$ //

5- $27^{99} = x \pmod{35}$

$27 \in \mathbb{Z}^+$
$35 \in \mathbb{Z}^+$
$mcd(27, 35) = 1$

↳ Euler → $27^{\phi(35)} = 1 \pmod{35}$

$\phi(35) = \phi(5)\,\phi(7)$
$= 4 \cdot 6 = 24$

$35 = 27 \times 1 + 8$
$27 = 8 \times 3 + 3$
$8 = 3 \times 2 + 2$
$3 = 2 \times 1 + 1$
$2 = 1 \times 2 + 0$
$mcd = 1$

$27^{99} \equiv 27^{24 \times 4 + 3}$
$\equiv (27^{24})^4 \cdot 27^3 \equiv 1^4 \cdot 27^3 \equiv 19683$
$\equiv 35 \times 562 + 13$
$\equiv 13 \pmod{35}$

R: $35 - 13 =$ Jaltan 22 cinöt