Points, lines, and planes

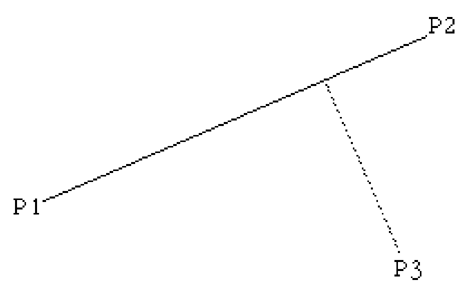
In what follows are various notes and algorithms dealing with points, lines, and planes.

Minimum Distance between a Point and a Line

Written by Paul Bourke
October 1988

This note describes the technique and gives the solution to finding the shortest distance from a point to a line or line segment. The equation of a line defined through two points **P1** (x1,y1) and **P2** (x2,y2) is

$$\mathbf{P} = \mathbf{P1} + u\,(\mathbf{P2} - \mathbf{P1})$$



The point **P3** (x3,y3) is closest to the line at the tangent to the line which passes through **P3**, that is, the dot product of the tangent and line is 0, thus

$$(\mathbf{P3} - \mathbf{P}) \text{ dot } (\mathbf{P2} - \mathbf{P1}) = 0$$

Substituting the equation of the line gives

$$[\mathbf{P3} - \mathbf{P1} - u(\mathbf{P2} - \mathbf{P1})] \text{ dot } (\mathbf{P2} - \mathbf{P1}) = 0$$

Solving this gives the value of u

$$u = \frac{(x3 - x1)(x2 - x1) + (y3 - y1)(y2 - y1)}{\|p2 - p1\|^2}$$

Substituting this into the equation of the line gives the point of intersection (x,y) of the tangent as

$$x = x1 + u\,(x2 - x1)$$
$$y = y1 + u\,(y2 - y1)$$

The distance therefore between the point **P3** and the line is the distance between (x,y) above and **P3**.

Notes

- The only special testing for a software implementation is to ensure that **P1** and **P2** are not coincident (denominator in the equation for u is 0)
- If the distance of the point to a line segment is required then it is only necessary to test that u lies between 0 and 1.
- The solution is similar in higher dimensions.

Source code contributions
C source from Damian Coventry: C source code
VBA from Brandon Crosby: VBA source code
Dephi from Graham O'Brien: Delphi version
"R" version from Gregoire Thomas: pointline.r
JAVA version from Pieter Iserbyt: DistancePoint.java
LabView implementation from Chris Dancer: Pointlinesegment.vi.zip
Right side distance by Orion Elenzil: rightside
VBA VB6 by Thomas Ludewig: vbavb6.txt

Minimum Distance between a Point and a Plane

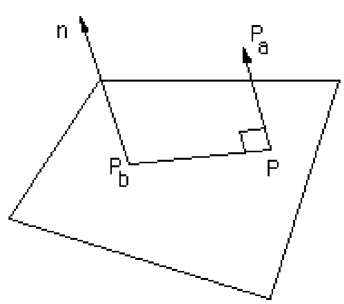
Written by Paul Bourke
March 1996

Let $\mathbf{P_a} = (x_a, y_a, z_a)$ be the point in question.

A plane can be defined by its normal $\mathbf{n} = (A, B, C)$ and any point on the plane $\mathbf{P_b} = (x_b, y_b, z_b)$

Any point $\mathbf{P} = (x,y,z)$ lies on the plane if it satisfies the following

$$A x + B y + C z + D = 0$$



The minimum distance between P_a and the plane is given by the absolute value of

$$(A x_a + B y_a + C z_a + D) / \text{sqrt}(A^2 + B^2 + C^2) \quad \ldots 1$$

To derive this result consider the projection of the line ($\mathbf{P_a}$ - $\mathbf{P_b}$) onto the normal of the plane **n**, that is just $||\mathbf{P_a} - \mathbf{P_b}||$ cos(theta), where theta is the angle between ($\mathbf{P_a}$ - $\mathbf{P_b}$) and the normal **n**. This projection is the minimum distance of $\mathbf{P_a}$ to the plane.

This can be written in terms of the dot product as

$$\text{minimum distance} = (\mathbf{P_a} - \mathbf{P_b}) \text{ dot } \mathbf{n} / ||\mathbf{n}||$$

That is

$$\text{minimum distance} = (A (x_a - x_b) + B (y_a - y_b) + C (z_a - z_b)) / \text{sqrt}(A^2 + B^2 + C^2) \qquad ...\ 2$$

Since point (x_b, y_b, z_b) is a point on the plane

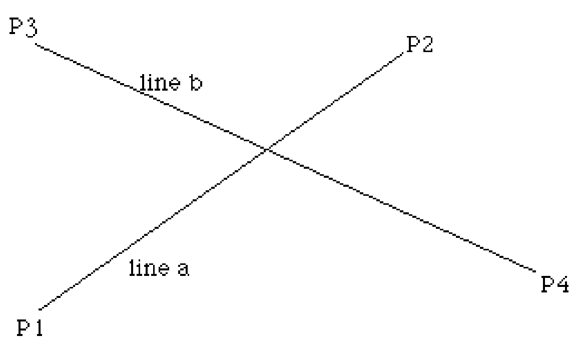
$$A x_b + B y_b + C z_b + D = 0 \qquad ...\ 3$$

Substituting equation 3 into equation 2 gives the result shown in equation 1.

Intersection point of two line segments in 2 dimensions

Written by Paul Bourke
April 1989

This note describes the technique and algorithm for determining the intersection point of two lines (or line segments) in 2 dimensions.



The equations of the lines are

$$\mathbf{P_a} = \mathbf{P1} + u_a (\mathbf{P2} - \mathbf{P1})$$

$$\mathbf{P_b} = \mathbf{P3} + u_b (\mathbf{P4} - \mathbf{P3})$$

Solving for the point where $\mathbf{P_a} = \mathbf{P_b}$ gives the following two equations in two unknowns (u_a and u_b)

$$x1 + u_a (x2 - x1) = x3 + u_b (x4 - x3)$$

and

$$y1 + u_a (y2 - y1) = y3 + u_b (y4 - y3)$$

Solving gives the following expressions for u_a and u_b

$$u_a = \frac{(x4 - x3)(y1 - y3) - (y4 - y3)(x1 - x3)}{(y4 - y3)(x2 - x1) - (x4 - x3)(y2 - y1)}$$
$$u_b = \frac{(x2 - x1)(y1 - y3) - (y2 - y1)(x1 - x3)}{(y4 - y3)(x2 - x1) - (x4 - x3)(y2 - y1)}$$

Substituting either of these into the corresponding equation for the line gives the intersection point. For example the intersection point (x,y) is

$$x = x1 + u_a (x2 - x1)$$

$$y = y1 + u_a (y2 - y1)$$

Notes:

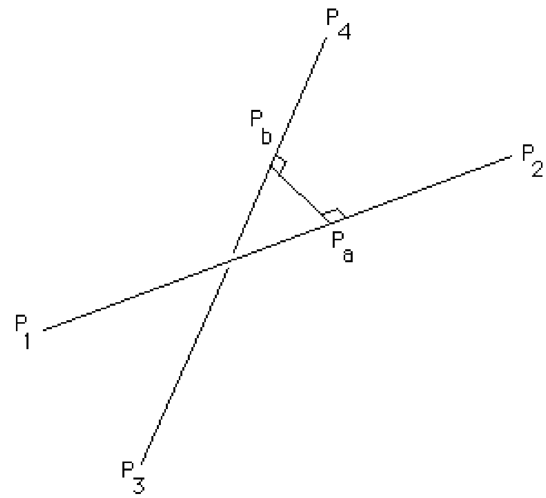
- The denominators for the equations for u_a and u_b are the same.
- If the denominator for the equations for u_a and u_b is 0 then the two lines are parallel.
- If the denominator and numerator for the equations for u_a and u_b are 0 then the two lines are coincident.
- The equations apply to lines, if the intersection of line segments is required then it is only necessary to test if u_a and u_b lie between 0 and 1. Whichever one lies within that range then the corresponding line segment contains the intersection point. If both lie within the range of 0 to 1 then the intersection point is within both line segments.

Source code contributions
Original C code by Paul Bourke.
C++ contribution by Damian Coventry.
LISP implementation by Paul Reiners.
C version for Rockbox firmware by Karl Kurbjun.
C# version by Olaf Rabbachin.
VB.net version by Olaf Rabbachin.
VBA implementation by Giuseppe Iaria.
Javascript version by Leo Bottaro.

The shortest line between two lines in 3D

Written by Paul Bourke
April 1998

Two lines in 3 dimensions generally don't intersect at a point, they may be parallel (no intersections) or they may be coincident (infinite intersections) but most often only their projection onto a plane intersect.. When they don't exactly intersect at a point they can be connected by a line segment, the shortest line segment is unique and is often considered to be their intersection in 3D.



The following will show how to compute this shortest line segment that joins two lines in 3D, it will as a bi-product identify parallel lines. In what follows a line will be defined by two points lying on it, a point on line "a" defined by points P_1 and P_2 has an equation.

$$P_a = P_1 + mu_a (P_2 - P_1)$$

similarly a point on a second line "b" defined by points P_4 and P_4 will be written as

$$P_b = P_3 + mu_b (P_4 - P_3)$$

The values of mu_a and mu_b range from negative to positive infinity. The line segments between P_1 P_2 and P_3 P_4 have their corresponding mu between 0 and 1.

There are two approaches to finding the shortest line segment between lines "a" and "b". The first is to write down the length of the line segment joining the two lines and then find the minimum. That is, minimise the following

$$\| P_b - P_a \|^2$$

Substituting the equations of the lines gives

$$\| P_1 - P_3 + mu_a (P_2 - P_1) - mu_b (P_4 - P_3) \|^2$$

The above can then be expanded out in the (x,y,z) components. There are conditions to be met at the minimum, the derivative with respect to mu_a and mu_b must be zero. Note: it is easy to convince oneself that the above function only has one minima and no other minima or maxima. These two equations can then be solved for mu_a and mu_b, the actual intersection points found by substituting the values of mu into the original equations of the line.

An alternative approach but one that gives the exact same equations is to realise that the shortest line segment between the two lines will be perpendicular to the two lines. This allows us to write two equations for the dot product as

$$(P_a - P_b) \text{ dot } (P_2 - P_1) = 0$$

$$(P_a - P_b) \text{ dot } (P_4 - P_3) = 0$$

Expanding these given the equation of the lines

$$(P_1 - P_3 + mu_a (P_2 - P_1) - mu_b (P_4 - P_3)) \text{ dot } (P_2 - P_1) = 0$$

$$(P_1 - P_3 + mu_a (P_2 - P_1) - mu_b (P_4 - P_3)) \text{ dot } (P_4 - P_3) = 0$$

Expanding these in terms of the coordinates (x,y,z) is a nightmare but the result is as follows

$$d_{1321} + mu_a d_{2121} - mu_b d_{4321} = 0$$

$$d_{1343} + mu_a d_{4321} - mu_b d_{4343} = 0$$

where

$$d_{mnop} = (x_m - x_n)(x_o - x_p) + (y_m - y_n)(y_o - y_p) + (z_m - z_n)(z_o - z_p)$$

Note that $d_{mnop} = d_{opmn}$

Finally, solving for mu_a gives

$$mu_a = (d_{1343} d_{4321} - d_{1321} d_{4343}) / (d_{2121} d_{4343} - d_{4321} d_{4321})$$

and back-substituting gives mu_b

$$mu_b = (d_{1343} + mu_a d_{4321}) / d_{4343}$$

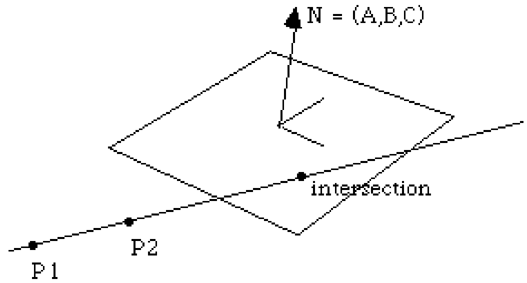
Source code contributions
Original C source code from the author: lineline.c
Contribution by Dan Wills in MEL (Maya Embedded Language): source.mel.
A Matlab version by Cristian Dima: linelineintersect.m.
A Maxscript function by Chris Johnson: LineLineIntersect.ms
LISP version for AutoCAD (and Intellicad) by Andrew Bennett: int1.lsp and int2.lsp
A contribution by Bruce Vaughan in the form of a Python script for the SDS/2 design software: L3D.py
C# version by Ronald Holthuizen: calclineline.cs
VBA VB6 version by Thomas Ludewig: vbavb6.txt
LabView implementation by John van Schaijk: V01_LineLine3D.vi.zip

Intersection of a plane and a line

Written by Paul Bourke
August 1991

Contribution by Bryan Hanson: Implementation in R

This note will illustrate the algorithm for finding the intersection of a line and a plane using two possible formulations for a plane.



Solution 1

The equation of a plane (points **P** are on the plane with normal **N** and point **P3** on the plane) can be written as

$$\mathbf{N} \text{ dot } (\mathbf{P} - \mathbf{P3}) = 0$$

The equation of the line (points **P** on the line passing through points **P1** and **P2**) can be written as

$$\mathbf{P} = \mathbf{P1} + u\,(\mathbf{P2} - \mathbf{P1})$$

The intersection of these two occurs when

$$\mathbf{N} \text{ dot } (\mathbf{P1} + u\,(\mathbf{P2} - \mathbf{P1})) = \mathbf{N} \text{ dot } \mathbf{P3}$$

Solving for u gives

$$u = \frac{\mathbf{N} \text{ dot } (\mathbf{P3} - \mathbf{P1})}{\mathbf{N} \text{ dot } (\mathbf{P2} - \mathbf{P1})}$$

Note

- If the denominator is 0 then the normal to the plane is perpendicular to the line. Thus the line is either parallel to the plane and there are no solutions or the line is on the plane in which case there are an infinite number of solutions
- If it is necessary to determine the intersection of the line segment between **P1** and **P2** then just check that u is between 0 and 1.

Solution 2

A plane can also be represented by the equation

$$A x + B y + C z + D = 0$$

where all points (x,y,z) lie on the plane.

Substituting in the equation of the line through points **P1** (x1,y1,z1) and **P2** (x2,y2,z2)

$$P = P1 + u\,(P2 - P1)$$

gives

$$A (x1 + u (x2 - x1)) + B (y1 + u (y2 - y1)) + C (z1 + u (z2 - z1)) + D = 0$$

Solving for u

$$u = \frac{A\ x1 + B\ y1 + C\ z1 + D}{A\ (x1 - x2) + B\ (y1 - y2) + C\ (z1 - z2)}$$

Note

- the denominator is 0 then the normal to the plane is perpendicular to the line. Thus the line is either parallel to the plane and there are no solutions or the line is on the plane in which case are infinite solutions
- if it is necessary to determine the intersection of the line segment between P1 and P2 then just check that u is between 0 and 1.

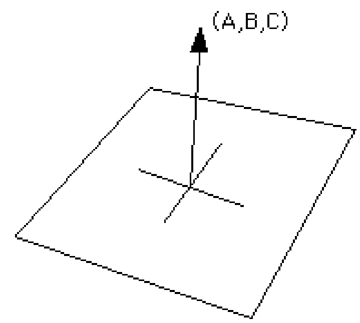
Equation of a plane

Written by Paul Bourke
March 1989

The standard equation of a plane in 3 space is

$$Ax + By + Cz + D = 0$$

The normal to the plane is the vector (A,B,C).



Given three points in space (x1,y1,z1), (x2,y2,z2), (x3,y3,z3) the equation of the plane through these points is given by the following determinants.

$$A = \begin{vmatrix} 1 & y1 & z1 \\ 1 & y2 & z2 \\ 1 & y3 & z3 \end{vmatrix} \quad B = \begin{vmatrix} x1 & 1 & z1 \\ x2 & 1 & z2 \\ x3 & 1 & z3 \end{vmatrix} \quad C = \begin{vmatrix} x1 & y1 & 1 \\ x2 & y2 & 1 \\ x3 & y3 & 1 \end{vmatrix} \quad D = - \begin{vmatrix} x1 & y1 & z1 \\ x2 & y2 & z2 \\ x3 & y3 & z3 \end{vmatrix}$$

Expanding the above gives
$A = y1 (z2 - z3) + y2 (z3 - z1) + y3 (z1 - z2)$
$B = z1 (x2 - x3) + z2 (x3 - x1) + z3 (x1 - x2)$
$C = x1 (y2 - y3) + x2 (y3 - y1) + x3 (y1 - y2)$
$- D = x1 (y2 z3 - y3 z2) + x2 (y3 z1 - y1 z3) + x3 (y1 z2 - y2 z1)$

Note that if the points are collinear then the normal (A,B,C) as calculated above will be (0,0,0).

The sign of $s = Ax + By + Cz + D$ determines which side the point (x,y,z) lies with respect to the plane. If $s > 0$ then the point lies on the same side as the normal (A,B,C). If $s < 0$ then it lies on the opposite side, if $s = 0$ then the point (x,y,z) lies on the plane.

Alternatively

If vector **N** is the normal to the plane then all points **p** on the plane satisfy the following

$\mathbf{N} \cdot \mathbf{p} = k$

where . is the dot product between the two vectors.

ie: $\mathbf{a} \cdot \mathbf{b} = (a_x,a_y,a_z) \cdot (b_x,b_y,b_z) = a_x b_x + a_y b_y + a_z b_z$

Given any point **a** on the plane

$\mathbf{N} \cdot (\mathbf{p} - \mathbf{a}) = 0$

The intersection of two planes

Written by Paul Bourke
February 2000

The intersection of two planes (if they are not parallel) is a line.

Define the two planes with normals N as

$$\mathbf{N_1} \cdot \mathbf{p} = d_1$$

$$\mathbf{N_2} \cdot \mathbf{p} = d_2$$

The equation of the line can be written as

$$\mathbf{p} = c_1 \mathbf{N_1} + c_2 \mathbf{N_2} + u \mathbf{N_1} * \mathbf{N_2}$$

Where "*" is the cross product, "." is the dot product, and u is the parameter of the line.

Taking the dot product of the above with each normal gives two equations with unknowns c_1 and c_2.

$$\mathbf{N_1} \cdot \mathbf{p} = d_1 = c_1 \mathbf{N_1} \cdot \mathbf{N_1} + c_2 \mathbf{N_1} \cdot \mathbf{N_2}$$

$$\mathbf{N_2} \cdot \mathbf{p} = d_2 = c_1 \mathbf{N_1} \cdot \mathbf{N_2} + c_2 \mathbf{N_2} \cdot \mathbf{N_2}$$

Solving for c_1 and c_2

$$c_1 = (d_1 \mathbf{N_2} \cdot \mathbf{N_2} - d_2 \mathbf{N_1} \cdot \mathbf{N_2}) / \text{determinant}$$

$$c_2 = (d_2 \mathbf{N_1} \cdot \mathbf{N_1} - d_1 \mathbf{N_1} \cdot \mathbf{N_2}) / \text{determinant}$$

$$\text{determinant} = (\mathbf{N_1} . \mathbf{N_1}) (\mathbf{N_2} . \mathbf{N_2}) - (\mathbf{N_1} . \mathbf{N_2})^2$$

Note that a test should first be performed to check that the planes aren't parallel or coincident (also parallel), this is most easily achieved by checking that the cross product of the two normals isn't zero. The planes are parallel if

$$\mathbf{N_1} * \mathbf{N_2} = 0$$

Intersection of three planes

Written by Paul Bourke
October 2001

A contribution by Bruce Vaughan in the form of a Python script for the SDS/2 design software: P3D.py.

The intersection of three planes is either a point, a line, or there is no intersection (any two of the planes are parallel).

The three planes can be written as

$$\mathbf{N_1} . \mathbf{p} = d_1$$

$$\mathbf{N_2} . \mathbf{p} = d_2$$

$$\mathbf{N_3} . \mathbf{p} = d_3$$

In the above and what follows, "." signifies the dot product and "*" is the cross product. The intersection point **P** is given by:

$$\mathbf{P} = \frac{d_1 (\mathbf{N}_2 * \mathbf{N}_3) + d_2 (\mathbf{N}_3 * \mathbf{N}_1) + d_3 (\mathbf{N}_1 * \mathbf{N}_2)}{\mathbf{N}_1 . (\mathbf{N}_2 * \mathbf{N}_3)}$$

The denominator is zero if $\mathbf{N}_2 * \mathbf{N}_3 = 0$, in other words the planes are parallel. Or if $\mathbf{N}_1$ is a linear combination of $\mathbf{N}_2$ and $\mathbf{N}_3$.

Equation of a line in polar coordinates

Written by Paul Bourke
March 2013

How might one represent a line in polar coordinates?
In Cartesian coordinates it can be represented as:

$$y = mx + c$$

The derivation is quite straightforward once one realises that for a point (r, theta) the x axis is simply r cos(theta) and the y axis is r sin(theta). Substituting those into the equation for the line gives the following result.

$$r = \frac{c}{\sin(\theta) - m\cos(\theta)}$$

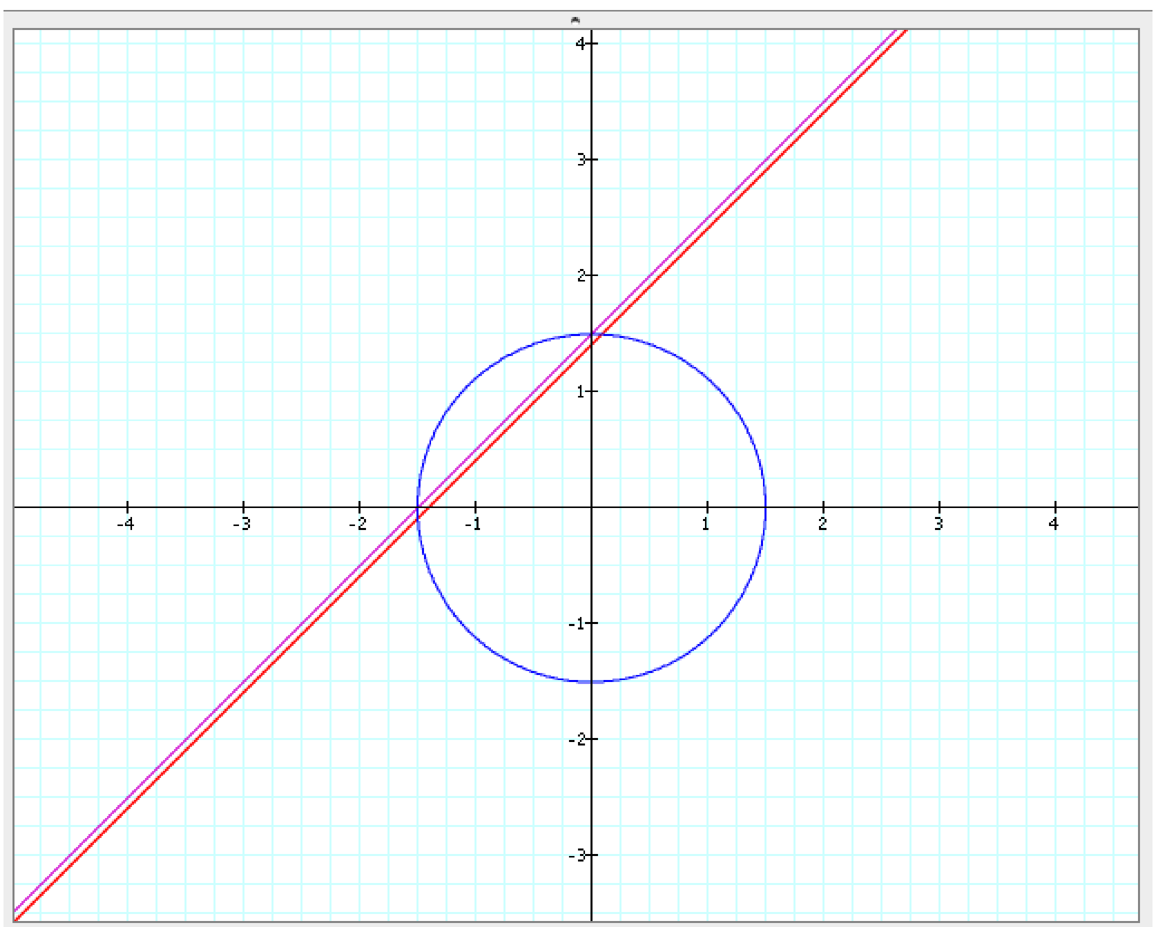
Example from the Graphing Calculator.

$$r = \frac{c}{\sin(\theta) - m\cos(\theta)}$$

$$c = 1.5, m = 1$$

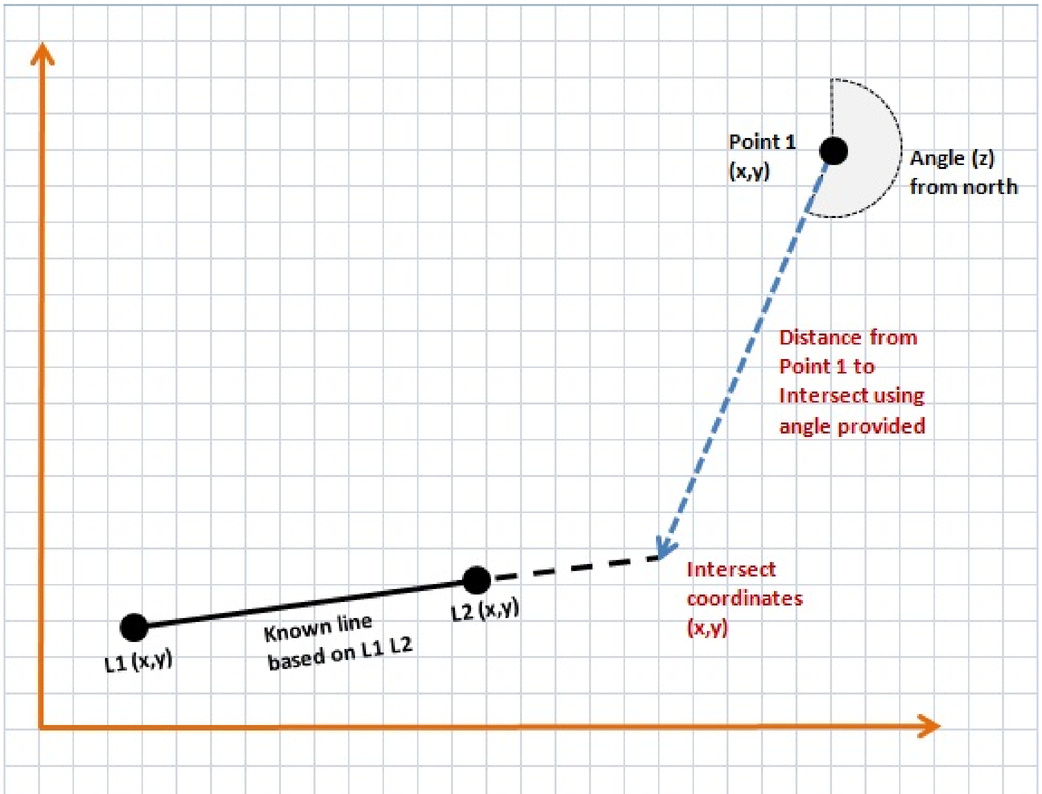
$$y = mx + (c - 0.1)$$

$$r = 1.5$$



Question

Given a line defined by two points L1 L2, a point P1 and angle z (bearing from north) find the intersection point between the direction vector from P1 to the line.



Short answer: choose a second point P2 along the direction vector from P1, say P2 = (x_{P1}+sin(z),y_{P1}+cos(z)). Apply the algorthm here for the intersection of two line segments. Perform the additional test that u_b must be greater than 0, the solution where u_b is less than 0 is the solution in the direction z+180 degrees.